Flux Compactifications and Feynman Integrals - Calabi-Yau Geometries in Physics

Dissertation
zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften

am Fachbereich Physik, Mathematik und Informatik
der Johannes Gutenberg-Universität
in Mainz

vorgelegt von
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geboren in Lübbecke

Mainz, den 19.07.2024

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Datum der mündlichen Prüfung: 26.09.2024

Abstract

Calabi-Yau manifolds define fascinating geometric structures that find several applications in theoretical physics. Naturally, this special class of manifolds appears in the context of compactifications of superstring, M- and F-theory.

This thesis begins with an examination of the existence of supersymmetric flux vacua in type IIB string and M-theory compactifications on a Calabi-Yau manifold. The vacuum conditions for non-trivial background fluxes on the compactification space are conveniently formulated in terms of $\mathcal{N} = 1$ supergravity. For type IIB string compactifications on a Calabi-Yau threefold we provide an equivalent description for the $\mathcal{N} = 1$ flux vacuum constraints in terms of Minkowski vacua of an associated gauged $\mathcal{N} = 2$ supergravity theory. These describe vacua in the landscape of UV consistent effective field theories with partially spontaneously broken supersymmetry.

The existence of supersymmetric flux vacua is related to an arithmetic property of the underlying Calabi-Yau manifold which is called modularity. Generalizing existing methods for Calabi-Yau threefolds, we derive an algorithm, which provides a systematic search for modular points on the corresponding complex structure moduli space of certain types of Calabi-Yau fourfolds with one complex structure modulus. Compactifying M-theory on such modular Calabi-Yau fourfolds may lead to non-trivial supersymmetric flux vacua. We demonstrate the application of this method for several examples. Most interestingly, we identify a modular Calabi-Yau fourfold within the family of Hulek-Verrill fourfolds and verify this observation by several independent consistency checks.

Furthermore, Calabi-Yau geometries appear prominently in the framework of multi-loop Feynman integrals. It is well-known that many multi-loop Feynman integrals can be realized in terms of period integrals of certain algebraic varieties such as Calabi-Yau manifolds and hyperelliptic curves. Using the construction of intermediate Jacobians, we derive a correspondence between families of Calabi-Yau threefolds and suitable families of genus-g curves that realize the same family of Feynman integrals. As an explicit example, we discuss this Calabi-Yau-to-curve correspondence for the four-loop equal mass banana integral which is realized by a one-parameter family of Hulek-Verrill Calabi-Yau threefolds.

Statement of Originality

Except where otherwise stated, the work presented in this thesis is a product of the following collaborations to which the author contributed substantially:

[1] **“Modular Calabi-Yau Fourfolds and Connections to M-Theory Fluxes“** by H. Jockers, S. Kotlewski, P. Kuusela, arXiv:2312.07611[hep-th]

[2] **“A Calabi-Yau-to-Curve Correspondence for Feynman Integrals“** by H. Jockers, S. Kotlewski, P. Kuusela, A. J. McLeod, S. Pögel, M. Sarve, X. Wang, S. Weinzierl, arXiv:2404.05785[hep-th]

[3] **“On the Geometry of $N = 2$ Minkowski Vacua of Gauged $N = 2$ Supergravity Theories in Four Dimensions“** by H. Jockers, S. Kotlewski, arXiv:2404.11655[hep-th], to appear in Physical Review D.

Any ideas, data, images or text resulting from the work of others are clearly identified as such within the work and attributed to the authors in the text or bibliography. This thesis has not been submitted for any other academic degree or professional qualification.

Contents

Chapter 1

Introduction

1.1 Motivation

The investigation of mathematical structures has always been essential for the understanding of the theoretical origin of physical phenomena. Most important, the framework of differential geometry has influenced the development of theoretical physics in its full spectrum from classical mechanics to modern physics in various fields. In this work, we focus on the application of Calabi-Yau geometries in physics and discuss, how the properties of this special class of manifolds influence the corresponding physical models.

Very prominent, Calabi-Yau manifolds appear in the context of compactifications of superstring, M- and F-theory. These theories, which provide a proposal of the fundamental constituents of high energy physics, are well-defined only in ten, eleven or twelve spacetime dimensions respectively. In order to make contact with observations in lower dimensional spacetimes, one obtains an effective theory by compactifying the additional dimensions on a compact geometry. For superstring compactifications it is convenient to choose a complex three-dimensional Calabi-Yau manifold in order to obtain an effective four-dimensional theory whereas M- and F-theory are conveniently compactified on complex four-dimensional Calabi-Yau manifolds which yields a three-dimensional and a four-dimensional effective field theory respectively.

The phenomenological models built from such Calabi-Yau compactifications are crucially dependent on the chosen Calabi-Yau geometry. In particular, these models lead usually to the problem of “moduli stabilization“ meaning that the moduli of the compactification space may vary while the theory evolves. In the effective theory, these dynamical moduli would be observable as light or even massless modes of the spectrum which have not been observed in any accelerator experiment. One mechanism to solve this issue is proposed by introducing non-trivial background fluxes on the internal Calabi-Yau manifold. Doing so, the moduli fields admit mass terms depending on the background fluxes which stabilize them to their vacuum expectation value in the low energy limit. For a given Calabi-Yau compactification it is a non-trivial problem to ask whether the internal geometry admits a non-trivial background flux F that gives rise to a stable vacuum of the effective theory.

This search for a non-trivial *supersymmetric flux vacua* of string, M- or F-theory compactifications can be translated into a Hodge theoretical problem for the compactification space. For type IIB string theory compactified on a Calabi-Yau threefold X_3, there need to be two independent background fluxes F, H which are given by elements of the integral middle cohomology $H^3(X_3, \mathbb{Z})$ of the Calabi-Yau manifold. Moreover, the vacuum constraints restrict F and H to have no purely holomorphic or anti-holomorphic contribution if we treat $H^3(X_3, \mathbb{Z})$ as a subset of the complex cohomology $H^3(X_3, \mathbb{C})$. Thus, the flux vacuum conditions imply that

$$F, H \in \left(H^{2,1}(X_3, \mathbb{C}) \oplus H^{1,2}(X_3, \mathbb{C})\right) \cap H^3(X_3, \mathbb{Z}) \tag{1.1}$$

generate a two-dimensional sublattice of $H^3(X, \mathbb{Z})$ that has a definite Hodge type. While each of the subsets in equation (1.1) is simple to construct, it is in general difficult to decide whether this intersection is trivial or not. For M- and F-theory flux compactifications on a Calabi-Yau fourfold X_4, the corresponding flux vector G needs to obey a similar vacuum condition which in this case is given by

$$G \in \left(H^{4,0}(X_4, \mathbb{C}) \oplus H^{2,2}(X_4, \mathbb{C}) \oplus H^{0,4}(X_4, \mathbb{C})\right) \cap H^3(X_4, \mathbb{Z}) \ . \tag{1.2}$$

Thus, a Calabi-Yau fourfold gives rise to a non-trivial flux compactification of M- or F-theory only if its integral middle cohomology admits a non-trivial sublattice that is of the given definite Hodge type.

In the language of algebraic geometry, these conditions are formulated equivalently by the question whether the Hodge structure of the rational middle cohomology $H^n(X_n, \mathbb{Q})$ is simple or whether it splits into a sum of two Hodge substructures. If one of these substructures realizes a two-dimensional subspace of $H^n(X_n, \mathbb{Q})$, the Calabi-Yau variety is said to be modular as this subspace of $H^n(X_n, \mathbb{Q})$ is characterized by a certain modular form. Beyond the cases of elliptic curves and rigid Calabi-Yau threefolds which have been proven to be modular, only few examples of modular Calabi-Yau manifolds have been identified so far. Recently, a method has been developed to systematically search for algebraic points on the complex structure moduli space of a given family of Calabi-Yau threefolds that correspond to modular manifolds [4–6]. By means of this analysis, it was possible to identify a rank-two attractor point on the family of Hulek-Verrill threefolds [7] and consistent flux configurations for type IIB string compactifications.

In this work, we extend these techniques to search for modular Calabi-Yau manifolds of complex dimension four which may serve as candidates for a consistent flux compactification of M- or F-theory. Beside presenting the abstract method which relys on tools from arithmetic geometry that are introduced in detail, we present its application to families of Calabi-Yau fourfolds that depend on one complex structure modulus. Most important, we identify a candidate for modularity within the family of Hulek-Verrill fourfolds that are defined in analogy to their three-dimensional cousins [8]. In order to verify that the rational point on the complex structure moduli space, which has been identified with the developed arithmetic techniques, corresponds indeed to a modular fourfold, we perform several consistency checks that are completely independent of the tools and assumptions we used in the first place. In particular, we manage to identify uniquely the corresponding modular form characterizing the split of the Hodge structure and deduce from its critical L-function values the explicit generators for the two-dimensional sublattice of $H^4(X_4, \mathbb{Z})$ that is of Hodge type $(3,1)+(1,3)$. Due to the special structure of the primary horizontal middle cohomology for this type of fourfolds, the existence of this sublattice is sufficient to argue that $H^4(X_4, \mathbb{Z})$ contains integral four-forms of Hodge type $(4,0)+(2,2)+(0,4)$ as well which suit as consistent fluxes for M- or F-theory compactifications. By direct computation, we identify one of these fluxes explicitly.

In addition to their importance for the construction of supersymmetric string compactifications, Calabi-Yau geometries recently showed up in the context of high precision multi-loop Feynman integral computations. Using the modern techniques of deriving differential equations for multi-loop Feynman integrals, it turns out that these can often be identified with period integrals of certain geometric objects. Among many additional examples, the identification of the two-loop banana integral with the period of a certain elliptic curve has pointed to this relation between Feynman integrals and geometry. Extending this observation even to higher loop integrals leads to period integrals of more complicated geometrical objects such as hyperelliptic curves or Calabi-Yau manifolds. It has been observed that some Feynman integrals enjoy both a representation in terms of period integrals of Calabi-Yau manifolds one in terms of hyperelliptic curves.

Analyzing this observation in more detail, we prove that this relation between the periods of certain Calabi-Yau threefolds and genus-g curves is not coincidential but can be formulated in terms of a "Calabi-Yau-to-curve correspondence" that realizes a holomorphic map connecting the moduli spaces of Calabi-Yau threefolds with the moduli space of stable genus-g curves. Applying this correspondence to the four-loop equal mass banana integral which is known to be represented by a suitable one-parameter family of Hulek-Verrill threefolds, we construct the corresponding family of hyperelliptic genus-two curves describing the same Feynman integral.

1.2 Outline

The first two chapters of this thesis are relegated to introduce the most important concepts from superstring theory and the algebraic geometry of Calabi-Yau manifolds to equip the reader with the relevant background knowledge which is necessary to follow the discussions of this work. Since our focus lies on the connection between Calabi-Yau geometries and physics, the physical introduction to superstring and M-theory in chapter 2 is restricted to mainly discuss the concept of compactification and its implications on the low energy spectrum of these theories. More extensively, we dicuss in chapter 3 the geometrical properties of Calabi-Yau n-folds and their moduli spaces.

The main part of this thesis is separated in three different research projects [1–3] which are discussed in chapter 4 - 6 respectively. In chapter 4 we introduce the idea of flux compactification for both, string and M-theory compactifications, highlighting its application to stabilize the moduli of the effective theories and discussing the explicit vacuum constraints on the flux vectors. We introduce the flux superpotential $W(\mathbf{z})$ which has been derived in the framework of $\mathcal{N} = 1$ supergravity [9]. While suitable for the low energy description of M-theory compactifications, type II string compactifications on a Calabi-Yau threefold give rise to an $\mathcal{N} = 2$ supergravity theory which cannot be equipped with such a superpotential. Following the discussions which we provide in sections 4.2 and 4.3, it is possible to reproduce similar vacuum constraints in $\mathcal{N} = 2$ supergravity by using the mechanism of gauging certain isometries on the target space of the theory. The choice of gauge charges for the scalar fields can be identified with the flux vectors from the flux compactification proposal and need to obey similar consistency conditions in order to give rise to a supersymmetric Minkowski vacuum of the gauged $\mathcal{N} = 2$ supergravity theory.

After discussing the constraints on the flux vectors for a consistent supersymmetric flux vacuum of M-theory compactifications in section 4.4, we turn to the arithmetic search for Calabi-Yau fourfolds which admit non-trivial four-form fluxes in chapter 5. To that end, we start by setting up the arithmetic toolbox for analyzing algebraic varieties over finite fields which in part is given by the local zeta function $\zeta_p(X, T)$ and its relation to the lifted Frobenius map on suitable p-adic cohomology groups. Moreover, section 5.2 provides a brief introduction to the phenomenon of modularity, revising Serre's modularity conjecture, discussing its application for Calabi-Yau fourfolds whose middle cohomology splits and possible implications for physical systems. After this rather abstract discussion, we continue with the derivation of the practical evaluation of the relevant parts of the local zeta function for families of Calabi-Yau fourfolds following a similar procedure as pioneered in [4, 7] for threefolds, focussing in section 5.4 on families of Calabi-Yau fourfolds that depend only on one complex structure modulus.

The remaing sections of chapter 5 are relegated to the presentation of the application of this method to explicit examples. As our first example, we discuss extensively the family of Hulek-Verrill fourfolds $\mathrm{HV}^4_{\mathbf{z}}$ which arises as a generalization of their three-dimensional cousins that have been introduced in [8]. Applying our method to compute the relevant pieces of the zeta function for a suitable one-dimensional subfamily of these, a candidate for a modular Calabi-Yau fourfold is found. In section 5.5.4, we discuss several independent consistency checks that verify the split of the primary horizontal middle cohomology of this manifold by using different techniques. For the additional families of Calabi-Yau fourfolds which are presented in section 5.6, our deformation method suggests that these do not admit any modular member.

With chapter 6 we dive into another physical application for Calabi-Yau geometries provided by the high precision computation of Feynman integrals. Starting with a brief introduction of the method of differential equations which provides an almost algorithmic way to compute Feynman integrals in terms of iterated integrals, we discuss that many multi-loop Feynman integrals enjoy a realization in terms of periods integrals of certain Calabi-Yau manifolds or hyperelliptic curves. For some examples, even both geometries give a valid description for the Feynman integral. Providing the necessary mathematical background on stable genus-g curves and intermediate Jacobians of algebraic varieties, we derive in section 6.4 an explicit correspondence between Calabi-Yau threefolds and genus-g curves which is defined such that their period integrals agree. To be more precise, we differentiate between two types of correspondences, one providing a real analytic bijection between suitable subsets of the corresponding complex structure moduli spaces and a second which is holomorphic but gives a proper identification of the period integrals only on a Lagrangian submanifold of the Calabi-Yau moduli space. Section 6.5 closes the discussion of the Calabi-Yau-to-curve correspondence by providing an explicit example given by the geometric realization of the four-loop equal mass banana integral.

In addition to the main text, this work is extended by five appendices. The first two of them review important mathematical concepts such as a brief introduction to p-adic numbers which is necessary for the construction of the local zeta function in chapter 5 and a review of the toric contstruction of algebraic varieties that are embedded in projective space. Appendices C and D contain the derivations of algorithms which provide an efficient computation of the geometric data which is used for the applications of our developed methods to explicit examples. Finally, appendix E collects tables of the polynomials $R_H(X,T)$ that appear in the local zeta function of the four families of Calabi-Yau fourfolds which are discussed in sections 5.5 and 5.6.

1.3 Notations and Conventions

Except where otherwise stated, we use the following notations and conventions. Vector-valued quantities are usually denoted by a bold-face symbol $\mathbf{v}$ whereas the corresponding entries are given by the corresponding normal-script symbols including an (upper) index v^i. We often use the Einstein summation convention imposing that we implicitly sum over upper and lower indices that appear with the same label. Usually, we denote by greek letters $\mu, \nu, \ldots = 0, 1, \ldots$ Lorentzian indices of a D-dimensional spacetime whereas latin indices $i, j, \ldots = 1, 2, \ldots$ denote the indices of the internal Riemannian manifolds. On a complex manifold, we denote by $i, j, \ldots$ the indices corresponding to holomorphic coordinates whereas $\bar{i}, \bar{j}, \ldots$ represent the corresponding anti-holomorphic directions.

In the following, we summarize the most important defintions and notations which appear frequently throughout this thesis.

Symbol	**Definition/Description**
$h^{p,q}$	The Hodge numbers $h^{p,q} := \dim_{\mathbb{C}}(H^{p,q}(X,\mathbb{C})$ of a Calabi-Yau n-fold X. $H^{p,q}(X,\mathbb{C})$ are the Dolbeault cohomology groups of X.
$X_{\mathbf{z}}$	A Calabi-Yau n-fold which is characterized by its complex structure moduli $\mathbf{z} = (z^1, \ldots, z^{h^{n-1,1}})$. Within this work we restrict mainly to the cases $n = 3$ and $n = 4$.
$X_{\mathbf{t}}$	A Calabi-Yau n-fold which is characterized by its Kähler moduli $\mathbf{t} = (t^1, \ldots, t^{h^{1,1}})$.
$\mathcal{M}$	The moduli space parametrizing a family of Calabi-Yau n-folds. Locally, $\mathcal{M}$ decomposes into the complex structure moduli space $\mathcal{M}_{C.S.}$ and the Kähler moduli space $\mathcal{M}_K$.
Ω	The nowhere vanishing holomorphic n-form of a Calabi-Yau n-fold.
$\Pi^a(\mathbf{z})$	The periods of the n-fold $X_{\mathbf{z}}$ with respect to an integral basis of $H^n(X_{\mathbf{z}},\mathbb{Z})$.
$\varpi^a(\mathbf{z})$	The periods of the n-fold $X_{\mathbf{z}}$ with respect to the Frobenius basis of $H^n(X_{\mathbf{z}},\mathbb{C})$.
$(X^a(\mathbf{z}), F_a(\mathbf{z}))$	The periods of the Calabi-Yau threefold $X_{\mathbf{z}}$ with respect to an symplectic integral basis of $H^3(X_{\mathbf{z}},\mathbb{Z})$.
$H^n_H(X_{\mathbf{z}},\mathbb{C})$	The primary horizontal subspace of the middle cohomology $H^n(X_{\mathbf{z}},\mathbb{C})$ of a Calabi-Yau n-fold $X_{\mathbf{z}}$.
$\mathcal{P}_\Lambda$	The triplet of Killing prepotentials corresponding to an isometry $\tilde{k}^u_\Lambda \partial_u$ of a quaternionic Kähler manifold.
$\zeta_p(X,T)$	The local zeta function of an algebraic variety X.
Fr_p	The Frobenius map $\mathrm{Fr}_p : H^k(X,\mathbb{Q}_p) \to H^k(X,\mathbb{Q}_p)$ acting on p-adic cohomology groups $H^k(X,\mathbb{Q}_p)$.
$U_p(\mathbf{z})$	The inverse of the matrix representation of Fr_p.
$R_k(X,T)$	The characteristic polynomial of $\mathrm{Fr}_p : H^k(X,\mathbb{Q}_p) \to H^k(X,\mathbb{Q}_p)$ which appears as a factor in $\zeta_p(X,T)$.
$R_H(X,T)$	The factor of $R_n(X,T)$ originating from the restricted Frobenius map to the primary horizontal subspace of the middle cohomology of the Calabi-Yau n-fold X.
Teich_p	The Teichmüller lift $\mathrm{Teich}_p : \mathbb{F}_p \hookrightarrow \mathbb{Z}_p$.
$E(\mathbf{z})$	The matrix containing the Frobenius periods $\varpi^a(\mathbf{z})$ and their derivatives $\Theta^j \varpi^a(\mathbf{z})$ of a Calabi-Yau n-fold $X_{\mathbf{z}}$.

Symbol	Definition/Description
$\mathrm{HV}^n_{\mathbf{z}}$	The family of Hulek-Verrill n-folds parametrized by $(n+2)$ complex structure moduli $\mathbf{z}$.
$\mathrm{H}\Lambda^n_{\mathbf{t}}$	The mirror of $\mathrm{HV}^n_{\mathbf{z}}$ paramterized by $(n+2)$ Kähler moduli $\mathbf{t}$.
HV^n_z	The one-dimensional subfamily of Hulek-Verrill n-folds which are invariant under the $\mathbb{Z}_{n+2}$-action.
$I_{\nu_1,\dots,\nu_\rho,b_i,\dots,b_\kappa}$	A family of Feynman integrals which are characterized by ρ propagators D_i, each raised to power ν_i, and κ scalar products S_j, each raised to power b_j.
$\mathcal{C}_g$	A genus-g curve.
$\bar{\mathcal{M}}_g$	The moduli space of stable genus-g curves.
$\mathcal{X}^i_k$	The A-periods of a genus-g curve.
$\mathcal{T}_{ik}$	The B-periods of a genus-g curve.
$J^1(\mathcal{C}_g)$	The first intermediate Jacobian of a stable genus-g curve $\mathcal{C}_g$.
$\mathcal{A}_g$	The moduli space of g-dimensional abelian varieties.
τ	The matrix $\tau = \mathcal{T}\mathcal{X}^{-1}$ characterizing $J^1(\mathcal{C}_g)$.
$J^2_G(X)$	The second Griffiths intermediate Jacobian of a Calabi-Yau threefold X.
$J^2_W(X)$	The second Weil intermediate Jacobian of a Calabi-Yau threefold X.
$J^2_{\Delta_\mathbb{R}}(X)$	The second polarized holomorphic intermediate Jacobian of a Calabi-Yau threefold X. This Jacobian is well-defined locally in the vicinity of a Lagrangian submanifold $\Delta_\mathbb{R}$
F_{ab}	The matrix containing all second derivatives of the prepotential $F(\mathbf{X})$ of a Calabi-Yau threefold X parametrized by the affine coordinates $\mathbf{X}$. This matrix characterizes the second Griffiths intermediate Jacobian $J^2_G(X)$.
N_{ab}	The matrix characterizing the second Weil intermediate Jacobian $J^2_W(X)$ of a Calabi-Yau threefold X.
H_{ab}	The matrix characterizing the second polarized holomorphic intermediate Jacobian $J^2_{\Delta_\mathbb{R}}(X)$ of a Calabi-Yau threefold X in the vicinity of a Lagrangian submanifold $\Delta_\mathbb{R}$.
$\mathcal{S}_g$	The Schottky-locus of genus-g curves.
$\Phi_\mathbb{R}$	The real analytic Calabi-Yau-to-curve correspondence
$\Phi^U_{\Delta_\mathbb{R}}$	The local holomorphic Calabi-Yau-to-curve correspondence
ϑ	The Riemann theta function

Chapter 2

String Theory in a Nutshell

The framework of string theory, first discussed in the 1960th as an alternative approach to Quantum Chromodynamics for the description of the strong nuclear forces, developed over the last decades to become one of the most important candidates for a consistent unification theory that combines the quantum physics of the standard model of particle physics with the macroscale physics of gravity given by Einstein's general relativity [10]. Since comprehensive introductions to string theory fill whole textbooks (e.g. [10–12]), this introductory chapter is meant to give a qualitative overview on the main concepts of string theory, including the unification of quantum field theory and gravity, the necessity of additional spacetime dimensions and the idea of string compactifications. Moreover, we introduce the concept of M-theory and its relation to string theory.

2.1 String Theory as a Grand Unification Theory

In contrast to oridinary quantum field theories which are characterized by point particles, the fundamental objects in string theory are one-dimensional extended objects which are called strings. The classical dynamics of the string is determined by minimizing the volume of a (two-dimensional) worldsheet Σ that is spanned by the string evolving in a D-dimensional spacetime. For a pure bosonic string, the quantized equations of motion lead to excitation modes for the string with squared mass proportional to the oscillation number N

$$m^2 = 4\pi T(N - a) \ . \tag{2.1}$$

The string tension T is the only tunable parameter of string theory and is typically set to be of order $T \sim (1/M_{pl})^2$. Here M_{pl} denotes the Planck mass[1]. Moreover, the (positive) integer a appears to be the zero-mode mass and can be renormalized to be $a = 1$. Since the zero-mode string ($N = 0$) leads to a tachyon, i.e. a state of negative mass, such a theory might be considered to be non-physical, however if we include fermionic degrees of freedom in a supersymmetric fashion, these non-physical states of negative mass can be projected

[1] In terms of fundamental physical constants, M_{pl} can be expressed as $M_{pl} = \sqrt{\frac{\hbar c}{G}}$.

out. This procedure is well-known in the string literature as the GSO-projection [13]. Hence, if we consider a superstring theory, the state spectrum contains massless states and additional towers of massive fields with mass of order M_{pl}. Since this mass scale is far beyond any probing regime of current particle accelerators, one considers only the massless string modes to be part of the physical spectrum. This can be achieved by defining an effective field theory on the revelant energy scale.

The main feature of this massless spectrum is that it contains vector bosons as massless modes of the open string and gravitons[2] as massless modes of the closed string. By this means, string theory naturally combines gauge interactions and gravity as different oscillation modes of the same object, the fundamental string.

So far, we have left the target space dimension D to be undetermined. As it turns out, it is not possible to tune the spacetime dimension of string theory as an input but instead it is constrained by consistency conditions from Lorentz invariance. In particular, superstring theory contains a conformal anomaly which is canceled only if we restrict the spacetime dimension to be $D = 10$ [11].

For a classification of superstring theories [10, 11] one should start by distinguishing between two fundamentally different types of strings, the open string and the closed string. For the latter, the topology of the worldsheet Σ is periodic in the space-like direction implying that there is a non-trivial *level-matching*[3] of left- and rightmoving excitation modes for the closed string, whereas these excitations for the open string remain unconstrained.

If we focus in the following on superstring theories on a ten-dimensional spacetime, the level-matching condition for the closed string allows for two different kinds GSO-projections that deviate in a different relative chirality between the left- and right-moving excitations. Hence, depending on the choice of GSO-projection, one obtains two different theories, denoted by type IIA and type IIB string theory, both describing a supersymmetric closed string. Since the open string is free of any level-matching condition, both parts of the excitation spectrum are completely independent of each other, hence a relative phase between left- and right-moving excitation leads to an equivalent theory. Consequently, the open superstring is described by a unique theory, called type I string theory, regardless of the explicit choice of GSO-projection. It should be noted that type I string theory comes always with an inherit $\mathrm{Spin}(32)/\mathbb{Z}_2$ gauge group [14] .

In addition to these three types of superstring theories, there exist additional, so-called heterotic, string theories that combine a left-moving sector of a superstring with the right-moving sector of a non-supersymmetric bosonic string. Since the bosonic string is extremal

[2] At this stage of the analysis a graviton is defined to be a bosonic massless mode of spin-two in the spectrum of the superstring theory.

[3] That means, for any left-moving excitation there must be a corresponding right-moving excitation of the same level.

in $\tilde{d} = 26$ spacetime dimensions, this rightmoving sector is strictly speaking given by the 10-dimensional compactification of a bosonic string on a 16-dimensional internal manifold. In order to obtain a consistent string theory, the internal manifold needs to be described by a torus that is characterized either by the Lie algebra $\mathfrak{e}_8 \times \mathfrak{e}_8$ or the Lie algebra $\mathfrak{so}(32)$. This combination of superstring and compactified bosonic string leads to a local Yang-Mills gauge symmetry whose gauge group corresponds to that of the underlying Lie algebra of the internal space. Following [14], the spin structure of the superstring implies that the latter case of Lie algebra $\mathfrak{so}(32)$ gives rise to the corresponding gauge group $\mathrm{Spin}(32)/\mathbb{Z}_2$.

Table 2.1 summarizes the most important properties, such as the supersymmetry algebra and the gauge group of these five different types of superstring theories. It should be noted that these exhaust already the full spectrum of supersymmetric string theories in ten spacetime dimensions.

string theory	string type	SUSY algbebra	Gauge group
Type I	open	$\mathcal{N} = 1$	$\mathrm{Spin}(32)/\mathbb{Z}_2$
Type IIA	closed	$\mathcal{N} = 2$	–
Type IIB	closed	$\mathcal{N} = 2$	–
Heterotic $E_8 \times E_8$	heterotic	$\mathcal{N} = 1$	$E_8 \times E_8$
Heterotic $SO(32)$	heterotic	$\mathcal{N} = 1$	$\mathrm{Spin}(32)/\mathbb{Z}_2$

Table 2.1: An overview on the five superstring theories in ten spacetime dimensions. The heterotic string theory with gauge group $\mathrm{Spin}(32)/\mathbb{Z}_2$ is historically called "heteroric $SO(32)$" since the corresponding Lie algebra is isomorphic to $\mathfrak{so}(32)$.

2.1.1 Spectrum of Type II String Theories

Throughout this thesis, we are interested in string theories that restore a maximal amount of supersymmetry, hence the focus of this work is on the type II superstring.

The massless type IIA and type IIB string spectra are obtained by combining the correct representations of left- and rightmoving string modes that have an opposite or aligned chirality respectively. Since the fermionic fields on both sectors are characterized by two different kinds of boundary conditions, the Ramond (R) and Neveu-Schwarz (NS) boundary conditions[4], there appear several different combinations which give rise to a rich spectrum for the closed string. In the following, we collect the result for the spectra of both, type IIA and type IIB string theory.

For the type IIA string, the bosonic massless sector is given by [10]

$$\tilde{b}^i_{-1/2}|0\rangle_{NS} \otimes b^j_{-1/2}|0\rangle_{NS} \oplus |+\rangle_R \otimes |-\rangle_R \tag{2.2}$$

[4]For a closed string, there are two separate choices $\psi(\sigma) = \pm\psi(\sigma+\pi)$ for the periodicity of the fermionic fields on the worldsheet. The positive sign belongs to the Ramond sector whereas the negative sign is known to be the Neveu-Schwarz boundary condition.

with $\tilde{b}^i_k$ and b^i_k denoting the raising operators of a level k fermionic excitation in the Neveu-Schwarz-sector. Moreover, $|+\rangle_R$ denotes an eight-component ground state spinor of the Ramond-sector that is of positive chirality. Decomposing these representations into irreducible one, we obtain in total 128 bosonic fields that can be summarized as follows.

- The representation of the NS-NS sector decomposes into an symmetric traceless two-form field G that is identified with the graviton, an antisymmetric two-form field B and one scalar dilaton field ϕ.
- The R-R sector decomposes into a one-form field C_1 and a three-form field C_3.

In addition, the spectrum obtains contributions from an NS-R sector and an R-NS sector. Both give rise to a spin-1/2 dilatino and a spin-3/2 gravitino wich just lead to the completion of the $\mathcal{N} = 2$ supersymmetry multiplets.

The type IIB string spectrum is given by a similar analysis. Deviations in contrast to the type IIA spectrum occur only in the R-R sector since the coupling of spinors with aligned chirality give rise to a different decomposition in terms of irreducible representations. In analogy to the type IIA spectrum, the bosonic modes are given by [10]

$$\tilde{b}^i_{-1/2}|0\rangle_{NS} \otimes b^j_{-1/2}|0\rangle_{NS} \oplus |+\rangle_R \otimes |+\rangle_R \ . \tag{2.3}$$

The decomposition into irreducible representations leads to the following fields.

- As for type IIA strings, the representation of the NS-NS sector decomposes into a symmetric traceless graviton G, an antisymmetric two-form field B and one scalar dilaton field ϕ.
- Due to the aligned chirality, the R-R sector decomposes into a scalar C_0, a two-form field C_2 and a four-form field C_4 which is subject to the additional self-duality constraint $dC_4 = \star dC_4$.

In the fermionic sector, the difference between type IIA and type IIB string theories is given by the fact that both, dilatini and gravitini have the same chirality for type IIB string theory whereas the two copies are of opposite chirality for type IIA string theory.

2.1.2 Type II Supergravity

So far, we considered only the massless spectrum of type II string theories. These are completed by infinite towers of massive states which need to be taken into account if we investigate full string theory at any energy scale. Since the typical mass scale of string theory is set to be of order M_{pl}, it is reasonable, to discuss the low energy limit of string theory that is given by integrating out all massive modes of the string spectrum. These effective field theories turn out to become supergravity theories [10, 15], that are supersymmetric quantum field theories, containing a gravity multiplet. In this notation, a gravity multiplet is defined to be a multiplet of the supersymmetry algebra, that contains a spin-2 boson, the graviton, the corresponding spin-3/2 gravitini and a scalar dilaton field.

Comparing this required field content with the massless excitations from the NS-NS sector and the NS-R and R-NS sectors, we find that type IIA and type IIB string theories naturally contain a gravity multiplet within its field content leading to an $\mathcal{N} = 2$ supergravity theory since there is always a double copy of the gravitino. Since supersymmetry is highly restrictive, the supergravity action of type IIA and type IIB string theory is entirely fixed by the given bosonic field content which we discussed in the previous section. For type IIA string theory, the bosonic part of the low energy supergravity action expanded up to second order in derivatives of the fields[5] reads [16, 17]

$$\begin{aligned} S_{\mathrm{IIA}} = \frac{1}{2} \int e^{-2\phi} \left(-R(G) \star 1 + 4 d\phi \wedge \star d\phi - \frac{1}{2} H_3 \wedge \star H_3 \right) \\ - \frac{1}{2} \int \left(F_2 \wedge \star F_2 + F_4 \wedge \star F_4 - H_3 \wedge C_3 \wedge dC_3 \right) \end{aligned} \tag{2.4}$$

with $R(G)$ being the Ricci curvature of the graviton field G and the field strength tensor fields are defined as

$$H_3 = dB \quad , \quad F_2 = dC_1 \quad , \quad F_4 = dC_3 - C_1 \wedge H_3 \ . \tag{2.5}$$

A similar expression holds for type IIB supergravity. The contribution for the fields from the NS-NS sector is identical to the previous terms and the R-R fields lead to similar kinetic terms for the contributing p-form fields. However, in contrast to the low energy limit of type IIA string theory, we recall that the four-form field C_4 was subject to a self-duality constraint. This constraint cannot be included into a well-defined action functional but has to be added by hand to the theory. Following [11, 17], we find that the dynamics of the fields for type IIB supergravity is described by a "pseudo-action"

$$\begin{aligned} S_{\mathrm{IIB}} = \frac{1}{2} \int e^{-2\phi} \left(-R(G) \star 1 + 4 d\phi \wedge \star d\phi - \frac{1}{2} H_3 \wedge \star H_3 \right) \\ - \frac{1}{2} \int \left(dC_0 \wedge \star dC_0 + F_3 \wedge \star F_3 + \frac{1}{2} F_5 \wedge \star F_5 - C_4 \wedge H_3 \wedge dC_2 \right) \end{aligned} \tag{2.6}$$

together with the additional self-duality constraint

$$\star F_5 = F_5 \ . \tag{2.7}$$

In this notation, the field strength tensors are defined as

$$H_3 = dB \quad , \quad F_3 = dC_2 - C_0 dB \quad \text{and} \quad F_5 = dC_4 - dB \wedge C_2 \ . \tag{2.8}$$

[5] Since we consider an effective field thoery, the low energy action may contain additional contributions which are of higher order in derivatives of the contributing fields and hence would lead to a non-local action. These higher-order terms are suppressed in an α'-expansion where $\alpha' \sim 1/T$ denotes the string coupling. Therefore, these higher order terms are neglected in the leading order supergravity approximation of type II superstring theory.

2.2 String Compactification

As discussed in the previous section, superstring theories are consistent only in ten spacetime dimensions. In order to make contact with the observable, four-dimensional world, one can deduce an effective theory in four dimensions from string theory by compactifying the remaining six dimensions on an internal space. Primary, this idea was not developed for string compactifications but has its origin in a work by Kaluza and Klein [18, 19], describing electromagnetism from a compactification of a five-dimensional gravitational theory on a circle. To get an idea of the compactification process, we follow the steps of Kaluza's and Klein's construction to see explicitly the compactification process at work in order to analyze afterwards the effect of compactifying string theory on a six-dimensional compact space. From this we deduce that the six-dimensional internal space needs to be a Calabi-Yau manifold, in order to obtain an effective four-dimensional field theory which is still $\mathcal{N} = 2$ supersymmetric. Comprehensive reviews on Kaluza-Klein compactifications of type II string theory can be found in refs. [15, 20–23].

2.2.1 Circle Compactification of a Massless Scalar Field

In order to discuss the general principle of Kaluza-Klein compactification, let us first consider the simple example of a free massless scalar field evolving on the five dimensional spacetime $M = \mathbb{R}^{1,3} \times S^1$ where $\mathbb{R}^{1,3}$ denotes the usual Minkowski space and S^1 is a circle of Radius R [22]. The action functional for the scalar field ϕ reads

$$S[\phi] = \int_M d^5x g_{\mu\nu} \partial^\mu \phi \partial^\nu \phi^* \tag{2.9}$$

where $g_{\mu\nu}$ is at this stage an arbitrary Lorentzian metric on M. From this we can read off the equations of motion to be

$$\Box_M \phi \coloneqq g_{\mu\nu} \partial^\mu \partial^\nu \phi = 0 \ . \tag{2.10}$$

To perform the compactification of this theory to four spacetime dimensions, let us assume at first that $g_{\mu\nu}$ decomposes into a block diagonal form[6] which is compatible with the product structure of M, i.e.

$$g_{\mu\nu} = \begin{pmatrix} g_{ab}(x) & 0 \\ 0 & R \end{pmatrix} \qquad \text{for local coordinates } \mathbf{x} \in \mathbb{R}^{1,3} \ , \ \theta \in S^1 \tag{2.11}$$

such that $g_{ab}(\mathbf{x})$ is a Lorentzian metric on $\mathbb{R}^{1,3}$. In more general settings, the block which corresponds to the internal space is any Riemannian metric on the compact space. Since we have chosen the S^1 to be of radius R, this fixes its metric uniquely. With this special choice for the metric, the equations of motion can be rewritten as

$$0 = \Box_M \phi = \Box_{\mathbb{R}^{1,3}} \phi(\mathbf{x}, \theta) + R \partial_\theta^2 \phi(\mathbf{x}, \theta) \tag{2.12}$$

[6] As a next step, we generalize in section 2.2.2 the idea of compactification to a general five-dimensional metric.

which can be solved by expanding $\phi(\mathbf{x},\theta)$ in spherical harmonics on S^1. Since S^1 is a compact space, this expansion has a countable basis[7] of eigenfunctions $\chi_\ell(\theta)$ obeying

$$\partial_\theta^2 \chi_\ell(\theta) = \lambda_\ell^2 \chi_\ell(\theta) \ . \tag{2.13}$$

Thus, in this expansion, one obtains

$$0 = \sum_{\ell=1}^{\infty} \left(\Box_{\mathbb{R}^{1,3}} \phi_\ell(\mathbf{x}) + R\lambda_\ell^2 \phi_\ell(\mathbf{x}) \right) \chi_\ell(\theta) \ . \tag{2.14}$$

From this result we find that starting with one massless scalar field $\phi(\mathbf{x},\theta)$ in five spacetime dimensions, compactifying the fifth dimension on a circle leads to an infinite tower of massive scalar fields $\phi_\ell(x)$ in four spacetime dimensions whose mass is given by

$$m_\ell^2 = -R\lambda_\ell \ .$$

The final ingredient to end up with a four dimensional theory is taking the limit $R \to 0$ which can be viewed as "shrinking" the extra-dimension. From the mass relation we see immediately that for any given finite energy scale, only the massless modes of this effective theory can be excited. Hence, the effective four dimensional field theory resulting from this compactification process is again that of a massless scalar field $\phi_0(x)$.

2.2.2 Kaluza-Klein Compactification

The original idea of Kaluza and Klein [18, 19] is to apply the compactification method we discussed above to pure Einstein-Hilbert gravity theory in five spacetime dimensions. As a result, the effective four-dimensional theory is given by the full Maxwell electromagnetism coupled to Einstein-Hilbert gravity and an uncharged massless scalar field. Hence, by the dimensional reduction, new fields evolve from the geometry of the higher dimensional theory. Since we will use a similar construction for the compactification of superstring theories, let us briefly review this type of Kaluza-Klein compactification [23, 24].

We start by considering a pure Einstein-Hilbert action in five spacetime dimensions. The action of this theory is given by

$$S_{5\mathrm{D}} = \int_{M_5} R(G(\mathbf{x})) \star 1 \tag{2.15}$$

with $R(G)$ being the Ricci curvature corresponding to the spacetime dependent Lorentzian metric $G(\mathbf{x})$ on the five-dimensional manifold M_5. In order to perform a dimensional reduction as in the previous example of the scalar field, we first decompose G into an effective four-dimensional Lorentzian metric g, a four-dimensional spacetime-vector A and a scalar field σ. To that end, let us parametrize M_5 by local coordinates $\mathbf{x} = (x^0, \ldots, x^4)$ and

[7] Since S^1 is compact, the spectral theorem implies that the Laplacian $\Delta_{S^1} = \partial_\theta^2$ has a discrete, non-negative spectrum and that every eigenspace is finite dimensional.

assume that the fifth direction x^4 will be compactified later on. Then we can decompose the metric G according to

$$ds^2 = G_{ij}dx^i dx^j = \left(g_{\mu\nu} + e^{2\sigma} A_\mu A_\nu\right) dx^\mu dx^\nu + 2e^{2\sigma} A_\mu dx^\mu dx^4 + e^{2\sigma} dx^4 dx^4 \ , \qquad (2.16)$$

with the greek indices μ, ν running from 0 to 3. In this way, the five-dimensional metric G decomposes into a metric $g_{\mu,\nu}$, a vector field A_μ and a scalar field σ in four spacetime dimensions. We note that the diffeomorphism invariance of the five-dimensional metric G implies that g is invariant under four-dimensional diffeomorphisms and hence gives rise to a well-defined metric. Moreover, one can conclude that the invariance of G under the transformation $x^4 \mapsto x^4 + \chi(x^\mu)$ implies that the vector A_μ has to transform according to an $U(1)$ gauge transformation $A_\mu \mapsto A_\mu - \partial_\mu \chi$. Defining the corresponding field strength tensor F in the usual fashion as

$$F := dA \qquad (2.17)$$

the Ricci curvature $R(G)$ can be rewritten as [24]

$$R(G) = R(g) + F_{\mu\nu}F^{\mu\nu} + g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma \ . \qquad (2.18)$$

Thus, by inserting this expression into the five-dimensional action S_{5D}, we can perform a similar Fourier decomposition as in the case of the scalar field by assuming that the fifth direction on M_5 is compactified. This results in an overall mode expansion of all fields which, after integrating out the massive modes in the small radius limit and performing a suitable Weyl rescaling, leads to an effective four-dimensional action of the form

$$S_{\text{4D}} = \int_{M_4} R(g) \star 1 + F \wedge \star F + d\sigma \wedge \star d\sigma \qquad (2.19)$$

which describes Maxwell's theory of electromagnetism together with an uncharged scalar field σ coupled to Einstein-Hilbert gravity in four spacetime dimensions.

2.2.3 Compactification of Type II String Theory

Similar to the Kaluza-Klein compactification of five-dimensional gravity theories, we can consider the reduction of ten-dimensional superstring theory to an effective field theory in four spacetime dimensions [15, 21–23]. As previously, let us assume that the ten-dimensional spacetime M_{10} decomposes topologically into a direct product

$$M_{10} = M_4 \times X_6 \qquad (2.20)$$

of a four-dimensional manifold M_4 and a six-dimensional compact internal space X_6. In the following, we focus on the low energy type II supergravity actions as introduced in equations (2.4) and (2.6) and investigate their effective four-dimensional field theories after compactification. Note that the ten-dimensional Lorentz group $SO(1,9)$ splits into the direct product $SO(1,3) \times SO(6)$. Hence, the contributing massless fields decompose according to this group structure.

Let us first consider the fermionic sector of the spectra. Recall that for both type II string theories, the massless fermionic spectrum contains two gravitini and two dilatini. The corresponding $\mathcal{N} = 2$ supersymmetry algebra is characterized by 32 supercharges. If we naively compactify this theory on a six-dimensional compact manifold X_6, the spinor representation for the supercharges decomposes according to the split of the Lorentz group. Thus, a ten-dimensional supercharge gives rise to an effective four-dimensional supercharge after compactification if and only if the six-dimensional remnant gives rise to a globally defined, covariantly constant section on the spinor bundle over X_6 [10, 24]. The number of independent covariant constant spinors on a compact manifold X_6 can be deduced from the holonomy group of the spinor bundle which we denote by $\mathrm{Hol}(X_6) \subseteq SO(6)$.

For a generic choice of compactification space X_6, the holonomy group is maximal, meaning that $\mathrm{Hol}(X_6) = SO(6)$. This implies that X_6 does not admit any covariant constant spinor. Hence, all supersymmetries of the ten-dimensional type II string theory get broken by the compactification. Thus, in order to preserve some supersymmetry, the choice of X_6 needs to be restricted to those compact manifolds whose holonomy group is a proper subgroup of $SO(6)$. This condition is highly restrictive. In total, there are three possible structures for the holonomy group of X_6 that give rise to an effective four-dimensional supersymmetric string compactification.

If $\mathrm{Hol}(X_6) = \{e\}$ is the trivial group, each spinor is covariantly constant and hence, all 32 superchrages are preserved. In this case, we obtain a four-dimensional field theory with 32 supercharges which leads to the maximal amount of $\mathcal{N} = 8$ supersymmetries. In six real dimensions, the torus $\mathbb{T}^6$ is the only compact manifold with trivial holonomy. Less constraining, we obtain an $\mathcal{N} = 4$ supersymmetric effective theory by compactifying type II string theory on a compact manifold X_6 with holonomy $\mathrm{Hol}(X_6) = SU(2)$. Again, this condition is sufficient to uniquely determine the topological structure of X_6 to be a product $\mathbb{T}^2 \times \mathrm{K3}$ of a two-trous and a K3 surface[8].

Most interesting for the purpose of this work is the remaining possibility that X_6 has $SU(3)$-holonomy. In this case, the effective four-dimensional theory has eight supercharges which realizes $\mathcal{N} = 2$ supersymmetry. In contrast to the former cases, $SU(3)$-holonomy does not fix the topology of X_6 entirely. However, it implies very strong constraints on X_6. In chapter 3, we define such compact manifolds to be *Calabi-Yau manifolds* and discuss their properties extensively.

From now on, let us assume that X_6 has this required structure of being a Calabi-Yau manifold. Let us now focus on the bosonic part of the compactified spectra. To that end, we perform a similar decomposition of the bosonic p-form fields appearing in the massless type II string spectra as for the metric in the classical Kaluza-Klein reduction. The resulting fields can be arranged in $\mathcal{N} = 2$ supermultiplets.

[8] A K3 surface is a complex two-dimensional Calabi-Yau manifold.

For type IIA string theory, the compactified fields are given by [23]

- One gravity multiplet: $(G_{\mu\nu}, (C_1)_\mu, \text{fermions})$
- n_V vector mulitplets: $((C_3)_{\mu i\bar{j}}, G_{i\bar{j}}, B_{i\bar{j}}, \text{fermions})$
- $n_H - 1$ hypermultiplets: $((C_3)_{ij\bar{k}}, \widetilde{G_{ij}}, \text{fermions})$
- One universal hypermultiplet: $((C_3)_{ijk}, \phi, \widetilde{B_{\mu\nu}}, \text{fermions})$

where greek indices $\mu, \nu = 0, \ldots, 3$ denote spacetime indices on M_4 whereas latin indices $i, j, k = 4, \ldots, 6$ and $\bar{i}, \bar{j} = 7, \ldots, 9$ are holomorphic and anti-holomorphic indices for the internal manifold X_6. Additional contributions from decomposing the ten-dimensional fields vanish due to the special structure of the cohomology groups $H^{(p,q)}(X_6, \mathbb{C})$ of Calabi-Yau manifolds which we discuss in more detail in section 3.1. The complex scalar field $\widetilde{B_{\mu\nu}}$ of the universal hypermultiplet defines the Poincaré dual of the antisymmetric two-form field $B_{\mu\nu}$. Moreover, $\widetilde{G_{ij}}$ denotes the Poincaré dual $(2,1)$+$(1,2)$-form corresponding to G_{ij}.

Counting only the four-dimensional effective degrees of freedom, it is reasonable to expand fields like $(C_3)_{\mu i\bar{j}}$ in terms of a basis $h^I_{i\bar{j}} \in H^{1,1}(X_6, \mathbb{C})$. Hence, the multiplicity of the multiplets can be easily read off from the index structure of the contributing fields. For Calabi-Yau threefolds, it holds that $H^{3,0}(X_6, \mathbb{C})$ is one-dimensional, hence the universal hypermultiplet contains indeed four real scalars. Moreover, we deduce that $n_V = \dim(H^{1,1}(X_6, \mathbb{C}))$ and $n_H = \dim(H^{2,1}(X_6, \mathbb{C})) + 1$.

For the type IIB string theory, we obtain again the same mulitplets of an $\mathcal{N} = 2$ supergravity theory as for type IIA string compactifications. However, their field contents originate from different sources. We find [21, 23]

- One gravity multiplet: $(G_{\mu\nu}, (C_4)_{\mu ijk}, \text{fermions})$
- n_V vector mulitplets: $((C_4)_{\mu ij\bar{k}}, \widetilde{G_{ij}}, \text{fermions})$
- $n_H - 1$ hypermultiplets: $(B_{i\bar{j}}, (C_2)_{i\bar{j}}, G_{i\bar{j}}, \widetilde{(C_4)_{\mu\nu i\bar{j}}}, \text{fermions})$
- One universal hypermultiplet: $(\widetilde{(C_2)_{\mu\nu}}, \phi, \widetilde{B_{\mu\nu}}, \text{fermions})$

Again, by counting the multiplicities, we find that type IIB string compactifications have $n_V = \dim(H^{2,1}(X_6, \mathbb{C}))$ vector multiplets and $n_H = \dim(H^{1,1}(X_6, \mathbb{C})) + 1$ hypermultiplets.

The corresponding action functionals for these multiplets can be deduced from the ten-dimensional actions (2.4) and (2.6) respectively. Inserting the explicit decomposition of the fields into the action functionals and evaluating the integrals over the compactification space X_6 leads to [15, 23]

$$S^4_{\text{IIA}} = \int_{M_4} \big(R \star 1 + \text{Re}(\mathcal{N})_{AB} F^A \wedge F^B + \text{Im}(\mathcal{N})_{AB} F^A \wedge \star F^B \\ - g_{a\bar{b}} dt^a \wedge \star d\bar{t}^{\bar{b}} - h_{uv} dq^u \wedge \star dq^v \big) \tag{2.21}$$

for the bosonic part of the action for the four-dimensional $\mathcal{N} = 2$ supergravity theory coming from type IIA string compactification on a Calabi-Yau manifold X_6. Here, the t^a collect all vector multiplet scalars and their complex conjugates $\bar{t}^b$, q^u denote all hypermultiplet scalars (including the universal hypermultiplet) and F^A is a collection of the field strength two-forms of all $n_V + 1$ vectors, i.e. $F^0 = d(C_1)$ whereas $F^a = d(A^a)$ with $A^a_\mu = (C_3)_{\mu i \bar{j}}$ for $a = 1, \ldots, h^{1,1}$ being the vector fields from the n_V vector multiplets. Moreover, the functions g_{ab} and $\mathcal{N}_{ab}$ depend only on the vector multiplet scalars t^a and h_{uv} depends on the hypermultiplet scalars only[9].

Similarly, we obtain for the bosonic part of the four-dimensional $\mathcal{N} = 2$ supergravity theory action coming from type IIB string compactification[10] [15, 23]

$$\begin{aligned} S^4_{\mathrm{IIB}} = \int_{M_4} \big(& R \star 1 + \mathrm{Re}(\mathcal{M})_{AB} F^A \wedge F^B + \mathrm{Im}(\mathcal{M})_{AB} F^A \wedge \star F^B \\ & - \tilde{g}_{a\bar{b}} dz^a \wedge \star d\bar{z}^{\bar{b}} - \tilde{h}_{uv} dq^u \wedge \star dq^v \Big) \ . \end{aligned} \tag{2.22}$$

Similarly to above, we use the notation that z^a collects the vector multiplet scalars, q^u the hypermultiplet scalars whereas F^A denotes the field strengths of the vector fields. Again, the functions $\tilde{g}_{a\bar{b}}$ and $\mathcal{M}_{AB}$ depend on the vector multiplet scalars z^a and $\tilde{h}_{uv}$ depends on the hypermultiplet scalars. A comprehensive derivation of these results, including the explicit form of the appearing functions, can be found in [15].

It is not a coincidence that type IIA and type IIB string theory compactified on a Calabi-Yau threefold give rise to a very similar spectrum and moreover a quite similar action. In particular, it is conjectured and tested for many examples that for any Calabi-Yau compactification of type IIA string theory there exists another Calabi-Yau manifold such that type IIB string theory compactified on this so-called *mirror partner* gives rise to an equivalent quantum field theory [25–28]. This observation is known under the name of *mirror symmetry* which turns out to be a very powerful tool for the investigation of string compactifications. We come back to discuss the geometrical implications of mirror symmetry on the pairs of Calabi-Yau manifolds in section 3.2.3. For introductory review article concerning the mirror conjecture, we refer to [29–32].

By analyzing the actions from equation (2.21) and (2.22), one can conclude that none of them allows an interaction between the vector multiplet sector (including the gravity multiplet) and the hypermultiplet sector. From a string phenomenological point of view, such an interaction is desirable as it provides a mechanism to (partially) stabilize the scalar fields for a unique semi-classical vacuum configuration. However, it is also possible

[9] As we will see in section 4.3, the functions g_{ab} and h_{uv} can be identified with metric tensor fields on the corresponding target spaces that are spanned by the vector multiplet and hypermultiplet scalars respectively.

[10] One may note that S^4_{IIB} is an honest action for a four-dimensional field theory. The self-duality constraint which needed to be imposed by hand in the ten-dimensional case does not occur any longer after compactifying type IIB string theory on a Calabi-Yau threefold.

to further break the $\mathcal{N}=2$ supersymmetry of the four-dimensional theory to $\mathcal{N}=1$ via a suitable orientifold construction [24]. This procedure is relevant when we discuss flux compactifications in chapter 4 since $\mathcal{N}=1$ supergravity in four dimensions allows for a coupling of the vector multiplet scalars with internal three-form fluxes of the Calabi-Yau manifold.

2.3 M-Theory

In the 1990th, a new proposal [33] led to a revolution of string theory by claiming that all ten-dimensional superstring theories from table 2.1 are related by duality transformations and moreover originate as different compactification limits of one unique eleven-dimensional supersymmetric field theory which is called M-Theory. Many evidences for such an underlying theory have been found [34, 35] - most promising, the existence of a unique eleven dimensional supergravity theory [36, 37], whose ten-dimensional compactifications can be identified with the supergravity theory originating from type IIA or type IIB string theory, if the compactified dimension is a circle. Moreover, compactifying the eleven-dimensional supergravity on a compact interval yields the $\mathcal{N}=1$ supergravity theory that is given as the low energy limit of type I string theory. Moreover, the investigation of dualities between all kinds of superstring theories guides toward a unification of the zoo of superstring theories.

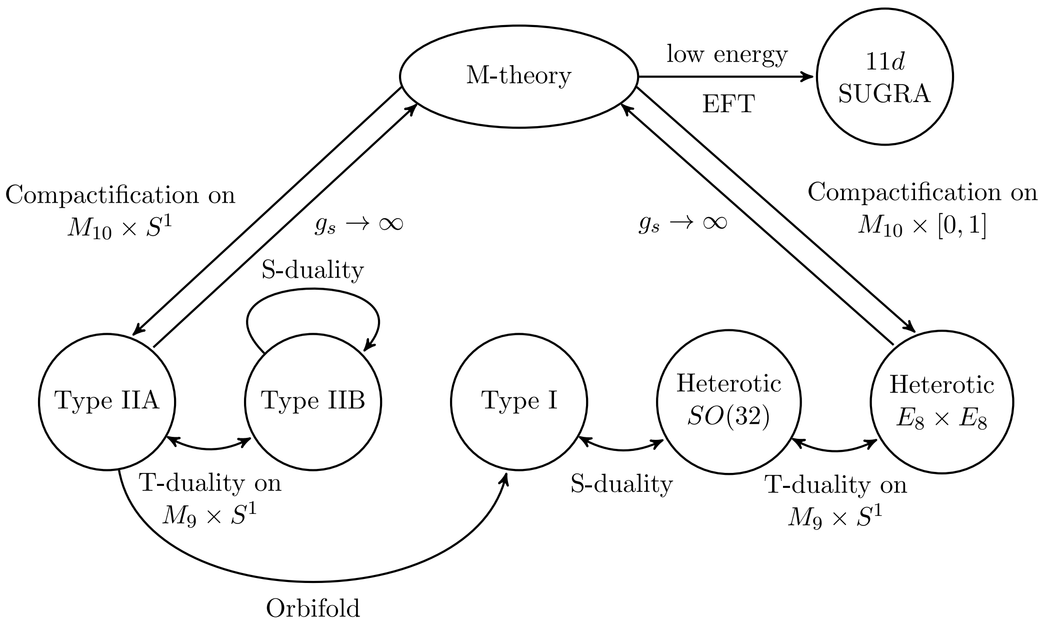

Figure 2.1: The web of ten-dimensional superstring theories which originate from a common eleven dimensional theory that is called M-theory. The connecting edges of this diagram represent either a compactification process, a duality transformation or an orbifolding of the theory. Figure inspired by [38].

Figure 2.1 provides a visualization of the connections between the various superstring theories and eleven-dimensional M-theory. Without going into details on the origin of the dualities, this schematic picture gives an impression on the special role of M-theory as a unifying theory of the string theory landscape.

Until today, there is no fundamental theory known, that describes M-theory at all energy scales. However, much progress has been made to discover not only the low energy supergravity behavior of M-theory but moreover to investigate also the high energy limit. Refs. [10, 11] provide a more comprehensive introduction to M-theory.

2.3.1 $\mathcal{N} = 1$ Supergravity in Eleven Dimensions

To begin with, let us briefly introduce $\mathcal{N} = 1$ supergravity in eleven spacetime dimensions. [39] provides an extensive introduction to eleven-dimensional supergravity. At first, this theory gained interest, as it is conjectured that there cannot be any interacting supersymmetric field theories beyond eleven dimensions. This is the case, since interactions can be described consistently only for fields that have a "superspin"-helicity of $|\lambda| \leq 2$ [11, 40]. It can be shown that the spectrum of any interacting supersymmetric field theory in $D \geq 12$ dimensions contains multiplets with fields of helicity beyond this threshold.

Moreover, if we consider a general eleven-dimensional supersymmetric field theory, it follows that the only multiplet with fields of helicity $|\lambda| \leq 2$ is given by the gravity multiplet

$$(G, C_3, \text{fermions}) \tag{2.23}$$

of $\mathcal{N} = 1$ supergravity, containing the eleven-dimensional graviton G which is a symmetric, traceless tensor of rank two, an anti-symmetric three-form C_3 and the corresponding fermionic partner fields.

The bosonic part of the action for this theory is given by [36, 41]

$$S_{\text{11-dim}} = \frac{1}{2} \int_{M_{11}} R(G) \star 1 - \frac{1}{2} F_4 \wedge \star F_4 - \frac{1}{6} C_3 \wedge F_4 \wedge F_4 \ . \tag{2.24}$$

Here, $F_4 = dC_3$ denotes the usual field strength of the three-form C_3. By this observations we can conclude that there is indeed only one unique non-trivial supersymmetric field theory in eleven spacetime dimensions which is precisely $\mathcal{N} = 1$ supergravity.

Previously we have claimed that the low energy supergravity theory originating from type IIA string theory is obtained by a circle compactification of this eleven-dimensional theory. On the level of the spectrum, this observation can be deduced by performing a decomposition of G and C_3 according to [11]

$$\begin{aligned} G_{ij} dx^i dx^j &= G_{\mu\nu} dx^\mu dx^\nu + e^{2\phi} (C_1)_\mu dx^\mu dx^{10} + e^{2\phi} dx^{10} dx^{10} \\ (C_3)_{ijk} dx^i dx^j dx^k &= (C_3)_{\mu\nu\rho} dx^\mu dx^\nu dx^\rho + B_{\mu\nu} dx^\mu dx^\nu dx^{10} \ . \end{aligned} \tag{2.25}$$

Note that all further contributions in the decomposition of C_3 vanish due to its anti-symmetry. Comparing this decomposition to the massless bosonic spectrum of type IIA string theory, we find perfect agreement. Moreover, $S_{11\text{-dim}}$ reduces to S_{IIA} by inserting these field decompositions and integrating out the x^{10}-direction which was chosen to be compactified on a circle[11].

2.3.2 Compactification of M-Theory

For the purpose of this work, we will not consider the framework of full eleven-dimensional M-theory but instead focus on an M-theory compactifications. As for the type II strings, we can perform a compactification of the eleven-dimensional spacetime to obtain a low-dimensional effective theory. As a first guess one might be interested in reducing the dimension by choosing a compactification space that is seven-dimensional such that the resulting effective theory becomes again four-dimensional and hence may make contact with the observable world.

Instead, we will analyze the effective three-dimensional theory that is obtained by compactifying M-theory on an eight-dimensional compact manifold X_8, more precisely on a Calabi-Yau manifold of complex dimension four [41–43]. The reason to consider this a priori non-physical theory is two-fold. First, it gives a natural extension of the theories that are obtained from compactifying type II string theories on a Calabi-Yau manifold of complex dimension three. Hence, it is possible to apply many tools that are developed for the investigation of string compactifications in order to gain insights on this three-dimensional theories that originate from M-theory compactifications.

Moreover, it has been observed [42, 44] that certain limits of twelve-dimensional F-theory[12] compactified on a Calabi-Yau fourfold can be equivalently described by such an M-theory compactification. Hence, analyzing M-theory compactifications on a Calabi-Yau fourfold give insights on certain F-theory compactifications which could be of interest for model building procedures. In particular, we argue in section 4.4 that flux compactifications of F-theory on a Calabi-Yau fourfold can be treated very similar to flux compactifications of M-theory on the same compactification space [9, 41].

Again, by performing a Kaluza-Klein reduction of the eleven-dimensional fields from the gravity multiplet of $\mathcal{N} = 1$ supergravity on a Calabi-Yau manifold of complex dimension four, we obtain an effective three-dimensional $\mathcal{N} = 2$ supergravity theory [42] with four

[11] A derivation of this identification can be found for instance in [11].

[12] F-theory became of interest as a twelve-dimensional representation of type IIB string theory which is constructed by treating the combination of dilaton ϕ and the axion field C_0 as an additional complexified geometrical dimension of the internal space [44–46]. Hence, F-theory is by definition given as a theory on an elliptically-fibred space. The evolution of F-theory provides a powerful tool for the construction of string phenomenologically interesting models. Throughout this work, we will not consider F-theory in more detail. Refs. [47–49] give a selection of review articles that provide the reader with a comprehensive introduction to F-theory.

unbroken supercharges[13] whose spectrum is given by [41, 50]

- One gravity multiplet: $(G, \text{fermions})$
- n_C chiral multiplets: $(N^\rho, Z^a, \text{fermions})$
- n_V vector multiplets: $(A^a_\mu, M^K, \text{fermions})$

where the vector fields A^J_μ and the complex scalars N^I originate from the decomposition of the eleven-dimensional three-form (C_3) whereas the three-dimensional graviton G, and the scalars Z^J and M^K are obtained from decomposing the eleven-dimensional graviton field. As for the string compactifications, the number of multiplets is completely fixed by the topology of the Calabi-Yau fourfold X_8 and can be read off from the field decompositions. One finds that

$$n_C = \dim(H^{3,1}(X_8,\mathbb{C})) + \dim(H^{2,1}(X_8,\mathbb{C})) \quad \text{and} \quad n_V = \dim(H^{1,1}(X_8,\mathbb{C})) \ . \tag{2.26}$$

The bosonic part of the effective three-dimensional action reads [41, 50]

$$\begin{aligned} S_{\text{3-dim}} = \frac{1}{2}\int_{M_3} \Big(& R \star 1 - 2G_{a\bar{b}} dZ^a \wedge \star d\bar{Z}^{\bar{b}} - \frac{1}{2}(V)^2 h_{IJ} F^I \wedge \star F^J \\ & - \tilde{G}_{\rho\bar{\sigma}} dN^\rho \wedge \star d\bar{N}^{\bar{\sigma}} - h_{IJ} dM^I \wedge \star dM^J - d\ln(\mathcal{V}) \wedge \star d\ln(\mathcal{V}) \Big) \end{aligned} \tag{2.27}$$

where $G_{a\bar{b}}$, $\tilde{G}_{\rho\bar{\sigma}}$ and h_{IJ} are functions depending on Z^a, N^ρ and M^I respectively. As for the string compactifications, these functions can be interpreted as metric tensors or the corresponding target spaces. The quantity $\mathcal{V}$ depends only on the M^I and has the interpretation of a quantum volume of the Calabi-Yau fourfold. As usual, we denote by $F^I = dA^I$ the field strength of the vector fields.

As we will see in section 4.4, this action can be equipped with an additional superpotential that originates from interactions with internal four-form fluxes of the Calabi-Yau manifold leading to a flux compactification of M-theory.

[13] At first view it might be unnatural from a physics perspective to consider an effective theory in three spacetime dimensions. However, since this theory has four unbroken supercharges, its supersymmetry algebra is similar to that of $\mathcal{N}=1$ supersymmetric theories in four spacetime dimensions and hence gives non-trivial insights on such four-dimensional theories [50].

Chapter 3

Calabi-Yau Manifolds

In section 2.2.3 we have discussed, how four dimensional field theories emerge from a superstring theory by performing a Kaluza-Klein-type compactification on a six-dimensional internal space X. Moreover, we have collected several properties which X has to obey such that the effective field theory is consistent and still $\mathcal{N} = 2$ supersymmetric. These properties can be formalized in requiring X to be a compact *Calabi-Yau manifold* (also Calabi-Yau n-fold) which we define as a compact Kähler manifold of (complex) dimension $\dim_{\mathbb{C}}(X) = n$ that has $SU(n)$ holonomy[14]. Equivalently, the latter condition can be stated by X admitting a Ricci-flat (Kähler) metric. In the following, we will always assume that X is a simply connected manifold.

Due to their relation to physics, this very special type of manifolds will be the key objects of consideration throughout this thesis. Therefore, we will discuss their properties in quite detail in the following sections. However, since this chapter should provide the reader with all necessary tools to follow the main text, we will focus on discussing the features of Calabi-Yau manifolds, highlighting the most important arguments without giving formal proofs. Readers interested in additional formal details may be referred to any textbook or review article on Calabi-Yau geometries such as [32, 51, 52].

3.1 Topology of Calabi-Yau Manifolds

When talking about Calabi-Yau n-folds, one should start by pointing out that the explicit construction of Calabi-Yau geometries is very difficult, since constructing a concrete Ricci-flat Kähler metric is highly non-trivial. In particular, beyond the case $n = 1$, no such metric is known[15], however there are numerical approximations to find those. This seem-

[14]Often in the physics literature, this constraint is relaxed such that the holonomy is any subgroup of $SU(n)$. However, throughout this thesis we will assume that the holonomy is always the full group $SU(n)$.

[15]Compact, connected Calabi-Yau one-folds are given by complex elliptic curves which have been studied extensively in mathematics literature as for example [53]. Since elliptic curves are given by complex tori $\mathcal{E} = \mathbb{C}/\Lambda$, a Ricci-flat Kähler metric on $\mathcal{E}$ is always constructable by inducing a metric on $\mathcal{E}$ from the flat metric of $\mathbb{C}$ [54].

ingly disappointing fact may lead to the impression, that there is no use of the description of Calabi-Yau manifolds in practical computations. However, it was conjectured by Eugenio Calabi [55, 56] and later proven by Shing-Tung Yau [57, 58] that for any compact Kähler manifold X a representative of the Kähler class can be deformed such that it gives rise to a Ricci-flat Kähler metric if and only if its first Chern class $c_1(X)$ vanishes. This condition gives a purely topological classification of those complex compact manifolds that admit a Ricci-flat Kähler metric. Although, the proof of this theorem is non-constructive, it is frequently used in order to decide whether a given compact manifold admits a Kähler class turning it into a Calabi-Yau manifold. If we refer in the following to a Kähler manifold X as a Calabi-Yau manifold, we mean it to obey the Calabi-Yau condition $c_1(X) = 0$ without specifying the concrete metric. In this sense, Calabi-Yau manifolds are definable purely in terms of topological quantities.

As we have discussed in section 2.2.3, we will be interested mostly in the Dolbeault-cohomology $H^{p,q}(X, \mathbb{C})$. The (complex) dimensions of these spaces are called *Hodge numbers* $h^{p,q} = \dim_{\mathbb{C}} H^{p,q}(X, \mathbb{C})$. The conditions for X to be a Calabi-Yau manifold restrict its Hodge numbers highly [10] as we will discuss in the following.

- For any complex n-dimensional manifold, there can be at most n holomorphic and n anti-holomorphic coordinates. Hence, by construction, $H^{p,q}(X, \mathbb{C}) = 0$ for $p > n$ or $q > n$. Thus, all non-trivial Hodge numbers can be arranged in a diamond.

- Assuming that X is connected, we have by definition that $H^{0,0}(X, \mathbb{C}) \cong \mathbb{C}$. Moreover, $H^{n,n}(X, \mathbb{C})$ is spanned only by the volume form on X. Thus $h^{0,0} = h^{n,n} = 1$. In addition, due to the $SU(n)$-holonomy of X, there exists (up to rescaling) only one unique holomorphic n-form which we denote in the following by $\Omega \in H^{n,0}(X, \mathbb{C})$, implying $h^{n,0} = 1$.

- Moreover, $SU(n)$-holonomy implies that $H^{k,0}(X, \mathbb{C}) = 0$ for $k = 1, \ldots, n-1$.

- Complex conjugation maps $H^{p,q}(X, \mathbb{C})$ isomorphically to $H^{q,p}(X, \mathbb{C})$. Thus, the Hodge numbers obey the symmetry relation $h^{p,q} = h^{q,p}$.

- The Hodge-star operator $\star : H^k(X, \mathbb{C}) \to H^{n-k}(X, \mathbb{C})$ respects the complex structure and hence induces isomorphisms $H^{p,q}(X, \mathbb{C}) \cong H^{n-p,n-q}(X, \mathbb{C})$. In terms of the Hodge numbers, these isomorphisms give rise to a symmetry $h^{p,q} = h^{n-p,n-q}$ which, combined with complex conjugation, can be viewed as mirroring the Hodge diamond along the horizontal line.

The following figure 3.1 summarizes the Hodge diamonds for Calabi-Yau n-folds with $n \leq 3$. Note that for K3 surfaces[16] the only non-trivial Hodge number $h^{1,1}$ is fixed to $h^{1,1} = 20$ [59]. Increasing the dimension of the Calabi-Yau manifold increases the number

[16] All compact, simply connected Calabi-Yau twofolds (with proper $SU(2)$ holonomy) can be constructed as K3 surfaces [59]. Hence, we will use the notations K3 surface and (compact, simply connected) Calabi-Yau twofold equivalently.

$$
\begin{array}{ccc}
 & 1 & \\
1 & & 1 \\
 & 1 &
\end{array}
\qquad
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Figure 3.1: The Hodge diamonds for Calabi-Yau n-folds with $n \leq 3$.

of undetermined Hodge numbers. Hence, even though the Hodge diamond is highly restricted, the classification of Calabi-Yau manifolds via their Hodge numbers becomes more complex with increasing dimension of the manifold. In particular, for Calabi-Yau fourfolds, the Hodge diamond as shown in Figure 3.2 has already four undetermined characterizing degrees of freedom.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & h^{1,1} & & 0 & & \\
 & 0 & & h^{2,1} & & h^{2,1} & & 0 & \\
1 & & h^{3,1} & & h^{2,2} & & h^{3,1} & & 1 \\
 & 0 & & h^{2,1} & & h^{2,1} & & 0 & \\
 & & 0 & & h^{1,1} & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

Figure 3.2: The Hodge diamond for Calabi-Yau fourfolds.

However, these four numbers are not independent of each other but obey the non-trivial linear relation [60]

$$0 = 44 + 4h^{1,1} - 2h^{2,1} + 4h^{3,1} - h^{2,2} \ . \tag{3.1}$$

Moreover, the Hodge numbers can be related to the most important topological invariant, the Euler characteristic χ which is defined to be the alternating sum of the betti numbers b^k.

$$\chi := \sum_{k=0}^{2n} (-1)^k b^k = \sum_{p+q=0}^{2n} (-1)^{p+q} h^{p,q} \ . \tag{3.2}$$

Inserting the symmetries discussed above, we obtain

$$\chi_{n=3} = 2(h^{1,1} - h^{2,1}) \tag{3.3}$$

for Calabi-Yau threefolds and

$$\chi_{n=4} = 6(8 + h^{1,1} - h^{2,1} + h^{3,1}) \tag{3.4}$$

for Calabi-Yau fourfolds where we have used equation (3.1) to eliminate $h^{2,2}$. These relations will play an important role when we discuss explicit examples since it is possible using methods from toric geometry to compute χ directly which for Calabi-Yau Fourfolds will allow to solve for $h^{2,1}$ in terms of $h^{3,1}$ and $h^{1,1}$ which are more accessible.

3.2 Moduli Spaces

Generically, Calabi-Yau manifolds appear in families which are connected by smooth parameters. These parameters are called the moduli of the corresponding family and form themselves local coordinates of a manifold which we call the moduli space of the given family of Calabi-Yau manifolds. More formally, one can view a family $\mathcal{X}_{\mathcal{M}}$ of Calabi-Yau manifolds as a fibre bundle over the moduli space $\mathcal{M}$ with fibre $X_{\mathbf{z}}$ being the Calabi-Yau manifold represented by the point $\mathbf{z} \in \mathcal{M}$ on the moduli space.

$$\begin{array}{ccc} X_{\mathbf{z}} & \longrightarrow & \mathcal{X}_{\mathcal{M}} \\ & & \downarrow \pi \\ & & \mathcal{M} \end{array} \tag{3.5}$$

As it turns out, these moduli are tightly connected to the topological data of any representative of the family. Moreover, from a physics perspective, these moduli spaces become very central when considering the low energy supergravity description of a string theory that is compactified on a Calabi-Yau manifold as the moduli space turns out to be the target space of the scalar degrees of freedom of this effective theory.

The following construction of the moduli space $\mathcal{M}$ for a given family of Calabi-Yau Manifolds goes back to the discussions of [10, 51, 61] defining the moduli space of Calabi-Yau threefolds. However, the arguments are not specific to the dimension of the Calabi-Yau manifolds and hence generalize for any Calabi-Yau n-fold (with $n \geq 3$). Recall that for a given family of Calabi-Yau manifolds $\mathcal{X}_{\mathcal{M}}$, each Calabi-Yau manifold $X_{\mathbf{z}}$ is defined to be a Kähler manifold admitting a Ricci-flat metric g. Locally, the corresponding moduli space $\mathcal{M}$ arises by considering all continuous deformations of the metric g of a generic representative $X_{\mathbf{z}}$ such that the result is still Ricci-flat. Locally, any such deformation δg can be written as

$$\delta g = \delta g_{ij}^{C.S.} dx^i dx^j + \delta g_{i\bar{j}}^{K} dx^i d\bar{x}^{\bar{j}} \tag{3.6}$$

where the x^i denote a set of complex coordinates on the Calabi-Yau n-fold $X_{\mathbf{z}}$. Now, Ricci-flatness of $g + \delta g$ imposes that δg needs to solve a Laplace equation which is called the *Lichnerowicz equation*. Due to the different type of holomorphic structure, the two contributions $\delta g^{C.S.}$ and δg^K can be treated separately. For δg^K we find immediately that solving the Lichnerowicz equation is equivalent to $\delta g^K \in H^{1,1}(X_z, \mathbb{C})$ being a real-valued harmonic $(1,1)$-form which can be interpreted as a deformation of the Kähler form J defining the original Kähler metric g

$$i\delta g^K = \delta J \ . \tag{3.7}$$

Hence, these deformations are called *Kähler deformations* of the Calabi-Yau manifold. On the other hand, the deformation $\delta g_{ij}^{C.S.}$ cannot be described by a form as it is symmetric in i and j. However, it is possible to relate it uniquely to a harmonic $(n-1,1)$-form $\eta \in H^{n-1,1}(X_{\mathbf{z}}, \mathbb{C})$ which is given by

$$\eta_{i_1,\ldots,i_{n-1},\bar{\ell}} = -\frac{1}{2}\Omega_{i_1,\ldots,i_n} g^{i_n\bar{m}} \delta g_{\bar{m}\bar{\ell}}^{C.S.} \ . \tag{3.8}$$

The Lichnerowicz equation for $\delta g^{C.S.}$ is equivalent to η being harmonic. Thus, we can treat the deformations $\delta g^{C.S.}$ equivalently as the n-forms $\eta \in H^{n-1,1}(X_{\mathbf{z}}, \mathbb{C})$.

Let us investigate the latter type of metric deformations in more detail. As we have seen, $\delta g^{C.S.}$ cannot be a $(2,0)$-form, hence the resulting deformed metric turns out to be non hermitian anymore. This can be cured by a non-holomorphic change of coordinates (mixing holomorphic and anti-holomorphic coordinates) paying the prize of changing the complex structure of the corresponding complex manifold. Consequently, we call these deformations the *complex structure deformations* of X.

From this discussion we can conclude that the moduli space $\mathcal{M}$ of a given family of Calabi-Yau manifolds can be locally described by two different types of moduli. Hence, $\mathcal{M}$ decomposes locally into the direct product

$$\mathcal{M} = \mathcal{M}_{C.S.} \times \mathcal{M}_K \tag{3.9}$$

with $\mathcal{M}_{C.S.}$ being the complex $h^{n-1,1}$-dimensional complex structure moduli space and $\mathcal{M}_K$ being the real $h^{1,1}$-dimensional moduli space of Kähler deformations.

3.2.1 Complexified Kähler Moduli Space

So far, we considered $\mathcal{M}$ only as a topological manifold with local coordinates defined by the complex structure and Kähler deformations. However, it is possible to assign a natural metric to this manifold. For the purpose of string compactifications, it is convenient to include the real-valued harmonic B-field from the string theory or M-theory spectrum into this discussion. Hence, we combine deformations of the metric on X with possible deformations of the internal B-field in a complexified $(1,1)$-form $\mathcal{J} = \delta(B + iJ) \in H^{1,1}(X, \mathbb{C})$ [51]. By this construction, $\mathcal{M}_K$ turns into a complex $h^{1,1}$-dimensional moduli space which we denote by $\mathcal{M}_{\mathbb{C}K}$ and call the *complexified Kähler moduli space.*

The complexified Kähler moduli space can be equipped with a natural metric that is given by

$$ds^2 = \frac{1}{2\mathrm{Vol}(X)} \int_X g^{a\bar{b}} g^{c\bar{d}} (\delta g_{a\bar{d}} \delta g_{b\bar{c}} - \delta B_{a\bar{d}} \delta B_{b\bar{c}}) d^{2n}x \ . \tag{3.10}$$

If we denote by $\{h_a\}$ a real basis of $H^{1,1}(X, \mathbb{C})$, the matrix representation of this metric becomes [62]

$$g_{a\bar{b}} = g(h_a, h_b) = \frac{1}{2\mathrm{Vol}(X)} \int h_a \wedge \star h_b = -\frac{1}{2}\partial_a \partial_{\bar{b}} \log(\mathrm{Vol}(X)) \tag{3.11}$$

which in turn can be deduced as a Kähler metric[17] from the Kähler potential[18]

$$K_{\mathbb{C}K} = -\log(\mathrm{Vol}(X)) \; . \tag{3.12}$$

Classically, the Volume of X is obtained by integrating over the volume form ω which is given for a complex n-dimensional compact Kähler manifold X by [63]

$$\mathrm{Vol}(X) = \frac{1}{n!}\int_X \omega \tag{3.13}$$

with volume form

$$\omega = \bigwedge^n J \;\in H^{n,n}(X,\mathbb{C}) \; . \tag{3.14}$$

However, if we consider $\mathcal{M}$ to be the moduli space of a physical quantum theory, such as string or M-theory, the classical volume $\mathrm{Vol}(X)$ obtains quantum corrections such that the volume form ω needs to be replaced by a *quantum volume* form

$$\omega_q := \star^n J \tag{3.15}$$

where

$$\star : QH^\star(X) \times QH^\star(X) \to QH^\star(X) \tag{3.16}$$

denotes the quantum product of the quantum cohomology ring $QH^\star(X)$ [64, 65]. In the large volume limit, the quantum corrections become negligible and therefore, the quantum product $\star$ reduces to the original wedge product. Hence, in this limit, the quantum volume coincides with the actual volume of the Calabi-Yau n-fold.

3.2.2 Complex Structure Moduli Space

While the (complexified) Kähler deformations have a nice interpretation in terms of the complexified Kähler form $B - iJ$, the impact of the complex structure parameters is not immediately obvious. However, if we assume that X is defined as some zero-locus of a set of polynomials, the complex structure deformations are partly realized by deforming the coefficients of these polynomials. With this identification at hand, the complex structure parameters can be interpreted as deformations of the shape rather than the size of the Calabi-Yau manifold. For any point in the complex structure moduli space, we obtain a fixed representative of the Calabi-Yau family with it's holomorphic $(n, 0)$-form Ω varying smoothly with the complex structure moduli.

[17] Here, we use the convention $g = \frac{1}{2}\partial\bar{\partial}K$.

[18] In the string theory literature [10, 11, 41] it is often convenient to add a (positive) prefactor to the Volume $\mathrm{Vol}(X)$ in the Kähler potential such that for instance the Kähler potential in the case of Calabi-Yau threefolds becomes $K_{\mathbb{C}K} = -\log(8\mathrm{Vol}(X))$. Since such a prefactor corresponds to a constant additive shift $K_{\mathbb{C}K} \mapsto K_{\mathbb{C}K} + c$ of the Kähler potential which is a special kind of Kähler transformation, it leaves the Kähler metric invariant.

In analogy to the complexified Kähler moduli space, also $\mathcal{M}_{C.S.}$ can be equipped with a Kähler metric [61, 66] given by

$$ds^2 = \frac{1}{2\mathrm{Vol}(X)}\int_X \sqrt{g}g^{a\bar{b}}g^{c\bar{d}}\delta g_{ab}\delta g_{\bar{c}\bar{d}}d^{2n}x \ . \tag{3.17}$$

Recalling from equation (3.8) that any complex structure deformation δg_{ab} can be equivalently described by $\eta \in H^{n-1,1}(X_z)$, it is possible to rewrite this metric in terms of $(n-1,1)$-forms. Choosing a local frame of $H^{n-1,1}(X_z)$ given by $\{\eta_1, \ldots, \eta_{h^{2,1}}\}$ and denoting by $\mathbf{z} = (z^1, \ldots, z^{h^{2,1}})$ the corresponding local coordinates of the complex structure moduli space, equation (3.8) can be inverted such that

$$\delta g_{ab} = -\frac{1}{\|\Omega\|^2}\Omega_{ai_1\ldots i_{n-1}}g^{i_1\bar{i}_1}\cdots g^{i_{n-1}\bar{i}_{n-1}}\bar{\eta}_{\bar{k}\bar{i}_1\ldots\bar{i}_{n-1}b}dz^k \tag{3.18}$$

where

$$\|\Omega\|^2 = \frac{1}{n!}\Omega_{i_1,\ldots,i_n}\bar{\Omega}^{i_1,\ldots,i_n} = \frac{1}{n!}\int_X \Omega\wedge\bar{\Omega} \tag{3.19}$$

gives a proper normalization of the expression. Inserting this expression in equation (3.17) describes the metric of $\mathcal{M}_{C.S.}$ in terms of the local coordinates $\mathbf{z}$. After integrating out the Volume of $X_{\mathbf{z}}$, the result becomes [61, 66]

$$ds^2 = g_{a\bar{b}}dz^a d\bar{z}^{\bar{b}} \quad , \quad g_{a\bar{b}} = -\frac{i\int_X \eta_a\wedge\bar{\eta}_{\bar{b}}}{i\int_X \Omega\wedge\bar{\Omega}} \ . \tag{3.20}$$

One may note that $g_{a\bar{b}}$ is a smooth local section of the cotangent bundle of $\mathcal{M}_{C.S.}$ and hence depends smoothly on the local coordinates $\mathbf{z}$ which is implicite as Ω and η_a depend on $\mathbf{z}$. A direct computation [32] shows that

$$K_{C.S.} = -\log\left(i\int_X \Omega\wedge\bar{\Omega}\right) \tag{3.21}$$

is a valid Kähler potential for this metric, hence $\mathcal{M}_{C.S.}$, equipped with this metric is indeed a Kähler manifold.

In contrast to the complexified Kähler moduli space it should be pointed out that for type IIB string theory $K_{C.S.}$ is protected from perturbative quantum corrections[19][10] leaving equation (3.21) unchanged if we consider $\mathcal{M}$ to be the moduli space of a quantum theory. Additional non-perturbative effects however need to be taken into account [67, 68].

To conclude this section on the moduli space geometry of Calabi-Yau families, we should point out that the local product structure of equation (3.9) extends from the framework

[19]Quantum corrections occur due to an expansion in the string coupling α'. Since this corresponds to an expansion in the Calabi-Yau volume, it can affect only the (complexified) Kähler moduli space which is sensible for the volume.

of topological manifolds to the geometry by combining the metrics defined on $\mathcal{M}_{\mathbb{C}K}$ and $\mathcal{M}_{C.S.}$ to obtain the (block diagonal) metric of the full moduli space as

$$ds^2 = \frac{1}{2\mathrm{Vol}(X)} \int_X g^{a\bar{b}} g^{c\bar{d}} (\delta g_{ab} \delta g_{\bar{c}\bar{d}} + \delta g_{a\bar{d}} \delta g_{b\bar{c}} - \delta B_{a\bar{d}} \delta B_{b\bar{c}}) d^6 z \ . \tag{3.22}$$

In terms of the quantum corrected Kähler potential we find that this metric on $\mathcal{M}$ is deduced from the Kähler potential

$$K = K_{\mathbb{C}K} + K_{C.S.} = -\log\left(\frac{1}{n!}\int_X \star^n J\right) - \log\left(i\int_X \Omega\wedge\bar{\Omega}\right) \ . \tag{3.23}$$

3.2.3 Mirror Symmetry

The mirror symmetry conjecture[20] [25–28] states a very powerful and useful duality between families of Calabi-Yau manifolds. Originally stated as a duality relating type IIA string theory compactified on a Calabi-Yau threefold X with an equivalent type IIB string theory compactified on the *mirror* Calabi-Yau threefold Y, mirror symmetry evolved to become a bijection between moduli spaces of Calabi-Yau manifolds. The main statement of the conjecture is the existence of a map

$$\phi : \mathcal{M} \mapsto \mathcal{W} \tag{3.24}$$

on the set of moduli spaces of Calabi-Yau n-folds, such that

$$\phi(\mathcal{M}_{C.S.}) = \mathcal{W}_{\mathcal{C}K} \quad , \quad \phi(\mathcal{M}_{\mathcal{C}K}) = \mathcal{W}_{C.S.} \tag{3.25}$$

where we understand each moduli space to be equipped with its Kähler metric and the map ϕ to be compatible with the metric structure on $\mathcal{M}$ and $\mathcal{W}$ respectively. In other words, mirror symmetry states that for any family of Calabi-Yau n-folds there exists a *mirror family* of Calabi-Yau n-folds such that the complex structure moduli space of the former is diffeomorphic to the complexified Kähler moduli space of the latter and vice versa. Refs. [29–32] provide comprehensive review articles on mirror symmetry.

As a first consequence of this symmetry, the roles of complex structure moduli and (complexified) Kähler moduli are interchanged. Hence, one sees immediately that the Hodge numbers of the corresponding Calabi-Yau n-folds are mirrored on the diagonal of the Hodge diamond (c.f. figure 3.3) which motivates the name of this symmetry.

In practice, this correspondence can be achieved by identifying the Kähler potentials on both sides of the mirror symmetry map (3.24) and obtaining the Kähler moduli $\mathbf{t}$ on $\mathcal{W}_{\mathbb{C}K}$ as locally diffeomorphic functions of the complex structure moduli $\mathbf{z}$ on $\mathcal{M}_{C.S.}$. This map is called the mirror map and its construction for Calabi-Yau threefolds and fourfolds is explained in section 3.3.4.

[20] In the physics literature, mirror symmetry is used as it was proven, since this conjecture has been tested in innumerable many examples. However, since there is so far no complete proof of statement in full generality, we will refer to mirror symmetry within this thesis as a conjecture.

$$
\begin{array}{ccccccc}
 & & & \mathcal{M} & & & \\
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\quad \xrightarrow{\phi} \quad
\begin{array}{ccccccc}
 & & & \mathcal{W} & & & \\
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{2,1} & & 0 & \\
1 & & h^{1,1} & & h^{1,1} & & 1 \\
 & 0 & & h^{2,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Figure 3.3: Mirror symmetry for a Calabi-Yau threefold.

3.2.4 Integral Periods

Before moving on to discuss the mirror map, let us first introduce another very important tool for the analysis of moduli spaces of Calabi-Yau n-folds. As we have seen, the Kähler potentials of the moduli spaces are given in terms of integrals over certain combinations of Ω and J. Usually it is convenient to expand these forms in an explicit basis of the corresponding cohomology groups. The (moduli dependent) coefficients of these expansions are called the *periods* and turn out to be the main players for any explicit computation.

More generally, let $\Gamma \in H_n(X, \mathbb{Z})$ be any integral homology n-cycle on the Calabi-Yau manifold X, then

$$\Pi^\Gamma = \int_\Gamma \Omega \tag{3.26}$$

is called an integral period of Ω with respect to the n-cycle Γ. Hence, if $\{\Gamma^1, \ldots, \Gamma^{b^n}\}$ is a basis of integral n-cycles, any integral period can be written as an integral linear combination of the periods $\Pi^1, \ldots, \Pi^{b^n}$. Moreover, if $\{\alpha_1, \ldots, \alpha_{b^n}\}$ describes the basis of $H^n(X, \mathbb{Z})$ dual to $\{\Gamma^1, \ldots, \Gamma^{b^n}\}$, it follows directly from equation (3.26) that Ω enjoys the basis expansion

$$\Omega = \sum_{a=1}^{b^n} \Pi^a \alpha_a \quad , \quad \Pi^a = \int_{\Gamma^a} \Omega \; . \tag{3.27}$$

One may note that the period Π^a vanishes whenever α_a is orthogonal to Ω with respect to the inner product

$$\langle \eta, \rho \rangle = \int_X \eta \wedge \bar{\rho} \; . \tag{3.28}$$

Thus, it is natural to restrict the b^n-dimensional period vector $\Pi := (\Pi^1, \ldots, \Pi^{b^n})^T$ by neglecting its trivially vanishing entries. With this period vector at hand, it is possible to rewrite the Kähler potential (3.21) for the complex structure moduli space in terms of Π and $\bar{\Pi}$ via

$$K_{C.S.} = -\log\left(i \Pi^T \Sigma \bar{\Pi}\right) \tag{3.29}$$

with the inner product

$$\Sigma^{ab} := \int_X \alpha^a \wedge \bar{\alpha}^b \ . \tag{3.30}$$

The notion of periods extends to (smooth) families of Calabi-Yau n-folds $\mathcal{X}_{\mathcal{M}}$. Since the homology groups $H_k(X_z, \mathbb{Z})$ and their dual cohomology groups are topological objects, they are locally constant. Hence, we can always choose a basis $\{\Gamma^1, \ldots, \Gamma^{b^n}\}$ of n-cycles that are (locally) constant as sections of $H_k(X_{\mathbf{z}}, \mathbb{Z})$. As a consequence, the inner product (3.30) is locally constant along the complex structure moduli space whereas the full dependence of the holomorphic n-form Ω on the coordinates $\mathbf{z}$ of the complex structure moduli space is encoded in the periods $\Pi^a(\mathbf{z})$ which are meromorphic[21] functions on $\mathbf{z}$.

In a similar construction it is possible to obtain the Kähler potential of the complexified Kähler moduli space in terms of a period vector. Therefore, recall that $J = \frac{1}{2i}\mathrm{Im}(\mathcal{J})$ for the (complex-valued) $(1,1)$-form $\mathcal{J} = B + iJ$. Thus, if we expand $\mathcal{J}$ in the local frame $\{h_1, \ldots, h_{h^{1,1}}\}$ introduced in equation (3.11) with local coordinates $\mathbf{t} = (t^1, \ldots, t^{h^{1,1}})$ on $\mathcal{M}_{\mathbb{C}K}$ as

$$\mathcal{J} = \sum_{i=1}^{h^{1,1}} t^i h_i$$

we might rewrite the classical Kähler potential (3.12) as

$$K_{\mathbb{C}K} = -\log\left(\int_X \bigwedge^n J\right) = -\log\left(\frac{1}{n!}\sum_{i_1,\ldots,i_n=1}^{h^{1,1}} Y_{i_1\ldots i_n}\mathrm{Im}(t^{i_1}\cdots t^{i_n})\right) \tag{3.31}$$

with the n-tuple intersection numbers

$$Y_{i_1\ldots i_n} := \int_X h_{i_1} \wedge \cdots \wedge h_{i_n} \ . \tag{3.32}$$

Again, we assume that the chosen local frame $\{h_1, \ldots, h_n\}$ of $H^{1,1}(X)$ is locally constant along the complexified Kähler moduli space and hence the intersection numbers $Y_{i_1\ldots i_n}$ can be treated as constants. Similarly to the complex structure moduli space, we can rewrite this expression by introducing the period vector

$$\Pi = \begin{pmatrix} 1 \\ t^i \\ -\frac{1}{2}Y_{i_1,\ldots i_n}t^{i_1}t^{i_2} \\ \vdots \\ (-1)^{n-1}\frac{1}{n!}Y_{i_1\ldots i_n}t^{i_1}\cdots t^{i_n} \end{pmatrix} \tag{3.33}$$

[21] On generic points of $\mathcal{M}_{C.S.}$ the periods are holomorphic. However, if the corresponding Calabi-Yau geometry becomes singular, certain periods may obtain poles and logarithmic singularities.

and a corresponding inner product Σ such that the classical contribution to the Kähler potential is given by

$$K_{\mathbb{C}K} = -\log\left(i\Pi^T\Sigma\bar{\Pi}\right) \ . \tag{3.34}$$

For Calabi-Yau threefolds, the inner product Σ is given by the symplectic pairing

$$\Sigma = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \tag{3.35}$$

where each block is of size $(h^{1,1}+1)\times(h^{1,1}+1)$. However, for Calabi-Yau fourfolds, Σ fails to be symplectic and the additional block of size $h^{2,2}\times h^{2,2}$ obtains contributions in terms of the inverse intersection numbers.

So far, this analysis was rescricted to the classical Kähler potential. If we take quantum corrections into account, the periods obtain additional perturbative corrections that originate from the quantum product $\star$. These corrections are of lower order in the Kähler parameters $\mathbf{t}$ and hence are suppressed with respect to the classical terms in the large volume limit $t^i \to \infty$.

3.2.5 Asymptotics of Integral Periods

It has been shown [69–73] that the leading order quantum corrections for the integral periods of the complexified Kähler moduli space can be computed using a characteristic class of the corresponding Calabi-Yau manifolds; the Γ-class. This discussion is based on the framework of B-branes in the language of K-theory. For a review on K-thoery we refer to [74, 75]. In the following, we will briefly review the results from [73] of this construction for the case of Calabi-Yau fourfolds as those will play the central role in the following part of this thesis, however the Γ-class representation can be applied equivalently to any Calabi-Yau n-fold and is used for example in the following section 3.2.6 to deduce the explicit values for the intersection numbers Y_{ijk} for Calabi-Yau threefolds in terms of topological quantities.

In the framework of string theory, the integral periods Π of the complexified Kähler moduli space can be associated to the central charge of certain B-branes that are wrapped along the internal directions of the compactification space. Given a B-brane $\mathcal{E}$, the corresponding period $\Pi_{\mathcal{E}}$ is given by [73]

$$\Pi_{\mathcal{E}}(\mathbf{t}) = \int_X e^{\mathcal{J}}\ \Gamma_{\mathbb{C}}(X)\ \mathrm{ch}(\mathcal{E}^\vee) + \mathcal{O}(e^{2\pi i t^k},\dots) \ . \tag{3.36}$$

Here, $\Gamma_{\mathbb{C}}$ denotes the Γ-class of the Calabi-Yau manifold X which is defined to be the multiplicative characteristic class[22] of X based upon the series $\Gamma_{\mathbb{C}}(z) = e^{\frac{z}{4}}\Gamma(1-\frac{z}{2\pi i})$.

[22]Following refs. [72, 76], a multiplicative characteristic class of a topological manifold X based upon a series $f(z) = 1+\sum_{k=1}^{\infty} a_k z^k$ is given by $f(X) = 1+\sum_{k=1}^{\infty} a_k c_k(X)$. Here $c_k(X)$ denotes the k^{th} Chern class of X. Since $f(X)$ is an expression in terms of cohomology classes of X, the series terminates after finitely many terms.

Since we are interested in Calabi-Yau n-folds with $n \leq 4$, it suffices to consider the contributions to the series expansion up to fourth order. If we impose in addition the Calabi-Yau condition $c_1(X) = 0$, the Γ-class reads

$$\Gamma_{\mathbb{C}}(X) = 1 + \frac{1}{24}c_2(X) + \frac{i}{8\pi^3}\zeta(3)c_3(X) + \frac{1}{5700}(7c_2(X)^2 - 4c_4(X)) \ . \tag{3.37}$$

Moreover, $\mathrm{ch}(\mathcal{E}^\vee)$ denotes the Chern character of the K-theoretical sheaf corresponding to the given B-brane. As indicated in equation (3.36), the Γ-class representation gives only the asymptotic contributions to the integral periods in the large volume limit[23]. All remaining quantum corrections are summarized in $\mathcal{O}(e^{2\pi i t^k})$ and cannot be computed with this method. However, this information is sufficient to determine the set of integral periods uniquely. As we will discuss in section 3.3.4, we can use the mirror map to obtain the full quantum corrected integral periods by relating the asymptotic behavior of $\Pi_{\mathcal{E}}$ to the complex structure periods of the mirror partner, which are computable exactly as solutions of the Picard-Fuchs differential ideal.

Equation (3.36) gives the explicit form of a set of integral periods for the complexified Kähler moduli space (up to quantum corrections) purely in terms of topological quantities of any member of the underlying family of Calabi-Yau n-folds. In particular, a lot can be said about the structure of the Chern characters of certain types of B-branes which fixes most of the periods in an universal fashion. Following [73], we find the following summary of contributing brane configurations for a given Calabi-Yau n-fold X:

- 0-brane: A 0-brane corresponds just to a point which is represented in K-theory by the skyskraper sheaf $\mathcal{O}_{\mathrm{pt}}$. As this sheaf gives rise to a Chern character that is a $2n$-form, the remaining contribution to the corresponding period becomes

$$\Pi_{\mathrm{pt}}(\mathbf{t}) = \int_X 1 \cdot \mathrm{ch}(\mathcal{O}_{\mathrm{pt}}) = 1 \tag{3.38}$$

  for any Calabi-Yau n-fold.

- 2-branes: Any 2-brane on X can be realized by an embedded Mori cone curve γ^i on the Calabi-Yau n-fold. The corresponding Chern character can be computed using the Grothendieck–Riemann–Roch formula to be the Poincaré dual of γ^i which is a $2(n-1)$-form on X. Hence, the contribution to the asymptotic period becomes

$$\Pi^i(\mathbf{t}) = \int_X t^j h_j \wedge [\gamma^i] = t^i \tag{3.39}$$

- $2n$-branes: These maximal branes correspond to the structure sheaf $\mathcal{O}_X$ of X which has Chern character $\mathrm{ch}(\mathcal{O}_X^\vee) = 1$. Hence, the corresponding integral period, which

[23] Note that this asymptotic behavior deviates from the pure classical contributions given in equation (3.33) by including also the subleading polynomial terms in $\mathbf{t}$. However, the so-called *instanton contributions* of order $\mathcal{O}(e^{2\pi i t^k})$ are still neglected in this asymptotic structure.

we will call the *top period*, is given by

$$\Pi_{\text{top}}(\mathbf{t}) = \int_X e^{\mathcal{J}} \Gamma_{\mathbb{C}}(X) + \mathcal{O}(e^{2\pi i t^k}) \ . \tag{3.40}$$

By expanding the exponential and the Γ-class, the top period for a Calabi-Yau fourfold becomes (summation convention assumed)

$$\Pi_{\text{top}}(\mathbf{t}) = Y_{ijk\ell} t^i t^j t^k t^\ell + Y_{00ij} t^i t^j + Y_{000i} t^i + Y_{0000} + \mathcal{O}(e^{2\pi i t^k}) \tag{3.41}$$

where

$$\begin{aligned} Y_{ijk\ell} &= \int_X h_i \wedge h_j \wedge h_k \wedge h_\ell \ , \ Y_{00ij} = \frac{1}{24} \int_X h_i \wedge h_j \wedge c_2(X) \quad , \\ Y_{000i} &= \frac{i\zeta(3)}{8\pi^3} \int_X h_i \wedge c_3(X) \ , \ Y_{0000} = \frac{1}{5700} \left(7 \int_X c_2(X)^2 - 4\chi(X) \right) \end{aligned} \tag{3.42}$$

encode the topological information of X. Note that in the expression for Y_{0000} we have used the relation

$$\int_X c_4(X) = \chi(X) \tag{3.43}$$

between the top Chern class and the Euler characteristic $\chi(X)$.

- $2(n-1)$-branes: These subleading-dimensional branes are associated to the hyperplane divisors h_i of the Kähler cone whose structure sheafs $\mathcal{O}_{h_i}$ are the corresponding K-theoretical objects. Their periods are computed by

$$\Pi_i(\mathbf{t}) = \int_X e^{\mathcal{J}} \Gamma_{\mathbb{C}}(X)(1 - \text{ch}(\mathcal{O}_X(h_i)) + \mathcal{O}(e^{2\pi i t^k}) \tag{3.44}$$

where the last term gives an one-form contribution to the integrand which is proportinal to h_i. Hence, we find for Calabi-Yau fourfolds

$$\Pi_i(\mathbf{t}) = N(Y_{ijk\ell} t^j t^k t^\ell + Y_{00ij} t^j + Y_{000i}) + \mathcal{O}(e^{2\pi i t^k}) \tag{3.45}$$

where N denotes the normalization constant of $1 - \text{ch}(\mathcal{O}_X^\vee(h_i)) = N h_i$.

In addition to these universal brane configurations, Calabi-Yau fourfolds have one further type of contributing branes that are given by 4-branes. The periods which correspond to this type of branes are leading order quadratic in the Kähler moduli. If the 4-brane is realized by intersecting two (distinct) hyperplane divisors h_i and h_j ($i \neq j$), the corresponding sheaf is given by $\mathcal{O}^\vee_{h_i \cap h_j}$ and its Chern character can be computed to be

$$\text{ch}(\mathcal{O}^\vee_{h_i \cap h_j}) = 1 - \text{ch}(\mathcal{O}_X(h_i)) - \text{ch}(\mathcal{O}_X(h_j)) + \text{ch}(\mathcal{O}_X(h_i + h_j)) \ . \tag{3.46}$$

3.2.6 Projective Special Kähler Geometry for Calabi-Yau Threefolds

In the special case of Calabi-Yau threefolds, we can make use of the very powerful property of $\mathcal{M}_{C.S.}$ and $\mathcal{M}_{\mathbb{C}K}$ being not only Kähler manifolds but in particular projective special Kähler manifolds [77, 78] which allows to associate a holomorphic function, called the prepotential, to each of these moduli spaces which determines the Kähler potential and hence the period vector completely.

A Kähler manifold M with Kähler potential K is called *projective special Kähler*[24] if there exists a holomorphic section $F(w)$ of homogenous degree two such that the Kähler potential can be written as

$$K = -\log\left(2\mathrm{Im}\left(\sum_k \frac{\partial \bar{F}}{\partial \bar{w}^k} w^k\right)\right) = -\log\left(i\sum_k\left(\frac{\partial F}{\partial w^k}\bar{w}^k - \frac{\partial \bar{F}}{\partial \bar{w}^k} z^k\right)\right) \tag{3.47}$$

for a set of projective coordinates $\mathbf{w} = (w^0, \ldots, w^{h^n})$ on M. The holomorphic section F is called the *prepotential*. In this notation, the $\mathbf{w}$ are holomorphic local coordinates for the Kähler manifold M. Although very restrictive, the projective special Kähler property gives a very powerful tool for the analysis of the geometry of the manifold M as it allows to express all geometrical quantities in terms of a single holomorphic section F which can be computed explicitly.

In the following we will give a proof that for Calabi-Yau threefolds both, $\mathcal{M}_{C.S.}$ and $\mathcal{M}_{\mathbb{C}K}$, are indeed projective special Kähler manifolds by constructing the corresponding prepotentials. First, let us discuss the complexified Kähler moduli space $\mathcal{M}_{\mathbb{C}K}$. Equation (3.31) gives already an impression, how the prepotential can be obtained. However, we should note, that the prepotential was defined with respect to projective coordinates $\mathbf{w}$, whereas the Kähler parameters $\mathbf{t}$ are affine. We can nevertheless define projective coordinates $\mathbf{w}$ on $\mathcal{M}_{\mathbb{C}K}$ by

$$t^i = \frac{w^i}{w^0} \tag{3.48}$$

Then, by comparing the expressions for the Kähler potential, we find that[25]

$$\frac{1}{3!}\sum_{ijk=1}^{h^{1,1}} Y_{ijk}\mathrm{Im}(t^i t^j t^k) = 2\mathrm{Im}\left(\sum_{k=0}^{h^{1,1}} \frac{\partial \bar{F}}{\partial \bar{w}^k} w^k\right) . \tag{3.49}$$

which is solved by setting [61]

$$F(\mathbf{w}) = -\frac{1}{3!}\sum_{i,j,k=1}^{h^{1,1}} Y_{ijk}\frac{w^i w^j w^k}{w^0} . \tag{3.50}$$

[24] In the physics literature, such geometries are often called *Special Geometry* or *Local Special Kähler manifolds*. We follow the notation of [79].

[25] Note that we are discussing in this section the concrete case of Calabi-Yau threefolds.

Moreover, we see that the period vector Π given in equation (3.33) can be expressed in terms of the prepotential as

$$\Pi = \begin{pmatrix} 1 \\ t^i \\ F_i \\ F_0 \end{pmatrix} \tag{3.51}$$

with

$$F_i = \frac{\partial F}{\partial w^i} \ , \quad i = 0, \dots, h^{1,1} \tag{3.52}$$

and the inner product Σ takes indeed the form of (3.35). So far, this prepotential contains only the classical contributions in the strict large volume limit. Even after including all quantum corrections, the complexified Kähler moduli space can be shown to be projective special Kähler. Thus, the structure as introduced above remains unchanged, however the prepotential F gets modified by the quantum corrections. It has been worked out [80–82] that for a smooth family of Calabi-Yau threefolds, the full quantum corrected prepotential for the complexified Kähler moduli space takes the form

$$F(\mathbf{w}) = \sum_{i,j,k=0}^{h^{1,1}} -\frac{1}{3!} Y_{ijk} \frac{w^i w^j w^k}{w^0} + (w^0)^2 F_{\text{Inst}}(q^1, \dots, q^{h^{1,1}}) \tag{3.53}$$

with

$$\begin{aligned} Y_{ijk} &= \int_X h_i \wedge h_j \wedge h_k \quad , \quad Y_{0ij} \in \left\{0, \frac{1}{2}\right\} \quad , \\ Y_{00i} &= -\frac{1}{12} \int_X c_2(X) \wedge h_i \quad , \quad Y_{000} = -\chi(X) \frac{3\zeta(3)}{(2\pi i)^3} \end{aligned} \tag{3.54}$$

being topological invariants of the family $\mathcal{X}_{\mathcal{M}}$ of Calabi-Yau threefolds that are computed in analogy to the Γ-class construction for Calabi-Yau fourfolds as described in section 3.2.5. Note, that we have used the identiy

$$\int_X c_3(X) = \chi(X) \tag{3.55}$$

which holds for any Calabi-Yau threefold X to express Y_{000} in terms of the Euler characteristic $\chi(X)$. Moreover,

$$F_{\text{Inst}}(q^1, \dots, q^{h^{1,1}}) = \frac{1}{(2\pi i)^3} \sum_{\gamma \in \mathbb{N}^{h^{1,1}}} n_\gamma \text{Li}_3(q^\gamma) \qquad q^k = e^{2\pi i t^k} \tag{3.56}$$

collects all contributions coming from worldsheet instantons[26]. The function $\text{Li}_s(z)$ denotes the polylogarithm with is defined to be the analytic continuation of the series

$$\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} \ . \tag{3.57}$$

[26]Here, we make use of the convenient short-hand notation $q^\gamma = (q^1)^{\gamma_1} \cdots (q^{h^{1,1}})^{\gamma_{h^{1,1}}}$ for any tuple of integers $\gamma = (\gamma_1, \dots, \gamma_{h^{1,1}}) \in \mathbb{N}^{h^{1,1}}$.

The integers n_γ are known as genus-zero instanton numbers[27] and have an enumerative interpretation by counting rational curves on the Calabi-Yau threefold [83–85].

A similar analysis can be performed for the complex structure moduli space $\mathcal{M}_{C.S.}$. Again, we make use of the fact that the setting is concretized to Calabi-Yau threefolds. In this case, it is always possible to find a symplectic basis of integral 3-cycles in $H_3(X, \mathbb{Z})$. Given such a symplectic basis (A^a, B_a) $(a \in \{0, \dots, h^{2,1}\})$ of 3-cycles[28] obeying

$$A^a \cap B_b = -\delta^a_b \qquad A^a \cap A^b = B_a \cap B_b = 0 \tag{3.58}$$

the dual basis of $H^3(X, \mathbb{Z})$ is defined by 3-forms α_a and β^b obeying

$$\int_X \alpha_a \wedge \omega = \int_{A^a} \omega \qquad \int_X \beta^b \wedge \omega = \int_{B_b} \omega \tag{3.59}$$

for any 3-form ω. Thus, by expanding the holomorphic 3-form $\Omega \in H^{3,0}(X)$ in this integral basis, we find

$$\Omega = X^a \alpha_a - F_b \beta^b \tag{3.60}$$

with

$$X^a = \int_{A^a} \Omega \quad , \quad F_b = \int_{B_b} \Omega \tag{3.61}$$

being integral periods of Ω with respect to the basis (A^a, B_a) of integral 3-cycles. Inserting this expansion in equation (3.21), the Kähler potential of $\mathcal{M}_{C.S.}$ reads

$$K_{C.S.} = -\log\left(i \sum_{a=0}^{h^{2,1}} (F_a \bar{X}^a - \bar{F}_a X^a) \right) \ . \tag{3.62}$$

Following [61, 86] it can be shown that for any family of Calabi-Yau threefold there exists a holomorphic function $F(X)$ such that $F_b = \frac{\partial F}{\partial X^b}$, hence F serves as a prepotential turning $\mathcal{M}_{C.S.}$ into a special Kähler manifold.

The symplectic structure of the moduli spaces is a special feature of Calabi-Yau threefolds. Already for Calabi-Yau fourfolds, this structure breaks down and it is impossible to describe the periods by a prepotential[29]. Hence, $\mathcal{M}_{C.S.}$ and $\mathcal{M}_{\mathbb{C}K}$ are not projective special Kähler manifolds for Calabi-Yau fourfolds. However, the structure of the integral periods can still be deduced using the Γ-class representation for the classical contributions to the periods and then applying the mirror map to obtain all quantum corrections.

[27] In the literature these are also called Gromov-Witten invariants.

[28] Recall that $H^3(X, \mathbb{C})$ and therefore also $H_3(X, \mathbb{Z})$ are $2(h^{2,1} + 1)$-dimensional.

[29] This observation follows trivially by observing that the Dolbeault decomposition of $H^4(X, \mathbb{C})$ as well as the decomposition of the even cohomology have a self-dual contribution by $H^{2,2}(X)$ that spoils the symplectic structure.

3.3 The Picard-Fuchs Ideal

So far, we have mainly discussed the structure of the periods for the complexified Kähler moduli space and even for those, the discussion was restricted to the asymtotic part, neglecting the instanton corrections. In the following, we will focus on the complex structure moduli space and deduce a method to explicitly compute its corresponding periods. Going back to an idea by Griffiths and Dwork [87–90], it is possible to derive a set of differential equations, the Picard-Fuchs ideal, whose space of solutions is precisely spanned by the integral periods $\Pi^a(z)$ from equation (3.27).

Recall that the periods $\Pi^a(\mathbf{z})$ are defined to be the coefficients of the holomorphic n-form $\Omega \in H^{n,0}(X_{\mathbf{z}}, \mathbb{C})$ in terms of a given integral basis $\{\alpha_a\} \subset H^n(X_{\mathbf{z}}, \mathbb{Z})$. The Picard-Fuchs ideal is constructed by successively taking derivatives of $\Omega(\mathbf{z})$ with respect to the complex structure moduli z^i, until this procedure gives a closed relation. Obviously, differentiating Ω gives again an n-form, however, following [61], this form is not necessarily holomorphic any longer but contains (at most) one anti-holomorphic direction. By iterating this argument, we find that the derivative operator is a map on the Hodge filtration

$$H^{n,0}(X_{\mathbf{z}}, \mathbb{C}) = F_0 \subset F_1 \subset \cdots \subset F_n = H^n(X_{\mathbf{z}}, \mathbb{C}) \tag{3.63}$$

of $H^n(X_{\mathbf{z}}, \mathbb{C})$ with the filtration spaces F_k being defined as

$$F_k = \oplus_{q=0}^{k} H^{n-q,q}(X_{\mathbf{z}}, \mathbb{C}) \tag{3.64}$$

such that the derivatives with respect to the complex structure moduli z^i act according to

$$\frac{\partial}{\partial z^i} : \begin{cases} F_k \to F_{k+1} & \text{for } 0 \leq k < n \\ F_n \to F_n & \text{else} \end{cases} \quad . \tag{3.65}$$

Since each of these filtration spaces is finite dimensional and $\dim(F_n) = b^n$, we find that at most b^n derivatives of Ω can be independent of each other. Any additional derivative of Ω can hence be written as a linear combination of the former b^n derivatives, leading to a differential equation for Ω. By definition (c.f. with equation (3.27)), this translates into a differential equation for the periods. By this procedure, one could in principle produce an infinite number of differential operators for the periods. However, collecting these by defining a differential ideal, which we call in the following the *Picard-Fuchs ideal*, it can be shown that this ideal is always finitely generated.

Following the construction by Griffiths and Dwork, there exists an algorithmic way to systematically reduce derivatives of Ω in order to construct the Picard-Fuchs ideal for any given family of Calabi-Yau manifolds [61, 91, 92].

3.3.1 The Primary Horizontal Subspace of $H^n(X_{\mathbf{z}},\mathbb{C})$

It is convenient to define the *primary horizontal subspace* of $H^n(X_{\mathbf{z}},\mathbb{C})$ [93–95] by

$$H^n_H(X_{\mathbf{z}},\mathbb{C}) =< \partial_I\Omega >_{I\in\mathcal{I}} \subseteq H^n(X_{\mathbf{z}},\mathbb{C}) \tag{3.66}$$

where $\mathcal{I} \cong \mathbb{N}_0^{h^{(n-1),1}}$ is an index set collecting all derivatives of $\Omega(z)$ with respect to the complex structure moduli $\mathbf{z}$. Note that $H^n_H(X_{\mathbf{z}},\mathbb{C})$ is always finite dimensional with $\dim(H^n_H(X_{\mathbf{z}},\mathbb{C})) \leq b^n$ as $H^n(X_{\mathbf{z}},\mathbb{C})$ is finite dimensional. Hence, it suffices to consider only a finite subset

$$\mathcal{J} = \{\mathrm{Id}, \mathcal{D}_1, \ldots, \mathcal{D}_{b-1}\} \quad , \quad b = \dim(H^n_H(X_z,\mathbb{C}) \tag{3.67}$$

of differential operators to generate $H^n_H(X_{\mathbf{z}},\mathbb{C})$ by acting on Ω, meaning

$$H^n_H(X_{\mathbf{z}},\mathbb{C}) = \langle \mathcal{D}\Omega \rangle_{\mathcal{D}\in\mathcal{J}} \ . \tag{3.68}$$

The Picard-Fuchs ideal for Ω is now given by rewriting all additional derivatives that do not belong to $\mathcal{J}$ in terms of these generating elements.

For any family of Calabi-Yau n-folds, we have that $\Omega \in H^n_H(X_{\mathbf{z}},\mathbb{C})$ as $0 \in \mathcal{I}$ represents the identity operator $\partial_0\Omega = \Omega$. It is natural to define the subspace $H^n_\perp(X_{\mathbf{z}},\mathbb{C})$ by the following decomposition of the middle cohomology of $X_{\mathbf{z}}$

$$H^n(X_{\mathbf{z}},\mathbb{C}) =: H^n_H(X_{\mathbf{z}},\mathbb{C}) \oplus H^n_\perp(X_{\mathbf{z}},\mathbb{C}) \ . \tag{3.69}$$

We can deduce that the periods $\Pi^a(\mathbf{z})$ which correspond to basis elements $\alpha_a \in H^n_\perp(X_{\mathbf{z}},\mathbb{C})$ need to vanish as these are by definition orthogonal to Ω with respect to the inner product (3.28). Thus, it is convenient to restrict the full period vector by neglecting the orthogonal directions and viewing it as the coefficient vector of $\Omega(\mathbf{z})$ expanded in an integral basis of $H^n_H(X_{\mathbf{z}},\mathbb{C})$.

It should be emphasized that the definition of the primary horizontal subspace is redundant for the discussion of families of Calabi-Yau threefolds. For a generic point $\mathbf{z}$ on the complex structure moduli space, the three-form $\Omega(\mathbf{z})$ and its the first derivatives $\partial_i\Omega(\mathbf{z})$ are independent. Moreover, the symplectic structure of $H^3(X_z,\mathbb{Z})$ enforces the complex conjugates of these $(h^{2,1}+1)$ three-forms to be additional independent directions. Since these are elements of the filtration spaces F^2 and F^3, they are generated by second and third derivatives of Ω. Thus, it follows that in total $2(h^{2,1}+1)$ derivatives of Ω are independent at $\mathbf{z}$, hence

$$\dim(H^n_H(X_{\mathbf{z}},\mathbb{C})) = 2(h^{2,1}+1) = b^3 \ . \tag{3.70}$$

Thus, due to the dimensionality, we obtain at generic points[30]

$$H^3_H(X_{\mathbf{z}},\mathbb{C}) = H^3(X_{\mathbf{z}},\mathbb{C}) \quad \text{and} \quad H^3_\perp(X_{\mathbf{z}},\mathbb{C}) = 0 \ . \tag{3.71}$$

[30] One should note that there exist non-generic points and also extended subloci on $\mathcal{M}_{C.S.}$ for families of Calabi-Yau threefolds for which this argument fails. This phenomon happens for example on the invariant locus of a discrete symmetry of $\mathcal{M}_{C.S.}$ since here several periods become equal[7, 96].

For families of Calabi-Yau fourfolds with $h^{3,1}$ complex structure parameters, the structure of $H^4(X_{\mathbf{z}}, \mathbb{C})$ is slightly more involved since it contains in addition to the complex structure deformations contributions from $H^{2,2}(X_{\mathbf{z}}, \mathbb{C})$. It turns out that this subspace plays a special role in the Hodge theoretical consideration of Calabi-Yau fourfolds as it obtains contributions from both, the horizontal middle cohomology of $H^4(X_{\mathbf{z}}, \mathbb{C})$ and the vertical intersection theory of the $H^{k,k}(X_{\mathbf{z}}, \mathbb{C})$[31]. Thus, in general we have

$$\langle h_i \wedge h_j \rangle_{i,j=1,\dots,h^{1,1}} \subseteq H^4_{\perp}(X_{\mathbf{z}}, \mathbb{C}) \tag{3.72}$$

for h_i being the generators of the cohomology group $H^{1,1}(X_{\mathbf{z}}, \mathbb{C})$. This implies that $H^4_{\perp}(X_{\mathbf{z}}, \mathbb{C}) \neq 0$. Hence, the middle cohomology is not completely determined by $\Omega(\mathbf{z})$ but we rather have to distinguish between $H^4(X_{\mathbf{z}}, \mathbb{C})$ and the primary horizontal subspace $H^4_H(X_{\mathbf{z}}, \mathbb{C})$.

3.3.2 Frobenius Periods for Calabi-Yau Threefolds

Solving the Picard-Fuchs ideal gives a complex b-dimensional vector space, which is generated by the integral periods $\Pi^a(z)$. Standard techniques for solving a system of partial differential equations give some generators $\varpi^a(z)$ of this vector space, however these usually do not correspond to integrals of $\Omega(z)$ over integral n-cycles. Nevertheless it turns out to be very useful in practice to solve the Picard-Fuchs system, leading to what we will call the *Frobenius Periods* ϖ^a in the following. The integral period vector can then be obtained by performing a suitable change of basis.

If a given family of Calabi-Yau threefolds $\mathcal{X}_{\mathcal{M}}$ has a mirror partner which enjoys a geometric realization, the complex structure moduli space of $\mathcal{X}_{\mathcal{M}}$ has a non-generic, singular point which allows to expand the periods in a very convenient way. Recall, that $\mathcal{W}_{\mathbb{C}K}$, the mirror of $\mathcal{M}_{C.S.}$, contains information about the size of certain branes on the mirror geometry. Hence, the limit $t^i \to \infty$, which is called the large volume point (LVP), corresponds to a non-generic point on $\mathcal{M}_{C.S.}$ which we call the large complex structure Point[32] (LCS). Following the method by Frobenius on solutions of differential equations in the vicinity of regular singular points, any general solution to the Picard-Fuchs ideal can be decomposed into certain basis functions which have a definite singularity structure. Choosing the complex structure parameters $\mathbf{z}$ such that the origin $\mathbf{z} = 0$ corresponds to the LCS-point, these basis functions can be expressed and even constrained due to the monodromy behavior of $z^i \mapsto e^{2\pi i} z^i$ around the LCS-point. For a family of Calabi-Yau threefolds with $h^{2,1}$ complex structure parameters, the vector of Frobenius periods reads

[31] In addition it is so far unclear, whether there is a third contribution to $H^{2,2}(X_{\mathbf{z}}, \mathbb{C})$ that originates neither from derivatives of Ω nor from intersecting $(1,1)$-forms. A more detailed discussion on these additional contributions can be found in refs. [94, 95].

[32] Since the complex structure is responsible for the shape of the corresponding manifold, the notion of "large" (or small) complex structure is meaningless in the geometric sense. This name refers only to the fact that it corresponds to a large volume point under mirror symmetry.

around the large complex structure point

$$\varpi(\mathbf{z}) = \begin{pmatrix} \varpi^0(\mathbf{z}) \\ \varpi^a(\mathbf{z}) \\ \varpi_a(\mathbf{z}) \\ \varpi_0(\mathbf{z}) \end{pmatrix} \tag{3.73}$$

with

$$\begin{aligned}
\varpi^0(\mathbf{z}) &= A(\mathbf{z}) \\
\varpi^a(\mathbf{z}) &= \log(z^a)A(\mathbf{z}) + B^a(\mathbf{z}) \\
\varpi_a(\mathbf{z}) &= Y_{abc}(\log(z^b)\log(z^c)A(\mathbf{z}) + 2\log(z^b)B^c(\mathbf{z}) + C^{bc}(\mathbf{z})) \\
\varpi_0(\mathbf{z}) &= \frac{Y_{abc}}{3!}\Big(\log(z^a)\log(z^b)\log(z^c)A(\mathbf{z}) + 3\log(z^a)\log(z^b)B^c(\mathbf{z}) \\
&\qquad +3\log(z^a)C^{bc}(\mathbf{z}) + D^{abc}(\mathbf{z})\Big) \ .
\end{aligned} \tag{3.74}$$

Here, $A(\mathbf{z})$, $B^a(\mathbf{z})$, $C^{ab}(\mathbf{z})$ and $D^{abc}(\mathbf{z})$ are holomorphic functions and the Y_{ijk} are the triple intersection numbers of the mirror manifold. These so far unknown holomorphic functions can be computed iteratively by expanding them as power series and imposing that each period is a solution to the Picard-Fuchs ideal. By this procedure one obtains recursion relations for the coefficients appearing in the series expansion.

3.3.3 Frobenius Periods Calabi-Yau Fourfolds

A similar discussion can be worked out for the Frobenius periods of Calabi-Yau fourfolds. However, since the primary horizontal subspace $H^4_H(X_{\mathbf{z}},\mathbb{C})$ is a proper subspace of $H^4(X_{\mathbf{z}},\mathbb{C})$, the explicit construction is more involved. As it turns out, a similar construction as in the previous section can be obtained by the method of Frobenius to solve the Picard-Fuchs system for Calabi-Yau fourfolds. For a family of Calabi-Yau fourfolds with $h^{3,1}$ complex structure moduli and $b = \dim(H^n_H(X_{\mathbf{z}},\mathbb{C})) = 2(h^{3,1}+1)+m$, this construction predicts a period vector of the form

$$\varpi(\mathbf{z}) = \begin{pmatrix} 1+\mathcal{O}(\mathbf{z}) \\ \log(z^a)\varpi^0(\mathbf{z}) + \mathcal{O}(\mathbf{z}) \\ \log(z^a)\log(z^b)\varpi^0(\mathbf{z}) + \mathcal{O}(log(\mathbf{z})\mathbf{z}) \\ Y_{abcd}\log(z^b)\log(z^c)\log(z^d)\varpi^0(\mathbf{z}) + \mathcal{O}(log^2(\mathbf{z})\mathbf{z}) \\ \frac{1}{4!}Y_{abcd}\log(z^a)\log(z^b)\log(z^c)\log(z^d)\varpi^0(\mathbf{z}) + \mathcal{O}(log^3(\mathbf{z})\mathbf{z}) \end{pmatrix} \tag{3.75}$$

with $\varpi^0(\mathbf{z})$ being a holomorphic function. Here, we have listed only the leading order contributions to the period functions. The additional subleading terms that are still logarithmic in z^a can be uniquely determined by the mirror map which is discussed in the next section and the semi-classical structure of the integral periods on the mirror partner of $X_{\mathbf{z}}$. While the leading order behavior of the $\sim \log^3(\mathbf{z})$-periods and the top period are determined by the quadruple intersection numbers Y_{abcd} of the mirror, such an identification is impossible for the m independent $\log^2(\mathbf{z})$-periods.

Throughout this work, we will be mainly interested in computing the periods for Calabi-Yau fourfolds in the special case of $h^{3,1} = 1$. Due to the simple structure of the appearing functions that depend only on one complex structure parameter z, the general structure of the period vector can be further concretized. If $h^{3,1} = 1$, there exists exactly one direction of derivatives ∂_z. By iterative differentiation of $\Omega(z)$ with respect to z, we obtain $b = \dim(H^n_H(X_z, \mathbb{C})$ independent four-forms $\partial_z^k \Omega(z)$ for $k = 0, \ldots, b-1$. However, on dimensional grounds, the b's derivative needs to be linearly dependent on the former one's. Hence, we obtain one single Picard-Fuchs operator[33] $\mathcal{L}$ of degree b. In analogy to the threefold case, the corresponding set of Frobenius periods expanded around the LCS-point is constrained by momodromy transformations. One should remark that the $b-4$ solutions of the type

$$\varpi_2^{(k)} = \log^2(z) A(z) + 2\log(z) B(z) + C^{(k)}(z) \ , \tag{3.76}$$

that correspond to the generators of $H^4_H(X_z, \mathbb{Z}) \cap H^{2,2}(X_z, \mathbb{C})$, deviate only in the holomorphic function $C^{(k)}(z)$. Hence, it is convenient to perform a change of basis for this part of the primary horizontal subspace such that the logarithmic contributions are absorbed for all but one $\varpi_2^{(k)}$. Thus, without loss of generality, the period vector for families of Calabi-Yau fourfolds with $h^{3,1} = 1$ is given by

$$\varpi(z) = \begin{pmatrix} \varpi^0(z) \\ \vdots \\ \varpi^4(z) \\ C^{(1)}(z) \\ \vdots \\ C^{(b-5)}(z) \end{pmatrix} \tag{3.77}$$

with

$$\varpi^i(z) = \sum_{k=0}^{i} \log^k(z) A_{i-k}(z) \quad \text{for } i = 0, \ldots, 4 \ . \tag{3.78}$$

Thus, the full period vector is characterized by b holomorphic functions $A_i(z)$ and $C^{(k)}(z)$ that can be solved by deducing a recursion relation for the coefficients in their power series expansion that originates from the Picard-Fuchs equation $\mathcal{L}\varpi(z) = 0$. We present a derivation of these recursion relations in appendix C.

3.3.4 The Mirror Map

To finish our general discussion of Calabi-Yau geometries and in particular of the moduli spaces and their periods, let us come back to the concept of mirror symmetry. So far, we have stated the existence of a map ϕ between the moduli spaces of a pair of families of

[33] Since any further derivative of Ω can be treated as a derivative of $\partial_z^b \Omega$, the corresponding differential operators always factorize into $\mathcal{L}$ and a remainder. Hence, the Picard-Fuchs ideal is generated only by this unique operator of minimal degree.

Calabi-Yau manifolds in section 3.2.3. This map interchanges the role of complex structure and complexified Kähler moduli space meaning in practice that it maps the integral periods of the complex structure moduli space $\mathcal{M}_{C.S.}$ to the periods of the complexified Kähler moduli space $\mathcal{W}_{\mathbb{C}K}$ of the mirror family. One can now collect all partial information about the periods on both sides that we have gathered in the previous sections, to obtain a set of quantum corrected integral periods.

To that end, let us recall that the asymptotic structure of the integral periods on $\mathcal{W}_{\mathbb{C}K}$ was fixed by the Γ-class representation given in section 3.2.5 whereas the Frobenius periods as solutions to the Picard-Fuchs ideal of $\mathcal{M}_{C.S.}$ give exact solutions (as the complex structure moduli space is protected from quantum corrections) but the Frobenius periods are generically not integral. The idea is now, to find proper linear combinations of the Frobenius periods that give rise to the correct asymptotic structure which is computed for the mirror partner. The strategy is as follows:

- The skyscraper period was set to be $\Pi_{\text{pt}} = 1$. Since we have the freedom, to rescale Ω by any holomorphic function, we can rescale it by the fundamental period $\varpi^0(\mathbf{z})$ such that the new, normalized vector of Frobenius periods becomes

$$\tilde{\varpi}(\mathbf{z}) = \frac{1}{\varpi^0(\mathbf{z})} \cdot \varpi(\mathbf{z}) = \begin{pmatrix} 1 \\ \frac{\varpi^a(\mathbf{z})}{\varpi^0(\mathbf{z})} \\ \vdots \\ \frac{\varpi_a(\mathbf{z})}{\varpi^0(\mathbf{z})} \\ \frac{\varpi_0(\mathbf{z})}{\varpi^0(\mathbf{z})} \end{pmatrix} . \tag{3.79}$$

- The 2-brane periods correspond just to the complexified Kähler parameters t^a. Hence, we impose the identification

$$2\pi i t^a = \frac{\varpi^a(\mathbf{z})}{\varpi^0(\mathbf{z})} = \log(z^a) + \frac{B^a(\mathbf{z})}{A(\mathbf{z})} \tag{3.80}$$

which gives the explicit mirror map $\phi : \mathcal{M}_{C.S.} \to \mathcal{W}_{\mathbb{C}K}$.

- The mirror map is a local diffeomorphism of the moduli spaces. Hence, we can invert the former expression to obtain $\phi^{-1} : \mathcal{W}_{\mathbb{C}K} \to \mathcal{M}_{C.S.}$. Inserting this invers mirror map for the logarithmic contributions of the remaining Frobenius periods, one can identify a unique set of linear combinations of those such that their asymptotic dependence on $\mathbf{t}$ matches with that of the integral periods on $\mathcal{W}_{\mathbb{C}K}$.

In practice, we find for Calabi-Yau threefolds that

$$\Pi(\mathbf{t}) = \begin{pmatrix} 1 \\ t^a \\ F_a \\ F_0 \end{pmatrix} = M \cdot \tilde{\varpi}(\mathbf{z}(\mathbf{t})) \tag{3.81}$$

with the change-of-basis matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathbb{1}_n & 0 & 0 \\ -\frac{1}{2}Y_{i00} & Y_{ij0} & -\mathbb{1}_n & 0 \\ -\frac{1}{3}Y_{000} & -\frac{1}{2}Y_{00i} & 0 & 1 \end{pmatrix} \nu^{-1} \ . \tag{3.82}$$

Here, ν denotes the diagonal matrix

$$\nu = \mathrm{diag}\left(1, 2\pi i \mathbb{1}_n, (2\pi i)^2 \mathbb{1}_n, (2\pi i)^3\right) \ . \tag{3.83}$$

with n being the dimension of $\mathcal{M}_{C.S.}$. Hence, the matrices M and ν have a total size of $2(n+1) \times 2(n+1)$.

For families of Calabi-Yau fourfolds, the asymptotic structure of the Frobenius periods can be obtained similarly by replacing

$$t^a \mapsto \frac{1}{2\pi i} \frac{\varpi^a(\mathbf{z})}{\varpi^0(\mathbf{z})} \ . \tag{3.84}$$

As soon as the structure of the logarithms is uniquely fixed by this procedure, the sub-leading[34] terms of the power series follow from recursion relations due to the Picard-Fuchs equations.

[34] In the large complex structure limit $z \to 0$

Chapter 4

Supersymmetric Flux Compactifications

Flux compactifications play a central role for string phenomenological constructions of effective field theories from string or M-Theory [9, 41, 67, 97–100] as they serve as possible candidates for solving the problem of “moduli stabilization“. For a given Calabi-Yau compactification of string or M-Theory, there is not a unique vacuum configuration but the space of possible vacua is spanned by the complex structure and complexified Kähler moduli of the Calabi-Yau manifold. Moreover, non-perturbative effects generate additional potentials beyond the low energy supergravity description that evolve the moduli to large volume and hence an extremely weakly coupled system [101, 102]. In order to obtain a quantum field theory with a unique vacuum and an internal compactification space of finite volume, a mechnanism to fix all moduli to a certain value is required.

One proposal for such a mechanism considers the coupling of the complex structure moduli to internal n-form fluxes $G \in H^n(X, \mathbb{Z})$ on the Calabi-Yau n-fold X subject to certain quantization and consistency conditions. The corresponding contribution to the $\mathcal{N} = 1$ supergravity superpotential reads [9]

$$W = \int_X G \wedge \Omega(\mathbf{z}) \tag{4.1}$$

and constrains for a given flux vector G the possible vacuum states enormously. Note that this superpotential originates from a semi-classical treatment of the flux compactification. In general, it obtains perturbative and non-perturbative quantum corrections [9, 42, 67, 68, 103] which we will not discuss further within this work.

In the following chapter, we discuss these flux compactifications for type IIB string theories compactified on a Calabi-Yau threefold in section 4.1 and for M-Theory compactified on a Calabi-Yau fourfold in section 4.4. In particular, we derive explicit constraints for supersymmetric flux vacua in both cases and deduce from them the necessary and sufficient types of the integral flux $G \in H^n(X, \mathbb{Z})$. Moreover, we provide an equivalent description of the flux vacuum constraints for type IIB flux compactifications in the language of $\mathcal{N}=2$ supergravity. Since supersymmetry prohibits the existence of a superpotential in this case, the framework of gauged $\mathcal{N}=2$ supergravity is required for this discussion.

4.1 Flux Vacua in Type IIB String Theory

For type IIB string theory compactified on a Calabi-Yau threefold $X_{\mathbf{z}}$, flux compactifications give rise to a coupling of vector multiplets and hypermultiplets in the corresponding low energy supergravity theory according to the flux superpotential (4.1) with the three-form flux [104]

$$G = F - \tau H \ . \tag{4.2}$$

Here, $F, H \in H^3(X_z, \mathbb{Z})$ are integral flux vectors on $X_{\mathbf{z}}$ and τ denotes the axio-dilaton field of the universal hypermultiplet. This superpotential furnishes a scalar potential [104]

$$V = e^K \left(\nabla_a W \overline{\nabla^a W} - 3|W|^2 \right) \tag{4.3}$$

on the moduli space

$$\mathcal{M}_{C.S.} \times \mathcal{M}_{\mathbb{C}K} \times \mathcal{H} \ . \tag{4.4}$$

The additional factor $\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \geq 0\}$, denoting the siegel upper half plane, describes the moduli space of the axio-dilaton τ. K denotes the Kähler potential of the moduli space which reads

$$K = K_{C.S.} + K_{\mathbb{C}K} - \log(2\mathrm{Im}(\tau)) \tag{4.5}$$

and the covariant derivative $\nabla_i = \partial_i + (\partial_i K)$ is a $h^{2,1} + h^{1,1} + 1$ dimensional vector with $h^{2,1}$ directions along the vector multiplet scalars $\mathbf{z}$, $h^{1,1}$ directions along the complexified Kähler moduli $\mathbf{t}$ and one additional direction along the axio-dilaton τ. Possible vacuum configuration of this theory are obtained if the scalar potential vanishes. If we require in addition that the vacuum solution is supersymmetric, the superpotential needs to vanish additionally [104, 105]. The scalar potential V obeys what is called a no-scale relation [67, 106, 107]. This implies that the contribution $\nabla_{t^k} W \overline{\nabla^{t^k} W}$ containing derivatives with respect to the Kähler moduli just cancels against the term $3|W|^2$. Hence, the scalar potential reduces to

$$V = e^K \left(\nabla_{z^i} W \overline{\nabla^{z^i} W} + |\nabla_\tau W|^2 \right) \ . \tag{4.6}$$

Thus, the vacuum constraints $V = 0$ and $W = 0$ can be combined into the following set of equations on W

$$\partial_{z^k} W = 0 \quad , \quad \partial_\tau W = 0 \quad , \quad W = 0 \ , \tag{4.7}$$

which we call the $\mathcal{N}=1$ *supersymmetric flux vacuum constraints*. Recalling the explicit form of W from equations (4.1) and (4.2)

$$W = \int_X \Omega(\mathbf{z}) \wedge (F - \tau H) = \Pi^b(\mathbf{z}) \Sigma_{ab} (F^b - \tau H^b) \tag{4.8}$$

these vacuum constraints can be analyzed explicitly on the level of the integral period vector $\Pi(\mathbf{z})$. The latter equality in equation (4.8) is a consequence of expanding the three-forms Ω, F and H in an intergral basis $\{\alpha^a\}$ of $H^3(X,\mathbb{Z})$, where F^b and H^b denote the coefficients of F and H in this basis-representation respectively. A direct computation shows that these equations are solved if and only if

$$\int_X \Omega \wedge F = 0 \quad , \quad \int_X \Omega \wedge H = 0 \tag{4.9}$$

and in addition

$$\int_X (\partial_{z^k}\Omega) \wedge (F - \tau H) = 0 \ . \tag{4.10}$$

In terms of coordinates with respect to the basis $\{\alpha_a\}$, these conditions become

$$\Pi^a \Sigma_{ab} F^b = 0 \quad , \quad \Pi^a \Sigma_{ab} H^b = 0 \tag{4.11}$$

and

$$(\partial_{z^k}\Pi^a)\Sigma_{ab}(F^b - \tau H^b) = 0 \ . \tag{4.12}$$

Note that the supersymmetric flux vacuum constraints, as they are given so far, only constrain the complex structure moduli and the axio-dilaton and leave the complexified Kähler moduli unconstrained. This observation is related to the fact that the superpotential in equation (4.1) gives a semi-classical approximation in the large volume limit. As soon as perturbative and non-perturbative quantum corrections are taken into account, the flux vacuum constraints need to get modified and become dependent on the complexified Kähler moduli as well [67, 68, 103].

The challenging task is to decide, whether a point[35] $(\mathbf{z},\tau) \in \mathcal{M}_{C.S.} \times \mathcal{M}_\tau$ in the moduli space[36] of a given family of Calabi-Yau threefolds corresponds to a manifold that has two independent integral flux vectors $F, H \in \mathbb{H}^3(X_\mathbf{z},\mathbb{Z})$ solving equations (4.9) and (4.10). By independence of F and H, we mean that

$$\int_X F \wedge H \neq 0 \ . \tag{4.13}$$

At this point, we should notice an important consequence of the flux equation (4.9) on the Hodge type of the three-form fluxes F and H. Since $H^n(X_\mathbf{z},\mathbb{Z}) \subset H^n(X_\mathbf{z},\mathbb{C})$, we can decompose the fluxes F and H with respect to the Hodge structure of $H^n(X_\mathbf{z},\mathbb{C})$. Recall, that $\Omega(\mathbf{z}) \in H^{3,0}(X_\mathbf{z},\mathbb{C})$ is defined to be the unique holomorphic three-form on $X_\mathbf{z}$. Hence, only the $(0,3)$-part of F and H contributes to the integrals in equation (4.11)

[35]In the following, we collect all complex structure in the short notation $\mathbf{z}$ which always denotes a vector of $h^{2,1}$ local coordinates of $\mathcal{M}_{C.S.}$. Moreover, we use $\partial_\mathbf{z}$ as a short notation for the $h^{2,1}$ dimensional gradient with respect to the complex structure moduli.

[36]In this discussion, we ignore any Kähler deformations, since the flux potential as stated in equation (4.1) is independent of those. If not stated differently, it will be assumed in the following that the Calabi-Yau threefold is equipped with a suitable Kähler structure.

and consequently needs to vanish. Moreover, by complex conjugating equations (4.11), we find that

$$\int_X \bar{\Omega} \wedge F = 0 \quad , \quad \int_X \bar{\Omega} \wedge H = 0 \tag{4.14}$$

which holds since F and H are chosen to be integral, hence in particular real three-forms. Similar to the previous argument, only the pure holomorphic part of F and H contributes to these integrals and therefore needs to vanish as well. The result of this discussion can be summarized by imposing that F and H span a two-dimensional integral lattice within $H^{2,1}(X_{\mathbf{z}}, \mathbb{C}) \oplus H^{1,2}(X_{\mathbf{z}}, \mathbb{C})$, i.e.

$$< F, H > \subseteq \left(H^{2,1}(X_{\mathbf{z}}, \mathbb{C}) \oplus H^{1,2}(X_{\mathbf{z}}, \mathbb{C})\right) \cap H^3(X_{\mathbf{z}}, \mathbb{Z}) \tag{4.15}$$

is a sublattice. Equation (4.13) guarantees that F and H are not parallel but span a proper two-dimensional lattice. The existence of such sublattices of $H^3(X_{\mathbf{z}}, \mathbb{Z})$ with definite Hodge type will become very important in the context of modular Calabi-Yau manifolds which we will discuss in section 5.2.

Already by a naive count of degrees of freedom, we can argue that this system of equations is overconstrained and hence a generic point $(\mathbf{z}, \tau)$ in the moduli space will not give rise to a supersymmetric flux vacuum solution. However, examples of Calabi-Yau manifolds with $h^{2,1} = 1$ have been identified [4, 96] using techniques from arithmetic geometry that inspired the analysis for fourfold fluxes presented in chapter 5.

Instead of discussing the solutions to the $\mathcal{N} = 1$ supersymmetric flux vacuum constraints, we focus in the following section on the attempt, discussed in [3] to reconstruct the flux vacuum equations (4.11) and (4.12) in an equivalent $\mathcal{N} = 2$ supergravity description.

4.2 Minkowski Vacua of Four-Dimensional Gauged $\mathcal{N} = 2$ Supergravity

Recall from section 2.2.3 that the low energy limit of type IIB string theory compactified on a Calabi-Yau threefold is given by an $\mathcal{N} = 2$ supergravity theory with one gravity multiplet ($G_{\mu\nu}, A_\mu$, fermions), n_V vector multiplets (A^i_μ, z^i, fermions) and n_H hypermultiplets (q^u, fermions). The dynamics of these multiplets is described by the action

$$\begin{aligned} S^4_{\mathrm{IIB}} = \int_{M_4} \big(& R \star 1 + \mathrm{Re}\mathcal{M}_{AB} F^A \wedge F^B + \mathrm{Im}\mathcal{M}_{AB} F^A \wedge \star F^B \\ & - \tilde{g}_{a\bar{b}} dz^a \wedge \star d\bar{z}^{\bar{b}} - \tilde{h}_{uv} dq^u \wedge \star dq^v \big) \ . \end{aligned} \tag{4.16}$$

The n_V (complex) vector multiplet scalars z^i can be identified with the complex structure moduli, whereas $4(n_h - 1)$ real hypermultiplet scalars are given by the complexified Kähler moduli t^k and the scalars that originate from internal R-R two-form and four-form fields. The additional hypermultiplet, which is called the *universal hypermultiplet*, is a common feature of all $\mathcal{N}=2$ supergravity theories that originate from type II string compactifications and will be of special interest for the construction of the vacuum constraints (4.11) and (4.12) in gauged $\mathcal{N}=2$ supergravity.

It is of special interest to analyze the target space of a given supergravity theory which is spanned only by the scalar fields z^i and q^u of the multiplets. Due to supersymmetry, the dynamics of the remaining fields is already characterized by that of the target space degrees of freedom. For supergravity theories that originate as low energy limits of string compactifications, we can conclude that the numbers of multiplets are fixed by the Calabi-Yau topology to be $n_v = h^{2,1}$ vector multiplets and $n_h = h^{1,1}+1$ hypermultiplets. Hence, the corresponding target space of this supergravity theory is a differentiable manifold of real dimension $2h^{2,1} + 4(h^{1,1}+1)$.

4.2.1 $\mathcal{N}=2$ Supergravity Target Space Geometry

Before introducing gaugings of $\mathcal{N}=2$ supergravity theories, let us briefly review the most important concepts of the target space geometry of $\mathcal{N}=2$ supergravity theories in four spacetime dimensions. For a very detailed and comprehensive discussion of $\mathcal{N}=2$ supergravity, we refer to [108–112].

Locally, the target space of $\mathcal{N}=2$ supergravity theories is given by the direct product

$$\mathcal{M}_V \times \mathcal{M}_H \tag{4.17}$$

of a projective special Kähler manifold $\mathcal{M}_V$ parametrized by the n_v complex scalars of the vector multiplets and a quaternionic Kähler manifold $\mathcal{M}_H$ parametrized by the $4n_H$ real scalars of the hypermultiplets. In analogy to the discussion of section 3.2.6, the geometry of $\mathcal{M}_V$ is governed by a prepotential $F(\mathbf{z})$ that defines the Kähler potential and hence the Kähler metric on $\mathcal{M}_V$ according to equation (3.47). In the following, we collect the projective coordinates ω^k of $\mathcal{M}_V$ and the derivatives of the prepotential F_k in the period vector $X^\Lambda = (\omega^k, F_k)$ of the projective special Kähler manifold $\mathcal{M}_V$.

We define the quaternionic Kähler manifold $\mathcal{M}_H$ to be a Riemannian manifold of real dimension $4n_H$ with $Sp(1)\times Sp(n_H)$ special holonomy[37]. Due to this property, the metric h_{uv} on $\mathcal{M}_H$ can be written in terms of a vielbein $V_u^{A\alpha}$ and moreover there exists a triplet of almost complex structures[38] $J^x : \mathrm{T}\mathcal{M}_H \to \mathrm{T}\mathcal{M}_H$ $(x=1,2,3)$ that transform in the adjoint representation of $Sp(1) \cong SU(2)$, that means

$$J^x J^y = -\delta^{xy}\mathrm{Id} + \varepsilon^{xyz} J^z \tag{4.18}$$

[37]Quaternionic Kähler manifolds should not be confused with hyperkähler manifolds which are Riemannian manifolds of real dimension $4n$, whose holonomy is given by a proper subgroup of $Sp(1)\times Sp(n_H)$. This seemingly harmless deviation causes the fact that hyperkähler manifolds are always Ricci-flat whereas this is not the case for quaternionic Kähler manifolds [108].

[38]An almost complex structure $J : T\mathcal{M} \to T\mathcal{M}$ is a smooth isomorphism of real vector bundles such that $J^2 = -\mathrm{Id}$. In contrast to almost complex structures, a complex structure I is an almost complex structure on a complex manifold that varies not only smoothly but holomorphically along the manifold.

and h_{uv} is a hermitian metric with respect to all J^x. Note that $\mathcal{M}_H$ is in general not a complex manifold as the J^x are generically honest almost complex structures. Any almost complex structure is equivalently expressed in terms of a Kähler form $K^x \in \Omega^2(\mathcal{M}_H)$ via

$$(J^x)_v^u =: h^{uw}(K^x)_{wv} \ . \tag{4.19}$$

Here, h^{uw} denotes conveniently the inverse metric on $\mathcal{M}_H$. Finally, these Kähler forms give rise to $Sp(1)$ connection one-forms ω^x on $T\mathcal{M}_H$ that are defined by

$$K^x = d\omega^x + \tfrac{1}{2}\varepsilon^{xyz}\omega^y\omega^z \ . \tag{4.20}$$

4.2.2 Minkowski Vacua of Gauged $\mathcal{N}=2$ Supergravity

As we can deduce from the lagrangian in equation (4.16), $\mathcal{N}=2$ supergravity is a free theory that prohibits couplings between the vector multiplets and the hypermultiplets. In particular, the invariance of equation (4.16) with respect to the supersymmetry algebra prohibits the existence of a non-trivial superpotential. However, if we assume that the target space geometry admits smooth isometries that are realized by non-trivial Killing vectors $k_\lambda^i\partial_i$ on $\mathcal{M}_V$ and $\tilde{k}_\lambda^u\partial_u$ on $\mathcal{M}_H$ respectively, it is possible, following refs. [108, 110, 111, 113, 114], to gauge these isometries by treating the graviphoton and the vector fields of the vector multiplets A_μ^k ($k=0,\dots,n_V$) as electric gauge fields. Moreover, we introduce dual magnetic gauge fields $B_{\mu,k}$ and pair them together with the electric gauge fields in a $2(n_V+1)$-dimensional vector $A_\mu^\Lambda = (A_\mu^k, B_{\mu,k})$. The gauged $\mathcal{N}=2$ supergravity is now obtained by introducing gauge covariant derivatives

$$D_\mu z^i = \partial_\mu z^i - A_\mu^\Lambda k_\Lambda^i \qquad , \quad k_\Lambda^i = \Theta_\Lambda^\lambda k_\lambda^i \tag{4.21}$$

on $\mathcal{M}_V$ and

$$D_\mu q^u = \partial_\mu q^u - A_\mu^\Lambda \tilde{k}_\Lambda^u \qquad , \quad \tilde{k}_\Lambda^u = \tilde{\Theta}_\Lambda^\lambda \tilde{k}_\lambda^u \tag{4.22}$$

on $\mathcal{M}_H$. Here, the embedding tensors Θ_Λ^λ and $\tilde{\Theta}_\Lambda^\lambda$ characterize the representation of the multiplets with respect to the gauge group of the isometry indexed by λ. The replacement of the ordinary derivatives in equation (4.16) by these covariant derivatives leads to a non-trivial scalar potential

$$\begin{aligned} V(\mathbf{z},\mathbf{q}) = e^K \Big(& 2||\bar{X}^\Lambda(\mathbf{z})k_\Lambda^j(\mathbf{z})||^2_{\mathcal{M}_V} + 4||\bar{X}^\Lambda(\mathbf{z})\tilde{k}_\Lambda^v(\mathbf{q})||^2_{\mathcal{M}_H} \\ & + \mathrm{Tr}\left(||\bar{\nabla}^{\bar{j}}\bar{X}^\Lambda(\mathbf{z})\mathcal{P}_\Lambda(\mathbf{q})||^2_{\mathcal{M}_V}\right) - \frac{3}{2}\mathrm{Tr}\left(|X^\Lambda(\mathbf{z})\mathcal{P}_\Lambda(\mathbf{q})|^2\right)\Big) \end{aligned} \tag{4.23}$$

for the scalar fields $\mathbf{z}=(z^1,\dots,z^{n_v})$ and $\mathbf{q}=(q^1,\dots,q^{4n_H})$ which describes non-trivial interactions of vector multiplets and hypermultiplets[39]. Here, the norms $||\cdot||_{\mathcal{M}_V}$ and $||\cdot||_{\mathcal{M}_H}$ are taken with respect to the corresponding metrics $g_{i\bar{j}} = \partial_i\partial_{\bar{j}}K$ on $\mathcal{M}_V$ and h_{uv}

[39] As we are only interested in the implications on the target space, we spare out the collection of the full lagrangian for a general gauged $\mathcal{N}=2$ supergravity theory at this place. A comprehensive summary of all additional contributions can be found in [108].

on $\mathcal{M}_H$ respectively. ∇_i denotes the Kähler-covariant derivative $\nabla_i = \partial_i + (\partial_i K)$ on $\mathcal{M}_V$ and $\mathcal{P}_\Lambda = (\mathcal{P}^x_\Lambda)_{x=1,2,3}$ denotes a triplet of $\mathfrak{su}(2)$ Lie algebra-valued Killing prepotentials for the Killing vector fields $\tilde{k}^u_\Lambda \partial_u$ obeying

$$\nabla_u \mathcal{P}^x_\Lambda = -2\tilde{k}^v_\Lambda K^x_{uv} \ . \tag{4.24}$$

∇_u denotes the covariant derivative with respect to the subgroup $Sp(1)$ of the holonomy group of $\mathcal{M}_H$ and is defined in terms of the connection one-form ω^x as

$$\nabla_u \mathcal{P}^x_\Lambda = \partial_u \mathcal{P}^x_\Lambda + \varepsilon^{xyz} \omega^y_u \mathcal{P}^c_\Lambda \ . \tag{4.25}$$

Finally, the trace operation $\mathrm{Tr}(\cdot)$ in equation (4.23) is meant to be with respect to the $SU(2)$ structure of the Killing prepotentials.

In contrast to the ungauged $\mathcal{N} = 2$ supergravity theory, this non-trivial scalar potential constrains possible supersymmetric vacua significantly, as those are given by minimizing $V(\mathbf{z}, \mathbf{q})$. Moreover, supersymmetry requires that minimizing the potential is not sufficient but in addition, $V(\mathbf{z}, \mathbf{q})$ needs to vanish on the vacuum states. Hence, a supersymmetric vacuum configuration of the scalar fields $(\mathbf{z}, \mathbf{q})$ of the gauged $\mathcal{N} = 2$ supergravity theory is obtained if

$$\partial_i V(\mathbf{z}, \mathbf{q}) = 0 \quad , \quad \partial_u V(\mathbf{z}, \mathbf{q}) = 0 \quad , \quad V(\mathbf{z}, \mathbf{q}) = 0 \ . \tag{4.26}$$

One can read off that equation (4.23) contains three positive definite terms and one additional negative definite contribution

$$-\frac{3}{2} \mathrm{Tr}\left(|X^\Lambda(\mathbf{z}) \mathcal{P}_\Lambda(\mathbf{q})|^2 \right) \sim \Lambda \ . \tag{4.27}$$

It can be shown [110, 111] that this term corresponds to the cosmological constant Λ of the vacuum state. Hence, if we restrict our analysis further to supersymmetric Minkowski vacua, the cosmological constant and therefore the term of equation (4.27) needs to vanish. Consequently, the remaining positive definite terms in equation (4.23) need to vanish separately in order to obey $V(\mathbf{z}, \mathbf{q}) = 0$. Thus, we conclude that a field configuration $(\mathbf{z}, \mathbf{q}) \in \mathcal{M}_V \times \mathcal{M}_H$ describes a supersymmetric Minkowski vacuum if and only if[40]

$$\begin{aligned} 0 &= \bar{X}^\Lambda(\mathbf{z}) \mathcal{P}_\Lambda(\mathbf{q}) \ , \quad 0 = \bar{\nabla}_{\bar{j}} \bar{X}^\Lambda(\mathbf{z}) \mathcal{P}_\Lambda(\mathbf{q}) \ , \\ 0 &= \bar{X}^\Lambda(\mathbf{z}) \tilde{k}^u_\Lambda(\mathbf{q}) \ , \quad 0 = \bar{X}^\Lambda(\mathbf{z}) k^j_\Lambda(\mathbf{z}) \ . \end{aligned} \tag{4.28}$$

An $\mathcal{N} = 2$ Minkowski vacuum of this gauged $\mathcal{N} = 2$ supergravity theory is hence specified by a set of expectation values for the scalar fields $(\mathbf{z}, \mathbf{q}) \in \mathcal{M}_V \times \mathcal{M}_H$ obeying equations (4.28). If the scalar potential $V(\mathbf{z}, \mathbf{q})$ admits m flat directions on $\mathcal{M}_V$ and $\tilde{m}$ flat directions on $\mathcal{M}_H$, meaning that its value vanishes constantly along those, these flat directions

[40] So far we have discussed that if $(\mathbf{z}, \mathbf{q})$ describes a supersymmetric Minkowski vacuum, then it obeys the equations (4.28). For the converse we need to ensure that these conditions are also sufficient to find that the gradients $\partial_i V(\mathbf{z}, \mathbf{q})$ and $\partial_u V(\mathbf{z}, \mathbf{q})$ vanish as well. This statement follows by an explicit computation of the derivatives, c.f. refs. [110, 111].

can be identified with deformations of the expectation values. Following [111], these deformations, together with the additional higher-rank field contents, can be collected in $\mathcal{N} = 2$ massless multiplets whereas the deformations orthogonal to the flat directions lead to $\mathcal{N} = 2$ massive multiplets. Integrating out the massive degrees of freedom, the low energy theory describing the dynamics of these massless multiplets is again an $\mathcal{N} = 2$ supergravity theory. The number of vector multiplets and hypermultiplets of this effective theory is reduced compared to the original gauged $\mathcal{N} = 2$ supergravity theory. The scalars of these multiplets span a submanifold of $\mathcal{M}_V \times \mathcal{M}_H$ since they appear as the flat directions of $V(\mathbf{z}, \mathbf{q})$ on $\mathcal{M}_V \times \mathcal{M}_H$. Hence, we claim [3] that semi-classically the target space of this low energy effective $\mathcal{N} = 2$ supergravity theory is locally given by

$$\mathcal{S}_V \times \mathcal{S}_H \tag{4.29}$$

where $\mathcal{S}_V \subset \mathcal{M}_V$ is a projective special Kähler manifold of complex dimension m and $\mathcal{S}_H \subset \mathcal{M}_H$ is a quaternionic Kähler manifold of real dimension $\tilde{m}$. One should note that the target space geometry of this effective field thoery receives additional corrections due to one-loop contributions which arise in the process of integrating out the massive multiplets, c.f. refs. [115, 116]. As our aim is to relate these Minkowski vacua to the flux vacua which have been discussed in section 4.1 only on a semi-classical level, we will not discuss these additional quantum corrections further within this work.

4.2.3 Projective Special Kähler Submanifolds

Before examining the Minkowski vacuum contraints given by equations (4.28) in more detail, let us discuss the general geometric structure of the moduli space $\mathcal{S}_V$ of the flat directions on the vector multiplet target space[41]. As proposed above, this manifold is a submanifold of $\mathcal{M}_V$, that is again projective special Kähler. This observation motivates the definition of a *projective special Kähler submanifold* in the following sense [3].

If $\mathcal{M}$ is a projective special Kähler manifold of dimension $n = \dim(\mathcal{M})$ and $\mathcal{S} \subset \mathcal{M}$ is a submanifold of $\mathcal{M}$ of dimension $s = \dim(\mathcal{S})$, then $\mathcal{S}$ is a projective special Kähler submanifold (of $\mathcal{M}$) if $\mathcal{S}$ is projective special Kähler and the prepotential $F_\mathcal{S}$ of $\mathcal{S}$ is compatible with the prepotential $F_\mathcal{M}$ of $\mathcal{M}$. With the latter condition we mean that if $\mathcal{S}$ is realized locally by the subspace

$$S = \{\omega^{s+1} = \cdots = \omega^n = 0\} \subset \mathcal{M} \tag{4.30}$$

for ω^k being a set of projective coordinates on $\mathcal{M}$, we have

$$F_\mathcal{S} = F_\mathcal{M}|_\mathcal{S} \text{ and } \left.\frac{\partial F_\mathcal{M}}{\partial \omega^k}\right|_\mathcal{S} = 0 \text{ for } k = s+1, \ldots, n \, . \tag{4.31}$$

[41]The geometric properties of $\mathcal{S}_H$ as a hyperkähler submanifold are of interest in their own rights. However, as the geometric structure of projective special Kähler manifolds is richer, we focus on this part of the full target space within this work.

One should note, that the existence of any non-trivial projective special Kähler submanifold is a highly non-trivial condition on $\mathcal{M}$. To investigate this statement further, we will give another equivalent definition of projective special Kähler submanifolds in terms of the $(n+1)\times(n+1)$ matrix

$$F_{IJ} = \frac{F_{\mathcal{M}}}{\partial\omega^I\partial\omega^J} \ . \tag{4.32}$$

We claim [3] that $\mathcal{S}$ is a projective special Kähler submanifold of $\mathcal{M}$ if and only if

- $F_{IJ}|_{\mathcal{S}}$ is block diagonal with block-matrices $(F_{IJ})_{\{I,J\leq s\}}$ of size $(s+1)\times(s+1)$ and $(F_{IJ})_{\{I,J>s\}}$ of size $(n-s)\times(n-s)$,
- the matrix $(\mathrm{Im}(F)_{IJ})_{\{(I,J)\leq s\}}$ has signature $(s,1)$ and the matrix $(\mathrm{Im}(F)_{IJ})_{\{(I,J)>s\}}$ is positive definite.

Note that the first condition is equivalent to

$$F_{IJ}|_{\mathcal{S}} = 0 \quad , \quad I = 0,\ldots,s \ , \ J = s+1,\ldots,n \tag{4.33}$$

which is a simple condition to check for any submanifold $\mathcal{S}\subset\mathcal{M}$. Moreover, the block-matrix $(F_{IJ})_{\{I,J\leq s\}}$ coinsides with the expression (4.32) for the prepotential $F_{\mathcal{S}}$, whereas the remaining block $(F_{IJ})_{\{(I,J)>s\}}$ encodes the geometry of the orthogonal complement $\mathcal{S}^\perp\subset\mathcal{M}$.

In the context of $\mathcal{N}=2$ supergravity theories that originate from type IIB string compactifications on a Calabi-Yau threefold X we recall that the vector multiplet target space is identified with the complex structure moduli space $\mathcal{M}_{C.S.}$ of X. Hence, if we assume that gauging this $\mathcal{N}=2$ supergravity gives rise to a non-trivial Minkowski vacuum that is described by a special Kähler submanifold $\mathcal{S}\subset\mathcal{M}_{C.S.}$ of dimension s, this means that $\mathcal{S}$ characterizes a sub-Hodge structure on $H^3(X_{\mathbf{z}},\mathbb{C})$ which is of dimension $2(s+1)$. Thus, we can conclude that a given type IIB string compactification on a familiy of Calabi-Yau threefolds $\mathcal{X}_{\mathcal{M}}$ can give rise to a non-trivial supersymmetric Minkowski vacuum of a gauged $\mathcal{N}=2$ supergravity theory only if there exists a sublocus on $\mathcal{M}_{C.S.}$ on which the variation of Hodge structures[42] of $H^3(\mathcal{X}_{\mathcal{M}},\mathbb{C})$ admits a variation of sub-Hodge structures.

Generically, a family of Calabi-Yau threefolds does not give rise to any variation of sub-Hodge structures. However, the following mechanisms allow for subloci on $\mathcal{M}_{C.S.}$ that have a sub-Hodge structure. If $\mathcal{M}_{C.S.}$ admits a discrete symmetry, the prepotential $F_{\mathcal{M}}$ is invariant under this symmetry. Hence, if we restrict the moduli to the invariant sublocus $\mathcal{S}_{inv}\subset\mathcal{M}_{C.S.}$, the symmetry implies that the prepotential becomes extremized on $\mathcal{S}_{inv}$ with respect to the normal directions and the restriction of $F_{\mathcal{M}}$ on $\mathcal{S}_{inv}$ gives rise to a suitable prepotential on $\mathcal{S}_{inv}$. Hence, any discrete symmetry on $\mathcal{M}_{C.S.}$ gives rise to a projective special Kähler submanifold.

[42]We refer to refs. [63, 117] for a comprehensive introduction to Hodge theory and variations of Hodge structures

In addition, we can investigate extended conifold singularities on $\mathcal{M}_{C.S.}$. In order to describe the family of Calabi-Yau threefolds by a smooth moduli space, such singularities are resolved either by topologically replacing the conifold[43] point by an S^2 of finite radius (small resolution) or by an S^3 (deformation) [51, 95, 118–120]. Since these two processes change the topology of the underlying Calabi-Yau threefolds, the corresponding moduli spaces are distinct and in particular have a different number of moduli. However, the $\mathcal{N} = 2$ supergravity description of the corresponding string compactification in the vicinity of the singular locus should be equivalent. Based on the work presented in [119, 121], this observation is interpreted as a topology changing transition between moduli spaces that is known as *extremal transitions*[44]. It has been shown [122] that for such extremal transitions the complex structure moduli space with lower dimension can always be identified with a sublocus of the higher dimensional complex structure moduli space in the sense of a projective special Kähler submanifold.

We will finish this section by providing a proof of the equivalence statement for the two definitions of projective special Kähler submanifolds. The main observation we need is that the prepotential[45] $F(\omega)$ is homogenous of degree two meaning

$$F(\lambda\omega^0, \dots, \lambda\omega^n) = \lambda^2 F(\omega^0, \dots, \omega^n) \tag{4.34}$$

for any $\lambda \in \mathbb{C}$. It follows directly from this relation that (summation convention assumed)

$$\omega^I F_I = 2F \quad , \quad \omega^I F_{IJ} = F_J \ . \tag{4.35}$$

Here, we use the short notation $F_I = \frac{\partial F}{\partial \omega^I}$. Let us now assume that $\mathcal{S} \subset \mathcal{M}$ is a projective special Kähler submanifold of dimension s parametrized by $z^1, \dots, z^s$, i.e. $F_J|_{\mathcal{S}} = 0$ for $J = s+1, \dots, n$ and $F|_{\mathcal{S}} = F_{\mathcal{S}}$ is the prepotential corresponding to $\mathcal{S}$. Then we find for $J = s+1, \dots, n$ using equation (4.35) that

$$0 = F_J|_{\mathcal{S}} = (\omega^I F_{IJ})|_{\mathcal{S}} = \sum_{k=0}^{s} (\omega^k F_{kJ})|_{\mathcal{S}} \tag{4.36}$$

as $\omega^{s+i} = 0$ on $\mathcal{S}$. Since the ω^k are projective coordinates for $\mathcal{S}$ and therefore in particular non-vanishing on a generic point of $\mathcal{S}$, this set of $(n-s)$ linear equations is solved only if $F_{kJ} = 0$ for $k = 0, \dots, s$ proving that F_{IJ} becomes block-diagonal on $\mathcal{S}$. The signatures of the block-matrices follow from the observation that the matrix $\mathrm{Im}(F)_{IJ}$ has signature $(n, 1)$ for any projective special Kähler manifold $\mathcal{M}$ of dimension n [123]. Consequently, the matrix $\mathrm{Im}((F_{\mathcal{S}}))_{k\ell}$ is of signature $(s, 1)$ and coincides with the first block-matrix of $\mathrm{Im}(F)_{IJ}|_{\mathcal{S}}$. Since the signature of a block-diagonal matrix behaves additive, it follows that the remaining second block is positive definite.

[43]Topologically, a conifold singularity is the limit of a cone with base $S^2 \times S^3$.

[44]Extremal transitions have been studies extensively in the sting theory literature. A comprehensive review can be found in [51].

[45]For simplifying the notation, we drop the index $\mathcal{M}$ in the following computation.

The converse statement follows directly by observing for $J = s+1, \dots, n$ that

$$F_J|_{\mathcal{S}} = (\omega^I F_{IJ})|_{\mathcal{S}} = \sum_{k=s+1}^{n} (\omega^k F_{kJ})|_{\mathcal{S}} = 0 \qquad (4.37)$$

where we used the assumption that both, ω^{s+i} and F_{kJ} $(k = 0, \dots, s)$ vanish on $\mathcal{S}$. Moreover, the signature of $(\mathrm{Im}(F)_{IJ})_{\{I,J\leq s\}}$ guarantees that $F_{\mathcal{M}}|_{\mathcal{S}}$ is a suitable prepotential for the submanifold $\mathcal{S}$.

4.2.4 Examining the Vacuum Constraints

After this formal excurse on projective special Kähler submanifolds and their characterization in terms of the matrix F_{IJ}, let us turn back to the discussion of supersymmetric Minkowski vacua in gauged $\mathcal{N} = 2$ supergravity. We observe that the former three constraints of equation (4.28) give rise to an interaction between the vector multiplet scalars z^i and the hypermultiplet scalars q^u whereas the latter gives a relation only among the vector multiplet scalars. Moreover, these constraints can be distinguished by the fact that only the fourth constraint depends on the isometries of $\mathcal{M}_V$ whereas the remaining three originate from isometries of $\mathcal{M}_H$. Thus, we will discuss vacuum solutions of vector multiplet isometries $k^i_\lambda \partial_i$ and of hypermultiplet isometries $\tilde{k}^u_\lambda \partial_u$ separately.

First, let us assume that there are only non-trivial isometries $k^j_\lambda \partial_j$ $(\lambda = 1, \dots, N)$ on the vector multiplet target space $\mathcal{M}_V$. In this case, only the fourth constraint of (4.28), given by

$$\bar{X}^\Lambda(\mathbf{z}) k^j_\Lambda(\mathbf{z}) = 0 \; , \qquad (4.38)$$

becomes relevant. Recall that $k^i_\Lambda = \Theta^\lambda_\Lambda k^j_\lambda$, hence this vector-valued equation gives N independent constraints on the local coordinates $\mathbf{z}$ of $\mathcal{M}_V$. It is known [111] that a gauging of isometries on $\mathcal{M}_V$ is possible only if the the local coordinates $\mathbf{z}$ transform in the adjoint representation of a non-Abelian gauge group implying that the gauge group breaks to its maximal torus if we restrict the theory to a generic vacuum locus obeying equation (4.38). Hence, the effective $\mathcal{N} = 2$ supergravity theory described by the flat directions of the scalar potential is given by the Coulomb branch of the corresponding $\mathcal{N} = 2$ supersymmetric gauge theory coupled to gravity. Moreover, the gauge fields that correspond to the broken isometries obtain a mass term from the Higgs mechanism and hence rearrange with the massive scalar fields to short massive BPS vector multiplets [111].

The Killing vectors in equation (4.38) depend on the local coordinates of $\mathcal{M}_V$ and hence are related to the periods X^Λ of $\mathcal{M}_V$. therefore, vacuum constraints of this type will be in general not linear in the period vector but give rise to non-linear constraints. Moreover, this type of isometries does not lead to an interaction of vector multiplet scalars and hypermultiplet scalars as it would be necessary to reproduce flux vacuum-like constraints as in equations (4.9) and (4.10). Hence, we turn now to the second possibility of isometries, given by non-trivial Killing vectors on the hypermultiplet target space $\mathcal{M}_H$. Recall from equation (4.28) that non-trivial Killing vectors $\tilde{k}^u_\lambda \partial_u$ ($\lambda = 1, \ldots, M$) give rise to three different types of vacuum constraints given by

$$0 = \bar{X}^\Lambda(\mathbf{z})\mathcal{P}_\Lambda(\mathbf{q}) \ , \quad 0 = \bar{X}^\Lambda(\mathbf{z})\tilde{k}^u_\Lambda(\mathbf{q}) \ , \ 0 = \bar{\nabla}_{\bar{j}}\bar{X}^\Lambda(\mathbf{z})\mathcal{P}_\Lambda(\mathbf{q}) \ . \tag{4.39}$$

Note, that these constraints can be brought in a more symmetric form by using the definition of the Killing prepotentials given in equation (4.24) and the fact that the Kähler 2-forms K^x are invertible[46]. Hence, we obtain

$$0 = X^\Lambda(\mathbf{z})\mathcal{P}_\Lambda(\mathbf{q}) \ , \quad 0 = X^\Lambda(\mathbf{z})\nabla_u\mathcal{P}_\Lambda(\mathbf{q}) \ , \ 0 = \nabla_j X^\Lambda(\mathbf{z})\mathcal{P}_\Lambda(\mathbf{q}) \ . \tag{4.40}$$

All quantities on the quaternionic Kähler manifold $\mathcal{M}_H$ are by definition real and therefore, complex conjugating the equations has no effect on $\mathcal{P}_\Lambda$ and derivatives thereof. Moreover, we can replace the covariant derivatives ∇_u and ∇_i by the ordinary partial derivatives ∂_u and ∂_i respectively, since the connection terms vanish automatically due to the first constraint. Therefore, we conclude that an $\mathcal{N} = 2$ Minkowski vacuum of a gauged $\mathcal{N} = 2$ supergravity theory with non-trivial isometries on $\mathcal{M}_H$ is characterized by the conditions

$$0 = X^\Lambda(\mathbf{z})\mathcal{P}_\Lambda(\mathbf{q}) \ , \quad 0 = \partial_u\left(X^\Lambda(\mathbf{z})\mathcal{P}_\Lambda(\mathbf{q})\right) \ , \ 0 = \partial_j\left(X^\Lambda(\mathbf{z})\mathcal{P}_\Lambda(\mathbf{q})\right) \ . \tag{4.41}$$

Compared to the $\mathcal{N} = 1$ flux vacuum conditions from the superpotential W, discussed in equation (4.7), we find a similar structure of constraints by identifying the expression of the superpotential with the term $X^\Lambda\mathcal{P}_\Lambda$. In the following section, we discuss this type of gaugings in more detail, explicitly considering $\mathcal{N} = 2$ supergravity theories that arise from type IIB string compactifications on a Calabi-Yau threefold.

4.3 Flux Vacua from Gauged $\mathcal{N} = 2$ Supergravity

As already briefly discussed in section 4.2, the low energy effective field theory of type IIB string theory compactified on a Calabi-Yau threefold X is given by an $\mathcal{N} = 2$ supergravity theory with $n_V = h^{2,1}$ vector multiplets and $n_H = h^{1,1}+1$ hypermultiplets. The additional hypermultiplet, which does not belong to any of the complexified Kähler moduli, is a universal feature of all such supergravities that arise from type IIB string theory and is hence called the *universal hypermultiplet*. Since it contains the dilaton field ϕ and the axion field σ, which in the framework of the $\mathcal{N} = 1$ flux superpotential (4.8) are combined to the complex axio-dilaton τ, this multiplet will be of special interest within the following discussion.

The strategy is as follows: Starting with the $\mathcal{N} = 2$ supergravity theory related to a generic choice for the Calabi-Yau manifold X, we consider a non-trivial gauging of the isometries only in the universal hypermultiplet sector. Since the semi-classical geometry of its target space is known, we can deduce the explicit structure of the vacuum constraints (4.41). A generic choice of gauge representations for the contributing fields is too

[46]Recall from equation (4.19) that K^x is expressible purely in terms of the invertible quantities J^x and h_{uv}.

constraining such that generically one expects no non-trivial solution for this model [110]. However, we will discuss in section 4.3.3 that on non-generic subspaces of the universal hypermultiplet target space, the constraints for a Minkowski vacuum just coincide with the supersymmetric flux vacuum constraints (4.11) and (4.12). Hence, demonstrate that it is possible to obtain moduli stabilizing constraints for the vacuum of the $\mathcal{N} = 2$ low energy effective supergravity theory arising from type IIB string theory compactified on a Calabi-Yau threefold, that are equivalent to those which are obtained from the $\mathcal{N} = 1$ flux superpotential (4.1).

4.3.1 Geometry of the Universal Hypermultiplet

To begin with, we review the geometry of the universal hypermultiplet target space and identify its isometries. The universal hypermultiplet has been studied extensively in the supergravity literature, as for instance refs. [109, 124–128]. Semi-classically, the geometry of its target space is given by the coset space

$$\mathcal{M}_{\mathrm{U}} = \frac{SU(2,1)}{S(U(2)\times U(1))} \tag{4.42}$$

having the special feature that this manifold is not only quaternionic Kähler but even a complex manifold that can be parametrized by two complex coordinates S and C. Physically, the complex coordinate C combines the real scalar field $\mathrm{Re}(C)$ dual to the B-field and the real scalar field $\mathrm{Im}(C)$ which is dual to the Ramond-Ramond two-form [126]. The coordinate S is given in terms of the dilaton ϕ and the Ramond-Ramond axion field σ by

$$S = e^{-\phi} + i\sigma + C\bar{C}\ . \tag{4.43}$$

Following [127], the metric on $\mathcal{M}_{\mathrm{U}}$ reads[47]

$$ds^2 = i\left(\partial_u\partial_v K_{\mathrm{U}}\right)dq^u dq^v = e^{2K_{\mathrm{U}}}\left(dSd\bar{S} - 2CdSd\bar{C} - 2\bar{C}d\bar{S}dC + 2(S+\bar{S})dCd\bar{C}\right) \tag{4.44}$$

with the Kähler potential

$$K_{\mathrm{U}} = -\log(S+\bar{S}-2C\bar{C})\ . \tag{4.45}$$

From the metric, one can deduce the $SU(2)$-connection one-forms ω^x corresponding to the almost complex structures $J^x : T\mathcal{M}_{\mathrm{U}} \to T\mathcal{M}_{\mathrm{U}}$. In terms of the local coordinates $\phi, \sigma, C, \bar{C}$ of $\mathcal{M}_{\mathrm{U}}$, these read (c.f. [109])

$$\begin{aligned}
\omega^1 &= 2ie^{\phi/2}(dC - d\bar{C})\\
\omega^2 &= 2e^{\phi/2}(dC + d\bar{C})\\
\omega^3 &= e^{\phi}(d\sigma + i(\bar{C}dC - Cd\bar{C}))\ .
\end{aligned} \tag{4.46}$$

[47] Note that there exist several different parametrizations (c.f. refs. [109, 128, 129]) of the universal hypermultiplet which are all equivalent. This choice of parametrization is very convenient for the identification with the axio-dilaton τ since this is usually written as $\tau = e^{-\phi} + i\sigma$ and hence can be identified with $S - C\bar{C}$.

Using eqaution (4.20), the triplet of Kähler two-forms K^x follows to be

$$\begin{aligned}
K^1 &= e^{\phi/2}\left(-id\phi\wedge(dC-d\bar{C})+e^{\phi}d\sigma\wedge(dC+d\bar{C})+ie^{\phi}(C+\bar{C})dC\wedge d\bar{C}\right)\\
K^2 &= e^{\phi/2}\left(-d\phi\wedge(dC+d\bar{C})+ie^{\phi}d\sigma\wedge(dC-d\bar{C})+e^{\phi}(C-\bar{C})dC\wedge d\bar{C}\right)\\
K^3 &= -e^{\phi}(d\phi\wedge d\sigma+id\phi\wedge(\bar{C}dC-Cd\bar{C})-idC\wedge d\bar{C})\ .
\end{aligned} \tag{4.47}$$

The Kähler metric (4.44) has four independent real isometries that can be identified by analyzing the structure of the Kähler potential K_{U}. First, we observe that K_{U} depends only on $\mathrm{Re}(S)$. Hence, we obtain a shift symmetry along the imaginary part of S. Moreover, the Kähler potential is invariant under phase-shift of the complex field C. These two obvious isometries give rise to the Killing vector fields

$$\tilde{k}_1 = i(\partial_S-\partial_{\bar{S}})\quad,\quad \tilde{k}_2=-(C\partial_C-\bar{C}\partial_{\bar{C}})\ . \tag{4.48}$$

Less obvious, there exist two additional real isometries given by the (complexified) transformation $S\to S+2C\bar{\varepsilon}+\varepsilon^2$ and $C\to C+\varepsilon$ for any $\varepsilon\in\mathbb{C}$ [128]. The corresponding Killing vector fields are given by

$$\begin{aligned}
\tilde{k}_3 &= \frac{1}{2}(\partial_C+\partial_{\bar{C}})-\frac{i}{2}\mathrm{Im}(C)(\partial_S-\partial_{\bar{S}})\\
\tilde{k}_4 &= -\frac{i}{2}(\partial_C-\partial_{\bar{C}})+\frac{i}{2}\mathrm{Re}(C)(\partial_S-\partial_{\bar{S}})\ .
\end{aligned} \tag{4.49}$$

Note that these Killing vector fields are manifest real-valued as they need to be since the complexification of the local coordinates is a remnant of the chosen parametrization and not a manifest property of the quaternionic Kähler manifold.

A direct computation shows that these Killing vector fields of the universal hypermultiplet obey the Lie algebra

$$\begin{gathered}
[\tilde{k}_1,\tilde{k}_2]=[\tilde{k}_1,\tilde{k}_3]=[\tilde{k}_1,\tilde{k}_4]=0\ ,\\
[\tilde{k}_2,\tilde{k}_3]=-\,i\tilde{k}_3\ ,\ [\tilde{k}_2,\tilde{k}_4]=i\tilde{k}_4\ ,\ [\tilde{k}_3,\tilde{k}_4]=-2\tilde{k}_1\ .
\end{gathered} \tag{4.50}$$

This structure can be interpreted as the central extension of the Euclidean Lie algebra of the two-dimensional real space $\mathbb{R}^2$. In this picture, $\tilde{k}_2$ describes the rotation of $\mathbb{R}^2$ whereas $\tilde{k}_3$ and $\tilde{k}_4$ are the translations. Finally, $\tilde{k}_1$ is the central element of the extended algebra.

Since the Kähler two-forms K^x are known explicitly, we continue by integrating the Killing vector fields according to equation (4.24) to determine the corresponding Killing prepotentials. In the following we use $i\sigma^x$ as generators for the Lie-algebra $\mathfrak{su}(2)$ with σ^x denoting the Pauli-matrices. Expanding the Killing prepotentials according to $\mathcal{P}_\lambda=\mathcal{P}^x_\lambda(i\sigma^x)$, the solutions to equation (4.24) are given by

$$\begin{aligned}
\mathcal{P}_1 &= \frac{1}{2}e^{\phi}i\sigma^3 \\
\mathcal{P}_2 &= -e^{\phi/2}\left(\mathrm{Re}(C)i\sigma^1 + \mathrm{Im}(C)i\sigma^2\right) + \frac{1}{2}\left(1 - e^{\phi}C\bar{C}\right)i\sigma^3 \\
\mathcal{P}_3 &= e^{\phi/2}i\sigma^2 + e^{\phi}\mathrm{Im}(C)i\sigma^3 \\
\mathcal{P}_4 &= e^{\phi/2}i\sigma^1 + e^{\phi}\mathrm{Re}(C)i\sigma^3 \quad .
\end{aligned} \tag{4.51}$$

4.3.2 Gauging of Universal Hypermultiplet Isometries

Now, we have collected all ingredients to discuss $\mathcal{N} = 2$ Minkowski vacua of a generic gauged $\mathcal{N} = 2$ supergravity theory containing an universal hypermultiplet. In the following, we assume that the four independent isometries of the universal hypermultiplet which have been discussed in section 4.3.1 are gauged with generic embedding tensors $\tilde{\Theta}^{\lambda}_{\Lambda}$ characterizing the gauge group representation of the vector multiplet scalars. In particular, we assume that there are no further isometries on the target space that are gauged. Supersymmetric Minkowski vacua are hence characterized by solutions to the constraints (4.41) with the Killing prepotenials $\mathcal{P}_{\Lambda} = \tilde{\Theta}^{\lambda}_{\Lambda}\mathcal{P}_{\lambda}(S, C)$ given by equation (4.51).

First, let us analyze the conditions

$$0 = X^{\Lambda}(\mathbf{z})\mathcal{P}_{\Lambda}(S, C) \quad , \quad 0 = X^{\Lambda}(\mathbf{z})\partial_u\mathcal{P}_{\Lambda}(S, C) \tag{4.52}$$

constraining the period vector $X^{\Lambda}(\mathbf{z})$ rather than its gradient. These $\mathfrak{su}(2)$-valued conditions, give rise to seven real constraints[48] on the target space coordinates[49] $(\mathbf{z}, S, C)$ as $\mathcal{P}_{\lambda}$ and $\partial_u\mathcal{P}_{\lambda}$ are generically lineraly independent. Upon inserting the concrete form for $\mathcal{P}_{\Lambda}$, a direct computation[50] shows that these seven conditions condense to the four equations

$$0 = X^{\Lambda}(\mathbf{z})\tilde{\Theta}^{\lambda}_{\Lambda} \quad , \quad \lambda = 1, \ldots, 4 \tag{4.53}$$

independent of the local coordinates (S, C) of the universal hypermultiplet. Note, that these conditions contrain only the vector multiplet scalars to a sublocus of generic codimension four, whereas the universal hypermultiplet target space remains fully unconstrained. In particular, these constraints are linear in the period vector and depend only on a choice for the embedding tensors $\tilde{\Theta}^{\lambda}_{\Lambda}$.

---

[48] Recall that the conditions containing the gradient of $\mathcal{P}_{\Lambda}$ originate from the $\dim(\mathcal{M}_{\mathrm{U}}) = 4$-dimensional vector-valued constraint $X^{\Lambda}\tilde{k}^u_{\Lambda} = 0$. Hence, only four of these equations can be independent.

[49] We neglect the additional hypermultiplet coordinates q^u as the Minkowski vacuum conditions do not constrain those.

[50] The equations (4.52) give rise to a liner system of equations on $\mathrm{Re}(S), \mathrm{Im}(S), \mathrm{Re}(C), \mathrm{Im}(C)$ and X^{Λ} that can be reduced to the four equations (4.53) by eliminating S and C. The limiting cases $\mathrm{Re}(S) \to 0$ and $\mathrm{Re}(S) \to \infty$ need to be treated separately in this derivation but lead to the same result.

The hypermultiplet target space requires constraints by considering the remaining vacuum conditions

$$0 = \partial_i X^\Lambda(\mathbf{z})\ \mathcal{P}_\Lambda(S,C) \tag{4.54}$$

dependent on the gradient of X^Λ. In total, these give $3n_V$ conditions on the expectation values of $(\mathbf{z}, S, C)$, since $\mathcal{P}_\Lambda$ is $\mathfrak{su}(2)$-valued and the gradient ∂_i has n_V components. However, this counting needs to be modified by observing that equation (4.53) gives already up to four directions, on which the period vector $X^\Lambda(\mathbf{z})$ is constantly vanishing. Hence, we expect for a generic choice of embedding tensors that the total number of derivative constraints is reduced by the number of constant directions due to equation (4.53). Inserting the explicit form of the Killing prepotentials, the conditions (4.54) are given by

$$\begin{aligned}
0 &= \left(\mathrm{Re}(C)\tilde{\Theta}^2_\Lambda + \frac{i}{2}(\tilde{\Theta}^3_\Lambda - \tilde{\Theta}^4_\Lambda)\right)\partial_i X^\Lambda(\mathbf{z}) \\
0 &= \left(\mathrm{Im}(C)\tilde{\Theta}^2_\Lambda - \frac{1}{2}(\tilde{\Theta}^3_\Lambda + \tilde{\Theta}^4_\Lambda)\right)\partial_i X^\Lambda(\mathbf{z}) \\
0 &= \left(\tilde{\Theta}^1_\Lambda + e^{-\phi}\tilde{\Theta}^2_\Lambda - i\mathrm{Re}(C)(\tilde{\Theta}^3_\Lambda - \tilde{\Theta}^4_\Lambda) + \mathrm{Im}(C)(\tilde{\Theta}^3_\Lambda + \tilde{\Theta}^4_\Lambda)\right)\partial_i X^\Lambda(\mathbf{z})
\end{aligned} \tag{4.55}$$

and hence give interactions among the vector multiplet and the hypermutliplet scalar fields. Note that these interactions are purely linear in the scalar fields of the universal hypermultiplet S and C.

4.3.3 Flux Vacua from Non-Generic Gaugings

Let us now make contact with the type IIB supersymmetric flux vacuum constraints from equations (4.9) and (4.10). To that end, we consider non-trivial gaugings along only two independent isometries whose Killing prepotentials are denoted in the following by $\mathcal{P}_{(1)}$ and $\mathcal{P}_{(2)}$. In general, these isometries are given by any linear combination of the Killing prepotentials (4.24). Moreover, we choose the embedding tensors $\tilde{\Theta}^{(1)}_\Lambda$ and $\tilde{\Theta}^{(2)}_\Lambda$ such that they coincide with the flux vectors F_Λ and H_Λ from section 4.1. For a generic point (S, C) on $\mathcal{M}_\mathrm{U}$, the Killing prepotentials $\mathcal{P}_{(1)}$ and $\mathcal{P}_{(2)}$ are linearly independent and hence, the Minkowski vacuum constraints result in the $2(n_V + 1)$ independent equations

$$\begin{aligned}
X^\Lambda(\mathbf{z})F_\Lambda = 0 \quad &, \quad X^\Lambda(\mathbf{z})H_\Lambda = 0\ , \\
\partial_i X^\Lambda(\mathbf{z})F_\Lambda = 0 \quad &, \quad \partial_i X^\Lambda(\mathbf{z})H_\Lambda = 0\ \ .
\end{aligned} \tag{4.56}$$

Compared to the supersymmetric flux constraints, this set of equations restricts possible vacuum configurations even further, as we obtain an additional set of equations for the gradient of the period vector. This obstruction can be cured by imposing the additional condition that the Killing prepotentials $\mathcal{P}_{(1)}$ and $\mathcal{P}_{(2)}$ should be parallel. This requirement is realized by constraining the expectation values of the universal hypermultiplet scalars S and C to the sublocus

$$\mathcal{T} = \{(S,C)\ |\ \mathcal{P}_{(1)}(S,C)\ ||\ \mathcal{P}_{(2)}(S,C)\} \subset \mathcal{M}_\mathrm{U}\ . \tag{4.57}$$

Since each Killing prepotential is $\mathfrak{su}(2)$-valued, this alignment requirement is realized by two real constraints on S and C, hence $\mathcal{T}$ is generically expected to be a two-dimensional sublocus on $\mathcal{M}_{\text{U}}$ matching with the two real degrees of freedom of the axio-dilaton in the framework of supersymmetric flux compactifications. One should notice that, although the Killing prepotentials become aligned on $\mathcal{T}$, the corresponding isometries $\tilde{k}_{(1)}$ and $\tilde{k}_{(2)}$ are still independent. Hence, the vacuum constraints (4.53) still remain independent. However, the constraints on $\partial_i X^\Lambda$ given by equation (4.54) become

$$0 = \partial_i X^\Lambda(\mathbf{z})(F_\Lambda - \tau_\mathcal{T}|_\mathcal{T}\, H_\Lambda) \tag{4.58}$$

with $\tau_\mathcal{T}(S, C)$ being the scalar factor relating $\mathcal{P}_{(1)}$ and $\mathcal{P}_{(2)}$ on the sublocus $\mathcal{T}$ according to

$$\mathcal{P}_{(2)}\big|_\mathcal{T} = -\tau_\mathcal{T}\mathcal{P}_{(1)}\big|_\mathcal{T} \ . \tag{4.59}$$

Equation (4.60) summarizes the constraints for an $\mathcal{N} = 2$ Minkowski vacuum that originate from this specific, non-generic choice of gauging and the restriction of the universal hypermultiplet on the two-dimensional sublocus $\mathcal{T}$.

$$X^\Lambda(\mathbf{z})F_\Lambda = 0 \ , \quad X^\Lambda(\mathbf{z})H_\Lambda = 0 \ , \ \partial_i X^\Lambda(\mathbf{z})(F_\Lambda - \tau_\mathcal{T}|_\mathcal{T}\, H_\Lambda) = 0 \ . \tag{4.60}$$

These just coincide with the constraints for a supersymmetric flux vacuum in type IIB flux compactifications given in equations (4.9) and (4.10), if we identify the function $\tau_\mathcal{T}$, containing two real degrees of freedom of the universal hypermultiplet, with the axio-dilaton τ. This observation motivates to call the set $\mathcal{T}$ the *axio-dilaton non-genericity constraint*.

Let us illustrate this construction with an explicit example. To that end, we choose

$$\mathcal{P}_{(1)} = \mathcal{P}_1 \text{ and } \mathcal{P}_{(2)} = -\mathcal{P}_2 \tag{4.61}$$

which is achieved by setting $\tilde{\Theta}^3_\Lambda = \tilde{\Theta}^4_\Lambda = 0$. The axio-dilaton non-genericity constraint for this choice is given by the set

$$\mathcal{T} = \{(S, 0) \mid S \in \mathbb{C}\} \tag{4.62}$$

that restricts C to vanish because the contributions of $\mathcal{P}_2$ in the directions of σ^1 and σ^2 cannot be compensated otherwise. Hence, from equation (4.51) we can read off that

$$\mathcal{P}_{(1)}\big|_\mathcal{T} = \mathcal{P}_1|_\mathcal{T} = e^\phi\, \mathcal{P}_2|_\mathcal{T} = -e^\phi\, \mathcal{P}_{(2)}\big|_\mathcal{T} \tag{4.63}$$

implying $\tau_\mathcal{T} = e^{-\phi}$. We conclude that this concrete choice of gauging the isometries of the universal hypermultiplet realizes the Minkowski vacuum constraints

$$X^\Lambda(\mathbf{z})F_\Lambda = 0 \ , \quad X^\Lambda(\mathbf{z})H_\Lambda = 0 \ , \ \partial_i X^\Lambda(\mathbf{z})(F_\Lambda - e^{-\phi}H_\Lambda) = 0 \ . \tag{4.64}$$

on the non-genericity sublocus $\mathcal{T} \subset \mathcal{M}_{\text{U}}$. Note, that this construction is indeed consistent with the full vacuum constraints (4.53) and (4.55) if we fix $\tilde{\Theta}^3_\Lambda = \tilde{\Theta}^4_\Lambda = 0$ and hence gives rise to a valid $\mathcal{N} = 2$ Minkowski vacuum of the gauged $\mathcal{N} = 2$ supergravity theory.

In contrast to the general discussion, the function $\tau_{\mathcal{T}}$, that should be identified with the axio-dilaton, depends for this example not on both remaining degrees of freedom on $\mathcal{T}$ but only on the real dilaton field ϕ. This result is not surprising if we recall that the second unconstrained degree of freedom of $\mathcal{T}$ is the real axion field σ. Since the whole gauged $\mathcal{N} = 2$ supergravity theory is invariant under a shift in σ, any functional dependence on σ is prohibited. Hence, for any choice of gauging, $\mathcal{T}$ will always contain σ as a real degree of freedom and $\tau_{\mathcal{T}}$ can never be dependent on it. This observation might contradict with our identification of the Minkowski vacua with the supersymmetric flux vacua at first view. However, one should recall that the constraints were formulated in a semi-classical approximation. Since the constraints for a supersymmetric flux vacuum are formulated in terms of an $\mathcal{N} = 1$ superpotential which necessarily appears as a holomorphic function of the corresponding chiral fields, it can be expected that $\tau_{\mathcal{T}}$ will depend on two real degrees of freedom that can be combined in a complex coordinate on $\mathcal{T}$, as soon as quantum corrections are taken into account.

4.4 Flux Vacua for M-Theory Compactifications

Following [9, 41, 43, 130], a similar flux compactification as discussed in section 4.1 can be applied to M-Theory compactified on a Calabi-Yau fourfold X. The result gives an $\mathcal{N} = 1$ supergravity theory in $d = 3$ spacetime dimensions with a superpotential of the form

$$W = \int_X \Omega \wedge G \tag{4.65}$$

leading to a scalar potential

$$V(z) = e^K \left(\nabla_a W \nabla^a W - 4|W|^2\right) \ . \tag{4.66}$$

In contrast to the flux compactification of type IIB string theory, the internal four-form flux G does not receive any dependences on the axio-dilaton on the semi-classical level but is a pure topological quantity. However, in order to cancel tadpole-contributions[51] arising from the three-form C_3 it is necessary that G obeys the consistency condition [41, 43, 98, 130]

$$\int_X G \wedge G = \frac{T_2 \kappa_{11}^2}{6} \chi(X) \ . \tag{4.67}$$

Thus, properly normalized, consistent fluxes are integral, i.e. $G \in H^4(X, \mathbb{Z})$. Similar to the former discussion, a vacuum of this supergravity theory is obtained if the scalar potential V vanishes. As for the flux compactifications of type IIB string theory, V is subject to a no-scale relation [41] that cancels the contributions in $\nabla_a W \nabla^a W$ originating from the derivatives with respect to the hypermultiplet scalars against the term $-4|W|^2$. Hence, we obtain a vacuum configuration if and only if [52]

$$\nabla_{\mathbf{z}} W = 0 \ . \tag{4.68}$$

[51] These may arise if we consider next-to-leading order terms in the action. Here, $T_2 = (2\pi)^{2/3}(2\kappa_{11}^2)^{-1/3}$ denotes the corresponding membrane tension [98].

[52] Recall the short notation $\partial_{\mathbf{z}}$ for the gradient with respect to all complex structure moduli.

Following [9, 41], this condition already implies $W = 0$ which means that consistent flux vacua of M-theory compactified on a Calabi-Yau fourfold are always supersymmetric.

Explicitly written in terms of the holomorphic $(4,0)$-form $\Omega \in H^4(X_{\mathbf{z}}, \mathbb{C})$, the flux vacuum conditions translate to

$$\int_X (\nabla_{\mathbf{z}}\Omega) \wedge G = 0 \ . \tag{4.69}$$

Recall that the covariant derivative ∇ as was defined by $\nabla_i = \partial_i + (\partial_i K)$ which implies [10] that $\nabla_i\Omega \in H^{3,1}(X_{\mathbf{z}}, \mathbb{C})$ because the connection part $(\partial_i K)$ of ∇_i is chosen such that it annihilates the $(4,0)$-component of $\partial_i\Omega$. A similar analysis as in section 4.1 on the Hodge structure of G implies that the $(3,1)$ and $(1,3)$ parts of G need to vanish in order to obey equation (4.69) and its complex conjugate. Hence, possible four-form fluxes that give rise to flux vacua of M-theory are characterized by one-dimensional sublattices

$$< G > \subset \big(H^{4,0}(X_{\mathbf{z}}, \mathbb{C}) \oplus H^{2,2}(X_{\mathbf{z}}, \mathbb{C}) \oplus H^{0,4}(X_{\mathbf{z}}, \mathbb{C})\big) \cap H^4(X_{\mathbf{z}}, \mathbb{Z}) \ . \tag{4.70}$$

Similar to the former case of type IIB string compactifications, this observation transforms the flux equations to the question whether a non-trivial lattice of integral four-forms of a definite Hodge type exists.

As already mentioned briefly in section 4.1, this problem can be addressed using techniques from arithmetic geometry which we discuss in the following chapter. We will return to the quest of finding Calabi-Yau manifolds with consistent flux configurations in section 5.2 by relating the existence of integral sublattices of definite Hodge type to the property of modularity and present a possible strategy to solve this problem for a certain class of Calabi-Yau fourfolds in section 5.4.

Chapter 5

Arithmetic Analysis of Calabi-Yau Fourfolds and Modularity

Arithmetic Geometry provides very powerful tools to investigate the Hodge theory of Calabi-Yau manifolds. As we have discussed in the previous chapter, this is directly related to the existence of consistent flux vectors $G \in H^n(X, \mathbb{C})$ in the middle cohomology of a given Calabi-Yau n-fold as these correspond to the existence of integral sublattices of $H^n(X, \mathbb{Z})$ that are of definite Hodge type. The relation between number theory, geometry and physics has raised interest and became a fast growing field of research within mathematical physics. Among many further references, [4–7, 131–133] give a first insight on important results on this field.

The key object for the arithmetic analysis of algebraic varieties is given by the local zeta function $\zeta_p(X, T)$. Established as the generating function for the number of points $N_{p^r}(X)$ of a given variety X defined over the finite fields $\mathbb{F}_{p^r}$ for a prime p, the zeta function enjoys several powerful properties which are collected in the Weil conjectures [134]. One important implication thereof for our analysis of the integral middle cohomology of Calabi-Yau varieties turns out to be that any sublattice of $H^n(X, \mathbb{Z})$ that is of definite Hodge type implies an integral factorization of a certain polynomial in the rational representation of the zeta function. Those factorizations are discussed in the context of modularity as it is conjectured that such factors correspond to certain modular forms.

This modular correspondence has beeen proven for elliptic curves [135–137] which have been discussed extensively in the mathematical literature. More recently, the arithmetic geometry of K3 surfaces [138, 139] and Calabi-Yau threefolds [4–7, 133, 140–142] has been explored and the connection between integral sublattices of definite Hodge type, modularity and a factorization of the local zeta function has been confirmed. In particular, this exploration became very useful for the search of Calabi-Yau threefolds that lead to a consistent non-trivial supersymmetric flux vacuum of type IIB string compactifications [4, 96] as well as in the context of the attractor mechanism of BPS black holes [131, 132] which we review briefly in section 5.2.

Based on [1], we extend this arithmetic analysis in the following chapter even further to Calabi-Yau fourfolds. We begin by introducing the local zeta function and discussing the Weil conjectures and implications thereof in section 5.1. Most importantly, section 5.2 relates properties of the zeta function of Calabi-Yau manifolds to the splitting of the integral middle cohomology and introduces the notion of modular Calabi-Yau manifolds.

Sections 5.3 and 5.4 provide an algorithm for the practical evaluation of the relevant polynomial factor $R_4^H(X,T)$ of the zeta function for a given family of Calabi-Yau fourfolds. While the discussion in section 5.3 remains general, we will focus in section 5.4 on Calabi-Yau fourfolds with one complex structure modulus. Conceptionally, the analysis can be extended to Calabi-Yau fourfolds with more complex structure moduli[53], however, this goes beyond the scope of this work.

We summarize this chapter by demonstrating the methods presented for a couple of examples. Among the considered families of Calabi-Yau fourfolds, we identify one example that admits a point in its complex structure moduli space that corresponds to a modular Calabi-Yau fourfold. We discuss this example in quite extense in section 5.5 and investigate the possible geometric origin of the splitting of the Hodge structure.

5.1 The Local Zeta Function

In the mathematical literature, the local zeta function of an algebraic variety arises in the context of point countings on varieties which are defined over finite fields. Thus, let us begin with defining such varieties over finite fields and formulating the point counting problem[54].

Recall that for any prime p there exists a field with p elements, which we denote by $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$. Usually, this field is identified with the set of integers $\{0, 1, \dots, p-1\}$ and computations $(+, \cdot)$ modulo p. Moreover, for each prime p and each positive integer $r \in \mathbb{N}$, there exists a unique field with p^r elements, denoted by $\mathbb{F}_{p^r}$ which can be realized by the disjoint union of all p^{th} roots of unity and their first $p-1$ multiples. Again, all computations in these fields will be performed modulo p. In the following we will use the convenient notation of $q := p^r$ for fixed prime p and $r \in \mathbb{N}$.

Assuming that a given algebraic variety (affine or projective) of algebraic dimension $\dim(X) = n$ is defined as the zero-locus of a certain set of polynomials with integer

[53]Recently, in [143], the arithmetic analysis of Calabi-Yau threefolds has been extended to the multiparameter case. A similar generalization is expected to be possible for Calabi-Yau fourfolds as well.

[54]For a review on p-adic analysis and the local zeta function we refer to the textbook [144] and the more physics oriented review article [133].

coefficients [55]

$$X := \{f_i(x_1,\dots,x_n)=0\} \subset \mathbb{A}^n(\text{ or } \mathbb{P}^{n-1}) \qquad f_i \in \mathbb{Z}[x_1,\dots,x_n], \tag{5.1}$$

we can define X over $\mathbb{F}_q$ by considering the canonical projection[56]

$$\mathbb{Z}[x_1,\dots,x_n] \to \mathbb{F}_p[x_1,\dots,x_n] \quad , \quad f_i \mapsto \bar{f}_i \tag{5.2}$$

of all defining polynomials and considering solutions to these equations in $\mathbb{F}_q$. We denote this variety, defined over the finite field $\mathbb{F}_q$ by $X/\mathbb{F}_q$. Thus,

$$X/\mathbb{F}_q := \{\bar{f}_i(x_1,\dots,x_n)=0\} \subset (\mathbb{F}_q)^n \ . \tag{5.3}$$

As $\mathbb{F}_q$ is a finite field, any subset of $(\mathbb{F}_q)^n$ is a finite set, hence any variety defined over a finite field is given by a collection of finitely many points. We denote the number of points in $X/\mathbb{F}_q$ by

$$N_q(X) := |X/\mathbb{F}_q| \ . \tag{5.4}$$

For a fixed prime p, the collection of these numbers $N_{p^r}(X)$ gives rise to the local zeta function $\zeta_p(X,T)$ which is defined to be the generating function of the numbers $N_{p^r(X)}$ given by the formal power series

$$\zeta_p(X,T) := \exp\left(\sum_{r=1}^{\infty} N_{p^r}(X)\frac{T^r}{r}\right) \ . \tag{5.5}$$

From a number theoretical perspective it would now be an interesting problem to determine the $N_{p^r}(X)$ up to a high order in r. Whereas for small primes (e.g. $p=2,3,5$) and small values for r, this point counting can be done even by hand, the problem becomes very challenging if any of p or r increases such that even numerical algorithms fail to perform the explicit point count in a suitable time. However, the zeta function enjoys very powerful and highly non-trivial properties which are collected by André Weil [134]. These give an alternative approach to perform an indirect point count by using cohomology theory.

5.1.1 The Weil Conjectures

The Weil conjectures[57] [134] give high constraints on the structure of the local zeta function and allow to bring it in a computable form. Originally stated by Weil in 1949 [134], the conjectures were proven much later by Dwork [145], Grothendiek [146] and Deligne [147, 148] based on the invention of étale or ℓ-adic cohomology theory. The beauty of these proofs belongs to the fact, that they prove Weil's conjectures for general smooth projective varieties turning them into a very powerful tool for the arithmetic analysis of algebraic varieties. The Weil conjectures can be summarized as follows:

[55] More general, we can allow for rational coefficients by multiplying all defining equations by the common denominator.

[56] For a polynomial $f \in \mathbb{Z}[x_1,\dots,x_n]$ we obtain its canonical projection $\bar{f}$ onto $\mathbb{F}_p[x_1,\dots,x_n]$ by reducing all coefficients of f modulo p.

[57] We should point out at this point that the properties of the local zeta function which are summarized under this name have been proven. The name Weil conjecture is a historical remnant giving credit to André Weil who formulated these properties of $\zeta_p(X,T)$ following observations on elliptic curves ten years before Dwork gave a formal proof for the first of these conjectures.

Rationality: For any algebraic variety X of complex dimension n and any prime p, the local zeta function $\zeta_p(X,T)$ is a rational function

$$\zeta_p(X,T) = \frac{R_1(X,T)\cdots R_{2n-1}(X,T)}{R_0(X,T)\cdots R_{2n}(X,T)} \quad , \quad R_K \in \mathbb{Z}[T] \ . \tag{5.6}$$

Moreover, the degree of the polynomial R_k is given by the k^{th} betti number of the variety X, $\deg(R_k) = b^k(X)$.

Functional Equation: For any algebraic variety X, the local zeta function obeys the identity

$$\zeta_p(X, p^{-n}T^{-1}) = \pm p^{\frac{n}{2}\chi} T^{\chi} \zeta_p(X,T) \tag{5.7}$$

with χ being the Euler characteristic of X.

Riemann Hypothesis: If X is a smooth algebraic variety, the polynomials $R_k(X,T)$ from equation (5.6) factorize over $\mathbb{C}$ as

$$R_k(X,T) = \prod_{i=1}^{b^k} (1 - \lambda_{ik} T) \tag{5.8}$$

where the $\lambda_{ik} \in \mathbb{C}$ are algebraic integers[58] with absolute value $|\lambda_{ik}| = p^{\frac{k}{2}}$.

Instead of giving the formal proofs for these properties[59], it should be emphasized here, that the following discussion will crucially need all of these properties. Thus, the importance of the Weil conjectures for the arithmetic analysis of Calabi-Yau manifolds cannot be strengthen enough at this point.

Applying the Weil conjectures to local zeta functions of Calabi-Yau varieties gives a strong restriction on those. In particular, one can make use of the highly constrained form of the Hodge diamonds (c.f. Figures 3.1 and 3.2) to deduce that many of the polynomials $R_k(X,T)$ become trivial. For any Calabi-Yau n-fold we have $b^0 = b^n = h^{0,0} = 1$. Hence, $R_0(X,T)$ and $R_{2n}(X,T)$ are linear in T. For this special case, the Riemann hypothesis implies that

$$R_0(X,T) = (1 - \lambda_0 T) \qquad , \qquad R_{2n}(X,T) = (1 - \lambda_{2n} T) \tag{5.9}$$

with the restriction that the coefficients λ_0 and λ_{2n} need to be integers as $R_k \in \mathbb{Z}[T]$.

[58]That means that there exists an algebraic equation over $\mathbb{Z}$ whose solution is given by λ_{ik}.

[59]The interested reader is referred to Refs. [147, 148] for a comprehensive review on the Weil conjectures. Ref. [149] provides a modern review article including the proofs of the rationality condition and the functional equation.

Moreover, the Riemann hypothesis restricts the absolut value of these coefficients[60] such that one obtains[61]

$$R_0(X,T) = (1-T) \qquad , \qquad R_{2n}(X,T) = (1-p^n T) \ . \tag{5.10}$$

In addition, we have for Calabi-Yau n-folds with $n > 1$ that $b^1(X) = b^{2n-1}(X) = 0$ impliying that the corresponding polynomials $R_1(X)$ and $R_{2n-1}(X)$ become constant. Due to the Riemann hypothesis, these constants are fixed to one. Moreover, if the Picard group[62] of X is generated by divisors which are defined over the finite field $\mathbb{F}_p$, the polynomials $R_2(X)$ and $R_{2n-2}(X)$ are computed to be [6, 149]

$$R_2(X) = (1-pT)^{b^2(X)} \qquad , \qquad R_{2n-2}(X) = (1-p^{n-1}T)^{b^{2n-2}(X)} \tag{5.11}$$

since their contribution is purely encoded in the intersection theory of the vertical cohomology $H^{1,1}(X,\mathbb{C})$ and $H^{(n-1),(n-1)}(X,\mathbb{C})$ respectively. In the following we will assume that all families of Calabi-Yau n-folds which we consider have this property.

Using these implications for $\zeta_p(X,T)$ following directly from the Weil conjectures, the local zeta function for a given Calabi-Yau n-fold simplifies drastically. In the following equations (5.12) – (5.14) we summarize the result for the local zeta functions of elliptic curves ($n = 1$), K3 surfaces ($n = 2$) and Calabi-Yau threefolds ($n = 3$).

$$n=1 \qquad \zeta_p(X) = \frac{1-a_pT+T^2}{(1-T)(1-pT)} \tag{5.12}$$

$$n=2 \qquad \zeta_p(X) = \frac{1}{(1-T)R_2(X)(1-p^2T)} \tag{5.13}$$

$$n=3 \qquad \zeta_p(X) = \frac{R_3(X)}{(1-T)(1-pT)^{b^2(X)}(1-p^2T)^{b^2(X)}(1-p^3T)} \ . \tag{5.14}$$

We should point out that for each prime p, the local zeta function for elliptic curves is already completely determined up to one remaining coefficient a_p, whereas even the zeta functions for K3 surfaces and Calabi-Yau threefolds are fixed up to only one unknown polynomial $R_2(X)$ and $R_3(X)$, respectively that is (for a given family of Calabi-Yau manifolds) of fixed degree and hence is characterized by a finite number of coefficients.

---

[60] Observe that the only integers $\lambda \in \mathbb{Z}$ with $|\lambda| = p^{\frac{k}{2}}$ are given by $\lambda = \pm p^{\frac{k}{2}}$.

[61] The relative minus sign is not a direct consequence of the Weil conjectures but follows from the representation of $R_k(X,T)$ as the characteristic polynomial of a certain linear operator Fr_p. We discuss this representation in detail in section 5.1.2.

[62] The Picard group $\mathrm{Pic}(X)$ of a complex manifold X is given by set of holomorphic line bundles over X modulo isomorphisms which obtains a group structure via the tensor product of line bundles [150]. Moreover, $\mathrm{Pic}(X)$ can be identified with the integral cohomology classes in $H^2(X,\mathbb{Z}) \cap H^{1,1}(X,\mathbb{C})$ modulo torsion.

As the final result of this section, we present the structure of the full local zeta function of Calabi-Yau fourfolds in equation (5.15).

$$\zeta_p(X,T) = \frac{R_3(X,T)R_5(X,T)}{(1-T)(1-pT)^{b^2(X)}R_4(X,T)(1-p^3T)^{b^2(X)}(1-p^4T)} \ . \tag{5.15}$$

Thus, we see that computing the local zeta function of a Calabi-Yau fourfold X reduces to the task of finding the polynomials $R_3(X,T)$, $R_4(X,T)$ and $R_5(X,T)$ for each prime p.

5.1.2 The Frobenius Map Fr_p

An important milestone on the way for proving the Weil conjectures is given by the observation that for any algebraic variety X, the point counts $N_{p^r}(X)$ are related to the trace of a certain linear operator

$$\mathrm{Fr}_p : H^k(X,\mathbb{Q}_p) \to H^k(X,\mathbb{Q}_p) \ . \tag{5.16}$$

Here, the $H^k(X,\mathbb{Q}_p)$ denote the b^k-dimensional k^{th} p-adic cohomology groups[63] of the variety X defined over the field $\mathbb{Q}_p$ of p-adic numbers[64]. Although the consistent construction of these cohomology groups is a very interesting subfield of arithmetic geometry, it turns out that we will not need any details of this framework in order to compute the relevant information of Fr_p for the point counts. We refer to [133, 153, 154] for an introduction to p-adic cohomology and its relevance for proving the Weil conjectures. Instead of p-adic cohomology it is also possible to consider the Frobenius map on étale or ℓ-adic cohomology groups, where ℓ is coprime to p as for instance in refs. [4, 149].

For any such cohomology theory, we can formulate the Lefschetz fixed-point theorem [155], stating that the number of fixed points of a continous map $f : X \to X$ is given by the alternating sum of traces of the induced maps $f_\star H^k(X,\mathbb{Q}_p) \to H^k(X,\mathbb{Q}_p)$ on suitable singular coholomogy groups

$$\Gamma^f = \sum_{k=0}^{2n}(-1)^k \mathrm{Tr}\left(f_\star : H^k(X,\mathbb{Q}_p) \to H^k(X,\mathbb{Q}_p)\right) \ . \tag{5.17}$$

Let us define the Frobenius map $\mathrm{fr}_p : X \to X$ acting on $X/\mathbb{Q}_p$ the algebraic variety X defined over field of p-adic numbers $\mathbb{Q}_p$ by

$$(x_1,\ldots,x_n) \mapsto (x_1^p,\ldots,x_n^p) \ . \tag{5.18}$$

By definition, the points on $X/\mathbb{F}_p$ are given by the fixed points fr_p on $X/\mathbb{Q}_p$. Hence, we find by applying the Lefschetz fixed-point theorem

$$N_p(X) = \sum_{k=0}^{2n}(-1)^k \mathrm{Tr}\left(\mathrm{Fr}_p : H^k(X,\mathbb{Q}_p) \to H^k(X,\mathbb{Q}_p)\right) \ . \tag{5.19}$$

[63] In the math literature, there many different constructions of suitable cohomology theories of varieties over finite fields as for instance [88, 151, 152] have been invented. In this work, we follow [6] and use the notation of p-adic cohomology as a collective name for any of these well-defined cohomology theories.

[64] Appendix A provides a brief introduction to p-adic numbers.

Here, $\mathrm{Fr}_p := (\mathrm{fr}_p)_\star$ denotes the lift of the Frobenius map to the chosen p-adic (or ℓ-adic) cohomology groups. By replacing p with p^r in this formula, we obtain the final result to express the number of points

$$N_{p^r}(X) = \sum_{k=0}^{2n} (-1)^k \mathrm{Tr}\left(\mathrm{Fr}_{p^r} : H^k(X, \mathbb{Q}_p) \to H^k(X, \mathbb{Q}_p)\right) \tag{5.20}$$

purely in terms of the lifted Frobenius map [133]. Inserting this expression for the point counts into the defining equation of the local zeta function (5.5), it follows[65] that

$$\zeta_p(X,T) = \prod_{k=0}^{2n} \det\left((\mathrm{Id} - T\mathrm{Fr}_p^{-1}) : H^k(X, \mathbb{Q}_p) \to H^k(X, \mathbb{Q}_p)\right)^{(-1)^k} \,. \tag{5.21}$$

This observation not only sketches the proof[66] of the first Weil conjecture but moreover gives an expression of the polynomials $R_k(X,T)$ in terms of the Frobenius lift Fr_p as

$$R_k(X,T) = \det\left((\mathrm{Id} - T\mathrm{Fr}_p^{-1}) : H^k(X, \mathbb{Q}_p) \to H^k(X, \mathbb{Q}_p)\right) \,. \tag{5.22}$$

This equation is the central result which allows to compute the polynomial $R_k(X,T)$ explicitly. Since $H^k(X, \mathbb{Q}_p)$ is finite-dimensional, the action of the lifted Frobenius map Fr_p can be encoded by fixing a basis on $H^k(X, \mathbb{Q}_p)$ in a representation matrix $F_p^k(X)$. Moreover, we use the convenient notation $U_p^k(X) = (F_p^k)^{-1}(X)$ for its inverse matrix. In this notation, equation (5.22) reads

$$R_k(X,T) = \det\left(\mathbb{1} - TU_p^k(X)\right) \,. \tag{5.23}$$

The task of computing the local zeta function hence reduced to the problem of finding an explicit expression of the matrices $U_p^k(X)$. From now on, let us specify the algebraic variety to be a Calabi-Yau n-fold $X(\mathbf{z})$ which corresponds to a smooth family $\mathcal{X}_\mathcal{M}$ of Calabi-Yau n-folds. The representation matrices $F_p^k(X_\mathbf{z})$ extend to locally holomorphic functions by choosing a locally constant frame on $\mathcal{H}_p^k(\mathcal{X}_\mathcal{M})$. To abbreviate our notation, we denote the Frobenius matrix $F_p^k(X_\mathbf{z})$ by $F_p^k(\mathbf{z})$ and use the analogous notation for its inverse U_p^k.

Recall from chapter 4, that we are mainly interested in the middle cohomology $H^n(X_\mathbf{z}, \mathbb{Z})$ of a given Calabi-Yau n-fold. Hence, we will focus from now on on the polynomial $R_n(X_\mathbf{z}, T)$ in the zeta function, as this encodes the relevant information that is required to identify integral sublattices of $H^n(X_\mathbf{z}, \mathbb{Z})$. In the remainder of this section we will collect all the properties of Fr_p acting on the middle cohomology, that are needed for the

[65] Here, we use the identity $e^{\mathrm{Tr}(A)} = \det(e^A)$ for any linear operator A on a finite dimensional vector space as well as $\mathrm{Fr}_{p^r} = (\mathrm{Fr}_p)^r$.

[66] One should note that the highly non-trivial statement is hidden in the definition of suitable p-adic cohomology groups that are finite-dimensional vector spaces and moreover give rise to a Lefschetz fixed-point theorem. This construction is one of the main achievements of the work [88] by Dwork.

computation of $R_n(X_{\mathbf{z}}, T)$.

First, we note that Fr_p is *Frobenius-linear* [6] which means that for any scalar function $c(\mathbf{z}) \in C^\infty(\mathcal{M}_{C.S.})$ and any vector $v \in \mathcal{H}^k$ we have the identity

$$\mathrm{Fr}_p(\mathbf{z})\left(c(\mathbf{z})v\right) = c(\mathbf{z}^p)\mathrm{Fr}_p(\mathbf{z})(v)\ . \tag{5.24}$$

For any family of Calabi-Yau manifolds, there exists a special connection ∇ on the vector bundle $H^n(\mathcal{X}_{\mathcal{M}}, \mathbb{C})$ over $\mathcal{M}_{C.S.}$ that is called the *Gauss-Manin connection*[67]. This connection gives an equivalent description of the Picard-Fuchs ideal by its defining property that $\Omega(\mathbf{z})$ and its derivatives $\mathcal{D}^i\Omega(\mathbf{z})$ are covariant constant sections on $H^n(\mathcal{X}_{\mathcal{M}}, \mathbb{C})$ with respect to ∇. For a given constant local frame $\{e^k\}$ on $\mathcal{H}^n(\mathcal{X}_{\mathcal{M}}, \mathbb{C})$ and any holomorphic section $X \in \Gamma(\mathcal{M}_{\mathcal{C}.S.})$, the action of ∇_X is determined by a matrix B_X such that[68]

$$\nabla_X v_k(\mathbf{z})e^k = X(v_k(\mathbf{z}))e^k - v_k(B_X)^k_j e^j \tag{5.25}$$

for any $v = v_k e^k \in \mathcal{H}^n(\mathcal{X}_{\mathcal{M}}, \mathbb{C})$. For our purposes, we can restrict the choice of vector fields X on $\mathcal{M}$ to the derivatives ∂_{z^i} as these generate $\Gamma(\mathcal{M}_{C.S.})$. It is convenient in the context of Calabi-Yau moduli spaces to introduce the logarithmic derivative $\Theta_i := z^i\partial_{z^i}$ and consider $\Gamma(\mathcal{C}_{C.S.})$ to be generated by those instead of the ordinary partial derivatives.

The crucial observation is that the lifted Frobenius map $\mathrm{Fr}_p(\mathbf{z})$ on $\mathcal{H}^n_p(\mathcal{X}_{\mathcal{M}})$ is compatible with the Gauss-Manin connection ∇ which extends naturally to the p-adic cohomology group[6]. That means, we have the identity

$$\nabla_X \mathrm{Fr}_p(\mathbf{z}) = p\mathrm{Fr}_p(\mathbf{z})\nabla_X\ . \tag{5.26}$$

Finally, the Frobenius map is compatible with the inner product on $H^n(X_{\mathbf{z}}, \mathbb{C})$ [6], hence we have the additional identity

$$\int_{X_{\mathbf{z}}} \mathrm{Fr}_p(\alpha) \wedge \mathrm{Fr}_p(\beta) = p^n \mathrm{Fr}_p\left(\int_{X_{\mathbf{z}}} \alpha \wedge \beta\right)\ . \tag{5.27}$$

It should be noted that the Frobenius map on the left side of this equation is given by $\mathrm{Fr}_p : H^n(X_{\mathbf{z}}, \mathbb{Q}_p) \to H^n(X_{\mathbf{z}}, \mathbb{Q}_p)$ in contrast to the Frobenius map $\mathrm{Fr}_p : H^{2n}(X_{\mathbf{z}}, \mathbb{Q}_p) \to H^{2n}(X_{\mathbf{z}}, \mathbb{Q}_p)$ on the right side.

As we will discuss in section 5.3, these three properties of the Frobenius map, given by equations (5.24), (5.26) and (5.27) suffice to compute the matrix $U_p(\mathbf{z})$ by solving a set of matrix-valued differential equations. Before discussing this strategy, which is called the *deformation method*, for the concrete case of Calabi-Yau fourfolds, we will first discuss in the following section the important connection between integral sublattices of $H^n(X_{\mathbf{z}}, \mathbb{Z})$ that are of definite Hodge type, as they appear in the context of flux compactifications, and factorizations of the polynomial $R_n(X_{\mathbf{z}}, T)$ appearing in the local zeta function.

[67] In general, one can define a Gauss-Manin connection on any vector bundle over an algebraic variety [63].

[68] Note that the matrix B_X acts from the right on the vector e^j. We have chosen this convenient notation following [6].

5.2 Modularity of Calabi-Yau Manifolds

Modularity of Calabi-Yau manifolds is one important aspect of the connection between the mathematical fields of algebraic geometry and number theory which is summarized in the *Langlands Programm* [156, 157]. Moreover, this connection raised interest in the physics literature [4, 158, 159] as it leads in particular to a relation between certain Calabi-Yau geometries and modular forms that appear in the context of rank-two attractor points or supersymmetric flux vacua. Before discussing these physics application of modularity in section 5.2.3, we briefly review the theory of modular forms and introduce the formal defintion of modular Calabi-Yau n-folds. This section is in parts inspired by the discussions in ref. [160].

It should be noted, that this section is not aimed to give a comprehensive review of modularity but rather provides a motivation, why modularity is a very fascinating field to study from a mathematical point of view and moreover argues for its applications in the search for non-trivial supersymmetric flux compactifications. More detailed work on the topic can be found in refs. [139, 161].

The modularity correspondence that will play a central role in this thesis, is given by a relation between two-dimensional representations of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and modular forms that originate as eigenvalues of so-called Hecke operators. As we will discuss in section 5.2.2, the representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ are characterized by the eigenvalues of the Frobenius map Fr_p acting on the étale cohomology groups $H^k_{\text{ét}}(X,\mathbb{Q})$ providing a direct relation between modular forms and properties of the local zeta function $\zeta_p(X,T)$.

5.2.1 Modular Forms

The following section gives a review on the basic definitions and concepts regarding modular forms. We follow the conventions presented in [162]. Let

$$\mathcal{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\} \tag{5.28}$$

denote the Siegel upper-half plane of the complex numbers. A Möbius transformation of $\mathcal{H}$ is given by

$$\tau \mapsto \gamma\tau = \frac{A\tau + B}{C\tau + D} \quad \text{for } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL_2(\mathbb{Z}) \ . \tag{5.29}$$

In the conext of modular forms, we often call such a transformation a modular transformation. A modular form of weight k is a holomorphic function $f : \mathcal{H} \to \mathcal{H}$ that is holomorphic at infinity[69] and transforms under a Möbius transformation according $\gamma \in SL_2(\mathbb{Z})$ to

$$f(\gamma\tau) = (C\tau + D)^k f(\tau) \ . \tag{5.30}$$

[69] A function $f : \mathcal{H} \to \mathcal{H}$ is holomorphic at infinity if $f(\tau)$ converges in the limit $\tau \to \infty$ and moreover, f has a convergent series expansion in $q = e^{2\pi i\tau}$ around $q = 0$.

We denote the vector space of all modular forms of weight k by $M_k(SL_2(\mathbb{Z}))$.

This special transformation behavior of modular forms is very constraining. It follows that $M_k(SL_2(\mathbb{Z})) = \{0\}$ for any odd integer k and for $k = 2$. Moreover, $M_n(SL_2(\mathbb{Z}))$ is one-dimensional[70] for $n = 0, 4, 6, 8, 10$ whereas in general $M_k(SL_2(\mathbb{Z}))$ remains finite dimensional for any $k \in \mathbb{N}$.

We note that $SL_2(\mathbb{Z})$ is generated by what are called S- and T-transformations

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{5.31}$$

which act on $\mathcal{H}$ as a reflection (S) and a translation (T)

$$S : \tau \mapsto -\frac{1}{\tau} \quad , \quad T : \tau \mapsto \tau + 1 \ . \tag{5.32}$$

Any two points on $\mathcal{H}$ that are not connected by a modular transformation can be collected in the *fundamental domain* $\mathcal{F}_1 \subset \mathcal{H}$ that is given by

$$\mathcal{F}_1 = \{\tau \in \mathcal{H} \mid |\tau| > 1 \ \wedge \ -1 \leq 2\mathrm{Re}(\tau) < 1\} \ . \tag{5.33}$$

Figure 5.1 sketches the fundamental domain $\mathcal{F}_1$ together with the connection of any further region on $\mathcal{H}$ to $\mathcal{F}_1$ via S- and T-transformations
This observation implies that any modular form f is fully characterized by its values on $\mathcal{F}_1$. In particular, f is one-periodic, as

$$f(\tau + 1) = f(T\tau) = (0\tau + 1)^k f(\tau) = f(\tau) \tag{5.34}$$

and hence can be expressed in a Fourier expansion[71]

$$f(\tau) = \sum_{k=0}^{\infty} a_n q^n \quad , \quad q = e^{2\pi i \tau} \ . \tag{5.35}$$

Note that this series converges on $\mathcal{F}_1$ and in particular in the limit $\tau \to \infty$ as f was assumed to be holomorphic at infinity. Any modular form $f \in M_k(SL_2(\mathbb{Z}))$ is hence characterized by its Fourier coefficients $f \mapsto (a_n)_{n \in \mathbb{N}}$.

The very constraining fact for modular forms given by equation (5.30) can be relaxed by restricting the modular group $SL_2(\mathbb{Z})$ to a subgroup thereof. Most important for the following discussions are the *Hecke congruence subgroups* of level N which are defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \mod N \right\} \tag{5.36}$$

[70] In these cases, $M_k(SL_2(\mathbb{Z}))$ is generated by the k^{th} Eisenstein series G_k.
[71] Following [163], this Fourier expansion can be identified with the Taylor expansion of f around infinity.

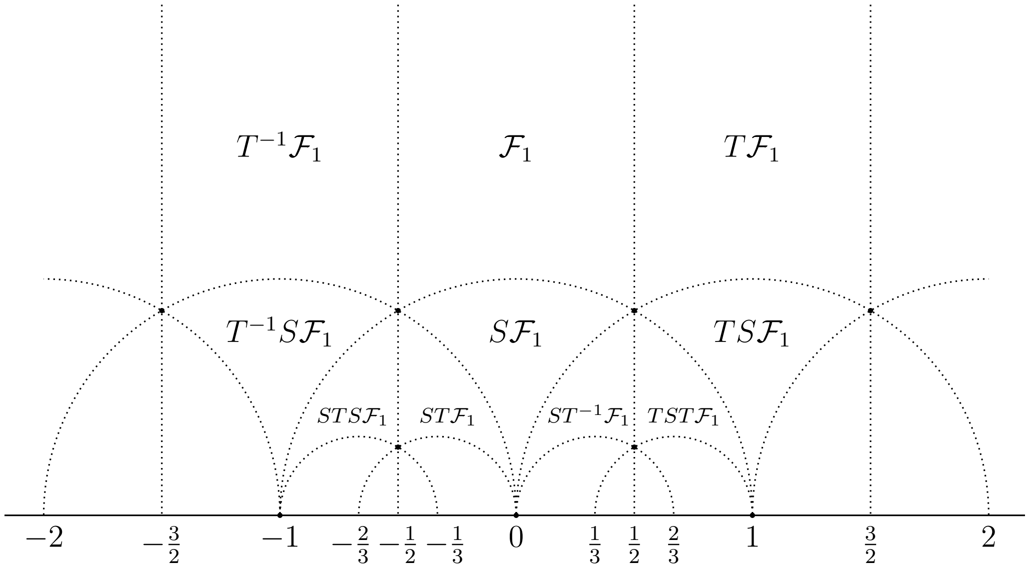

Figure 5.1: The fundamental domain $\mathcal{F}_1$ of the Siegel upper-half plane. For all remaining regions on $\mathcal{H}$ it is indicated how they evolve from $\mathcal{F}_1$ by appropriate S- and T-transformations. This figure is taken from [163].

for any $N \in \mathbb{N}$. In analogy to the previous definition, we call $f : \mathcal{H} \to \mathcal{H}$ a modular form of the group $\Gamma_0(N)$ of weight k, if f is holomorphic on $\mathcal{H}$ and obeys equation (5.30) for all $\gamma \in \Gamma_0(N)$. We denote the space of modular forms of $\Gamma_0(N)$ of weight k by $M_k(\Gamma_0(N))$.

It follows directly that $M_k(SL_2(\mathbb{Z}))$ is a sub vector space of $M_k(\Gamma_0(N))$ but moreover $M_k(\Gamma_0(N))$ is non-zero for any $k \in \mathbb{N}$. However, the transformation behavior is still constraining enough to guarantee that $M_k(\Gamma_0(N))$ remains finite-dimensional for all k and N. One should note that $T \in \Gamma_0(N)$ for all N. Hence, all modular forms of $\Gamma_0(N)$ are still one-periodic and therefore have a fourier expansion at infinity according to equation (5.35).

Finally a *cusp form* of weight k is a modular form $f \in M_k(SL_2(\mathbb{Z}))$ whose coefficient a_0 in the Fourier expansion (5.35) vanishes. Obviously, the space of all cusp forms of weight k which we denote by $S_k(SL_2(\mathbb{Z}))$ is a sub vector space of $M_k(SL_2(\mathbb{Z}))$. In analogy we define $S_k(\Gamma_0(N))$ to be the space of all cusp forms in $M_k(\Gamma_0(N))$.

5.2.2 The Modularity Conjecture

Since the modularity conjecture is formulated in terms of representations of the Galois group $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, we start this section by briefly introducing Galois theory of field extensions.

Let us consider a general field extension L/K. Its Galois group $\text{Gal}(L/K)$ is defined to be the group of automorphisms on L that leave K pointwise invariant [164], i.e.

$$\mathrm{Gal}(L/K) = \{\phi \in \mathrm{Aut}(L) \mid \phi(k) = k \quad \text{for all } k \in K\} \ . \tag{5.37}$$

In order to relate the Hodge theory of the p-adic middle cohomology $H^n(X, \mathbb{Q}_p)$ of a Calabi-Yau manifold X defined over $\mathbb{F}_p$ to Galois theory, let us observe that for any field extension $L/\mathbb{F}_p$, the Frobenius map $\mathrm{fr}_p : L \to L$ is a natural Galois element, $\mathrm{fr}_p \in \mathrm{Gal}(L/\mathbb{F}_p)$. In particular, if $L = \bar{\mathbb{F}}_p$ is the algebraic closure of $\mathbb{F}_p$, then the generated subgroup $\langle \mathrm{fr}_p \rangle \subset \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ is shown to be dense in $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$. The corresponding lift $\mathrm{Fr}_p : H^n(X, \mathbb{Q}_p) \to H^n(X, \mathbb{Q}_p)$ on the middle p-adic cohomology group defines a representation ρ of $\mathrm{fr}_p \in \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ [165].

This defines as well a representation ρ of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ that is of dimension $b^n(X)$ because any representation of $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ can be lifted to a representation on $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ via the diagram

$$\begin{array}{ccc} \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) & \overset{\iota}{\hookrightarrow} & \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \\ \Big\downarrow{\scriptstyle \pi} & \nearrow & \\ \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) & & \end{array} \ . \tag{5.38}$$

Now we can formulate Serre's modularity conjecture[72] [170, 171] which states that for any two-dimensional representation $\rho : V \to V$ of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, there exists a corresponding modular form $f_\rho \in M_k(\Gamma_0(N))$ that is the eigenform of all Hecke operators[73].

Thus, we can conclude that if the representation ρ on the middle p-adic cohomology $H^n(X, \mathbb{Q}_p)$ is two-dimensional, this representation defines uniquely a certain modular form. As the representation is completely characterized by the action of the Frobenius maps Fr_p, this modular form necessarily needs to be encoded in the action of Fr_p.

Modularity is best understood in the case of elliptic curves $\mathcal{E}$. In this case, we have by definition that the middle cohomology $H^1(X, \mathbb{Q})$ is fixed to be two-dimensional. Hence, from Serre's conjecture, we expect for any elliptic curve a corresponding modular form. Recall from equation (5.12) that the characteristic polynomial $R_1(\mathcal{E}, T)$ of the Frobenius map is given by the quadratic polynomial

$$R_1(\mathcal{E}, T) = 1 - a_p T + T^2 \tag{5.39}$$

[72]This conjecture has been proven by Dieulefait, Khare, Wintenberger, and Kisin [166–169].

[73]Hecke operators define a certain type of linear operators acting on $M_k(\Gamma_0(N))$. It can be shown that these commute among each other, hence it is possible to find a basis of modular forms that are simultaneos eigenforms of all Hecke operators. Ref. [162] provides the interested reader with a comprehensive introduction to Hecke theory.

for some coefficients $a_p \in \mathbb{N}$. In particular, for elliptic curves, these coefficients can be identified directly with the point counts as

$$a_p = p + 1 - N_p(\mathcal{E}) \ . \tag{5.40}$$

It is proven [135–137] that these coefficients a_p are precisely the Fourier coefficients of a weight-two modular form which realizes a Hecke eigenform.

Another very prominent example are so-called *rigid* Calabi-Yau threefolds, which are Calabi-Yau threefolds with $h^{2,1} = 0$. For these, $H^3(X, \mathbb{Q}_p)$ is two-dimensional and it has been proven in [172] that the coefficients of the quadratic polynomial $R_3(X, T)$ give rise to the Fourier expansion of a modular form.

For a general Calabi-Yau n-fold X, the middle cohomology group $H^n(X, \mathbb{Q}_p)$ is $b^n(X)$-dimensional where $b^n(X) > 2$ as soon as we exclude rigid Calabi-Yau manifolds from the discussion. Hence, the representation ρ of these manifolds, which we have introduced above, does not seem to contain any information on modular forms due to Serre's conjecture. However, let us assume that $H^n(X, \mathbb{Q}_p)$ factorizes over $\mathbb{Q}_p$ into a two-dimensional subspace Λ of definite Hodge type[74] and a remaining $(b^n - 2)$-dimensional subspace Σ according to

$$H^n(X, \mathbb{Q}_p) = \Lambda \oplus \Sigma \ . \tag{5.41}$$

Then, the Hodge conjectures [173] imply that the representation ρ of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ splits into a two-dimensional representation ρ_Λ on Λ and a $(b^n(X) - 2)$-representation ρ_Σ on Σ [4]. Moreover, the Frobenius map on $H^n(X, \mathbb{Q}_p)$ becomes block diagonal implying that the polynomial $R_n(X, T)$ in the zeta function factorizes for all primes into a quadratic polynomial $R_\Lambda(X, T)$ and a remaining polynomial $R_\Sigma(X, T)$ of degree $b^n(X) - 2$ as

$$R_n(X, T) = R_\Lambda(X, T) R_\Sigma(X, T) \ . \tag{5.42}$$

We call this a *persistent factorization* of the poylnomial $R_n(X, T)$. Now, we can apply Serre's conjecture to the former representation ρ_Λ leading to a corresponding modular form whose Fourier coefficients can be identified with the coefficients a_p of the polynomials $R_\Lambda(X, T)$ for all primes p.

In this discussion we have neglected the subtlety that ρ_Λ and ρ_Σ are proven to be representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/L)$ where $L/\mathbb{Q}$ is in general some field extension of $\mathbb{Q}$ [4]. Hence, we need to include the additional assumption that $L = \mathbb{Q}$ in order to apply Serre's conjecture.

Summarizing this discussion, we can formulate the following version of the *modularity conjecture* for Calabi-Yau n-folds [4, 5]: Let X be a Calabi-Yau n-fold. Assuming that

[74] Here and in the following, we refer to a subspace Λ of $H^n(X, \mathbb{Q}_p)$ as being of definite Hodge type if it gives rise to a consistent Hodge substructure. That means $\Lambda \otimes \mathbb{C}$ decomposes into a direct sum of complex spaces $\Lambda^{p,q}$ with $p + q = n$ such that $\bar{\Lambda^{p,q}} = \Lambda^{q,p}$. Moreover, $\Lambda^{p,q} \subset H^{p,q}(X, \mathbb{C})$ for each of these components. If we say that Λ is of definite Hodge type $(p, q) + (q, p)$, we mean that this decomposition contains only subspaces of precisely these Dolbeault signatures.

$H^n(X,\mathbb{Q})$ factorizes over $\mathbb{Q}$ into a two-dimensional subspace Λ of definite Hodge type and a remaining subspace Σ, then the characteristic polynomial $R_n(X,T)$ of the Frobenius map $\mathrm{Fr}_p : H^n(X,\mathbb{Q}_p) \to H^n(X,\mathbb{Q}_p)$ factorizes for all primes p of "good reduction" into a quadratic polynomial $R_\Lambda(X,T)$ and a remainder $R_\Sigma(X,T)$. In this context, we say that p is a prime of good reduction for an algebraic variety X if $X/\mathbb{F}_p$ is non-singular. If $R_n(X,T)$ factorizes in this fashion. the quadratic polynomial $R_\Lambda(X,T)$ is given by

$$R_\Lambda(X,T) = 1 - a_p p^\alpha T + p^\beta T^2 \tag{5.43}$$

for some $\alpha, \beta \in \mathbb{N}$ that depend on the dimension of X and are fixed by the Weil conjectures. The coefficients a_p can be identified with the Fourier coefficients of a corresponding modular form f_Λ.

Note that conversely, this conjecture gives a sufficient criterion on the existence of a two-dimensional subspace $\Lambda \subset H^n(X,\mathbb{Q})$ of the middle cohomology that is of definite Hodge type purely in terms of the persistent factorization of the polynomial $R_n(X,T)$. Such a subspace always defines a two-dimensional sublattice of $H^n(X,\mathbb{Z})$ of the same definite Hodge type. Hence, the local zeta function (or more precise the characteristic polynomial of Fr_p) gives a very powerful tool to search for Calabi-Yau n-folds whose integral middle cohomology splits off a two-dimensional sublattice of definite Hodge type.

5.2.3 Modularity and Physics

The modularity conjecture can be used to investigate several problems that occur in mathematical physics whenever integral sublattices of the middle cohomology $H^n(x,\mathbb{Z})$ of a Calabi-Yau n-fold appear. In the following section, we collect such problems and discuss the origin and the structure of the corresponding sublattices.

Supersymmetric Flux Vacua of type IIB string compactifications:
As we have discussed in section 4.1, flux compactifications of type IIB string theory on a Calabi-Yau threefold X give rise to consistent supersymmetric vacuum configurations, only if the integral flux vectors $F, H \in H^3(X,\mathbb{Z})$ are contained in $H^{2,1}(X,\mathbb{C}) \oplus H^{1,2}(X,\mathbb{C})$. Hence, X has a non-trivial flux configuration if and only if its integral middle cohomology $H^3(X,\mathbb{Z})$ has a two-dimensional sublattice of definite Hodge type $(2,1)+(1,2)$. Following the strategy as presented in the previous section, this sublattice is equivalently characterized by a split of the rational middle cohomology

$$H^3(X,\mathbb{Q}) = \Lambda_{\text{flux}} \oplus \Sigma \tag{5.44}$$

with

$$\Lambda_{\text{flux}} \subset H^3(X,\mathbb{Q}) \cap \big(H^{2,1}(X,\mathbb{C}) \oplus H^{1,2}(X,\mathbb{C})\big) \tag{5.45}$$

being two-dimensional. According to the modularity conjecture, such a splitting should be spottable by a factorization of the polynomial $R_3(X,T)$ appearing in the local zeta function $\zeta_p(X,T)$ for every prime p of good reduction. This method has been applied for instance in refs. [4, 96] in order to find type IIB compactifications with non-trivial flux configurations.

Moreover, a basis for Λ_{flux} and hence a suitable choice of fluxes $F, H \in H^3(X,\mathbb{Z})$ can be determined by observing that the derivative of the integral period vector $\Pi(\mathbf{z})$ along a certain direction z^i in the complex structure moduli space becomes purely of Hodge type $(2,1)+(1,2)$. Thus, suitably normalized, the two independent vectors

$$F = \text{Re}(\partial_i \Pi(\mathbf{z}_{\text{flux}})) \quad \text{and} \quad H = \text{Im}(\partial_i \Pi(\mathbf{z}_{\text{flux}})) \tag{5.46}$$

are integral and in addition of pure Hodge type $(2,1)+(1,2)$ [96]. Here, $\mathbf{z}_{\text{flux}}$ denotes the specific point on the complex structure moduli space, that corresponds to the Calabi-Yau threefold which admits the persistent factorization of $R_3(X_{\mathbf{z}_{\text{flux}}}, T)$.

Rank-two Attractor Points:
The attractor mechanism, established in [174, 175], describes the dynamics of BPS black hole solutions of vector multiplets in $\mathcal{N}=2$ supergravity theories that arise from type IIB string compactifications on a Calabi-Yau threefold X. Following [7, 175], these are characterized by a quantized charge vector $Q \in \mathbb{H}^3(X,\mathbb{Z})$ that defines the central charge

$$Z(Q) = \frac{\int_X Q \wedge \Omega}{\int_X \Omega \wedge \bar{\Omega}} \tag{5.47}$$

of the black hole. The dynamics of the black hole is governed by a set of differential equations. It has been shown that these differential equations lead to a supersymmetric solution only if $Z(Q)$ has a critical point on the horizon of the black hole. This point on the complex structure moduli space is called an *attractor point* as this value gives rise to a minimum of the absolute value of the central charge $|Z(Q)|$.

The critiality of $Z(Q)$ as defined in equation (5.47) at the attractor point $\mathbf{z}_{\text{att}}$ implies the so-called *alignment condition*

$$0 = \int_X Q \wedge \partial_j \Omega(\mathbf{z}_{\text{att}}) \tag{5.48}$$

if $Z(Q)$ is non-vanishing at the attractor point and the *orthogonality condition*

$$0 = \int_X Q \wedge \Omega(\mathbf{z}_{\text{att}}) \tag{5.49}$$

if $Z(Q)$ vanishes at the attractor point. The alignment equation can be reexpressed as

$$\text{Im}(C\Omega) = Q \tag{5.50}$$

for some $C \in \mathbb{C}$ which implies that

$$Q \in H^{3,0}(X,\mathbb{C}) \oplus H^{0,3}(X,\mathbb{C}) \ . \tag{5.51}$$

Since Q is in addition quantized and hence (in suitable units) an integral three-form, we obtain the condition that Q spans a one-dimensional sublattice

$$\langle Q \rangle \subset H^3(X,\mathbb{Z}) \cap \left(H^{3,0}(X,\mathbb{C}) \oplus H^{0,3}(X,\mathbb{C})\right) \ . \tag{5.52}$$

Note that the subspace $H^{3,0}(X,\mathbb{C}) \oplus H^{0,3}(X,\mathbb{C})$ is a real two-dimensional plane, however for the existence of an attractor point it suffices that this plane is intersected by the integral three-form cohomology only along a one-dimensional line that is spanned by Q. In this case we say that the attractor point is of *rank one*. The special cases of attractor points for which the lattice

$$\Lambda_{\text{att}} = H^3(X,\mathbb{Z}) \cap \left(H^{3,0}(X,\mathbb{C}) \oplus H^{0,3}(X,\mathbb{C})\right) \tag{5.53}$$

is proper two-dimensional are called attractor points of *rank two*.

If the central charge $Z(Q)$ vanishes at the attractor point, the orthogonality condition implies that Q cannot have a contribution from $H^{0,3}(X,\mathbb{C})$. Moreover, by complex conjugating this relation, we find that the integral three-form Q is as well orthogonal to $\bar{\Omega}$ and hence does not contain a $(3,0)$-contribution. In summary, the orthogonality condition restricts Q to span a sublattice

$$\langle Q \rangle \subset H^3(X,\mathbb{Z}) \cap \left(H^{2,1}(X,\mathbb{C}) \oplus H^{1,2}(X,\mathbb{C})\right) \ . \tag{5.54}$$

Supersymmetric Flux Vacua of M-Theory and F-Theory compactifications:
Finally, let us consider the relation between non-trivial fluxes of M-theory compactified on a Calabi-Yau fourfold X and integral sublattices of the middle cohomology $H^4(X,\mathbb{Z})$. To that end, let us recall from section 4.4 that a consistent non-trivial G-flux needs to obey the condition

$$G \in H^4(X,\mathbb{Z}) \cap \left(H^{4,0}(X,\mathbb{C}) \oplus H^{2,2}(X,\mathbb{C}) \oplus H^{0,4}(X,\mathbb{C})\right) \ . \tag{5.55}$$

The existence of such non-trivial G-fluxes on X can be investigated by using the modularity conjecture as well. Since the Dobeault cohomology group $H^{2,2}(X,\mathbb{C})$ is self-dual and hence requires special treatment, we restrict the discussion in the following on two different kinds of splits of $H^4(X,\mathbb{Q})$. In analogy to the rank-two attractor points, a Calabi-Yau fourfold is called *attractive of rank-two* if its rational middle cohomology $H^4(X,\mathbb{Q})$ factorizes as

$$H^4(X,\mathbb{Q}) = \Lambda_{\text{att}} \oplus \Sigma \tag{5.56}$$

where

$$\Lambda_{\text{att}} \subset H^4(X,\mathbb{Q}) \cap \left(H^{4,0}(X,\mathbb{C}) \oplus H^{0,4}(X,\mathbb{C})\right) \tag{5.57}$$

is a two-dimensional subspace of Hodge type $(4,0)+(0,4)$. The corresponding point in the complex structure moduli space is called a *rank-two attractor point*. Moreover, we say that X is an *attractive K3* Calabi-Yau fourfold, if the factorization of $H^4(X,\mathbb{Q})$ contains a two-dimensional subspace

$$\Lambda_{\text{AK3}} \subset H^4(X,\mathbb{Q}) \cap \left(H^{3,1}(X,\mathbb{C}) \oplus H^{1,3}(X,\mathbb{C})\right) \ . \tag{5.58}$$

The corresponding points on $\mathcal{M}_{C.S.}$ is called an *attractive K3 point.* This notation is motivated by the fact that such two-dimensional subspaces can be identified with Tate twists [176] of subspaces $\Lambda_{\mathrm{K3}} \subset H^2(Y, \mathbb{Q})$ of the middle cohomology of a K3 surface Y that are of Hodge type $(2,0) + (0,2)$. This defines Y to be an attractive K3 surface.

Note that a rank-two attractor point always corresponds to a Calabi-Yau fourfold that admits a non-trivial integral four-form[75] G that is of Hodge type $(4,0) + (0,4)$ and hence supports a non-trivial supersymmetric flux vacuum of M-theory compactifications.

In addition, the modularity conjecture allows to search for integral four-form fluxes of Hodge type $(4,0) + (2,2) + (0,4)$ in the special case of Calabi-Yau fourfolds with $h^{3,1} = 1$ complex structure modulus. Recall that in this case, $H^{3,1}(X, \mathbb{C})$ and $H^{1,3}(X\mathbb{C})$ are one-dimensional. Hence, if X is attractive K3, then $H^4(X, \mathbb{Q})$ factorizes according to

$$H^4(X, \mathbb{Q}) = \Lambda_{\mathrm{AK3}} \oplus \Sigma \; . \tag{5.59}$$

In this special case, the remainder Σ is necessarily a subspace of $H^4(X, \mathbb{Q})$ that is of definite Hodge type $(4,0) + (2,2) + (0,4)$. Thus, in this way, attractive K3 points of one-dimensional complex structure moduli spaces of Calabi-Yau fourfolds admit a sublattice of the integral middle cohomology that is of the required Hodge type to admit a non-trivial supersymmetric flux vacuum of M-theory compactifications.

Since this discussion is based only on the geometric properties of the Calabi-Yau fourfold X, this analysis can also be applied to flux compactifications of twelve-dimensional F-theory on an elliptically fibred Calabi-Yau fourfold. In this context, the conditions on non-trivial four-form fluxes can be formulated similarly to the M-theory flux condition and hence allow an analysis by arithmetic techniques in a similar way.

5.3 A Deformation Method for Calabi-Yau Fourfolds

In order to apply the modularity conjecture, as stated in section 5.2.2, to search for Calabi-Yau manifolds whose middle cohomology splits, it is necessary to compute the characteristic polynomial

$$R_n(X,T) = \det\left(\mathrm{Id} - T(\mathrm{Fr}_p^{-1}) : H^n(X, \mathbb{Q}_p) \to H^n(X, \mathbb{Q}_p)\right) \tag{5.60}$$

of the lifted Frobenius map on the middle p-adic cohomology group $H^n(X, \mathbb{Q}_p)$ for as many primes p as possible. In particular, we need an algorithm that efficiently computes the matrix $U_p(X)$ which is the inverse of the representation matrix $F_p(X)$ of the Frobenius map with respect to any chosen basis of $H^n(X, \mathbb{Q}_p)$. It was realized by Dwork [87, 88] that for a family of quintic Calabi-Yau threefolds, the matrix-valued function $F_p(X_{\mathbf{z}})$ can be computed by solving a set of differential equations whose solutions turn out to be given

[75] In particular, there exist two independent such four-forms as Λ_{att} is two-dimensional.

in terms of the period matrix $E(\mathbf{z})$ of the family[76]. In [4–7, 133, 143] this observation has been used to develop a method for the computation of the local zeta function and in particular the polynomial $R_3(X,T)$ for more general families of Calabi-Yau threefolds. In the following, we extend this method to the analysis of the polynomial $R_4(X_{\mathbf{z}},T)$ for families of Calabi-Yau fourfolds. Although the conceptional steps follow in analogy to the threefold-case, the appearance of the self-dual factor $H^{2,2}(X_{\mathbf{z}},\mathbb{C})$ in $H^4(X_{\mathbf{z}},\mathbb{C})$ causes several subtleties that need to be taken care of.

For the remainder of this chapter, $X_{\mathbf{z}}$ always denotes a Calabi-Yau fourfold with complex structure moduli $\mathbf{z}\in\mathcal{M}_{C.S.}$ that is a member of a smooth family of Calabi-Yau fourfolds. Moreover, we use the notation $F_p(\mathbf{z}) := F_p(X_{\mathbf{z}})$ and $U_p(\mathbf{z}) := U_p(X_{\mathbf{z}})$ for the matrices representing the Frobenius map.

5.3.1 Restriction to the Primary Horizontal Subspace

As stated above, the deformation method is based on the idea of expressing the Frobenius map in terms of the period vector and its derivatives. Hence, this method is not sensitive to the full middle cohomology $H^4(X_{\mathbf{z}},\mathbb{C})$ but only to the primary horizontal subspace[77] $H^n(X,\mathbb{Q}_p)$

$$H^4(X_{\mathbf{z}},\mathbb{C}) = H^4_H(X_{\mathbf{z}},\mathbb{C}) \oplus H^4_\perp(X_{\mathbf{z}},\mathbb{C}) \ . \tag{5.61}$$

For a Calabi-Yau fourfold $X_{\mathbf{z}}$, the orthogonal subspace $H^4_\perp(X_{\mathbf{z}},\mathbb{C})$ consists of additional $(2,2)$-forms that are not captured by the Picard-Fuchs ideal.

In order to make progress, let us assume[78] that the action of the Frobenius map Fr_p is compatible with the decomposition (5.61) of $H^n(X_{\mathbf{z}},\mathbb{Q}_p)$. By this assumption we mean that Fr_p is reducible and hence the matrices F_p and U_p become block-diagonal. The reducibility of Fr_p implies that its characteristic polynomial $R_4(X_{\mathbf{z}},T)$ factorizes accordingly such that

$$R_4(X_{\mathbf{z}},T) = R_H(X_{\mathbf{z}},T)R_\perp(X_{\mathbf{z}},T) \ . \tag{5.62}$$

Here, $R_H(X_{\mathbf{z}},T)$ denotes the characteristic polynomial of that factor of Fr_p, that acts on the primary horizontal subspace of $H^n(X_{\mathbf{z}},\mathbb{Q}_p)$. For the discussions in [4, 133] on the deformation method for Calabi-Yau threefolds, this subtlety does not appear because $H^3(Y,\mathbb{C}) = H^3_H(Y,\mathbb{C})$ for any Calabi-Yau threefold Y. Hence, for Calabi-Yau threefolds, the deformation method allows the computation of the complete polynomial $R_3(Y,T)$ and not only of a factor thereof.

[76]A modern mathematical discussion of the deformation method can be found in Refs. [177, 178].

[77]Recall the definition of the primary horizontal subspace from section 3.3.1

[78]There is no proof that the Frobenius map is indeed compatible with the decomposition of $H^n(X_{\mathbf{z}},\mathbb{Q}_p)$ into the primary horizontal subspace and its orthogonal complement. However, for the examples discussed in section 5.5 and 5.6, this assumption passes non-trivial consistency checks that indicate a posteriori a justification of this assumption.

In the following, we continue by computing the factor $R_H(X_{\mathbf{z}}, T)$ of the full characteristic polynomial and analyze its factorization behavior. Since a persistent factorization of $R_H(X_{\mathbf{z}}, T)$ implies trivially a persistent factorization of $R_4(X_{\mathbf{z}}, T)$, the modularity conjecture can still be applied to this discussion. However, it is not guaranteed that the orthogonal subspace $R_\perp(X_{\mathbf{z}}, T)$ will not contribute to any further factorizations which are not detectable with this method.

5.3.2 A Differential Equation for $U_p(\mathbf{z})$

In this section, we derive a set of differential equations for the matrix $U_p^H(\mathbf{z})$ that represents the inverse of the Frobenius map on the primary horizontal subspace $H_H^4(X_{\mathbf{z}}, \mathbb{Q}_p)$. Recall from section 5.1 that the cohomology groups form vector bundles $H_H^4(\mathcal{X}_{\mathcal{M}}, \mathbb{Q}_p)$ over the complex structure moduli space $\mathcal{M}_{C.S.}$ and that Fr_p obeys several compatibility conditions with respect to the geometric structure on $H_H^4(\mathcal{X}_{\mathcal{M}}, \mathbb{Q}_p)$.

Let us fix a local constant frame $\{e^k\}$ on $H_H^4(\mathcal{X}_{\mathcal{M}}, \mathbb{Q}_p)$. Then, the compatibility of Fr_p with the Gauss-Manin connection, as stated in equation (5.26), can be rewritten as

$$\nabla_X(F_p(\mathbf{z})(e^k)) = p\mathrm{Fr}_p(\mathbf{z})(\nabla_X e^k) = p\mathrm{Fr}_p(\mathbf{z})(e^j(B_X)^k_j) = pB^k_j(\mathbf{z}^p)(F_p)^j_\ell(\mathbf{z})e^\ell \tag{5.63}$$

under application of the Frobenius-linearity given in equation (5.24). On the other side, this term can be expressed by using the Leibniz rule for ∇_X to be

$$\begin{aligned}\nabla_X(Fr_p(\mathbf{z})(e^k)) &= X((F_p)^k_j(\mathbf{z}))e^j + (F_p)^k_j(\mathbf{z})\nabla_X(e^j) \\ &= X((F_p)^k_j(\mathbf{z}))e^j + (F_p)^k_j(\mathbf{z})B^j_\ell e^\ell \ .\end{aligned} \tag{5.64}$$

Combining both expression we find that $F_p(\mathbf{z})$ obeys a matrix-valued differential equation

$$X(F_p(\mathbf{z})) = pF_p(\mathbf{z})B_X(\mathbf{z}^p) - B_X(\mathbf{z})F_p(\mathbf{z}) \ . \tag{5.65}$$

Note that this expression gives a set of differential equations as it need to be obeyed by F for any choice of $X \in \Gamma(\mathcal{M}_{C.S.})$. Due to the linearity of the covariant derivative, it suffices to consider a basis for these vector fields which is conveniently chosen to be the set of all logarithmic derivatives $\Theta_i = z_i\partial_{z_i}$ with respect to the local coordinates on $\mathcal{M}_{C.S.}$. Hence, we are looking for the general solution to the set of differential equations

$$\Theta_i(F_p(\mathbf{z})) = pF_p(\mathbf{z})B_i(\mathbf{z}^p) - B_i(\mathbf{z})F_p(\mathbf{z}) \ . \tag{5.66}$$

To proceed at this stage, let us recall that the Gauss-Manin connection was defined by requiring that $\Omega(\mathbf{z})$ and its derivatives $\mathcal{D}^i\Omega(\mathbf{z})$ are covariantly constant. This requirement can be collected in the matrix-valued equation

$$\nabla_i E(\mathbf{z}) = \Theta_i E(\mathbf{z}) - E(\mathbf{z})B_i(\mathbf{z}) = 0 \quad \text{for all } i = 1, \ldots, h^{2,1} \tag{5.67}$$

for the period matrix

$$(E(\mathbf{z}))_{ij} = \mathcal{D}^i\varpi^j(\mathbf{z}) \ . \tag{5.68}$$

Here, $\varpi^j(\mathbf{z})$ denote a set of the Frobenius periods as defined in section 3.3 and the $\mathcal{D}^i$ are the differential operators $\mathcal{D}^i \in \mathcal{J}$ that generate the primary horizontal subspace $H^n_H(X_{\mathbf{z}}, \mathbb{C})$ by acting on $\Omega(\mathbf{z})$.

Using this defining property of ∇_i, it follows that the general solution to equation (5.66) is given by

$$F_p(\mathbf{z}) = E(\mathbf{z})^{-1} \tilde{F}_p(0) E(\mathbf{z}^p) \ . \tag{5.69}$$

Thus, the matrix $F_p(\mathbf{z})$, representing the Frobenius map on $H^4_H(X_{\mathbf{z}}, \mathbb{Q}_p)$, is obtained by deforming $\tilde{F}(0)$ smoothly along the complex structure moduli space. Moreover, we obtain by inverting this result that

$$U_p(\mathbf{z}) = E(\mathbf{z}^p)^{-1} V_p(0) E(\mathbf{z}) \tag{5.70}$$

where, $V_p(0) = \tilde{F}_p^{-1}(0)$ characterizes the action of the inverse Frobenius map at $\mathbf{z} = 0$.

While true for Calabi-Yau threefolds, the period matrix $E(\mathbf{z})$ is in general not normalized such that $E(0)^{-1} V_p(0) E(0) = V_p(0)$. Hence, in general the matrix $V_p(0)$ does not coincide with the matrix $U_p(0)$ but contains in addition the conjugation with the matrix $E(0)$. This initial value $V_p(0)$ is highly constrained by the Weil conjectures and the compatibility condition (5.27) which is used in section 5.4 to restrict the structure of $V_p(0)$. Moreover, one can use the fact that $U_p(z)$ needs to be a rational function in $\mathbf{z}$, to deduce a suitable system of linear equations among the entries of $V_p(0)$ to solve for them up to any given p-adic accuracy. While being algorithmic, these methods need to be applied to any given family of Calabi-Yau n-folds separately, as the system of equations depends strongly on the concrete Picard-Fuchs system. Except for the special case of Calabi-Yau threefolds with $h^{2,1} = 1$ complex structure modulus [6], no exact formula for the entries of this matrix is known.

5.3.3 The Teichmüller Lift

At this point, we should pause to examine the algebraic structure of equation (5.70). Recall that the Frobenius map Fr_p acts on p-adic cohomology groups $H^k(X, \mathbb{Q}_p)$, here in particular on $H^4_H(X, \mathbb{Q}_p)$. However, when discussing $F_p(\mathbf{z})$ to be the solution of a differential equation, we treated $H^4(\mathcal{X}_{\mathcal{M}}, \mathbb{Q}_p)$ as a vector bundle over $\mathbb{C}$ since the complex structure moduli $\mathbf{z}$ take complex values. Strictly speaking, the p-adic cohomology groups are vector spaces over the field $\mathbb{Q}_p$ of p-adic numbers rather than over $\mathbb{C}$. Practically, this is achieved by restricting[79] the complex structure moduli $\mathbf{z}$ to integers $\mathbf{z} \in \mathbb{F}_p$ in the finite field $\mathbb{F}_p$ and find a suitable embedding of $\mathbb{F}_p$ into $\mathbb{Q}_p$. Following [133] it is convenient to choose the *Teichmüller lift*

$$\mathrm{Teich}_p : \mathbb{F}_p \hookrightarrow \mathbb{Z}_p \subset \mathbb{Q}_p \tag{5.71}$$

[79] One should note that many complex numbers have representatives in $\mathbb{F}_p$. Thus, as soon as we iterate the arithmetic analysis over all (or in practical computations many) primes p, every algebraic number $x \in \mathbb{Q}_{\mathrm{alg}} \subset \mathbb{C}$ is represented by this construction.

which has the defining property that

$$\mathrm{Teich}_p(x) \equiv x \mod p \tag{5.72}$$

for all $x \in \mathbb{F}_p$ and moreover its image is invariant under the Frobenius map fr_p that is

$$\mathrm{fr}_p(\mathrm{Teich}_p(x)) = \mathrm{Teich}_p(x)^p = \mathrm{Teich}_p(x) \tag{5.73}$$

for all $x \in \mathbb{F}_p$. Following from Hensel's lemma, these two properties define the group homeomorphism Teich_p uniquely [144]. For the practical computation of $\mathrm{Teich}_p(x)$, we note that

$$\mathrm{Teich}_p(x) = \lim_{n\to\infty} \bar{x}^{p^n} \tag{5.74}$$

for $\bar{x} \in \mathbb{Z}$ being any integral representative of x. This limit in $\mathbb{Q}_p$ is considered with respect to the p-adic norm $|\cdot|_p$ and hence converges to a p-adic integer obeying the defining properties of the Teichmüller lift. Thus, the sequence $\bar{x}^{p^n}$ provides a useful iterative method to compute the Teichmüller lift up to any finite p-adic accuracy $\mathcal{O}(p^n)$.

Using the Teichmüller representative, equation (5.70) reads[80]

$$U_p(\mathbf{z}) = E(\mathbf{x}^p)^{-1} V_p(0) E(\mathbf{x})\big|_{\mathbf{x}=\mathrm{Teich}_p(\mathbf{z})} \tag{5.75}$$

for any $\mathbf{z} \in (\mathbb{F}_p)^{h^{2,1}}$. To ease the notation, we use from now on the Teichmüller lift implicitly and write $\mathbf{z} \in \mathbb{Q}_p$ whenever we mean more precisely $\mathrm{Teich}_p(\mathbf{z})$.

This expression for $U_p(\mathbf{z})$, together with the observation that $\mathrm{Teich}_p(\mathbf{z})$ is invariant under the Frobenius map fr_p, gives the impression that $U_p(\mathbf{z})$ is obtained from $V_p(0)$ simply by conjugation with the matrix $E(\mathbf{z})$ and hence that the corresponding characteristic polynomial $R_H(X_{\mathbf{z}}, T)$ would be constant all over the complex structure moduli space. However, it is important to note [6] that the matrix $E(\mathbf{z})$ is given in terms of a power series in $\mathbf{z} \in \mathbb{Q}_p$ and hence has a finite radius of convergence $\rho = 1$ meaning that the values for $\mathbf{z}$ are restricted by $|z^i|_p < 1$ for $E(\mathbf{z})$ being convergent. Conversely, for any $z \in \mathbb{F}_p$, the image of the Teichmüller lift $\mathrm{Teich}_p(z)$ is a p-adic integer, $\mathrm{Teich}_p(z) \in \mathbb{Z}_p \subset \mathbb{Q}_p$ and hence $|\mathrm{Teich}_p(z)|_p = 1$. Therefore, we are not allowed to insert $\mathbf{x} = \mathrm{Teich}_p(\mathbf{z})$ in each term of equation (5.75). Instead it is necessary to compute

$$E(\mathbf{x}^p)^{-1} V_p(0) E(\mathbf{x}) \tag{5.76}$$

as a formal power series in $\mathbf{x}$ with coefficients in $\mathbb{Q}_p$ before substituting $\mathbf{x} = \mathrm{Teich}(\mathbf{z})$ in the final expression which is proven to be a power series in $\mathbb{Q}_p$ with radius of convergence $\rho' = 1 + \delta$ for some $\delta > 0$ [6]. Thus, the full expression $U_p(\mathbf{z})$ turns out to be convergent for all p-adic integer-valued $z^i \in \mathbb{Z}_p$.

[80] Here and in the following, the action of Teich_p on a vector $\mathbf{z} \in (\mathbb{F}_p)^{h^{2,1}}$ is understood to be componentwise.

5.3.4 Practical Inversion of $E(\mathbf{z})$

In order to compute $U_p(\mathbf{z})$ from equation (5.70), it is required to invert the $(b \times b)$-matrix $E(\mathbf{z}^p)$. Here, $b = \dim(H^4_H(X_{\mathbf{z}}, \mathbb{C}))$ denotes the effective betti number of the primary horizontal middle cohomology. If we were able to first insert explicit values for the complex structure moduli $z^i \in \mathbb{Q}_p$, an inversion of $E(\mathbf{z})$ could easily be done numerically or even analytically for small b. However, as we have discussed in the previous section, it is necessary to treat $E(\mathbf{z}^p)^{-1}$ as a formal power series in $\mathbf{z}$ and peform the matrix multiplication before inserting a certain value in order to restore the convergence behavior of $U_p(\mathbf{z})$ on an open p-adic disk of radius $\rho' > 1$. Hence, we need to find an effective way to invert a matrix, whose entries are formal power series.

Motivated from the computations of $U_p(\mathbf{z})$ for Calabi-Yau threefolds [179], we define the Wronskian matrix

$$W^{ij}(\mathbf{z}) = \int_{X_z} \mathcal{D}^i\Omega(\mathbf{z}) \wedge \mathcal{D}^j\Omega(\mathbf{z}) = \mathcal{D}^i\varpi^k(\mathbf{z})\sigma_{k\ell}\mathcal{D}^j\varpi^\ell(\mathbf{z}) = (E(\mathbf{z})^T\sigma E(\mathbf{z}))^{ij} \tag{5.77}$$

and find by inverting this expression that

$$E(\mathbf{z})^{-1} = W(\mathbf{z})^{-1}\sigma E(\mathbf{z})^T \tag{5.78}$$

implying that the inverse of $E(\mathbf{z})$ is expressible in terms of $W^{-1}(\mathbf{z})$ and the original matrix $E(\mathbf{z})$. Moreover, the matrix σ representing the wedge-product on $H^4_H(X_{\mathbf{z}}, \mathbb{C})$ in the chosen locally constant basis $\{e_i\}$ of $H^4_H(X_{\mathbf{z}}, \mathbb{C})$ is given by

$$\sigma_{ij} = \int_{X_z} e_i \wedge e_j \ . \tag{5.79}$$

This matrix is purely topological and can be computed directly for any given family of Calabi-Yau fourfolds.

The great benefit of this identity relies in the fact that the entries of the matrix $W(\mathbf{z})$ are given in terms of rational functions in $\mathbf{z}$ that can be computed by using the Picard-Fuchs ideal to derive differential equations for each entry of $W(\mathbf{z})$ that can be solved in terms of a closed rational expression. Hence, we have traded the inversion of a matrix whose entries are power series in $\mathbf{z}$ by the inversion of a matrix whose entries are rational functions. Since the latter can be done very efficiently, this computational trick speeds up the evaluation of $U_p(\mathbf{z})$ drastically.

The entries of $W(\mathbf{z})$ can be computed algorithmically as solutions to a certain set of differential equations which originate from the Picard-Fuchs ideal. For certain types of Calabi-Yau fourfolds that depend on one complex structure modulus, we provide a derivation of these differential equations in appendix D.

5.4 The Zeta Function of One-parameter Calabi-Yau Fourfolds

So far, the deformation method for computing $R_H(X_z, T)$ has been discussed for a generic family of Calabi-Yau fourfolds. As noted in section 5.3.2, the missing ingredient for a full solution of the inverse Frobenius matrix

$$U_p(z) = E(z^p)^{-1} V_p(0) E(z) \tag{5.80}$$

is given by the initial value matrix $V_p(0)$.

To be more specific, we focus the analysis to families of Calabi-Yau fourfolds with one complex structure modulus which we will denote by z in the following. Moreover, we restrict the discussion to fourfolds, whose primary horizontal subspace $H^4(X_z, \mathbb{C})$ is either five or six dimensional. As a short hand notation to distinguish between these cases, let us define the primary horizontal Hodge type of a given family of fourfolds by the row vector $(h^{4,0}, h^{3,1}, h^{2,2}_H, h^{1,3}, h^{0,4})$ with

$$h^{2,2}_H = \dim(H^{2,2}(X_z, \mathbb{C}) \cap H^4_H(X_z, \mathbb{C})) \ . \tag{5.81}$$

Thus, since we fixed $h^{3,1} = 1$ by requiring that the family depends only on one complex structure modulus, we will discuss in the following families of primary horizontal Hodge type $(1, 1, 1, 1, 1)$ (if $H^4_H(X_z, \mathbb{C})$ is five-dimensional) and $(1, 1, 2, 1, 1)$ (if $H^4_H(X_z, \mathbb{C})$ is six-dimensional).

It should be noted that the method, presented in the following section, extend naturally to families of Calabi-Yau fourfolds which are of primary horizontal Hodge type $(1, 1, k, 1, 1)$ for any $k \in \mathbb{N}$. Since the dimensional growth of the appearing structures increases the computation time, we restricted the analysis in [1] to the simplest two non-trivial cases of $k = 1, 2$. As the result of this discussion, we obtain the full structure of the matrix $V_p(0)$ which, as it turns out, depends only on two parameters in the first case and on three independent parameters in the second case. Moreover, we discuss, how these remaining coefficients can be computed numerically. Finally we discuss, how the Weil conjectures can be used to constrain the structure of $R_H(X_z, T)$ in both cases and derive the practical method of computing this polynomial in terms of the matrix $U_p(z)$.

5.4.1 Fourfolds of Hodge type $(1, 1, 1, 1, 1)$

To begin with, let us consider a one-parameter family of Calabi-Yau fourfolds X_z, which is of primary horizontal Hodge type $(1, 1, 1, 1, 1)$. That means, $b := \dim(H^4_H(X_z, \mathbb{C})) = 5$ and following the discussion of section 3.3, the period vector $\Pi(z)$ is characterized by a single Picard-Fuchs operator

$$\mathcal{L} = \sum_{k=0}^{b} f_k(z) \Theta^k \ . \tag{5.82}$$

Since we are considering a one-parameter family, we use the short notation $\Theta := \Theta_z$ for the logarithmic derivative. Moreover, it follows that $H^4_H(X_z, \mathbb{C})$ is generated by

$$H^4_H(X_z, \mathbb{C}) = \langle \Theta^k \Omega(z) \rangle_{k=0,\dots,4} \ . \tag{5.83}$$

Hence, the generating derivatives of $H^4_H(X_z, \mathbb{C})$ are given by $\mathcal{D}^i = \Theta^i$ for $i = 0, \dots, 4$ and the matrix $E(z)$ takes the form

$$E(z) = \begin{pmatrix} \varpi^0(z) & \Theta\varpi^0(z) & \Theta^2\varpi^0(z) & \Theta^3\varpi^0(z) & \Theta^4\varpi^0(z) \\ \varpi^1(z) & \Theta\varpi^1(z) & \Theta^2\varpi^1(z) & \Theta^3\varpi^1(z) & \Theta^4\varpi^1(z) \\ \varpi^2(z) & \Theta\varpi^2(z) & \Theta^2\varpi^2(z) & \Theta^3\varpi^2(z) & \Theta^4\varpi^2(z) \\ \varpi^3(z) & \Theta\varpi^3(z) & \Theta^2\varpi^3(z) & \Theta^3\varpi^3(z) & \Theta^4\varpi^3(z) \\ \varpi^4(z) & \Theta\varpi^4(z) & \Theta^2\varpi^4(z) & \Theta^3\varpi^4(z) & \Theta^4\varpi^4(z) \end{pmatrix} \tag{5.84}$$

where $\varpi^i(z)$ denotes the i^{th} Frobenius period of X_z. Following the discussion in section 3.3.3, the Frobenius periods for this class of fourfolds take the form

$$\varpi^i(z) = \sum_{k=0}^{i} \log^k(z) A_{i-k}(z) \tag{5.85}$$

for holomorphic functions $A_k(z)$ that are normalized such that $A_k(0) = \delta_{k0}$. Moreover, the matrix σ that represents the wedge product in equation (5.77) is evaluated to be

$$\sigma = \frac{\kappa}{(2\pi i)^4} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.86}$$

where κ is a normalization constant that can be fixed by enforcing σ to coincide with the intersection form Σ of the vertical cohomology of the mirror partner Y of X. It follows that

$$\kappa = \int_Y h \wedge h \wedge h \wedge h \tag{5.87}$$

where $h \in H^{1,1}(Y, \mathbb{C})$ is the (unique) generator of the second cohomology of Y.

The matrix $B(z)$ characterizing the Gauss-Manin connection ∇_Θ can be expressed explicitly in this basis by enforcing

$$\nabla_\Theta E(z) = \Theta E(z) - E(z) B(z) = 0 \tag{5.88}$$

and yields

$$B(z) = \begin{pmatrix} 0 & 0 & 0 & 0 & -\frac{f_0(z)}{f_5(z)} \\ 1 & 0 & 0 & 0 & -\frac{f_1(z)}{f_5(z)} \\ 0 & 1 & 0 & 0 & -\frac{f_2(z)}{f_5(z)} \\ 0 & 0 & 1 & 0 & -\frac{f_3(z)}{f_5(z)} \\ 0 & 0 & 0 & 1 & -\frac{f_4(z)}{f_5(z)} \end{pmatrix} \tag{5.89}$$

away from the locus $f_5(z) = 0$. One observes that the first four columns of this matrix-valued equation are trivially obeyed whereas the last column reproduces exactly the Picard-Fuchs equation (5.82). The condition of $z = 0$ being a point of maximal unipotent monodromy requires the polynomials $f_k(z)$ for $k < 5$ to vanish at $z = 0$, whereas $f_5(0)$ remains finite. Hence, it is convenient to define $\varepsilon := B(0)$. For the case at hand, it follows that

$$\varepsilon = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} . \tag{5.90}$$

We can now use this information, together with the compatibility conditions (5.26) and (5.27) of the Frobenius map Fr_p applied to the concrete point $z = 0$ in the complex structure moduli space to obtain constraints on the matrix $V_p(0)$. First, we note that the differential equation (5.65) evaluated at $z = 0$ reads

$$\Theta(F_p(0)) = pF_p(0)B(0) - B(0)F_p(0) . \tag{5.91}$$

Since $\Theta(F_p(0)) = 0$, we find that the matrices $F_p(0)$ and $\varepsilon = B(0)$ commute up to a rescaling by p. Consequently, $U_p(0) = E(0)^{-1}V_p(0)E(0)$ obeys

$$U_p(0)\varepsilon = p\varepsilon U_p(0) . \tag{5.92}$$

In order to identify constraints on $V_p(0)$, one should notice that the period matrix $E(z)$ becomes singular in the limit $z \to 0$ as the periods contain logarithms of z. However, it is argued that $U_p(z)$ is well-defined and remains finite in this limit [6]. Therefore, all terms in the matrix product (5.70) that contain logarithms need to cancel identically. Thus, it is convenient to rewrite the deformation expression for $U_p(z)$ in terms of the *logarithm-free* period matrix

$$\tilde{E}(z) = E(z)\Big|_{\log(z)=0} \tag{5.93}$$

in the form

$$U_p(z) = E(z^p)^{-1}V_p(0)E(z) = \tilde{E}(z^p)^{-1}V_p(0)\tilde{E}(z) . \tag{5.94}$$

The strategy to compute the inverse of $E(z)$ from section 5.3.4 can be adopted in analogy to compute $\tilde{E}(z)$. We note that the matrix $\tilde{E}(0)$ becomes the unit matrix for Calabi-Yau fourfolds that are of primary horizontal Hodge type $(1, 1, 1, 1, 1)$. Hence, for this special case, we find that

$$U_p(0) = \tilde{E}(0)^{-1}V_p(0)\tilde{E}(0) = V_p(0) \tag{5.95}$$

implying that $V_p(0)$ agrees with $U_p(0)$. Moreover, we can insert this identification in equation (5.92) to obtain constraints on $V_p(0)$ restricting its general form to

$$V_p(0) = u \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \alpha p & p & 0 & 0 & 0 \\ \beta p^2 & \alpha p^2 & p^2 & 0 & 0 \\ \gamma p^3 & \beta p^3 & \alpha p^3 & p^3 & 0 \\ \delta p^4 & \gamma p^4 & \beta p^4 & \alpha p^4 & p^4 \end{pmatrix} . \tag{5.96}$$

These five parameters $\alpha, \beta, \gamma, \delta, u \in \mathbb{Q}_p$ can be further constrained by realizing the compatibility condition (5.27) of the Frobenius map Fr_p and the wedge procduct on $H^4(X_0)$ in terms of the representation matrix $F_p(0)$. For $v, w \in H^4_H(X_0)$ we find

$$\int_{X_0} (\mathrm{Fr}_p v) \wedge (\mathrm{Fr}_p w) = (F_p(0)v)^T \sigma(F_p(w)) \ . \tag{5.97}$$

Hence, we obtain from equation (5.27) the identity

$$(F_p(0)v)^T \sigma(F_p(0)w) = p^{-4} \left(v^T \sigma w\right) \quad \text{for all } v, w \in H^4_H(X_0) \tag{5.98}$$

since the action of Fr_p on the cohomology group $H^8(X_0, \mathbb{C})$, which is spanned by the volume form, is given by a rescaling with the factor p^4 [7]. For the inverse of $F_p(0)$, we hence deduce the compatibility condition

$$U_p(0) \sigma U_p(0)^T = p^4 \sigma \tag{5.99}$$

which is solved only if

$$u = \pm 1 \quad , \quad \beta = \frac{\alpha^2}{2} \quad , \quad \delta = \alpha\left(\gamma - \frac{1}{8}\alpha^3\right) \ . \tag{5.100}$$

Since we are interested in persistent factorizations of the characteristic polynomial

$$R_H(X_z, T) = \det(\mathbb{1} - T U_p(z)) \tag{5.101}$$

the overall sign u can always be absorbed in the formal variable T without changing the factorization behavior of $R_H(X_z, T)$ over $\mathbb{Q}$. Thus, we can set without loss of generality $u = +1$. By this, we are left with only two unknown coefficients α and γ whose concrete values are so far undeterminable for a generic family of Calabi-Yau fourfolds that is of Hodge type $(1, 1, 1, 1, 1)$. However, for a given example and a fixed prime p, it is possible to determine both, α and γ up to any necessary accuracy in the p-adic expansion by the following strategy.

Recall that the entries of $U_p(z)$ are given by rational functions on z. Following [7], the rational structure of $U_p(z)$ can be determined for the analogous Calabi-Yau threefold analysis up to a given p-adic accuracy[81] by

$$U_p(z) = \frac{\mathcal{U}_p^n(z)}{P_n(z^p)} + \mathcal{O}(p^n) \ . \tag{5.102}$$

Here, $\mathcal{U}_p^n(z)$ denotes a matrix with polynomial entries and $P_n(z)$ consists of certain powers of the discriminant $\Delta(z)$ of the Picard-Fuchs ideal and the denominator of the matrix

[81] Since the coefficients α, γ are p-adic numbers, contributions of the order $\mathcal{O}(p^n)$ are small with respect to the p-adic norm.

$W^{-1}(z)$ which we denote by $\mathcal{W}(z)$. In [1], we conjecture that this structure is true as well for Calabi-Yau fourfolds. Concretely, we assume that the denominator takes the form

$$P_n(z) = (\Delta(z))^{n-4}\mathcal{W}(z)\ . \tag{5.103}$$

Using this assumption, we compute $\mathcal{U}_p^n(z) := P_n(z^p)U_p(z)$ up to a given accuracy $\mathcal{O}(p^n)$ and enforce that every entry of this matrix is a polynomial in z. In practice, we force every contribution $a_k z^k$ in the power series expansion of $\mathcal{U}_p^n(z)$ with k sufficiently large to vanish. This procedure imposes a highly overconstrained system of equations on the coefficients α and γ which however has always a unique solution if k is chosen large enough. Increasing the order of n, this gives an iterative method to compute α and γ up to a maximal expansion order in n that can be set as large as it is needed.

The structure of the appearing system of equations is strongly dependent on the explicit solution $E(z)$ to the Picard-Fuchs ideal and hence cannot be analyzed for a generic family of Calabi-Yau fourfolds. Instead, a case by case computation needs to be performed for computing the remaining two coefficients of $U_p(0)$. One should note that the permanent existence of solutions to this overconstrained system of equations gives an a posteriori, highly non-trivial consistency check on the assumption that the denominator of $U_p(z)$ is indeed given by equation (5.103).

Since we cannot give a closed formula for α and γ as p-adic numbers depending on the topology of X_z and the prime p but rather provide a p-adic expansion of both, it is necessary to discuss whether there exists a sufficient order of the expansion such that the result for the polynomial $R_H(X_z, T)$ is still exact. As it turns out, the Weil conjectures, in particular the functional equation and the Riemann hypothesis, provide an upper bound on the absolute value of the coefficients appearing in the polynomial $R_H(X_z, T)$. From these, we can deduce that the result for $R_H(X_z, T)$ is still exact, even if we know the matrix $U_p(z)$ or equivalently the constants α and γ only up to a finite p-adic accuracy. In the remainder of this section we will derive these bounds for Calabi-Yau fourfolds of primary horizontal Hodge type $(1, 1, 1, 1, 1)$ explicitly.

First, let us recall that $R_H(X_z, T)$ is a polynomial of degree $\deg(R_H(X_z, T)) = 5$ if X_z is of primary horizontal Hodge type $(1, 1, 1, 1, 1)$. The Riemann hypothesis implies that the polynomial factorizes over $\mathbb{C}$ into linear factors of the type $(1 - \lambda_i T)$, $i = 1, \dots, 5$ where the coefficients $\lambda_i \in \mathbb{C}$ have absolute value $|\lambda_i| = p^2$. Since $R_H(X_z, T)$ is a polynomial in $\mathbb{Z}[T]$, the coefficients λ_i need to be either real, hence $\lambda_i = \pm p^2$, or appear in conjugate pairs $(\lambda_i, \bar{\lambda}_i)$. Since $\deg(R_H(X_z, T))$ is odd, there must be at least one such coefficients that is real. Therefore, the most general form of $R_H(X_z, T)$ is given by

$$R_H(X_z, T) = (1 - \varepsilon p^2 T)(1 - p^2 e^{i\theta_1} T)(1 - p^2 e^{-i\theta_1} T)(1 - p^2 e^{i\theta_2} T)(1 - p^2 e^{-i\theta_2} T) \tag{5.104}$$

with $\varepsilon = \pm 1$ and $\theta_1, \theta_2 \in [0, 2\pi)$ being the phases of the coefficients. Factorized over the real numbers, we obtain

$$R_H(X_z, T) = (1 - \varepsilon p^2 T)(1 - 2p^2 \cos\theta_1 T + p^4 T^2)(1 - 2p^2 \cos\theta_2 T + p^4 T^2)\ . \tag{5.105}$$

This polynomial seems to factorize always into two polynomials of degree two and a linear factor. However, one needs to keep in mind that the modularity conjecture states a relation between the existence of a modular form and a factorization of $R_H(X_z, T)$ over $\mathbb{Q}$. The expression in equation (5.105) factorizes generically only into the linear factor $(1 - \varepsilon p^2 T)$ and a remaining degree-four polynomial over $\mathbb{Q}$ as $\cos(\theta_i)$ is generically not rational.

Expanding equation (5.105) as a formal polynomial over $\mathbb{Z}$, it turns out that the appearing coefficients obey a symmetry relation such that

$$R_H(X_z, T) = 1 + a_p T + b_p p T^2 + \varepsilon b_p p^3 T^3 + \varepsilon a_p p^6 T^4 + \varepsilon p^{10} T^5 \tag{5.106}$$

with

$$\begin{aligned} a_p &= -p^2(2\cos(\theta_1) + 2\cos(\theta_2) - \varepsilon) \\ b_p &= \frac{1}{2}\left(a_p^2 - p^4(2\cos(2\theta_1) + 2\cos(2\theta_2) + 1)\right) \ . \end{aligned} \tag{5.107}$$

This symmetry of the coefficients is not a coincidence but is equivalent to the functional equation of the Weil conjectures which is given by

$$R_H(X_z, T) = \varepsilon p^{10} T^5 R_H(X_z, T^{-1}) \tag{5.108}$$

for this concrete type of degree-five polynomials with linear factor $(1 - \varepsilon p^2 T)$.

Using the triangle inequality for the absolute value and the upper estimate $|\cos(\theta)| \leq 1$, the expressions (5.106) and (5.107) give upper bounds for the coefficients a_p and b_p as

$$|a_p| \leq 5p^2 \quad , \quad \left| b_p - \frac{a_p^2}{2p} \right| \leq \frac{5}{2} p^3 < 3p^3 \ . \tag{5.109}$$

Since $R_H(X_z, T) = \det(\mathbb{1} - T U_p(z))$, these coefficients are immediately related to $U_p(z)$. In particular, we find

$$\begin{aligned} a_p &= -\mathrm{Tr}(U_p(z)) \\ p b_p &= \frac{1}{2}\Big([\mathrm{Tr}(U_p(z))]^2 - \mathrm{Tr}(U_p^2(z))\Big) \end{aligned} \tag{5.110}$$

and moreover, the sign ε is related to $U_p(z)$ by expressing the third non-trivial coefficient $\varepsilon b_p p^3$ in terms of $U_p(z)$ as

$$\varepsilon b_p p^3 = -\frac{1}{6}\Big([\mathrm{Tr}(U_p(z))]^3 - 3\mathrm{Tr}(U_p^2(z))\mathrm{Tr}(U_p(z)) + 2\mathrm{Tr}(U_p^3(z))\Big) \ . \tag{5.111}$$

Note, that the expressions for a_p and b_p together with their upper bounds imply

$$\left| \varepsilon b_p p^3 + \frac{a_p^3}{3p^3} - \frac{a_p b_p}{p^2} \right| \leq \frac{5}{2} p^3 < 3p^3 \ . \tag{5.112}$$

From this analysis we obtain two main results. First, we find that the coeffiecients of $R_H(X_z,T)$ can be computed from traces of powers of the matrix $U_p(z)$ whose numerical implementation is more efficient than the naive computation of the full characteristic polynomial. Second, the bounds on a_p, b_p and $\varepsilon b_p p^3$ show that for primes $p>5$ it suffices to compute the matrix $U_p(z)$ up to p-adic accuracy $\mathcal{O}(p^4)$ in order to obtain exact results for $R_H(X_z,T)$.

5.4.2 Fourfolds of Hodge type (1, 1, 2, 1, 1)

Let us now repeat the construction of $R_H(X_z,T)$ for a family of Calabi-Yau fourfolds X_z that are of primary horizontal Hodge type $(1,1,2,1,1)$. Since the conceptional arguments are similar as in the previous case, we abbreviate the discussion by mainly stating the results for the contributing expressions. Since there are some subtleties that are related to the additional $(2,2)$-form appearing in the primary horizontal subspace, we focus on these throughout this section.

Everything that is presented here should extend naturally to the more general case of families that are of Hodge type $(1,1,k,1,1)$ for $k\geq 2$. However, since there is no conceptional difference to the case of $k=2$ but instead, the dimensions of the relevant matrices increase, we restrict the discussion to this specific case.

As previously, the period vector of X_z is characterized by a Picard-Fuchs operator

$$\mathcal{L}=\sum_{k=0}^{b} f_k(z)\Theta^k \tag{5.113}$$

which is in this case of rank $b=\dim(H^4_H(X,\mathbb{C}))=6$. Again, the generating derivatives of $H^4_H(X,\mathbb{C})$ are given by $\mathcal{D}^i=\Theta^i$ for $i=0,\ldots,5$. This Picard-Fuchs operator results in six independent periods $\varpi^i(z)$. From the Hodge structure, one naively expects two independent periods whose leading term behaves as $\log^2(z)$. However, since we are in the situation that $h^{2,2}_H>\frac{1}{2}h^{3,1}(h^{3,1}+1)$, it is possible to eliminate all logarithmic terms in one of these double-logarithmic periods. Hence, a full set of Frobenius solutions to $\mathcal{L}$ is given by

$$\varpi^i(z)=\sum_{k=0}^{i}\log^k(z)A_{i-k}(z) \text{ for } i=0,\ldots,4 \tag{5.114}$$

and one additional holomorphic period $\varpi^5(z)$. Consequently, the period matrix $E(z)$ is given by

$$E(z)=\begin{pmatrix} \varpi^0(z) & \Theta\varpi^0(z) & \Theta^2\varpi^0(z) & \Theta^3\varpi^0(z) & \Theta^4\varpi^0(z) & \Theta^5\varpi^0(z) \\ \varpi^1(z) & \Theta\varpi^1(z) & \Theta^2\varpi^1(z) & \Theta^3\varpi^1(z) & \Theta^4\varpi^1(z) & \Theta^5\varpi^1(z) \\ \varpi^2(z) & \Theta\varpi^2(z) & \Theta^2\varpi^2(z) & \Theta^3\varpi^2(z) & \Theta^4\varpi^2(z) & \Theta^5\varpi^2(z) \\ \varpi^3(z) & \Theta\varpi^3(z) & \Theta^2\varpi^3(z) & \Theta^3\varpi^3(z) & \Theta^4\varpi^3(z) & \Theta^5\varpi^3(z) \\ \varpi^4(z) & \Theta\varpi^4(z) & \Theta^2\varpi^4(z) & \Theta^3\varpi^4(z) & \Theta^4\varpi^4(z) & \Theta^5\varpi^4(z) \\ \varpi^5(z) & \Theta\varpi^5(z) & \Theta^2\varpi^5(z) & \Theta^3\varpi^5(z) & \Theta^4\varpi^5(z) & \Theta^5\varpi^5(z) \end{pmatrix}. \tag{5.115}$$

Recalling that the wedge product on $H_H^4(X,\mathbb{C})$ is highly dependent on the Hodge structure of the contributing four-form, the evaluation of σ naturally extends to this case up to the fact, that the two independent four-forms on $H_H^{2,2}(X,\mathbb{C})$ give rise to a non-trivial (2×2)-block. Without loss of generality, one can always find a basis of $(2,2)$-forms such that this block diagonalizes such that σ reads

$$\sigma = \frac{\kappa}{(2\pi i)^4}\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_{55} \end{pmatrix} \tag{5.116}$$

where again $\kappa = \int_{X^\vee} h\wedge h\wedge h\wedge h$. The coefficient σ_{55} is a topological quantity that needs to be computed explicitly for evey family of Calabi-Yau fourfold which arises since the two independent integral $(2,2)$-forms in the chosen basis are of different norm.

The most important difference to fourfolds of Hodge type $(1,1,1,1,1)$ is the fact that $\tilde{E}(z) = E(z)|_{\log(z)=0}$ becomes singular in the limit $z\to 0$, since $\varpi^5(z)$ vanishes in this limit. At first view, this subtlety seems to be a minor point, however it points out that we cannot perform the analysis of $U_p(0)$ as presented in equation (5.95) as $\tilde{E}^{-1}(0)$ is not well-defined. In particular, this implies that $U_p(0)\neq V_p(0)$ for this type of Calabi-Yau fourfolds. However, we can still obtain some information on the matrix $V_p(0)$ by making use of the fact that $U_p(z)$ is known to be a rational function. Hence, for any $z\in\mathbb{Q}$, the logarithmic contributions in

$$U_p(z) = E(z^p)^{-1}V_p(0)E(z) \tag{5.117}$$

need to vanish identically. In the previous discussion we have used this observation in order to simply replace $E(z)$ by $\tilde{E}(z)$. Here, we do not perform this replacement but rather use the rationality of $U_p(z)$ in its original expression as a constraint on the entries of $V_p(z)$. Enforcing all logarithmic contributions in equation (5.117) to vanish restricts $V_p(0)$ to take the form

$$V_p(0) = u\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & v_{05} \\ \alpha p & p & 0 & 0 & 0 & v_{15} \\ \beta p^2 & \alpha p^2 & p^2 & 0 & 0 & v_{25} \\ \gamma p^3 & \beta p^3 & \alpha p^3 & p^3 & 0 & v_{35} \\ \delta p^4 & \gamma p^4 & \beta p^4 & \alpha p^4 & p^4 & v_{45} \\ \eta & 0 & 0 & 0 & 0 & v_{55} \end{pmatrix} . \tag{5.118}$$

Finally, the compatibility condition (5.27) gives, in analogy to the previous case that

$$U_p(z)\sigma U_p(z)^T = p^4\sigma\ . \tag{5.119}$$

Inserting the deformation expression for $U_p(z)$, this relation can be rewritten as

$$V_p(0)E(z)\sigma E(z)^T V_p(0)^T = p^4 E(z^p)\sigma E(z^p)^T \tag{5.120}$$

which is a relation that is independent of $E^{-1}(z)$. Hence, the limit $z \to 0$ is well-defined for this equation and results in

$$V_p(0)\tilde{\sigma}V_p(0)^T = p^4\tilde{\sigma} \tag{5.121}$$

for

$$\tilde{\sigma} = E(0)\sigma E(0)^T \ . \tag{5.122}$$

This condition can now be used to restrict the coefficients of $V_p(0)$ further. The solution to this compatibility equation is given by

$$u = \pm 1 \quad , \quad \beta = \frac{\alpha^2}{2} \quad , \quad \delta = \alpha\left(\gamma - \frac{1}{8}\alpha^3\right) - \frac{\sigma_{55}}{2}\varepsilon^2 \tag{5.123}$$

and in addition

$$v_{05} = v_{15} = v_{25} = v_{35} = 0 \quad , \quad v_{45} = p^2\varepsilon \quad , \quad \eta = -v\sigma_{55}\varepsilon \quad , \quad v_{55} = vp^2 \tag{5.124}$$

for $v = \pm 1$. Thus, $V_p(0)$ depends in total on three topological constants $\alpha, \gamma, \varepsilon \in \mathbb{Q}_p$ and two phases $u, v \in \{\pm 1\}$. The strategy to compute these remaining constants is now in analogy to the previous case given by assuming that $U_p(z)$ takes the rational form as conjectured in equation (5.102) for any given p-adic accuracy $\mathcal{O}(p^n)$ and computing $\alpha, \gamma, \varepsilon$ and the signs u, v by forcing the entries of $U_p(z)P_n(z)$ to be polyonomials of finite degree. This method gives again results for all unkown coefficients of $V_p(0)$ up to any given p-adic accuracy.

To infer a bound on a sufficient p-adic expansion order that leads to exact results for the characteristic polynomial $R_H(X_z, T)$, we perform a similar analysis as in section 5.4.1 by rewriting $R_H(X_z, T)$ in terms of its (complex) roots and using the Weil conjectures to get suitable bounds. In contrast to the former case, the polynomial is now of degree $\deg(R_H(X_z, T)) = 6$, hence we distinguish the following two cases. If there are three conjugated pairs of inverse roots $(\lambda_i, \bar{\lambda}_i)$ that have each by the Riemann hypothesis an absolute value of p^2, we can characterzie the polynomial $R_H(X_z, T)$ by three independent phases $\theta_i \in [0, 2\pi)$ such that $\lambda_i = p^2 e^{i\theta_i}$. Consequently, we obtain

$$R_H(X_z, T) = (1 - 2p^2\cos\theta_1 T + p^4T^2)(1 - 2p^2\cos\theta_2 T + p^4T^2)(1 - 2p^2\cos\theta_3 T + p^4T^2) \tag{5.125}$$

after pairing the conjugated linear factors in order to obtain a representation of $R_H(X_z, T)$ with real coefficients. For a generic Calabi-Yau fourfold X_z of Hodge type $(1, 1, 2, 1, 1)$ we do not expect this polynomial to factorize over $\mathbb{Q}$ at all since the three phases θ_i are generically unrelated.

To be distinguished from this case, let us consider the situation that two (inverse) roots of $R_H(X_z, T)$ are real-valued. In this case, $R_H(X_z, T)$ factorizes over $\mathbb{Q}$ into two linear factors and a remainder of degree four according to

$$R_H(X_z, T) = (1 - p^2T)(1 + p^2T)(1 - 2p^2\cos\theta_1 T + p^4T^2)(1 - 2p^2\cos\theta_2 T + p^4T^2) \ . \tag{5.126}$$

Again, the integral coefficients of the polynomial $R_H(X_z, T)$ obey a symmetry relation which can either be deduced from the functional equation of the Weil conjectures or can be identified from the structure in equations (5.125) and (5.126) explicitly. We find that

$$R_H(X_z, T) = 1 + a_p T + b_p p T^2 + c_p p^3 T^3 + \varepsilon b_p p^5 T^4 + \varepsilon a_p p^8 T^5 + \varepsilon p^{12} T^6 \tag{5.127}$$

for $\varepsilon = \pm 1$. Moreover, the functional equation, which reads

$$R_H(X_z, T) = \varepsilon p^{12} T^6 R_H(X_z, T^{-1}) \tag{5.128}$$

for fourfolds of this Hodge type, implies that $c_p = 0$ if $\varepsilon = -1$. Note that the structure of equation (5.126) is compatible with this form only for $\varepsilon = -1$ and therefore $c_p = 0$.

For the former case, the coefficients are given by

$$\begin{aligned} a_p &= -2p^2 \left(\cos(\theta_1) + \cos(\theta_2) + \cos(\theta_3)\right) \ , \\ p b_p &= \frac{1}{2} a_p^2 - p^4 \left(\cos(2\theta_1) + \cos(2\theta_2) + \cos(2\theta_3)\right) \ , \\ p^3 c_p &= -\frac{1}{3} a_p^3 + p a_p b_p - \frac{2}{3} p^6 \left(\cos(3\theta_1) + \cos(3\theta_2) + \cos(3\theta_3)\right) \ , \\ p^5 d_p &= \frac{1}{4} a_p^4 - p a_p^2 b_p + \frac{1}{2} p^2 b_p^2 + p^2 a_p c_p - \frac{1}{8} p^8 \left(\cos(4\theta_1) + \cos(4\theta_2) + \cos(4\theta_3)\right) \end{aligned} \tag{5.129}$$

where we define $d_p = \varepsilon b_p$ which is necessary to be computed in order to deduce the sign ε. This implies the following bounds on the magnitude of these coefficients

$$\begin{gathered} |a_p| \leq 6p^2 \quad , \quad \left| b_p - \frac{a_p^2}{2p} \right| \leq 3p^3 \quad , \quad \left| c_p - \frac{a_p^2}{3p^3} - \frac{a_p b_p}{p^2} \right| \leq 2p^3 \\ \left| d_p - \frac{a_p^4}{4p^5} + \frac{a_p^2 b_p}{p^4} - \frac{b_p^2}{2p^3} - \frac{a_p c_p}{p^3} \right| \leq \frac{3}{2} p^3 < 2p^3 \ . \end{gathered} \tag{5.130}$$

For the second case, these relations[82] slightly simplify to

$$\begin{aligned} a_p &= -2p^2 \left(\cos(\theta_1) + \cos(\theta_2)\right) \ , \\ p b_p &= \frac{1}{2} a_p^2 - p^4 \left(\cos(2\theta_1) + \cos(2\theta_2) + 1\right) \ , \\ p^3 c_p &= -\frac{1}{3} a_p^3 + p a_p b_p - \frac{2}{3} p^6 \left(\cos(3\theta_1) + \cos(3\theta_2)\right) \ , \\ p^5 d_p &= \frac{1}{4} a_p^4 - p a_p^2 b_p + \frac{1}{2} p^2 b_p^2 + p^2 a_p c_p - \frac{1}{8} p^8 \left(\cos(4\theta_1) + \cos(4\theta_2) + 1\right) \end{aligned} \tag{5.131}$$

[82] Recall that $\varepsilon = -1$ and $c_p = 0$ are fixed in this case by the Weil conjectures. However, since we cannot distinguish a priori between these two cases in the computation of $R_H(X_z, T)$, it is necessary to compute c_p and ε in any case.

implying the (slightly) stronger bounds

$$\begin{gathered}|a_p| \leq 4p^2 \quad , \quad \left|b_p - \frac{a_p^2}{2p}\right| \leq 3p^3 \quad , \quad \left|c_p - \frac{a_p^2}{3p^3} - \frac{a_p b_p}{p^2}\right| \leq \frac{4}{3}p^3 < 2p^3 \\ \left|d_p - \frac{a_p^4}{4p^5} + \frac{a_p^2 b_p}{p^4} - \frac{b_p^2}{2p^3} - \frac{a_p c_p}{p^3}\right| \leq \frac{3}{2}p^3 < 2p^3\end{gathered} \tag{5.132}$$

on the magnitudes of the coefficients.

Hence, computing the coefficients α,γ and ε up to p-adic order $\mathcal{O}(p^4)$ for primes $p \geq 7$ is sufficient in both cases to obtain an exact result for the polynomial $R_H(X_z, T)$. Recalling that it is our aim to compute the characteristic polynomial of $U_p(z) = F_p^{-1}(z)$, we can deduce similar relations between a_p, b_p, c_p and d_p and the matrix $U_p(z)$ as in equations (5.110) and (5.111). For the given Hodge type $(1, 1, 2, 1, 1)$, these relations read

$$\begin{aligned}a_p &= -\mathrm{Tr}(U(z)) \\ pb_p &= \frac{1}{2}\Big([\mathrm{Tr}(U_p(z))]^2 - \mathrm{Tr}(U_p^2(z)\Big) \\ p^3 c_p &= -\frac{1}{6}\Big([\mathrm{Tr}(U_p(z))]^3 - 3\mathrm{Tr}(U_p^2(z))\mathrm{Tr}(U_p(z)) + \mathrm{Tr}(U_p^3(z))\Big) \\ p^5 d_p &= \frac{1}{24}\Big([\mathrm{Tr}(U_p(z))]^4 - 6\,[\mathrm{Tr}(U_p(z))]^2\,\mathrm{Tr}(U_p^2(z)) + 3\,\big[\mathrm{Tr}(U_p^2(z))\big]^2 \\ &\qquad + 8\mathrm{Tr}(U_p^3(z))\mathrm{Tr}(U_p(z)) - 6\mathrm{Tr}(U_p^4(z))\Big)\end{aligned} \tag{5.133}$$

and allow for an efficient computation of $R_H(X_z, T)$ in terms of $U_p(z)$.

5.5 Modularity of Hulek-Verrill Fourfolds

Let us demonstrate the arithmetic techniques that have been developed in the previous sections by analyzing the modular properties of a concrete family of Calabi-Yau fourfolds. As our first example, we consider a specific one-dimensional subspace of the full moduli space of so-called Hulek-Verrill Calabi-Yau fourfolds HV^4 which we construct in analogy to their three-dimensional cousins that are introduced in [8]. This family of threefolds has raised certain attention as its moduli space admits a modular point that has been identified to be a rank-two attractor point [7, 8, 96].

We begin by constructing the full family of Hulek-Verrill fourfolds $\mathrm{HV}^4_{\mathbf{z}}$ and discussing its topological properties. In particular, it turns out that this family is described by a six-dimensional complex structure moduli space that posseses a $\mathbb{Z}_6$-symmetry. As for the threefold case, we might expect that the quotient $\mathrm{HV}^4_{\mathbf{z}}/\mathbb{Z}_6$ or more precise[83], the $\mathbb{Z}_6$-invariant subspace of the moduli space gives rise to a proper one-dimensional subfamily

[83]As we discuss in the following section, the quotient manifold turns out to be singular as the $\mathbb{Z}_6$ action acts non-freely.

of Hulek-Verrill fourfolds. This family is of primary horizontal Hodge type $(1,1,1,1,1)$, hence we can apply the discussion from section 5.4.1 in order to search for points of persistent factorization of the polynomial $R_H(\mathrm{HV}^4_{\mathbf{z}}, T)$.

This family of fourfolds is of special interest, as its arithmetic analysis will lead to a point of persistent factorization on $\mathcal{M}_{C.S.}$. In section 5.5.4, we verify this result by constructing the corresponding two-dimensional integral sublattice on the primary horizontal cohomology and performing several independent consistency checks.

5.5.1 Families of Hulek-Verrill Manifolds

Families of Calabi-Yau manifolds that are of Hulek-Verrill type appear in any dimension. First discussed in [8], the family of three-dimensional Hulek-Verrill manifolds has been intensively studied as it gives rise to a non-trivial rank-two attractor point which has been discovered in [7] using the analog arithmetic techniques that we developed in the previous sections for families of Calabi-Yau fourfolds. The full family of Hulek-Verrill n-folds turn out to be characterized by $n+2$ complex structure moduli. However, the varieties admit a discrete $\mathbb{Z}_{n+2}$-symmetry that allows to treat the invariant sublocus of this symmetry[84] effectively as a one-dimensional submoduli space. Moreover, if the symmetry acts freely on the ambient space of the n-folds, as it is the case for Hulek-Verrill threefolds, the quotient of these manifolds by this discrete symmetry group leads to an honest family of Calabi-Yau n-folds with only one complex structure modulus.

Hulek-Verrill manifolds appear as well very prominently in the framework of high precision computations for Feynman integrals which we discuss in chapter 6. It has been shown [180] that the n-loop banana integral[85] corresponds to the geometry of the family of Hulek-Verrill $n+1$-folds.

For the construction of Hulek-Verrill n-folds, which are described by a complete intersection in a toric variety, we follow in analogy to [8] the construction of Batyrev and Borisov [181–183]. As this construction is formulated in the framework of toric geometry we relegate a brief introduction to toric geometry and the tools required to define complete intersections in terms of lattice polytopes to appendix B. A comprehensive review on toric geometry can be found in [184–186].

Following [8], we define the family of Hulek-Verrill n-folds by constructing the lattice polyhedron

$$\Delta = \mathrm{Conv}\left(\{0\} \cup \{e_1, -e_1, \ldots, e_{n+2}, -e_{n+2}\} \cup \{e_i - e_j\}_{i \neq j}\right) \subset N \otimes \mathbb{R} \tag{5.134}$$

where $\{e_1, \ldots, e_{n+2}\}$ denotes a basis of the $(n+2)$-dimensional lattice N and considering

[84]Recall, that the invariant sublocus of a discrete symmetry action is one of the mechanisms that give rise to a proper projective special Kähler submanifold as we discussed in section 4.2.3.

[85]We introduce the banana integral and its connection to Calabi-Yau geometries in section 6.5.

the nef-partition[86] of $\Delta = \text{Mink}(\Delta, \Delta')$ with

$$\Delta = \text{Conv}\left(\{0, e_1, \ldots, e_{n+2}\}\right) \quad , \quad \Delta' = \text{Conv}\left(\{0, -e_1, \ldots, -e_{n+2}\}\right) \tag{5.135}$$

being suitable lattice polyhedra. Introducting local coordinates $\mathbf{x} = (x^1, \ldots, x^{n+2})$, any such n-fold can be realized by the subspace

$$\widetilde{\text{HV}}^n_{\mathbf{z}} := \left\{ \left(\frac{z^1}{x_1} + \cdots + \frac{z^{n+2}}{x_{n+2}} \right) (x_1 + \cdots + x_{n+2}) = 1 \right\} \subset \mathbb{T}^{n+2} \ . \tag{5.136}$$

Here, we have already excluded the points $x_k = 0$ from the ambient space by introducing the projective $(n+2)$-dimensional torus

$$\mathbb{T}^{n+2} = \mathbb{P}^{n+1} \backslash \bigcup_{k=1}^{n+2} \{x_k = 0\} \ . \tag{5.137}$$

The $n+2$ complex structure moduli are collected in the vector

$$\mathbf{z} := (z^1, \ldots, z^{n+2}) \ . \tag{5.138}$$

It is important to mention that this construction leads to Calabi-Yau n-folds that admit so-called nodal singularities. These singularities can be resolved by using different techniques from toric geometry[87]. Obviously, such resolutions change the geometry of the manifolds, however the result will still be birational[88] to the singular manifold. Since the arithmetic analysis which we presented above is sensitive only to the rational cohomology of the manifolds which is invariant under such birational transformations, the birational but smooth resolutions contain the equivalent information about possible modular points in the moduli space.

Observe that we can equivalently write the defining equation (5.136) by introducing an additional projective coordinate $y \in \mathbb{P}^1$ as

$$\sum_{k=1}^{n+2} \frac{z^k}{x_k} + \frac{1}{y} = 0 \quad , \quad \sum_{k=1}^{n+2} x_k + y = 0 \ . \tag{5.139}$$

Thus, away from the singular points, $\widetilde{\text{HV}}^n_{\mathbf{z}}$ is birational to the manifold

$$\text{HV}^n_{\mathbf{z}} := \left\{ \sum_{k=1}^{n+2} \frac{z^k}{x_k} + \frac{1}{y} = 0 \quad , \quad \sum_{k=1}^{n+2} x_k + y = 0 \right\} \subset \mathbb{T}^{n+2} \times \mathbb{P}^1 \tag{5.140}$$

[86]For details on the construction of Calabi-Yau varieties via nef-partitions of reflexive polyhedrons we refer to the work of Batyrev and Borisov [183].

[87]Originally in [8] this has been achieved by a small resolution of the singularities. Following [142] we use a different strategy that nevertheless leads to a birational description of the singular geometry.

[88]A birational map between algebraic varieties X, Y is an invertible, locally rational map $X \to Y$.

which we call in the following the resolved Hulek-Verrill n-fold.

Recall that an algebraic variety V which is defined by rational functions $f_i(\mathbf{x})$ is singular at $\mathbf{x} \in V$, if the Jacobian $\left(\partial_{x_j} f_i\right)$ becomes singular at $\mathbf{x}$. Applying this condition to the defining equation (5.140) leads to the singularity condition

$$\frac{z^1}{x_1^2} = \cdots = \frac{z^{n+2}}{x_{n+2}^2} \; . \tag{5.141}$$

For a generic choice of complex structure moduli $\mathbf{z}$, these conditions intersect trivially with $\mathrm{HV}^n_{\mathbf{z}}$. However, there exist non-generic points on the moduli space for which the explicit geometry is still singular. In particular, each family contains a disciminant locus on its complex structure moduli space which gives rise to conifold singularities. These additional conifold singularities appear for those points $\mathbf{z}$ on the complex structure moduli space, which obey [8, 142]

$$\prod_{\varepsilon_k=\pm1}\left(1+\sum_{k=1}^{n+2}\varepsilon_k\sqrt{z^k}\right) = 0 \; . \tag{5.142}$$

From the representation given by equation (5.140), we can read off that the family of Hulek-Verrill n-folds $\mathrm{HV}^n_{\mathbf{z}}$ is characterized by $n+2$ complex structure moduli z^k. Moreover, the Batyrev-Borisov construction allows to compute the non-trivial Hodge numbers $h^{p,q}$ of these n-folds. Table 5.1 summarizes the relevant topological data for Hulek-Verrill n-folds with $n \leq 4$.

n	Notation	χ	Non-trivial Hodge numbers $h^{p,q}$
1	$\mathcal{E}_{\mathbf{z}}$	0	
2	$\mathrm{K3}_{\mathbf{z}}$	42	
3	$\mathrm{HV}^3_{\mathbf{z}}$	80	$h^{2,1}=5$ and $h^{1,1}=45$
4	$\mathrm{HV}^4_{\mathbf{z}}$	720	$h^{3,1}=6$, $h^{1,1}=106$, $h^{2,1}=0$ and $h^{2,2}=492$

Table 5.1: Hulek-Verrill n-folds for $n=1,\ldots,4$ together with their Euler characteristic χ and their non-trivial Hodge numbers. Since the one- and two-dimensional manifolds are elliptic curves and K3 surfaces respectively, we introduce a special notation. The Hodge numbers for these two examples are completely fixed by the Calabi-Yau requirement, hence there are no non-trivial Hodge numbers for $n \leq 2$.

For Hulek-Verrill n-folds with $n \geq 3$, there exists yet another birational description which provides a geometric interpretation in terms of an elliptically fibred product. For $\mathrm{HV}^3_{\mathbf{z}}$ it has been observed in [8] that the defining equations (5.140) can be rewritten as

$$\begin{aligned} \frac{1}{\lambda_0} &:= y + x_1 + x_2 &&= -(x_3+x_4+x_5) \\ \lambda_1 &:= \frac{1}{y} + \frac{z^1}{x_1} + \frac{z^2}{x_2} &&= -\left(\frac{z^3}{x_3}+\frac{z^4}{x_4}+\frac{z^5}{x_5}\right) \; . \end{aligned} \tag{5.143}$$

for two complex constants $\lambda_0, \lambda_1 \in \mathbb{C}$ that parametrize an additional projective line $[\lambda_0; \lambda_1] \in \mathbb{P}^1$. For the following argument it is convenient to define the corresponding projective coordinate $\lambda := \frac{\lambda_1}{\lambda_0}$. If we now perform a change of coordinates according to

$$\begin{aligned} \tilde{x_0} &= -\frac{1}{y}(y + x_1 + x_2) \quad , \quad \tilde{x_1} = \frac{x_1}{y} \quad , \quad \tilde{x_2} = \frac{x_2}{y} \\ \tilde{x_3} &= -\frac{1}{x_3}(x_3 + x_4 + x_5) \quad , \quad \tilde{x_4} = \frac{x_4}{x_3} \quad , \quad \tilde{x_5} = \frac{x_5}{x_3} \end{aligned} \tag{5.144}$$

we find

$$\begin{aligned} 1 + \tilde{x}_0 + \tilde{x}_1 + \tilde{x}_2 = 0 \quad &, \quad 1 + \frac{z^1}{\tilde{x}_1} + \frac{z^2}{\tilde{x}_2} + \frac{\lambda}{\tilde{x}_0} = 0 \\ 1 + \tilde{x}_3 + \tilde{x}_4 + \tilde{x}_5 = 0 \quad &, \quad 1 + \frac{z^4/z^3}{\tilde{x}_4} + \frac{z^5/z^3}{\tilde{x}_5} + \frac{\lambda/z^3}{\tilde{x}_3} = 0 \; . \end{aligned} \tag{5.145}$$

Note that the two equations on the left hand are a trivial fact from the change of coordinates whereas the remaining two follow from equation (5.143). Comparing these with equation (5.140), it follows that the coordinates $(\tilde{x}_0, \tilde{x}_1, \tilde{x}_2) \in \mathbb{T}^3$ and $(\tilde{x}_3, \tilde{x}_4, \tilde{x}_5) \in \mathbb{T}^3$ parametrize each an Hulek-Verrill elliptic curve whose complex structure moduli are given in terms of $\mathbf{z}$ and the additional projective parameter λ. Following the arguments in [8], this change of coordinates extends to a birational map

$$\mathrm{HV}^3_{(z^1,z^2,z^3,z^4,z^5)} \overset{\text{rat.}}{\longrightarrow} \mathcal{E}_{(\lambda,z^1,z^2)} \times_{\mathbb{P}^1} \mathcal{E}_{\left(\frac{\lambda}{z^3},\frac{z^4}{z^3},\frac{z^5}{z^3}\right)} \tag{5.146}$$

relating the given Hulek-Verrill threefold to a double-fibred Calabi-Yau threefold whose base is a projective line $\mathbb{P}^1$ parametrized by λ and its fibres are given by two Hulek-Verrill elliptic curves.

An analog construction holds true for the family of Hulek-Verrill fourfolds. If we include the additional sixth coordinate in either of the elliptic curves, we obtain

$$\begin{aligned} 1 + \tilde{x}_0 + \tilde{x}_1 + \tilde{x}_2 = 0 \quad &, \quad 1 + \frac{z^1}{\tilde{x}_1} + \frac{z^2}{\tilde{x}_2} + \frac{\lambda}{\tilde{x}_0} = 0 \\ 1 + \tilde{x}_3 + \tilde{x}_4 + \tilde{x}_5 + \tilde{x}_6 = 0 \quad &, \quad 1 + \frac{z^4/z^3}{\tilde{x}_4} + \frac{z^5/z^3}{\tilde{x}_5} + \frac{z^6/z^3}{\tilde{x}_6} + \frac{\lambda/z^3}{\tilde{x}_3} = 0 \; . \end{aligned} \tag{5.147}$$

With the new coordinates

$$\begin{aligned} \tilde{x_0} &= -\frac{1}{y}(y + x_1 + x_2) \quad , \quad \tilde{x_1} = \frac{x_1}{y} \quad , \quad \tilde{x_2} = \frac{x_2}{y} \\ \tilde{x_3} &= -\frac{1}{x_3}(x_3 + x_4 + x_5 + x_6) \quad , \quad \tilde{x_4} = \frac{x_4}{x_3} \quad , \quad \tilde{x_5} = \frac{x_5}{x_3} \quad , \quad \tilde{x_6} = \frac{x_6}{x_3} \end{aligned} \tag{5.148}$$

we obtain again a birational map to a double-fibred product over a projective line

$$\mathrm{HV}^4_{(z^1,z^2,z^3,z^4,z^5,z^6)} \overset{\text{rat.}}{\longrightarrow} \mathcal{E}_{(\lambda,z^1,z^2)} \times_{\mathbb{P}^1} \mathrm{K3}_{\left(\frac{\lambda}{z^3},\frac{z^4}{z^3},\frac{z^5}{z^3},\frac{z^6}{z^3}\right)} \; . \tag{5.149}$$

In contrast to the previous case, the second factor now becomes a Hulek-Verrill K3 surface instead of an elliptic curve.

In order to obtain the correct linear combinations of integral periods for the middle cohomology later on, it is necessary to construct the mirror geometries $\mathrm{H}\Lambda^n$ of the Hulek-Verrill n-folds. For families of complete intersections, this can be achieved by a similar Batyrev-Borisov construction as before by changing the lattice polyhedron to its dual

$$\nabla = \mathrm{Conv}\left(\{0\} \cup \{\varepsilon_1 e_1^\star + \cdots + \varepsilon_{n+2} e_{n+2}^\star \mid \varepsilon_i = \pm 1\}\right) \subset M^\star \otimes \mathbb{R} \,. \tag{5.150}$$

As it turns out, the result is given by a complete intersection on the direct product of $n+2$ projective lines

$$\mathrm{H}\Lambda^n \cong \begin{matrix} \mathbb{P}^1 \\ \vdots \\ \mathbb{P}^1 \end{matrix} \begin{bmatrix} 1,1 \\ \vdots \\ 1,1 \end{bmatrix} \,. \tag{5.151}$$

This convenient notation is meant to describe the family of zero loci of two polynomials on the direct product $\left(\mathbb{P}^1\right)^{n+2}$ that are both homogenous of degree one in each projective coordinate. A combinatorial counting of all possible parameters describing this type of polynomials gives a consistency check as it reproduces exactly the Hodge numbers $h^{n-1,1}$ of the Hulek-Verrill n-folds discussed above. Moreover, the number of Kähler parameters is trivially given by $n+2$ as the Kähler deformations of the ambient space induce those of the Calabi-Yau variety. Since any $\mathbb{P}^1$ contributes with one Kähler deformation as $\dim(H^{1,1}(\mathbb{P}^1,\mathbb{C})) = 1$, we end up with in total $n+2$ Kähler moduli $\mathbf{t} := (t^1, \ldots, t^{n+2})$ on $\mathrm{H}\Lambda^n$. We denote the corresponding generators of the spaces $H^{1,1}(\mathbb{P}^1,\mathbb{C})$ by h_i. Here, the index i is relevant to distinguish between the $n+2$ copies of $\mathbb{P}^1$ of the complete intersection. As previously, we use the notation $\mathrm{H}\Lambda^n_{\mathbf{t}}$ for that mirror Hulek-Verrill n-fold which is characterized by the Kähler moduli $\mathbf{t}$.

The aim for the remainder of this section is to compute the asymptotic structure of the integral periods $\Pi^a(\mathbf{t})$ for both, the Hulek-Verrill threefolds and fourfolds. Recalling the discussions from sections 3.2.5 and 3.2.6 we need for the appropriate Γ-class analysis to compute the Chern classes of $\mathrm{H}\Lambda^3_{\mathbf{t}}$ and $\mathrm{H}\Lambda^4_{\mathbf{t}}$ respectively. Since these manifolds are given in terms of complete intersections on a direct product of projective lines, we can express the Chern classes in terms of the hyperplane classes h_i of the individual $\mathbb{P}^1$-factors as[89]

$$c(\mathrm{H}\Lambda^n_{\mathbf{t}}) = \frac{\prod_{k=1}^{n+2}(1+h_i)^2}{\left(1+\sum_{k=1}^{n+2} h_i\right)^2} \,. \tag{5.152}$$

[89] In general, if X is defined as a complete intersection on the direct product of projective spaces $\mathbb{P}^{n_k}$ by polynomials P_i that have homogenous degree m_{ik} on the k-th projective factor, the Chern class can be computed using the adjunction formula to be [187]

$$c(X) = \frac{\prod_k (1+h_k)^{n_k+1}}{\prod_i (1+\sum_k m_{ik} h_k)}$$

where h_k denotes the corresponding hyperplane class.

For a threefold $\mathrm{H}\Lambda^3_{\mathbf{t}}$, an expansion of this identity yields

$$c(\mathrm{H}\Lambda^3_{\mathbf{t}}) = 1 + 2\sum_{i<j} h_i \wedge h_j - 4\sum_{i<j<k} h_i \wedge h_j \wedge h_k \tag{5.153}$$

from which we compute the topological invariants Y_{ijk} according to equation (3.54) to be

$$\begin{aligned} Y_{ijk} &= \begin{cases} 2 & \text{if } i,j,k \text{ are pairwise distinct} \\ 0 & \text{otherwise} \end{cases} \\ Y_{0ij} = 0 \quad , \quad & Y_{00i} = -2 \quad , \quad Y_{000} = 240\frac{\zeta(3)}{(2\pi i)^3} \ . \end{aligned} \tag{5.154}$$

These characterize the asymptotic structure of the prepotential F and hence the leading order contributions of the integral periods. We find

$$\begin{aligned} \Pi^0(\mathbf{t}) &= 1 \\ \Pi^i(\mathbf{t}) &= t^i \\ \Pi_i(\mathbf{t}) &= -s_2^4(\hat{\mathbf{t}}_i) + 1 + \cdots \\ \Pi_0(\mathbf{t}) &= s_3^5(\mathbf{t}) + s_1^5(\mathbf{t}) - 80\frac{\zeta(3)}{(2\pi i)^3} + \cdots \end{aligned} \tag{5.155}$$

where $s_l^m(\mathbf{t})$ denotes the l^{th} symmetric polynomial in m variables $(t^1,\ldots,t^m)$. Moreover we denote by $\hat{\mathbf{t}}_i$ the set of Kähler moduli $(t^1,\ldots,\hat{t^i},\ldots,t^5)$ with t^i being removed.

For $\mathrm{H}\Lambda^4_{\mathbf{t}}$, this construction is slightly more complicated as we need in addition to compute the Chern characters $\mathrm{ch}(\mathcal{O}_{h_i\cap h_j})$ that belong to 4-branes. Using the expansion

$$c(\mathrm{H}\Lambda^4_{\mathbf{t}}) = 1 + 2\sum_{i<j} h_i \wedge h_j - 4\sum_{i<j<k} h_i \wedge h_j \wedge h_k + 24\sum_{i<j<k<\ell} h_i \wedge h_j \wedge h_k \wedge h_\ell \tag{5.156}$$

for the Chern class and $\mathrm{ch}(\mathcal{O}_{h_i\cap h_j}) = h_i \wedge h_j$, the asymptotic structure of the integral periods follows to be

$$\begin{aligned} \Pi^0(\mathbf{t}) &= 1 \\ \Pi^i(\mathbf{t}) &= t^i \\ \Pi_{ij}(\mathbf{t}) &= 2s_2^4(\hat{\mathbf{t}}_{ij}) + 1 + \cdots \\ \Pi_i(\mathbf{t}) &= -2s_3^5(\hat{\mathbf{t}}_i) - s_1^5(\hat{\mathbf{t}}_i) + 80\frac{\zeta(3)}{(2\pi i)^3} + \cdots \\ \Pi_0(\mathbf{t}) &= 2s_4^6(\mathbf{t}) + s_2^6(\mathbf{t}) - 80\frac{\zeta(3)}{(2\pi i)^3}s_1(\mathbf{t}) + \frac{3}{8} + \cdots \ . \end{aligned} \tag{5.157}$$

Again, we denote by $s_\ell^m(\mathbf{t})$ the ℓ^{th} symmetric polynomial in m variables and define the short notations $\hat{\mathbf{t}}_i := (t^1,\ldots,\hat{t^i},\ldots,t^6)$ and $\hat{\mathbf{t}}_{ij} := (t^1,\ldots,\hat{t^i},\ldots,\hat{t^j},\ldots,t^6)$ to be the sets of Kähler moduli with t^i and t^i, t^j being removed respectively.

5.5.2 One-parameter Families of Hulek-Verrill n-folds

The geometry of Hulek-Verrill n-folds admits a natural $\mathbb{Z}_{n+2}$ group action that is given by cyclically permuting its coordinates x_i. $\mathrm{HV}^n_{\mathbf{z}}$ is symmetric under this group action if and only if the complex structure moduli are given by the diagonal subspace $z^1 = \ldots = z^{n+2}$ on the complex structure moduli space. If we define z to parametrize this diagonal subspace, we can define a one-parameter subfamily within the full family of Hulek-Verrill n-folds by

$$\mathrm{HV}^n_z := \mathrm{HV}^n_{(z,\ldots,z)} \ . \tag{5.158}$$

One should note that we have simply restricted the $(n+2)$-dimensional complex structure moduli space to a one-dimensional subspace by this procedure. This has no effect on the geometry of any explicit member HV^n_z of the family. In particular, its Hodge number $h^{n-1,1}$ is still $n+2$ and hence disagrees with the number of varying complex structure moduli. This is owed to the fact that the diagonal $z^1 = \ldots = z^{n+2} = z$ does not cover all possible complex structure deformations of these manifolds but fixes $n-1$ of them.

A natural construction to obtain an honest one-parameter family of Calabi-Yau n-folds with $h^{n-1,1} = 1$ is given by quotienting the manifold HV^n_z by the $\mathbb{Z}_{n+2}$ group action. While this construction can be done for any algebraic variety and any discrete symmetry group, it turns out that the quotient manifold $\mathrm{HV}^n_z/\mathbb{Z}_{n+2}$ obtains orbifold singularities whenever the group does not act freely[90] on HV^n_z [188]. For HV^3_z the fixed-point locus of the $\mathbb{Z}_5$-action on the ambient space $\mathbb{T}^5$ is given by

$$\left(\mathbb{T}^5\right)^{\mathbb{Z}_5} = \{[x_1;\ldots;x_5] \in \mathbb{T}^5 \mid x_1 = \ldots = x_5\} \ . \tag{5.159}$$

Equation (5.141) implies that this fixed-point locus intersects with HV^3_z only if $(z,\ldots,z)$ is a singular point on the complex structure moduli space. By inserting this diagonal locus into the equation (5.142) for the discriminant locus, we obtain that there are five conifold singularities on the diagonal sublocus of the complex structure moduli space which are given by the roots of the discriminant

$$\Delta = (1-z)(1-9z)(1-25z) \tag{5.160}$$

together with the large complex structure point $z=0$ and $z=\infty$. From equation (5.136) it follows that an intersection of $\left(\mathbb{T}^5\right)^{\mathbb{Z}_5}$ with HV^3_z is possible only at the conifold singularity $z = \frac{1}{25}$. Thus, the symmetry group $\mathbb{Z}_5$ indeed acts freely on HV^3_z outside the singular points and hence the corresponding family of quotient manifolds gives rise to a family of Calabi-Yau threefolds with one complex structure modulus which is smooth for $z \notin \{0, \frac{1}{25}, \frac{1}{9}, 1, \infty\}$.

The mirror family of $\mathrm{HV}^3_z/\mathbb{Z}_5$ is obtained from $\mathrm{H}\Lambda^3_{\mathbf{t}}$ by the similar construction. Again, the manifolds that are described by the diagonal sublocus $t := t^1 = \ldots = t^5$ are invariant

[90] Recall that a group G acts freely on a set X, if for all $x \in X$ and $g \in G$ the condition $g.x = x$ implies that $g = e$ is the identity element of G. In other words, G acts freely on X if and only if no non-trivial group element has a fixed point in X.

under a $\mathbb{Z}_5$-action. Hence, the corresponding quotient gives rise to a family of Calabi-Yau threefolds with one Kähler modulus. On the level of periods, this procedure leads to an "averaging" which in practice is obtained by inserting the diagonal locus $t^i = t$ into the expressions (5.155) for the periods and rescaling the result by the inverse size of the group orbits which is in this case $\frac{1}{5}$.

Let us now turn to the analog one-parameter subfamily of Hulek-Verrill fourfolds. On first view, one might guess that it is possible to perform a similar construction by quotiening with resprect to the $\mathbb{Z}_6$-symmetry group of HV^4_z. In contrast to the threefold case, this symmetry group is not simple but decomposes into the two subgroups $\mathbb{Z}_2$ and $\mathbb{Z}_3$ whose generators can be chosen[91] to be the permutations

$$\begin{aligned} \mathbb{Z}_2 &\cong \langle (x_1, x_4)(x_2, x_5)(x_3, x_6) \rangle \\ \mathbb{Z}_3 &\cong \langle (x_1, x_3, x_5)(x_2, x_4, x_6) \rangle \ . \end{aligned} \tag{5.161}$$

Obviously, the $\mathbb{Z}_2$-action has an extended three-dimensional fixed-point locus on the ambient space $\mathbb{T}^6$ that is given by

$$\left(\mathbb{T}^6\right)^{\mathbb{Z}_2} := \{[x_1; \ldots; x_6] \in \mathbb{T}^6 \mid x_1 = x_4 \ , \ x_2 = x_5 \ , \ x_3 = x_6\} \tag{5.162}$$

whereas the $\mathbb{Z}_3$-action gives rise to a two-dimensional fixed-point locus

$$\left(\mathbb{T}^6\right)^{\mathbb{Z}_3} := \{[x_1; \ldots; x_6] \in \mathbb{T}^6 \mid x_1 = x_3 = x_5 \ , \ x_2 = x_4 = x_6\} \ . \tag{5.163}$$

These intersect non-trivially on

$$\left(\mathbb{T}^6\right)^{\mathbb{Z}_6} := \left(\mathbb{T}^6\right)^{\mathbb{Z}_2} \cap \left(\mathbb{T}^6\right)^{\mathbb{Z}_3} = \{[x_1; \ldots; x_6] \in \mathbb{T}^6 \mid x_1 = \ldots = x_6\} \tag{5.164}$$

which is the fixed-point locus of the full $\mathbb{Z}_6$-action. As in the threefold case, equation (5.141) implies that this fixed-point locus intersects with the Hulek-Verrill fourfold HV^4_z only, if $(z, \ldots, z)$ is a singular point. In particular, an insertion into the defining equation (5.136) of the fourfolds shows that there is such an intersection only for $z = \frac{1}{36}$ which is exactly one of the roots of the discriminant Δ that becomes

$$\Delta = (1 - 4z)(1 - 16z)(1 - 36z) \tag{5.165}$$

along the diagonal $z^i = z$. The main difference to the previously discussed threefold case is that the fixed-point loci of the subgroups cause additional orbifold singularities if we consider the quotient $\mathrm{HV}^4_z/\mathbb{Z}_6$. Unlike the fixed-point locus of the full $\mathbb{Z}_6$-action, $\left(\mathbb{T}^6\right)^{\mathbb{Z}_2}$ and $\left(\mathbb{T}^6\right)^{\mathbb{Z}_3}$ intersect HV^4_z for generic z and hence the quotient $\mathrm{HV}^4_n/\mathbb{Z}_6$ obtains orbifold singularities for any $z \in \mathbb{C}$.

It follows that $\left(\mathbb{T}^6\right)^{\mathbb{Z}_2}$ intersects HV^4_z only at discrete points. Hence, the corresponding orbifold singularities on the quotient have a crepant resolution [188]. However, $\left(\mathbb{T}^6\right)^{\mathbb{Z}_3}$

[91] Any other decomposition of the $\mathbb{Z}_6$ action is equivalent to the chosen one.

intersects the Hulek-Verrill fourfold along an extended line which prohibits a crepant resoultion of the caused orbifold singularities. From this discussion we can conclude that the quotient $\mathrm{HV}_z^4/\mathbb{Z}_6$ contains for any $z \in \mathbb{C}$ orbifold singularities that cannot be resolved, hence in contrast to the threefold case this quotient turns out to be a singular manifold.

We can nevertheless discuss the arithmetic properties of this one-parameter subfamily of Hulek-Verrill fourfolds by applying the methods that have been presented in section 5.3 to a subfamily

$$\mathrm{HV}_z^4 = \mathrm{HV}_{(z,\ldots,z)}^4 \tag{5.166}$$

of the full family of Hulek-Verrill fourfolds. If we would have been able to construct a corresponding honest one-parameter family of smooth fourfolds by quotienting with the symmetry group, it would follow directly that the primary horizontal subspace $H_H^4(\mathrm{HV}_z^4/\mathbb{Z}_6, \mathbb{C})$ would be generated by $\Omega(z) := \Omega(z, \ldots, z)$ and an appropriate number of derivatives $\Theta^k\Omega(z)$. Since these quotient manifolds turn out to be singular, we need to find an analog description within the full horizontal cohomology. We note that, due to the $\mathbb{Z}_6$-symmetry along the diagonal locus, the holomorphic four-form $\Omega(z)$ is invariant on this diagonal under a cyclic permutation of the complex structure moduli. Hence, the derivatives $\Theta_i\Omega(z)$ should coincide on the diagonal. By this observation one might expect that $H_H^4(\mathrm{HV}_z^4, \mathbb{Q})$ admits a natural sub Hodge structure along the diagonal $z^i = z$ which effectively corresponds to a one-parameter subfamily of Calabi-Yau fourfolds[92].

To finish the discussion of the one-parameter models, we now turn to the derivation of the Picard-Fuchs operators which characterize the periods of $\mathrm{HV}_z^3/\mathbb{Z}_5$ and of the sub Hodge structure on HV_z^4 respectively. Since we follow in both cases an analog procedure which has been performed for the threefold case in [142], the computations will be kept general in order to cover both cases simultaneously.

We recall that the Picard-Fuchs ideal for any Calabi-Yau n-fold is obtained by iterative differentiation of the holomorphic n-form with respect to the complex structure moduli and reducing the result modulo exact forms until one obtains a closed expression in terms of a differential equation for $\Omega(z)$. This iteration, which is known as Griffiths-Dwork reduction[93] [89, 90], is guaranteed to succeed as the primary horizontal subspace $H_H^n(X, \mathbb{C})$ is known to be finite dimensional. For a Calabi-Yau n-fold X that is given by a complete intersection in a projective ambient space $\mathbb{P}^{n+2}$, Ω enjoys an explicit expression in terms of a residue integral [62]

$$\Omega = \int_\gamma \frac{\Delta}{\prod_{k=1}^N P_k(x)} \; . \tag{5.167}$$

[92] This observation has been expectable from the discussions of the threefold case, since the periods of the $\mathbb{Z}_5$-quotient $\mathrm{HV}_z^3/\mathbb{Z}_5$ just coincide (up to an overall normalization) with those of HV_z^3 on the diagonal subspace.

[93] For a comprehensive review on the Griffiths-Dwork reduction procedure we refer to [185].

Here, the $P_i(x)$ denote the defining polynomials of X and

$$\Delta = \sum_{k=1}^{n+2} (-1)^k x_k dx^1 \wedge \cdots \wedge \hat{dx_k} \wedge \cdots \wedge dx^{n+2} \ . \tag{5.168}$$

Moreover, γ defines a torus, surrounding the Calabi-Yau n-fold in its ambient space. Note that for a smooth family of Calabi-Yau manifolds, the polynomials P_i become smoothly dependent on the complex structure moduli.

Let us now apply this residue formula to the $\mathbb{Z}_{n+2}$-invariant subfamily[94] of Hulek-Verrill n-folds HV_z^n. The defining equation (5.136) implies that we can write $\Omega(z)$ as

$$\Omega(z) = \int_\gamma \frac{\Delta}{\left(\sum_{k=1}^{n+2} \frac{z}{x^k}\right)\left(\sum_{k=1}^{n+2} x_k\right) - 1} \ . \tag{5.169}$$

For this special family, it is even possible to perform this integration explicitly such that we obtain a closed expression for the fundamental period without knowing the Picard-Fuchs ideal yet. Following [142], we find that the fundamental period corresponding to $\Omega(z)$ is obtained by choosing the integration along a torus. In the vicinity of the large complex structure point $z = 0$, the denominator is expandable in terms of a geometric series and hence we obtain[95]

$$\begin{aligned} \varpi^0(z) &= -\frac{1}{(2\pi i)^{n+2}} \int \prod_{i=1}^{n+2} \frac{dx^i}{x^i} \frac{1}{\left(\sum_{k=1}^{n+2} \frac{z}{x^k}\right)\left(\sum_{k=1}^{n+2} x_k\right) - 1} \\ &= \frac{1}{(2\pi i)^{n+2}} \int \prod_{i=1}^{n+2} \frac{dx^i}{x^i} \sum_{\ell=0}^{\infty} \left(\sum_{k=1}^{n+2} \frac{z}{x^k}\right)^\ell \left(\sum_{k=1}^{n+2} x_k\right)^\ell \\ &= \frac{1}{(2\pi i)^{n+2}} \sum_{\ell=0}^{\infty} \sum_{|\mathbf{j}|=\ell} \sum_{|\mathbf{k}|=\ell} \begin{pmatrix} \ell \\ \mathbf{j} \end{pmatrix} \begin{pmatrix} \ell \\ \mathbf{k} \end{pmatrix} \prod_{i=1}^{n+2} z^{k_i} \int_{S^1} \frac{dx^i}{x^i} x_i^{j_i - k_i} \ . \end{aligned} \tag{5.170}$$

Here, $\mathbf{j} = (j_1, \ldots, j_{n+2})$ and $\mathbf{k} = (k_1, \ldots, k_{n+2})$ denote $(n+2)$-dimensional multi-indices,

$$\begin{pmatrix} \ell \\ \mathbf{k} \end{pmatrix} := \frac{\ell!}{k_1! \cdots k_{n+2}!} \tag{5.171}$$

are the multinomial coefficients that count the multiplicity of the appearing factors and $|\mathbf{j}| = \sum_{i=1}^{n+2} j_i$ is the absolute value of the multi-index $\mathbf{j}$. The remaining integrals can be computed easily by Cauchy's integral formula to be

$$\int_{S^1} \frac{dx^i}{x^i} x_i^{j_i - k_i} = 2\pi i \delta_{j_i, k_i} \ . \tag{5.172}$$

[94] At this point we could have started with the full $n+2$-dimensional moduli space as well and perform the analog computation. However, since we are interested in the Picard-Fuchs operator for this one-dimensional subfamily, we restrict the following discussion to this special case.

[95] The prefactor of $-(2\pi i)^{-n+2}$ is chosen such that $\varpi^0(z)$ is properly normalized to $\varpi^0(0) = 1$.

Hence, the fundamental period for the one-parameter subfamily HV_z^n reads

$$\varpi^0(z) = \sum_{k=0}^{\infty} \sum_{|\mathbf{j}|=k} \begin{pmatrix} k \\ \mathbf{j} \end{pmatrix}^2 z^k \tag{5.173}$$

in a vicinity of the large complex structure point.

Since we know that the primary horizontal subspace $H_H^n(\mathrm{HV}_z^n, \mathbb{Q})$ has a substructure that is spanned by $\Omega(z)$ and a finite number of its derivatives and hence can be treated as a one-parameter family of Calabi-Yau n-folds, it is assured that the Picard-Fuchs ideal is generated only by a single differential operator $\mathcal{L}$. This Picard-Fuchs operator $\mathcal{L}$ can be constructed by requiring it to be of minimal degree such that

$$\mathcal{L}\varpi^0(z) = 0 \ . \tag{5.174}$$

In practice, $\mathcal{L}$ can be deduced by finding a recursion relation for the coefficients

$$a_k = \sum_{|j|=k} \begin{pmatrix} k \\ \mathbf{j} \end{pmatrix}^2 \tag{5.175}$$

and translating this into a differential equation for $\varpi^0(z)$.

For the Hulek-Verrill threefolds we find that the coefficients a_k obey the recursion relation

$$\begin{aligned} a_n =& \frac{1}{n^4} \Big((35n^4 - 70n^3 + 63n^2 - 28n + 5) a_{n-1} \\ & - (n-1)^2 (259n^2 - 518n + 285) a_{n-2} + 225 (n-1)^2 (n-2)^2 a_{n-3} \Big) \end{aligned} \tag{5.176}$$

implying that $\varpi^0(z)$ is annihilated by the degree-four differential operator[96]

$$\mathcal{L} = \sum_{k=0}^{4} f_k(z) \Theta^k \tag{5.177}$$

with

$$\begin{gathered} f_4(z) = (1-z)(1-9z)(1-25z) \ , \\ f_3(z) = -2z(675z^2 - 518z + 35) \quad , \quad f_2(z) = -z(2925z^2 - 1580z + 63) \ , \\ f_1(z) = -4z(675z^2 - 272z + 7) \quad , \quad f_0(z) = -5z(180z^2 - 57z + 1) \ . \end{gathered} \tag{5.178}$$

[96] Note that this operator has been identified as operator AESZ34 in the list [189] to be the Calabi-Yau operator of a one-parameter family of Calabi-Yau threefolds.

In analogy, the coefficients a_k for the one-parameter subfamily HV^4_z are characterized by the recursion relation

$$\begin{aligned} a_n =& \frac{1}{n^5}\Big(2(2n-1)(14n^4-28n^3+28n^2-14n+3)a_{n-1} \\ & -4(n-1)^3(196n^2-392n+255)a_{n-2}+1152(n-2)^2(n-1)^2(2n-3)a_{n-3}\Big) \end{aligned} \tag{5.179}$$

that gives rise to a degree-five differential operator

$$\mathcal{L} = \sum_{k=0}^{5} f_k(z)\Theta^k \ . \tag{5.180}$$

with

$$\begin{aligned} f_5(z) &= (1-4z)(1-16z)(1-36z) \ , \ f_4(z) = -20z(864z^2-196z+7) \ , \\ f_3(z) &= -12z(4224z^2-673z+14) \ , \ f_2(z) = -4z(18144z^2-2137z+28) \ , \\ f_1(z) &= -4z(12672z^2-1157z+10) \ , \ f_0(z) = -6z(2304z^2-170z+1) \ . \end{aligned} \tag{5.181}$$

From this result, we first note that the Hodge substructure of $H^4_H(\mathrm{HV}^4_z,\mathbb{Q})$ on the $\mathbb{Z}_6$-invariant subfamily is of Hodge type $(1,1,1,1,1)$ as it is decribed by one effective complex structure modulus z and moreover, the Picard-Fuchs operator of degree five ensures that there is only one $(2,2)$-forms contributing to the primary horizontal subspace.

With the Picard-Fuchs operator at hand it is now straight forward to compute the additional periods $\varpi^i(z)$ in the Frobenius basis given by equation (3.73) for the threefold case and equation (3.77) for the fourfold case. In practice, this is achieved by expanding all holomorphic functions in a series expansion and deriving recursion relations for their coefficients from the condition $\mathcal{L}\varpi^i(z)=0$. In appendix C we collect a summary of these recursion relations for general one-parameter families of Calabi-Yau threefolds and in addition for families of Calabi-Yau fourfolds that are of primary horizontal Hodge type $(1,1,\ell,1,1)$ for some $\ell \geq 1$.

Having computed the Frobenius periods by this method (up to a given precision), we can use the mirror map and the period structure we have discussed for the mirror manifolds in the previous section to obtain the correct linear combinations of the Frobenius periods that lead to a set of integral periods on $H^n_H(\mathrm{HV}^n_z,\mathbb{C})$. In analogy to the previous threefold discussion, we can associate as a mirror of the one-dimensional subfamily HV^4_z the subfamily

$$\mathrm{H}\Lambda^4_t := \mathrm{H}\Lambda^4_{(t,t,t,t,t,t)} \tag{5.182}$$

that is defined on the diagonal of the complexified Kähler moduli space of $\mathrm{H}\Lambda^4_{\mathbf{t}}$. The integral periods on this subfamily are hence given by inserting the diagonal $t^i = t$ in the expressions from equation (5.157) and averaging over all periods that correspond to the same type of branes. We obtain asymptotically that

$$\begin{aligned}
\Pi^0(t) &= 1 \\
\Pi^1(t) &= \frac{1}{6}\sum_{i=1}^{6}\Pi^i(t,\ldots,t) = t \\
\Pi^2(t) &= \frac{1}{6}\sum_{i=1}^{3}\Pi_{i,i+3}(t,\ldots,t) = t^2 + \frac{1}{2} + \cdots \\
\Pi_i(t) &= \frac{1}{6}\sum_{i=1}^{6}\Pi_i(t,\ldots,t) = -20t^3 - 5t + 80\frac{\zeta(3)}{(2\pi i)^3} + \cdots \\
\Pi_0(t) &= \frac{1}{6}\Pi_0(t,\ldots,t) = 5t^4 + \frac{5}{2}t^2 - 80\frac{\zeta(3)}{(2\pi i)^3}t + \frac{1}{16} + \cdots \ .
\end{aligned} \tag{5.183}$$

5.5.3 Arithmetic Search for Modular Hulek-Verrill Fourfolds

Now we have collected all ingredients to analyze the modular structure of the primary horizontal middle cohomology of the Hulek-Verrill fourfolds HV^4_z. We follow the strategy as discussed in section 5.4.1.

To begin with, we compute the matrix $W(z)$ which is needed to perform the efficient inversion of the period matrix $E(z)$. Due to the representation

$$W^{ij}(z) = \int_X \Theta^i\Omega(z) \wedge \Theta^j\Omega(z) \tag{5.184}$$

in terms of the generating derivatives $\mathcal{D}^i = \Theta^i$ of the primary horizontal subspace, $W(z)$ can be computed by using the Picard-Fuchs equation to derive a set of differential equations for the functions $W^{ij}(z)$ which turn out to be solved in terms of rational functions. In appendix D.1 we derive the concrete differential equations for $W^{ij}(z)$ for any family of Calabi-Yau fourfolds of Hodge type $(1,1,1,1,1)$ in terms of the functions $f_k(z)$ appearing in the Picard-Fuch operator $\mathcal{L}$. For the one-parameter subfamily HV^4_z of Hulek-Verrill fourfolds, the Picard-Fuchs operator (5.180) implies by solving equation (D.16) that

$$W^{04}(z) = -\frac{c}{\Delta} \quad , \quad \Delta = -(1-4z)(1-16z)(1-36z) \tag{5.185}$$

which fixes the additional entries of $W(z)$. The overall integration constant c depends only on the chosen normalization for $\Omega(z)$ and can be fixed by comparing the first terms of the series expansion of $W(z)$ around $z=0$ with the matrix $E(z)^T\sigma E(z)$ leading for our conventions to $c = \frac{\kappa}{(2\pi i)^4}$. Since the entries of $W(z)$ have a closed expression in terms of rational functions, this matrix can be easily inverted. One finds that $W^{-1}(z)$ becomes

$$\frac{-4z}{c}\begin{pmatrix} 3(z(768z-85)+1) & (3456z-451)+7 & 3(1632z-281)+7) & 3(4z(216z-49)+7) & \frac{\Delta}{4z} \\ (3456z-451)+7 & -(z(2304z-451)+14) & -(4z(216z-49)+7) & -\frac{\Delta}{4z} & 0 \\ 3(1632z-281)+7) & -(4z(216z-49)+7) & \frac{\Delta}{4z} & 0 & 0 \\ 3(4z(216z-49)+7) & -\frac{\Delta}{4z} & 0 & 0 & 0 \\ \frac{\Delta}{4z} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In particular, the denominator of $W^{-1}(z)$ is constant and therefore trivially given by $\mathcal{W}(z) = 1$. Hence, we expect that the entries of $U_p(z)$ have the common denominator

$$P_n(z) = \Delta(z^p)^{n-4} = (4z^p - 1)^{n-4}(16z^p - 1)^{n-4}(36z^p - 1)^{n-4} \tag{5.186}$$

in their p-adic expansion up to order $\mathcal{O}(p^n)$. By computing the periods $\varpi^i(z)$ to sufficient high order in their series expansions[97] and enforcing that

$$U_p(z) = E^{-1}(z^p)V_p(0)E(z) \tag{5.187}$$

is a rational function whose denominator is given by equation (5.186) up to p-adic precision $\mathcal{O}(p^n)$, we obtain for any prime p a set of linear equations that can be solved for the coefficients α and γ that determine the matrix $V_p(0)$. We perform this computation up to the p-adic precision[98] $\mathcal{O}(p^9)$ and primes $7 \leq p \leq 733$. For all examined primes we find

$$\alpha = 0 + \mathcal{O}(p^8) \ . \tag{5.188}$$

Note that the precision order is reduced by one as $U_p(z)$ depends on the combination αp which has been computed to p-adic accuracy $\mathcal{O}(p^9)$. The observations that α vanishes seems to be a general feature which we encounter for all examined examples (c.f. section 5.6). For Calabi-Yau threefolds a similar structure has been observed and explained [143] to be always achievable by choosing a certain set of coordinates on the complex structure moduli space. We expect a similar reason to apply for equation (5.188) to hold for general fourfolds as well. The coefficient γ is non-vanishing for all examined primes. Table 5.2 displays the results for γ up to $p \leq 103$.

p	$\gamma + \mathcal{O}(p^6)$	p	$\gamma + \mathcal{O}(p^6)$	p	$\gamma + \mathcal{O}(p^6)$	p	$\gamma + \mathcal{O}(p^6)$
7	2909	29	17091144	53	2282970	79	2722587862
11	4767	31	8281113	59	448576980	83	2262384100
13	158297	37	30272565	61	807455087	89	3172624216
17	1359627	41	62998329	67	58131285	97	5859814943
19	1609854	43	62219334	71	518484801	101	8620742110
23	1824119	47	53913209	73	1091760013	103	6279625446

Table 5.2: The coefficient γ for the one-parameter subfamily HV_z^4 of Hulek-Verrill fourfolds for the primes $7 \leq p \leq 103$. Since $U_p(z)$ depends on $p^3\gamma$, a computation of $U_p(z)$ up to order $\mathcal{O}(p^9)$ leads to a precision of $\mathcal{O}(p^6)$ for γ.

Since this analysis fixes all coefficients of $V_p(0)$, we can now proceed to compute the full matrix $U_p(z)$ for any given prime $p \geq 7$ as a formal series expansion in z. Using equations

[97]For the following investigation, we computed the power series $A_i(z)$ characterizing in the Frobenius periods up to order $N_{\max} = 6000$ using the algorithm which is derived in appendix C.

[98]Recall from section 5.4.1 that is suffices to compute $U_p(z)$ up to order $\mathcal{O}(p^4)$ for primes $p \geq 7$ in order to obtain exact results for the polynomial $R_H(\mathrm{HV}_z^4, T)$.

(5.110) and (5.111) we can use this result to obtain the coefficients a_p, b_p and ε for the characteristic polynomial $R_H(\mathrm{HV}^4_z, T)$ of the Frobenius map for any value $z \in \mathbb{F}_p$. Recall that z needs to be replaced by its Teichmüller lift $\mathrm{Teich}_p(z)$ in the series expression for $U_p(z)$ due to the convergence property of this power series.

Performing a systematic computation of $R_H(\mathrm{HV}^4_z, T)$ for all primes $7 \leq p \leq 733$ and all values $z \in \mathbb{F}_p$, we can search for points $z \in \mathbb{C}$ of persistent factorization. A summary of all characteristic polynomials $R_H(\mathrm{HV}^4_z, T)$ for the primes $7 \leq p \leq 37$ and all $z \in \mathbb{F}_p$ for which $\mathrm{HV}^4_z/\mathbb{F}_p$ is non-singular[99] is collected in appendix E.1.

As expected from the general discussion in section 5.4.1, we can confirm that the polynomials $R_H(\mathrm{HV}^4_z, T)$ factorize into a linear factor and a remaining degree four polynomial. Moreover, already the small selection of primes given in appendix E.1 shows that for each prime, there is at least one $z \in \mathbb{F}_p$ such that $R_H(\mathrm{HV}^4_z, T)$ factorizes further into two quadratic factors and the omnipresent linear factor. Figure 5.2 presents a histogram[100] counting the number of points $z \in \mathbb{F}_p$ for each prime $7 \leq p \leq 733$ that lead to a quadratic factorization.

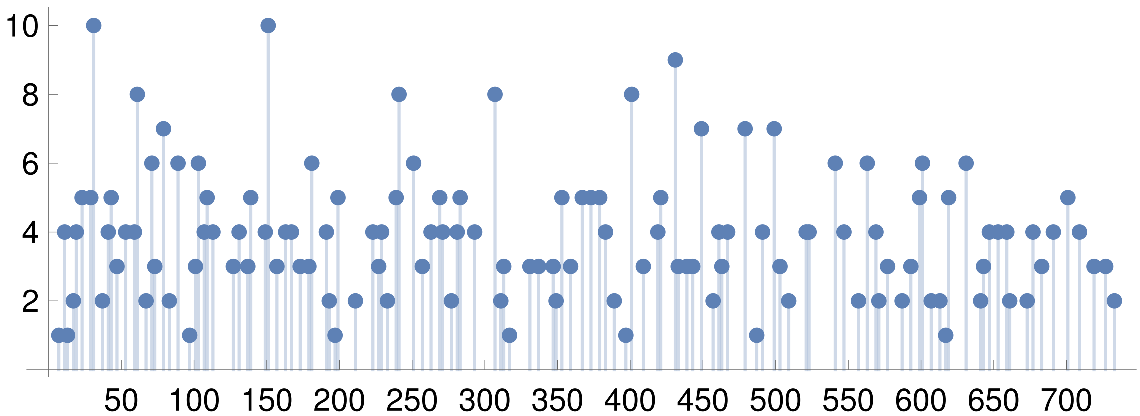

Figure 5.2: Histogram of the number of points $z \in \mathbb{F}_p$ that lead to a quadratic factorization of the characteristic polynomial $R_H(\mathrm{HV}^4_z, T)$ for all primes $7 \leq p \leq 733$. This figure is taken from [1].

This observation gives a strong indication that HV^4_z might admit a point $z \in \mathcal{M}_{C.S.}$ of persistent factorization. In addition, we summarize in table 5.3 all values $z \in \mathbb{F}_p$ for which $R_H(\mathrm{HV}^4_z, T)$ has such a quadratic factorization for the first primes $7 \leq p \leq 29$.

[99] Note that $\mathrm{HV}^4_z/\mathbb{F}_p$ is singular if $\Delta(z) \equiv 0 \mod p$.

[100] Introduced in [7], these histograms are a convenient method to decide whether a family of Calabi-Yau n-folds X_z admits a point $z \in \mathbb{M}_{C.S.}$ that leads to a persistent factorization of the characteristic polynomial $R_H(X_z, T)$.

p	z	$R_H(\mathrm{HV}^4_z, T)$
11	1	$(1+p^2T)^2(1-p^2T)^3$
11	6	$(1+p^2T)(1-p^2T)^2(1-122T+p^4T^2)$
11	8	$(1-p^2T)(1+p^2T)^2(1-62T+p^4T^2)$
11	10	$(1+p^2T)(1-p^2T)^2(1+178T+p^4T^2)$
13	1	$(1+p^2T)(1-p^2T)^2(1+310T+p^4T^2)$
17	1	$(1+p^2T)(1+70T+p^4T^2)(1+14pT+p^4T^2)$
17	15	$(1+p^2T)(1-p^2T)^2(1+110T+p^4T^2)$
19	1	$(1+p^2T)(1-338T+p^4T^2)(1+22pT+p^4T^2)$
19	2	$(1+p^2T)(1-p^2T)^2(1+22T+p^4T^2)$
19	7	$(1-p^2T)(1+p^2T)^2(1-122T+p^4T^2)$
19	17	$(1+p^2T)(1-p^2T)^2(1+262T+p^4T^2)$
23	1	$(1-p^2T)(1+1010T+p^4T^2)(1-34pT+p^4T^2)$
23	4	$(1-p^2T)(1+p^2T)^2(1-410T+p^4T^2)$
23	5	$(1+p^2T)(1-p^2T)^2(1+830T+p^4T^2)$
23	12	$(1+p^2T)(1-962T+p^4T^2)(1-10pT+p^4T^2)$
29	1	$(1-p^2T)(1+p^2T)^2(1-1178T+p^4T^2)$
29	6	$(1-p^2T)(1+p^2T)^2(1-482T+p^4T^2)$
29	11	$(1-p^2T)(1+482T+p^4T^2)(1+38pT+p^4T^2)$
29	24	$(1+p^2T)(1-p^2T)^2(1-698T+p^4T^2)$

Table 5.3: All points $z \in \mathbb{F}_p$ for $7 \leq p \leq 29$ for which $R_H(\mathrm{HV}^4_z, T)$ factorizes into two quadratic and one linear factor.

To confirm that the family HV^4_z has indeed a point of persistent factorization, we need to analyze whether for each prime p one of the factorization points originates from a reduction of the same complex number $z \in \mathbb{C}$ modulo p. If so, this point z describes a point of persistent factorization. Our strategy to identify such a point in the complex structure moduli space follows in analogy to [7]. This procedure is given by an iterative scan over algebraic numbers $z \in \bar{\mathbb{Q}} \subset \mathbb{C}$ in increasing order of its defining polynomial. For the k^{th} iteration step, we search for a set of integers $(a_0, \ldots, a_k) \in \mathbb{Z}^{k+1}$ such that for each prime p, there exists a point $z_p \in \mathbb{F}_p$ obeying

$$a_k z_p^k + \cdots + a_1 z_p + a_0 \equiv 0 \mod p \tag{5.189}$$

for which $R_H(\mathrm{HV}^4_z, T)$ has a quadratic factorization or HV^4_z is singular. If we find such a set of integers for a given iteration step, the algebraic roots $z \in \bar{\mathbb{Q}}$ of the resulting polynomial $p(z) = a_k z^k + \cdots + a_1 z + a_0$ describe a point z of persistent factorization in the complex structure moduli space of HV^4_z. In addition to equation (5.189), we need to include the condition that $z \in \bar{\mathbb{Q}}$ has a well-defined representative in $\mathbb{F}_p$. Hence, for a given set of integers (a_i), we ignore the prime p in equation (5.189), if the roots of $p(z)$ have no representative in $\mathbb{F}_p$.

The first iteration step $k = 1$ includes all rational numbers $z = -\frac{a_0}{a_1} \in \mathbb{Q}$. A rational number has a representative in $\mathbb{F}_p$ if and only if

$$a_1 \not\equiv 0 \mod p \; . \tag{5.190}$$

For the practical evaluation, we scan in each iteration step over all polynomials with $|a_i| \leq 1000$. For the one-parameter subfamily HV^4_z of Hulek-Verrill fourfolds the first iteration step $k = 1$ leads to four solutions that are given by

$$z = \frac{1}{36} \quad , \quad z = \frac{1}{16} \quad , \quad z = \frac{1}{4} \quad , \quad z = 1 \; . \tag{5.191}$$

Note that the former three points correspond exactly to the three actual conifold points that are described by the discriminant locus given by equation (5.165) whereas $z = 1$ is a point on the complex structure moduli space for which the geometry of HV^4_z is smooth. From the data of table 5.3 we can verify that $R_H(\mathrm{HV}^4_1, T)$ factorizes for each prime. Hence, we can conclude that $z = 1$ is indeed a point of persistent factorization.

After identifying this first point $z = 1$ of persistent factorization, it is of interest to analyze whether the family HV^4_z admits additional modular points. By extending the search for modular points beyond rational points using the next iteration step of the algebraic search, we can rule out additional modular points in any quadratic field extension of $\mathbb{Q}$. Moreover, we observe that after excluding those factorization points from the histogram that correspond to representatives of $z = 1$, the remaining points of factorization appear only for sporadic primes.

This argument does not completely rule out the possibility that there are further modular points. Assume that $\tilde{z} \in \mathbb{C}$ is another modular point on the complex structure moduli space of HV^4_z. Since modularity implies a factorization of $R_H(\mathrm{HV}^4_{\tilde{z}}, T)$ for each prime p for which $\tilde{z}$ has a representative in $\mathbb{F}_p$ and $\mathrm{HV}^4_{\tilde{z}}$ is non-singular, the existence of an additional modular point $\tilde{z}$ would imply that the sporadic primes of factorization correspond to such primes that admit a representative of $\tilde{z}$ in $\mathbb{F}_p$ whereas for all other primes p', the point $\tilde{z}$ does not have a representative in $\mathbb{F}_{p'}$.

Following the arguments of [142], we argue in the following that the possibility of an additional modular point is very unlikely. Let us first consider the case of an additional rational modular point $\tilde{z} \in \mathbb{Q}$. As we have discussed above, $\tilde{z} = -\frac{a_0}{a_1}$ does not have a representative in $\mathbb{F}_p$ if and only if $a_1 \equiv 0 \mod p$. Hence, a_1 has to be divisible by all primes p for which $\tilde{z}$ does not have a representative. Using the data from table 5.3, this argument leads to a lower bound on the size of a_1. Increasing the data set to that of figure 5.2, we can even increase this bound to

$$|a_1| > 10^{29} \; . \tag{5.192}$$

Assuming that $\tilde{z}$ is a “reasonable“ rational number, this bound on its denominator rules out that such an additional rational modular point exists. Adapting the same argument for additional modular points in a quadratic field extension of $\mathbb{Q}$ leads to a similar consistency bound that is expressible by $\Delta(\tilde{z}) \equiv 0 \mod p$ for all primes p without representative for $\tilde{z}$. Again, applying the data of figure 5.2, we obtain a lower bound on Δ given by

$$|\Delta(\tilde{z})| > 10^{50} \ . \tag{5.193}$$

Following [142], this argument is extendible to all iteration steps of the algebraic search for points of persistent factorization. Hence, we can conclude that there are no further algebraic modular points $\tilde{z} \in \bar{\mathbb{Q}}$ on the complex structure moduli space of HV^4_z.

5.5.4 Verification of the Modularity of HV^4_1

Let us now turn to the verification of the modularity conjecture for the manifold HV^4_1 which we have identified as a manifold that admits a persistent factorization of the polynomial $R_H(\mathrm{HV}^4_1, T)$. From table 5.3 and the additional data for larger primes, we can read off the coefficients of the quadratic factors of $R_H(\mathrm{HV}^4_1, T)$. Note that the coefficients are of fixed order in the corresponding prime p such that we can write the polynomial as

$$R_H(\mathrm{HV}^4_1, T) = (1 \pm p^2 T)(1 - a_p T + p^4 T^2)(1 - b_p p T + p^4 T^2) \tag{5.194}$$

where the coefficients a_p and b_p are collected in table 5.4.

p	11	13	17	19	23	29	31	37	41	43	47	53
a_p	0	-310	-70	338	-1010	1178	1322	2570	-338	1010	4130	-1730
b_p	0	0	-14	-22	34	0	2	0	0	0	-14	-86

Table 5.4: The coefficients a_p and b_p of the quadratic factors of $R_H(\mathrm{HV}^4_1, T)$ for all primes $11 \leq p \leq 53$. One should note that the prime $p = 7$ is not included to this table as it admits a singular structure of $\mathrm{HV}^4_1/\mathbb{F}_p$.

The modularity conjecture states that the coefficients a_p should correspond to the Fourier coefficients of a weight-three modular form in $M_3(\Gamma_0)$. The “L-Functions and modular Forms Database“ (LMFDB) [190] provides a large database of more than a million modular forms, including those which are Hecke eigenforms[101] in $M_k(\Gamma_0(N))$. Assuming that the coefficients a_p from table 5.4 correspond to a modular form of weight three, the Fourier expansion

$$f(\tau) = \sum_p a_p q^p \quad , \quad q = e^{2\pi i \tau} \tag{5.195}$$

can be identified with a unique modular form that is listed in [190] with the label 15.3.d.b. This identification holds for all primes $11 \leq p \leq 733$ that have been tested. The existence of this unique modular form whose Fourier coefficients coincide with the coefficients of the degree-two factor of $R_H(\mathrm{HV}^4_1, T)$ is in full accordance with the modularity conjecture.

[101] To be more specific, we are interested in so-called *Hecke Newforms*.

Thus, it gives a strong consistency check that HV_1^4 is indeed a modular Calabi-Yau fourfold and hence admits a split of its primary horizontal cohomology.

In order to verify this behavior, we compute the integral periods $\Pi(z)$ at the modular point $z = 1$ and check explicitly, whether the integral primary cohomology has a two-dimensional sublattice. To compute the periods at $z = 1$, we need to analytically continue the integral periods that have been computed in the vicinity of the large complex structure point $z = 0$. Since there are conifold singularities at the real values for which $\Delta(z) = 0$, the analytic continuation is performed in the complex plane. Apart from the singular points, the periods are holomorphic, hence by choosing expansion points with overlapping areas of convergence, the solutions at $z = 0$ can be continued to the point $z = 1$. One should note that this continuation is unique only up to a monodromy transformation depending on whether the path of continuation passes a conifold singularity on different sides.

By choosing the path to be in the upper half plane, $\mathrm{Im}(z) \geq 0$, we compute the integral period vector $\Pi(1)$ at $z = 1$ within a numerical precision of 100 digits. By computing the ratios of several entries of the period vector, it can be verified that its real part has a rational structure of the form

$$\mathrm{Re}(\Pi(1)) = C \begin{pmatrix} 1 \\ 1 \\ 8 \\ -10 \\ 2 \end{pmatrix} \tag{5.196}$$

with the irrational coefficient

$$C = -0.15616717244022644275494677677108499738008955066963 3... \,. \tag{5.197}$$

However, this observation seems to fail for its imaginary part $\mathrm{Im}(\Pi(1))$. Within the given numerical precision of 100 digits we can exclude that the ratio of the imaginary parts of any two entries of $\Pi(1)$ is a rational number with a height larger than 10^{50}. The height of a rational number $q = \frac{r}{s}$ is given by $\mathrm{ht}(q) = (|r| + |s|)$ and gives a good measure whether a numerical result is likely to represent a rational number. Since the ratios of the imaginary parts of the numerically computed periods are that large, we conclude that $\mathrm{Im}(\Pi(1))$ does not have a rational structure like its corresponding real part.

If HV_1^4 would be an attractive fourfold, a two-dimensional integral sublattice Λ that is of Hodge type $(4,0) + (0,4)$ would split off the primary horizontal integral cohomology $H^4(\mathrm{HV}_1^4, \mathbb{Z})$. Hence, $\Omega(1)$ and $\overline{\Omega(1)}$ need to be given in terms of integral vectors. The previous observation that $\mathrm{Im}(\Pi(1))$ cannot be written in terms of a rational vector shows that $H^4(\mathrm{HV}_1^4, \mathbb{Z})$ does not admit a two-dimensional sublattice of the required Hodge type.

Let us now consider the second possible case. If HV_1^4 is of attractive K3 type, the primary horizontal intgeral cohomology is required to have a two-dimensional sublattice that is of

Hodge type $(3,1)+(1,3)$. To verify the existence of such a sublattice, we compute the Kähler covariant derivative

$$\nabla_z \Pi(z) = (\partial_z - \partial_z K_{C.S.}(z))\Pi(z) \tag{5.198}$$

of the integral period vector evaluated at $z=1$. Up to the same numerical precision of 100 digits we find

$$\nabla_z \Pi(1) = A \begin{pmatrix} 12 \\ 5 \\ 20 \\ -50 \\ 10 \end{pmatrix} + iB \begin{pmatrix} 0 \\ 5 \\ 24 \\ -80 \\ 20 \end{pmatrix} \tag{5.199}$$

for two real constants A, B whose values are given by

$$\begin{aligned} A &= -0.011783350037859430679992523295962605881917251716994... \\ B &= -0.003042447897277474018298818684307367137030444536707... \, . \end{aligned} \tag{5.200}$$

Note that ∇_z was defined such that $\nabla_z \Omega(z) \in H^{3,1}(\mathrm{HV}^4_z, \mathbb{C})$. Hence, we find within the given numerical precision that

$$\Lambda_{\mathrm{AK3}} := \langle \nabla_z \Omega(1), \overline{\nabla_z \omega(1)} \rangle = \left(H^{3,1}(\mathrm{HV}^4_1, \mathbb{C}) \oplus H^{1,3}(\mathrm{HV}^4_1, \mathbb{C}) \right) \cap H^4(\mathrm{HV}^4_1, \mathbb{Z}) \tag{5.201}$$

defines a two-dimensional sublattice of the integral middle cohomology $H^4(\mathrm{HV}^4_1, \mathbb{Z})$ that is of definite Hodge type $(3,1)+(1,3)$. From this analysis, we can verify that HV^4_1 is a modular Calabi-Yau fourfold which is of attractive K3 type. Moreover, the expression (5.199) gives the explicit generators of the two-dimensional sublattice in terms of the integral periods.

We should note that although the modularity conjecture detected an integral sublattice of Hodge type $(3,1)+(1,3)$, this Calabi-Yau fourfold admits a non-trivial flux compactification of M-theory. Since $H^4_H(\mathrm{HV}^4_1, \mathbb{C})$ is given by the five-dimensional subspace which is spanned by $\Omega(1)$ and its first four derivatives, we have that the remaining three-dimensional lattice Σ of the decomposition

$$H^4_H(\mathrm{HV}^4_1, \mathbb{Z}) = \Lambda_{\mathrm{AK3}} \oplus \Sigma \tag{5.202}$$

is by definition of Hodge type $(4,0)+(2,2)+(0,4)$. Hence, it provides three independent four-forms that are in accordance with the supersymmetric flux vacuum constraint (4.70) of M-theory compactifications. In particular, equation (5.196) realizes one of these consistent integral flux vectors as the real part of the period vector $\Pi(1)$.

We can perform yet another strong consistency check for the modularity of HV^4_1 that is completely independent of the arithmetic analysis we presented above. Following [6, 165, 191, 192], the integral periods corresponding to the two-dimensional sublattice

Λ_{AK3} which splits off the primary horizontal cohomology can be written as a rational linear combination of two critical values of the L-function that corresponds to the modular form $f(\tau)$ of equation (5.195). This observation goes back to Deligne's conjecture [193]. In the following, we state Deligne's conjecture for the purpose of our analysis for the middle cohomology of Calabi-Yau fourfolds. Without going into any details, the following discussion gives a brief introduction to Deligne's conjecture and states its implication on the periods of the modular fourfold HV_1^4. The reader interested in details of this derivation, including a proper definition of critical motives, is referred to [1].

Deligne's conjecture is formulated in the language of *motives*. For our case it is essentially based on the observation that we can define two different rational structures on the middle cohomology, for instance by choosing $H^4_{\text{sing}}(X,\mathbb{Q})$ and $H^4_{dR}(X,\mathbb{Q})$ whose complexifications can be identified isomorphically. In practical computations, this *comparison isomorphism*

$$I_\infty : H^4_{\text{sing}}(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \to H^4_{\text{dR}}(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \tag{5.203}$$

is obtained by the change of basis matrix between the Frobenius basis and the integral basis of the period vector

$$I_\infty = (2\pi i)^4 M\, E(z)^T \tag{5.204}$$

where M denotes the change of basis matrix which is defined in analogy to the three-dimensional matrix given by equation (3.81). The idea of Deligne is now to associate two *periods*[102] $c^\pm$ to this map which are given by restricting I_∞ to the eigenspaces $M^\pm_{\text{sing}}$ of the complex conjugation map on $H^4_{\text{sing}}(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$ with eigenvalues ± 1 and setting

$$c^\pm := \det(I_\infty : M^\pm_{\text{sing}} \to M^\pm_{\text{dR}}) \ . \tag{5.205}$$

Here, $M^\pm_{\text{dR}}$ denotes the image of $M^\pm_{\text{sing}}$ under I_∞. It is a non-trivial but true statement that these restricted maps are again isomorphisms and hence, the determinant is well-defined.

Now, Deligne's conjecture states that if $H^4(X,\mathbb{Q})$ defines a so-called *critical motive*, then the quotient

$$\frac{L(M^\pm_{\text{sing},0})}{c^+ M^\pm_{\text{sing}}} \tag{5.206}$$

is a rational number. Here, L denotes the motivic L-function of the middle cohomology that can be computed from the polynomials $R_4(X,T)$ appearing in the local zeta function of X.

For the given two-dimensional subspace of $H^4_H(\text{HV}_1^4,\mathbb{Q})$ which is spanned by $\nabla_z \Pi(1)$ and its complex conjugate, Deligne's conjecture implies[103] that the coefficients A and B are

[102]The name period is introduced by Kontsevich and Zagier [194]. In particular, it is independent of the periods Π that characterize the holomorphic four-form Ω.

[103]It should be noted that the motive which is given by this two-dimensional subspace is not critical, hence Deligne's conjecture does not seem to be applicable. However, by performing a suitable *Tate twist* $\Lambda_{\text{AK3}}(\mathbb{Q}) \mapsto \Lambda_{\text{AK3}}(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}(m)$, the motive can be arranged to be critical [1]. In practice, this operation shifts the evaluation point for the motivic L-functions from 0 to $m-1$.

given by

$$A = \frac{r}{s}\frac{L_3(1)}{(2\pi i)^2} \quad , \quad iB = \frac{r'}{s'}\frac{L_3(2)}{(2\pi i)^3} \tag{5.207}$$

for two rational numbers $\frac{r}{s}$ and $\frac{r'}{s'}$. The corresponding motivic L-function values can be read off from the LMFDB database [190] to be

$$\begin{aligned} L_3(1) &= 0.54271934916842485520176378379613163082128362217397375229323... \\ L_3(2) &= 0.88045982535822981044968910894132568513898932226249847773441... \ . \end{aligned} \tag{5.208}$$

Inserting the numerical values for A and B from equation (5.200), we can deduce the rational prefactors such that

$$A = -4\frac{L_3(1)}{(2\pi i)^2} \quad , \quad iB = 60\frac{L_3(2)}{(2\pi i)^3} \ . \tag{5.209}$$

Hence, we can conclude that

$$\nabla_z \Pi(1) = -4\frac{L_3(1)}{(2\pi i)^2}\begin{pmatrix} 12 \\ 5 \\ 20 \\ -50 \\ 10 \end{pmatrix} + 60\frac{L_3(2)}{(2\pi i)^3}\begin{pmatrix} 0 \\ 5 \\ 24 \\ -80 \\ 20 \end{pmatrix} \tag{5.210}$$

which is in full agreement with Deligne's conjecture for modular Calabi-Yau manifolds. Since Deligne's conjecture gives a correspondence between the weight-three modular form $f(\tau)$ and the two-dimensional integral sublattice Λ_{AK3} that is completely independent of the arithmetic analysis, it provides an excellent and very strong a posteriori consistency check for the arithmetic method, which we discussed in sections 5.1 to 5.4.

One should note that the modularity for Hulek-Verrill n-folds has been discussed already in [195, 196]. Here, the authors present a different approach to check whether a given geometry is modular and to compute the corresponding Hecke eigenform $f(\tau)$. This independent analysis, which uses the notion of periods and quasiperiods of modular forms, leads to the same result of HV^4_1 being modular with the corresponding modular form 15.3.d.b. which gives yet another consistency check.

As a final remark, we note that, in accordance with Hodge-like conjectures[104], it is possible to assign rational four-cycles to HV^4_1 that are responsible for the splitting of the primary horizontal cohomology. To that end, we recall that each Hulek-Verrill fourfold $\mathrm{HV}^4_{\mathbf{z}}$ has a birational model in terms of the elliptically fibred product

$$\mathrm{HV}^4_{(z^1,z^2,z^3,z^4,z^5,z^6)} \xrightarrow{\text{rat.}} \mathcal{E}_{(\lambda,z^1,z^2)} \times_{\mathbb{P}^1} \mathrm{K3}_{\left(\frac{\lambda}{z^3},\frac{z^4}{z^3},\frac{z^5}{z^3},\frac{z^6}{z^3}\right)} \ . \tag{5.211}$$

[104] Roughly speaking, these generalizations of the Hodge conjecture [173] imply that any rational Hodge substructure $V \subset H^k(X,\mathbb{Q})$ has a geometrical origin in terms of a corresponding set of dual rational k-cycles. For additional details on these generalized Hodge-like conjectures we refer to [7, 141, 196]

For the one-parameter subfamily HV^4_z, the corresponding birational model is given by

$$\mathrm{HV}^4_z \xrightarrow{\text{rat.}} \mathcal{E}_{(\lambda,z,z)} \times_{\mathbb{P}^1} \mathrm{K3}_{\left(\frac{\lambda}{z},1,1,1\right)} \ . \tag{5.212}$$

The elliptic curve of this decomposition enjoys the long Weierstraß form

$$\mathcal{E}_{(\lambda,z,z)} = \{(x,y) \in \mathbb{C} \mid y^2 = P_{\lambda,z}(x)\} \tag{5.213}$$

with the cubic polynomial

$$P_{\lambda,z}(x) = x\left(x + \frac{2(\lambda^2+1+2z^2) - (\lambda+1+2z)^2}{8}\right)^2 - \frac{\prod_{\delta_i=\pm1}(\lambda - (1+\delta_1\sqrt{z}+\delta_2\sqrt{z}))}{64}x \ . \tag{5.214}$$

Transforming this expression into the isomorphic short Weierstraß form[105]

$$y^2 = x^3 + a_{\lambda,z}x + b_{\lambda,z} \ , \tag{5.215}$$

the discriminant of the elliptic curve $\mathcal{E}_{(\lambda,z,z)}$ can be computed to be

$$\Delta(\lambda,z) := -4a^3_{\lambda,z} - 27b^2_{\lambda,z} = (\lambda-1)^2\lambda^2z^4(\lambda^2 - 2\lambda(4z+1) + (1-4z)^2) \ . \tag{5.216}$$

Thus, we conlcude that $\mathcal{E}_{(\lambda,z,z)}$ is singular if and only if

$$\lambda \in \{0, 1, \lambda_+(z), \lambda_-(z), \infty\} \quad \text{for } \lambda_\pm(z) := 1 + 4z \pm 4\sqrt{z} \ . \tag{5.217}$$

According to the work of Hulek and Verrill [8], the modularity of certain Hulek-Verrill threefolds can be explained by the existence of a singular elliptic fibre in the birational model (5.146) for each modular point on the complex structure moduli space. If this singular fibre is of Kodeira type[106] I_n with $n > 1$, it decomposes into more than one irreducible $\mathbb{P}^1$ components. In this case, the two rational three-cycles of Hodge type $(2,1)$ and $(1,2)$ which give rise to the splitting of the middle cohomology can be identified to be the real one-cycle of the smooth elliptic fibre and one of the irreducible components of the singular fibre [96].

Applying a similar analysis to the present example of the Hulek-Verrill fourfolds HV^4_z, we note from the structure of $\Delta(\lambda,z)$ that for generic z the singular fibres for $\lambda = 0$ and $\lambda = 1$ are of Kodeira type I_2 whereas $\lambda = \lambda_\pm$ provide a singular fibre of type I_1. However,

[105]Note that for any elliptic curve $\mathcal{E}$ which is defined by the solution space of the equation $y^2 = P(x)$ with $P(x) = x^3 + a_2x^2 + a_4x + a_6$, the isomorphism $x \mapsto x + \frac{1}{3}a_2$ transforms $P(x)$ into the desired short Weierstraß form with $a = a_4 - \frac{1}{3}a_2^2$ and $b = a_6 - \frac{1}{3}a_2a_4 + \frac{2}{27}a_2^3$.

[106]The Kodeira classification [197] of elliptic fibres provides a distinction of different types of singularities with respect to the number of irreducible $\mathbb{P}^1$ components and intersection points thereof. I_0 defines a smooth elliptic fibre whereas in general I_n describes a fibre that consists of n irreducible components which intersect in n distinct points.

there exist special non-generic values for z such that the fibres $\lambda = \lambda_\pm$ coincide with either $\lambda = 0$ or $\lambda = 1$ which enhances the singular structure to I_3. Solving $\lambda_\pm \in \{0, 1\}$ yields

$$\begin{aligned} z = 0 &\Leftrightarrow \lambda_+ = 1 \\ z = \frac{1}{4} &\Leftrightarrow \lambda_- = 0 \\ z = 1 &\Leftrightarrow \lambda_- = 1 \ . \end{aligned} \tag{5.218}$$

We recall that HV^4_z is singular for $z = 0$ and $z = \frac{1}{4}$ whereas HV^4_z is a smooth geometry for $z = 1$ which is exactly that Calabi-Yau fourfold which has been identified to be modular. Thus, the enhancement of the singular structure of the elliptic fibre $\mathcal{E}_{\lambda,1,1}$ for $\lambda = 1$ coincides exactly with the observation of HV^4_1 being modular. In order to identify the associated rational cycles of Hodge type $(3, 1)$ and $(1, 3)$, we need to investigate the second fibre of (5.211) which is the K3 surface $\mathrm{K3}_{1,1,1,1}$. If this fibre were to be modular and in particular attractive, meaning that the $(2, 0) + (0, 2)$ part of its primary horizontal middle cohomology splits of, the two-cycles that correspond to this Hodge substructure would serve as possible rational cycles to geometrically explain the sublattice Λ_{AK3} of the fourfold HV^4_1 similarly to the threefold discussions in [8].

The one-parameter family of K3 surfaces $\mathrm{K3}_{z,z,z,z} =: \mathrm{K3}_z$ which realizes the K3 surface of interest for $z = 1$ has been studied extensively in [198]. In this work it is shown that each member of this family has Picard number $\rho = 19$ which implies that the primary horizontal middle cohomology $H^2_H(\mathrm{K3}_z, \mathbb{Q})$ is three-dimensional and generated by $\Omega(z)$ and its first two derivatives. Following the analysis of section 5.5.2, the fundamental period of $\Omega(z)$ reads

$$\varpi^0(z) = \sum_{k=0}^{\infty} \sum_{|\mathbf{j}|=k} \begin{pmatrix} k \\ \mathbf{j} \end{pmatrix}^2 z^k \tag{5.219}$$

for $\mathbf{j} \in \mathbb{N}^4$. The corresponding degree-three Picard-Fuchs operator is deduced to be

$$\mathcal{L} = \Theta^3 + 64z^2(\Theta + 1)^3 - 2z(2\Theta + 1)(5\Theta(\Theta + 1) + 2) \ . \tag{5.220}$$

In order to check whether $\mathrm{K3}_1$ is modular, it is necessary to compute the characteristic polynomial $R_H(\mathrm{K3}_1, T)$ which realizes the horizontal factor of the polynomial $R_2(\mathrm{K3}_1, T)$ that appears in the local zeta function for the K3 surface. For each member of this family, the polynomial $R_H(\mathrm{K3}_z, T)$ is of degree three and moreover, in analogy to the fourfold discussion in section 5.4.1 the Weil conjectures imply that

$$R_H(\mathrm{K3}_z, T) = (1 \pm pT)(1 - b_p T + p^2 T^2) \ . \tag{5.221}$$

Thus, a persistent quadratic factorization of $R_H(\mathrm{K3}_z, T)$ is generic and not special to a point z for which the middle cohomology furnishes a two-dimensional split. However, if the coefficients b_p can be identified with the Fourier coefficients of a weight-three modular form $f(q)$, we would obtain a modular K3 surface. Performing an analogous analysis as for fourfolds of primary horizontal Hodge type $(1, 1, 1, 1, 1)$ which was presented in section

5.4.1, it is possible to compute this factor of the zeta function for the family of K3 surfaces for any given complex structure modulus z. For $z = 1$, this computation yields that the coefficients b_p of this polynomial just coincide with the coefficients b_p of the polynomials $R_H(\mathrm{HV}_1^4, T)$ which are collected in table 5.4. Thus, we can identify the same modular form with the label 15.3.d.b in the list [190].

Finally, to verify that this modular point corresponds to an attractive K3 surface, we construct the explicit two-dimensional sublattice

$$\Gamma_{\text{att}} \subset \left(H^{2,0}(\mathrm{K3}_1, \mathbb{C}) \oplus H^{0,2}(\mathrm{K3}_1, \mathbb{C})\right) \cap H^2(\mathrm{K3}_1, \mathbb{Z}) \tag{5.222}$$

by computing the integral period vector $\Pi(z)$ at $z = 1$. In analogy to the structure of the fourfold periods at the AK3 point, we expect the period vector to be given in terms of critical L-function values corresponding to the modular form $f(q)$. Performing an analytic continuation of the integral periods around $z = 0$ to the point $z = 1$, we find up to 100 digits of numerical accuracy that

$$\Pi(1) = \frac{1}{4}\left(\frac{L_3(2)}{(2\pi i)^2}A - \frac{L_3(1)}{2\pi i}B\right) \tag{5.223}$$

for the integral vectors

$$A = \begin{pmatrix} 24 \\ 24 \\ 7 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 24 \\ 8 \\ 1 \end{pmatrix} \tag{5.224}$$

which hence span the lattice Γ_{att}. This observation gives a strong argument that $\mathrm{K3}_{1,1,1,1}$ is indeed an attractive K3 surface and hence admits a two-dimensional integral sublattice of $H^2(\mathrm{K3}_{1,1,1,1}, \mathbb{Z})$ of Hodge type $(2,0)+(0,2)$. Following the threefold discussion in [8], it is hence possible to construct two rational four-cycles that are responsible for the splitting of the middle cohomology of HV_1^4 by tate twisting the two-cycles of $\mathrm{K3}_{1,1,1,1}$ that are responsible for Γ_{att} using one of the $\mathbb{P}^1$ components of the singular elliptic fibre of Kodeira type I_3.

5.6 Non-modular Families of Calabi-Yau Fourfolds

In section 5.5 we have extensively studied the arithmetic properties of the one-parameter family of Hulek-Verrill fourfolds HV_z^4 and identified that HV_1^4 is a modular fourfold. Since the procedure as presented in section 5.4 is rather algorithmic, it can be easily applied to further examples with one complex structure modulus. In the following, we investigate three additional families of Calabi-Yau fourfolds and search for points of persistent factorizations of $R_H(X_z, T)$ on the corresponding complex structure moduli space. Since the computational steps are essentially repetitive, we will abbreviate this discussion slightly by mainly stating the important quantities and the results from the arithmetic computation.

5.6.1 The Mirror Family of the Complete Intersection $\mathbb{P}^7[2,2,4]$

The complete intersection $\mathbb{P}^7[2,2,4]$ of two quadratics and a quartic in the projective space $\mathbb{P}^7$ defines a family of Calabi-Yau fourfolds with Hodge numbers

$$h^{3,1} = 263 \quad , \quad h^{2,2} = 1100 \quad , \quad h^{2,1} = 0 \quad , \quad h^{1,1} = 1 \tag{5.225}$$

implying that its mirror family which we denote by $\mathbb{P}^7[2,2,4]^\vee$ serves as a suitable one-parameter family that is smooth outside its conifold locus. Choosing z to be a local coordinate on the complex structure moduli space of $\mathbb{P}^7[2,2,4]^\vee$ with $z = 0$ being the large complex structure point, the holomorphic period $\varpi^0(z)$ can be read of directly to be

$$\varpi^0(z) = \sum_{k=0}^{\infty} a_n z^n \quad , \quad a_n = \frac{((2n)!)^2(4n)!}{(n!)^8} \ . \tag{5.226}$$

The coefficients a_n obey the recursion relation

$$a_n = \frac{32}{n^5}(2n-1)^3(4n-1)(4n-3)a_{n-1} \tag{5.227}$$

leading to the Picard-Fuchs operator[107]

$$\mathcal{L} = \Theta^5 - 32z(2\Theta+1)^3(4\Theta+1)(4\Theta+3) \tag{5.228}$$

with discriminant

$$\Delta(z) = 1 - 2^{12}z \ . \tag{5.229}$$

Since this operator is of degree five, we can deduce that $\mathbb{P}^7[2,2,4]^\vee$ is of primary horizontal Hodge type $(1,1,1,1,1)$ hence we can continue in analogy to the one-parameter subfamily of Hulek-Verrill fourfolds.

The matrix $W(z)$ is computed again using the algorithm described in appendix D.1 leading to

$$W^{-1}(z) = \begin{pmatrix} -192z & -896z & -3840z & -6144z & \Delta(z) \\ -896z & 1792z & 2048z & -\Delta(z) & 0 \\ -3840z & 2048z & \Delta(z) & 0 & 0 \\ -6144z & -\Delta(z) & 0 & 0 & 0 \\ \Delta(z) & 0 & 0 & 0 & 0 \end{pmatrix}$$

which allows to compute $E^{-1}(z)$. Moreover, we find again that the denominator of $W^{-1}(z)$ is trivially given by $\mathcal{W}(z) = 1$ and therefore, the expected denominator of $U_p(z)$ up to p-adic precision $\mathcal{O}(p^n)$ reads

$$P_n(z) = \Delta(z^p)^{n-4} = (1 - 2^{12}z^p)^{n-4} \ . \tag{5.230}$$

[107]This Picard-Fuchs operator could also be obtained by an explicit Griffiths-Dwork reduction.

With this result at hand, we can now continue to compute the coefficients α and γ that characterize the matrix $V_p(0)$. Computing $U_p(z)$ up to p-adic accuracry $\mathcal{O}(p^7)$, we find as for the one-parameter subfamily of Hulek-Verrill fourfolds for all primes $7 \leq p \leq 103$ that

$$\alpha = 0 + \mathcal{O}(p^6) \ . \tag{5.231}$$

Moreover, the coefficients γ are again non-vanishing and prime dependent. Table 5.5 summarizes the values for γ for all primes $7 \leq p \leq 103$ up to p-adic order $\mathcal{O}(p^4)$.

p	$\gamma + \mathcal{O}(p^4)$	p	$\gamma + \mathcal{O}(p^4)$	p	$\gamma + \mathcal{O}(p^4)$	p	$\gamma + \mathcal{O}(p^4)$
7	1481	29	653995	53	3282110	79	14759236
11	10561	31	746306	59	8446428	83	7348764
13	15053	37	924799	61	5962873	89	23593156
17	3266	41	1679522	67	17160797	97	76590606
19	92220	43	581369	71	13278982	101	38600178
23	185490	47	3642167	73	194924	103	64347008

Table 5.5: The coefficient γ for the one-parameter family $\mathbb{P}^7[2,2,4]^\vee$ computed up to order $\mathcal{O}(p^4)$ for the primes $7 \leq p \leq 103$.

From this data we can compute the characteristic polynomials $R_H((\mathbb{P}^7[2,2,4]^\vee)_z, T)$ for this range of primes whenever the reduced manifold over $\mathbb{F}_p$ is non-singular. Instead of presenting a long list of these polynomials at this point, we summarize the results by visualizing the number of points that lead to a quadratic factorization per prime in the histogram given by figure 5.3. For completeness, the a list of polynomials for the first primes $7 \leq p \leq 37$ is collected in appendix E.2.

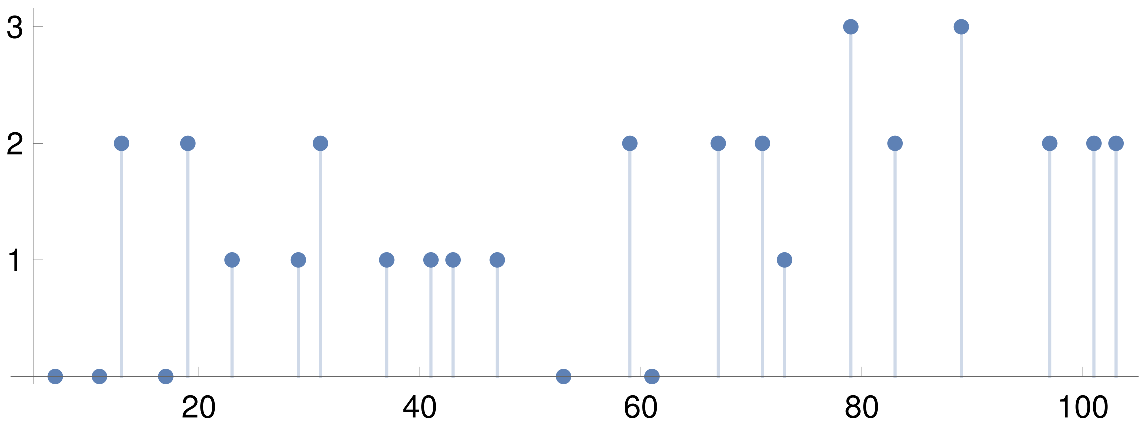

Figure 5.3: Histogram of the number of points $z \in \mathbb{F}_p$ that lead to a quadratic factorization of the characteristic polynomial $R_H((\mathbb{P}^7[2,2,4]^\vee), T)$ for all primes $7 \leq p \leq 107$.

As discussed in section 5.5.3, a point of persistent factorization leads to at least one quadratic factorization for almost each prime. However, figure 5.3 shows that there exist primes for which no factorization occurs at all. This observation indicates that the complex structure moduli space of $\mathbb{P}^7[2,2,4]^{\vee}_z$ has no point of persistent factorization and hence this family of fourfolds does not have a modular member.

We support this fact by increasing the range for p up to $p \leq 317$ as presented in figure 5.4 which verifies that there appear frequently additional primes for which not quadratic factorization of $R_H((\mathbb{P}^7[2,2,4]^{\vee})_z, T)$ is observed.

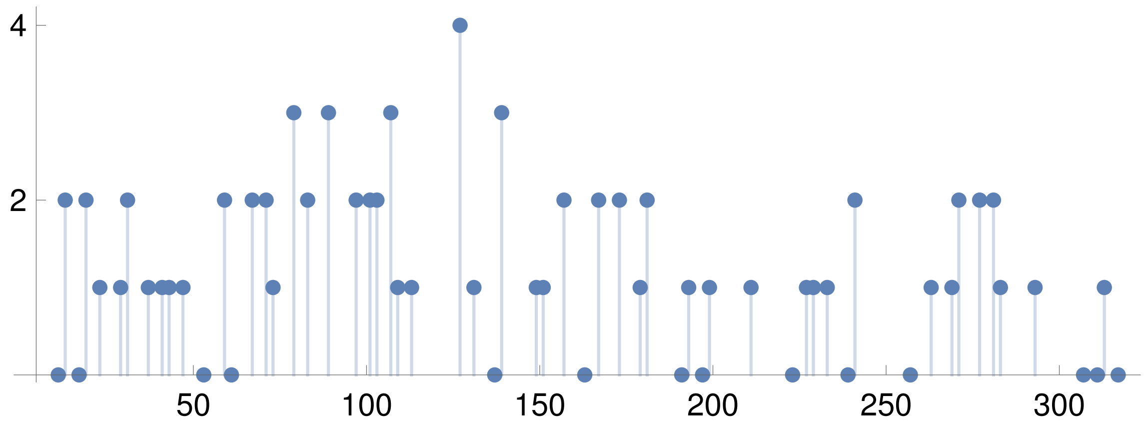

Figure 5.4: Histogram of the number of points $z \in \mathbb{F}_p$ that lead to a quadratic factorization of the characteristic polynomial $R_H((\mathbb{P}^7[2,2,4]^{\vee})_z, T)$ for all primes $7 \leq p \leq 317$. This figure is taken from [1].

Performing a similar analysis as in section 5.5.3 we derive a lower bound on the height of possible rational or algebraic points z on the complex structure moduli space. Using the given data, and taking into account only those primes p' for which no factorization is observed, we find that the denominator of z is at least of order

$$|a_0| > \prod_{p'} p' \sim 10^{29} \ . \tag{5.232}$$

For any point in a quadratic field extension $\mathbb{Q}(a)$ with $a^2 \in \mathbb{Q}$ the equivalent bound on the discriminant $\Delta(z)$ is given by $\Delta(z) > 10^{50}$. Again, we can conclude from this discussion that there cannot be a point of persistent factorization within $\mathbb{Q}$ and any quadratic field extension. Further iterating this analysis might rule out also field extensions of higher order.

5.6.2 The Mirror of the Complete Intersection $X_{1,4} \subset Gr(2,5)$

Beside complete intersections of projective varieties, a second standard construction of Calabi-Yau fourfolds is given by (the mirrors of) complete intersections of hypersurfaces in Grassmannian varieties. The following example realizes a one-parameter family that is of primary horizontal Hodge type $(1,1,2,1,1)$. Hence, this family gives an opportunity to demonstrate the extended algorithm given by section 5.4.2.

The family of interest is given by the intersection of a degree-one and a degree-four hypersurface $X_{1,4}$ within the Grassmannian Gr$(2,5)$. Following [73], the corresponding manifolds are Calabi-Yau fourfolds with non-trivial Hodge numbers

$$h^{3,1} = 299 \quad , \quad h^{2,2} = 1244 \quad , \quad h^{2,1} = 0 \quad , \quad h^{1,1} = 1 \; . \tag{5.233}$$

Again, this structure shows that the mirror family $X_{1,4}^\vee$ realizes a family of fourfolds that have one complex structure modulus. The Picard-Fuchs operator for this family

$$\begin{aligned} \mathcal{L} = \Theta^6 - \Theta^5 - 8z(2\Theta+1)(4\Theta+1)(4\Theta+3)(11\Theta^2+11\Theta+3)\Theta \\ - 64z^2(2\Theta+1)(2\Theta+3)(4\Theta+1)(4\Theta+5)(4\Theta+7) \end{aligned} \tag{5.234}$$

is of degree six and hence shows that $X_{1,4}^\vee$ is indeed a one-parameter family of primary horizontal Hodge type $(1,1,2,1,1)$. Moreover, the discriminant locus of this operator is given by

$$\Delta(z) = 1 - 2816z - 65536z^2 \; . \tag{5.235}$$

The main difference to the previous examples is that the period vector contains a second additional holomorphic period. The coefficients of the two holomorphic periods obey the recursion relation

$$\begin{aligned} a_n^{1/2} = & \frac{8}{n^5(n-1)} \big(2(4n-1)(4n-2)(4n-3)(4n-5)(4n-6)(4n-7)a_{n-2} \\ & + (4n-1)(4n-2)(4n-3)(4n-4)(11n^2-11n+3)a_{n-1}\big) \; . \end{aligned} \tag{5.236}$$

As initial values for the two holomorphic solutions we choose $a_0^1 = 1$ for the fundamental period whereas $a_0^2 = 0$ and $a_1^2 = 1$ are chosen for the second holomorphic solution.

The strategy to compute the characteristic polynomials $R_H((X_{1,4}^\vee)_z, T)$ is essentially analogous to that of the previous examples. For the matrix $W(z)$ we find a similar set of differential equations as in the former case. The explicit relations are summarized in appendix D.2 and lead for $W^{-1}(z)$ to the result

$$W^{-1}(z) = M(z) \tag{5.237}$$

where the entries of the symmetric matrix $M(z)$ are given by

$$
\begin{array}{llll}
M^{00}(z) = 9 + 5040z(1 + 144z) & , & M^{01}(z) = 99 + 72z(679 + 83072z) \\
M^{02}(z) = 419 + 16z(11149 + 1074688z) & , & M^{03}(z) = 866 + 64z(4999 + 333312z) \\
M^{04}(z) = 881 + 256z(1139 + 45824z) & , & M^{05}(z) = \frac{(1+288z)(-1+256z(11+256z)}{8z} \\
M^{11}(z) = 9(121 + 16z(3221 + 341568z)) & , & M^{12}(z) = 4609 + 32z(51399 + 4412416z) \\
M^{13}(z) = 9525 + 64z(44501 + 2729472z) & , & M^{14}(z) = 80(11 + 512z)(11 + 2336z) \\
M^{15}(z) = \frac{(11+2336z)(-1+256z(11+256z))}{8z} & , & M^{22}(z) = \frac{2}{9}(87785 + 128z(197669 + 14255872z)) \\
M^{23}(z) = \frac{2}{9}(433 + 37376z)(419 + 60416z) & , & M^{24}(z) = \frac{80}{9}(11 + 512z)(419 + 60416z) \\
M^{25}(z) = \frac{(419+60416z)(-1+256z(11+256z))}{72z} & , & M^{33}(z) = \frac{4}{9}(433 + 37376z)^2 \\
M^{34}(z) = \frac{160}{9}(11 + 512z)(433 + 37376z) & , & M^{35}(z) = \frac{(433+37376z)(-1+256z(11+256z))}{36z} \\
M^{44}(z) = \frac{6400}{9}(11 + 512z)^2 & , & M^{45}(z) = \frac{10(11+512z)(-1+256z(11+256z))}{9z} \\
M^{45}(z) = \frac{(1-256z(11+256z))^2}{576z^2} & . &
\end{array}
$$

Note that for this example the (maximal) denominator of $W^{-1}(z)$ is non-trivial but reads $\mathcal{W}(z) = z^2$. Hence, the denominator of the matrix $U_p(z)$ is expected to be

$$
P_n(z) = \mathcal{W}^p(z)\Delta(z^p)^{n-4} = z^{2p}(1 - 2816z^p - 65536z^{2p})^{n-4} \tag{5.238}
$$

within the p-adic precision $\mathcal{O}(p^n)$.

The intersection matrix σ contains an additional free parameter σ_{55} that depends on the internal structure of the $(2,2)$-part of the period vector. Since we have computed the matrix $W(z)$ by other means, we can use the expression

$$
W(z) = E^T(z)\sigma E(z) \tag{5.239}
$$

to determine this unknown coefficient. For the case given, we find that the intersection matrix σ takes the form

$$
\sigma = \frac{20}{(2\pi i)^4} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 576 \end{pmatrix} . \tag{5.240}
$$

Again, we compute the coefficients α, γ and ε by requiring that $U_p(z)$ takes the desired rational form with denominator $P_n(z)$ given by equation (5.238). For all primes in the range $7 \leq p \leq 103$ we find

$$
\alpha = 0 + \mathcal{O}(p^6) \quad , \quad \varepsilon = 0 + \mathcal{O}(p^7) \tag{5.241}
$$

whereas γ is again non-vanishing. The values for γ computed up to p-adic precision $\mathcal{O}(p^4)$ are listed in table 5.6.

p	$\gamma + \mathcal{O}(p^4)$	p	$\gamma + \mathcal{O}(p^4)$	p	$\gamma + \mathcal{O}(p^4)$	p	$\gamma + \mathcal{O}(p^4)$
7	2358	29	304795	53	4323681	79	26512660
11	10901	31	838034	59	7742559	83	6736367
13	25699	37	66832	61	6619787	89	42541140
17	16914	41	2010522	67	3975910	97	25943415
19	84535	43	1957422	71	7937120	101	87413697
23	30112	47	2932013	73	19110841	103	21467797

Table 5.6: The coefficient γ for the one-parameter family $X_{1,4}^\vee$ computed up to order $\mathcal{O}(p^4)$ for the primes $7 \leq p \leq 103$.

Computing the characteristic polynomials $R_H((X_{1,4})_z, T)$ as discussed in section 5.4.2, we find that $R_H((X_{1,4})_z, T)$ factorizes frequently into a quadratic term and a four-dimensional remainder according to

$$R_H((X_{1,4})_z, T) = (1 - p^2 T)(1 + p^2 T)\tilde{R}(T) \ . \tag{5.242}$$

This special quadratic factor is not a consequence of a split in the Hodge structure but rather an obstruction of the Weil conjectures. The explicit polynomials for $7 \leq p \leq 37$ are listed in appendix E.3. In order to spot a point of persistent factorization, we search for points $z \in \mathbb{F}_p$ that admit an additional quadratic factorization. Figure 5.5 presents a histogram of the number of such additional factorizations for all primes $7 \leq p \leq 103$.

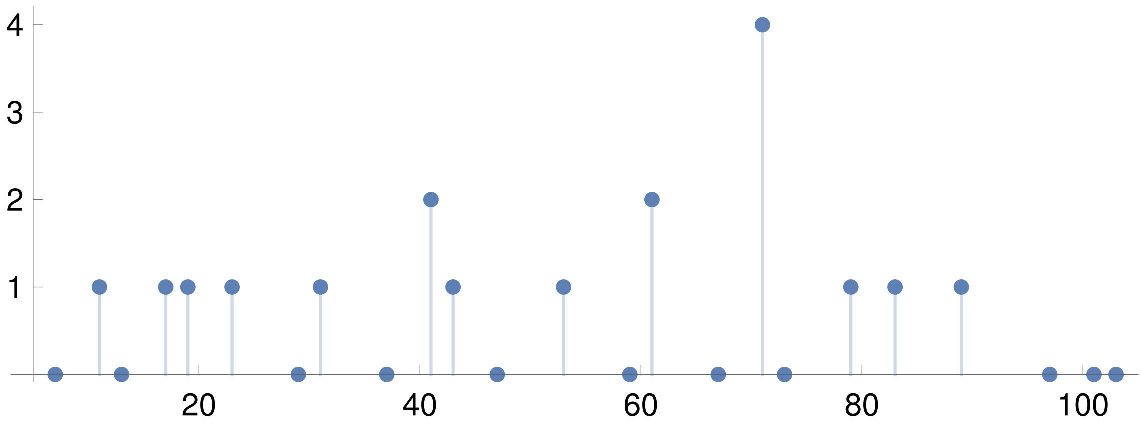

Figure 5.5: Histogram of the number of points $z \in \mathbb{F}_p$ that lead to a quadratic factorization of the characteristic polynomial $R_H((X_{1,4}^\vee), T)$ for all primes $7 \leq p \leq 103$.

This result shows that the existence of a point of persistent factorization is highly unlikely since about half of all considered primes do not show any further quadratic factorization.

5.6.3 The Mirror of the Family of Sextic Fourfolds $\mathbb{P}^5[6]$

As the last example within this discussion, we investigate the mirror family of the family of sextic fourfolds $\mathbb{P}^5[6]$. Since this family is the direct extension of the family of quintic threefolds $\mathbb{P}^4[5]$ to the four-dimensional setting, it is one of the most studied families of Calabi-Yau fourfolds . The Hodge numbers of $\mathbb{P}^5[6]$ are given by [95]

$$h^{3,1} = 426 \quad , \quad h^{2,2} = 1752 \quad , \quad h^{2,1} = 0 \quad , \quad h^{1,1} = 1 \; , \tag{5.243}$$

hence the mirror $\mathbb{P}^5[6]^\vee$ is again a family that depends on one complex structure modulus. In analogy to the mirror quintic, the fundamental period of $\mathbb{P}^5[6]^\vee$ reads

$$\varpi^0(z) = \sum_{n=0}^{\infty} a_n z^n \quad , \quad a_n = \frac{(6n)!}{(n!)^6} \tag{5.244}$$

which implies a recursion relation

$$a_n = \frac{1}{n^5} 6(6n-1)(6n-2)(6n-3)(6n-4)(6n-5) \tag{5.245}$$

and hence gives the Picard-Fuchs operator

$$\mathcal{L} = \Theta^5 - 6z(6\Theta+1)(6\Theta+2)(6\Theta+3)(6\Theta+4)(6\Theta+5) \; . \tag{5.246}$$

The discriminant locus of this operator is given by

$$\Delta(z) = (1 - 6^6 z) \; . \tag{5.247}$$

From the Picard-Fuchs operator we can read off that $\mathbb{P}^5[6]^\vee$ is again of primary horizontal Hodge type $(1,1,1,1,1)$. Using the algorithm of appendix D.1 we compute the matrix $W(z)$ whose inverse reads

$$W^{-1}(z) = \begin{pmatrix} -1440z & -8424z & -40176z & -69984z & \Delta(z) \\ -8424z & 16848z & 23328z & -\Delta(z) & 0 \\ -40176z & 23328z & \Delta(z) & 0 & 0 \\ -69984z & -\Delta(z) & 0 & 0 & 0 \\ \Delta(z) & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.248}$$

and has again a trivial denominator $\mathcal{W}(z) = 1$. Thus the denominator of the rational form of $U_p(z)$ is given by

$$P_n(z) = \Delta(z^p)^{n-4} = (1 - 6^6 z^p)^{n-4} \; . \tag{5.249}$$

As previously, we compute the coefficients α and γ for all primes $7 \leq p \leq 103$. Again, α vanishes for all considered primes

$$\alpha = 0 + \mathcal{O}(p^6) \tag{5.250}$$

whereas the values for γ are listed in table 5.7.

p	$\gamma + \mathcal{O}(p^4)$	p	$\gamma + \mathcal{O}(p^4)$	p	$\gamma + \mathcal{O}(p^4)$	p	$\gamma + \mathcal{O}(p^4)$
7	518	29	198223	53	2997420	79	17081051
11	2741	31	483604	59	400693	83	21433895
13	880	37	42269	61	4699692	89	26985211
17	23446	41	2543805	67	18146383	97	2066065
19	8333	43	3120160	71	9083403	101	60553652
23	121251	47	456985	73	19500689	103	37610932

Table 5.7: The coefficient γ for the one-parameter family $\mathbb{P}^5[6]^\vee$ computed up to order $\mathcal{O}(p^4)$ for the primes $7 \leq p \leq 103$.

Before we move on to discuss the factorization bahavior of the characteristic polynomial $R_H((\mathbb{P}^5[6]^\vee)_z, T)$, it should be noted that $\mathbb{P}^5[6]$ has a well-known modular point [199] which is identified to be the Fermat point of the moduli space. In the chosen coordinates, this point would correspond to $z = \infty$ and hence cannot be reached by our arithmetic analysis. Nevertheless it is of interest, whether the family of mirror sextics has additional members that are modular Calabi-Yau fourfolds. Hence, we continue as for the previous examples by computing the polynomials $R_H((\mathbb{P}^5[6]^\vee)_z, T)$ for all $z \in \mathbb{F}_p$ and for all primes $7 \leq p \leq 103$ and provide the histogram of the number of factorizations per prime in figure 5.6. The corresponding results for the characteristic polynomials for $7 \leq p \leq 37$ are listed in appendix E.4.

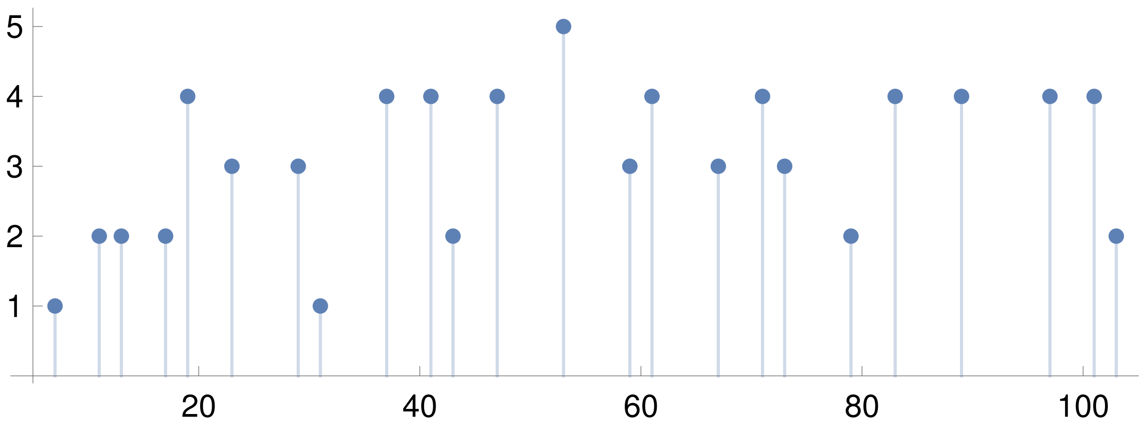

Figure 5.6: Histogram of the number of points $z \in \mathbb{F}_p$ that lead to a quadratic factorization of the characteristic polynomial $R_H((\mathbb{P}^5[6]^\vee), T)$ for all primes $7 \leq p \leq 103$.

We observe that each prime p in the considered range has at least one point $z \in \mathbb{F}_p$ for which $R_H((\mathbb{P}^5[6]^\vee), T)$ factorizes into two quadratic and one linear factor which may indicate the possibility of the existence of a point $z \in \bar{\mathbb{Q}}$ that is of persistent factorization.

However, if we try to reconstruct this point by the iterative procedure described in section 5.5.3, neither a consistent rational point nor a consistent point in a quadratic field extension of $\mathbb{Q}$ has been identified if we search for a corresponding defining polynomial (5.189) with coefficients a_i with $|a_i| \leq 1000$.

To conclude this discussion, we can neither confirm nor rule out whether $\mathbb{P}^5[6]^\vee$ has additional modular members beside the Fermat sextic which is obtained for $z = \infty$. The factorization histogram 5.6 suggests that such a point of persistent factorization on the complex structure moduli space could exist. It might either be that this point is an element of a field extension of $\mathbb{Q}$ that is of higher degree than two or that the chosen interval for the search of the defining polynomial needs to be extended. In practice, the algorithm we provide to search for points of persistent factorization cannot proof the non-existence of a modular point in this case, as the finite computation power always enforces to truncate the search for algebraic numbers at a finite degree.

Chapter 6

A Calabi-Yau-to-Curve Correspondence

Now we turn to a second application of Calabi-Yau geometries in the context of mathematical physics which arose in the last decades. Beside their importance for supersymmetric string compactifications, Calabi-Yau geometries or to be more precise, the residue integrals defining the periods $\varpi(z)$ of Calabi-Yau n-folds can be identified with certain multi-loop Feynman integrals [200–204]. For some of these integrals, there exists an equivalent representation in terms of linear combinations of the periods corresponding to the g holomorphic one-forms of a genus-g curve [205, 206, 206, 207]. In the following chapter, we show that this observation is not coincidental but can be formulated via an explicit correspondence. Following [3], we focus on a special class of Feynman integrals which have a representation in terms of the periods of a family of Calabi-Yau threefolds X with $h^{2,1}$ complex structure moduli.

This "Calabi-Yau-to-curve correspondence", which we derive in section 6.4, is given as a bijective map between certain subsets of the complex structure moduli space of families of Calabi-Yau threefolds and the moduli space of stable genus-$(h^{2,1}+1)$ curves. This map is based on the identification of suitable second intermediate Jacobians of the Calabi-Yau threefolds with the first intermediate Jacobian of the genus-g curves. After providing a brief overview on the relation between Calabi-Yau geometries and Feynman integrals in section 6.1, we start the discussion of this correspondence by introducing the moduli space of stable genus-g curves and the definition of their first intermediate Jacobians. Then we turn to the Calabi-Yau side by introducing the corresponding geometric objects which are the second intermediate Jacobians. An identification of the Jacobians on both sides leads to a bijection which maps a Calabi-Yau threefold uniquely to a stable genus-g curve. Finally, in section 6.5, we demonstrate the correspondence for the explicit example of the four-loop equal mass banana integral whose maximal cut is described by the periods of the one-parameter family of $\mathbb{Z}_5$-quotients of Hulek-Verrill threefolds $\mathrm{HV}^3/\mathbb{Z}_5$ which have been introduced already in section 5.5.

6.1 Calabi-Yau Geometries from Feynman Integrals

In order to achieve a high precision for the computation of scattering amplitudes in perturbative quantum field theory, it is necessary to develop efficient tools for solving multi-loop Feynman intergrals. Intense investigations of the analytical structure of these integrals have led to the success that many multi-loop Feynman integrals can be solved analytically in terms of polylogarithms and generalizations thereof. However, starting already at two-loop order, first intergrals appear, for which no description in terms of these "special functions" is possible. Instead, tools from algebraic geometry have been used to identify such intergrals with periods of elliptic curves and for higher loops even as certain linear combinations of the periods of hyperelliptic curves and Calabi-Yau geometries. The following discussion is meant to give a brief overview on this relation between geometry and Feynman integrals. For a comprehensive introduction to the high precision computation of Feynman integrals using techniques from algebraic geometry, the reader is referred to review articles like [208–210].

In the following, we discuss the general scenario that we are interested in the solution of an arbitrary Feynman integral I of any well-defined perturbative quantum field theory in d spacetime dimensions. First, we note that for any field content of the spectrum, such a Feynman integral can be reduced to a sum of *scalar* Feynman integrals that are multiplied with a suitable tensor structure [211]. Such a *tensor decomposition* can always be achieved by introducing suitable projection operators. Assuming that the Feynman integral depends on n external fields with momenta $\mathbf{p}_i$ and has in addition ℓ loops with loop momenta $\mathbf{k}_i$, the general structure of the remaining scalar Feynman integrals is schematically given in momentum space by

$$I = \int \prod_{i=1}^{\ell} \frac{d^d k_i}{(2\pi)^d} \frac{\mathcal{P}(p_i, k_i)}{D_1^{\nu_1} \cdots D_\rho^{\nu_\rho}} \tag{6.1}$$

where the D_i denote the internal (scalar) propagators, $\nu_i \in \mathbb{N}_0$ and $\mathcal{P}$ is a polynomial that can depend on all possible (Lorentz-invariant) scalar products of the internal and external momenta. Treating the propagators as generating terms of this expression, one may note that ρ scalar products of momenta can be expressed in terms of the internal propagators. Generically, the number of Lorentz-invariant scalars is larger than the number of different propagators, hence one is left with κ additional irreducible scalar products S_j. Thus, the polynomial $\mathcal{P}$ can be expressed as a polynomial $\tilde{\mathcal{P}}$ in terms of the propagators D_i and the irreducible scalar products S_j. By expanding this polynomial $\tilde{\mathcal{P}}$, the Feynman integral I decomposes into a finite sum of terms that are given up to combinatorical factors by

$$I_{\nu_1,\ldots,\nu_\rho,b_i,\ldots,b_\kappa} := \int \prod_{i=1}^{\ell} \frac{d^d k_i}{(2\pi)^d} \frac{S_1^{b_1} \cdots S_\kappa^{b_\kappa}}{D_1^{\nu_1} \cdots D_\rho^{\nu_\rho}} \tag{6.2}$$

where $\nu_i \in \mathbb{Z}$ defines the power to which the propagator D_i contributes and the $b_j \in \mathbb{N}_0$ give the corresponding power of the irreducible scalar product S_j. By such a decomposition, any Feynman integral is decomposable into a finite number of integrals that are of the type given by equation (6.2). The goal is thus, to compute all types of integrals $I_{\nu_1,\ldots,\nu_\rho,b_i,\ldots,b_\kappa}$ that appear in the decomposition of a given Feynman integral I.

Two great advantages of this method should be emphasized at this point. On the one hand, these integrals serve as bulding blocks for many different Feynman integrals. It suffices to compute them once and then build up several multi-loop Feynman integrals out of them. On the other hand, the seemingly infinite number of building blocks for given numbers ρ and κ of propagators and irreducible scalar products are not independent but span a finite dimensional vector space as we will discuss in section 6.1.1. Any basis of this vector space consists of so-called *Master integrals* which turn out to be the solutions of a coupled system of differential equations [212]. Hence, very schematically, the computation of any Feynman integral reduces to the (still non-trivial) quest of solving finite systems of differential equations.

6.1.1 Integration-by-parts Identities and Master Integrals

Integration-by-parts Identities (IBPs) [213, 214] provide a powerful tool to reduce for a fixed number of external momenta and loops the infinite number of Feynman Integrals $I_{\nu_1,\ldots,\nu_\rho,b_i,\ldots,b_\kappa}$ with $\nu_i \in \mathbb{Z}$ and $b_j \in \mathbb{N}_0$ to a finite basis of *Master Integrals.* In order to derive the IBPs, let us consider a loop-integral

$$I = \int \prod_{i=1}^{\ell} \frac{d^d k_i}{(2\pi)^d} f(\mathbf{k}, \mathbf{p}) \tag{6.3}$$

of a generic function $f(\mathbf{k}, \mathbf{p})$ that depends on the external momenta $\mathbf{p}$ and the internal loop momenta $\mathbf{k}$. Now, by performing a shift of one internal momentum k_i according to

$$k_i^\mu \mapsto k_i^m u + \alpha v^\mu \tag{6.4}$$

by either any internal momentum $v = k_j$ or any of the external momenta $v = p_j$, we obtain

$$f(\mathbf{k}, \mathbf{p}) \mapsto f(\mathbf{k}, \mathbf{p}) + \alpha v^\mu \partial_{k_i^\mu} f(\mathbf{k}, \mathbf{p}) + \mathcal{O}(\alpha^2) \ . \tag{6.5}$$

Here, $\alpha \in \mathbb{R}$ serves as an arbitrary expansion parameter. By imposing that I is invariant under any such shift for arbitrary $\alpha \in \mathbb{R}$, we find that the correction terms in each power of α need to vanish separately. In particular, the first order term yields

$$\int \prod_{i=1}^{\ell} \frac{d^d k_i}{(2\pi)^d} v^\mu \partial_{k_i^\mu} f(\mathbf{k}, \mathbf{p}) = 0 \tag{6.6}$$

for all chosen values of the vector field v^u. If we now insert the integrand of equation (6.2) for the function f, this result gives the *Integration-by-parts identities* for Feynman Integrals

$$0 = \int \prod_{i=1}^{\ell} \frac{d^d k_i}{(2\pi)^d} v^\mu \partial_{k_i^\mu} \frac{S_1^{b_1} \cdots S_\kappa^{b_\kappa}}{D_1^{\nu_1} \cdots D_\rho^{\nu_\rho}} \tag{6.7}$$

which translate into linear combinations

$$\sum_{\nu_i,b_j} C_{\nu_i,b_j} I_{\nu_1,\ldots,\nu_\rho,b_i,\ldots,b_\kappa} = 0 \tag{6.8}$$

for finitely non-vanishing real coefficients C_{ν_i,b_j}. Since these IBPs exist for any member of the family of integrals, they can be used to systematically reduce each integral to a finite number of independent integrals which are called the *Master integrals* of this family.

To solve these remaining master integrals of a given family $I_{\nu_1,\ldots,\nu_\rho,b_i,\ldots,b_\kappa}$ of Feynman integrals, it is convenient instead of performing the integration explicitly to derive a set of differential equations for them. If one considers the derivative of any master integral with respect to an external momentum p_i^μ, the result will be given in terms of a linear combination of Feynman integrals that belong to the same family with coefficients that are polynomial in p_i^μ. This is true, since a derivative of the integrand leads to a sum of similar expressions but with shifted powers of the contributing propagators and scalar products. If one reduces these Feynman integrals again using the IBPs, the result gives a set of first order differential equations for the master integrals $m_j(\mathbf{p})$

$$\partial_{p_i^\mu} m_j(\mathbf{p}) = (\tilde{A}_{i,\mu})_{jk}(\mathbf{x}) m_k(\mathbf{p}) \ . \tag{6.9}$$

Since the Feynman integrals do not depend on all vector-valued external momenta independently but only on the independent scalar kinematic variables, which we denote in the following by $\mathbf{x}$, this set of differential equations reduces to

$$\partial_{x_i} \mathbf{m}_j(\mathbf{x}) = (A_i)_{jk}(\mathbf{x}) m_k(\mathbf{x}) \tag{6.10}$$

for certain matrices $A_i(\mathbf{x})$.

The key observation to construct the solution to this differential equation is given by the conjecture that it is always possible to find a set of master integrals such that in dimensional regularization the matrices A_i depend only by an overall scale on the dimensional regularization parameter ε

$$A_i(\mathbf{x}) = \varepsilon^N B_i(\mathbf{x}) \tag{6.11}$$

with $N \in \mathbb{N}$ and $B_i(\mathbf{x})$ being independent of ε The master integrals in this *ε-factorized* form [215] enjoy a solution in terms of a path-ordered exponential

$$\mathbf{m} = \mathcal{P}\left(e^{\varepsilon^N \int_\gamma B(\mathbf{x}) d\ell}\right) \mathbf{m}_0 \tag{6.12}$$

for γ being a path on the phase space of the kinematic variables. By this expression, the master integrals can be solved systematically order by order in ε which gives a series expansion in ε for the master integrals in terms of iterated integrals.

6.1.2 Example: The One-loop Self Energy Integral

In the following, let us demonstrate the rather abstract discussion from the previous section with the example of the one-loop contribution to the self-energy of a scalar field with mass m whose Feynman diagram is presented in figure 6.1. The consideration of this example is based on ref. [210, 216].

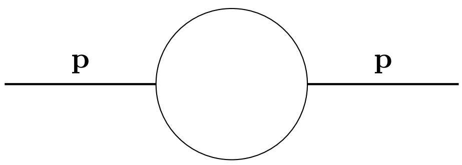

Figure 6.1: The Feynman diagram corresponding to the one-loop contribution of the self-energy of a scalar field with loop momentum $\mathbf{k}$ and external momentum $\mathbf{p}$.

Up to a normalization constant, the corresponding Feynman Integral reads

$$I = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2 + i\delta)((k+p)^2 - m^2 + i\delta)} \; . \tag{6.13}$$

Using the notation introduced in equation (6.2), we find that this integral belongs to the family I_{ν_1,ν_2} with the two internal propagators given by

$$D_1(\mathbf{k},\mathbf{p}) = k^2 - m^2 + i\delta \quad , \quad D_2(\mathbf{k},\mathbf{p}) = (k+p)^2 - m^2 + i\delta \; . \tag{6.14}$$

In particular, we find that the self-energy integral is given by the member $I_{1,1}$ of this family.

In order to deduce a set of master integrals for this family of Feynman integrals, we observe that these integrals obey the symmetry

$$I_{\nu_1,\nu_2} = I_{\nu_2,\nu_1} \tag{6.15}$$

which is realized by changing the internal momentum from $\mathbf{k}$ to $\mathbf{k} - \mathbf{p}$. Moreover, we can compute the IBPs for this family of integrals. Since there is only one internal momentum and one external momentum, we find two independent IBPs that arise from choosing the shift vector v^μ to be either p^μ or k^μ. The corresponding relations read [216]

$$\begin{aligned} 0 &= \mathbf{p}^2(\nu_2 p^2 I_{\nu_1,\nu_2+1} - \nu_1 I_{\nu_1+1,\nu_2}) + \nu_1 I_{\nu_1+1,\nu_2-1} - \nu_2 I_{\nu_1-1,\nu_2+1} + (\nu_2 - \nu_1) I_{\nu_1,\nu_2} \\ 0 &= 2\nu_1 m^2 I_{\nu_1+1\nu_2} + \nu_2(\mathbf{p}^2 + 2m^2) I_{\nu_1,\nu_2+1} - \nu_2 I_{\nu_1-1,\nu_2+1} + (D - 2\nu_1 - \nu_2) I_{\nu_1,\nu_2} \end{aligned} \tag{6.16}$$

and hence provide a set of recursion relations that determine together with the symmetry property any member of the family I_{ν_1,ν_2} in terms of the two initial values $I_{1,0}$ and $I_{1,1}$.

For example, we find

$$\begin{aligned} I_{2,0} &= -\frac{D-2}{2m^2} I_{1,0} \\ I_{2,1} &= -\frac{1}{\mathbf{p}^2+4m^2}\left(\frac{D-2}{2m^2} I_{1,0} + (D-3) I_{1,1}\right) \end{aligned} \tag{6.17}$$

and similar recursive expressions for any additional Feynman integral I_{ν_1,ν_2} with $\nu_1+\nu_2 \geq 4$.

The remaining task is hence to compute the two master integrals $I_{1,0}$ and $I_{1,1}$ of this family. Lorentz invariance implies that any two-point Feynman integral can depend only on the kinematic variable $s = \mathbf{p}^2$. In this simple situation, we have

$$\partial_s = \frac{1}{2s} p^\mu \partial_{p^\mu} \tag{6.18}$$

which allows to derive a set of differential equations for the master integrals as functions of s. From the explicit integral representations we compute

$$\begin{aligned} \partial_s I_{1,0} &= 0 \\ \partial_s I_{1,1} &= \frac{1}{2s}\left(I_{2,0} - I_{1,1}\right) - \frac{1}{2} I_{2,1} = \frac{1}{2}\left(\frac{D-3}{s+4m^2} - \frac{1}{s}\right) I_{1,1} - \frac{D-2}{s(s+4m^2)} I_{1,0} \ . \end{aligned} \tag{6.19}$$

For the last identity we have used the IBPs (6.17) to reduce the integrals $I_{2,0}$ and $I_{2,1}$ to linear combinations of the master integrals. In vector notation, this system of coupled differential equations can be combined as

$$\partial_s \mathbf{m} = A(s)\mathbf{m} \ , \quad \mathbf{m} = \begin{pmatrix} I_{1,0} \\ I_{1,1} \end{pmatrix} \tag{6.20}$$

with the connection matrix

$$A(s) = \begin{pmatrix} 0 & 0 \\ -\frac{D-2}{s(s+4m^2)} & \frac{D-3}{s+4m^2} - \frac{1}{2s} \end{pmatrix} . \tag{6.21}$$

In this example, we see that $A(s)$ is explicitly dependent on the spacetime dimension D. Thus, to obtain a solution to this differential equation in terms of an expansion in the dimensional regularization parameter which is conveniently given by $\varepsilon = (2-D)/2$ for this family of Feynman integrals, it is necessary to perform a change of basis such that $A(s)$ is given in ε-factorized form. This can be achieved by rescaling the master integrals according to [216]

$$\mathbf{m} = \frac{2-D}{2}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{s(s+4m^2)} \end{pmatrix} \mathbf{M} \ . \tag{6.22}$$

In this new basis, the differential equation simplifies to

$$\partial_s \mathbf{M} = \varepsilon B(s) \mathbf{M} \ , \quad B(s) = \begin{pmatrix} 0 & 0 \\ \frac{2}{\sqrt{s(s+4m^2)}} & \frac{1}{s+4m^2} \end{pmatrix} \tag{6.23}$$

which gives indeed an ε-factorized form that can be systematically integrated by using a path ordered exponential of $\int_\gamma B(s)ds$.

We may note that the Feynman integral $I_{1,0}$ corresponds yet to another family of Feynman integrals. Defining

$$I_\nu := \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 - m^2 + i\delta)^\nu} \tag{6.24}$$

we observe that $I_{1,0} = I_1$. Performing a similar analysis of the IBPs for this family, one finds that I_1 is its only master integral and moreover, a direct integration shows that this integral yields the value

$$I_1 = \frac{4\Gamma\left(\frac{6-D}{2}\right) m^{D-2}}{(D-2)(D-4)} \tag{6.25}$$

which is finite for $D < 3$. Thus, we can treat the differential equation (6.19) as an inhomogenous differential equation for the remaining master integral $I_{1,1}$ with a constant inhomogeneity.

This observation is very generic and can be applied whenever a family $I_{\nu_1,\dots,\nu_\rho,b_i,\dots,b_\kappa}$ has a master integral for which one of the exponents ν_i or b_i vanishes. These so-called sub sectors of the family can hence be computed first, either by direct integration or by iteratively applying the method of deriving differential equations from the IBPs before using them as inhomogenities for the differential equations of the remaining master integrals of the full family. The subset of master integrals which cannot be reduced to any sub sector is called the *top sector* of the family of Feynman integrals.

6.1.3 Relating Master Integrals to Geometry

In the previous example of the one-loop contribution to the self-energy, we have argued that the non-trivial master integral $I_{1,1}$ is the solution of an inhomogenous first order differential equation. In ε-factorized form, this differential equation is given in a so-called *dlog-form* meaning that the coefficients of the matrix $B(s)$ are derivatives of logarithms of algebraic functions. For differential equations of this type, the iterated integrals that appear in the ε-expansion of $I_{1,1}$ enjoy a solution in terms of multiple polylogarithms [217].

However, starting at two-loops, there exist Feynman integrals for which the ε-factorized form cannot be brought in such a dlog-form. In particular, the decoupling of the vector-valued differential equations for these master integrals leads to higher order differential equations for each of the master integrals in the top sector which cannot be solved in terms of multiple polylogarithms any longer. Following [218, 219], it has been argued that the general homogenous solution to these differential equations can be reconstructed from the *maximal cut* of the Feynman integral that emerges by replacing the propagators by suitable δ-functions [219]. Thus, the integration of maximal cuts of multi-loop Feynman integrals provides a first milestone in determining their full solution.

Following [219], any ℓ-loop Feynman integral that depends on n different propagators can be parametrized in terms of projective coordinates such that it becomes

$$I_{\nu_1,\ldots,\nu_\rho} = \int_\gamma \frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{F}^\omega}\Delta \tag{6.26}$$

with the so-called first and second Symanzik polynomials $\mathcal{U}$ and $\mathcal{F}$ that can be computed in terms of the propagators and Δ being defined as in equation (5.168). Moreover, the exponent ω is given by

$$\omega = \sum_{i=1}^{\rho} \nu_i - \ell\frac{D}{2} \tag{6.27}$$

and the domain of integration γ is given by

$$\gamma = \{[x_1;\ldots;x_n] \in \mathbb{P}^{n-1} \mid x_i \geq 0\}\ . \tag{6.28}$$

The corresponding maximal cut integral is characterized by the same integrand but a deformed integration contour such that

$$I_{\text{max cut}} = \int_{\mathbb{T}^n} \frac{\mathcal{U}^{\omega-\frac{D}{2}}}{\mathcal{F}^\omega}\Delta \tag{6.29}$$

with $\mathbb{T}^n \subset \mathbb{P}^{n-1}$ being an n-torus. The Symanzik polynomial $\mathcal{F}$ can be shown to be homogenous of degree $\ell-1$ with ℓ being the number of loops and hence is a special type of toric polynomials. The key insight for the relation of Feynman integrals to geometry is given by the observation that integrals of this type emerge as periods of holomorphic forms of certain projective varieties.

Historically, the first geometric identifications of this type have been made between the maximal cuts of two-loop Feynman integrals and certain periods of elliptic curves for which $\mathcal{F}$ is always of homogenous degree one [219]. Extending this observation to Feynman integrals with yet more loops, there exist two canonical generalizations of the corresponding geometry. Increasing the complex dimension of the corresponding variety, we obtain the higher dimensional analogs of elliptic curves which are in turn Calabi-Yau varieties. A second possibility to generalize the notion of elliptic curves is given by increasing the (arithmetic) genus of the curve. In this way, one obtains complex one-dimensional curves with a non-trivial topological structure leading as well to a higher dimensional middle cohomology. Recently [205, 206], Feynman integrals have been discovered whose toric representation translates into the description of periods of such higher genus hyperellitpic curves.

Among others, a very interesting example is given by the ℓ-loop banana integral which we discuss in detail in section 6.5. For any $\ell \geq 2$, the toric parametrization of this integral has been identified [180] with the period integral corresponding to the family of Hulek-Verrill Calabi-Yau $(\ell-1)$-folds whereas moreover, the four-loop equal mass banana integral can be identified in addition with the periods of a certain family of genus-two hyperelliptic curves. It is this identification of period integrals that originate from very different geometric objects, which we are interested in for the following discussion of a "Calabi-Yau-to-curve correspondence".

6.2 Stable Genus-g Curves

The geometry of Calabi-Yau threefolds and their moduli spaces has been explored extensively already in chapter 3. Thus, we focus in the following section on a brief introduction to genus-g curves which may be seen as a different type of generalization of elliptic curves. Formally, a genus-g curve is a Riemann surface, i.e. a connected, complex one-dimensional manifold, that has an arithmetic genus of $g \in \mathbb{N}$ which is smooth for all but finitely many double points [150, 220]. Hence, a genus-g curve $\mathcal{C}$ is given by the connected sum of g distinct two-tori equipped with a complex structure which is smooth on $\mathcal{C}$ except for its double points. Hence, genus-g curves can be understood as the natural generalization of elliptic curves which correspond in this notation to the case of $g = 1$. Note that moreover any hyperelliptic curve is a genus-g curve whereas the opposite statement holds true only for genus $g \leq 2$ [150].

In the following, we are interested in genus-g curves that behave well in terms of the geometric invariant theory. Such curves are called *stable* and obey the additional property that the group of automorphisms $G : \mathcal{C}_g \to \mathcal{C}_g$ is finite. Since every genus-g curve with $g \geq 2$ is by definition stable [150], we will not be concerned about this technical condition too much. For the special cases of $g = 0$ and $g = 1$, any curve can be stabilized by including a suitable number of marked points on the curve which reduces the number of automorphisms to finitely many.

6.2.1 Periods of Genus-g Curves

To each genus-g curve $\mathcal{C}_g$ we can associate its corresponding de Rham cohomology groups $H^k(\mathcal{C}_g, \mathbb{C})$. Since $\mathcal{C}_g$ is complex one-dimensional, these groups decompose into Dolbeault cohomology groups $H^{p,q}(\mathcal{C}_g, \mathbb{C})$ whose complex dimensions $h^{p,q}$ can be collected in a Hodge diamond. For a genus-g curve, the entries of the Hodge diamond are completely fixed by its genus according to figure 6.2.

$$\begin{matrix} & h^{0,0} & \\ h^{1,0} & & h^{0,1} \\ & h^{1,1} & \end{matrix} \quad = \quad \begin{matrix} & 1 & \\ g & & g \\ & 1 & \end{matrix}$$

Figure 6.2: The Hodge diamond of a genus-g curve.

We find in particular that the middle cohomology $H^1(\mathcal{C}_g, \mathbb{C})$ is even-dimensional and admits a symplectic structure. In analogy to the middle cohomology of Calabi-Yau threefolds, we define an integral basis (α_i, β^i) of $H^1(\mathcal{C}_g, \mathbb{Z})$ by choosing a canonical symplectic basis (a^i, b_i) of 1-cycles $H_1(\mathcal{C}_g, \mathbb{Z})$ obeying

$$a^i \cap b_j = \delta^i_j \quad , \quad a^i \cap a^j = b_i \cap b_j = 0 \ . \tag{6.30}$$

Moreover, $H^{1,0}(\mathcal{C}_g, \mathbb{C})$ is characterized by g independent holomorphic one-forms

$$H^{1,0}(\mathcal{C}_g, \mathbb{C}) = \langle \omega_1, \ldots, \omega_g \rangle \ . \tag{6.31}$$

The coefficients of the basis expansion of ω_k according to the cohomology basis (α_i, β^i) are given by period integrals of the type

$$\mathcal{X}^i_k = \int_{a^i} \omega_k \quad , \quad \mathcal{T}_{ik} = \int_{b_i} \omega_k \tag{6.32}$$

such that ω_k is given by

$$\omega_k = \sum_{i=1}^{g} \mathcal{X}^i_k \alpha_i - \mathcal{T}_{ik} \beta^i \ . \tag{6.33}$$

Combining the holomorphic periods to a period vector $\omega = (\omega^1, \ldots, \omega^g)^T$, it holds that

$$\omega = \left((\alpha_i, \beta^i) \Sigma \begin{pmatrix} \mathcal{X} \\ \mathcal{T} \end{pmatrix} \right) \quad , \quad \Sigma = \begin{pmatrix} 0 & \mathbb{1}_d \\ -\mathbb{1}_d & 0 \end{pmatrix} \ . \tag{6.34}$$

The matrix Σ is called the symplectic intersection pairing. Depending on the choice of cycles, we call these the A-periods and the B-periods of ω_k respectively.

As for Calabi-Yau manifolds, stable genus-g curves appear in families that are characterized by smooth deformations. Thus, by collecting these deformations in a moduli space, we find an analog fibre bundle structure for the family of genus-g curves over its moduli space as for families of Calabi-Yau manifolds. In contrast to the large landscape of distinct families of Calabi-Yau manifolds, it holds that all curves $\mathcal{C}_g$ with a fixed genus g belong to the same family. Hence, for each genus g, there exists a unique moduli space characterizing all genus-g curves. In the following we restrict the discussion on the complex structure deformations of $\mathcal{C}_g$. We denote the complex structure moduli space of all stable genus-g curves by $\bar{\mathcal{M}}_g$. If we exclude the cases of genus $g = 0$ and $g = 1$, it holds [150] that $\bar{\mathcal{M}}_g$ is of dimension

$$\dim(\bar{\mathcal{M}}_g) = 3g - 3 \ . \tag{6.35}$$

$H^{1,0}(\mathcal{C}_g, \mathbb{C})$ extends to a holomorphic vector bundle over the complex structure moduli space $\bar{\mathcal{M}}_g$. Thus, the periods turn into meromorphic functions of the complex structure moduli. As for the periods of Calabi-Yau manifolds, it is possible to derive a Picard-Fuchs differential ideal whose solutions are precisely given by the $2g^2$ periods. The existence of such a finitely generated ideal is again ensured by the finiteness of the middle cohomology.

6.2.2 First Intermediate Jacobian of Stable Genus-g Curves

The middle cohomology of a genus-g curve $\mathcal{C}_g$ is characterizable by a g-dimensional complex torus $J^1(\mathcal{C}_g)$ which is called the *first intermediate Jacobian* of $\mathcal{C}_g$. For a given complex algebraic variety X, an n^{th} intermediate Jacobian is given by the complex torus [150, 221, 222]

$$J^n(X) = \frac{H^{2n-1}(X,\mathbb{R})}{H^{2n-1}(X,\mathbb{Z})} \cong \frac{H^{2n-1}(X,\mathbb{C})/V}{H^{2n-1}(X,\mathbb{Z})} \tag{6.36}$$

where V is a real subspace of $H^{2n-1}(X,\mathbb{C})$ of half dimension such that

$$V + \overline{V} = H^{2n+1}(X,\mathbb{C}) \ . \tag{6.37}$$

The choice of V is equivalent to choosing a complex structure that turns

$$\frac{H^{2n-1}(X,\mathbb{R})}{H^{2n-1}(X,\mathbb{Z})} \tag{6.38}$$

into a complex torus. Depending on the choice of complex structure, these Jacobian varieties behave rather different.

From the Hodge diamond of $\mathcal{C}_g$ it is obvious that the only non-trivial intermediate Jacobian of a genus-g curve is given by $J^1(\mathcal{C}_g)$. Moreover, this Jacobian characterizes the middle cohomology of $\mathcal{C}_g$ and hence contains in particular the information which is encoded in the A- and B-periods. The $2g$-dimensional space $H^1(\mathcal{C}_g,\mathbb{C})$ admits two canonical choices for the vector space V to be either $H^{1,0}(\mathcal{C}_g,\mathbb{C})$ or $H^{0,1}(\mathcal{C}_g,\mathbb{C})$. Since both choices lead to isomorphic tori, we consider in the following without loss of generality the case

$$V = H^{1,0}(\mathcal{C}_g,\mathbb{C}) \ . \tag{6.39}$$

It is convenient to express the complex torus $J^1(\mathcal{C}_g)$ in terms of an integral lattice $\Lambda \subset \mathbb{C}^g$ such that

$$J^1(\mathcal{C}_g) = \frac{\mathbb{C}^g}{\Lambda} \ . \tag{6.40}$$

This lattice can be constructed in terms of the A- and B-periods by explicitly computing the action of the quotient $H^1(\mathcal{C}_g,\mathbb{C})/V$ on the basis elements (α_i,β^i) of the cohomology group. We note that this quotient can be equivalently described in terms of g equivalence relations which read

$$0 \sim \omega_k = \sum_{i=1}^{g} \alpha_i \mathcal{X}^i_k - \beta^i \mathcal{T}_{ik} \quad , \quad k = 1,\ldots,g \ . \tag{6.41}$$

Rewriting these expressions, we find that the basis elements α_i can be eliminated in favor of the β^i by

$$\alpha_i \sim \beta^j \tau_{ij} \quad , \quad \tau_{ij} = \mathcal{T}_{ik}(\mathcal{X}^{-1})^k_j \ . \tag{6.42}$$

From this exercise we can conclude that

$$H^1(\mathcal{C}_g, \mathbb{C})/V \cong \mathbb{C}^g \tag{6.43}$$

via the identification $\beta^i \mapsto e^i$ with $\{e^i\}$ being the standard basis of $\mathbb{C}^g$. Finally, the intermediate Jacobian is obtained by moreover modding out the $2g$-dimensional lattice that is given by all integral multiples of the basis elements α_i and β^i. Under the isomorphism (6.43), the latter translate to all integral multiples of the basis vectors e^i whereas the former need to be transformed into expressions of the β^i by using the equivalence relation (6.42). Thus, under the isomorphism (6.43) we find that

$$H^1(\mathcal{C}_g, \mathbb{Z}) \cong \mathbb{Z}^g + \mathbb{Z}^g \tau_{ij} \tag{6.44}$$

implying that the first intermediate Jacobian $J^1(\mathcal{C}_g)$ of any stable genus-g curve $\mathcal{C}_g$ is given by

$$J^1(\mathcal{C}_g) \cong \frac{\mathbb{C}^g}{\mathbb{Z}^g + \mathbb{Z}^g \tau_{ij}} \tag{6.45}$$

with the $(g \times g)$-matrix τ_{ij} being defined in equation (6.42).

In the following, we will discuss the most important properties of $J^1(\mathcal{C}_g)$. First, we note that τ_{ij} depends purely on the periods which are meromorphic functions of the complex structure moduli. Hence, $J^1(\mathcal{C}_g)$ furnish a family of complex tori that varies holomorphically over the complex structure moduli space. This could be seen from the very beginning, since the vector spaces $V = H^{1,0}(\mathcal{C}_g, \mathbb{C})$ extend to a holomorphic vector bundle over $\bar{\mathcal{M}}_g$. Thus, the complex structure on the family of complex tori $J^1(\mathcal{C}_g)$ coincides with that of the underlying moduli space.

Moreover, it holds true that $J^1(\mathcal{C}_g)$ is an abelian variety for all smooth stable genus-g curves. We recall [150] that a complex torus $T = \mathbb{C}^g/\Lambda$ is an abelian variety if it is embeddable into a projective space $\mathbb{P}^N$ for some $N \in \mathbb{N}$. Following [150], T has such an embedding if and only if it admits an ample line bundle. If the lattice Λ is given in terms of a matrix $\mathcal{M}$ as

$$\Lambda = \mathbb{Z}^g + \mathbb{Z}^g \mathcal{M} \tag{6.46}$$

with $\mathrm{Im}(\mathcal{M})$ being non-degenerate, T has an ample line bundle and hence is an abelian variety if and only if $\mathrm{Im}(\mathcal{M})$ is positive definite[108] [150]. Thus, in order to prove that $J^1(\mathcal{C}_g)$ is an abelian variety, it suffices to show that the matrix $\mathrm{Im}(\tau)_{ij}$ is positive definite. To that end, we define the intersection pairing of one-forms $\eta, \rho \in H^1(\mathcal{C}_g, \mathbb{C})$ by

$$\mathcal{Q}(\eta, \rho) = i \int \eta \wedge \bar{\rho} \ . \tag{6.47}$$

We observe that $\mathcal{Q}$ has signature (g, g) if the curve $\mathcal{C}_g$ is stable and smooth. The maximal subspace for which $\mathcal{Q}$ is positive definite is given by $H^{1,0}(\mathcal{C}_g, \mathbb{C})$. Moreover, if we expand

[108] Equivalently, $\mathcal{M}$ can be chosen to be negative definite. The additional sign can be absorbed in a redefinition of the lattice because $\mathbb{Z}^g = -\mathbb{Z}^g$.

any holomorphic one-form in terms of the periods ω_k, the matrix representation of this sesquilinear form $\mathcal{Q}$ becomes

$$\mathcal{Q}(v^k\omega_k, w^\ell\omega_\ell) = v^k i \int \omega_k \wedge \bar{\omega}_\ell \bar{w}^\ell = v^k \mathcal{Q}_{k\ell}\bar{w}^\ell \tag{6.48}$$

with

$$\mathcal{Q}_{k\ell} = -\int \omega_k \wedge \bar{\omega}_\ell = 2\mathrm{Im}(\tau_{k\ell}) \ . \tag{6.49}$$

Thus, we conclude that $\mathrm{Im}(\tau_{ij})$ is a positive definite matrix whenever $\mathcal{C}_g$ is a stable and smooth curve and hence, $J^1(\mathcal{C}_g)$ is an abelian variety. If $\mathcal{C}_g$ is a singular curve, no statement on the signature of $\mathcal{Q}$ can be made. In order to exclude these singular cases from the discussion, we refine the definition of the moduli space $\bar{\mathcal{M}}_g$ by introducing

$$\bar{\mathcal{M}}_g^{\mathcal{A}} := \{\mathcal{C}_g \in \bar{\mathcal{M}}_g \mid J^1(\mathcal{C}_g) \text{ is an abelian variety}\} \ . \tag{6.50}$$

We conclude that for any genus-g curve $\mathcal{C}_g \in \bar{\mathcal{M}}_g^{\mathcal{A}}$ the corresponding first intermediate Jacobian is by definition an abelian variety and varies holomorphically if we deform the complex structure moduli. Since we have excluded only singular points by the restriction to the submoduli space $\bar{\mathcal{M}}_g^{\mathcal{A}} \subset \bar{\mathcal{M}}_g$, we have excluded only isolated points of $\bar{\mathcal{M}}_g$ from the discussion.

Recall from section 5.2 that we have introduced the Siegel upper half-plane $\mathcal{H}$ to be the set of complex numbers with positive imaginary part. It is conventient to extend this notation by introducing the g^{th} Siegel upper half-space

$$\mathcal{H}_g = \{\tau \in \mathrm{Sym}(g,\mathbb{C}) \mid \mathrm{Im}(\tau) \text{ is positive definite}\} \ . \tag{6.51}$$

Hence, for each genus-g curve $\mathcal{C}_g \in \bar{\mathcal{M}}_g^{\mathcal{A}}$ we can conclude that $J^1(\mathcal{C}_g) \in \mathcal{H}_g$. Note that for $g = 1$, the first Siegel upper half-space reduces to the ordinary Siegel upper half plane and moreover, the first intermediate Jacobian of an elliptic curve (i.e. a stable genus-one curve) is given by a torus whose lattice is determined by precisely one complex structure modulus $\tau \in \mathcal{H}$ that defines the ratio of the two basis vectors of the lattice [163].

Finally, let us take a look at the transformation behavior of the matrix τ_{ij} if we perform a change of the symplectic basis (α_i, β^i). In order to respect the symplectic pairing Σ, any such transformation $\Gamma \in GL_{2g}(\mathbb{Z})$ needs to obey

$$\Gamma^T \Sigma \Gamma = \Sigma \tag{6.52}$$

which restricts Γ to be a symplectic matrix $\Gamma \in Sp_g(\mathbb{Z})$ meaning that

$$\Gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{6.53}$$

with $(g \times g)$-block matrices A, B, C and D obeys that AC^T and BD^T are symmetric and moreover

$$D^T A - B^T C = \mathbb{1}_g \ . \tag{6.54}$$

Such a symplectic change of basis transforms the matrix τ_{ab} according to a modular-like transformation

$$\Gamma : \tau \mapsto (A\tau + B)(C\tau + D)^{-1} \ . \tag{6.55}$$

We note, that this transformation behavior generalizes again the observations for elliptic curves for which it is shown that any two complex structure moduli τ and τ' that describe the same elliptic curve are connected by a modular transformation (5.29).

6.3 Second Intermediate Jacobians of Calabi-Yau Threefolds

Now we turn to the other side of the claimed correspondence and investigate the intermediate Jacobians of Calabi-Yau threefolds. For our purpose, the second intermediate Jacobians $J^2(X)$ turn out to be the main player on the threefold-side as these encode the information of the middle cohomology and hence of the holomorphic three-form Ω.

Since each Calabi-Yau threefold is a member of a family $\mathcal{X}_{\mathcal{M}}$ of threefolds that varies holomorphically over the $h^{2,1}$-dimensional complex structure moduli space, one might expect that the corresponding second intermediate Jacobians arrange to a holomorphic family over $\mathcal{M}_{C.S.}$ as it happens for the first intermediate Jacobians of genus-g curves. Since each of these complex tori comes with an inherent complex structure, this expectation can only be true, if the complex structure on the intermediate Jacobian coincides with that of the complex structure moduli space. Such intermediate Jacobians are given by *Griffiths intermediate Jacobians* $J^2_G(X)$ [221, 222].

Griffiths intermediate Jacobians have a downside as they generically do not define an abelian variety meaning that the complex torus $J^2_G(X)$ cannot be embedded in a projective space. This observation leads to a problem if we want to identify the second intermediate Jacobian of a Calabi-Yau threefold with a first intermediate Jacobian of a stable genus-g curve as we have seen that $J^1(\mathcal{C}_g)$ is an abelian variety. This issue can be resolved by deforming the complex structure of the underlying torus such that the second intermediate Jacobian of the threefold becomes an abelian variety. This new intermediate Jacobian $J^2_W(X)$, named after André Weil [223], however does not vary holomorphically on $\mathcal{M}_{C.S.}$ anymore.

To sort out this situation, we define a third type of intermediate Jacobians which is both, an abelian variety and holomorphically dependent on the complex structure moduli. The prize to pay for this construction is given by the fact, that these *polarized holomorphic Jacobians* cannot be defined globally on the full moduli space but only on an open subset of $\mathcal{M}_{C.S.}$. Moreover, it turns out that it characterizes the periods of the threefolds only on a Lagrangian submanifold of $\mathcal{M}_{C.S.}$. In the context of Feynman Integral computations, this restriction is rather harmless as the physical momenta $\mathbf{z}$ define always a Lagrangian submanifold.

6.3.1 Griffiths Intermediate Jacobian

First, let us define the second Griffiths intermediate Jacobian of a Calabi-Yau threefold $X_{\mathbf{z}}$ which depends holomorphically on $h^{2,1}$ complex structure moduli $\mathbf{z} = (z^1, \dots, z^{h^{2,1}})$. In order to construct a family of second intermediate Jacobians that varies holomorphically with the complex structure moduli, we enforce that the vector spaces $V_{\mathbf{z}}$ are given as fibres of a holomorphic vector bundle $\mathcal{V}$ over $\mathcal{M}_{C.S.}$. Hence, the quotients

$$J^2(X_{\mathbf{z}}) \cong \frac{H^3(X_{\mathbf{z}}, \mathbb{C})/V_{\mathbf{z}}}{H^3(X_{\mathbf{z}}, \mathbb{Z})} \tag{6.56}$$

define a holomorphic family of second intermediate Jacobians. Recall that the Hodge filtrations

$$F_k := \bigoplus_{q=0}^{k} H^{3-q,q}(X_{\mathbf{z}}, \mathbb{C}) \tag{6.57}$$

which we introduced in section 3.3 vary holomorphically over $\mathcal{M}_{C.S.}$ and hence provide the required vector bundle structure. On dimensional grounds, we find that F_2 is a suitable choice for $V_{\mathbf{z}}$ as it is the only filtration step that is half-dimensional[109]. Thus, we define the second Griffiths intermediate Jacobian of the Calabi-Yau threefold $X_{\mathbf{z}}$ to be [221, 222]

$$J^2_G(X_{\mathbf{z}}) \cong \frac{H^3(X_{\mathbf{z}}, \mathbb{C})/F_2}{H^3(X_{\mathbf{z}}, \mathbb{Z})} \ . \tag{6.58}$$

This notation is still rather abstract. To make contact with well-known geometrical quantities, as for example the period vector, we observe that $J^2_G(X_{\mathbf{z}})$ is a complex torus of dimension $d = h^{2,1} + 1$. Thus, the Griffiths intermediate Jacobian can be identified with the quotient

$$J^2_G(X_{\mathbf{z}}) \cong \frac{\mathbb{C}^d}{\Lambda(\mathbf{z})} \tag{6.59}$$

where $\Lambda(\mathbf{z})$ is an integral lattice of dimension $2d$. From the structure of $J^2_G(X_{\mathbf{z}})$ in terms of the Hodge filtration it is possible to deduce an explicit expression for the lattice $\Lambda(\mathbf{z})$ in terms of two $(d \times d)$-matrices $\mathcal{X}(\mathbf{z})$ and $\mathcal{F}(\mathbf{z})$ which vary holomorphically with the complex structure moduli.

To determine these matrices, we proceed as in the genus-g analysis. Recall that F_2 is generated by the holomorphic three-form $\Omega(\mathbf{z})$ and its derivatives $\partial_i \Omega(\mathbf{z})$. Thus, if we choose a locally constant integral symplectic basis (α_a, β^a) of $H^3(X_{\mathbf{z}}, \mathbb{Z})$ such that $\Omega(\mathbf{z})$ is given according to equation (3.60) with period vector $(X^a(\mathbf{z}), F_a(\mathbf{z}))$, the quotient $H^3(X_{\mathbf{z}}, \mathbb{C})/F_2$ is characterized by d equivalence relations

[109] Equivalently, one could use $\bar{F}_2$ which leads to an isomorphic complex torus.

$$
\begin{aligned}
0 \sim \Omega(\mathbf{z}) &= \sum_{a=1}^{h^{2,1}} \alpha_a X^a(\mathbf{z}) - \beta^a F_a(\mathbf{z}) \\
0 \sim \partial_i \Omega(\mathbf{z}) &= \sum_{a=1}^{h^{2,1}} \alpha_a (\partial_i X^a(\mathbf{z})) - \beta^a (\partial_i F_a(\mathbf{z})) \ .
\end{aligned} \tag{6.60}
$$

Summarizing these in terms of a matrix-relation, we obtain

$$
0 \sim (\alpha_a, \beta^a) \Sigma \begin{pmatrix} \mathcal{F}(\mathbf{z})_{i,b} \\ \mathcal{X}(\mathbf{z})_i^b \end{pmatrix} \tag{6.61}
$$

with

$$
\Sigma = \begin{pmatrix} 0 & \mathbb{1}_d \\ -\mathbb{1}_d & 0 \end{pmatrix} \tag{6.62}
$$

representing the symplectic pairing on $H^3(X_{\mathbf{z}}, \mathbb{C})$ for the chosen basis. Moreover, the matrices $\mathcal{X}(\mathbf{z})$ and $\mathcal{F}(\mathbf{z})$ are defined purely in terms of the periods and are given by

$$
\begin{aligned}
\mathcal{X}(\mathbf{z})_i^a &:= \begin{cases} X^a(\mathbf{z}) & \text{if } i = 0 \\ \partial_i X^a(\mathbf{z}) & \text{if } i = 1, \ldots, h^{2,1} \end{cases} , \\
\mathcal{F}(\mathbf{z})_{i,a} &:= \begin{cases} F_a(\mathbf{z}) & \text{if } i = 0 \\ \partial_i F_a(\mathbf{z}) & \text{if } i = 1, \ldots, h^{2,1} \end{cases} .
\end{aligned} \tag{6.63}
$$

Equivalently, we can use these equivalence relations to eliminate the α_a basis vectors in favor of the β^a basis vectors. The quotient $H^3(X_{\mathbf{z}}, \mathbb{C})/F_2$ is therefore given by

$$
H^3(X_{\mathbf{z}}, \mathbb{C})/F_2 \cong \mathbb{C}^d \tag{6.64}
$$

via the identification $\beta^a \mapsto e^a$ with $\{e^a\}$ being the standard basis of $\mathbb{C}^d$. In addition, the lattice $H^3(X_{\mathbf{z}}, \mathbb{Z})$ consists of all integral multiples of the basis vectors α_a and β^a. In analogy to the previous discussion for the genus-g curves we find that under the isomorphism (6.64), the latter basis elements translate trivially into integral multiples of the basis vectors e^a whereas the former need to be transformed into expressions of the β^a by using the equivalence relation (6.61) leading to all integral multiples of the vectors $(e^a \mathcal{F} \mathcal{X}^{-1})$. In total, we obtain that under the isomorphism (6.64)

$$
H^3(X_{\mathbf{z}}, \mathbb{Z}) \cong \mathbb{Z}^d + \mathbb{Z}^d(\mathcal{F}(\mathbf{z})\mathcal{X}^{-1}(\mathbf{z})) =: \Lambda \subset \mathbb{C}^d \ . \tag{6.65}
$$

Thus, $J_G^2(X_{\mathbf{z}})$ has the explicit form

$$
J_G^2(X_{\mathbf{z}}) \cong \frac{\mathbb{C}^d}{\mathbb{Z}^d \mathcal{X}(\mathbf{z}) + \mathbb{Z}^d \mathcal{F}(\mathbf{z})} \ . \tag{6.66}
$$

Here, we have chosen a slightly different but equivalent representation of the lattice Λ in favor of a symmetric description. This expression for $J_G^2(X_{\mathbf{z}})$ provides an additional argument to show that the Griffiths intermediate Jacobian varies holomorphically in the complex structure moduli, since $\mathcal{X}(\mathbf{z})$ and $\mathcal{F}(\mathbf{z})$ are purely holomorphic functions in terms of the periods and their derivatives.

More conveniently, the Griffiths intermediate Jacobian is expressed by parametrizing the complex structure moduli space $\mathcal{M}_{C.S.}$ in terms of projective coordinates [224]. This analysis can be performed by recalling from section 3.2.6 that for Calabi-Yau threefolds the complex structure moduli space $\mathcal{M}_{C.S.}$ is a projective special Kähler manifold whose projective coordinates are given by the periods (X^a) themselves. Changing the local coordinates on $\mathcal{M}_{C.S.}$ according to $\mathbf{z} \mapsto \mathbf{X}(\mathbf{z}) = (X^0(\mathbf{z}), \ldots, X^{h^{2,1}}(\mathbf{z}))$, the holomorphic three-form Ω enjoys the expansion

$$\Omega(\mathbf{X}) = \alpha_a X^a - \beta^a F_a(\mathbf{X}) \tag{6.67}$$

with $F_a = \partial_a F(\mathbf{X})$ being the gradient of the prepotential $F(\mathbf{X})$ that characterizes the projective special Kähler geometry. Moreover, $\Omega(\mathbf{X})$ obeys the relation

$$\sum_{b=0}^{h^{2,1}} X^b \partial_b \Omega(\mathbf{X}) = \sum_{b=0}^{h^{2,1}} X^b \partial_b (\alpha_a X^a - \beta^a F_a(\mathbf{X})) = X^b \alpha_b - \beta^a \sum_{b=0}^{h^{2,1}} X^b F_{ab}(\mathbf{X}) = \Omega(\mathbf{X}) \tag{6.68}$$

which is true because $F_a(\mathbf{X})$ is homogenous of degree one in $\mathbf{X}$ and therefore we have the identity

$$\sum_{b=0}^{h^{2,1}} X^b F_{ab}(\mathbf{X}) = F_a(\mathbf{X}) \ . \tag{6.69}$$

Here and in the following we denote by $F_{ab}(\mathbf{X}) = \partial_a \partial_b F(\mathbf{X})$ the second derivative of the prepotential. Thus, $\Omega(\mathbf{X})$ is generated by its gradient and hence the Hodge filtration space F_2 is generated by

$$F_2 = \langle \partial_a \Omega(\mathbf{X}) \rangle_{a=0,\ldots,h2,1} \ . \tag{6.70}$$

Again, translating this expression into an equivalence relation, we find

$$0 \sim (\alpha_a, \beta^a) \Sigma \begin{pmatrix} F_{ab}(\mathbf{X}) \\ \mathbb{1}_d \end{pmatrix} \ . \tag{6.71}$$

Using this special coordinate frame, the second Griffiths intermediate Jecobian reads

$$J_G^2(X_{\mathbf{z}}) \cong \frac{\mathbb{C}^d}{\mathbb{Z}^d + \mathbb{Z}^d (F_{ab})(\mathbf{X})} \ . \tag{6.72}$$

Let us now discuss, whether $J_G^2(X_{\mathbf{z}})$ describes an abelian variety. We recall that $J_G^2(X_{\mathbf{z}})$ is an abelian variety, if and only if $\mathrm{Im}(F_{ab}(\mathbf{X}))$ is positive definite.

Similar to the previous section, the determination of the signature of $\mathrm{Im}(F_{ab}(\mathbf{X}))$ can be achieved by identifying this matrix with the matrix representation of a sesquilinear form

$\mathcal{Q} : F_2 \times F_2 \to \mathbb{R}$ whose signature can be deduced by other arguments. We observe that the intersection pairing on $H^3(X\mathbf{z}, \mathbb{C})$ given by

$$\mathcal{Q} : (\eta, \rho) \mapsto i \int \eta \wedge \bar{\rho} \quad , \quad \eta, \rho \in H^3(X_{\mathbf{z}}, \mathbb{C}) \tag{6.73}$$

restricts to a pairing on the Hodge filtration space F_2 which can be expressed in the basis (6.70) according to

$$\mathcal{Q}(A^a \partial_a \Omega, B^a \partial_b \Omega) = i \int A^a \partial_a \Omega \wedge \bar{B}^b \overline{\partial_b \Omega} = A^a \mathcal{Q}_{ab} \bar{B}^b \tag{6.74}$$

with matrix representation

$$\mathcal{Q}_{ab} = i \int \partial_a \Omega \wedge \overline{\partial_b \Omega} = 2\mathrm{Im}(F_{ab}) \ . \tag{6.75}$$

The last identity follows by inserting the expression (6.67) and using the canonical properties of the symplectic basis vectors (α_i, β^i) which follow from equation (3.58). Hence, we have shown that $\mathrm{Im}(F_{ab})$ is the matrix representation of $\mathcal{Q}$ restricted to the vector space F_2. The pairing $\mathcal{Q}$ is either positive or negative definite on each of the Dolbeault cohomology groups $H^{p,3-p}(X_{(z)}, \mathbb{C})$. Obviously, the sign of the signature alternates in p because $\wedge$ is antisymmetric for three-forms and hence on the level of differentials we find

$$\int (dx_i \wedge dx_j \wedge dx_k) \wedge (d\bar{x_m} \wedge d\bar{x_n} \wedge d\bar{x_p}) = - \int (dx_i \wedge dx_j \wedge d\bar{x_m}) \wedge (d\bar{x_n} \wedge d\bar{x_p} \wedge dx_k) \tag{6.76}$$

and similar expressions for other combinations of holomorphic and anti-holomorphic differentials. Moreover, we note that

$$\omega = i((dx_1 \wedge dx_2 \wedge dx_3) \wedge (d\bar{x_1} \wedge d\bar{x_2} \wedge d\bar{x_3})) \tag{6.77}$$

defines the volume form of $X_{\mathbf{z}}$ and hence, $\mathcal{Q}(\omega, \omega) = \mathrm{Vol}(X_{\mathbf{z}}) > 0$ which proves that $\mathcal{Q}$ is a positive definite pairing on $H^{3,0}(X_{\mathbf{z}}, \mathbb{C})$. To combine these observations, we can conclude that $\mathcal{Q} : H^{p,3-p}(X_{\mathbf{z}}, \mathbb{C}) \times H^{p,3-p}(X_{\mathbf{z}}, \mathbb{C}) \to \mathbb{R}$

- is positive definite if $p = 1, 3$ is odd and
- is negative definitie if $p = 0, 2$ is even .

From this result, we can read of that the sesquilinear form $\mathcal{Q}$ restricted to the Hodge filtration space $F_2 = H^{3,0}(X_{\mathbf{z}}, \mathbb{C}) \oplus H^{1,3}(X_{\mathbf{z}}, \mathbb{C})$ is indefinite and hence $\mathrm{Im}(F_{ab}(\mathbf{X}))$ is an indefinite matrix with signature $(h^{2,1}, 1)$. This completes the proof that $J^2_G(X_{\mathbf{z}})$ is not an abelian variety.

6.3.2 Weil Intermediate Jacobian

Starting from the Griffiths intermediate Jacobian, one can deform its complex structure in such a way that the result turns out to become an abelian variety which is called the Weil intermediate Jacobian. From the discussion in the previous section 6.3.1 we have learned that this goal can be achieved only if the vector space $V_{\mathbf{z}}$ in the quotient of $J^2(X_{\mathbf{z}})$ is half-dimensional and moreover, the intersection pairing $\mathcal{Q}$ is positive (or equivalently negative) definite if we restrict it on $V_{\mathbf{z}}$.

For the definition of the Griffiths intermediate Jacobian, we have chosen a combination of Dolbeault cohomology groups $H^{p,3-p}(X_{\mathbf{z}},\mathbb{C})$ that does not obey this property. However, this can be cured if we consider the combination

$$V_{\mathbf{z}} = H^{3,0}(X_{\mathbf{z}},\mathbb{C}) \oplus H^{1,2}(X_{\mathbf{z}},\mathbb{C}) \ . \tag{6.78}$$

Following the same logic as above, it follows directly that the intersection pairing $\mathcal{Q}$ acts positive definite on $V_{\mathbf{z}}$ and hence, the Weil intermediate Jacobian [223]

$$J^2_W(X_{\mathbf{z}}) \cong \frac{H^3(X_{\mathbf{z}},\mathbb{C})/V_{\mathbf{z}}}{H^3(X_{\mathbf{z}},\mathbb{Z})} \tag{6.79}$$

is an abelian variety. Again, this intermediate Jacobian can be described in terms of a $d = (h^{2,1}+1)$- dimensional complex torus

$$J^2_W(X_{\mathbf{z}}) \cong \frac{\mathbb{C}^d}{\tilde{\Lambda}(\mathbf{z})} \tag{6.80}$$

whose lattice $\tilde{\Lambda}(\mathbf{z})$ is characterized by a $(d\times d)$-matric $N(\mathbf{z})$ such that

$$\tilde{\Lambda}(\mathbf{z}) = \mathbb{Z}^d + \mathbb{Z}^d(N_{ab}(\mathbf{z})) \ . \tag{6.81}$$

In the following, we derive the explicit form of this matrix in analogy to the previous computation for the Griffiths intermediate Jacobian.

As the first step, it is necessary to define a suitable basis for the space $V_{\mathbf{z}}$. To that end, we recall that the Dolbeault cohomology groups obey the property

$$H^{p,q}(X_{\mathbf{z}},\mathbb{C}) = \overline{H^{q,p}}(X_{\mathbf{z}},\mathbb{C}) \ . \tag{6.82}$$

Thus, it is convenient to define a basis of $H^{2,1}(X_{\mathbf{z}},\mathbb{C})$ and consider its complex conjugate. From equation (6.70) we have learned that the ordinary derivatives of Ω with respect to the projective coordinates $\mathbf{X}$ form a basis of F_2. Hence, by projecting out the $(3,0)$-contributions to these derivatives, we obtain a basis of $H^{2,1}(X_{\mathbf{z}},\mathbb{C})$. Recalling that the Kähler covariant derivative

$$\nabla_a := \partial_a + (\partial_a K_{C.S.}(\mathbf{X})) \tag{6.83}$$

which we have introduced in chapter 4 was defined such that $\nabla_a\Omega(\mathbf{X}) \in H^{2,1}(X_{\mathbf{z}},\mathbb{C})$, we find after complex conjugation that

$$H^{1,2}(X_{\mathbf{z}},\mathbb{C}) = \langle \overline{\nabla}_a\, \overline{\Omega}(\overline{\mathbf{X}})\rangle_{a=0,\ldots,h^{2,1}} \ . \tag{6.84}$$

On dimensional grounds, this set of generators cannot be a basis of the $h^{2,1}$-dimensional space $H^{1,2}(X_{\mathbf{z}},\mathbb{C})$. In particular, we can again use the homogeneity of the periods $F_a(\mathbf{X})$ to compute that

$$\sum_{a=0}^{h^{2,1}} \bar{\mathbf{X}}^a \overline{\nabla}_a\, \overline{\Omega}(\overline{\mathbf{X}}) = 0 \ . \tag{6.85}$$

Hence, equation (6.84) becomes a basis of $H^{1,2}(X_{\mathbf{z}},\mathbb{C})$ if we consider the set of generators modulo this additional contraint.

To extend this basis to a basis of $V_{\mathbf{z}}$, we could simply include $\Omega(\mathbf{X})$ as this form generates the one-dimensional space $H^{3,0}(X_{\mathbf{z}},\mathbb{C})$ of holomorphic three-forms. However, it is convenient, to use the trivial observation

$$\overline{\nabla}_a\Omega(\mathbf{X}) = \partial_{\bar{a}}\Omega(\mathbf{X}) + (\partial_{\bar{a}}K_{C.S.}(\mathbf{X}))\Omega(\mathbf{X}) = (\partial_{\bar{a}}K_{C.S.}(\mathbf{X}))\Omega(\mathbf{X}) \tag{6.86}$$

implying that $\overline{\nabla}_a\Omega(\mathbf{X}) \in H^3(X_{\mathbf{z}},\mathbb{C})$ serves as a generator for the space of holomorphic three-forms as well. We conclude that a suitable basis for $V(\mathbf{z})$ is given by

$$V_{\mathbf{z}} = \langle \overline{\nabla}_a\left(\Omega(\mathbf{X}) + \overline{\Omega}(\overline{\mathbf{X}})\right)\rangle_{a=0,\ldots,h^{2,1}} \ . \tag{6.87}$$

This result allows to describe the quotient $H^3(X_{\mathbf{z}},\mathbb{C})/V(\mathbf{z})$ by the set

$$0 \sim \overline{\nabla}_a\left(\Omega(\mathbf{X}) + \overline{\Omega}(\overline{\mathbf{X}})\right) = \sum_{b=0}^{h^{2,1}} (\overline{\nabla}_a(X^b + \overline{X}^b))\alpha_b - (\overline{\nabla}_a(F_b(\mathbf{X}) + \overline{F}_b(\overline{\mathbf{X}}))\beta^b \tag{6.88}$$

of d equivalence relations. In analogy to the derivation of $J^2_G(X_{\mathbf{z}})$, we can use these relations to express the basis elements α_a according to

$$\alpha_a = -N_{ab}(\mathbf{X})\beta^b \quad , \quad N(\mathbf{X}) = -\mathcal{N}(\mathbf{X})\mathcal{Z}^{-1}(\mathbf{X}) \tag{6.89}$$

with

$$\begin{aligned} \mathcal{N}_{ab}(\mathbf{X}) &= \overline{\nabla}_b(F_a(\mathbf{X}) + \overline{F}_a(\mathbf{X})) = \overline{F}_{ab}(\mathbf{X}) + (F_a(\mathbf{X}) + \overline{F}_a(\overline{\mathbf{X}})\overline{\partial_b K_{C.S.}(\mathbf{X})}, \\ \mathcal{Z}^a_b(\mathbf{X}) &= \overline{\nabla}_b(X^a + \overline{X}^a) = \delta^a_b + (X^a + \overline{X}^a)\overline{\partial_b K_{C.S.}(\mathbf{X})} \ . \end{aligned} \tag{6.90}$$

Note that $\mathcal{Z}(\mathbf{X})$ is an invertible matrix whose inverse is given by

$$(\mathcal{Z}^{-1})^a_b(\mathbf{X}) = \delta^a_b - \frac{(X^a + \overline{X}^a)\overline{\partial_b K_{C.S.}(\mathbf{X})}}{X^c\overline{\partial_c K_{C.S.}(\mathbf{X})}} \tag{6.91}$$

Hence, the Weil intermediate Jacobian reads[110]

$$J^2_W(X_{\mathbf{z}}) \cong \frac{\mathbb{C}^d}{\mathbb{Z}^d + \mathbb{Z}^d(N_{ab})(\mathbf{X}(\mathbf{z}))} \ . \tag{6.92}$$

As discussed at the beginning of this section, this complex torus defines an abelian variety. Hence, we have a chance to identify it pointwise with the first intermediate Jacobian of a genus-$(h^{2,1}+1)$ curve. However, from the expression for the matrix N_{ab} we can conclude that $J^2_W(X_{\mathbf{z}})$ does not vary holomorphically along the complex structure moduli space but contains anti-holomorphic contributions $\bar{z}^i$ that originate from the complex conjugation of the periods.

Before we continue in section 6.3.3 to solve this issue by introducing yet another kind of intermediate Jacobians, let us note that the matrix N_{ab} is well-known in the context of four-dimensional supergravity theories that arise from string compactifications on a Calabi-Yau threefold. Recall that the action of the low energy supergravity theory of type II string compactifications given by equations (2.21) and (2.22) respectively contains a term

$$S^4_{\mathrm{IIA/IIB}} \supset \int_{M_4} \mathcal{N}_{AB} F^A \wedge F^B + \mathrm{Im}\mathcal{N}_{AB} F^A \wedge \star F^B \tag{6.93}$$

that describes the kinetic coupling of the vector fields. In section 2.2.3 we have not specified the *gauge kinetic coupling matrix* $\mathcal{N}$ except for stating that it is a function of the vector multiplet scalars and hence of the periods X^a. As it turns out [108, 112, 113, 225], this matrix is (up to a conventional overall sign) precisely the matrix N_{ab} which we have introduced to describe the Weil intermediate Jacobian. In this context, the definite signature of the matrix $\mathrm{Im}(N_{ab})$ is a crucial property for $S^4_{\mathrm{IIA/IIB}}$ being a well-defined action. In the supergravity literature, this matrix is conveniently written in the form[111]

$$N_{ab}(\mathbf{X}) = -\overline{F}_{ab} - 2i \frac{\mathrm{Im}(F_{ac}) X^c \mathrm{Im}(F_{bd}) X^d}{X^e X^f \mathrm{Im}(F_{ef})} \tag{6.94}$$

which can be shown to be equivalent to the expression (6.89). To make this argument, we recall that the Kähler potential (3.62) of the complex structure moduli space was given by

$$K_{C.S.}(\mathbf{X}) = -\log(i \sum_{a=0}^{h^{2,1}} \left(F_a \overline{X}^a - \overline{F}_a X^a\right)) \ . \tag{6.95}$$

Thus, using the convenient summation conventions, we arrive at

$$\partial_a K_{C.S.}(\mathbf{X}) = \frac{\overline{F}_a - \overline{X}^b F_{ab}}{F_c \overline{X}^c - \overline{F}_c X^c} \tag{6.96}$$

[110] Here, we have absorbed the minus sign from equation (6.89) into the lattice via the isomorphism $\mathbb{Z}^d \cong -\mathbb{Z}^d$. This redefinition is convenient in order to ensure that $\mathrm{Im}(N_{ab})$ is positive definite.

[111] Note that our conventions agree with the supergravity literature up to an overall sign for N_{ab}.

implying that the matrix $\mathcal{Z}^{-1}(\mathbf{X})$ takes the values

$$\mathcal{Z}^{-1}_{ab}(\mathbf{X}) = \delta_{ab} - \frac{(X^a + \overline{X}^a)\mathrm{Im}(F_{bc})X^c}{X^e X^f \mathrm{Im}(F_{ef})} \ . \tag{6.97}$$

With this result at hand, the matrix product $\mathcal{N}\mathcal{Z}^{-1}$ can be evaluated explicitly to agree with equation (6.94).

6.3.3 Polarized Holomorphic Intermediate Jacobians

In the previous sections, we have discussed two special choices for a complex structure on the second intermediate Jacobian $J^2(X)$ of a Calabi-Yau threefold X that either vary holomorphically along the complex structure moduli space using the construction of Griffiths or realize abelian varieties in the case of Weil. The conceptional difference of their definitions was given by the different choice of the half-dimensional vector space $V \subset H^3(X,\mathbb{C})$ which defines the complex structure on $J^2(X)$. In the following, we generalize these constructions and hence obtain a general class of second intermediate Jacobians containing $J^2_G(X)$ and $J^2_W(X)$ as two limiting cases.

Recall that the vector space V was required to be of real dimension $\dim_{\mathbb{R}}(V) = d$ if $d = h^{2,1} + 1$ is the complex dimension of the intermediate Jacobian and moreover, V together with its complex conjugate need to generate the full middle cohomology according to

$$H^3(X,\mathbb{C}) = V \oplus \overline{V} \ . \tag{6.98}$$

Most generally, such a vector space is generated by d generic three-forms which expanded in the symplectic basis (α_i, β^i) take the form

$$\eta_k := \alpha_i f^i_k - \beta^i g_{ik} \quad k = 1, \dots, d \ . \tag{6.99}$$

So far, f^i_k and g_{ik} are complex coefficients that need to be chosen such that the matrices

$$\mathfrak{f} = (f^i_k) \quad , \quad \mathfrak{g} = (g_{ik}) \tag{6.100}$$

have full rank and hence the η_k define in total d independent three-forms. Following the analog steps as before, the quotient $H^3(X,\mathbb{C})/V$ can be expressed in terms of the equivalence relations

$$\alpha_a = -\mathcal{M}_{ab}\beta^b \quad , \quad \mathcal{M} = -\mathfrak{g}\,\mathfrak{f}^{-1} \ . \tag{6.101}$$

Here, we introduced the additional minus sign in the definition of $\mathcal{M}$ in analogy to the definition of the matrix N_{ab} for the Weil intermediate Jacobian to ensure that $\mathrm{Im}(\mathcal{M})$ has a suitable signature. Again, this sign is absorbed in a redefinition of the lattice $\mathbb{Z}^d$ when we characterize the second intermediate Jacobian $J^2(X,V)$ that corresponds to the chosen vector space V by

$$J^2(X,V) := \frac{H^3(X,\mathbb{C})/V}{H^3(X,\mathbb{Z})} \cong \frac{\mathbb{C}^d}{\mathbb{Z}^d + \mathbb{Z}^d\mathcal{M}} \ . \tag{6.102}$$

We note that all relevant properties of $J^2(X,V)$ are encoded in the matrix $\mathcal{M}$ which is defined in terms of the coefficients f_k^i and g_{ik} of the chosen basis vectors of V.

As for the genus-g curves, the matrix $\mathcal{M}$ is defined only up to a change of the symplectic basis. By similar arguments we find that a basis transformation $\Gamma \in Sp_{2d}(\mathbb{Z})$ with $(d \times d)$-block matrices

$$\Gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{6.103}$$

transforms $\mathcal{M}$ according to

$$\mathcal{M} \mapsto -(A\mathcal{M} - B)(C\mathcal{M} - D)^{-1} \tag{6.104}$$

which can be more conveniently rewritten in terms of the matrix $(-\mathcal{M})$ as a modular-like transformation

$$(-\mathcal{M}) \mapsto (A(-\mathcal{M}) + B)(C(-\mathcal{M}) + D)^{-1} \ . \tag{6.105}$$

As before, the definition of $J^2(X,V)$ extends to a family of intermediate Jacobians that varies over the complex structure moduli space. In practice, this is achieved by choosing the coefficients f_k^i and g_{ik} to be smooth functions depending on the complex structure moduli $\mathbf{z}$ and therefore imposing that

$$\begin{array}{ccc} V_{\mathbf{z}} & \longrightarrow & \mathcal{V} \\ & & \big\downarrow \pi \\ & & \mathcal{M}_{C.S.} \end{array} \tag{6.106}$$

becomes a complex vector bundle[112]. By this means, for each member $X_{\mathbf{z}}$ of a given family of Calabi-Yau threefolds $\mathcal{X}_{\mathcal{M}_{C.S.}}$, the corresponding vector space $V_{\mathbf{z}}$ defines an intermediate Jacobian $J^2(X_{\mathbf{z}}, V_{\mathbf{z}})$ which is a member of the family $J^2(\mathcal{X}_{\mathcal{M}_{C.S.}}, \mathcal{V})$ such that

$$\begin{array}{ccc} J^2(X_{\mathbf{z}}, V_{\mathbf{z}}) & \longrightarrow & J^2(\mathcal{X}_{\mathcal{M}_{C.S.}}, \mathcal{V}) \\ & & \big\downarrow \pi \\ & & \mathcal{M}_{C.S.} \end{array} \tag{6.107}$$

is a smooth family of intermediate Jacobians. Note that the members of such families generically do not depend holomorphically on the complex structure moduli but are rather real analytically dependent of the moduli $\mathbf{z}$. $J^2(\mathcal{X}_{\mathcal{M}_{C.S.}}, \mathcal{V})$ extends to a holomorphic family of complex tori only if $\mathcal{V}$ admits the structure of a holomorphic vector bundle over $\mathcal{M}_{C.S.}$. In this case, we call $J^2(\mathcal{X}_{\mathcal{M}_{C.S.}}, \mathcal{V})$ a family of *holomorphic intermediate Jacobians.*

[112]A vector bundle is complex if the fibres V are complex vector spaces and the transition functions are real analytic (i.e. $\mathcal{C}^\infty$-functions). In particular, we do not require the underlying structure to be holomorphic. If this is the case, the vector bundle would be called a holomorphic vector bundle.

By choosing

$$f^a_b = \begin{cases} X^a & \text{if } b = 0 \\ \partial_{z^b} X^a & \text{if } b \neq 0 \end{cases} \quad , \quad g_{ab} = \begin{cases} F_a & \text{if } b = 0 \\ \partial_{z^b} F_a & \text{if } b \neq 0 \end{cases} \tag{6.108}$$

we obtain that $V_{\mathbf{z}} = F_2$ and hence $J^2(X_{\mathbf{z}}, V_{\mathbf{z}}) = J^2_G(X_{\mathbf{z}})$ realizes the second Griffiths intermediate Jacobian. As discussed in section 6.3.1, this special choice of the vector bundle $\mathcal{V}$ indeed furnishes a family of holomorphic intermediate Jacobians.

Moreover, for the choice

$$f^a_b = \nabla_b X^a \quad , \quad g_{ab} = \nabla_b F_a \tag{6.109}$$

the vector space $V_{\mathbf{z}}$ realizes the second Weil intermediate Jacobian $J^2(X_{\mathbf{z}}, V_{\mathbf{z}}) = J^2_W(X_{\mathbf{z}})$ which, as discussed in section 6.3.2, does not extend to a holomorphic family. However, this special case for $J^2(\mathcal{X}_{\mathcal{M}_{C.S.}}, \mathcal{V})$ leads to a real analytic family of intermediate Jacobians whose members $J^2(X_{\mathbf{z}}, V_{\mathbf{z}})$ are abelian varieties that are in particular polarizable [63].

The following definition finally combines the properties of both, the Griffiths and the Weil intermediate Jacobian. For a family $\mathcal{X}_{\mathcal{M}_{C.S.}}$ of Calabi-Yau threefolds that admits a complex vector bundle $\mathcal{V}$ whose fibres $V_{\mathbf{z}}$ are defined according to equation (6.98), we call $J^2(\mathcal{X}_{\mathcal{M}_{C.S.}}, \mathcal{V})$ a family of *polarized holomorphic second Jacobians* of $\mathcal{X}_{\mathcal{M}_{C.S.}}$ if

- $\mathcal{V}$ is a holomorphic vector bundle over $\mathcal{M}_{C.S.}$ and
- each member $J^2(X_{\mathbf{z}}, V_{\mathbf{z}})$ of this family is a (polarizable) abelian variety.

If we were to have given such a family of polarized holomorphic Jacobians, it hence realizes a $h^{2,1}$-dimensional complex subspace of the moduli space of abelian varieties $\mathcal{A}_d$ that are of dimension $d = h^{2,1} + 1$. Moreover, we recall from section 6.2 that the first intermediate Jacobians $J^1(\mathcal{C}_g)$ of stable genus-g curves $\mathcal{C}_g \in \bar{\mathcal{M}}^{\mathcal{A}}_g$ give rise to a $3g-3$-dimensional complex subspace of $\mathcal{A}_g$. Thus, by choosing $g = d$ we have a chance that these two subspaces indeed intersect and hence we obtain a correspondence between the periods of Calabi-Yau threefolds with $h^{2,1} = d - 1$ and those of genus-d curves.

Unfortunately, it is too much to hope for a globally defined family of polarized holomorphic Jacobians on the full complex structure moduli space. Thus, in order to make progress, we restrict the full moduli space to an open neighborhood U of suitable submoduli spaces $\Delta_{\mathbb{R}}$ which are Lagrangian submanifolds of $\mathcal{M}_{C.S.}$. The following construction provides a family of locally defined polarized holomorphic intermediate Jacobians on U.

For a symplectic manifold M of real dimension $2n$, a Lagrangian submanifold $\Delta_{\mathbb{R}} \subset M$ is defined to be an n-dimensional real submanifold of M such that the restricted symplectic two-form $h \in H^2(X, \mathbb{R})$ vanishes on $\Delta_{\mathbb{R}}$. For a Kähler manifold, its Kähler form serves as a canonical symplectic two-form for which we can apply the definition of Lagrangian submanifolds. Since the Kähler two-form defines the imaginary part of the Kähler metric with respect to the chosen complex structure of M, any Lagrangian submanifold $\Delta_{\mathbb{R}} \subset M$

obtains a well-defined structure of a Riemannian real manifold of dimension n whose Riemannian metric is obtained by pulling back the hermitian metric of the complex manifold.

Let us now apply this procedure to the complex structure moduli space $\mathcal{M}_{C.S.}$ of a family of Calabi-Yau threefolds $\mathcal{X}_{\mathcal{M}_{C.S.}}$ to construct a suitable family of polarized holomorphic Jacobians. Let us assume that $\mathcal{M}_{C.S.}$ or, to be more specific, an open disc $\mathcal{D} \subset \mathcal{M}_{C.S.}$ of it[113], admits a Lagrangian submanifold $\Delta_{\mathbb{R}}$. For each point $\mathbf{w} \in \Delta_{\mathbb{R}}$ we can now consider the corresponding Weil intermediate Jacobian $J^2_W(X_{\mathbf{w}})$ together with its corresponding vector space

$$V_{\mathbf{w}} = H^{3,0}(X_{\mathbf{w}}, \mathbb{C}) \oplus H^{1,2}(X_{\mathbf{w}}, \mathbb{C}) \ . \tag{6.110}$$

These vector spaces can be combined to a complex vector bundle $\mathcal{V}_{\Delta_{\mathbb{R}}}$ over the Lagrangian submanifold $\Delta_{\mathbb{R}}$ because $\mathcal{V}_{\Delta_{\mathbb{R}}} \subset \mathcal{V}$ is a sub vector bundle of the complex vector bundle $\mathcal{V}$ characterizing the family of Weil intermediate Jacobians on the full moduli space $\mathcal{D}$. In particular, we find that the fibres $V_{\mathbf{w}}$ vary analytically with respect to the real coordinates $\mathbf{w}$ on $\Delta_{\mathbb{R}}$.

The key insight is that it is possible to holomorphically continue a real analytic structure that is defined on a Lagrangian submanifold to an open neighborhood $\tilde{U} \supset \Delta_{\mathbb{R}}$ of the submanifold. In a local analysis, this observation follows directly as $V_{\mathbf{w}}$ is defined in terms of the real analytic functions $f^i_k(\mathbf{w})$ and $g_{ik}(\mathbf{w})$ on $\Delta_{\mathbb{R}}$ that can be continued to holomorphic functions $f^i_k(\mathbf{z})$ and $g_{ik}(\mathbf{z})$ on $\tilde{U}$ by naively replacing the real coordinates $\mathbf{w}$ in their power series expansions by the corresponding holomorphic coordinates $\mathbf{z} \in \tilde{U}$. One needs to choose $\tilde{U}$ small enough such that these complex valued power series still converge. In this way, we have constructed a holomorphic vector bundle $\mathcal{V}_{\tilde{U}}$ on the open set $\tilde{U} \subset \mathcal{D}$ whose restriction on $\Delta_{\mathbb{R}}$ coincides with $\mathcal{V}_{\Delta_{\mathbb{R}}}$. By definition, the corresponding family of intermediate Jacobians $J^2(\mathcal{X}_{\tilde{U}}, \mathcal{V}_{\tilde{U}})$ is holomorphically dependent on the local coordinates of $\tilde{U}$ and moreover, its members on $\Delta_{\mathbb{R}}$ are abelian varieties and hence polarizable. In local coordinates $\mathbf{z}$ of $\tilde{U}$, these intermediate Jacobians are given by

$$J^2_{\Delta_{\mathbb{R}}}(X_{\mathbf{z}}) := J^2(X_{\mathbf{z}}, V_{\mathbf{z}}) = \frac{\mathbb{C}^d}{\mathbb{Z}^d + \mathbb{Z}^d H_{ab}(\mathbf{z})} \tag{6.111}$$

with $H_{ab}(\mathbf{z})$ being the holomorphic continuation of the real analytic matrix

$$N_{ab}(\mathbf{w}) = \left(-\bar{F}_{ab} - 2i \frac{\mathrm{Im}(F_{ac}) X^c \mathrm{Im}(F_{bd}) X^d}{X^e X^f \mathrm{Im}(F_{ef})} \right)\Bigg|_{\Delta_{\mathbb{R}}} \tag{6.112}$$

defining the Weil intermediate Jacobian $J^2_W(X_{\mathbf{w}})$ on the Lagrangian submanifold. It is important to note that the matrix $H_{ab}(\mathbf{z})$ does not coincide with $N_{ab}(\mathbf{z})$ outside the Lagrangian submanifold $\Delta_{\mathbb{R}}$ which can be seen directly since $H_{ab}(\mathbf{z})$ was defined to be holomorphic whereas $N_{ab}(\mathbf{z})$ contains contributions that depend on $\bar{\mathbf{z}}$.

[113] By restricting the discussion to a local description on the disc $\mathcal{D}$, we do not have to worry about the global structure of $\mathcal{M}_{C.S.}$ and in particular its singularities.

With this construction, we have obtained a family of holomorphic intermediate Jacobians $J^2_{\Delta_{\mathbb{R}}}(\mathcal{X}_{\tilde{U}})$ whose members are known to be abelian varieties on the submnifold $\Delta_{\mathbb{R}}$. Recalling that an intermediate Jacobian is an abelian variety if and only if the imaginary part of its defining matrix is positive definite, it holds that this is an open property[114] and hence can be extended to an open neighborhood $U_{\mathcal{A}}$ of $\Delta_{\mathbb{R}}$. Thus, on the intersection

$$U = \tilde{U} \cap U_{\mathcal{A}} \supset \Delta_{\mathbb{R}} \tag{6.113}$$

of the two open neighborhoods $\tilde{U}$ and $U_{\mathcal{A}}$, $J^2_{\Delta_{\mathbb{R}}}(X_{\mathbf{z}})$ is a holomorphic family of intermediate Jacobians that are abelian varieties. Hence, $J^2_{\Delta_{\mathbb{R}}}(\mathcal{X}_U)$ is a family of polarized holomorphic intermediate Jacobians that is not defined on the full complex structure moduli space but rather on some open neighborhood U of the Lagrangian submanifold $\Delta_{\mathbb{R}}$.

To close this section, let us briefly discuss some important properties of $J^2_{\Delta_{\mathbb{R}}}(\mathcal{X}_U)$. First, it is important to note that this construction is very sensitive to the choice of the Lagrangian submanifold $\Delta_{\mathbb{R}}$. If we were to have two such Lagrangian submanifolds $\Delta_{\mathbb{R}}$ and $\Delta'_{\mathbb{R}}$ their corresponding intermediate Jacobians will in general not coincide on the intersection $U \cap U'$ of the open neighborhoods U and U' of $\Delta_{\mathbb{R}}$ and $\Delta'_{\mathbb{R}}$ respectively. In particular, since $J^2_{\Delta_{\mathbb{R}}}(X_{\mathbf{z}})$ arises as the holomorphic continuation of $J^2_W(X_{\mathbf{w}})$ on $\Delta_{\mathbb{R}}$, it is entirely determined by the geometry of those threefolds $X_{\mathbf{w}}$ that are characterized by $\mathbf{w} \in \Delta_{\mathbb{R}}$.

Moreover, we should note that the matrix $H_{ab}(\mathbf{z})$ is well-defined which means that any symplectic transformation $\Gamma \in Sp_{2d}(\mathbb{Z})$ of the cohomology basis (α_i, β^i) commutes with the holomorphic continuation of $N_{ab}(\mathbf{w})$. This can be seen by recalling that both, the matrices $N(\mathbf{w})$ and $H(\mathbf{z})$ transform according to equation (6.104). If we now perform the holomorphic continuation of $\Gamma(N)$ to an open neighborhood U of $\Delta_{\mathbb{R}}$, this is by definition a holomorphic function $h(\Gamma(N))$ that agrees on $\Delta_{\mathbb{R}}$ with $\Gamma(N)$. Moreover, if we first holomorphically continue N to H and then apply Γ, we find that H agrees with N on $\Delta_{\mathbb{R}}$ and hence $\Gamma(H) = \Gamma(N)$ on $\Delta_{\mathbb{R}}$. Thus, we have two holomorphic functions $h(\Gamma(N))$ and $\Gamma(H)$ that both agree with $\Gamma(N)$ on the Lagrangian submanifold $\Delta_{\mathbb{R}}$. Since holomorphic continuations are unique, we conclude that $h(\Gamma(N)) = \Gamma(H)$ which proves that Γ and the holomorphic continuation commute.

6.4 The Calabi-Yau-to-Curve Correspondence

After this excurse into the algebraic geometry of intermediate Jacobians we have collected all the ingredients that are necessary to state and derive the Calabi-Yau-to-curve correspondence which identifies a given Calabi-Yau threefold with a stable genus-g curve. Since this correspondence is meant to give pairs of geometries that are related by identifying their periods suitably, it can be stated in terms of intermediate Jacobians.

[114] If $\mathrm{Im}(H_{ab})(\mathbf{z})$ is positive definite for all $\mathbf{z} \in \Delta_{\mathbb{R}}$, it is positive definite on an open neighborhood of $\Delta_{\mathbb{R}}$ as its eigenvalues vary holomorphically with $\mathbf{z}$.

We recall that the first intermediate Jacobian of a smooth and stable genus-g curve $\mathcal{C}_g$ was given by an abelian variety $J^1(\mathcal{C}_g) \in \mathcal{A}^g$ of complex dimension g that is described as the complex g-torus

$$J^1(\mathcal{C}_g) = \frac{\mathbb{C}^g}{\mathbb{Z}^g + \mathbb{Z}^g \tau} \tag{6.114}$$

with τ being a $(g \times g)$ matrix that characterizes the complex structure of $\mathcal{C}_g$. On the other side, for a family of Calabi-Yau threefolds $X_{\mathbf{z}}$ with $h^{2,1}$ complex structure moduli $\mathbf{z}$, the corresponding second Weil intermediate Jacobians describe a real analytic family of abelian varieties $J^2_W(X_{\mathbf{z}}) \in \mathcal{A}_{(h^{2,1}+1)}$. Thus, we can obtain a correspondence between both sides by requiring that

$$J^2_W(X_{\mathbf{z}}) = J^1(\mathcal{C}_g) \ . \tag{6.115}$$

As a first constraint, we find that both Jacobians live in the same space $\mathcal{A}_g$ only if the genus of the curve is fixed by the dimension of $H^{2,1}(X_{\mathbf{z}}, \mathbb{C})$ to be

$$g = h^{2,1} + 1 \ . \tag{6.116}$$

Moreover, this correspondence has chance to work only if the $(3g-3)$ dimensional moduli space $\bar{M}^{\mathcal{A}}_g$ of stable genus-g curves and the $h^{2,1}$-dimensional moduli space of the family of Calabi-Yau threefolds give rise to a non-trivial intersection of the corresponding family of intermediate Jacobians on $\mathcal{A}_g$. The question whether a given abelian variety is realized as the first intermediate Jacobian of a genus-g curve is formalized in the unsolved Riemann-Schottky problem.

6.4.1 The Riemann-Schottky Problem

Independent of the identification among intermediate Jacobians of different geometries, the question whether a given abelian variety $A \in \mathcal{A}_g$ can be realized as the first intermediate Jacobian of a genus-g curve $\mathcal{C}_g$ has been prominently discussed in the mathematical literature starting in the 19^{th} century with the work of Riemann [226] and Schottky [227]. For a modern review on the Riemann-Schottky problem we refer to [228, 229]. This problem can be formalized as follows.

Define the holomorphic map

$$J^1 : \bar{\mathcal{M}}^{\mathcal{A}}_g \to \mathcal{A}_g \tag{6.117}$$

by mapping the curve $\mathcal{C}_g$ to its first intermediate Jacobian $J^1(\mathcal{C}_g)$. The period matrix τ characterizes the complex structure of $\mathcal{C}_g$ completely[115] and hence, $J^1(\mathcal{C}_g)$ determines $\mathcal{C}_g$. Due to this fact, the map J^1 is injective.

We now define the Schottky-locus $\mathcal{S}_g \subset \mathcal{A}_g$ to be the image of J^1

$$\mathcal{S}_g := J^1\left(\bar{\mathcal{M}}^{\mathcal{A}}_g\right) \ . \tag{6.118}$$

[115]This observation goes back to Torelli's theorem which can be reviewed in ref. [150].

For general g, the determination of this subspace of $\mathcal{A}_g$ turns out to be very challenging and is known as the *Riemann-Schottky problem.*

To get a first catch at this problem, let us perform a dimensional analysis. Since an abelian variety $A \in \mathcal{A}_g$ is given by a complex g-torus that is embeddable in a projective space, its complex structure is completely encoded by the choice of a lattice $\Lambda \subset \mathbb{C}^g$ which is uniquely characterized up to a symplectic change of basis by a $(g \times g)$-matrix in the Siegel upper half-space $\mathcal{H}_g$. Hence, the moduli space of abelian varieties is given by

$$\mathcal{A}_g \cong \frac{\mathcal{H}_g}{Sp_{2g}(\mathbb{Z})} \tag{6.119}$$

which is of complex dimension

$$\dim(\mathcal{A}_g) = \frac{g(g+1)}{2} \ . \tag{6.120}$$

Comparing this expression (for $g \geq 2$) with the dimension of the moduli space of stable genus-g curves, we find that

$$\dim(\bar{\mathcal{M}}_g^{\mathcal{A}}) \leq \dim(\mathcal{A}_g) \tag{6.121}$$

for all $g \geq 2$. Equality holds exactly for $g = 2$ and $g = 3$.

We conclude that for $g = 2$ and $g = 3$, the Riemann-Schottky problem becomes trivial as the Schottky-locus is already given by the full moduli space of abelian varieties

$$\mathcal{S}_2 = \mathcal{A}_2 \quad , \quad \mathcal{S}_3 = \mathcal{A}_3 \tag{6.122}$$

whereas for $g > 3$, the Schottky-locus turns out to be a proper subspace of $\mathcal{A}_g$. It should be noted that for $g > 3$ no Schottky locus has been constructed explicitly.

6.4.2 The Real Analytic Correspondence

In analogy to the map J^1 which maps a genus-g curve to its first intermediate Jacobian, we can define the real analytic map

$$J_W^2 : \mathcal{M}_{C.S.} \to \mathcal{A}_g \quad g = h^{2,1} + 1 \tag{6.123}$$

which maps a Calabi-Yau threefold $X_{\mathbf{z}}$ to its second Weil intermediate Jacobian. Due to Torelli's theorem [150], the matrix $N_{ab}(\mathbf{z})$ determines uniquely the complex structure moduli $\mathbf{z}$ and hence this map is again injective. The correspondence which is proposed by equation (6.115) can now be formalized by defining a map $\Phi_{\mathbb{R}}$ that is given by first applying J_W^2 on a point $\mathbf{z} \in \mathcal{M}_{C.S.}$ to obtain an abelian variety and then using $(J^1)^{-1}$ to find the corresponding point in the moduli space of stable genus-g curves that leads to the same Jacobian variety. To ensure that this map is well-defined, we have to ensure that the image of $\mathbf{z}$ under J^2_W lies on the Schottky-locus $\mathcal{S}_g$ for which the inverse of J^1 exists. Hence, we have to restrict the domain of this map to the subspace

$$\tilde{\mathcal{S}}_g := (J^2_W)^{-1}\,(\mathcal{S}_g) \subseteq \mathcal{M}_{C.S.} \ . \tag{6.124}$$

The *real analytic Calabi-Yau-to-curve correspondence* is hence defined by the real analytic map

$$\Phi_{\mathbb{R}} : \tilde{\mathcal{S}}_g \to \bar{\mathcal{M}}_g^{\mathcal{A}} \tag{6.125}$$

from the Calabi-Yau complex structure moduli space to the moduli space of stable genus-g curves that is given by

$$\Phi_{\mathbb{R}}(\mathbf{z}) := ((J^1)^{-1} \circ J^2_W)(\mathbf{z}) = (J^1)^{-1}\big(J^2_W(X_{\mathbf{z}})\big) \ . \tag{6.126}$$

It should be noted that Φ is an injective map because both J^2_W and $(J^1)^{-1}$ are injective and hence gives rise to a real analytic bijection onto its image $\Phi_{\mathbb{R}}(\tilde{\mathcal{S}}_g) \subset \bar{\mathcal{M}}_g^{\mathcal{A}}$. The following diagram summarizes the action of the map $\Phi_{\mathbb{R}}$

$$\begin{array}{ccccc} \mathcal{M}_{C.S.} & \xhookrightarrow{J^2_W} & \mathcal{A}_g & & \bar{\mathcal{M}}_g^{\mathcal{A}} \\ \cup\!\! \mid & & \cup\!\! \mid & \swarrow_{J^1} & \uparrow \\ \tilde{\mathcal{S}}_g & \xhookrightarrow{J^2_W} & \mathcal{S}_g & & \\ & & \underset{\Phi_{\mathbb{R}}}{\curvearrowright} & & \end{array} \ . \tag{6.127}$$

For families of Calabi-Yau threefolds with $h^{2,1} \leq 2$, the Schottky-locus is trivially given by $\mathcal{S}_g = \mathcal{A}_g$. Thus, we find that $\tilde{S}_g$ is given by the full complex structure moduli space $\mathcal{M}_{C.S.}$ of the family of Calabi-Yau threefolds and hence, the real analytic Calabi-Yau to-curve correspondence extends to a global identification

$$\Phi_{\mathbb{R}}\mathcal{M}_{C.S.} \to \bar{\mathcal{M}}_g^{\mathcal{A}} \tag{6.128}$$

that furnishes an $h^{2,1}$-dimensional image in the moduli space of stable genus-g curves.

For $h^{2,1} > 2$ this correspondence cannot be applied generically to any member of the family $\mathcal{X}_{\mathcal{M}_{C.S.}}$ anymore but only to the subfamily $\tilde{S}_g$. Thus, to compute the correspondence map $\Phi_{\mathbb{R}}$ in practice, it would be necessary in this case to construct the Schottky-locus $\mathcal{S}_g$ in order to find the domain $\tilde{S}_g$ of $\Phi_{\mathbb{R}}$. In particular, it should be noted that for certain families of Calabi-Yau threefolds $\tilde{S}_g$ might be the empty set and hence $\Phi_{\mathbb{R}}$ becomes the trivial map. In this case, no member $X_{\mathbf{z}}$ of this family corresponds to a genus-g curve.

The following counting argument gives an interesting bound on $h^{2,1}$ above which the existence of a non-trivial Calabi-Yau-to-curve correspondence becomes unlikely. Consider a family of Calabi-Yau threefolds with $h^{2,1}$ complex structure moduli. The corresponding family of Weil intermediate Jacobians realizes a real subfamily of $\mathcal{A}_{h^{2,1}+1}$ that is of real dimension $2h^{2,1}$. On the other side, the Schottky-locus on $\mathcal{A}_{h^{2,1}+1}$ is

$$\dim_{\mathbb{C}}(\mathcal{S}_{h^{2,1}+1}) = 3(h^{2,1}+1) - 3 = 3h^{2,1} \tag{6.129}$$

dimensional. In the worst case, these two subspaces of $\mathcal{A}_{h^{2,1}+1}$ intersect transversely which gives the lower bound

$$\dim_{\mathbb{C}}(S_{h^{2,1}+1} \cap J^2_W(X_{\mathbf{z}})) \geq \frac{1}{2}h^{2,1}(5 - h^{2,1}) - 1 \tag{6.130}$$

on the dimension of this intersection. Equality is obtained if and only if the subspaces intersect transversely. It follows that for $h^{2,1} \leq 4$ the preimage of this intersection $\tilde{S}_{h^{2,1}+1} \subset \mathcal{M}_{C.S.}$ is at least one-dimensional and hence, the Calabi-Yau-to-curve correspondence is non-trivially applicable on a subspace of the full complex structure moduli space of the family of threefolds. However, for $h^{2,1} \geq 5$, this result states that $\tilde{S}_{h^{2,1}+1}$ may become empty and hence $\Phi_{\mathbb{R}}$ turns out to be the trivial map. For such families, the Calabi-Yau-to-curve correspondence as defined above does not give any relations. It would be an interesting project to study in detail, whether the transversality assumption is generic which we have used to derive the bound (6.130) on the dimension of $\tilde{S}_{h^{2,1}+1}$. In particular, it would be of interest, whether there exist families of Calabi-Yau threefolds with more than four complex structure moduli that can be identified with a corresponding genus-g curve. The existence of such those would contradict the transversality assumption.

6.4.3 The Local Holomorphic Correspondence

So far, we have constructed a correspondence between Calabi-Yau threefolds and genus-g curves that defines a real analytic bijection $\Phi_{\mathbb{R}}$ between certain subspaces of $\mathcal{M}_{C.S.}$ and $\bar{\mathcal{M}}_g^{\mathcal{A}}$. In the following, we use the construction of a family of polarized holomorphic Jacobians from section 6.3.3 to find a similar Calabi-Yau-to-curve correspondence which is moreover a holomorphic map. For this purpose, we recall that for a chosen Lagrangian submanifold $\Delta_{\mathbb{R}}$ of the complex structure moduli space and a suitable open neighborhood $U \supset \Delta_{\mathbb{R}}$, the construction of the polarized holomorphic intermediate Jacobian $J^2_{\Delta_{\mathbb{R}}}(X_{\mathbf{z}})$ defines a holomorphic map

$$J^2_{\Delta_{\mathbb{R}}} : \mathcal{M}_{C.S.} \supset U \to \mathcal{A}_g \ , \qquad g = h^{2,1} + 1 \ . \tag{6.131}$$

By similar arguments as before, this map is injective and hence

$$\begin{aligned} \Phi^U_{\Delta_{\mathbb{R}}} : \bar{\mathcal{S}}_g &\longrightarrow \bar{\mathcal{M}}_g^{\mathcal{A}} \\ \mathbf{z} &\mapsto (J^1)^{-1}\left(J^2_{\Delta_{\mathbb{R}}}(X_{\mathbf{z}})\right) \end{aligned} \tag{6.132}$$

defines a holomorphic map that is bijective onto its image. We call this holomorphic bijection the *local holomorphic Calabi-Yau-to-curve correspondence.* As before, the space $\bar{\mathcal{S}}_g \subset U$ denotes the preimage of the Schottky-locus under $J^2_{\Delta_{\mathbb{R}}}$, meaning

$$\bar{\mathcal{S}}_g := \left(J^2_{\Delta_{\mathbb{R}}}\right)^{-1}(\mathcal{S}_g) \subseteq U \ . \tag{6.133}$$

In analogy to the real analytic version of the correspondence, the map $\Phi^U_{\Delta_\mathbb{R}}$ can become trivial if the preimage of the Schottky-locus $\bar{\mathcal{S}}_g$ is empty which can happen for $h^{2,1} \geq 5$ by the same arguments as above. Moreover, for $h^{2,1} \in \{1, 2\}$ it again holds that $\bar{\mathcal{S}}_g = U$ as the Riemann-Schottky problem becomes trivial for these cases and hence, the local holomorphic correspondence extends to the full open neighborhood U for which $J^2_{\Delta_\mathbb{R}}$ can be defined.

Finally, we note that for $\mathbf{z} \in \Delta_\mathbb{R}$

$$\Phi^U_{\Delta_\mathbb{R}}(\mathbf{z}) = \Phi_\mathbb{R}(\mathbf{z}) \tag{6.134}$$

which follows directly from the observation that by construction the corresponding second intermediate Jacobians $J^2_{\Delta_\mathbb{R}}(X_\mathbf{z})$ and $J^2_W(X_\mathbf{z})$ coincide on thie Lagrangian submanifold $\Delta_\mathbb{R}$. However, this equality fails if $\mathbf{z}$ leaves the Lagrangian submanifold. Thus, although their construction looks very similar, the two maps $\Phi^U_{\Delta_\mathbb{R}}$ and $\Phi_\mathbb{R}$ lead to different correspondences of Calabi-Yau threefolds and genus-g curves. However, if one is interested only in the values of the period integrals on a Lagrangian sublocus of $\mathcal{M}_{C.S.}$ as it is the case for the physical values of the complex structure moduli in Feynman integral computations, both descriptions coincide.

6.5 Example: The Four-Loop Equal Mass Banana Integral

Let us close the discussion of the Calabi-Yau-to-curve correspondence by presenting its application to an explicit example. Among many Feynman integrals that have been related to a Calabi-Yau geometry, one of the most studied examples is the family of n-loop *banana integrals*[116] [180, 203, 230] which describe a higher-loop contribution to the propagator self-energy. Without specifying the explicit fields, the schematic Feynman diagram for an n-loop banana intergal is shown in figure 6.3 and generalizes the one-loop contribution which we have discussed in section 6.1.2 as an explicit example.

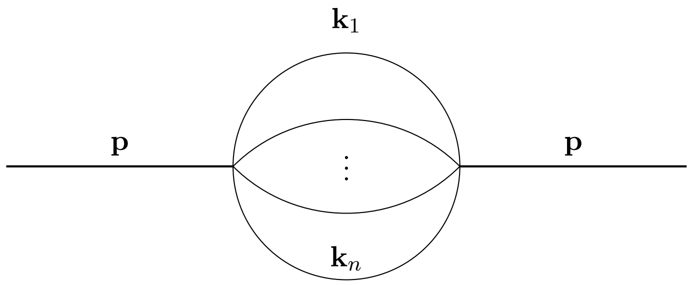

Figure 6.3: The schematic Feynman diagram that corresponds to the n-loop banana Integral with external momentum $\mathbf{p}$ and internal momenta $\mathbf{k}_i$ obeying the momentum conservation condition $\mathbf{k}_1 + \ldots + \mathbf{k}_n = \mathbf{p}$.

[116]These Integrals are also known as *sunrise Integrals* in the physics literature.

It is well-known [180, 203] that an n-loop banana Integral has a geometric description in terms of periods of a Calabi-Yau $n-1$ fold. Thus, to apply the Calabi-Yau-to-curve correspondence which has been introduced in section 6.4, we focus in the following on the case of $n = 4$. Following [230], the general family of four-loop banana intergals with external momentum $\mathbf{p}$ and internal masses m_1 to m_4 is defined in dimensional regularization by

$$I_{\nu_1,\ldots,\nu_5} := e^{4\varepsilon\gamma_E}(\mathbf{p}^2)^{\nu-2D} \int \prod_{i=1}^{5} \left(\frac{d^D\mathbf{k}_i}{(i\pi)^{D/2}} \right) (i\pi)^{D/2} \delta^D \left(\mathbf{p} - \sum_{j=1}^{5} \mathbf{k}_j \right) \frac{1}{D_1^{\nu_1}\cdots D_5^{\nu_5}} \tag{6.135}$$

in $D = 2 - 2\varepsilon$ spacetime dimensions. The propagator D_i corresponding to the internal momentum $\mathbf{k}_i$ is given by

$$D_i := -\mathbf{k}_i^2 + m_i^2 \tag{6.136}$$

and $\nu = \nu_1 + \cdots + \nu_5$ counts the total number of propagators.
To be more concrete, we consider from now on the case that all four internal masses become equal

$$m_1 = \ldots = m_5 =: m \ . \tag{6.137}$$

For this subclass, the Feynman Integrals $I_{\nu_1,\ldots,\nu_5}$ simplify to be dependent only on the remaining dimensionless mass parameter $z = m^2/\mathbf{p}^2$. Expressing the integrals in terms of a Feynman parameter representation, we find [203]

$$I_{\nu_1,\ldots,\nu_5} = \frac{e^{4\varepsilon\gamma_E}\Gamma(\nu-2D)}{\prod_{i=1}^{5}\Gamma(\nu_i)} \int_{a_i\geq 0} d^5a \ \delta\left(1 - \sum_{j=1}^{5} a_j\right) \prod_{k=1}^{5} a_k^{\nu_k-1} \frac{\mathcal{U}(a)^{\nu-5D/2}}{\mathcal{F}(a)^{\nu-2D}} \tag{6.138}$$

with the graph polynomials $\mathcal{U}(a)$ and $\mathcal{F}(a)$

$$\mathcal{U}(a) = \prod_{i=1}^{5} a_i \left(\sum_{j=1}^{5} \frac{1}{a_j} \right) \quad , \quad \mathcal{F}(a) = \prod_{i=1}^{5} a_i \left[z \left(\sum_{j=1}^{5} a_j \right) \left(\sum_{k=1}^{5} \frac{1}{a_k} \right) - 1 \right] \ . \tag{6.139}$$

Using integration-by-parts identities (IBP), this one-parameter family of Feynman integrals is shown to be generated by five master intergrals. One suitable choice for such generators is given by

$$I_{11110} \ , \quad I_{11111} \ , \quad I_{11112} \ , \quad I_{11113} \ , \quad I_{11114} \tag{6.140}$$

which form a system of coupled differential equations that can be deduced by additional IBPs. Since $I_{1,1,1,1,0}$ realizes a subsector of this family of Feynman integrals, the non-trivial four-loop master integral with the smallest degree of the denominator is given by $I_{1,1,1,1,1}$ which obeys an inhomogeneous fourth-order differential equation [231]

$$\mathcal{L}_\varepsilon I_{1,1,1,1,1} = \varepsilon^4 I_{1,1,1,1,0} \tag{6.141}$$

for some differential operator $\mathcal{L}_\varepsilon$ in the kinematic variable z that depends polynomially on the dimensional regularization parameter ε. In the following, we are interested in the solution to the maximal cut of this master integral. Following [203], the maximal cut is given by the homogenous solution to the leading order ε^0-contribution of the differential operator $\mathcal{L}_\varepsilon$. For the four-loop equal mass banana intergal, this implies that the maximal cut of $I_{1,1,1,1,1}$ obeys the differential equation

$$\mathcal{L}^{(0)} I^{\text{max cut}}_{1,1,1,1,1} = 0 \tag{6.142}$$

with the degree-four differential operator

$$\mathcal{L}^{(0)} = \frac{1}{z^4(1-z)(1-9z)(1-25z)} \sum_{k=0}^{4} f_k(z)\Theta^k \quad , \quad \Theta = z\partial_z \tag{6.143}$$

with coefficient functions

$$\begin{aligned} f_4(z) &= (1-z)(1-9z)(1-25z) \ , \\ f_3(z) = -2z(675z^2-518z+35) \quad &, \quad f_2(z) = -z(2925z^2-1580z+63) \ , \\ f_1(z) = -4z(675z^2-272z+7) \quad &, \quad f_0(z) = -5z(180z^2-57z+1) \ . \end{aligned} \tag{6.144}$$

It is important to note that this operator coincides up to a normalization with the Picard-Fuchs operator (5.177) of the $\mathbb{Z}_5$-quotient of Hulek-Verrill threefolds which is given by equation (5.177). Thus, we can conclude that, up to a normalization constant, the maximal cut of the four-loop equal mass banana integral $I_{1,1,1,1,1}$ is given in terms of the periods $\varpi^a(z)$ of the one-parameter family HV^3_z of Hulek-Verrill Calabi-Yau threefolds[117].

In the following discussion we apply the Calabi-Yau-to-curve correspondence to this family of Calabi-Yau threefolds in order to construct the corresponding one-parameter subfamily of genus-two curves.

6.5.1 The Prepotential $F(z)$ of the Family HV^3_z

Since it will simplify the computations of its Griffiths and Weil intermediate Jacobians, we start by constructing the prepotential F of this family of Calabi-Yau threefolds from its mirror construction. We recall from section 5.5 that $\text{H}\Lambda^3_t$ emerges as the $\mathbb{Z}_5$-quotient of the full five-dimensional family of mirror Hulek-Verrill threefolds $\text{H}\Lambda^3_{\mathbf{t}}$ for which we have computed the asymptotic structure of the integral periods to be

$$\begin{aligned} \Pi^0(t) &= 1 \\ \Pi^i(t) &= t^i \\ \Pi_i(t) &= -s_2^4(\hat{t}_i) + 1 + \cdots \\ \Pi_0(t) &= s_3^5(t) + s_1^5(t) - 80\frac{\zeta(3)}{(2\pi i)^3} + \cdots \end{aligned} \tag{6.145}$$

[117]One may note that the $\varepsilon \to 0$ limit of $I_{1,1,1,1,1}$ realizes precisely the torus integral (5.170) of the periods $\varpi^a(z)$ characterizing the holomorphic three-form $\Omega(z)$ of this one-paramter family of Hulek-Verrill threefolds.

with the mirror map

$$t^i = \frac{1}{2\pi i}\frac{X^i(z)}{X^0(z)} = \frac{1}{2\pi i}\log(z^i) + \cdots \ . \qquad (6.146)$$

In analogy to the fourfold discussion in section 5.4, we find that the integral periods of the $\mathbb{Z}_5$-quotient $\mathrm{H}\Lambda^3_z$ are obtained from these by

$$\begin{aligned}
\Pi^0(t) &= 1 \\
\Pi^1(t) &= \frac{1}{5}\sum_{i=1}^{5}\Pi^i(t,\ldots,t) = t \\
\Pi^2(t) &= \frac{1}{5}\sum_{i=1}^{5}\Pi_i(t,\ldots,t) = -12t^2 + 1 + \cdots \\
\Pi^3(t) &= \frac{1}{5}\Pi_0(t,\ldots,t) = 4t^3 + t - 16\frac{\zeta(3)}{(2\pi i)^3} + \cdots \ .
\end{aligned} \qquad (6.147)$$

This asymptotic structure of the integral periods allows to determine the leading order contributions to the prepotential $F(t)$ which characterizes the projective special Kähler geometry of the moduli space. From equation (6.147), we deduce the topological coefficients Y_{ijk} of this one-parameter family of Calabi-Yau threefolds to be

$$Y_{111} = 24 \ , \ Y_{110} = 0 \ , \ Y_{001} = 1 \ , \ Y_{000} = 16\frac{\zeta(3)}{(2\pi i)^3} \qquad (6.148)$$

which result in the prepotential

$$F(t) = -4t^3 + t - 8\frac{\zeta(3)}{(2\pi i)^3} - \frac{1}{(2\pi i)^3}\sum_{k=1}^{\infty} n_k \mathrm{Li}_3(q^k) \ , \quad q = e^{2\pi i t} \ . \qquad (6.149)$$

The instanton contributions which are encoded in the integer coefficients n_k can be computed up to any given finite order by requiring that the corresponding period vector $\Pi(t)$ is a solution to the Picard-Fuchs equation (6.142). The instanton numbers n_k for $k \leq 20$ are listed in table 6.1.

k	n_k	k	n_k	k	n_k	k	n_k
1	24	6	62816	11	4035075768	16	553885878032448
2	48	7	516336	12	41309494400	17	6187279738200480
3	224	8	4539696	13	432744979608	18	69914895876133904
4	1248	9	42022520	14	4623051088128	19	798164568524432088
5	8400	10	405055200	15	50231067390600	20	9196286679263451840

Table 6.1: The genus-zero instanton numbers n_k for the one-parameter family of $\mathbb{Z}_5$-quotients for Hulek-Verrill threefolds $\mathrm{H}\Lambda^3_t$ for $k \leq 20$.

We note that equation (6.149) provides an expression for the prepotential in terms of the affine coordinate

$$t = \frac{X^1(z)}{X^0(z)} \ . \qquad (6.150)$$

However, the expressions (6.72) and (6.92), which we derived for the second Griffiths and Weil intermediate Jacobians respectively, were given in terms of the projective coordinates $X^i = X^i(z)$. We recall that in projective coordinates, the prepotential was defined to be homogenous of degree two. This property allows to relate the prepotentials in these two chosen coordinate systems by the simple relation

$$F(X^0, X^1) = (X^0)^2 F\left(1, \frac{X^1}{X^0}\right) = (X^0)^2 F(t) \ , \quad F(t) := F(1,t) \ . \tag{6.151}$$

Inserting the explicit expression for $F(t)$, the prepotential in projective coordinates becomes

$$F(X^0, X^1) = -4\frac{(X^1)^3}{X^0} + X^0 X^1 - 8\frac{\zeta(3)}{(2\pi i)^3}(X^0)^2 - \frac{(X^0)^2}{(2\pi i)^3}\sum_{k=1}^{\infty} n_k \mathrm{Li}_3(q^k) \ . \tag{6.152}$$

6.5.2 Griffiths and Weil Intermediate Jacobian of HV_z^3

In order to apply the Calabi-Yau-to-curve correspondence, we need the explicit value for the matrix $N(X)$ that characterizes the Weil intermediate Jacobian $J_W^2(\mathrm{HV}_z^3)$. Before stating the result for $N(X)$ which is obtained by equation (6.94), we first compute the corresponding Griffiths intermediate Jacobian. The motivation for this excurse is twofold. First, the entries of the matrix $F_{ab}(X)$ defining the Griffiths intermediate Jacobian appear as contributions to $N(X)$ anyways and second, we can use this computation to demonstrate that $J_G^2(\mathrm{HV}_z^3)$ has indeed the proposed properties of not being an algebraic variety but describing a holomorphic family of complex tori over $\mathcal{M}_{C.S.}$.

We recall from section 6.3.1 that the second Griffiths intermediate Jacobian of a Calabi-Yau threefold $X_{\mathbf{z}}$ was given by

$$J_G^2(X) \cong \frac{\mathbb{C}^d}{\mathbb{Z}^d + \mathbb{Z}^d(F_{ab})(X)} \ , \quad d = h^{2,1} + 1 \tag{6.153}$$

where the symmetric $(d \times d)$-matrix $F_{ab}(X)$ is given in terms of second derivatives of the projective prepotential $F(X)$ according to

$$F_{ab}(X) = \partial_{X^a}\partial_{X^b} F(X) \ . \tag{6.154}$$

Reexpressing this matrix in terms of the affine coordinates $t^i = X^i/X^0$ and the corresponding affine prepotential $F(\mathbf{t})$ we find

$$F_{ab}(\mathbf{t}) = \begin{pmatrix} F_{00}(\mathbf{t}) & F_{0i}(\mathbf{t}) \\ F_{0j}(\mathbf{t}) & F_{ij}(\mathbf{t}) \end{pmatrix} \tag{6.155}$$

for $i, j = 1, \dots, d-1$ with

$$\begin{aligned} F_{00}(\mathbf{t}) &:= 2F(\mathbf{t}) - 2t^i\partial_i F(\mathbf{t}) + t^i t^j \partial_i\partial_j F(\mathbf{t}) \\ F_{0i}(\mathbf{t}) &:= \partial_i F(\mathbf{t}) + t^j \partial_i\partial_j F(\mathbf{t}) \\ F_{ij}(\mathbf{t}) &:= \partial_i\partial_j F(\mathbf{t}) \ . \end{aligned} \tag{6.156}$$

Concrete for the one-parameter family HV_z^3 of Hulek-Verrill threefolds, these expressions for the (2×2)-matrix F_{ab} are evaluated to be

$$(F_{ab})(t) = \begin{pmatrix} F_{00}(t) & F_{01}(t) \\ F_{10}(t) & F_{11}(t) \end{pmatrix} = \begin{pmatrix} -8t^3 - 16\frac{\zeta(3)}{(2\pi i)^3} & 12t^2+1 \\ 12t^2+1 & -24t \end{pmatrix} \tag{6.157}$$
$$+ \begin{pmatrix} -t^2 & t \\ t & -1 \end{pmatrix} \frac{1}{2\pi i} \sum_{k=1}^{\infty} k^2 n_k \mathrm{Li}_1(q^k) + \begin{pmatrix} 2t & -1 \\ -1 & 0 \end{pmatrix} \frac{1}{(2\pi i)^2} \sum_{k=1}^{\infty} k n_k \mathrm{Li}_2(q^k) \ .$$

From this result, we can read off directly that $(F_{ab})(z)$ is a matrix-valued holomorphic function of the complex structure modulus z which is encoded in $t(z)$ via the holomorphic mirror map. This shows that $J_G^2(\mathrm{HV}_z^3)$ yields indeed a holomorphic family of complex tori over the complex structure moduli space $\mathcal{M}_{C.S.}$ of HV_z^3.

In order to show that $J_G^2(\mathrm{HV}_z^3)$ is not polarized, we have seen that it suffices to discuss whether $\mathrm{Im}(F_{ab})$ is positive definite. To that end, we recall that the expansion of the periods in terms of the Frobenius basis and hence, the expression (6.157) are valid only in the vicinity of the large complex structure limit $z \to 0$. From the asymptotic structure of $X^1(z)$ and $X^0(z)$ we can infer that the large complex structure limit translates into $\mathrm{Im}(t) \to \infty$. Thus, the expression for $F_{ab}(t)$ is valid in the regime for which the affine coordinate t has a large, positive imaginary part which supresses the quantum corrections that are encoded in the instanton sums over the polylogarithms $\mathrm{Li}_n(q^k)$. In this limit, the leading order term

$$(F_{ab})(t) = \begin{pmatrix} -8t^3 - 16\frac{\zeta(3)}{(2\pi i)^3} & 12t^2+1 \\ 12t^2+1 & -24t \end{pmatrix} + \mathcal{O}(q) \tag{6.158}$$

dominates the behavior of the Griffiths intermediate Jacobian. Its imaginary part can be computed simply in terms of the real and imaginary part t yielding

$$\mathrm{Im}(F_{ab})(t) = 8\mathrm{Im}(t) \begin{pmatrix} \mathrm{Im}(t)^2 - 3\mathrm{Re}(t)^2 & 3\mathrm{Re}(t) \\ 3\mathrm{Re}(t) & -3 \end{pmatrix} - \begin{pmatrix} 2\frac{\zeta(3)}{\pi^3} & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{O}(q) \tag{6.159}$$

which is an indefinite matrix in the regime of large $\mathrm{Im}(t)$. By this argument, we can conclude that at least in the vicinity of the large complex structure point, $J_G^2(\mathrm{HV}_z^3)$ does not describe an abelian variety and hence cannot be identified with the first intermediate Jacobian of a genus-two curve.

This result was of course to be expected from the abstract discussion of section 6.3.1 but nevertheless it provides a non-trivial demonstration of these general statements for the example of the one-parameter family of Hulek-Verrill threefolds. We continue by deriving the concrete form of the matrix $N(t)$ that characterizes the corresponding second Weil intermediate Jacobian

$$J_W^2(X) \cong \frac{\mathbb{C}^d}{\mathbb{Z}^d + \mathbb{Z}^d N(X)} \ , \quad d = h^{2,1} + 1 \ . \tag{6.160}$$

From equation (6.94) we recall that the entries of N can be computed from the Griffiths intermediate Jacobian according to

$$N_{ab}(X) = -\bar{F}_{ab} - 2i\frac{\mathrm{Im}(F_{ac})X^c\mathrm{Im}(F_{bd})X^d}{X^eX^f\mathrm{Im}(F_{ef})} \tag{6.161}$$

which is again expressed in terms of the projective coordinates X^i. As for the Griffiths intermediate Jacobian, we rewrite this result in terms of the affine coordinate t by inserting the expression for $F_{ab}(t)$ from equation (6.157). Restricting the result again to the leading order contribution in the large complex structure regime, an explicit evaluation shows that $N(t)$ is indeed a positive definite matrix and hence, $J^2_W(\mathrm{HV}^3_z)$ defines a family of algebraic varieties. On an open neighborhood of the line segement

$$\Delta = \{t \in \mathbb{C} \mid \mathrm{Re}(t) = 0\ ,\ \mathrm{Im}(t) > \tau_0\} \tag{6.162}$$

this argument can be made even analytically without evaluating the concrete terms. Here, τ_0 denotes a lower bound for $\mathrm{Im}(t)$ such that t is close enough to the large complex structure point for the power series of $\Pi(t)$ to converge. We note that for $t \in \Delta$, the expression for $F_{ab}(t)$ simplifies to

$$F_{ab}(t)|_\Delta = \begin{pmatrix} 0 & -12\tau^2+1 \\ -12\tau^2+1 & 0 \end{pmatrix} + i\begin{pmatrix} 8\tau^3 - \frac{2\zeta(3)}{\pi^3} & 0 \\ 0 & -24\tau \end{pmatrix} + \mathcal{O}(q) \tag{6.163}$$

with $\tau = \mathrm{Im}(t)$ parametrizing the line segment Δ. Thus, in the large complex structure-limit, the matrix $N(t)$ simplifies to

$$N(t)|_\Delta = \begin{pmatrix} 4i\tau^2 - i\frac{\zeta(3)}{4\pi^3} & -1 \\ -1 & 12i\tau \end{pmatrix} + \frac{\frac{9\zeta(3)}{4\pi^3}}{\tau^3 - \frac{\zeta(3)}{16\pi^3}}\begin{pmatrix} -i\frac{\zeta(3)}{16\pi^3} & \tau^2 \\ \tau^2 & i\tau \end{pmatrix} + \mathcal{O}(q) \tag{6.164}$$

implying that $\mathrm{Im}(N(t))$ is positive definite for all $t \in \Delta$ if we choose τ_0 sufficiently large. Since the entries of $N(t)$ vary smoothly with t, this condition extends at least to an open neighborhood of the line segment Δ. By extensive numerical checks for points close to this special line segments, we verify that $\mathrm{Im}(N(t))$ is positive definite for all t that are sufficiently close to the large complex structure point.

Again, from the general discussions in section 6.3.2, this observation is not very surprising since we have argued by general arguements that $J^2_W(t)$ defines always an algebraic variety for any family of Calabi-Yau threefolds.

6.5.3 The Calabi-Yau-to-Curve Correspondence for HV^3_z

For the purpose of computing the maximal cut of the four-loop equal mass banana integral $I_{1,1,1,1,1}(z)$, we are mainly interested in values for the complex structure modulus z, that describe physical momenta. Recalling that

$$z = \frac{m^2}{p^2} \tag{6.165}$$

for m being the mass of the internal propagators and p being the external momentum, we can conclude that $I_{1,1,1,1,1}(z)$ corresponds to a physical process only if

$$z \in (0, 1/5) =: \Delta_{\mathbb{R}} \tag{6.166}$$

which desribes the energy conservation condition $p^2 > 5m^2$ for this process. One may note that the boundary point $z = 1/5$ realizes precisely a conifold singularity of the complex structure moduli space $\mathcal{M}_{C.S.}$ of HV^3_z.

Via the mirror map

$$t = \frac{1}{2\pi i}\left(\frac{X^1(z)}{X^0(z)}\right) = \frac{1}{2\pi i}\left(\log(z) + \frac{B(z)}{A(z)}\right) \tag{6.167}$$

with A, B being holomorphic functions in z whose series expansion around $z = 0$ is given by real-valued coefficients, we find that the affine coordinate t is purely imaginary on the physical line segment. To be precise, we find that

$$t \in i(t_0, \infty)\ ,\ t_0 = \frac{1}{2\pi}\lim_{z\to(1/5)}\left(\log(z) + \frac{B(z)}{A(z)}\right)\ . \tag{6.168}$$

The notation in equation (6.166) is not chosen coincidentially but indeed the line segment $\Delta_{\mathbb{R}}$ defines a Lagrangian submanifold of the complex structure moduli space of HV^3_z. This property can be seen by recalling that the Kähler two-form on $\mathcal{M}_{C.S.}$ was defined by the imaginary part of the Kähler metric

$$h_{i\bar{j}} := \mathrm{Im}(\partial_i \partial_{\bar{j}} K(z, \bar{z})) \tag{6.169}$$

with the Kähler potential

$$K(z) = -\log\left(2\mathrm{Im}\left(\sum_{a=0}^{h^{2,1}} X^a F_a\right)\right)\ . \tag{6.170}$$

For a one-dimensional complex structure moduli space, the only one non-trivial component of $h_{i\bar{j}}$ is given by

$$h_{1,\bar{1}} = -\mathrm{Im}(\partial_z \partial_{\bar{z}} \log\left(2\mathrm{Im}(X^0 F_0 + X^1 F_1)\right) \tag{6.171}$$

which vanishes for $z \in \Delta_{\mathbb{R}}$ to any order in the z-expansion of the period functions. This observation implies that $\Delta_{\mathbb{R}}$ is a Lagrangian submanifold of $\mathcal{M}_{C.S.}$.

Moreover, we note that the matrix $F_{ab}(t(w))$ decomposes on this physical line segment very nicely into its real and imaginary part since

$$F_{00}(t(w)) \in i\mathbb{R}\ ,\quad F_{01}(t(w)) \in \mathbb{R}\ ,\quad F_{11}(t(w)) \in i\mathbb{R} \tag{6.172}$$

for $t(w)$ being restricted to $w \in \Delta_{\mathbb{R}}$ which holds since $F(X^0, X^1)$ is a homogenous function of degree two that is purely imaginary on the physical sublocus. To leading order in t, this general result has been found already in equation (6.163).

Using this decomposition of $F_{ab}(t(w))$ into real and imaginary parts, the expression for the matrix $N(t(w))$ on $\Delta_{\mathbb{R}}$ can be further simplified by making use of the trivial algebraic identity $\overline{F_{ab}(t)} = F_{ab}(t) - 2\mathrm{Im}(F_{ab})(t)$. Since $\mathrm{Im}(F_{01})(t) = 0$ on $\Delta_{\mathbb{R}}$, we find that

$$N(w) = \left.\begin{pmatrix} -F_{00}(t)\dfrac{F_{00}(t) - t^2F_{11}(t)}{F_{00}(t) + t^2F_{11}(t)} & -F_{01}(t) - \dfrac{2tF_{00}(t)F_{11}(t)}{F_{00}(t) + t^2F_{11}(t)} \\ -F_{01}(t) - \dfrac{2tF_{00}(t)F_{11}(t)}{F_{00}(t) + t^2F_{11}(t)} & F_{11}(t)\dfrac{F_{00}(t) - t^2F_{11}(t)}{F_{00}(t) + t^2F_{11}(t)} \end{pmatrix}\right|_{t=t(w)} \tag{6.173}$$

for any $w \in \Delta_{\mathbb{R}}$. This computational trick allows to holomorphically continue the restricted, real-analytic function $N(t(w))|_{\Delta_{\mathbb{R}}}$ to a holomorphic function $H(t(z))$ on an open neighborhood $U \supset \Delta_{\mathbb{R}}$ by replacing the coordinate $w \in \Delta_{\mathbb{R}}$ with $z \in U$. By this construction, we realize a local family of polarized holomorphic intermediate Jacobians

$$J^2_{\Delta_{\mathbb{R}}}(\mathrm{HV}^3_z) \cong \frac{\mathbb{C}^2}{\mathbb{C}^2 + \mathbb{C}^2 H(t(z))} \tag{6.174}$$

which is characterized by the matrix

$$H(t(z)) = \left.\begin{pmatrix} -F_{00}(t)\dfrac{F_{00}(t) - t^2F_{11}(t)}{F_{00}(t) + t^2F_{11}(t)} & -F_{01}(t) - \dfrac{2tF_{00}(t)F_{11}(t)}{F_{00}(t) + t^2F_{11}(t)} \\ -F_{01}(t) - \dfrac{2tF_{00}(t)F_{11}(t)}{F_{00}(t) + t^2F_{11}(t)} & F_{11}(t)\dfrac{F_{00}(t) - t^2F_{11}(t)}{F_{00}(t) + t^2F_{11}(t)} \end{pmatrix}\right|_{t=t(z)} \tag{6.175}$$

and coincides with the Weil intermediate Jacobian $J^2_W(\mathrm{HV}^3_z)$ on the physical line segment $z \in \Delta_{\mathbb{R}} \subset U$.

Since $\mathrm{Im}(N(t(w)))$ is positive definite on $\Delta_{\mathbb{R}}$, it follows that $\mathrm{Im}(H(t(z)))$ is positive definite if we choose U to be a suitable open subset for which the holomorphic continuation $H(t(z))$ is well-defined and the minimal eigenvalue of $\mathrm{Im}(H(t(z)))$ is still positive. To identify a corresponding subfamily of genus-two curves according to the holomorphic Calabi-Yau-to-curve correspondence

$$\Phi^U_{\Delta_{\mathbb{R}}} : \bar{\mathcal{S}}_2 \longrightarrow \bar{\mathcal{M}}_2^{\mathcal{A}} \tag{6.176}$$

we note that for $g = 2$ the Schottky-locus $\mathcal{S}_2 \subseteq \bar{\mathcal{M}}_2^{\mathcal{A}}$ coincides with the full moduli space of stable genus-two curves. Hence, the domain

$$\bar{\mathcal{S}}_2 := \left(J^2_{\Delta_{\mathbb{R}}}\right)^{-1}(\mathcal{S}_2) \tag{6.177}$$

coincides with U. Thus, it is guaranteed that for any $z \in U$ we find a corresponding genus-two curve

$$\mathcal{C}_2(z) := \Phi^U_{\Delta_{\mathbb{R}}}(z) \tag{6.178}$$

whose first intermediate Jacobian $J^1(\mathcal{C}_2(z))$ agrees with the second polarized holomorphic Jacobian $J^2_{\Delta_\mathbb{R}}(\mathrm{HV}^3_z)$. In the following, we aim for the explicit construction of this subfamily of genus-two curves $\mathcal{C}_2(z)$ by inversion of the map

$$J^1 : \bar{\mathcal{M}}_2^{\mathcal{A}} \to \mathcal{A}_2 \ . \tag{6.179}$$

For this special case of $g = 2$, we are in the nice situation that any stable genus-two curve $\mathcal{C}_2$ is a hyperelliptic curve [150, 232] implying that any stable genus-two curve $\mathcal{C}_2$ can be described as the zero-locus of a polynomial

$$y^2 = (x - \xi_1)(x - \xi_2)(x - \xi_3)(x - \xi_4)(x - \xi_5)(x - \xi_6) \tag{6.180}$$

in $\mathbb{C}^2 \cup \{\infty\}$ that is characterized by the branch points $\xi_i \in \mathbb{C}$. As for elliptic curves, it is possible to use the automorphisms of $\bar{\mathcal{M}}_2$ to fix three of the branch points which justifies the previous observation that $\bar{\mathcal{M}}_2$ is complex three-dimensional with local coordinates being the remaining branch points. A convenient choice [232] is given by

$$\xi_1 = 0 \ , \quad \xi_5 = 1 \ , \quad \xi_6 = \infty \tag{6.181}$$

implying that any stable genus-two curve $\mathcal{C}_2$ is defined by

$$\mathcal{C}_2(\xi_2, \xi_3, \xi_4) = \{(x, y) \in \mathbb{C}^2 \cup \{\infty\} \mid y^2 = x(x-1)(x-\xi_2)(x-\xi_3)(x-\xi_4)\} \tag{6.182}$$

and hence characterized by the three complex moduli ξ_2, ξ_3, ξ_4. For $h^{2,1} = 1$ and consequently $g = 2$, the holomorphic Calabi-Yau-to-curve correspondence $\Phi^U_{\Delta_\mathbb{R}}$ is effectively given by a set of three holomorphic functions $\xi_i(z)$ for $i = 2, 3, 4$ such that

$$\Phi^U_{\Delta_\mathbb{R}}(z) = \mathcal{C}_2(\xi_2(z), \xi_3(z), \xi_4(z)) \ . \tag{6.183}$$

Since the Riemann-Schottky problem has remained as a non-trivial open problem in algebraic geometry for a long time, we can use results from this research which are reviewed in [232] to deduce these holomorphic functions. To that end, we introduce the notation of the Riemann theta function

$$\vartheta(\mathbf{z}, T) = \sum_{\mathbf{n} \in \mathbb{Z}^g} e^{i\pi(\mathbf{n}^T T \mathbf{n} + 2\mathbf{n}^T \mathbf{z})} \tag{6.184}$$

for $\mathbf{z} \in \mathbb{C}^g$ and $T \in \mathcal{H}_g$. One may note that the restriction of T to the Siegel upper half-space is necessary in order to assure that this defining series converges. The Riemann theta function can be generalized to theta functions with characteristic. For $\mathbf{a}, \mathbf{b} \in \mathbb{Q}^g$ we define

$$\vartheta \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} (\mathbf{z}, T) := e^{i\pi(\mathbf{a}^T T \mathbf{a} + 2\mathbf{a}^T(\mathbf{z}+\mathbf{b}))} \vartheta(\mathbf{z} + T\mathbf{a} + \mathbf{b}, T) \ . \tag{6.185}$$

For any hyperelliptic genus-two curve $\mathcal{C}_2$ with corresponding first intermediate Jacobian

$$J^1(\mathcal{C}_2) \cong \frac{\mathbb{C}^2}{\mathbb{C}^2 + \mathbb{C}^2 \tau} \ , \tag{6.186}$$

the lemma of Picard [232] states that the parametrization of $\mathcal{C}_2$ can be chosen such that the branch points ξ_2, ξ_3 and ξ_4 of the curve are given in terms of the matrix $\tau \in \mathcal{H}_2$ by

$$\xi_2(\tau) = \frac{\vartheta_5(\tau)^2\vartheta_6^2(\tau)}{\vartheta_1^2(\tau)\vartheta_4^2(\tau)} \ , \ \xi_3(\tau) = \frac{\vartheta_6(\tau)^2\vartheta_7^2(\tau)}{\vartheta_4^2(\tau)\vartheta_8^2(\tau)} \ , \ \xi_4(\tau) = \frac{\vartheta_5(\tau)^2\vartheta_7^2(\tau)}{\vartheta_1^2(\tau)\vartheta_8^2(\tau)} \tag{6.187}$$

where we use the convenient notation $\vartheta_i(T)$ for the special theta functions [232]

$$\begin{aligned}
\vartheta_1(T) &:= \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0,T) \quad , \quad \vartheta_2(T) := \vartheta \begin{bmatrix} 0 & 0 \\ 1/2 & 1/2 \end{bmatrix} (0,T) \ , \\
\vartheta_3(T) &:= \vartheta \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix} (0,T) \quad , \quad \vartheta_4(T) := \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1/2 \end{bmatrix} (0,T) \ , \\
\vartheta_5(T) &:= \vartheta \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix} (0,T) \quad , \quad \vartheta_6(T) := \vartheta \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} (0,T) \ , \\
\vartheta_7(T) &:= \vartheta \begin{bmatrix} 0 & 1/2 \\ 0 & 0 \end{bmatrix} (0,T) \quad , \quad \vartheta_8(T) := \vartheta \begin{bmatrix} 1/2 & 1/2 \\ 0 & 0 \end{bmatrix} (0,T) \ , \\
\vartheta_9(T) &:= \vartheta \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix} (0,T) \quad , \quad \vartheta_{10}(T) := \vartheta \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} (0,T) \ .
\end{aligned} \tag{6.188}$$

Thus, if we indentify a Hulek-Verrill threefold parametrized by the complex structure modulus $z \in U$ and a stable genus-two curve $\mathcal{C}_2$ by imposing the condition

$$J^2_{\Delta_{\mathbb{R}}}(\mathrm{HV}^3_z) = J^1(\mathcal{C}_2) \tag{6.189}$$

this identification translates into

$$H(t(z)) = \tau(\xi_2, \xi_3, \xi_4) \ , \tag{6.190}$$

for a certain choice of symplectic integral basis on $H^1(\mathcal{C}_2, \mathbb{Z})$. From this identity, we can apply the lemma of Picard to infer the branch points ξ_i as holomorphic functions of z via

$$\xi_i(z) = \xi_i(H(t(z))) \ , \ i = 2, 3, 4 \ . \tag{6.191}$$

By construction, these holomorphic maps provide an explicit description of the action of the local holomorphic Calabi-Yau-to-curve correspondence $\Phi^U_{\Delta_{\mathbb{R}}}$ on the open neighborhood U of $\Delta_{\mathbb{R}}$.

To obtain the actual identification of intermediate Jacobians, it is necessary to take care of the subtlety that equation (6.190) holds true only for one specific choice of the symplectic integral basis on $H^1(\mathcal{C}_2, \mathbb{Z})$. Thus, for a generic choice of symplectic basis, this identification needs to be corrected by a suitable symplectic transformation $\Gamma \in Sp_2(\mathbb{Z})$ such that

$$H(t(z)) = \Gamma\tau(\xi_2(z), \xi_3(z), \xi_4(z)) \tag{6.192}$$

where $\Gamma\tau$ is defined to be the transformation of equation (6.55). Thus, the full data of the Calabi-Yau-to-curve correspondence is given by computing the branch points $\xi(H(t(w)))$

according to their ϑ-function representation and moreover computing the modular transformation Γ such that the matrices $H(t(z))$ and $\tau(\xi_2(z), \xi_3(z), \xi_4(z))$ agree after fixing any symplectic basis for $H^1(\mathcal{C}_2, \mathbb{Z})$.

To finish this discussion, let us demonstrate, how the periods (X^a, F_a) of the Hulek-Verrill threefolds are related to the periods $(\mathcal{X}_k^i, \mathcal{T}_{ik})$ of the corresponding genus-two curve. To that end, we observe that the two holomorphic one-forms of a hyperelliptic genus-two curve can be chosen to be

$$\omega_0 = \frac{dx}{\sqrt{P(x)}} \ , \quad \omega_1 = \frac{xdx}{\sqrt{P(x)}} \tag{6.193}$$

with the polynomial $P(x)$ being defined as

$$P(x) = x(x-1)(x-\xi_2)(x-\xi_3)(x-\xi_4) \ . \tag{6.194}$$

From these one-forms, the period matrices $\mathcal{X}$ and $\mathcal{T}$ can be computed according to equation (6.32) if we specify a symplectic basis (a_i, b^i) of a- and b-cycles generating $H_1(\mathcal{C}_2, \mathbb{Z})$. It can be shown [233] that the periods $\mathcal{X}_i^k$ enjoy an expression in terms of theta functions of the corresponding first intermediate Jacobian according to

$$\mathcal{X} = 2\frac{\vartheta_1\vartheta_4\vartheta_8}{\vartheta_2\vartheta_3\vartheta_9\vartheta_{10}} \begin{pmatrix} \frac{\vartheta_1\vartheta_4\vartheta_8}{\vartheta_5\vartheta_6\vartheta_7}\partial_0\vartheta_{16} & \partial_0\vartheta_{11} \\ \frac{\vartheta_1\vartheta_4\vartheta_8}{\vartheta_5\vartheta_6\vartheta_7}\partial_1\vartheta_{16} & \partial_1\vartheta_{11} \end{pmatrix} \tag{6.195}$$

with

$$\partial_i\vartheta_{11}(T) := \partial_{z^i}\vartheta\begin{bmatrix} 0 & 1/2 \\ 0 & 1/2 \end{bmatrix}(\mathbf{z}, T)\Big|_{\mathbf{z}=0} \ , \ \partial_i\vartheta_{16}(T) := \partial_{z^i}\vartheta\begin{bmatrix} 1/2 & 1/2 \\ 0 & 1/2 \end{bmatrix}(\mathbf{z}, T)\Big|_{\mathbf{z}=0} \tag{6.196}$$

for $i = 0, 1$. It should be noted that these theta functions are invariant under any symplectic transformation of the matrix τ defining the intermediate Jacobian, hence $\mathcal{X}$ can be computed directly from the values of $H(t(z))$. In order to compute the b-cycle periods $\mathcal{T}_{ik}$, we make use of the identification of the intermediate Jacobians

$$H(t(z)) = \Gamma\tau = \Gamma(\mathcal{T}\mathcal{X}^{-1}) \tag{6.197}$$

which holds up to the symplectic change of basis Γ for the a- and b-cycles of the curve that can be computed by enforcing equation (6.192). Thus, the b-cycle periods are obtained by

$$\mathcal{T} = \left(\Gamma^{-1}H(t(z))\right)\mathcal{X} \ . \tag{6.198}$$

Let us finally restrict this identification to physical values for the complex structure modulus z. That means, we assume $z \in \Delta_\mathbb{R}$ and hence, the matrix $H(t(z))$ coincides with the matrix $N(t(z))$ defining the Weil intermediate Jacobian. Recalling that

$$H(t(z)) = \left.\begin{pmatrix} -F_{00}(t)\dfrac{F_{00}(t) - t^2F_{11}(t)}{F_{00}(t) + t^2F_{11}(t)} & -F_{01}(t) - \dfrac{2tF_{00}(t)F_{11}(t)}{F_{00}(t) + t^2F_{11}(t)} \\ -F_{01}(t) - \dfrac{2tF_{00}(t)F_{11}(t)}{F_{00}(t) + t^2F_{11}(t)} & F_{11}(t)\dfrac{F_{00}(t) - t^2F_{11}(t)}{F_{00}(t) + t^2F_{11}(t)} \end{pmatrix}\right|_{t=t(z)} \tag{6.199}$$

we can conclude by direct computation[118] that the integral period vector Π of the Calabi-Yau threefold is expressible as

$$\Pi = \begin{pmatrix} X^0 \\ X^1 \\ F_0(\mathbf{X}) \\ F_1(\mathbf{X}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -H_{00} & -H_{01} \\ -H_{01} & -H_{11} \end{pmatrix} \begin{pmatrix} X^0 \\ X^1 \end{pmatrix} . \tag{6.200}$$

Inserting the corresponding matrix $\tau = \mathcal{T}\mathcal{X}^{-1}$, we obtain a direct identification of the Calabi-Yau periods Π and the genus-two periods $\mathcal{X}$ and $\mathcal{T}$ which is explicitly given by

$$\Pi(X^0, X^1) = \begin{pmatrix} \mathbb{1}_2 \\ \Gamma\tau \end{pmatrix} \begin{pmatrix} X^0 \\ X^1 \end{pmatrix} . \tag{6.201}$$

This result concludes the discussion of the explicit Calabi-Yau-to-curve correspondence for the four-loop equal mass banana intergral. To summarize, this constructions provides a local identification of the $\mathbb{Z}_5$-quotient of Hulek-Verrill threefolds $\mathrm{HV}^3_z/\mathbb{Z}_5$ with a one-dimensional family of genus-two curves that are characterized by the branch points (6.191) in the vicinity of the Lagrangian submanifold $\Delta_{\mathbb{R}}$ which describes the physical values for z. In particular, equation (6.201) provides the connection of the integral periods Π of the Calabi-Yau threefold with the corresponding periods of the genus-two curve which are encoded in the matrix τ.

[118] Recall that the prepotential $F(X^0, X^1)$ was homogeneous of degree two. This condition implies in particular that $F_a = X^b F_{ab}$.

Chapter 7

Conclusions

In this thesis, we have studied various tools from algebraic geometry and number theory in order to analyze Calabi-Yau gometries in the context of physical applications. After providing a brief introduction to the concept of string and M-theory compactifications on a Calabi-Yau manifold in chapter 2 and discussing the most important properties of Calabi-Yau manifolds and their moduli spaces in chapter 3, this thesis focuses on two concrete applications of these geometries.

First, we investigate the existence of supersymmetric flux vacua on type IIB string theory and M-theory compactifications. Since these arise as vacuum configurations of flux compactifications which provide a promising approach for realistic string models, it is of great interest to characterize whether a Calabi-Yau manifold admits a non-trivial flux vector. The vacuum constraints on quantized fluxes are originally stated in terms of a superpotential in $\mathcal{N} = 1$ supergravity. However, type IIB string theory compactified on a Calabi-Yau threefold gives rise to an $\mathcal{N} = 2$ supergravity theory whose supersymmetry algebra prohibits the existence of any superpotential. In chapter 4, we present an alternative description of the convenient flux vacuum constraints which is formulated in the framework of gauged $\mathcal{N} = 2$ supergravity by gauging suitable isometries of its target space. In particular, any $\mathcal{N} = 2$ supergravity theory originating as the low energy limit of type IIB string theory compactified on a Calabi-Yau threefold contains a universal hypermultiplet whose isometries can be gauged in such a way that the resulting interactions between vector multiplets and hypermultiplets lead to an $\mathcal{N} = 2$ Minkowski vacuum locus that is equivalent to the flux vacuum locus in $\mathcal{N} = 1$ supergravity.

Since any Calabi-Yau compactification of type IIB string theory leads to a universal hypermultiplet, the given construction is valid independently of the choice of Calabi-Yau manifold. It should be noted, however, that the process of gauging yields a Higgs mechanism that transforms some contributing fields into massive modes which need to rearrange into massive multiplets. While this rearrangement is obvious from the abstract representation theoretic analysis, it would be very interesting to see this happen by explicitly constructing the corresponding massive multiplets. It may be suggestive that it is not sufficienct to consider a pure gauging of the universal hypermultiplet alone, but that one needs to include additional hypermultiplets in the discussion. This suggestion is supported by the observation [121] that, for instance, extremal transitions between topologically distinct Calabi-Yau threefolds can be described as vacua of the corresponding gauged $\mathcal{N}=2$ supergravity theory. These transitions are well understood in the local limit of supersymmetric quantum field theories [120], but a realization in terms of global $\mathcal{N}=2$ supergravity requires a careful treatment of the full target space including all hypermultiplets.

Similar to type IIB flux compactifications, it is possible to construct three-dimensional theories by compactifying M-theory on a Calabi-Yau fourfold X admitting an analogous flux superpotential. The corresponding flux vacuum constraints restrict the quantized four-form flux $G \in H^4(X,\mathbb{Z})$ to be of specific Hodge type $(4,0)+(2,2)+(0,4)$. Thus, a Calabi-Yau fourfold gives rise to a flux compactification of M-theory with a well-defined vacuum only if its integral middle cohomology contains a sublattice

$$\Lambda \subset H^4(X,\mathbb{Z}) \cap \left(H^{4,0}(X,\mathbb{C}) \oplus H^{2,2}(X,\mathbb{C}) \oplus H^{0,4}(X,\mathbb{C})\right) \ .$$

In chapter 5 we argue that the existence of such a sublattice is related to the modularity of the Calabi-Yau fourfold. In particular, we claim that Serre's conjecture implies under certain assumptions that X is modular if its integral middle cohomology admits a two-dimensional sublattice that is of definite Hodge type. Thus, by identifying modular Calabi-Yau fourfolds, one obtains candidates of fourfolds whose integral middle cohomology has a sublattice containing a suitable four-form flux. Modularity provides a fascinating correspondence between modular forms and the number of points on a variety that is defined over the finite field $\mathbb{F}_{p^r}$. While elliptic curves and rigid Calabi-Yau threefolds are proven to be modular, it is argued that generic Calabi-Yau n-folds are not modular.

We present a method to compute the factor $R_H(X,T)$ of the local zeta function $\zeta_p(X,T)$ corresponding to the primary horizontal subspace of the middle cohomology for families of Calabi-Yau fourfolds that depends on a single complex structure modulus. We argue that a persistent rational factorization of this factor into a quadratic polynomial and a remainder for all primes p of good reduction indicates a modular Calabi-Yau fourfold. Focusing on Calabi-Yau fourfolds that are of primary horizontal Hodge type $(1,1,1,1,1)$ and $(1,1,2,1,1)$ respectively, we concretize these abstract discussions in terms of an explicit algorithm and demonstrate its applicability with several examples.

As the main example, we dicuss the one-dimensional subfamily of Hulek-Verrill fourfolds HV^4_z. Using the deformation method in order to compute the polynomial $R_H(\mathrm{HV}^4_z,T)$ for many primes, we argue that $z=1$ describes a point of persistent factorization. We verify this result by determining the concrete modular form that characterizes this modular Calabi-Yau fourfold. Moreover, using insights that are based on Deligne's conjecture, we compute the integral four-forms that span the sublattice of definite Hodge type of the primary horizontal middle cohomology. By this identification, we show that the point $z=1$ is what we call an attractive K3-point, meaning that this sublattice is of Hodge type $(3,1)+(1,3)$. For this special class of fourfolds with one complex structure modulus, the existence of such a sublattice implies that its complement is a sublattice of Hodge type $(4,0)+(2,2)+(0,4)$ implying that HV_1^4 realizes an M-theory compactification with non-trivial four-form fluxes that lead to a consistent vacuum. By numerical computations, we verify that $G \sim \mathrm{Re}(\Pi(1))$ defines a suitable integral flux vector on HV_1^4.

The applicability of the proposed deformation method for Calabi-Yau fourfolds is based on the assumption that the Frobenius map Fr_p is block-diagonal with respect to the decomposition of the middle cohomology into the primary horizontal subspace and its complement. This assumption implies in particular that the characteristic polynomial $R_4(X,T)$ factorizes into a horizontal part $R_H(X,T)$ and a remaining polynomial. While we do not have a formal argument that this assumption holds, the considered examples give rise to results which are in full accordance with the Weil conjectures. This observation, together with the independent consistency checks for the modular Hulek-Verrill fourfold suggests that this assumption holds for all examples that have been examined. It would be very interesting to investigate further, whether this assumption is generically obeyed for a broader class of Calabi-Yau fourfolds, or even whether it can be proven formally. As a first step, a systematic application of the deformation method to many families of Calabi-Yau fourfolds could guide towards an answer of this open question. Moreover, it would be very helpful to discover independent methods to compute the full zeta function $\zeta_p(X,T)$ for certain Calabi-Yau fourfolds, in order to compare whether they contain the polynomial $R_H(X,T)$ as a factor.

Another path to follow in future work is provided by the natural generalization of this deformation method to multi-parameter families of Calabi-Yau fourfolds. This extension is of particular interest for physical applications in the context of M- and F-theory flux compactifications as these often involve internal Calabi-Yau fourfolds with many complex structure moduli. The recent generalization to the multi-parameter case in the context of modular Calabi-Yau threefolds [96] promises a good chance that a similar extension can be achieved for fourfolds as well.

As discussed, modularity is related to the existence of two-dimensional sublattices of the integral middle cohomology of a Calabi-Yau manifold. State of the art methods, like the presented deformation method, usually search for single modular points $z \in \mathcal{M}_{C.S.}$ on the complex structure moduli space. It would be a very interesting task to develop an efficient method searching not for single modular points but for hyperplanes, in the simplest case one-dimensional curves, on $\mathcal{M}_{C.S.}$ which consistently admit modular Calabi-Yau manifolds. For example, considering an extremal transition between Calabi-Yau threefold moduli spaces that differ by exactly one complex structure modulus, we expect that the middle cohomology of the threefolds X on the transition locus splits into a direct sum

$$H^3(X,\mathbb{Q}) = H^3(Y,\mathbb{Q}) \oplus V \tag{7.1}$$

with Y being the corresponding Calabi-Yau threefold with less complex structure moduli and V being a two-dimensional remaining subspace that has to be of Hodge type $(2,1)+(1,2)$. Thus, any such extremal transition should realize a persistent split of the middle cohomology and hence modularity on the transition locus. Observing the existence of such extended loci of persistent modularity on a Calabi-Yau moduli space would hence indicates a non-trivial topological behavior of the corresponding manifolds such as a topology changing transition.

For the discussions in chapter 6, we leave the landscape of string compactifications and modular Calabi-Yau manifolds by investigating the appearance of geometry in the context of Feynman integrals. It is well-known that certain multi-loop Feynman integrals enjoy a realization in terms of period integrals of geometric objects such as Calabi-Yau varieties or hyperelliptic curves. The four-loop equal mass banana integral is a particular example which can be realized by both, the periods of a family of Calabi-Yau threefolds and the periods of a family of genus-two hyperelliptic curves. We show that this simultaneous realization is not coincidential but can be formalized in terms of a correspondence between Calabi-Yau threefolds with $g-1$ complex structure moduli and genus-g curves. Observing that the relevant information on the periods is encoded in the intermediate Jacobians of both geometries, this correspondence can be made explicit by identifying a suitable second intermediate Jacobian of the Calabi-Yau threefold with the first intermediate Jacobian of a genus-g curve. As a concrete example, we construct the one-parameter family of genus-two curves whose periods realize the maximal cut of the four-loop equal mass banana integral by applying this correspondence to the $\mathbb{Z}_5$-quotient of Hulek-Verrill threefolds.

This construction has two subtleties. First, we observe that choosing the second Weil intermediate Jacobian on the Calabi-Yau side leads to a non-holomorphic correspondence between the moduli spaces. In order to restore holomorphicity, we propose a new type of intermediate Jacobians which we call polarized holomorphic Jacobians turning the correspondence into a holomorphic map between the moduli spaces. This new class of Jacobians cannot be defined globally on the full complex structure moduli space of the threefolds but provides a well-defined abelian variety only in the vicinity of a Lagrangian submanifold $\Delta_{\mathbb{R}}$ of the complex structure moduli space. Moreover, it should be noted that the periods of the Calabi-Yau threefolds and the corresponding genus-g curves coincide only on this Lagrangian submanifold. The second subtlety is given by the fact that for $g>3$ not every abelian variety is realized as the intermediate Jacobian of a genus-g curve. The subspace of all intermediate Jacobians of genus-g curves is called the Schottky-locus. Thus, for Calabi-Yau threefolds with more than two complex structure moduli the Calabi-Yau-to-curve correspondence is restricted to those threefolds whose intermediate Jacobian lies on the Schottky-locus. One should note that, except for the trivial cases $g=2,3$, the determination of the Schottky-locus is an open problem of algebraic geometry.

From a conceptional point of view, it would be interesting to construct examples of this correspondence on submoduli spaces of Calabi-Yau threefolds with more than two complex structure moduli for which the non-trivial Riemann-Schottky problem becomes relevant.

In particular, if we increase the number of complex structure moduli beyond five, the dimensional discussion suggests that the domain of the Calabi-Yau-to-curve correspondence may even be empty. Investigating whether this case is generic would give interesting insights on possible intersections of the intermediate Jacobians of these threefolds and the Schottky-locus.

For the definition of the intermediate Jacobians of Calabi-Yau threefolds, it is essential that the middle cohomology has a symplectic structure. This property is obeyed for all Calabi-Yau n-folds where n is odd implying that it is possible to define similar intermediate Jacobians for such higher-dimensional Calabi-Yau n-folds as well. Thus, it is suggestive to extend the Calabi-Yau-to-curve correspondence which is presented in this work to these higher dimensional manifolds, which would open a way to identify geometries of Calabi-Yau n-folds and genus-g curves that realize an even broader class of Feynman integrals.

Appendix A

The Field of p-adic Numbers $\mathbb{Q}_p$

The discussion of the local zeta function $\zeta_p(X,T)$ and its relation to the Frobenius map $\mathrm{Fr}_p : H^n(X,\mathbb{Q}_p) \to H^n(X,\mathbb{Q}_p)$ is based on the framework of p-adic analysis and uses several properties that are special for the non-archimedean space of p-adic numbers and power series thereof. This review chapter is neither meant to be a comprehensive nor a pedagogical review article on the mathematical field of p-adic analysis but rather introduces the main concepts which are necessary for the discussions in chapter 5. Additional details can be found in the introductory textbooks [144, 234] on p-adic numbers. Moreover, we refer to the review article [235] which provides a well-written introduction on p-adic numbers.

A natural starting point for this discussion is the very basic fact that for a given natural number $k > 1$, we can represent any non-negative integer $n \in \mathbb{Z}$ by a finite number of digits $n_a \in \{0, \ldots, k-1\}$ such that

$$n = \sum_{a=0}^{N} n_a k^a \ . \tag{A.1}$$

For $k = 2$ this representation provides the famous *binary numeral system*, for $k = 10$ we obtain the common decimal numeral system and for $k = p$ being any prime, we call this representation the p-adic representation of the non-negative integer n. In the following we wish to extend this representation to rational numbers and even a suitable completion thereof which can be achieved by defining a new norm on the field of rational numbers for which the extension of the p-adic representation to an infinite series in powers of p converges.

From now on, p is fixed to be a certain prime. To define the p-adic norm $|\cdot|_p$ on $\mathbb{Q}$, it is necessary to introduce the valuation

$$\nu_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\} \tag{A.2}$$

by the following construction. Any integer $n \in \mathbb{Z}$ has a unique p-adic expansion given by

$$n = \pm \sum_{a=0}^{N} n_a p^a \ . \tag{A.3}$$

For $n \neq 0$ we set

$$\nu_p(n) := \min\left(0 \leq a \leq N \mid n_a \neq 0\right) \tag{A.4}$$

to be the minimal degree of p that contributes to the p-adic expansion[119] of n. For $n = 0$ we set $\nu_p(0) = \infty$. This definition extends to any rational number $\frac{r}{s} \in \mathbb{Q}$ by

$$\nu_p\left(\frac{r}{s}\right) := \nu_p(r) - \nu_p(s) \tag{A.5}$$

which is independent of the choice of representation of the rational number and hence is well-defined.

Given the valuation, we define the p-adic norm $|\cdot|_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}$ by

$$\begin{aligned} &|x|_p := p^{-\nu_p(x)} \text{ for any } x \in \mathbb{Q}\backslash\{0\} \\ &|0|_p := 0 \ . \end{aligned} \tag{A.6}$$

It is easy to proof that $|\cdot|_p$ indeed defines a norm on $\mathbb{Q}$ which has two counterintuitive but very important properties. First, we note that $|\cdot|_p$ is non-Archimedian which means that the triangular inequality is extremized in the sense that

$$|x + y|_p \leq \max(|x|_p, |y|_p) \tag{A.7}$$

and second, it holds that the sequence $(p^n)_{n\in\mathbb{N}}$ converges to zero with respect to this norm

$$\lim_{n\to\infty} p^n = 0 \ . \tag{A.8}$$

These two simple observations deviate drastically from the naive intuition that we have for the field of rational numbers equipped with the standard absolute value $|\cdot|$.

Given a normed field, it is natural to construct its completion by including all limits of Cauchy sequences to the field. For the given case of $\mathbb{Q}$ equipped with the p-adic norm $|\cdot|_p$, we denote the corresponding completion by $\mathbb{Q}_p$. This number field is called the field of *p-adic numbers*. One may note that the p-adic numbers realize a different completion of $\mathbb{Q}$ than the real numbers. In particular, there exist real numbers, such as π, for which no p-adic analog exists and vice versa. It is important to note that any p-adic number $x \in \mathbb{Q}_p$ can be represented in terms of a p-adic expansion

$$x = \sum_{n=n_0}^{\infty} x_n p^n \tag{A.9}$$

[119] In other words, $\nu_p(n)$ is the minimal integer such that $p \nmid n p^{-\nu_p(n)}$.

with $n_0 \in \mathbb{Z}$ and integer coefficients $0 \leq x_n \leq p-1$ with $x_{n_0} \neq 0$ [235]. We note that the valuation of any p-adic number x agrees with the minimal exponent n_0 appearing in the p-adic expansion of x.

An important subset of $\mathbb{Q}_p$ is given by the p-adic integers $\mathbb{Z}_p$ which are defined such that $\mathbb{Q}_p$ appears as the field of fractions of $\mathbb{Z}_p$. It follows that a p-adic number is a p-adic integer if and only if its valuation is non-negative. This observation implies that the ring of p-adic integers is given by

$$\mathbb{Z}_p = \left\{ \sum_{n=0}^{\infty} x_n p^n \mid 0 \leq x_n \leq p-1 \right\} = \{ x \in \mathbb{Q}_p \mid |x|_p \leq 1 \} \ . \tag{A.10}$$

In particular, this classification shows that any integer $n \in \mathbb{Z}$ is a p-adic integer. The opposite statement is not true as for instance any rational number whose denominator is coprime to p is also a p-adic integer.

For the practical computation with p-adic integers it is important to note that $x \in \mathbb{Z}_p$ obeys the relation

$$x \equiv \sum_{n=0}^{N-1} x_n p^n \mod p^N \ . \tag{A.11}$$

Performing an iteration over N, this relation allows to systematically compute the coefficients x_n of the p-adic expansion of any p-adic integer x up to any p-adic precision that is required. Noting that $p^N \to 0$ for $N \to \infty$, contributions of the order $\mathcal{O}(p^N)$ become "small" with respect to the p-adic norm if N increases.

Appendix B

Toric Geometry

Toric geometry provides a very powerful framework for the construction and analysis of certain types of algebraic varieties. In particular, the Calabi-Yau manifolds which are discussed in this thesis enjoy a realization as toric varieties. The following introduction is intended to give a brief overview of the basic concepts of toric geometry, as in particular the construction of toric varieties via fans and lattice polytopes. The content of this appendix is mainly based on the review articles [51, 236, 237] and the discussions within [142, 160]. Comprehensive textbook discussions of toric geometry from a mathematical point of view are given by refs. [184, 186].

B.1 Toric Varieties, Cones and Fans

Following [236], a d-dimensional *toric variety* X is defined as an algebraic variety containing the complex d-dimensional torus $\mathbb{T}^d = (\mathbb{C}^\star)^d$ as a dense subset such that the natural action

$$\mathbb{T}^d \times \mathbb{T}^d \to \mathbb{T}^d \; , \tag{B.1}$$

which is given by componentwise multiplication, extends to a group action on X. Toric varieties enjoy a realization in terms of a quotient space

$$X \cong \frac{\mathbb{C}^n \backslash Z_\Delta}{G} \tag{B.2}$$

for some $n > d$ and $Z_\Delta \subset \mathbb{C}^d$ being a set of points. Moreover, $G = \mathbb{C}^{n-d} \times \Gamma$ realizes a group action on $\mathbb{C}^n$ with Γ being an abelian discrete group. Recalling that the projective space $\mathbb{P}^n$ was defined as

$$\mathbb{P}^n := \frac{\mathbb{C}^{n+1} \backslash \{0\}}{\mathbb{C}^\star} \tag{B.3}$$

we conclude that the definition of toric varieties generalizes the notion of projective spaces.

In order to characterize a toric variety, it hence suffices to specify the point set Z_Δ and the corresponding discrete group Γ. For the analysis of toric varieties it is convenient to define the notion of *cones* and *fans* which can be shown to contain the equivalent information as Z_Δ and Γ and hence provide an alternative description for toric varieties. Let $N \cong \mathbb{Z}^m$ be a lattice of rank m and denote the associated real vector space by $N_\mathbb{R} = N \otimes \mathbb{R}$. A cone[120] of $N_\mathbb{R}$ is a subset $\sigma \subset N_\mathbb{R}$ that is defined by[121]

$$\sigma = \left\{ \sum_{i=1}^{k} a_i v_i \mid a_i \geq 0 \right\} \tag{B.4}$$

for v_i being a finite set of lattice vectors such that $\sigma \cap (-\sigma) = \{0\}$.

Given a lattice N, we can define its dual lattice $M := \text{Hom}(N, \mathbb{R})$ together with the canonical pairing $\langle \cdot, \cdot \rangle : M \times N \to \mathbb{R}$ given by

$$\langle m, n \rangle := m(n) \; . \tag{B.5}$$

Given a cone $\sigma \subset N_\mathbb{R}$, we define the corresponding dual cone $\sigma^\star \subset M$ to be the cone in $M_\mathbb{R}$ which is given by

$$\sigma^\star = \{ m \in M \mid \langle m, n \rangle \geq 0 \text{ for all } n \in \sigma \} \subset M_\mathbb{R} \; . \tag{B.6}$$

For any lattice point $m \in M$, we define the associated hyperplane

$$H_m := \{ n \in \mathbb{N}_\mathbb{R} \mid \langle m, n \rangle = 0 \} \subset N_\mathbb{R} \tag{B.7}$$

and say that H_u supports a cone $\sigma \subset N_\mathbb{R}$ if

$$\sigma \subset H_m^+ := \{ n \in \mathbb{N}_\mathbb{R} \mid \langle m, n \rangle \geq 0 \} \tag{B.8}$$

In that sense, any supporting hyperplane H_m of a cone σ characterizes a *face* $\tau \subset \sigma$ of a cone by defining it to be the intersection of σ with H_m. We note that each (polyhedral) cone has only a finite number of distinct faces and furthermore, each face $\tau \subset \sigma$ is again a cone in $N_\mathbb{R}$. Finally, we define a *fan* Δ in $N_\mathbb{R}$ to be a collection of cones such that

- for $\sigma \in \Delta$ and $\tau \subset \sigma$ a face of σ, we have that $\tau \in \Delta$ and
- if $\sigma, \sigma' \in \Delta$, then their intersection $\sigma \cap \sigma' =: \tau \in \Delta$ defines a face of both cones.

Given such a fan Δ, the following construction allows to associate an affine variety to it which in particular turns out to be a toric variety. Conversely, for each toric variety X it is possible to derive an associated fan Δ [51]. Thus, each toric variety is uniquely characterized by specifying the corresponding fan Δ.

[120]To be precise, we mean by this a strongly convex rational polyhedral cone. Since we are not interested in other kinds of cones within this discussion, we drop the adjectives “strongly convex“, “rational“ and “polyhedral“ and refer to σ simply as a cone.

[121]One my note that this definition is equivalent to σ obeying the properties of being polyhedral, meaning that σ is bounded by a finite number of hyperplanes and strongly convex which means that $\sigma \cap (-\sigma) = \{0\}$. Moreover, σ is rational in the sense that its edges are spanned by a finite number of lattice points.

For each cone $\sigma_i \in \Delta$, let us consider its dual cone $\sigma_i^\star$. Since $\sigma_i^\star$ is generated by finitely many lattice vectors on M, we can find a finite set of lattice points $m_{ij} \in M$ such that

$$\sigma_i^\star \cap M = \left\{ \sum_{j=1}^{k_i} a_j m_{ij} \mid a_k \in \mathbb{N}_0 \right\} \tag{B.9}$$

is a finitely generated commutative semi-group. Hence, for each cone σ_i we can define an associated affine algebraic variety

$$U_{\sigma_i} := \mathrm{Spec}(\mathbb{C}[\sigma_i^\star \cap M]) \ . \tag{B.10}$$

Using the generators m_{ij}, it is possible to express U_{σ_i} in local coordinates $\mathbf{x} = (x_1, \ldots, x_{k_i})$. Let us assume that the generators m_{ij} obey non-trivial relations in M that are generated by a finite number of relations

$$\sum_{j=1}^{k_i} \alpha_{ij}^r m_{ij} = 0 \quad r = 1, \ldots, R \ . \tag{B.11}$$

Then, we can define the coordinate ring $\mathbb{C}[\sigma_i^\star \cap M]$ of $U_{\sigma i}$ to be

$$\mathbb{C}[\sigma_i^\star \cap M] \cong \frac{\mathbb{C}[x_1, \ldots, x_{k_i}]}{\left\langle \left(\prod_{j=1}^{k_i} x_j^{\alpha_{ij}^r} - 1 \right) \right\rangle_{r=1,\ldots,R}} \ . \tag{B.12}$$

Suppose that we have associated such an affine variety to each cone $\sigma \in \Delta$. Then, we would like to define the toric variety corresponding to Δ by suitably gluing together the local patches U_{σ_i}. We note that for any face $\tau \subset \sigma$, it turns out that $U_\tau \hookrightarrow U_\sigma$ is a subspace. Thus, if σ_i and σ_j intersect in a common face $\tau = \sigma_i \cap \sigma_j$, we need to define transition functions

$$u_{ij} : U_{\sigma_i} \cap U_\tau \to U_{\sigma_j} \cap U_\tau \tag{B.13}$$

that follow from relations

$$\sum_{\ell=1}^{k_i} \alpha_{i\ell}^r m_{i\ell} = \sum_{p=1}^{k_j} \alpha_{jp}^r m_{jp} \quad r = 1, \ldots, \tilde{R} \tag{B.14}$$

among the generators of the semi-groups. These give rise to relations

$$\prod_{\ell=1}^{k_i} x_\ell^{\alpha_{i\ell}^r} - \prod_{p=1}^{k_j} y_p^{\alpha_{jp}^r} = 0 \tag{B.15}$$

for the local coordinates $\mathbf{x}$ on U_{σ_i} and $\mathbf{y}$ on U_{σ_j} respectively. Having computed all non-trivial transition functions between the U_{σ_i} in this way, the collection of the local patches U_{σ_i} for all $\sigma_i \in \Delta$, together with the transition functions on common faces, provides the definition of a toric variety which we denote by $\mathbb{P}_\Delta$.

B.2 Polytopes, Normal Fans and Minkowski Sums

Conveniently, a fan can be constructed via polytopes Δ which can be formulated in terms of the *convex hull* of a finite set of lattice points. Given a finite set $S \subset N$ of lattice points in N, then the convex hull of S is defined to be [186]

$$\mathrm{Conv}(S) := \left\{ \sum_{p \in S} \lambda_p p \;\middle|\; \lambda_p \geq 0 \ , \ \sum_{p \in S} \lambda_p = 1 \right\} \subseteq N_{\mathbb{R}} \ . \tag{B.16}$$

$\Delta \subseteq N_{\mathbb{R}}$ is a lattice polytope in $N_{\mathbb{R}}$ if there exists a finite set of lattice points $S \subset N$ such that $\Delta = \mathrm{Conv}(S)$. We call Δ integral, if it contains all of its vertices. In analogy to the discussion of cones, we define an affine hyperplane corresponding to $m \in M_{\mathbb{R}}$ to be

$$H_{m,b} := \{ n \in N_{\mathbb{R}} \mid \langle m, n \rangle = b \} \tag{B.17}$$

for some $b \in \mathbb{R}$ and moreover, we say that $H_{m,b}$ is supporting the polytope Δ if $\Delta \subset H^+_{m,b}$ for

$$H^+_{m,b} := \{ n \in N_{\mathbb{R}} \mid \langle m, n \rangle \geq b \} \ . \tag{B.18}$$

If $H_{m,b}$ is a supporting affine hyperplane of Δ, we call

$$\mathcal{Q} = H_{m,b} \cap \Delta \tag{B.19}$$

a face of Δ.

Given a lattice polytope Δ, it is possible to define a canocial fan $\Sigma(\Delta)$, called the *normal fan* of Δ. The cones of $\Sigma(\Delta)$ are characterized by the faces $\mathcal{Q}$ of Δ and are given by

$$\sigma_{\mathcal{Q}} := \{ \lambda n \mid n \in \mathcal{Q} \ , \ \lambda > 0 \} \ . \tag{B.20}$$

A proof that this collection of cones realizes a well-defined fan is not entirely trivial and can be found for instance in [186]. Thus, any lattice polytope realizes a toric variety by means of constructing its normal fan $\Sigma(\Delta)$ and considering the variety $\mathbb{P}_{\Sigma(\Delta)}$.

Finally, we introduce the Minkowski sum of two polytopes. For two given polytopes Δ and Δ', it is possible to define a new polytope by "adding" Δ and Δ'. This procedure is done by computing the comvex hull of the Minkowski sum of the underlying point sets S and S'. More precisely, if $\Delta = \mathrm{Conv}(S)$ and $\Delta' = \mathrm{Conv}(S')$ then

$$\mathrm{Mink}(\Delta, \Delta') := \mathrm{Conv}(S + S') \tag{B.21}$$

where

$$S + S' := \{ s + s' \mid s \in S \ , \ s' \in S' \} \tag{B.22}$$

is again a finite point set and hence $\mathrm{Mink}(\Delta, \Delta')$ defines a polytope in $N_{\mathbb{R}}$.

Appendix C

Computing Frobenius Periods from Picard-Fuchs Ideals

The following derivation provides an efficient method to compute the vector of Frobenius periods $\varpi^a(z)$ in an open neighborhood of the Large complex structure (LCS) point that correspond to a family of one-parameter Calabi-Yau n-folds for $n = 3, 4$. In sections 3.3.2 and 3.3.3 we have stated the general structure of the periods around the LCS point $z = 0$ in terms of holomorphic functions $A_i(z)$. In the following, the strategy is to expand these holomorphic functions in power series and deduce recursion relations for these that arise from the Picard-Fuchs equation. Given these recursion relations and suitable initial values, the periods can be computed numerically very efficiently and to a high precision that is limited only by the memory storage of the computational device.

This method can well be extended beyond the case of one-parameter families. Here, the recursion relations become more involved because the Picard-Fuchs ideal is generically not generated by a single operator of minimal degree.

C.1 One-parameter Families of Calabi-Yau Threefolds

To begin with, let us consider a given one-parameter family of Calabi-Yau threefolds X_z which is characterized by a Picard-Fuchs operator of the type

$$\mathcal{L} = \sum_{k=0}^{4} f_k(z)\Theta^k \qquad f_k \in \mathbb{Z}[z] \tag{C.1}$$

for $\Theta = z\partial_z$ being the logarithmic derivative. Note that this form of the Picard-Fuchs operator is very generic [189]. We choose the local coordinates on $\mathcal{M}_{C.S.}$ such that the Large complex structure point is located at $z = 0$. Since this singular point does not have a conifold structure, we find that $f_4(0) \neq 0$ which is the only additional requirement we assert to the operator $\mathcal{L}$.

From equation (3.73) we recall that the general structure of the Frobenius periods for a one-parameter family of threefolds is given by

$$\begin{aligned}\varpi^0(z) &= A(z)\\ \varpi^1(z) &= \log(z)A(z) + B(z)\\ \varpi^2(z) &= Y_{111}(\log^2(z)A(z) + 2\log(z)B(z) + C(z))\\ \varpi^3(z) &= \frac{Y_{111}}{3!}\left(\log^3(z)A(z) + 3\log^2(z)B(z) + 3\log(z)C(z) + D(z)\right) \ .\end{aligned} \tag{C.2}$$

with $A(z)$, $B(z)$, $C(z)$ and $D(z)$ being holomorphic functions that obey the normalization condition

$$A(0) = 1 \quad , \quad B(0) = C(0) = D(0) = 0 \ . \tag{C.3}$$

In a neighborhood of $z = 0$, these functions can be expressed in terms of power series

$$A(z) = \sum_{k=0}^{\infty} a_k z^k \ , \ B(z) = \sum_{k=0}^{\infty} b_k z^k \ , \ C(z) = \sum_{k=0}^{\infty} c_k z^k \ , \ D(z) = \sum_{k=0}^{\infty} d_k z^k \ . \tag{C.4}$$

First, let us analyze how $\mathcal{L}$ acts on monomials and logarithms. These building blocks will help in the following discussions to keep track of all contributions to the recursion relations. For any $m \in \mathbb{N}$ we observe that the action of $\mathcal{L}$ on the monomial z^m is given by

$$\mathcal{L}z^m = P_m(z)z^m \quad , \quad P_m(z) := \sum_{k=0}^{4} f_k(z)m^k \ . \tag{C.5}$$

The polynomials $P_m(z)$ have a constant maximal degree of

$$k_{\max} := \deg(P_m) = \max\left(\deg(f_k)\right) \tag{C.6}$$

that is independent of m. Moreover, if we denote the coefficients of these polynomials by p_k^m, we find that

$$\mathcal{L}z^m = \sum_{k=0}^{k_{\max}} p_k^m z^{m+k} = \sum_{k=m}^{k_{\max}+m} p_{k-m}^m z^k \ . \tag{C.7}$$

The logarithmic derivative Θ is defined such that

$$\Theta^k \log(z) = \begin{cases} \log(z) & \text{if } k = 0 \\ 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases} \ . \tag{C.8}$$

In the following, we apply these two rules for the action of $\mathcal{L}$ on the Frobenius periods $\varpi^a(z)$ and deduce recursion relations from the Picard-Fuchs equation $\mathcal{L}\varpi^a(z) = 0$.

The fundamental period
The action of $\mathcal{L}$ on the fundamental period $\varpi^0(z)$ provides a power series

$$\begin{aligned}\mathcal{L}\varpi^0(z) &= \sum_{m=0}^{\infty} a_m \mathcal{L} z^m = \sum_{m=0}^{\infty} a_m \sum_{k=0}^{k_{\max}} p_k^m z^{m+k} = \sum_{k=0}^{k_{\max}} \sum_{n=k}^{\infty} a_{n-k} p_k^{n-k} z^n \\ &\stackrel{\star}{=} \sum_{k=0}^{k_{\max}} \sum_{n=0}^{\infty} a_{n-k} p_k^{n-k} z^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{k_{\max}} a_{n-k}\, p_k^{n-k} \right) z^n\end{aligned} \tag{C.9}$$

whose coefficients are given by finite linear combinations of the coefficients a_k of the holomorphic function $A(z)$. Note that $\star$ holds since $A(z)$ was defined to be a holomorphic function and hence $a_k = 0$ for $k < 0$. The Picard-Fuchs equation $\mathcal{L}\varpi^0(z)$ thus implies order by order that

$$\sum_{k=0}^{k_{\max}} a_{n-k}\, p_k^{n-k} = 0 \tag{C.10}$$

which can be solved in terms of a recursion relation for the coefficients a_n that reads

$$a_n = -\frac{1}{p_0^n} \sum_{k=1}^{k_{\max}} a_{n-k} p_k^{n-k} \quad , \quad a_0 = 1\ . \tag{C.11}$$

We note that the necessary and sufficient condition for this recursion relation to be applicable is that $p_0^n \neq 0$ for all $n \geq 1$. This property is guaranteed if only $f_4(z)$ has a constant contribution which is the case for all Picard-Fuchs operators of the AESZ list [189]. In this case we find that

$$p_0^n = f_4(0) n^4 \tag{C.12}$$

which is non-zero for $n \geq 1$.

The single-log period
It is convenient to rewrite the single-log period $\varpi^1(z)$ in terms of the fundemental period $\varpi^0(z)$ and the remainder $B(z)$ and to use the knowledge that $\mathcal{L}$ annihilates both, $\varpi^1(z)$ and $\varpi^0(z)$.

$$\mathcal{L}\varpi^1(z) = \mathcal{L}\left(\log(z)\varpi^0(z)\right) + \mathcal{L}B(z) = \sum_{k=0}^{4} f_k(z)\Theta^k \left(\log(z)\varpi^0(z)\right) + \mathcal{L}B(z)\ . \tag{C.13}$$

By iterative application of the Leibniz-rule for the first term, we note that only those two terms contribute for which $\Theta^\ell \log(z)$ is non-vanishing. Moreover, the second term can be rewritten in analogy to the previous derivation in terms of a power series whose coefficients are determined by the p_k^m. In total we arrive at the following expression

$$\begin{aligned}\mathcal{L}\varpi^1(z) &= \log(z) \sum_{k=0}^{4} f_k(z)\Theta^k \varpi^0(z) + \sum_{k=1}^{4} f_k(z) k \Theta^{k-1}\varpi^0(z) + \sum_{n=0}^{\infty} \left(\sum_{k=0}^{k_{\max}} b_{n-k}\, p_k^{n-k} \right) z^n \\ &= \log(z) \underbrace{\mathcal{L}\varpi^0(z)}_{=0} + \sum_{k=1}^{4} k f_k(z)\Theta^{k-1}\varpi^0(z) + \sum_{n=0}^{\infty} \left(\sum_{k=0}^{k_{\max}} b_{n-k}\, p_k^{n-k} \right) z^n\ .\end{aligned} \tag{C.14}$$

Thus, the Picard-Fuchs equation $\mathcal{L}\varpi^1(z) = 0$ for the single-log period $\varpi^1(z)$ provides again a recursion relation for the coefficients b_n that is given by

$$b_n = -\frac{1}{p_0^n} \sum_{k=1}^{k_{\max}} b_{n-k} p_k^{n-k} + \Gamma_n \quad , \quad b_0 = 0 \ . \tag{C.15}$$

The homogenous part of this recursion relation, which originates from the term $\mathcal{L}B(z)$ equals to the recursion relation for the fundamental period whereas this relation gets modified by an inhomogeneity Γ_n that is defined by

$$\sum_{n=0}^{\infty} \Gamma_n z^n := \sum_{k=1}^{4} k f_k(z) \Theta^{k-1} \varpi^0(z) \tag{C.16}$$

in terms of the coefficients a_n of the fundamental period.

The double-log period
In analogy to the computation for the single-log period, we obtain an inhomogenous recursion relation for the coefficients c_n of the holomorphic function $C(z)$ by analyzing the Picard-Fuchs equation for $\varpi^2(z)$. Performing a similar reduction of $\varpi^2(z)$ using the expressions for $\varpi^1(z)$ and $\varpi^0(z)$, this derivation leads to

$$c_n = -\frac{1}{p_0^n} \sum_{k=1}^{k_{\max}} c_{n-k} p_k^{n-k} + \Lambda_n \quad , \quad c_0 = 0 \tag{C.17}$$

with the inhomogeneity coefficients Λ_n defined by

$$\sum_{n=0}^{\infty} \Lambda_n z^n := 2 \sum_{k=1}^{4} k f_k(z) \Theta^{k-1} B(z) + \sum_{k=2}^{4} k(k-1) f_k(z) \Theta^{k-2} A(z) \tag{C.18}$$

which depends of the coefficients a_n and b_n of the holomorphic functions $A(z)$ and $B(z)$ respectively. One may note that Λ_0 vanishes and hence the recursion relation gives nontrivial coefficients c_n only for $n \geq 2$.

The top period
As expected, the Picard-Fuchs equation for the top period $\varpi^3(z)$ provides a similar recursion relation for the coefficients d_n of the holomorphic function $D(z)$. Again leaving out the details of this lengthy computation, the result reads

$$d_n = -\frac{1}{p_0^n} \sum_{k=1}^{k_{\max}} d_{n-k} p_k^{n-k} + \Sigma_n \quad , \quad d_0 = 0 \tag{C.19}$$

with inhomogenities

$$\begin{aligned} \Sigma_n = & \sum_{k=3}^{4} k(k-1)(k-2) f_k(z) \Theta^{k-3} A(z) + 3 \sum_{k=2}^{4} k(k-1) f_k(z) \Theta^{k-2} B(z) \\ & + 3 \sum_{k=1}^{4} k f_k(z) \Theta^{k-1} C(z) \ . \end{aligned} \tag{C.20}$$

This recursion relation gives trivial results for the first coefficients as $\Sigma_0 = \Sigma_1 = 0$. Thus, d_n is generically non-zero only for $n \geq 3$.

The four recursion relations (C.11), (C.15), (C.17) and (C.19) provide an efficient algorithm to compute the Frobenius periods $\varpi^a(z)$ for any smooth one-parameter family of Calabi-Yau threefolds up to any given expansion order $n \leq N_{\max}$. Usually, the degree of the polynomials $f_k(z)$ that defines the Picard-Fuchs operator $\mathcal{L}$ is rather small[122] such that the recursion relations boil down to a sum of a handable number of terms.

C.2 One-parameter Families of Calabi-Yau Fourfolds

Next, we perform a similar analysis for the higher dimensional case of one-parameter families of Calabi-Yau fourfolds that are of primary horizontal Hodge type $(1, 1, \ell, 1, 1)$ for some $\ell \geq 1$. The conceptional ideas are similar to the threefold case and even the explicit computations follow straight in analogy. Thus, we restrict the following discussion on providing the resulting expressions for the recursion relations. In addition, we highlight at the end of this section the main difference to the threefold discussion which appears for $\ell > 1$ by the existence of the additional $\ell - 1$ holomorphic solutions.

We use the analog notation as for the previous derivation. Here, the Picard-Fuchs operator

$$\mathcal{L} = \sum_{k=0}^{b} f_k(z)\Theta^k \quad f_k(z) \in \mathbb{Z}[z] \tag{C.21}$$

is a degree-b differential operator with $b = \dim(H^4_H(X_z, \mathbb{C})) = 4+\ell$ denoting the dimension of the primary horizontal subspace. From equation (3.77) we recall that the corresponding vector of Frobenius periods around the large complex structure point $z = 0$ can be expanded according to

$$\begin{aligned}
\varpi^0(z) &= A(z) \\
\varpi^1(z) &= \log(z)A(z) + B(z) \\
\varpi^2(z) &= \log^2(z)A(z) + 2\log(z)B(z) + C(z) \\
\varpi^3(z) &= \log^3(z)A(z) + 3\log^2(z)B(z) + 3\log(z)C(z) + D(z) \\
\varpi^4(z) &= \log^4(z)A(z) + 4\log^3(z)B(z) + 6\log^2(z)C(z) + 4\log(z)D(z) + E(z) \\
\varpi^a(z) &= H^a(z)
\end{aligned} \tag{C.22}$$

for $a = 5, \ldots, b-1$.

[122]Typically degrees for the polynomials $f_k(z)$ that define the Picard-Fuchs operator of a given one-parameter family of Calabi-Yau threefolds is given by $k_{\max} \leq 3$.

Again, $A(z)$, $B(z)$, $C(z)$, $D(z)$, $E(z)$ and the $H^a(z)$ denote holomorphic functions that have a series expansion

$$\begin{aligned} A(z) = \sum_{k=0}^{\infty} a_k z^k \ , \ B(z) = \sum_{k=0}^{\infty} b_k z^k \ , \ C(z) = \sum_{k=0}^{\infty} c_k z^k \ , \ D(z) = \sum_{k=0}^{\infty} d_k z^k \\ E(z) = \sum_{k=0}^{\infty} e_k z^k \ , \ H^a(z) = \sum_{k=0}^{\infty} h_k^a z^k \end{aligned} \tag{C.23}$$

in an open neighborhood around $z = 0$. The initial values for these functions can be chosen such that

$$\begin{aligned} &A(0) = 1 \ , \ B(0) = C(0) = D(0) = E(0) = 0 \\ &\partial_z^k H^a(0) = 0 \text{ for all } k < a - 5 \ . \end{aligned} \tag{C.24}$$

These inital conditions for the additional holomorphic solutions $H^a(z)$ are chosen such that the holomorphic periods become linearly independent.

As for the threefolds, we observe that $\mathcal{L}$ acts on a monomial z^m by multiplication with a polynomial $P_m(z)$ that is determined by the $f_k(z)$ according to

$$\mathcal{L} z^m = P_m(z) z^m \quad , \quad P_m(z) := \sum_{k=0}^{b} f_k(z) m^k \ . \tag{C.25}$$

For the recursion relations it is again convenient to define the coefficients p_k^m of the polynomials $P_m(z)$ as well as their degree

$$k_{\max} := \deg(P_m) = \max\left(\deg(f_k)\right) \tag{C.26}$$

which is again independent of m.

Now, we have set the stage to compute and state the recursion relations for the coefficients of the holomorphic functions that appear in the period expansions. In analogy to the threefold discussion, we obtain the homogenous relation

$$a_n = -\frac{1}{p_0^n} \sum_{k=1}^{k_{\max}} a_{n-k} p_k^n \quad , \quad a_0 = 1 \tag{C.27}$$

for the coefficients of the fundamental period which is meaningful if $p_0^n \neq 0$ for all $n \geq 1$. For all examined examples, this condition is trivially obeyed as only the polynomial $f_b(z)$ which corresponds to the highest order derivative in $\mathcal{L}$ has a non-vanishing constant term and hence

$$p_0^n = n^b f_b(0) \neq 0 \tag{C.28}$$

for $n \geq 1$. By the same computations as in the previous section, we find for the coefficients b_n, c_n, d_n and e_n similar inhomogenous recursion relations which are summarized to be

$$\begin{aligned}
b_n &= -\frac{1}{p_0^n} \sum_{k=1}^{k_{\max}} b_{n-k} p_k^{n-k} + \Gamma_n \quad , \quad b_0 = 0 \\
c_n &= -\frac{1}{p_0^n} \sum_{k=1}^{k_{\max}} c_{n-k} p_k^{n-k} + \Lambda_n \quad , \quad c_0 = 0 \\
d_n &= -\frac{1}{p_0^n} \sum_{k=1}^{k_{\max}} d_{n-k} p_k^{n-k} + \Sigma_n \quad , \quad d_0 = 0 \\
e_n &= -\frac{1}{p_0^n} \sum_{k=1}^{k_{\max}} e_{n-k} p_k^{n-k} + \Xi_n \quad , \quad e_0 = 0 \ .
\end{aligned} \tag{C.29}$$

The inhomogenities Γ_n, Λ_n, Σ_n and Ξ_n are determined in terms of the coefficients of the other holomorphic solutions and can hence be computed iteratively. Explicitly, we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \Gamma_n z^n &:= \sum_{k=1}^{b} k f_k(z) \Theta^{k-1} A(z) \\
\sum_{n=0}^{\infty} \Lambda_n z^n &:= 2 \sum_{k=1}^{b} k f_k(z) \Theta^{k-1} B(z) + \sum_{k=2}^{b} k(k-1) f_k(z) \Theta^{k-2} A(z) \\
\sum_{n=0}^{\infty} \Sigma_n z^n &:= 3 \sum_{k=1}^{b} k f_k(z) \Theta^{k-1} C(z) + 3 \sum_{k=2}^{b} k(k-1) f_k(z) \Theta^{k-2} B(z) \\
&\quad + \sum_{k=3}^{b} k(k-1)(k-2) f_k(z) \Theta^{k-3} A(z) \\
\sum_{n=0}^{\infty} \Xi_n z^n &:= 4 \sum_{k=1}^{b} k f_k(z) \Theta^{k-1} D(z) + 6 \sum_{k=2}^{b} k(k-1) f_k(z) \Theta^{k-2} C(z) \\
&\quad + 4 \sum_{k=3}^{b} k(k-1)(k-2) f_k(z) \Theta^{k-3} B(z) \\
&\quad + \sum_{k=4}^{b} k(k-1)(k-2)(k-3) f_k(z) \Theta^{k-4} A(z) \ .
\end{aligned} \tag{C.30}$$

For families of Calabi-Yau fourfolds that are of primary horizontal Hodge type $(1,1,1,1,1)$, these recursion relations determine the vector of Frobenius periods entirely. However, for the more general setup of families that are of Hodge type $(1,1,\ell,1,1)$ we have $\ell - 1$ additional holomorphic solutions $H^a(z)$ subject to $\mathcal{L} H^a(z) = 0$. Thus, the coefficients of these additional holomorphic periods obey the same differential equation as the fundamental period $\varpi^0(z) = A(z)$ and hence, the coefficients h_n^a are subject to the same recursion relation

$$h_n^a = -\frac{1}{p_0^n} \sum_{k=1}^{k_{max}} h_{n-k}^a \, p_k^n \tag{C.31}$$

for all $a = 5, \ldots, b-1$. Since the $H^a(z)$ and $A(z)$ are by definition linearly independent holomorphic functions, the initial conditions for the $H^a(z)$ can be chosen such that their leading order contribution is given by

$$H^a(z) = z^{a-4} + \mathcal{O}(z^{a-3}) \tag{C.32}$$

which justifies the initial conditions that have been stated already in equation (C.24).

Appendix D

Differential Equations for the Wronskian Matrix $W(z)$

For a family Calabi-Yau n-folds $X_{\mathbf{z}}$ whose primary horizontal subspace $H_H^n(X_{\mathbf{z}},\mathbb{C})$ is generated by the finite set of derivatives $\mathcal{D}^a\Omega(\mathbf{z})$ of the holomorphic n-form, we have defined the Wronskian matrix $W(\mathbf{z})$ in section 5.3.4 by

$$W^{ab}(\mathbf{z}) = \int_{X_{\mathbf{z}}} \mathcal{D}^a\Omega(\mathbf{z}) \wedge \mathcal{D}^b\Omega(\mathbf{z}) \ . \tag{D.1}$$

This matrix plays a central role for the efficient inversion of the period matrix $E(\mathbf{z})$. In the following, we derive a set of differential equations for the components of this matrix in the special case of Calabi-Yau fourfolds with one complex structure modulus that are of primary horizontal Hodge type $(1,1,1,1,1)$ and $(1,1,2,1,1)$. The differential equations can be formulated explictly in terms of the Picard-Fuchs operator $\mathcal{L}$ and hence give rise to an algorithmic possibility to compute $W(\mathbf{z})$ analytically for all families of these types. It should be noted that the results will depend on one and three normalization constants respectively. These constants are determined by the chosen normalization of the holomorphic n-form and hence of the period vector. By inserting the Frobenius solutions for the periods into the identity

$$W(\mathbf{z}) = E^T(\mathbf{z})\sigma E(\mathbf{z}) \ , \tag{D.2}$$

these integration constants can be identified uniquely.

D.1 $W(z)$ for Calabi-Yau Fourfolds of Hodge type $(1,1,1,1,1)$

First, let us consider the Wronskian $W(z)$ of a family of Calabi-Yau fourfolds that are of primary horizontal Hodge type $(1,1,1,1,1)$. We recall that a suitable set of generators for $H_H^4(X_z,\mathbb{C})$ was given by

$$H_H^4(X_z,\mathbb{C}) = \langle \Theta^a\Omega(z)\rangle_{a=0,\dots,4} \tag{D.3}$$

with $\Theta = z\partial_z$ being the logarithmic derivative with respect to the complex structure modulus z. Thus, in this basis of $H_H^4(X_z,\mathbb{C})$, the components of the (5×5)-Wronskian matrix read

$$W^{ab}(z) = \int_{X_z} \Theta^a\Omega(z) \wedge \Theta^b\Omega(z) \ , \quad a,b = 0,\dots,4 \ . \tag{D.4}$$

For the following derivation, we use only two general facts about the holomorphic n-form $\Omega(z)$. First, we note that the logarithmic derivative Θ acts as a map $\Theta : F^i \to F^{i+1}$ among the Hodge filtration of the middle cohomology. Thus, we find in ascending order of the derivatives that

$$\begin{aligned}
\Omega(z) &\in H^{4,0}(X_z,\mathbb{C}) \\
\Theta\Omega(z) &\in H^{4,0}(X_z,\mathbb{C}) \oplus H^{3,1}(X_z,\mathbb{C}) \\
\Theta^2\Omega(z) &\in H^{4,0}(X_z,\mathbb{C}) \oplus H^{3,1}(X_z,\mathbb{C}) \oplus H^{2,2}(X_z,\mathbb{C}) \\
\Theta^3\Omega(z) &\in H^{4,0}(X_z,\mathbb{C}) \oplus H^{3,1}(X_z,\mathbb{C}) \oplus H^{2,2}(X_z,\mathbb{C}) \oplus H^{1,3}(X_z,\mathbb{C}) \\
\Theta^4\Omega(z) &\in H^{4,0}(X_z,\mathbb{C}) \oplus H^{3,1}(X_z,\mathbb{C}) \oplus H^{2,2}(X_z,\mathbb{C}) \oplus H^{1,3}(X_z,\mathbb{C}) \oplus H^{0,4}(X_z,\mathbb{C}) \ .
\end{aligned} \tag{D.5}$$

The integral over X_z in the definition of $W(z)$ gives a contribution only if the integrand is proportial to the volume form $\omega \in H^{4,4}(X_z,\mathbb{C})$. Thus, we can conclude

$$W^{ab}(z) = 0 \quad \text{if } a+b < 4 \tag{D.6}$$

since in these cases, there are at most three anti-holomorphic directions in each appearing wedge product.

Moreover, we find that $W^{ab}(z)$ is a symmetric matrix since

$$\eta \wedge \rho = \rho \wedge \eta \tag{D.7}$$

for any four-forms $\rho, \eta \in H^4(X_z,\mathbb{C})$. These observations reduce the Wronskian matrix to take the form

$$W(z) = \begin{pmatrix} 0 & 0 & 0 & 0 & W^{04}(z) \\ 0 & 0 & 0 & W^{13}(z) & W^{14}(z) \\ 0 & 0 & W^{22}(z) & W^{23}(z) & W^{24}(z) \\ 0 & W^{13}(z) & W^{23}(z) & W^{33}(z) & W^{34}(z) \\ W^{04}(z) & W^{14}(z) & W^{24}(z) & W^{34}(z) & W^{44}(z) \end{pmatrix} \tag{D.8}$$

which depends on 10 rational functions that are so far independent and undetermined.

The second fact about $\Omega(z)$ we use is that it is by definition annihilated by the degree five Picard-Fuchs operator

$$\mathcal{L} = \sum_{k=0}^{5} f_k(z)\Theta^k \tag{D.9}$$

that is characterized by six rational functions $f_k(z)$. Away from the conifold locus that is given by $f_5(z) = 0$, we can use this operator to derive the explicit basis expansion of

$\Theta^5\Omega(z)$ in terms of the generators of $H^4_H(X_z, \mathbb{C})$ which reads

$$\Theta^5\Omega(z) = -\sum_{k=0}^{4} \frac{f_k(z)}{f_5(z)} \Theta^k \Omega(z) \tag{D.10}$$

Now, the strategy is to systematically apply derivatives to the known entries of $W(z)$ in order to derive relations among these. The derivatives of $W^{ab}(z)$ for $a + b \leq 2$ do not provide any additional information as these produce relations among those entries with $a + b \leq 3$ that have been determined already to vanish. However, starting with $a + b = 3$, we obtain the following two non-trivial relations:

$$\begin{aligned} \Theta W^{03}(z) &= \Theta \int_X \Omega(z) \wedge \Theta^3\Omega = \int_X \Theta\Omega(z) \wedge \Theta^3\Omega(z) + \int_X \Omega(z) \wedge \Theta^4\Omega(z) \\ &= W^{13}(z) + W^{04}(z) \\ \Theta W^{12}(z) &= \Theta \int_X \Theta\Omega(z) \wedge \Theta^2\Omega = \int_X \Theta^2\Omega(z) \wedge \Theta^2\Omega(z) + \int_X \Theta\Omega(z) \wedge \Theta^3\Omega(z) \\ &= W^{22}(z) + W^{13}(z) \ . \end{aligned} \tag{D.11}$$

Since both, $W^{03}(z)$ and $W^{12}(z)$ are given by the constant zero-function, their derivatives vanish as well and hence we find the identities

$$W^{22}(z) = -W^{13}(z) = W^{04}(z) \tag{D.12}$$

for the entries on the anti-diagonal. Next, we examine the derivatives of all intependent entries $W^{ab}(z)$ with $a + b = 4$. In analogy to the previous computations we find three non-trivial relations

$$\begin{aligned} \Theta W^{04}(z) &= W^{14}(z) + \int_X \Omega(z) \wedge \Theta^5\Omega(z) \\ \Theta W^{13}(z) &= W^{23}(z) + W^{14}(z) \\ \Theta W^{22}(z) &= W^{23}(z) + W^{32}(z) = 2W^{23}(z) \ . \end{aligned} \tag{D.13}$$

For the first equation we can use the Picard-Fuchs operator $\mathcal{L}$ to rewrite

$$\int_X \Omega(z) \wedge \Theta^5\Omega(z) = -\sum_{k=0}^{4} \frac{f_k(z)}{f_5(z)} W^{0k}(z) = -\frac{f_4(z)}{f_5(z)} W^{04}(z) \ . \tag{D.14}$$

Moreover, we note that equation (D.12) allows to express the left handside of these equations purely in terms of $\Theta W^{04}(z)$. The second and third relation lead to

$$W^{23}(z) = \frac{1}{2}\Theta W^{04}(z) \quad , \quad W^{14}(z) = -\frac{3}{2}\Theta W^{04}(z) \tag{D.15}$$

whereas the first relation gives an additional constraint on $W^{04}(z)$ that reads

$$\Theta W^{04}(z) = -\frac{2}{5}\frac{f_4(z)}{f_5(z)} W^{04}(z) \ . \tag{D.16}$$

Before continuing, let us pause to analyze the current results. So far, we managed to express all non-trivial entries $W^{ab}(z)$ with $a+b \leq 5$ in terms of the single function $W^{04}(z)$. Moreover, equation (D.16) provides an ordinary first order differential equation for $W^{04}(z)$ that determines $W^{04}(z)$ up to an integration constant. This differential equation follows directly from the Picard-Fuchs operator $\mathcal{L}$ and has a solution for each family of Calabi-Yau fourfolds away from the conifold locus. As we will see now, the remaining entries of $W(z)$ are also given in terms of derivatives of $W^{04}(z)$. Thus, it remains to solve the differential equation (D.16) in order to compute the full analytic expression for the Wronskian matrix $W(z)$ up to an overall integration constant.

We continue the systematic analysis of derivatives of known entries of $W(z)$ and consider in the next step all entries with $a+b=5$. Here, we obtain two non-trivial relations that read

$$\begin{aligned} \Theta W^{14}(z) &= W^{24}(z) - \frac{f_4(z)}{f_5(z)} W^{14}(z) - \frac{f_3(z)}{f_5(z)} W^{13}(z) \\ \Theta W^{23}(z) &= W^{33}(z) + W^{24}(z) \ . \end{aligned} \tag{D.17}$$

Again, we have applied the reduction formula (D.10) that was derived from the Picard-Fuchs operator for the first relation. Inserting the expressions for $W^{14}(z)$ and $W^{23}(z)$ these relations provide an expression for $W^{24}(z)$ and $W^{33}(z)$ purely in terms of derivatives of $W^{04}(z)$ that is given by

$$\begin{aligned} W^{24}(z) &= -\frac{3}{2}\Theta^2 W^{04}(z) - \frac{3}{2}\frac{f_4(z)}{f_5(z)}\Theta W^{04}(z) - \frac{f_3(z)}{f_5(z)} W^{04}(z) \\ W^{33}(z) &= \quad 2\Theta^2 W^{04}(z) + \frac{3}{2}\frac{f_4(z)}{f_5(z)}\Theta W^{04}(z) + \frac{f_3(z)}{f_5(z)} W^{04}(z) \ . \end{aligned} \tag{D.18}$$

The remaining two entries of $W(z)$ follow similarly by

$$\begin{aligned} \Theta W^{33}(z) &= 2W^{34}(z) \\ \Theta W^{34}(z) &= W^{44}(z) - \sum_{k=1}^{4} \frac{f_k(z)}{f_5(z)} W^{3k}(z) \ , \end{aligned} \tag{D.19}$$

which provide the expressions

$$\begin{aligned} W^{34}(z) &= \Theta^3 W^{04}(z) + \frac{3}{4}\Theta\left(\frac{f_4(z)}{f_5(z)}\Theta W^{04}(z)\right) + \frac{1}{2}\Theta\left(\frac{f_3(z)}{f_5(z)} W^{04}(z)\right) \\ W^{44}(z) &= \Theta W^{34}(z) + \frac{f_4(z)}{f_5(z)} W^{34}(z) + 2\frac{f_3(z)}{f_5(z)}\Theta^2 W^{04}(z) \\ &\quad + \frac{1}{2}\left(3\frac{f_3(z)f_4(z)}{f_5^2(z)} + \frac{f_2(z)}{f_5(z)}\right)\Theta W^{04}(z) + \left(\frac{f_2(z)f_3(z)}{f_5^2(z)} - \frac{f_1(z)}{f_5(z)}\right) W^{04}(z) \ . \end{aligned} \tag{D.20}$$

The latter is understood to be dependent only on $W^{04}(z)$ and its derivatives by inserting the result for $W^{34}(z)$ for the first two terms. Thus, we reduced all non-trivial entries of the

Wronskian matrix $W(z)$ for families of Calabi-Yau fourfolds that are of primary horizontal Hodge type $(1,1,1,1,1)$ to expressions that depend only on the rational function $W^{04}(z)$ and its derivatives. Moreover, we have observed that this function is the solution of a first order differential equation that is formulated in terms of the data of the Picard-Fuchs operator. Thus, if the Picard-Fuchs operator of a given family of such fourfolds is known, we can determine $W(z)$ up to an overall normalization constant that remains from solving the differential equation (D.16). This constant can be fixed by comparing this analytic expression to the series expansion of the matrix product

$$W(z) = E^T(z)\sigma E(z) \tag{D.21}$$

containing the period matrix $E(z)$.

D.2 $W(z)$ for Calabi-Yau Fourfolds of Hodge type $(1,1,2,1,1)$

Slightly more involved in its derivation but conceptionally analogous, we now provide the solution to $W(z)$ for families of Calabi-Yau fourfolds that are of primary horizontal Hodge type $(1,1,2,1,1)$. Again, we will find that $W(z)$ is determined in terms of the data that is encoded in the Picard-Fuchs operator. In contrast to the former case, the modified Hodge structure will lead to a third order differential equation for $W^{04}(z)$ and hence we are left with three unknown intergration constants. Since the derivation follows in analogy to the previous section, we abbreviate this discussion by simply stating the results of each computation.

For families of this Hodge type, the primary horizontal middle cohomology is generated by

$$H^4_H(X_z,\mathbb{C}) = \langle \Theta^a \Omega(z)\rangle_{a=0,\dots,5} \tag{D.22}$$

leading to a similar expression for $W^{ab}(z)$ as in equation (D.4) but for these families, $W(z)$ extends to a (6×6)-matrix. Using the same argument as in the former case, the Hodge filtration gives

$$W^{ab}(z) = 0 \quad \text{if } a+b<4\ . \tag{D.23}$$

In contrast to the previous discussion, this observation does not restrict $W(z)$ to be an anti-triangular matrix but leads one additional off-diagonal undetermined. Using the symmetry of the wedge product, the undetermined entries of $W(z)$ read

$$W(z) = \begin{pmatrix} 0 & 0 & 0 & 0 & W^{04}(z) & W^{05}(z) \\ 0 & 0 & 0 & W^{13}(z) & W^{14}(z) & W^{15}(z) \\ 0 & 0 & W^{22}(z) & W^{23}(z) & W^{24}(z) & W^{25}(z) \\ 0 & W^{13}(z) & W^{23}(z) & W^{33}(z) & W^{34}(z) & W^{35}(z) \\ W^{04}(z) & W^{14}(z) & W^{24}(z) & W^{34}(z) & W^{44}(z) & W^{45}(z) \\ W^{05}(z) & W^{15}(z) & W^{25}(z) & W^{35}(z) & W^{45}(z) & W^{55}(z) \end{pmatrix} . \tag{D.24}$$

Again, the systematic consideration of the derivatives of known entries of $W(z)$ provide useful reduction formulas for the additional contributions of $W(z)$. Away from the conifold locus, the degree-six Picard-Fuchs operator $\mathcal{L}$ which characterizes the periods gives the relation

$$\Theta^6\Omega(z) = -\sum_{k=0}^{5}\frac{f_k(z)}{f_6(z)}\Theta^k\Omega(z) \tag{D.25}$$

that allows to reduce higher order derivatives of $\Omega(z)$ suitably. Increasing the total number of derivatives on Ω we find

$$\begin{aligned}
\Theta W^{12}(z) &= W^{22}(z) + W^{13}(z)\\
\Theta W^{03}(z) &= W^{13}(z) + W^{04}(z)\\
\Theta W^{04}(z) &= W^{05}(z) + W^{15}(z)\\
\Theta W^{13}(z) &= W^{23}(z) + W^{14}(z)\\
\Theta W^{22}(z) &= 2W^{23}(z)
\end{aligned} \tag{D.26}$$

for the first two non-trivial anti-diagonals of $W(z)$. Since $W^{12}(z) = W^{03}(z) = 0$ are constant, the former two relations further simplify. Again, these relations allow to rewrite every appearing entry in terms of one unknown function which we choose to be $W^{04}(z)$. It follows that

$$\begin{aligned}
W^{22}(z) = -W^{13}(z) = W^{04}(z)\\
W^{23}(z) = -\frac{1}{3}W^{14}(z) = \frac{1}{5}W^{05}(z) = \frac{1}{2}\Theta W^{04}(z)\ .
\end{aligned} \tag{D.27}$$

Starting from now, the reduction formula that arises from the Picard-Fuchs ideal becomes relevant. For the off-diagonal that is characterized by $a + b = 6$ we find the relations

$$\begin{aligned}
&\Theta W^{05}(z) = -\frac{f_4(z)}{f_6(z)}W^{04}(z) - \frac{f_5(z)}{f_6(z)}W^{05}(z) + W^{15}(z)\\
&\Theta W^{14}(z) = W^{24}(z) + W^{15}(z)\\
&\Theta W^{23}(z) = W^{33}(z) + W^{24}(z)\ .
\end{aligned} \tag{D.28}$$

Naively one might expect to find a differential equation of degree-two for $W^{04}(z)$ already at this stage of the reduction relations as the dimension of the matrix increased by one in comparison to the former case. However, since there are three non-trivial entries on this first off-diagonal, all relations are needed to express the new entries according to

$$\begin{aligned}
W^{15}(z) &= \quad\Theta^2 W^{04}(z) + \frac{f_5(z)}{f_6(z)}\Theta W^{04}(z) + \frac{2}{5}\frac{f_4(z)}{f_6(z)}W^{04}(z)\\
W^{24}(z) &= -\frac{5}{2}\Theta^2 W^{04}(z) - \frac{f_5(z)}{f_6(z)}\Theta W^{04}(z) - \frac{2}{5}\frac{f_4(z)}{f_6(z)}W^{04}(z)\\
W^{33}(z) &= \quad 3\Theta^2 W^{04}(z) + \frac{f_5(z)}{f_6(z)}\Theta W^{04}(z) + \frac{2}{5}\frac{f_4(z)}{f_6(z)}W^{04}(z)
\end{aligned} \tag{D.29}$$

and hence there is no relation left to determine $W^{04}(z)$ in terms of a differential equation. This observation is supported from a different point of view. For families of Hodge type

$(1,1,1,1,1)$ we found that the relations for the second non-trivial anti-diagonal, $W^{ab}(z)$ with $a+b=5$, give rise to a first order differential equation for the entry $W^{04}(z)$. For the present case, the number of independent entries of $W^{ab}(z)$ for each anti-diagonal $a+b \geq 5$ is increased by one. By counting the number of entries and relations, we hence expect a differential equation when analyzing the second off-diagonal which is given by $a+b=7$ in this case. Indeed, the reduction relations for this diagonal

$$\begin{aligned}
\Theta W^{15}(z) &= -\frac{f_3(z)}{f_6(z)} W^{13}(z) - \frac{f_4(z)}{f_6(z)} W^{14}(z) - \frac{f_5(z)}{f_6(z)} W^{15}(z) + W^{25}(z) \\
\Theta W^{24}(z) &= W^{34}(z) + W^{25}(z) \\
\Theta W^{33}(z) &= 2 W^{34}(z)
\end{aligned} \tag{D.30}$$

do not only characterize the corresponding entries in terms of $W^{04}(z)$ according to

$$\begin{aligned}
W^{25}(z) &= \Theta W^{15}(z) + \frac{f_5(z)}{f_6(z)} W^{15}(z) - \frac{3}{2} \frac{f_4(z)}{f_6(z)} \Theta W^{04} - \frac{f_3(z)}{f_6(z)} W^{04} \\
W^{34}(z) &= -\frac{3}{2} \Theta^3 W^{04}(z) - 2\Theta W^{15}(z) - \frac{f_5(z)}{f_6(z)} W^{15}(z) + \frac{3}{2} \frac{f_4(z)}{f_6(z)} \Theta W^{04} + \frac{f_3(z)}{f_6(z)} W^{04}
\end{aligned} \tag{D.31}$$

but moreover provide an additional relation that can be used to deduce the differential equation

$$\Theta^3 W^{04}(z) + \Theta W^{15}(z) + \frac{2}{5} \frac{f_2(z)}{f_6(z)} W^{15}(z) - \frac{3}{5} \frac{f_4(z)}{f_6(z)} \Theta W^{04}(z) - \frac{2}{5} \frac{f_3(z)}{f_6(z)} W^{04}(z) = 0 \tag{D.32}$$

for $W^{04}(z)$. This equation can be solved uniquely up to three integration constants. These relations are understood to be given purely in terms of $W^{04}(z)$ by inserting the explicit expression for $W^{15}(z)$.

The remaining entries of the matrix $W(z)$ can now be computed recursively from further derivatives. In the following, we summarize these iterative relations that provide the full solution for the Wronskian matrix $W(z)$ in terms of the rational function $W^{04}(z)$

$$\begin{aligned}
W^{35}(z) &= \Theta W^{25}(z) + \sum_{k=2}^{5} \frac{f_k(z)}{f_6(z)} W^{2k}(z) \\
W^{44}(z) &= \Theta W^{34}(z) - W^{35}(z) \\
W^{45}(z) &= \frac{1}{2} \Theta W^{44}(z) \\
W^{55}(z) &= \Theta W^{45}(z) + \sum_{k=0}^{5} \frac{f_k(z)}{f_6(z)} W^{k4}(z) \ .
\end{aligned} \tag{D.33}$$

The reduction of these formulas with respect to $W^{04}(z)$ leads to quite lengthy expressions. Since we do not need their explicit structure for the practical computation of $W(z)$ but rather use these recursion relations, we do not provide the full exclicit expressions in terms of $W^{04}(z)$ here. Instead, it should be mentioned that there are additional relations among these functions. For instance, $W^{45}(z)$ could be expressed in additon by

$$W^{45}(z) = \Theta W^{35}(z) + \sum_{k=1}^{5} \frac{f_k(z)}{f_6(z)} W^{3k} \ . \tag{D.34}$$

Expressing these additional relations in terms of $W^{04}(z)$ we find, as expected, that these do not provide any new information as they reduce to the differential equation (D.32). Thus, this method provides an explicit and analytic result for the Wronskian matrix $W(z)$ that is determined up to three integration constants which can be fixed by comparing this analytic expression with the series expansion of

$$W(z) = E^T(z)\sigma E(z) \tag{D.35}$$

in terms of the period matrix $E(z)$.

Appendix E

Tables of Frobenius Polynomials

In the following, we collect tables of the characteristic polynomials $R_H(X_z, T)$ for the four one-parameter families of Calabi-Yau fourfolds which have been investigated in chapter 5. For each considered prime $p \geq 7$ we list $R_H(X_z, T)$ for all $z \in \mathbb{F}_p \backslash \{0\}$ for which $\Delta(z) \not\equiv 0 \mod p$ and hence X_z is a smooth variety over $\mathbb{F}_p$. Since $z = 0$ corresponds for all primes to the large complex structure point and hence to a singular geometry, it is excluded from these tables.

These tables are meant to give an overview on the general structure of the characteristic polynomials rather than providing a complete list of the full data that has been aquired for the histograms presented in chapter 5. To that end, we restricted the range of these tables to the primes $7 \leq p \leq 37$ for the considered families of Calabi-Yau fourfolds. The extension of this data to larger primes $p > 37$ follows straight forward by the methods which are provided in section 5.4.

E.1 The One-parameter Family HV_z^4 of Hulek-Verrill Fourfolds

$p = 7$		
z	smooth/singular	$R_H(\mathrm{HV}_z^4, T)$
1	singular	
2	singular	
3	smooth	$(1 - p^2T)(1 - 70T + p^4T^2)(1 + 58T + p^4T^2)$
4	singular	
5	smooth	$(1 - p^2T)(1 + 48T + 586pT^2 + 48p^4T^3 + p^8T^4)$
6	smooth	$(1 - p^2T)(1 + 8T + 426pT^2 + 8p^4T^3 + p^8T^4)$

$p = 11$		
z	smooth/singular	$R_H(\mathrm{HV}_z^4, T)$
1	smooth	$(1+p^2T)^2(1-p^2T)^3$
2	smooth	$(1-p^2T)(1+96T+626pT^2+96p^4T^3+p^8T^4)$
3	singular	
4	singular	
5	smooth	$(1+p^2T)(1-144T+1946pT^2-144p^4T^3+p^8T^4)$
6	smooth	$(1+p^2T)(1-p^2T)^2(1-122T+p^4T^2)$
7	smooth	$(1+p^2T)(1+36T+146pT^2+36p^4T^3+p^8T^4)$
8	smooth	$(1-p^2T)(1+p^2T)^2(1-62T+p^4T^2)$
9	singular	
10	smooth	$(1+p^2T)(1-p^2T)^2(1+178T+p^4T^2)$

$p = 13$		
z	smooth/singular	$R_H(\mathrm{HV}_z^4, T)$
1	smooth	$(1+p^2T)(1-p^2T)^2(1+310T+p^4T^2)$
2	smooth	$(1+p^2T)(1-292T+3774pT^2-292p^4T^3+p^8T^4)$
3	smooth	$(1-p^2T)(1+368T+5334pT^2+368p^4T^3+p^8T^4)$
4	singular	
5	smooth	$(1+p^2T)(1-352T+4974pT^2-352p^4T^3+p^8T^4)$
6	smooth	$(1+p^2T)(1-88T-1746pT^2-88p^4T^3+p^8T^4)$
7	smooth	$(1+p^2T)(1-88T+54pT^2-88p^4T^3+p^8T^4)$
8	smooth	$(1+p^2T)(1-112T-1266pT^2-112p^4T^3+p^8T^4)$
9	singular	
10	singular	
11	smooth	$(1-p^2T)(1+168T+2374pT^2+168p^4T^3+p^8T^4)$
12	smooth	$(1-p^2T)(1+288T+4054pT^2+288p^4T^3+p^8T^4)$

$p = 17$		
z	smooth/singular	$R_H(\mathrm{HV}_z^4, T)$
1	smooth	$(1+p^2T)(1+70T+p^4T^2)(1+14pT+p^4T^2)$
2	smooth	$(1+p^2T)(1-312T+5966pT^2-312p^4T^3+p^8T^4)$
3	smooth	$(1-p^2T)(1+528T+9086pT^2+528p^4T^3+p^8T^4)$
4	smooth	$(1-p^2T)(1+528T+10526pT^2+528p^4T^3+p^8T^4)$
5	smooth	$(1+p^2T)(1-88T-4674pT^2-88p^4T^3+p^8T^4)$
6	smooth	$(1+p^2T)(1-36pT+13286pT^2-36p^5T^3+p^8T^4)$
7	smooth	$(1+p^2T)(1-592T+12246pT^2-592p^4T^3+p^8T^4)$
8	smooth	$(1+p^2T)(1+4pT+1686pT^2+4p^5T^3+p^8T^4)$
9	singular	
10	smooth	$(1-p^2T)(1+288T+5726pT^2+288p^4T^3+p^8T^4)$

11	smooth	$(1+p^2T)(1-568T+11646pT^2-568p^4T^3+p^8T^4)$
12	smooth	$(1+p^2T)(1-888T+20726pT^2-888p^4T^3+p^8T^4)$
13	singular	
14	smooth	$(1+p^2T)(1-24pT+4406pT^2-24p^5T^3+p^8T^4)$
15	smooth	$(1+p^2T)(1-p^2T)^2(1+110T+p^4T^2)$
16	singular	

$p=19$		
z	smooth/singular	$R_H(\mathrm{HV}_z^4, T)$
1	smooth	$(1+p^2T)(1-338T+p^4T^2)(1+22pT+p^4T^2)$
2	smooth	$(1+p^2T)(1-p^2T)^2(1+22T+p^4T^2)$
3	smooth	$(1-p^2T)(1+344T+354pT^2+344p^4T^3+p^8T^4)$
4	smooth	$(1+p^2T)(1-460T+12402pT^2-460p^4T^3+p^8T^4)$
5	singular	
6	singular	
7	smooth	$(1-p^2T)(1+p^2T)^2(1-122T+p^4T^2)$
8	smooth	$(1-p^2T)(1+560T+13962pT^2+560p^4T^3+p^8T^4)$
9	singular	
10	smooth	$(1+p^2T)(1-700T+16482pT^2-700p^4T^3+p^8T^4)$
11	smooth	$(1-p^2T)(1+344T-6pT^2+344p^4T^3+p^8T^4)$
12	smooth	$(1-p^2T)(1-376T+10434pT^2-376p^4T^3+p^8T^4)$
13	smooth	$(1-p^2T)(1+480T+5122pT^2+480p^4T^3+p^8T^4)$
14	smooth	$(1-p^2T)(1+564T+11554pT^2+564p^4T^3+p^8T^4)$
15	smooth	$(1-p^2T)(1+480T+5122pT^2+480p^4T^3+p^8T^4)$
16	smooth	$(1-p^2T)(1+204T-2126pT^2+204p^4T^3+p^8T^4)$
17	smooth	$(1+p^2T)(1-p^2T)^2(1+262T+p^4T^2)$
18	smooth	$(1+p^2T)(1-340T+4962pT^2-340p^4T^3+p^8T^4)$

$p=23$		
z	smooth/singular	$R_H(\mathrm{HV}_z^4, T)$
1	smooth	$(1-p^2T)(1+1010T+p^4T^2)(1-34pT+p^4T^2)$
2	smooth	$(1-p^2T)(1+528T+9554pT^2+528p^4T^3+p^8T^4)$
3	smooth	$(1+p^2T)(1+692T+21594pT^2+692p^4T^3+p^8T^4)$
4	smooth	$(1-p^2T)(1+p^2T)^2(1-410T+p^4T^2)$
5	smooth	$(1+p^2T)(1-p^2T)^2(1+830T+p^4T^2)$
6	singular	
7	smooth	$(1+p^2T)(1-892T+20154pT^2-892p^4T^3+p^8T^4)$
8	smooth	$(1-p^2T)(1-432T+7634pT^2-432p^4T^3+p^8T^4)$
9	smooth	$(1-p^2T)(1+108T+15194pT^2+108p^4T^3+p^8T^4)$
10	smooth	$(1+p^2T)(1-652T+9114pT^2-652p^4T^3+p^8T^4)$
11	smooth	$(1+p^2T)(1-468T+2954pT^2-468p^4T^3+p^8T^4)$

12	smooth	$(1+p^2T)(1-962T+p^4T^2)(1-10pT+p^4T^2)$
13	singular	
14	smooth	$(1-p^2T)(1+648T+5834pT^2+648p^4T^3+p^8T^4)$
15	smooth	$(1-p^2T)(1+648T+5834pT^2+648p^4T^3+p^8T^4)$
16	singular	
17	smooth	$(1-p^2T)(1+1008T+27434pT^2+1008p^4T^3+p^8T^4)$
18	smooth	$(1-p^2T)(1+1010T+p^4T^2)(1-34pT+p^4T^2)$
19	smooth	$(1+p^2T)(1-492T-406pT^2-492p^4T^3+p^8T^4)$
20	smooth	$(1+p^2T)(1-468T+2954pT^2-468p^4T^3+p^8T^4)$
21	smooth	$(1+p^2T)(1+272T-2286pT^2+272p^4T^3+p^8T^4)$
22	smooth	$(1+p^2T)(1-252T+11594pT^2-252p^4T^3+p^8T^4)$

$p=29$		
z	smooth/singular	$R_H(\mathrm{HV}_z^4,T)$
1	smooth	$(1-p^2T)(1+p^2T)^2(1-1178T+p^4T^2)$
2	smooth	$(1+p^2T)(1+200T+45462pT^2+200p^4T^3+p^8T^4)$
3	smooth	$(1-p^2T)(1+864T+7094pT^2+864p^4T^3+p^8T^4)$
4	smooth	$(1-p^2T)(1+384T-17866pT^2+384p^4T^3+p^8T^4)$
5	smooth	$(1-p^2T)(1-216T-9466pT^2-216p^4T^3+p^8T^4)$
6	smooth	$(1-p^2T)(1+p^2T)^2(1-482T+p^4T^2)$
7	smooth	$(1-p^2T)(1-336T-6826pT^2-336p^4T^3+p^8T^4)$
8	smooth	$(1+p^2T)(1-900T+17822pT^2-900p^4T^3+p^8T^4)$
9	smooth	$(1+p^2T)(1-1322T+p^4T^2)(1+38pT+p^4T^2)$
10	smooth	$(1+p^2T)(1-900T+17822pT^2-900p^4T^3+p^8T^4)$
11	smooth	$(1-p^2T)(1+482T+p^4T^2)(1+38pT+p^4T^2)$
12	smooth	$(1-p^2T)(1+864T+7094pT^2+864p^4T^3+p^8T^4)$
13	smooth	$(1+p^2T)(1-300T-25618pT^2-300p^4T^3+p^8T^4)$
14	smooth	$(1-p^2T)(1+864T+48374pT^2+864p^4T^3+p^8T^4)$
15	smooth	$(1+p^2T)(1-600T+15182pT^2-600p^4T^3+p^8T^4)$
16	smooth	$(1-p^2T)(1+1260T+27902pT^2+1260p^4T^3+p^8T^4)$
17	smooth	$(1-p^2T)(1+2040T+71942pT^2+2040p^4T^3+p^8T^4)$
18	smooth	$(1+p^2T)(1-1720T+53142pT^2-1720p^4T^3+p^8T^4)$
19	smooth	$(1+p^2T)(1-160T-34098pT^2-160p^4T^3+p^8T^4)$
20	singular	
21	smooth	$(1-p^2T)(1+1260T+27902pT^2+1260p^4T^3+p^8T^4)$
22	singular	
23	smooth	$(1-p^2T)(1+1320T+30182pT^2+1320p^4T^3+p^8T^4)$
24	smooth	$(1+p^2T)(1-p^2T)^2(1-698T+p^4T^2)$
25	singular	
26	smooth	$(1-p^2T)(1+1260T+27902pT^2+1260p^4T^3+p^8T^4)$
27	smooth	$(1+p^2T)(1-600T-4978pT^2-600p^4T^3+p^8T^4)$

28	smooth	$(1-p^2T)(1-480T+40982pT^2-480p^4T^3+p^8T^4)$

$p=31$		
z	smooth/singular	$R_H(\mathrm{HV}_z^4, T)$
1	smooth	$(1+p^2T)(1-1322T+p^4T^2)(1-2pT+p^4T^2)$
2	singular	
3	smooth	$(1-p^2T)(1+336T-12614pT^2+336p^4T^3+p^8T^4)$
4	smooth	$(1-p^2T)(1-144T-3614pT^2-144p^4T^3+p^8T^4)$
5	smooth	$(1+p^2T)(1-784T+10626pT^2-784p^4T^3+p^8T^4)$
6	smooth	$(1+p^2T)(1-p^2T)(1-2pT+p^4T^2)$
7	smooth	$(1+p^2T)(1-1084T+16266pT^2-1084p^4T^3+p^8T^4)$
8	singular	
9	smooth	$(1-p^2T)(1+2840T+117138pT^2+2840p^4T^3+p^8T^4)$
10	smooth	$(1+p^2T)(1-p^2T)^2(1+418T+p^4T^2)$
11	smooth	$(1+p^2T)(1-304T-6174pT^2-304p^4T^3+p^8T^4)$
12	smooth	$(1+p^2T)(1-364T-33414pT^2-364p^4T^3+p^8T^4)$
13	smooth	$(1-p^2T)(1+660T+5338pT^2+660p^4T^3+p^8T^4)$
14	smooth	$(1+p^2T)(1-1024T+5346pT^2-1024p^4T^3+p^8T^4)$
15	smooth	$(1+p^2T)(1-p^2T)^2(1+598T+p^4T^2)$
16	smooth	$(1-p^2T)(1+80pT+94818pT^2+80p^5T^3+p^8T^4)$
17	smooth	$(1+p^2T)(1-1202T+p^4T^2)(1+58pT+p^4T^2)$
18	smooth	$(1-p^2T)(1+1760T+50178pT^2+1760p^4T^3+p^8T^4)$
19	smooth	$(1+p^2T)(1-1744T+71586pT^2-1744p^4T^3+p^8T^4)$
20	smooth	$(1+p^2T)(1+596T-18774pT^2+596p^4T^3+p^8T^4)$
21	smooth	$(1+p^2T)(1-p^2T)^2(1+418T+p^4T^2)$
22	smooth	$(1+p^2T)(1-3004T+130266pT^2-3004p^4T^3+p^8T^4)$
23	smooth	$(1-p^2T)(1+1918T+p^4T^2)(1-2pT+p^4T^2)$
24	smooth	$(1+p^2T)(1-p^2T)^2(1+118T+p^4T^2)$
25	singular	
26	smooth	$(1-p^2T)(1+922T+p^4T^2)(1+38pT+p^4T^2)$
27	smooth	$(1+p^2T)(1-1804T+55866pT^2-1804p^4T^3+p^8T^4)$
28	smooth	$(1+p^2T)(1-p^2T)^2(1+1738T+p^4T^2)$
29	smooth	$(1+p^2T)(1-724T-8934pT^2-724p^4T^3+p^8T^4)$
30	smooth	$(1-p^2T)(1+576T-16934pT^2+576p^4T^3+p^8T^4)$

$p=37$		
z	smooth/singular	$R_H(\mathrm{HV}_z^4, T)$
1	smooth	$(1-p^2T)(1+p^2T)^2(1-2570T+p^4T^2)$
2	smooth	$(1+p^2T)(1-2008T+101286pT^2-2008p^4T^3+p^8T^4)$
3	smooth	$(1-p^2T)(1+3308T+163086pT^2+3308p^4T^3+p^8T^4)$
4	smooth	$(1-p^2T)(1+768T-35834pT^2+768p^4T^3+p^8T^4)$

5	smooth	$(1+p^2T)(1-1972T+81486pT^2-1972p^4T^3+p^8T^4)$
6	smooth	$(1+p^2T)(1-3472T+157566pT^2-3472p^4T^3+p^8T^4)$
7	singular	
8	smooth	$(1+p^2T)(1-1552T+50046pT^2-1552p^4T^3+p^8T^4)$
9	smooth	$(1-p^2T)(1+1088T+10566pT^2+1088p^4T^3+p^8T^4)$
10	smooth	$(1+p^2T)(1-2452T+93966pT^2-2452p^4T^3+p^8T^4)$
11	smooth	$(1+p^2T)(1-1852T+49566pT^2-1852p^4T^3+p^8T^4)$
12	smooth	$(1-p^2T)(1-2162T+p^4T^2)(1+50pT+p^4T^2)$
13	smooth	$(1+p^2T)(1-1732T+32046pT^2-1732p^4T^3+p^8T^4)$
14	smooth	$(1-p^2T)(1+2448T+80806pT^2+2448p^4T^3+p^8T^4)$
15	smooth	$(1-p^2T)(1+2768T+119526pT^2+2768p^4T^3+p^8T^4)$
16	smooth	$(1+p^2T)(1-4pT-67314pT^2-4p^5T^3+p^8T^4)$
17	smooth	$(1+p^2T)(1+392T+28446pT^2+392p^4T^3+p^8T^4)$
18	smooth	$(1+p^2T)(1+632T-18114pT^2+632p^4T^3+p^8T^4)$
19	smooth	$(1+p^2T)(1-2008T+90846pT^2-2008p^4T^3+p^8T^4)$
20	smooth	$(1-p^2T)(1+2048T+66246pT^2+2048p^4T^3+p^8T^4)$
21	smooth	$(1-p^2T)(1+2208T+64486pT^2+2208p^4T^3+p^8T^4)$
22	smooth	$(1+p^2T)(1-1492T+11406pT^2-1492p^4T^3+p^8T^4)$
23	smooth	$(1-p^2T)(1+1008T-2714pT^2+1008p^4T^3+p^8T^4)$
24	smooth	$(1+p^2T)(1-1528T+31206pT^2-1528p^4T^3+p^8T^4)$
25	smooth	$(1+p^2T)(1-2908T+148446pT^2-2908p^4T^3+p^8T^4)$
26	smooth	$(1-p^2T)(1-1192T+6486pT^2-1192p^4T^3+p^8T^4)$
27	smooth	$(1+p^2T)(1-1228T+24126pT^2-1228p^4T^3+p^8T^4)$
28	singular	
29	smooth	$(1+p^2T)(1-3928T+191526pT^2-3928p^4T^3+p^8T^4)$
30	smooth	$(1+p^2T)(1+128T-13434pT^2+128p^4T^3+p^8T^4)$
31	smooth	$(1-p^2T)(1+1368T+2326pT^2+1368p^4T^3+p^8T^4)$
32	smooth	$(1-p^2T)(1+3308T+163086pT^2+3308p^4T^3+p^8T^4)$
33	smooth	$(1+p^2T)(1-172T-57474pT^2-172p^4T^3+p^8T^4)$
34	smooth	$(1-p^2T)(1+1968T+48166pT^2+1968p^4T^3+p^8T^4)$
35	smooth	$(1-p^2T)(1+1868T+56526pT^2+1868p^4T^3+p^8T^4)$
36	singular	

E.2 The Mirror of the Complete Intersection $\mathbb{P}^7[2,2,4]$

$p=7$		
z	smooth/singular	$R_H(\mathbb{P}^7[2,2,4]^\vee,T)$
1	singular	
2	smooth	$(1+p^2T)(1-32T+274pT^2-32p^4T^3+p^8T^4)$
3	smooth	$(1+p^2T)(1-58T+254pT^2-58p^4T^3+p^8T^4)$
4	smooth	$(1-p^2T)(1-4pT-54pT^2-4p^5T^3+p^8T^4)$
5	smooth	$(1+p^2T)(1-26T-194pT^2-26p^4T^3+p^8T^4)$
6	smooth	$(1-p^2T)(1+32T+338pT^2+32p^4T^3+p^8T^4)$

$p=11$		
z	smooth/singular	$R_H(\mathbb{P}^7[2,2,4]^\vee,T)$
1	smooth	$(1+p^2T)(1-144T+1018pT^2-144p^4T^3+p^8T^4)$
2	smooth	$(1-p^2T)(1+48T-582pT^2+48p^4T^3+p^8T^4)$
3	singular	
4	smooth	$(1+p^2T)(1-48T+442pT^2-48p^4T^3+p^8T^4)$
5	smooth	$(1-p^2T)(1+16T+826pT^2+16p^4T^3+p^8T^4)$
6	smooth	$(1+p^2T)(1-234T+3286pT^2-234p^4T^3+p^8T^4)$
7	smooth	$(1+p^2T)(1-138T+662pT^2-138p^4T^3+p^8T^4)$
8	smooth	$(1+p^2T)(1-202T+2070pT^2-202p^4T^3+p^8T^4)$
9	smooth	$(1-p^2T)(1+212T+3026pT^2+212p^4T^3+p^8T^4)$
10	smooth	$(1-p^2T)(1+400T+6202pT^2+400p^4T^3+p^8T^4)$

$p=13$		
z	smooth/singular	$R_H(\mathbb{P}^7[2,2,4]^\vee,T)$
1	singular	
2	smooth	$(1-p^2T)(1+140T+3870pT^2+140p^4T^3+p^8T^4)$
3	smooth	$(1+p^2T)(1+12T-2658pT^2+12p^4T^3+p^8T^4)$
4	smooth	$(1-p^2T)(1+204T+278p^2T^2+204p^4T^3+p^8T^4)$
5	smooth	$(1-p^2T)(1+12T-482pT^2+12p^4T^3+p^8T^4)$
6	smooth	$(1+p^2T)(1+162T+p^4T^2)(1-16pT+p^4T^2)$
7	smooth	$(1+p^2T)(1-206T+970pT^2-206p^4T^3+p^8T^4)$
8	smooth	$(1+p^2T)(1-366T+5514pT^2-366p^4T^3+p^8T^4)$
9	smooth	$(1-p^2T)^2(1+p^2T)(1-34T+p^4T^2)$
10	smooth	$(1-p^2T)(1+112T-1226pT^2+112p^4T^3+p^8T^4)$
11	smooth	$(1-p^2T)(1+460T+7710pT^2+460p^4T^3+p^8T^4)$
12	smooth	$(1+p^2T)(1-84T-1058pT^2-84p^4T^3+p^8T^4)$

$p = 17$		
z	smooth/singular	$R_H(\mathbb{P}^7[2,2,4]^\vee, T)$
1	smooth	$(1-p^2T)(1+192T-994pT^2+192p^4T^3+p^8T^4)$
2	smooth	$(1+p^2T)(1-36pT+14182pT^2-36p^5T^3+p^8T^4)$
3	smooth	$(1-p^2T)(1+572T+10022pT^2+572p^4T^3+p^8T^4)$
4	smooth	$(1+p^2T)(1-132T-3290pT^2-132p^4T^3+p^8T^4)$
5	smooth	$(1+p^2T)(1-830T+18786pT^2-830p^4T^3+p^8T^4)$
6	smooth	$(1+p^2T)(1-318T+1634pT^2-318p^4T^3+p^8T^4)$
7	smooth	$(1-p^2T)(1+252T+38pT^2+252p^4T^3+p^8T^4)$
8	smooth	$(1-p^2T)(1+636T+12582pT^2+636p^4T^3+p^8T^4)$
9	smooth	$(1+p^2T)(1-100T+3942pT^2-100p^4T^3+p^8T^4)$
10	smooth	$(1+p^2T)(1-350T+2082pT^2-350p^4T^3+p^8T^4)$
11	smooth	$(1+p^2T)(1-286T+7586pT^2-286p^4T^3+p^8T^4)$
12	smooth	$(1-p^2T)(1+828T+18470pT^2+828p^4T^3+p^8T^4)$
13	smooth	$(1+p^2T)(1-260T+7206pT^2-260p^4T^3+p^8T^4)$
14	smooth	$(1-p^2T)(1+124T+2854pT^2+124p^4T^3+p^8T^4)$
15	smooth	$(1-p^2T)(1-5730pT^2+p^8T^4)$
16	singular	

$p = 19$		
z	smooth/singular	$R_H(\mathbb{P}^7[2,2,4]^\vee, T)$
1	smooth	$(1-p^2T)(1+20T+3426pT^2+20p^4T^3+p^8T^4)$
2	smooth	$(1-p^2T)(1+784T+19210pT^2+784p^4T^3+p^8T^4)$
3	smooth	$(1-p^2T)(1+432T+11914pT^2+432p^4T^3+p^8T^4)$
4	smooth	$(1-p^2T)^2(1+p^2T)(1+290T+p^4T^2)$
5	smooth	$(1+p^2T)(1-432T+10890pT^2-432p^4T^3+p^8T^4)$
6	smooth	$(1-p^2T)(1+528T+11530pT^2+528p^4T^3+p^8T^4)$
7	singular	
8	smooth	$(1+p^2T)(1-42T-11834pT^2-42p^4T^3+p^8T^4)$
9	smooth	$(1-p^2T)(1+276T-2206pT^2+276p^4T^3+p^8T^4)$
10	smooth	$(1-p^2T)(1+272T-1782pT^2+272p^4T^3+p^8T^4)$
11	smooth	$(1+p^2T)(1+368T+5066pT^2+368p^4T^3+p^8T^4)$
12	smooth	$(1+p^2T)(1+86T+9926pT^2+86p^4T^3+p^8T^4)$
13	smooth	$(1+p^2T)(1-1002T+24646pT^2-1002p^4T^3+p^8T^4)$
14	smooth	$(1+p^2T)(1-234T-4538pT^2-234p^4T^3+p^8T^4)$
15	smooth	$(1-p^2T)(1+48T-5494pT^2+48p^4T^3+p^8T^4)$
16	smooth	$(1+p^2T)(1+174T+p^4T^2)(1-10pT+p^4T^2)$
17	smooth	$(1-p^2T)(1+144T-3062pT^2+144p^4T^3+p^8T^4)$
18	smooth	$(1+p^2T)(1-714T+17286pT^2-714p^4T^3+p^8T^4)$

$p = 23$		
z	smooth/singular	$R_H(\mathbb{P}^7[2,2,4]^\vee, T)$
1	smooth	$(1+p^2T)(1-416T+13234pT^2-416p^4T^3+p^8T^4)$
2	smooth	$(1+p^2T)(1-192T+17266pT^2-192p^4T^3+p^8T^4)$
3	smooth	$(1-p^2T)(1+1216T+38002pT^2+1216p^4T^3+p^8T^4)$
4	smooth	$(1-p^2T)(1+60pT+40170pT^2+60p^5T^3+p^8T^4)$
5	smooth	$(1+p^2T)(1-762T+13918pT^2-762p^4T^3+p^8T^4)$
6	smooth	$(1-p^2T)(1+100T-3350pT^2+100p^4T^3+p^8T^4)$
7	smooth	$(1+p^2T)(1-54pT+33950pT^2-54p^5T^3+p^8T^4)$
8	smooth	$(1-p^2T)(1+868T+19690pT^2+868p^4T^3+p^8T^4)$
9	smooth	$(1-p^2T)(1+448T-1422pT^2+448p^4T^3+p^8T^4)$
10	smooth	$(1-p^2T)(1+800T+15602pT^2+800p^4T^3+p^8T^4)$
11	smooth	$(1-p^2T)(1+288T-5902pT^2+288p^4T^3+p^8T^4)$
12	singular	
13	smooth	$(1+p^2T)(1-800T+12978pT^2-800p^4T^3+p^8T^4)$
14	smooth	$(1+p^2T)(1-154T-1762pT^2-154p^4T^3+p^8T^4)$
15	smooth	$(1+p^2T)(1+262T+9822pT^2+262p^4T^3+p^8T^4)$
16	smooth	$(1+p^2T)(1-192T-13454pT^2-192p^4T^3+p^8T^4)$
17	smooth	$(1-p^2T)(1+448T+5490pT^2+448p^4T^3+p^8T^4)$
18	smooth	$(1-p^2T)^2(1+p^2T)(1-894T+p^4T^2)$
19	smooth	$(1-p^2T)(1+32T+5362pT^2+32p^4T^3+p^8T^4)$
20	smooth	$(1+p^2T)(1-570T+17374pT^2-570p^4T^3+p^8T^4)$
21	smooth	$(1+p^2T)(1-474T+12958pT^2-474p^4T^3+p^8T^4)$
22	smooth	$(1-p^2T)(1+832T+14194pT^2+832p^4T^3+p^8T^4)$

$p = 29$		
z	smooth/singular	$R_H(\mathbb{P}^7[2,2,4]^\vee, T)$
1	smooth	$(1-p^2T)(1+2124T+74814pT^2+2124p^4T^3+p^8T^4)$
2	smooth	$(1-p^2T)(1+588T+7230pT^2+588p^4T^3+p^8T^4)$
3	smooth	$(1-p^2T)(1+1612T+51774pT^2+1612p^4T^3+p^8T^4)$
4	smooth	$(1+p^2T)(1-1364T+36606pT^2-1364p^4T^3+p^8T^4)$
5	smooth	$(1-p^2T)(1+972T+14654pT^2+972p^4T^3+p^8T^4)$
6	smooth	$(1+p^2T)(1-852T+3838pT^2-852p^4T^3+p^8T^4)$
7	smooth	$(1-p^2T)^2(1+p^2T)(1-354T+p^4T^2)$
8	smooth	$(1+p^2T)(1-46T-26902pT^2-46p^4T^3+p^8T^4)$
9	smooth	$(1-p^2T)(1+240T-3882pT^2+240p^4T^3+p^8T^4)$
10	smooth	$(1+p^2T)(1-1742T+57258pT^2-1742p^4T^3+p^8T^4)$
11	smooth	$(1+p^2T)(1-238T-15766pT^2-238p^4T^3+p^8T^4)$
12	smooth	$(1-p^2T)(1+1420T+57918pT^2+1420p^4T^3+p^8T^4)$
13	smooth	$(1+p^2T)(1+428T+1174p^2T^2+428p^4T^3+p^8T^4)$

14	smooth	$(1+p^2T)(1-526T+5930pT^2-526p^4T^3+p^8T^4)$
15	smooth	$(1+p^2T)(1-750T+28522pT^2-750p^4T^3+p^8T^4)$
16	smooth	$(1-p^2T)(1+60pT+54078pT^2+60p^5T^3+p^8T^4)$
17	smooth	$(1+p^2T)(1-1614T+58282pT^2-1614p^4T^3+p^8T^4)$
18	smooth	$(1-p^2T)(1+60pT+57918pT^2+60p^5T^3+p^8T^4)$
19	smooth	$(1-p^2T)(1+396T-15810pT^2+396p^4T^3+p^8T^4)$
20	smooth	$(1-p^2T)(1-80T-16554pT^2-80p^4T^3+p^8T^4)$
21	smooth	$(1-p^2T)(1+1036T+19774pT^2+1036p^4T^3+p^8T^4)$
22	smooth	$(1+p^2T)(1-340T-18690pT^2-340p^4T^3+p^8T^4)$
23	smooth	$(1+p^2T)(1-2036T+77502pT^2-2036p^4T^3+p^8T^4)$
24	smooth	$(1-p^2T)(1+688T+41302pT^2+688p^4T^3+p^8T^4)$
25	singular	
26	smooth	$(1-p^2T)(1+268T-23234pT^2+268p^4T^3+p^8T^4)$
27	smooth	$(1+p^2T)(1-2158T+78698pT^2-2158p^4T^3+p^8T^4)$
28	smooth	$(1+p^2T)(1-1620T+66814pT^2-1620p^4T^3+p^8T^4)$

$p = 31$		
z	smooth/singular	$R_H(\mathbb{P}^7[2,2,4]^\vee, T)$
1	smooth	$(1-p^2T)(1+2020T+82042pT^2+2020p^4T^3+p^8T^4)$
2	smooth	$(1+p^2T)(1-32pT+34690pT^2-32p^5T^3+p^8T^4)$
3	smooth	$(1-p^2T)(1+608T-11070pT^2+608p^4T^3+p^8T^4)$
4	smooth	$(1-p^2T)(1+128T+50754pT^2+128p^4T^3+p^8T^4)$
5	smooth	$(1+p^2T)(1-1536T+37698pT^2-1536p^4T^3+p^8T^4)$
6	smooth	$(1-p^2T)(1+608T-21310pT^2+608p^4T^3+p^8T^4)$
7	smooth	$(1-p^2T)(1+2020T+69754pT^2+2020p^4T^3+p^8T^4)$
8	singular	
9	smooth	$(1+p^2T)(1-1632T+42114pT^2-1632p^4T^3+p^8T^4)$
10	smooth	$(1+p^2T)(1-352T-37246pT^2-352p^4T^3+p^8T^4)$
11	smooth	$(1-p^2T)(1-160T+27842pT^2-160p^4T^3+p^8T^4)$
12	smooth	$(1+p^2T)(1-1562T+39310pT^2-1562p^4T^3+p^8T^4)$
13	smooth	$(1+p^2T)(1-730T+4110pT^2-730p^4T^3+p^8T^4)$
14	smooth	$(1-p^2T)(1+1764T+75386pT^2+1764p^4T^3+p^8T^4)$
15	smooth	$(1+p^2T)(1-698T-14258pT^2-698p^4T^3+p^8T^4)$
16	smooth	$(1-p^2T)^2(1+p^2T)(1-382T+p^4T^2)$
17	smooth	$(1+p^2T)(1+1286T+26574pT^2+1286p^4T^3+p^8T^4)$
18	smooth	$(1+p^2T)(1-2016T+83842pT^2-2016p^4T^3+p^8T^4)$
19	smooth	$(1-p^2T)(1+256T-39614pT^2+256p^4T^3+p^8T^4))$
20	smooth	$(1-p^2T)(1+2432T+95298pT^2+2432p^4T^3+p^8T^4)$
21	smooth	$(1-p^2T)(1+1422T+p^4T^2)(1-18pT+p^4T^2))$
22	smooth	$(1+p^2T)(1-2042T+69070pT^2-2042p^4T^3+p^8T^4)$
23	smooth	$(1-p^2T)(1+2432T+93506pT^2+2432p^4T^3+p^8T^4)$

24	smooth	$(1+p^2T)(1-2458T+94862pT^2-2458p^4T^3+p^8T^4)$
25	smooth	$(1-p^2T)(1+484T-20358pT^2+484p^4T^3+p^8T^4))$
26	smooth	$(1+p^2T)(1-1658T+61646pT^2-1658p^4T^3+p^8T^4)$
27	smooth	$(1+p^2T)(1+102T-14706pT^2+102p^4T^3+p^8T^4)$
28	smooth	$(1+p^2T)(1-1280T+70978pT^2-1280p^4T^3+p^8T^4)$
29	smooth	$(1-p^2T)(1+1792T+67906pT^2+1792p^4T^3+p^8T^4)$
30	smooth	$(1-p^2T)(1+640T-19646pT^2+640p^4T^3+p^8T^4)$

$p=37$		
z	smooth/singular	$R_H(\mathbb{P}^7[2,2,4]^\vee,T)$
1	smooth	$(1-p^2T)(1+780T-35122pT^2+780p^4T^3+p^8T^4)$
2	smooth	$(1+p^2T)(1-366T-67494pT^2-366p^4T^3+p^8T^4)$
3	smooth	$(1-p^2T)(1+524T-35634pT^2+524p^4T^3+p^8T^4)$
4	smooth	$(1+p^2T)(1-3604T+180878pT^2-3604p^4T^3+p^8T^4)$
5	smooth	$(1+p^2T)(1+1234T+46810pT^2+1234p^4T^3+p^8T^4)$
6	smooth	$(1-p^2T)(1+1868T+48590pT^2+1868p^4T^3+p^8T^4)$
7	smooth	$(1-p^2T)(1+3632T+168486pT^2+3632p^4T^3+p^8T^4)$
8	smooth	$(1+p^2T)(1-3086T+154906pT^2-3086p^4T^3+p^8T^4)$
9	smooth	$(1-p^2T)(1-464T+37414pT^2-464p^4T^3+p^8T^4)$
10	singular	
11	smooth	$(1-p^2T)(1+1292T-4402pT^2+1292p^4T^3+p^8T^4)$
12	smooth	$(1+p^2T)(1-180T+12366pT^2-180p^4T^3+p^8T^4)$
13	smooth	$(1-p^2T)(1+1548T+16846pT^2+1548p^4T^3+p^8T^4)$
14	smooth	$(1-p^2T)(1+588T-55858pT^2+588p^4T^3+p^8T^4)$
15	smooth	$(1+p^2T)(1-2670T+106074pT^2-2670p^4T^3+p^8T^4)$
16	smooth	$(1+p^2T)(1-2612T+124750pT^2-2612p^4T^3+p^8T^4)$
17	smooth	$(1-p^2T)(1+844T-25394pT^2+844p^4T^3+p^8T^4)$
18	smooth	$(1+p^2T)(1-1806T+32538pT^2-1806p^4T^3+p^8T^4)$
19	smooth	$(1-p^2T)(1+1804T+95694pT^2+1804p^4T^3+p^8T^4)$
20	smooth	$(1-p^2T)(1+1164T+23246pT^2+1164p^4T^3+p^8T^4)$
21	smooth	$(1-p^2T)(1+368T-90pT^2+368p^4T^3+p^8T^4)$
22	smooth	$(1-p^2T)(1+2764T+109006pT^2+2764p^4T^3+p^8T^4)$
23	smooth	$(1+p^2T)(1-2734T+130778pT^2-2734p^4T^3+p^8T^4)$
24	smooth	$(1+p^2T)(1-22pT+37338pT^2-22p^5T^3+p^8T^4)$
25	smooth	$(1+p^2T)(1-2068T+80526pT^2-2068p^4T^3+p^8T^4)$
26	smooth	$(1-p^2T)(1+1420T+13774pT^2+1420p^4T^3+p^8T^4)$
27	smooth	$(1+p^2T)(1+2028T+81550pT^2+2028p^4T^3+p^8T^4)$
28	smooth	$(1+p^2T)(1-1556T+93838pT^2-1556p^4T^3+p^8T^4)$
29	smooth	$(1+p^2T)(1+946T+6298pT^2+946p^4T^3+p^8T^4)$
30	smooth	$(1+p^2T)(1-276T-43890pT^2-276p^4T^3+p^8T^4)$
31	smooth	$(1-p^2T)(1+332T-17202pT^2+332p^4T^3+p^8T^4)$

32	smooth	$(1+p^2T)(1-1070T-22054pT^2-1070p^4T^3+p^8T^4)$
33	smooth	$(1-p^2T)^2(1+p^2T)(1+638T+p^4T^2)$
34	smooth	$(1+p^2T)(1-564T+71502pT^2-564p^4T^3+p^8T^4)$
35	smooth	$(1-p^2T)(1-564T+20174pT^2-564p^4T^3+p^8T^4)$
36	smooth	$(1-p^2T)(1+1392T+8102pT^2+1392p^4T^3+p^8T^4)$

E.3 The Mirror of the Complete Intersection $X_{1,4} \subset \mathbf{Gr}(2,5)$

$p=7$		
z	smooth/singular	$R_H(X_{1,4}^\vee, T)$
1	smooth	$1-45T-215pT^2+90p^4T^3-215p^5T^4-45p^8T^5+p^{12}T^6$
2	smooth	$(1-p^2T)(1+p^2T)(1+30T+110pT^2+30p^4T^3+p^8T^4))$
3	smooth	$(1-p^2T)(1+p^2T)(1+5pT+352pT^2+5p^5T^3+p^8T^4))$
4	smooth	$(1-p^2T)(1+p^2T)(1+40T+90pT^2+40p^4T^3+p^8T^4))$
5	smooth	$1-60T+205pT^2+205p^5T^4-60p^8T^5+p^{12}T^6$
6	smooth	$1+50T+269pT^2+380p^3T^3+269p^5T^4+50p^8T^5+p^{12}T^6$

$p=11$		
z	smooth/singular	$R_H(X_{1,4}^\vee, T)$
1	smooth	$1+73T-831pT^2-2206p^3T^3-831p^5T^4+73p^8T^5+p^{12}T^6$
2	smooth	$(1+p^4T^2)(1+142T+p^4T^2)(1-10pT+p^4T^2)$
3	smooth	$(1-p^2T)(1+p^2T)(1-58T+318pT^2-58p^4T^3+p^8T^4)$
4	singular	
5	smooth	$(1-p^2T)(1+p^2T)(1+12pT+1258pT^2+12p^5T^3+p^8T^4)$
6	smooth	$(1-p^2T)(1+p^2T)(1-83T+1068pT^2-83p^4T^3+p^8T^4)$
7	singular	
8	smooth	$(1-p^2T)(1+p^2T)(1+7pT+228pT^2+7p^5T^3+p^8T^4)$
9	smooth	$1-43T-927pT^2+14p^4T^3-927p^5T^4-43p^8T^5+p^{12}T^6$
10	smooth	$1-102T-31pT^2+1244p^3T^3-31p^5T^4-102p^8T^5+p^{12}T^6$

$p=13$		
z	smooth/singular	$R_H(X_{1,4}^\vee, T)$
1	smooth	$1-205T+955pT^2-70p^4T^3+955p^5T^4-205p^8T^5+p^{12}T^6$
2	smooth	$1-40T-1685pT^2+80p^4T^3-1685p^5T^4-40p^8T^5+p^{12}T^6$
3	smooth	$-((-1+p^2T)(1+p^2T)(1+150T+3570pT^2+150p^4T^3+p^8T^4)$
4	smooth	$1+255T-497pT^2-3630p^3T^3-497p^5T^4+255p^8T^5+p^{12}T^6$
5	smooth	$1-30T-1029pT^2+2540p^3T^3-1029p^5T^4-30p^8T^5+p^{12}T^6$
6	smooth	$1+30T-865pT^2-3300p^3T^3-865p^5T^4+30p^8T^5+p^{12}T^6$
7	smooth	$-((-1+p^2T)(1+p^2T)(1+5pT+918pT^2+5p^5T^3+p^8T^4)$
8	smooth	$-((-1+p^2T)(1+p^2T)(1+225T+1510pT^2+225p^4T^3+p^8T^4)$
9	smooth	$-((-1+p^2T)(1+p^2T)(1+10T+3690pT^2+10p^4T^3+p^8T^4)$
10	smooth	$-((-1+p^2T)(1+p^2T)(1+90T+2538pT^2+90p^4T^3+p^8T^4)$
11	smooth	$-((-1+p^2T)(1+p^2T)(1-230T+4018pT^2-230p^4T^3+p^8T^4)$
12	smooth	$-((-1+p^2T)(1+p^2T)(1-150T+1098pT^2-150p^4T^3+p^8T^4)$

$p = 17$		
z	smooth/singular	$R_H(X_{1,4}^{\vee}, T)$
1	smooth	$(1-p^2T)(1+p^2T)(1-290T+5282pT^2-290p^4T^3+p^8T^4)$
2	smooth	$1+245T+111p^2T^2+3670p^3T^3+111p^6T^4+245p^8T^5+p^{12}T^6$
3	smooth	$(1-p^2T)(1+p^2T)(1+15pT+7002pT^2+15p^5T^3+p^8T^4)$
4	smooth	$1+270T+111p^2T^2+2820p^3T^3+111p^6T^4+270p^8T^5+p^{12}T^6$
5	smooth	$(1-p^2T)(1+p^2T)(1-310T+9130pT^2-310p^4T^3+p^8T^4)$
6	smooth	$1+100T-3741pT^2-1120p^3T^3-3741p^5T^4+100p^8T^5+p^{12}T^6$
7	smooth	$(1-p^2T)(1+p^2T)(1+15pT+5930pT^2+15p^5T^3+p^8T^4)$
8	smooth	$1+245T-3761pT^2-490p^4T^3-3761p^5T^4+245p^8T^5+p^{12}T^6$
9	smooth	$(1-p^2T)^2(1+203T-1324pT^2+203p^4T^3+p^8T^4)$
10	smooth	$(1-p^2T)(1+p^2T)(1-290T+7250pT^2-290p^4T^3+p^8T^4)$
11	smooth	$(1-p^2T)(1+p^2T)(1+175T+5370pT^2+175p^4T^3+p^8T^4)$
12	smooth	$(1-p^2T)(1+p^2T)(1+75T+3970pT^2+75p^4T^3+p^8T^4)$
13	smooth	$(1-p^2T)(1+p^2T)(1+140T-4090pT^2+140p^4T^3+p^8T^4)$
14	smooth	$1+140T-925pT^2+2640p^3T^3-925p^5T^4+140p^8T^5+p^{12}T^6$
15	smooth	$1-95T-2145pT^2-2050p^3T^3-2145p^5T^4-95p^8T^5+p^{12}T^6$
16	smooth	$(1-p^2T)(1+p^2T)(1-250T+50pT^2-250p^4T^3+p^8T^4)$

$p = 19$		
z	smooth/singular	$R_H(X_{1,4}^{\vee}, T)$
1	smooth	$1-435T-6359pT^2+15930p^3T^3-6359p^5T^4-435p^8T^5+p^{12}T^6$
2	smooth	$1-280T-6359pT^2+11240p^3T^3-6359p^5T^4-280p^8T^5+p^{12}T^6$
3	smooth	$1-4T-5435pT^2+1400p^3T^3-5435p^5T^4-4p^8T^5+p^{12}T^6$
4	singular	
5	smooth	$(1-p^2T)(1+p^2T)(1+22T-4154pT^2+22p^4T^3+p^8T^4)$
6	smooth	$1+85T+265pT^2+9418p^3T^3+265p^5T^4+85p^8T^5+p^{12}T^6$
7	smooth	$(1-p^2T)(1+p^2T)(1+382T+13278pT^2+382p^4T^3+p^8T^4)$
8	smooth	$(1-p^2T)(1+p^2T)(1+364T+14818pT^2+364p^4T^3+p^8T^4)$
9	smooth	$1+298T-2683pT^2-5276p^3T^3-2683p^5T^4+298p^8T^5+p^{12}T^6$
10	smooth	$(1-p^2T)(1+p^2T)(1-63T+7788pT^2-63p^4T^3+p^8T^4)$
11	smooth	$1+249T-3959pT^2-5262p^3T^3-3959p^5T^4+249p^8T^5+p^{12}T^6$
12	smooth	$(1-p^2T)(1+p^2T)(1-3T+3148pT^2-3p^4T^3+p^8T^4)$
13	smooth	$1+356T-5003pT^2-10840p^3T^3-5003p^5T^4+356p^8T^5+p^{12}T^6$
14	smooth	$(1-p^2T)(1+p^2T)(1-331T+10268pT^2-331p^4T^3+p^8T^4)$
15	smooth	$(1-p^2T)(1+p^2T)(1-131T+6604pT^2-131p^4T^3+p^8T^4)$
16	smooth	$1-21pT-3255pT^2+7970p^3T^3-3255p^5T^4-21p^9T^5+p^{12}T^6$
17	smooth	$(1-p^2T)(1+p^2T)(1+492T+p^4T^2)(1-10pT+p^4T^2)$
18	singular	

$p = 23$		
z	smooth/singular	$R_H(X_{1,4}^{\vee}, T)$
1	smooth	$(1-p^2T)(1+p^2T)(1-30T+20110pT^2-30p^4T^3+p^8T^4)$
2	smooth	$(1-p^2T)(1+p^2T)(1+200T+930pT^2+200p^4T^3+p^8T^4)$
3	smooth	$(1-p^2T)(1+p^2T)(1-490T+12870pT^2-490p^4T^3+p^8T^4)$
4	smooth	$1-65T-7479pT^2+12110p^3T^3-7479p^5T^4-65p^8T^5+p^{12}T^6$

5	smooth	$1 + 420T - 4419pT^2 - 3840p^3T^3 - 4419p^5T^4 + 420p^8T^5 + p^{12}T^6$
6	smooth	$(1 + p^2T)^2(1 - 1388T + 39914pT^2 - 1388p^4T^3 + p^8T^4)$
7	smooth	$(1 - p^2T)(1 + p^2T)(1 - 885T + 25240pT^2 - 885p^4T^3 + p^8T^4)$
8	smooth	$1 - 105T - 8455pT^2 - 290p^3T^3 - 8455p^5T^4 - 105p^8T^5 + p^{12}T^6$
9	smooth	$(1 - p^2T)(1 + p^2T)(1 + 880T + 24434pT^2 + 880p^4T^3 + p^8T^4)$
10	smooth	$1 - 90T - 4167pT^2 + 13740p^3T^3 - 4167p^5T^4 - 90p^8T^5 + p^{12}T^6$
11	smooth	$(1 - p^2T)(1 + p^2T)(1 + 95T + 4440pT^2 + 95p^4T^3 + p^8T^4)$
12	smooth	$(1 - p^2T)(1 + p^2T)(1 - 100T + 10034pT^2 - 100p^4T^3 + p^8T^4)$
13	smooth	$1 - 325T - 775pT^2 - 5530p^3T^3 - 775p^5T^4 - 325p^8T^5 + p^{12}T^6$
14	smooth	$1 + 250T - 5667pT^2 - 18100p^3T^3 - 5667p^5T^4 + 250p^8T^5 + p^{12}T^6$
15	smooth	$(1 - p^2T)(1 + p^2T)(1 + 495T + 5408pT^2 + 495p^4T^3 + p^8T^4)$
16	smooth	$(1 - p^2T)(1 + p^2T)(1 - 290T + 22838pT^2 - 290p^4T^3 + p^8T^4)$
17	smooth	$1 - 100T - 2167pT^2 + 10200p^3T^3 - 2167p^5T^4 - 100p^8T^5 + p^{12}T^6$
18	smooth	$(1 - p^2T)(1 + p^2T)(1 + 40pT + 1206p^2T^2 + 40p^5T^3 + p^8T^4)$
19	smooth	$1 - 240T + 12425pT^2 - 1280p^3T^3 + 12425p^5T^4 - 240p^8T^5 + p^{12}T^6$
20	smooth	$(1 - p^2T)(1 + p^2T)(1 + 1100T + 1350p^2T^2 + 1100p^4T^3 + p^8T^4)$
21	smooth	$1 + 610T + 10941pT^2 + 18140p^3T^3 + 10941p^5T^4 + 610p^8T^5 + p^{12}T^6$
22	smooth	$1 - 1390T + 44541pT^2 - 2140p^4T^3 + 44541p^5T^4 - 1390p^8T^5 + p^{12}T^6$

$p = 29$		
z	smooth/singular	$R_H(X_{1,4}^\vee, T)$
1	smooth	$(1 - p^2T)(1 + p^2T)(1 + 744T + 28038pT^2 + 744p^4T^3 + p^8T^4)$
2	smooth	$(1 - p^2T)(1 + p^2T)(1 - 2063T + 81798pT^2 - 2063p^4T^3 + p^8T^4)$
3	smooth	$1 + 354T + 11211pT^2 + 10668p^3T^3 + 11211p^5T^4 + 354p^8T^5 + p^{12}T^6$
4	smooth	$1 - 349T + 235p^2T^2 - 32950p^3T^3 + 235p^6T^4 - 349p^8T^5 + p^{12}T^6$
5	smooth	$1 + 931T + 44447pT^2 + 38730p^3T^3 + 44447p^5T^4 + 931p^8T^5 + p^{12}T^6$
6	smooth	$(1 - p^2T)(1 + p^2T)(1 - 656T + 24278pT^2 - 656p^4T^3 + p^8T^4)$
7	smooth	$(1 - p^2T)(1 + p^2T)(1 - 434T + 43490pT^2 - 434p^4T^3 + p^8T^4)$
8	smooth	$1 + 452T - 19189pT^2 - 22216p^3T^3 - 19189p^5T^4 + 452p^8T^5 + p^{12}T^6$
9	smooth	$(1 - p^2T)(1 + p^2T)(1 + 674T + 21538pT^2 + 674p^4T^3 + p^8T^4)$
10	smooth	$1 - 480T + 11pT^2 + 19040p^3T^3 + 11p^5T^4 - 480p^8T^5 + p^{12}T^6$
11	smooth	$(1 - p^2T)(1 + p^2T)(1 + 61T - 2850pT^2 + 61p^4T^3 + p^8T^4)$
12	smooth	$(1 - p^2T)(1 + p^2T)(1 - 263T + 4278pT^2 - 263p^4T^3 + p^8T^4)$
13	singular	
14	smooth	$(1 - p^2T)(1 + p^2T)(1 + 1182T + 58458pT^2 + 1182p^4T^3 + p^8T^4)$
15	smooth	$(1 - p^2T)(1 + p^2T)(1 - 246T + 26354pT^2 - 246p^4T^3 + p^8T^4)$
16	smooth	$1 + 731T - 20853pT^2 - 47870p^3T^3 - 20853p^5T^4 + 731p^8T^5 + p^{12}T^6$
17	smooth	$1 + 714T - 1489pT^2 + 2788p^3T^3 - 1489p^5T^4 + 714p^8T^5 + p^{12}T^6$
18	smooth	$1 - 312T - 8833pT^2 - 8392p^3T^3 - 8833p^5T^4 - 312p^8T^5 + p^{12}T^6$
19	smooth	$1 + 328T - 16693pT^2 - 1008p^4T^3 - 16693p^5T^4 + 328p^8T^5 + p^{12}T^6$
20	smooth	$1 + 1195T + 22155pT^2 + 17378p^3T^3 + 22155p^5T^4 + 1195p^8T^5 + p^{12}T^6$
21	smooth	$1 - 880T - 17589pT^2 + 39040p^3T^3 - 17589p^5T^4 - 880p^8T^5 + p^{12}T^6$
22	smooth	$(1 - p^2T)(1 + p^2T)(1 + 588T + 12918pT^2 + 588p^4T^3 + p^8T^4)$
23	smooth	$(1 - p^2T)(1 + p^2T)(1 - 32pT + 10998pT^2 - 32p^5T^3 + p^8T^4)$
24	singular	
25	smooth	$(1 - p^2T)(1 + p^2T)(1 - 1082T + 24178pT^2 - 1082p^4T^3 + p^8T^4)$

26	smooth	$(1-p^2T)(1+p^2T)(1-371T-19842pT^2-371p^4T^3+p^8T^4)$
27	smooth	$1+1336T+49087pT^2+67160p^3T^3+49087p^5T^4+1336p^8T^5+p^{12}T^6$
28	smooth	$(1-p^2T)(1+p^2T)(1-222T-11998pT^2-222p^4T^3+p^8T^4)$

$p=31$		
z	smooth/singular	$R_H(X_{1,4}^\vee,T)$
1	smooth	$1-757T-28175pT^2+43766p^3T^3-28175p^5T^4-757p^8T^5+p^{12}T^6$
2	smooth	$(1-p^2T)(1+p^2T)(1+2150T+92718pT^2+2150p^4T^3+p^8T^4)$
3	smooth	$(1-p^2T)(1+p^2T)(1+156T+55346pT^2+156p^4T^3+p^8T^4)$
4	smooth	$1+23T-2655pT^2+27246p^3T^3-2655p^5T^4+23p^8T^5+p^{12}T^6$
5	smooth	$(1-p^2T)^2(1+1200T+16418pT^2+1200p^4T^3+p^8T^4)$
6	smooth	$(1-p^2T)(1+p^2T)(1-1133T+48672pT^2-1133p^4T^3+p^8T^4)$
7	singular	
8	smooth	$(1-p^2T)(1+p^2T)(1-114T+16742pT^2-114p^4T^3+p^8T^4)$
9	smooth	$1-1133T-12687pT^2+46854p^3T^3-12687p^5T^4-1133p^8T^5+p^{12}T^6$
10	smooth	$(1-p^2T)(1+p^2T)(1-616T+37610pT^2-616p^4T^3+p^8T^4)$
11	singular	
12	smooth	$(1-p^2T)(1+p^2T)(1-1737T+74224pT^2-1737p^4T^3+p^8T^4)$
13	smooth	$1+1772T+28133pT^2+4688p^3T^3+28133p^5T^4+1772p^8T^5+p^{12}T^6$
14	smooth	$1+298T+8945pT^2+1396p^3T^3+8945p^5T^4+298p^8T^5+p^{12}T^6$
15	smooth	$(1-p^2T)(1+p^2T)(1-7pT+35664pT^2-7p^5T^3+p^8T^4)$
16	smooth	$(1-p^2T)(1+p^2T)(1+274T+1730p^2T^2+274p^4T^3+p^8T^4)$
17	smooth	$1-18pT-22587pT^2+29404p^3T^3-22587p^5T^4-18p^9T^5+p^{12}T^6$
18	smooth	$1-877T-16991pT^2+1754p^4T^3-16991p^5T^4-877p^8T^5+p^{12}T^6$
19	smooth	$1+807T-5007pT^2-21202p^3T^3-5007p^5T^4+807p^8T^5+p^{12}T^6$
20	smooth	$1+367T+4913pT^2+37438p^3T^3+4913p^5T^4+367p^8T^5+p^{12}T^6$
21	smooth	$(1-p^2T)(1+p^2T)(1-984T+32626pT^2-984p^4T^3+p^8T^4)$
22	smooth	$(1-p^2T)(1+p^2T)(1+1631T+67512pT^2+1631p^4T^3+p^8T^4)$
23	smooth	$1-532T-635pT^2-784p^4T^3-635p^5T^4-532p^8T^5+p^{12}T^6$
24	smooth	$(1-p^2T)(1+p^2T)(1+1779T+75240pT^2+1779p^4T^3+p^8T^4)$
25	smooth	$(1-p^2T)(1+p^2T)(1+546T+24062pT^2+546p^4T^3+p^8T^4)$
26	smooth	$1-662T-11295pT^2+20436p^3T^3-11295p^5T^4-662p^8T^5+p^{12}T^6$
27	smooth	$(1-p^2T)(1+p^2T)(1+891T+40792pT^2+891p^4T^3+p^8T^4)$
28	smooth	$(1-p^2T)(1+p^2T)(1+4T+55410pT^2+4p^4T^3+p^8T^4)$
29	smooth	$1-1168T+4833pT^2+928p^4T^3+4833p^5T^4-1168p^8T^5+p^{12}T^6$
30	smooth	$1-188T-19547pT^2+16768p^3T^3-19547p^5T^4-188p^8T^5+p^{12}T^6$

$p=37$		
z	smooth/singular	$R_H(X_{1,4}^\vee,T)$
1	smooth	$1+950T-40205pT^2-60220p^3T^3-40205p^5T^4+950p^8T^5+p^{12}T^6$
2	smooth	$(1-p^2T)(1+p^2T)(1-850T+60650pT^2-850p^4T^3+p^8T^4))$
3	smooth	$(1-p^2T)(1+p^2T)(1-620T-5250pT^2-620p^4T^3+p^8T^4))$
4	smooth	$1-350T-15885pT^2+48780p^3T^3-15885p^5T^4-350p^8T^5+p^{12}T^6$
5	smooth	$(1-p^2T)(1+p^2T)(1+2265T+80102pT^2+2265p^4T^3+p^8T^4)$
6	smooth	$(1-p^2T)(1+p^2T)(1+2370T+97690pT^2+2370p^4T^3+p^8T^4)$
7	smooth	$(1-p^2T)(1+p^2T)(1-530T-38894pT^2-530p^4T^3+p^8T^4)$

8	smooth	$(1-p^2T)(1+p^2T)(1-1135T+14822pT^2-1135p^4T^3+p^8T^4)$
9	smooth	$(1-p^2T)(1+p^2T)(1-930T+40882pT^2-930p^4T^3+p^8T^4)$
10	smooth	$(1-p^2T)(1+p^2T)(1-980T-18194pT^2-980p^4T^3+p^8T^4)$
11	smooth	$(1-p^2T)(1+p^2T)(1-1740T+94582pT^2-1740p^4T^3+p^8T^4)$
12	smooth	$1-65T-825p^2T^2-24630p^3T^3-825p^6T^4-65p^8T^5+p^{12}T^6$
13	smooth	$(1-p^2T)(1+p^2T)(1-170T-7838pT^2-170p^4T^3+p^8T^4)$
14	smooth	$(1-p^2T)(1+p^2T)(1+745T-42090pT^2+745p^4T^3+p^8T^4)$
15	smooth	$1+170T-34381pT^2-8260p^3T^3-34381p^5T^4+170p^8T^5+p^{12}T^6$
16	smooth	$1+395T-7821pT^2-17550p^3T^3-7821p^5T^4+395p^8T^5+p^{12}T^6$
17	smooth	$1+160T-34701pT^2+480p^4T^3-34701p^5T^4+160p^8T^5+p^{12}T^6$
18	smooth	$1-190T-40141pT^2+29900p^3T^3-40141p^5T^4-190p^8T^5+p^{12}T^6$
19	smooth	$(1-p^2T)(1+p^2T)(1-1855T+91830pT^2-1855p^4T^3+p^8T^4)$
20	smooth	$1+190T+14455pT^2+82300p^3T^3+14455p^5T^4+190p^8T^5+p^{12}T^6$
21	smooth	$(1-p^2T)(1+p^2T)(1+590T+33370pT^2+590p^4T^3+p^8T^4)$
22	smooth	$1+1460T-15245pT^2-37480p^3T^3-15245p^5T^4+1460p^8T^5+p^{12}T^6$
23	smooth	$(1-p^2T)(1+p^2T)(1-935T-8458pT^2-935p^4T^3+p^8T^4)$
24	smooth	$1+1680T+58759pT^2+81960p^3T^3+58759p^5T^4+1680p^8T^5+p^{12}T^6$
25	smooth	$1-685T-13881pT^2+14010p^3T^3-13881p^5T^4-685p^8T^5+p^{12}T^6$
26	smooth	$(1-p^2T)(1+p^2T)(1+1300T+72190pT^2+1300p^4T^3+p^8T^4)$
27	smooth	$(1-p^2T)(1+p^2T)(1+2650T+122682pT^2+2650p^4T^3+p^8T^4)$
28	smooth	$1-25pT-29465pT^2+30170p^3T^3-29465p^5T^4-25p^9T^5+p^{12}T^6$
29	smooth	$1-590T-12521pT^2+12340p^3T^3-12521p^5T^4-590p^8T^5+p^{12}T^6$
30	smooth	$(1-p^2T)(1+p^2T)(1+1200T+74406pT^2+1200p^4T^3+p^8T^4)$
31	smooth	$1+840T-30425pT^2-21960p^3T^3-30425p^5T^4+840p^8T^5+p^{12}T^6$
32	smooth	$(1-p^2T)(1+p^2T)(1-655T+60390pT^2-655p^4T^3+p^8T^4)$
33	smooth	$1-25pT-16461pT^2+53890p^3T^3-16461p^5T^4-25p^9T^5+p^{12}T^6$
34	smooth	$(1-p^2T)(1+p^2T)(1+360T+86870pT^2+360p^4T^3+p^8T^4)$
35	smooth	$1-1520T+475p^2T^2+2920p^3T^3+475p^6T^4-1520p^8T^5+p^{12}T^6$
36	smooth	$1-305T-47481pT^2+28770p^3T^3-47481p^5T^4-305p^8T^5+p^{12}T^6$

E.4 The Mirror of the Family of Sextic Fourfolds $\mathbb{P}^5[6]$

$p=7$		
z	smooth/singular	$R_H(\mathbb{P}^5[6]^\vee, T)$
1	singular	
2	smooth	$(1+p^2T)(1-48T+286pT^2-48p^4T^3+p^8T^4)$
3	smooth	$(1+p^2T)(1-54T+118pT^2-54p^4T^3+p^8T^4)$
4	smooth	$(1-p^2T)(1+73T+p^4T^2)(1-2pT+p^4T^2)$
5	smooth	$(1+p^2T)(1-24T-302pT^2-24p^4T^3+p^8T^4)$
6	smooth	$(1-p^2T)(1+54T+295pT^2+54p^4T^3+p^8T^4)$

$p = 11$		
z	smooth/singular	$R_H(\mathbb{P}^5[6]^\vee, T)$
1	smooth	$(1-p^2T)^2(1+p^2T)(1+126T+p^4T^2)$
2	smooth	$(1+p^2T)(1+1226pT^2+p^8T^4)$
3	smooth	$(1+p^2T)(1-126T+785pT^2-126p^4T^3+p^8T^4)$
4	smooth	$(1-p^2T)(1-43T-8pT^2-43p^4T^3+p^8T^4)$
5	smooth	$(1-p^2T)(1+216T+2666pT^2+216p^4T^3+p^8T^4)$
6	smooth	$(1-p^2T)(1-64T+2491pT^2-64p^4T^3+p^8T^4)$
7	smooth	$(1-p^2T)^2(1+p^2T)(1+30T+p^4T^2)$
8	smooth	$(1-p^2T)(1+146T+991pT^2+146p^4T^3+p^8T^4)$
9	singular	
10	smooth	$(1+p^2T)(1+70T-314pT^2+70p^4T^3+p^8T^4)$

$p = 13$		
z	smooth/singular	$R_H(\mathbb{P}^5[6]^\vee, T)$
1	smooth	$(1-p^2T)^2(1+p^2T)(1+175T+p^4T^2)$
2	smooth	$(1-p^2T)(1+440T+7062pT^2+440p^4T^3+p^8T^4)$
3	smooth	$(1-p^2T)(1+209T+1440pT^2+209p^4T^3+p^8T^4)$
4	smooth	$(1+p^2T)(1-219T+3652pT^2-219p^4T^3+p^8T^4)$
5	smooth	$(1+p^2T)(1-64T-1758pT^2-64p^4T^3+p^8T^4)$
6	smooth	$(1+p^2T)(1-288T+4969pT^2-288p^4T^3+p^8T^4)$
7	smooth	$(1+p^2T)(1-150T-419pT^2-150p^4T^3+p^8T^4)$
8	smooth	$(1-p^2T)(1+90T-758pT^2+90p^4T^3+p^8T^4)$
9	smooth	$(1-p^2T)(1-62T+p^4T^2)(1+22pT+p^4T^2)$
10	smooth	$(1+p^2T)(1-99T+1252pT^2-99p^4T^3+p^8T^4)$
11	smooth	$(1-p^2T)(1-112T+2694pT^2-112p^4T^3+p^8T^4)$
12	singular	

$p = 17$		
z	smooth/singular	$R_H(\mathbb{P}^5[6]^\vee, T)$
1	smooth	$(1+p^2T)(1+64T+3550pT^2+64p^4T^3+p^8T^4)$
2	smooth	$(1-p^2T)(1+135T-3472pT^2+135p^4T^3+p^8T^4)$
3	smooth	$(1+p^2T)(1-128T+601pT^2-128p^4T^3+p^8T^4)$
4	smooth	$(1-p^2T)^2(1+p^2T)(1-147T+p^4T^2)$
5	smooth	$(1+p^2T)(1-296T+1210pT^2-296p^4T^3+p^8T^4)$
6	smooth	$(1-p^2T)(1+54T+6410pT^2+54p^4T^3+p^8T^4)$
7	smooth	$(1-p^2T)(1-100T+4198pT^2-100p^4T^3+p^8T^4)$
8	smooth	$(1+p^2T)(1-333T+3224pT^2-333p^4T^3+p^8T^4)$
9	smooth	$(1+p^2T)(1+171T+5528pT^2+171p^4T^3+p^8T^4)$
10	smooth	$(1+p^2T)(1-594T+11693pT^2-594p^4T^3+p^8T^4)$

11	smooth	$(1-p^2T)(1+p^2T)^2(1+126T+p^4T^2)$
12	smooth	$(1+p^2T)(1+36T-191p^2T^2+36p^4T^3+p^8T^4)$
13	smooth	$(1-p^2T)(1+224T-1310pT^2+224p^4T^3+p^8T^4)$
14	smooth	$(1-p^2T)(1+180T+7958pT^2+180p^4T^3+p^8T^4)$
15	singular	
16	smooth	$(1-p^2T)(1+110T-2837pT^2+110p^4T^3+p^8T^4)$

$p=19$		
z	smooth/singular	$R_H(\mathbb{P}^5[6]^\vee, T)$
1	smooth	$(1-p^2T)(1+297T+1168pT^2+297p^4T^3+p^8T^4)$
2	smooth	$(1-p^2T)(1+56T-4866pT^2+56p^4T^3+p^8T^4)$
3	smooth	$(1-p^2T)(1+686T+13719pT^2+686p^4T^3+p^8T^4)$
4	smooth	$(1+p^2T)(1-697T+p^4T^2)(1-26pT+p^4T^2)$
5	smooth	$(1+p^2T)(1-726T+17545pT^2-726p^4T^3+p^8T^4)$
6	smooth	$(1-p^2T)(1-62T+p^4T^2)(1+37pT+p^4T^2)$
7	singular	
8	smooth	$(1+p^2T)(1-40T+3354pT^2-40p^4T^3+p^8T^4)$
9	smooth	$(1-p^2T)(1+941T+23064pT^2+941p^4T^3+p^8T^4)$
10	smooth	$(1-p^2T)(1+332T+12123pT^2+332p^4T^3+p^8T^4)$
11	smooth	$(1+p^2T)(1-704T+p^4T^2)(1+16pT+p^4T^2)$
12	smooth	$(1+p^2T)(1-396T+12130pT^2-396p^4T^3+p^8T^4)$
13	smooth	$(1+p^2T)(1-222T-482pT^2-222p^4T^3+p^8T^4)$
14	smooth	$(1+p^2T)(1-330T+1174pT^2-330p^4T^3+p^8T^4)$
15	smooth	$(1-p^2T)(1+386T+999pT^2+386p^4T^3+p^8T^4)$
16	smooth	$(1+p^2T)(1-312T-1562pT^2-312p^4T^3+p^8T^4)$
17	smooth	$(1-p^2T)(1+392T+2778pT^2+392p^4T^3+p^8T^4)$
18	smooth	$(1-p^2T)^2(1+p^2T)(1+70T+p^4T^2)$

$p=23$		
z	smooth/singular	$R_H(\mathbb{P}^5[6]^\vee, T)$
1	smooth	$(1-p^2T)(1+599T+17620pT^2+599p^4T^3+p^8T^4)$
2	singular	
3	smooth	$(1+p^2T)(1-434T-2183pT^2-434p^4T^3+p^8T^4)$
4	smooth	$(1-p^2T)^2(1+p^2T)(1+165T+p^4T^2)$
5	smooth	$(1+p^2T)(1-738T+20414pT^2-738p^4T^3+p^8T^4)$
6	smooth	$(1+p^2T)(1-854T+17137pT^2-854p^4T^3+p^8T^4)$
7	smooth	$(1-p^2T)(1+864T+21035pT^2+864p^4T^3+p^8T^4)$
8	smooth	$(1+p^2T)(1-1368T+41294pT^2-1368p^4T^3+p^8T^4)$
9	smooth	$(1-p^2T)(1+788T+26314pT^2+788p^4T^3+p^8T^4)$
10	smooth	$(1+p^2T)(1-108T-8566pT^2-108p^4T^3+p^8T^4)$
11	smooth	$(1+p^2T)(1-686T+11110pT^2-686p^4T^3+p^8T^4)$

12	smooth	$(1-p^2T)(1+p^2T)^2(1-14T+p^4T^2)$
13	smooth	$(1-p^2T)(1-265T+5092pT^2-265p^4T^3+p^8T^4)$
14	smooth	$(1+p^2T)(1+252T+16346pT^2+252p^4T^3+p^8T^4)$
15	smooth	$(1+p^2T)(1-38T-7034pT^2-38p^4T^3+p^8T^4)$
16	smooth	$(1-p^2T)(1+p^2T)^2(1+165T+p^4T^2)$
17	smooth	$(1-p^2T)(1-576T+22070pT^2-576p^4T^3+p^8T^4)$
18	smooth	$(1+p^2T)(1-248T-11954pT^2-248p^4T^3+p^8T^4)$
19	smooth	$(1-p^2T)(1+218T-3281pT^2+218p^4T^3+p^8T^4)$
20	smooth	$(1+p^2T)(1+144T-3310pT^2+144p^4T^3+p^8T^4)$
21	smooth	$(1-p^2T)(1-90T+7247pT^2-90p^4T^3+p^8T^4)$
22	smooth	$(1-p^2T)(1+494T-1385pT^2+494p^4T^3+p^8T^4)$

$p=29$		
z	smooth/singular	$R_H(\mathbb{P}^5[6]^\vee,T)$
1	smooth	$(1-p^2T)(1+702T+2963pT^2+702p^4T^3+p^8T^4)$
2	smooth	$(1+p^2T)(1+1078T+40573pT^2+1078p^4T^3+p^8T^4)$
3	smooth	$(1-p^2T)(1+576T+12854pT^2+576p^4T^3+p^8T^4)$
4	smooth	$(1-p^2T)^2(1+p^2T)(1+351T+p^4T^2)$
5	smooth	$(1+p^2T)(1-1260T+31214pT^2-1260p^4T^3+p^8T^4)$
6	smooth	$(1+p^2T)(1-1575T+49484pT^2-1575p^4T^3+p^8T^4)$
7	smooth	$(1-p^2T)(1+981T+10424pT^2+981p^4T^3+p^8T^4)$
8	smooth	$(1+p^2T)(1-896T+27490pT^2-896p^4T^3+p^8T^4)$
9	smooth	$(1+p^2T)(1-1175T+34924pT^2-1175p^4T^3+p^8T^4)$
10	smooth	$(1-p^2T)(1+486T-8926pT^2+486p^4T^3+p^8T^4)$
11	smooth	$(1+p^2T)(1-1260T+34049pT^2-1260p^4T^3+p^8T^4)$
12	smooth	$(1-p^2T)^2(1+p^2T)(1+386T+p^4T^2)$
13	smooth	$(1+p^2T)(1-1035T+11684pT^2-1035p^4T^3+p^8T^4)$
14	smooth	$(1-p^2T)(1+164T+36910pT^2+164p^4T^3+p^8T^4)$
15	smooth	$(1+p^2T)(1-2660T+107689pT^2-2660p^4T^3+p^8T^4)$
16	smooth	$(1-p^2T)(1+752T+17338pT^2+752p^4T^3+p^8T^4)$
17	smooth	$(1-p^2T)(1+1746T+64154pT^2+1746p^4T^3+p^8T^4)$
18	smooth	$(1-p^2T)(1+392T+2182pT^2+392p^4T^3+p^8T^4)$
19	smooth	$(1-p^2T)(1+1772T+68398pT^2+1772p^4T^3+p^8T^4)$
20	smooth	$(1-p^2T)^2(1+p^2T)(1-405T+p^4T^2)$
21	smooth	$(1+p^2T)(1-1386T+32285pT^2-1386p^4T^3+p^8T^4)$
22	smooth	$(1-p^2T)(1+1025T+14272pT^2+1025p^4T^3+p^8T^4)$
23	singular	
24	smooth	$(1-p^2T)(1+596T-11906pT^2+596p^4T^3+p^8T^4)$
25	smooth	$(1+p^2T)(1+673T+9388pT^2+673p^4T^3+p^8T^4)$
26	smooth	$(1+p^2T)(1+21449pT^2+p^8T^4)$
27	smooth	$(1-p^2T)(1+864T+1910pT^2+864p^4T^3+p^8T^4)$

28	smooth	$(1-p^2T)(1+2096T+74554pT^2+2096p^4T^3+p^8T^4)$

$p=31$		
z	smooth/singular	$R_H(\mathbb{P}^5[6]^\vee, T)$
1	singular	
2	smooth	$(1+p^2T)(1-1630T+41721pT^2-1630p^4T^3+p^8T^4)$
3	smooth	$(1+p^2T)(1-2370T+96766pT^2-2370p^4T^3+p^8T^4)$
4	smooth	$(1-p^2T)(1+1547T+37932pT^2+1547p^4T^3+p^8T^4)$
5	smooth	$(1+p^2T)(1+264T-25250pT^2+264p^4T^3+p^8T^4)$
6	smooth	$(1+p^2T)(1-216T+12610pT^2-216p^4T^3+p^8T^4)$
7	smooth	$(1-p^2T)(1-289T+49236pT^2-289p^4T^3+p^8T^4)$
8	smooth	$(1+p^2T)(1-1822T+p^4T^2)(1+46pT+p^4T^2)$
9	smooth	$(1+p^2T)(1-900T+5626pT^2-900p^4T^3+p^8T^4)$
10	smooth	$(1+p^2T)(1-816T+49810pT^2-816p^4T^3+p^8T^4)$
11	smooth	$(1+p^2T)(1-2400T+94018pT^2-2400p^4T^3+p^8T^4)$
12	smooth	$(1-p^2T)(1+1250T+48543pT^2+1250p^4T^3+p^8T^4)$
13	smooth	$(1-p^2T)(1+1052T+36267pT^2+1052p^4T^3+p^8T^4)$
14	smooth	$(1-p^2T)(1+1916T+70026pT^2+1916p^4T^3+p^8T^4)$
15	smooth	$(1+p^2T)(1+764T-9750pT^2+764p^4T^3+p^8T^4)$
16	smooth	$(1-p^2T)(1+324T-11270pT^2+324p^4T^3+p^8T^4)$
17	smooth	$(1+p^2T)(1+108T-7478pT^2+108p^4T^3+p^8T^4)$
18	smooth	$(1-p^2T)(1+131T-32964pT^2+131p^4T^3+p^8T^4)$
19	smooth	$(1+p^2T)(1-1446T+33745pT^2-1446p^4T^3+p^8T^4)$
20	smooth	$(1+p^2T)(1-1497T+63232pT^2-1497p^4T^3+p^8T^4)$
21	smooth	$(1+p^2T)(1+318T-20498pT^2+318p^4T^3+p^8T^4)$
22	smooth	$(1-p^2T)(1-184T-38949pT^2-184p^4T^3+p^8T^4)$
23	smooth	$(1-p^2T)(1-54T-27209pT^2-54p^4T^3+p^8T^4)$
24	smooth	$(1-p^2T)(1+32pT+28818pT^2+32p^5T^3+p^8T^4)$
25	smooth	$(1-p^2T)(1+1316T+22266pT^2+1316p^4T^3+p^8T^4)$
26	smooth	$(1+p^2T)(1+54T+57070pT^2+54p^4T^3+p^8T^4)$
27	smooth	$(1-p^2T)(1-126T-8345pT^2-126p^4T^3+p^8T^4)$
28	smooth	$(1-p^2T)(1+41pT+22356pT^2+41p^5T^3+p^8T^4)$
29	smooth	$(1+p^2T)(1-100T+8826pT^2-100p^4T^3+p^8T^4)$
30	smooth	$(1-p^2T)(1+432T-18398pT^2+432p^4T^3+p^8T^4)$

$p=37$		
z	smooth/singular	$R_H(\mathbb{P}^5[6]^\vee, T)$
1	smooth	$(1-p^2T)^2(1+p^2T)(1-1550T+p^4T^2)$
2	smooth	$(1-p^2T)(1+1478T+86466pT^2+1478p^4T^3+p^8T^4)$
3	smooth	$(1-p^2T)(1+3260T+140718pT^2+3260p^4T^3+p^8T^4)$
4	smooth	$(1+p^2T)(1-1599T+72508pT^2-1599p^4T^3+p^8T^4)$

5	smooth	$(1+p^2T)(1-864T-34487pT^2-864p^4T^3+p^8T^4)$
6	smooth	$(1-p^2T)(1+p^2T)^2(1-1190T+p^4T^2)$
7	smooth	$(1+p^2T)(1-963T+28756pT^2-963p^4T^3+p^8T^4)$
8	smooth	$(1-p^2T)(1+194T-38550pT^2+194p^4T^3+p^8T^4)$
9	smooth	$(1-p^2T)(1+2129T+75840pT^2+2129p^4T^3+p^8T^4)$
10	smooth	$(1-p^2T)(1+1800T+75994pT^2+1800p^4T^3+p^8T^4)$
11	smooth	$(1-p^2T)(1+2349T+86920pT^2+2349p^4T^3+p^8T^4)$
12	smooth	$(1+p^2T)(1-1428T+36814pT^2-1428p^4T^3+p^8T^4)$
13	smooth	$(1+p^2T)(1-3384T+175618pT^2-3384p^4T^3+p^8T^4)$
14	smooth	$(1+p^2T)(1-288T+52306pT^2-288p^4T^3+p^8T^4)$
15	smooth	$(1-p^2T)(1+1310T+474p^2T^2+1310p^4T^3+p^8T^4)$
16	smooth	$(1+p^2T)(1-1323T+6004pT^2-1323p^4T^3+p^8T^4)$
17	smooth	$(1+p^2T)(1-3900T+190969pT^2-3900p^4T^3+p^8T^4)$
18	smooth	$(1+p^2T)(1-1344T+31858pT^2-1344p^4T^3+p^8T^4)$
19	smooth	$(1+p^2T)(1-84T-9407pT^2-84p^4T^3+p^8T^4)$
20	smooth	$(1-p^2T)(1+4394T+230250pT^2+4394p^4T^3+p^8T^4)$
21	smooth	$(1+p^2T)(1-3171T+133540pT^2-3171p^4T^3+p^8T^4)$
22	smooth	$(1+p^2T)(1-1914T+91813pT^2-1914p^4T^3+p^8T^4)$
23	smooth	$(1+p^2T)(1+32T-48999pT^2+32p^4T^3+p^8T^4)$
24	smooth	$(1-p^2T)(1+2090T+53418pT^2+2090p^4T^3+p^8T^4)$
25	smooth	$(1-p^2T)(1+2840T+133818pT^2+2840p^4T^3+p^8T^4)$
26	smooth	$(1-p^2T)(1+2574T+91195pT^2+2574p^4T^3+p^8T^4)$
27	smooth	$(1-p^2T)(1+3078T+137491pT^2+3078p^4T^3+p^8T^4)$
28	smooth	$(1+p^2T)(1-1575T+16444pT^2-1575p^4T^3+p^8T^4)$
29	smooth	$(1-p^2T)(1+180T-59762pT^2+180p^4T^3+p^8T^4)$
30	smooth	$(1+p^2T)(1-1884T+47518pT^2-1884p^4T^3+p^8T^4)$
31	smooth	$(1+p^2T)(1-1516T+22785pT^2-1516p^4T^3+p^8T^4)$
32	smooth	$(1-p^2T)(1+1124T-17730pT^2+1124p^4T^3+p^8T^4)$
33	smooth	$(1-p^2T)(1+1664T+105930pT^2+1664p^4T^3+p^8T^4)$
34	smooth	$(1+p^2T)(1-2209T+p^4T^2)(1-26pT+p^4T^2)$
35	smooth	$(1-p^2T)(1-1886T+p^4T^2)(1+58pT+p^4T^2)$
36	singular	

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