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Kac-Moody symmetries and gauged supergravity

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On the cover: the root system of the hyperbolic Kac-Moody algebra E_{10} up to height 124 projected onto the Coxeter plane of a regular E_8 subalgebra.

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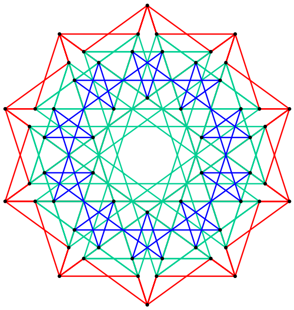

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Introduction

1.1 Physical background

The two great successes of physics in the last century were the discovery of General Relativity by Einstein and the construction of the Standard Model. Both were guided by what may be called the principle of symmetry. For Einstein this implied that the laws of physics should be the same for all observers, whether they are upside down or not, and standing still or accelerating [86]. For the Standard Model it means that its predictions are invariant under a big set of transformations known as $SU(3) \times SU(2) \times U(1)$ [18].

Perhaps the great failure of physics of the last century has been the unsuccessfulness of combining the two into one single theory. This theory would ideally unify the four forces of nature (the strong and weak interaction, electromagnetism, and gravity) into one single description. Although not ultimately successful, one of the more promising candidates for unification is *string theory* [39, 40]. The basic idea of string theory is to replace the point-like particles of the Standard Model by one-dimensional objects known as strings. All the known elementary particles should then correspond to different vibrations of the string. The biggest attraction of string theory, besides 'smoothing out' the infinities that are inherit to the framework of

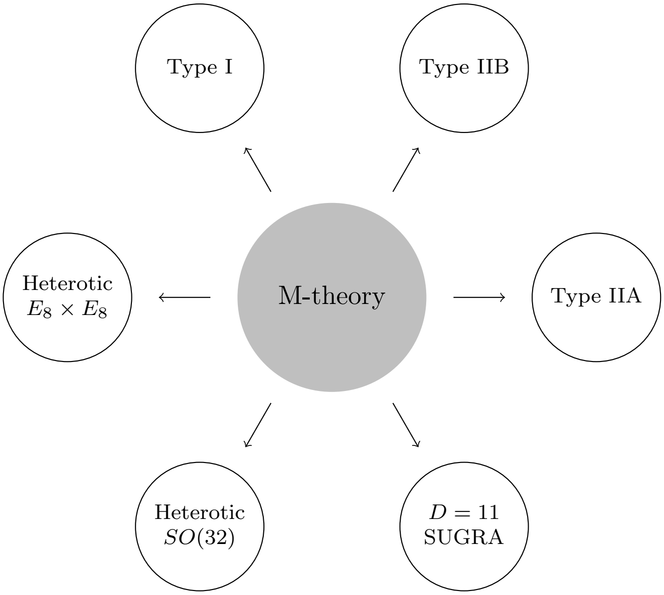


Figure 1.1: The limits of M-theory. The limits are known, M-theory is not.

the Standard Model, is that one of its vibrations gives rise to the graviton. As such, string theory naturally incorporates gravity.

For a theory that combines all the forces of nature, you would expect there to be only one. However, there is no one single unique string theory. Instead there are no less than five self-consistent string theories. They go by the (not very poetic) names of Type I, Type IIA, Type IIB, Heterotic $E_8 \times E_8$, and Heterotic $SO(32)$. They all carry some degree of supersymmetry (a symmetry between bosons (forces) and fermions (matter)), and all live in ten space-time dimensions. This did not bode well for string theory, until it was realized that these five theories are related by some form of symmetry, known as string dualities [46]. Furthermore, it was conjectured that they were all some limit of a yet unknown theory in eleven dimensions, dubbed *M-theory* [43]. Little was, and still is, known about M-theory. All we (think we) know is its low-energy limit: the unique eleven-dimensional supersymmetric gravity (supergravity) theory.

Like the theory itself, it remains a guess as to what its symmetries are. But once they are known, the principle of symmetry may guide our search for a concrete formulation of M-theory. One requirement of the symmetries of M-theory is that they should at least contain the dualities that tie the five string theories together. A further hint towards the symmetries of M-theory may come from supergravity:

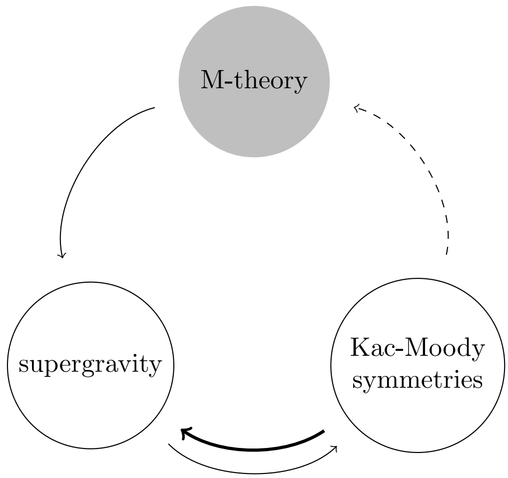


Figure 1.2: The interrelations between M-theory, supergravity, and Kac-Moody symmetries. This thesis focuses on the bold arrow from Kac-Moody symmetries to supergravity.

in certain lower-dimensional limits supergravity exhibits an infinite amount of symmetry [42]. These symmetries go by the collective name of *Kac-Moody symmetries*. As they contain the duality symmetries of string theory [54], Kac-Moody symmetries may be conjectured to describe the symmetry of M-theory [23, 87]. However, in this thesis we will not go that far (see also Figure 1.2). Instead, we will show that Kac-Moody symmetries play a unifying role in supergravity, and consequently may contain little bits and pieces of information on M-theory.

1.2 From symmetry to groups

Symmetry. Not only makes it our world round, but it's also what makes it *go* round. From the perfect circular wheels on our bikes and cars that deliver an enjoyable ride, to the error-correction protocols that keep e-mails from turning into junk [64]; it's literally all around us. It's also symmetry that dictates the laws of nature. On the small scale the symmetry group $SU(3) \times SU(2) \times U(1)$ of the Standard Model controls the interactions in molecules, atoms, and nuclei. On the large scale gravity is governed by Einstein's symmetry principle of our space-time.

But what is symmetry exactly? Let us first consider the principle of symmetry in physics. It can be formulated as the fact that a physical process remains a physical process after it has been transformed by a symmetry. In Figure 1.3 person 1 is

(a) Normal time **(b)** Reversed time

Figure 1.3: Person 1 throwing a ball to person 2 (a), and the other way around if we flip the time (b). Both are valid physical processes.

throwing a ball to person 2. If we reverse the time direction, the process is altered: it is now person 2 who's throwing a ball to person 1. A different process, but a valid one nonetheless. We can therefore say that Newton's laws of physics have a time reversal symmetry.

In the more mathematical sense, symmetry is an action on an object that, once you're done performing it, does not change that object. This is a very abstract definition, but we can try to illustrate it with a simple equilateral triangle. The triangle (see Figure 1.4a) has 6 symmetries. There are two different rotations (over 120° and 240°), three reflections, and finally the action of doing nothing at all, called the *identity*. After performing any of these actions you end up with the same triangle in the same position.

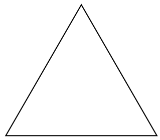
What's more, if we perform any two of these actions in a row, we will always end up with a third action. This is known as *closure* of the symmetry actions. For instance, if we rotate first over 120° and then over 240°, the net result is the identity operation. A concise way to write this is

$$a \cdot b = e, \tag{1.1}$$

where e is the identity, a and b are rotations over 120° and 240°, respectively. The result of all possible combinations of rotational symmetries are summarized in Figure 1.4b.

In fact, the combined symmetry actions form a mathematical object known as a *group*. A group G has four defining characteristics:

1. **Identity element** There exists an element $e \in G$, such that for all elements $a \in G$, the equation $e \cdot a = a \cdot e = a$ holds.
2. **Closure** For all $a, b \in G$, their product $a \cdot b$ is also in G.
3. **Inverse** For any $a \in G$ there exists an element a^{-1} such that $a \cdot a^{-1} = e$.
4. **Associativity** The equation $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ holds for any $a, b, c \in G$.



(a)

$\cdot$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

(b)

Figure 1.4: The equilateral triangle (a) and the group multiplication table (b) of its rotational symmetries. e is the identity, a and b are rotations over $120°$ and $240°$, respectively.

We have already seen that closure holds for the triangle; it is not hard to see that the other three properties also hold. But this is not only restricted to the symmetries of the triangle. The importance of group theory lies in the fact that any symmetry you can think of can be described as a group, and that conversely all groups describe a symmetry.

1.3 From groups to algebras

If we examine the symmetries of the circle (see Figure 1.5), you will notice that a rotation over any arbitrary angle leaves it invariant. This means that the circle has an infinite amount of rotational symmetry. The mathematical object that describes these symmetries is still a group, but no longer a discrete (i.e. finite) one. The symmetry group of the circle is *continuous*: every angle between e.g. $120°$ and $240°$ corresponds to a symmetry. This is not so for the triangle: in that case there are 'gaps' between the rotations. The symmetry of the triangle is therefore called *discrete*.

Continuous groups are known as *Lie groups*. They contain an infinite amount of elements. But because they're continuous we can parameterize the elements in one or more parameters. For the circle we can write any rotation $R(\theta)$ over an angle θ as

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{1.2}$$

which is the rotation matrix in two dimensions. With a bit of work it follows that

$$R(\theta_1) \cdot R(\theta_2) = R(\theta_1 + \theta_2). \tag{1.3}$$

This is what you would expect if you were to perform two rotations in a row: namely, the angles simply add up.

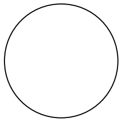


Figure 1.5: The circle.

Because of this property, we can go from the identity to a rotation over an arbitrary angle θ by repeatedly applying rotations over an infinitesimally small angle $d\theta$. In fact, the amount of change described by such a small rotation at the identity encodes almost all the information we need to describe the full group. This particular amount of change is encoded in a new object T:

$$T \equiv \frac{dR(\theta)}{d\theta}\bigg|_{\theta=0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.4}$$

Indeed, we recover all rotations by exponentiating T,

$$R(\theta) = e^{\theta T}. \tag{1.5}$$

We say that T generates the symmetry group of the circle. It is therefore also called a *generator*. This single object captures almost all of the important properties of the infinite symmetry group. It is not part of the group, but lives in the tangent space at the identity, known as the *Lie algebra* of the group.

Now the group of rotational symmetries of the circle is a particularly simple one. But every continuous group, even if it is horrendously complicated, has an associated Lie algebra. Thus studying the Lie algebra of a particular symmetry is sufficient to uncover most of its properties.

1.4 Infinite Lie algebras

The Lie algebras studied in this thesis, known as *Kac-Moody algebras*, are slightly more complicated than the one described above. While the Lie algebra of the rotational symmetries of the circle has only one generator, T, Kac-Moody algebras have an infinite number of them. Bear in mind that every single generator 'generates' an infinite amount of symmetry by means of the exponential mapping (1.5). This means that Kac-Moody algebras describe a symmetry that is infinitely many times infinite.

It is then not so surprising that most of the interesting Kac-Moody algebras are hard to describe in full. What one usually does is focus on a small portion (a *subalgebra*), and see how the whole algebra behaves with respect to that.

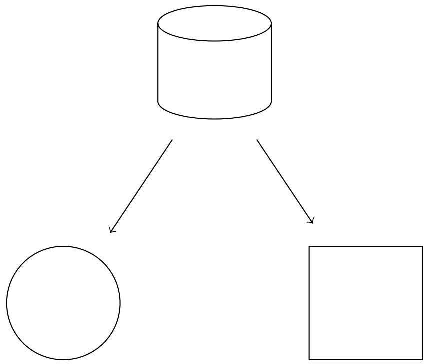


Figure 1.6: Two different projections of the same cylinder.

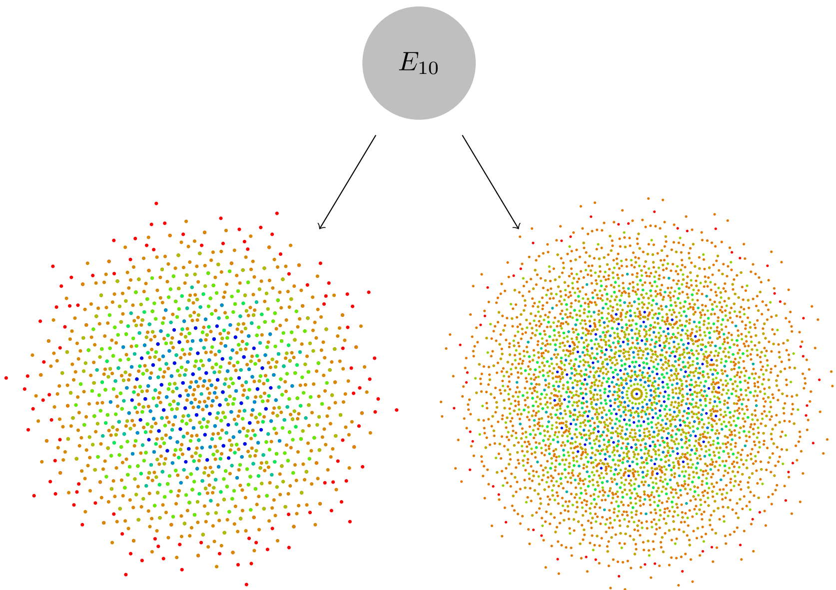


Figure 1.7: Two different slices of the Kac-Moody algebra E_{10} produce two different projections.

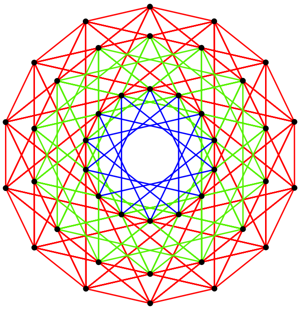
To illustrate the concept, take a look at for example a cylinder (Figure 1.6). The cylinder can be thought of as a stack of circles on top of each other, and thus it carries the symmetry of the circle. But rotating the circles is not the only symmetric operation we can perform on the cylinder. We can also interchange the circles in the stack. These two distinct symmetries can be thought of as two different projections of the cylinder: one produces a circle, and the other a square. Both are aspects of the full symmetry of the cylinder.

This is a simplification of what happens for Kac-Moody algebras. Because we cannot describe their symmetry in full, we must resort to finite subalgebras and 'slice' with respect to those (see Figure 1.7). The resulting projections then tell us something of what the full Kac-Moody algebra looks like.

Things get interesting when the subalgebras with respect to which we slice are chosen such that they match symmetries of physical theories, namely supergravities. Not only do the slicings then tell us something about the full Kac-Moody algebras, they can then also be used to construct maps from the Kac-Moody side to the physical side. The maps in question relate the various physical fields, for instance the graviton, to sets of generators of the Kac-Moody algebra.

How to do these slicings, and the process of matching Kac-Moody algebras to physical theories, is the main topic of this thesis.

2



Lie algebras

In this chapter the necessary mathematical concepts for describing infinite dimensional Lie algebras will be introduced. For a thorough treatment on the subject, see for example [35, 15, 50]. All the results are valid for both finite and infinite Lie algebras, unless stated otherwise.

2.1 Lie algebras

2.1.1 Basic definitions

A *Lie algebra* $\mathfrak{g}$ is a vector space with an additional bilinear operation $[\cdot,\cdot]$ that sends two generic elements in $\mathfrak{g}$ to another element in $\mathfrak{g}$:

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}. \tag{2.1}$$

This operation is called the *Lie bracket*. A defining feature of it is that the Lie bracket of an element with itself vanishes,

$$[x, x] = 0 \quad \forall\, x \in \mathfrak{g}. \tag{2.2}$$

Because of bilinearity the Lie bracket is automatically anti-symmetric, $[x, y] = -[y, x]$. Besides the Lie bracket, the *Jacobi identity*

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \quad \forall\, x, y, z \in \mathfrak{g}. \tag{2.3}$$

holds. The Lie bracket and the Jacobi identity are, as simple as they may appear, enough to endow a vector space with the rich and complex structure of a Lie algebra. In what follows, concepts and definitions indispensable to the study of Lie algebras will be introduced.

For a fixed but arbitrary element x of $\mathfrak{g}$ the *adjoint action* is given by

$$\mathrm{ad}_x(y) = [x, y]. \tag{2.4}$$

This is a map from $\mathfrak{g}$ onto itself, $\mathrm{ad}_x : \mathfrak{g} \mapsto \mathfrak{g}$. For finite Lie algebra the *Cartan-Killing form* is defined by taking traces over the adjoint action:

$$\langle x|y\rangle \propto \mathrm{Tr}(\mathrm{ad}_x\, \mathrm{ad}_y). \tag{2.5}$$

The Cartan-Killing form is an inner product on the Lie algebra. It sends two generic elements of $\mathfrak{g}$ to a number:

$$\langle\cdot|\cdot\rangle : \mathfrak{g} \times \mathfrak{g} \mapsto F, \tag{2.6}$$

where F is the field over which $\mathfrak{g}$ is a vector space.

A subspace $\mathfrak{s} \subseteq \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is called a Lie *subalgebra* if it is a Lie algebra in itself. In particular, it must close onto itself:

$$[\mathfrak{s}, \mathfrak{s}] \subseteq \mathfrak{s}, \tag{2.7}$$

which is a shorthand notation for $[x, y] \in \mathfrak{s}$ for all $x, y \in \mathfrak{s}$. If we impose the stronger condition

$$[\mathfrak{s}, \mathfrak{g}] \subseteq \mathfrak{s} \tag{2.8}$$

then $\mathfrak{s}$ is called an *ideal* of the Lie algebra $\mathfrak{g}$. Proper ideals and subalgebras are those which are not equal to $\mathfrak{g}$ or $\{0\}$.

A Lie algebra is *abelian* if its Lie bracket vanishes, $[\mathfrak{g}, \mathfrak{g}] = 0$. A *simple* Lie algebra contains no proper ideals and is not abelian. Finally, a *semi-simple* Lie algebra is a direct sum of simple ones.

The dimension $d = \dim \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ is the dimension of $\mathfrak{g}$ considered as a vector space. Thus we can find a basis of $\mathfrak{g}$ consisting of d linearly independent elements t_α, which are called *generators* of $\mathfrak{g}$. Expanding in terms of the generators, the Lie bracket reads

$$[t_\alpha, t_\beta] = f_{\alpha\beta}{}^{\gamma} t_\gamma. \tag{2.9}$$

The numbers $f_{\alpha\beta}{}^{\gamma}$ are called the *structure constants* and characterize $\mathfrak{g}$ completely. But because the indices α,β, and γ run over the dimension of $\mathfrak{g}$, the structure constants become quite untractable when dealing with infinite dimensional Lie algebras.

Instead, one usually adopts a rather different approach, in which the Lie algebra is completely defined by the so-called Cartan matrix.

Example 2.1: $\mathfrak{sl}(2)$

The Lie algebra $\mathfrak{gl}(n)$ is the vector space of $n \times n$ matrices with Lie bracket

$$[x, y] = x \cdot y - y \cdot x, \tag{2.10}$$

where $\cdot$ stands for ordinary matrix multiplication. In this case the Jacobi identity is automatically satisfied.

If we restrict to the space of matrices with vanishing trace, the Lie algebra is denoted by $\mathfrak{sl}(n)$. The smallest non-trivial example of such a Lie algebra is $\mathfrak{sl}(2)$. One particular basis for it is

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.11a}$$

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{2.11b}$$

$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.11c}$$

If we calculate the Lie brackets by means of (2.10), we find

$$[h, e] = 2e, \tag{2.12a}$$

$$[h, f] = -2f, \tag{2.12b}$$

$$[e, f] = h, \tag{2.12c}$$

with all other brackets vanishing.

2.1.2 The Cartan matrix

A *Cartan matrix* $A = (A_{ij})$ is a square $n \times n$ matrix with integer values, satisfying the following conditions:

$$A_{ii} = 2, \tag{2.13a}$$

$$A_{ij} \in \mathbb{Z}_{\leq 0} \quad \text{for } i \neq j, \tag{2.13b}$$

$$A_{ij} = 0 \Leftrightarrow A_{ji} = 0, \tag{2.13c}$$

$$\det A > 0, \tag{2.13d}$$

$$M(A) > 0. \tag{2.13e}$$

Here and in the following the indices i and j run over the size of the matrix, $i, j = 1, \ldots, n$. Note that the Einstein summation convention has been suspended; A_{ii} denotes the diagonal entries of the Cartan matrix, not its trace. In (2.13e) the expression $M(A)$ denotes all principal minors of A. A principal minor is the determinant of a submatrix obtained by the simultaneous removal of the same set of rows and columns. Conditions (2.13d) and (2.13e) together imply that A is positive definite. If they are dropped, the Cartan matrix is called *generalized*.

The Cartan matrix is decomposable if it can be rewritten in the form

$$A = \begin{pmatrix} A^{(1)} & 0 \\ 0 & A^{(2)} \end{pmatrix} \tag{2.14}$$

by simultaneously reordering rows and columns. Furthermore, A is assumed to be symmetrizable. This means there exists an invertible diagonal matrix $D = \operatorname{diag}(\epsilon_1, \ldots, \epsilon_n)$ such that

$$A = BD, \tag{2.15}$$

where B is a symmetric matrix.

We can associate a Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ to the Cartan matrix A by considering the $3n$-tuple of generators $\{h_i, e_i, f_i\}$ subject to the relations

$$[h_i, h_j] = 0, \tag{2.16a}$$
$$[h_i, e_j] = A_{ji} e_j, \tag{2.16b}$$
$$[h_i, f_j] = -A_{ji} f_j, \tag{2.16c}$$
$$[e_i, f_j] = \delta_{ij} h_i, \tag{2.16d}$$

and

$$(\mathrm{ad}_{e_i})^{1-A_{ji}} e_j = 0, \tag{2.17a}$$
$$(\mathrm{ad}_{f_i})^{1-A_{ji}} f_j = 0. \tag{2.17b}$$

The h_i form a maximal abelian subalgebra $\mathfrak{h} \in \mathfrak{g}$ known as the *Cartan subalgebra*. The e_i and f_i are called the *Chevalley generators*. Lastly, the equations (2.17) go by the name of the *Serre relations*.

The *rank* of $\mathfrak{g}$ is defined as the dimension of $\mathfrak{h}$. This in turn is equal to the size of the Cartan matrix if it is non-degenerate:

$$\operatorname{rank}(\mathfrak{g}) = \dim \mathfrak{h} = n. \tag{2.18}$$

The full Lie algebra $\mathfrak{g}(A)$ is constructed by considering multiple commutators of the form

$$[e_i, [\cdots [e_j, e_k]] \cdots], \tag{2.19a}$$
$$[f_i, [\cdots [f_j, f_k]] \cdots]. \tag{2.19b}$$

These multiple commutators correspond to additional generators. Together with the $3n$-tuple $\{h_i, e_i, f_i\}$ they form a basis of $\mathfrak{g}$. However, we must keep in mind that the Serre relations (2.17) will put certain commutators to zero, and that others are related to each other by the Jacobi identity (2.3).

A generic element of $\mathfrak{g}$ that is not part of $\mathfrak{h}$ is either a combination of multiple commutators of the e_i or of multiple commutators of the f_i. Thus the Lie algebra $\mathfrak{g}$ possesses a triangular decomposition:

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+. \tag{2.20}$$

The negative part $\mathfrak{n}_-$ consists of commutators of the form (2.19b), whereas the positive part $\mathfrak{n}_+$ consists of commutators of the form (2.19a). The negative and positive parts Chevalley generators can be interchanged by means of the *Chevalley involution* ω, which is defined as

$$\omega(e_i) = -f_i, \tag{2.21a}$$
$$\omega(f_i) = -e_i, \tag{2.21b}$$
$$\omega(h_i) = -h_i. \tag{2.21c}$$

The Chevalley involution can consistently be extended to the whole of $\mathfrak{g}$, exchanging $\mathfrak{n}_+$ and $\mathfrak{n}_-$. It therefore suffices to study only one of the two.

In the Chevalley basis introduced above, the Cartan-Killing form (2.5) reads

$$\langle h_i | h_j \rangle = \epsilon_i A_{ij} = \epsilon_i \epsilon_j B_{ij}, \tag{2.22a}$$
$$\langle e_i | f_j \rangle = \delta_{ij} \epsilon_i. \tag{2.22b}$$

All other combinations vanish. Here ϵ_i is the i^{th}diagonal entry of the matrix D in equation (2.15). The Cartan-Killing form can be uniquely extended by induction to the whole of $\mathfrak{g}$ by requiring that is is invariant, i.e. $\langle [x, y] | z \rangle = \langle x | [y, z] \rangle$.

The relations (2.16) and (2.17) summarize the structure of $\mathfrak{g}$ in a very compact form, namely in the Cartan matrix A. The procedure of constructing the Lie algebra from the Cartan matrix is known as the *Serre construction.* A Lie algebra $\mathfrak{g}$ obtained in this way is always finite and semi-simple. It is simple if the Cartan matrix A is indecomposable. If conditions (2.13d) and (2.13e) are dropped, the Lie algebra may become infinite. Although it is not immediately clear from the discussion here, the argument also works the other way around: *any* finite simple Lie algebra can be completely described in terms of a Cartan matrix. The task of classifying all possible finite simple Lie algebras thus boils down to finding all possible indecomposable matrices that satisfy (2.13). The result is the so-called Cartan classification. But before stating it, it is useful to introduce the concept of a Dynkin diagram.

Example 2.2: Serre construction

The simplest Cartan matrix is the one of rank 1:

$$A = \begin{pmatrix} 2 \end{pmatrix} . \tag{2.23}$$

The resulting Lie algebra has three generators, h, e, and f. This is in fact $\mathfrak{sl}(2)$, the same Lie algebra as in Example 2.1. If we inspect (2.16) more closely, we see that the building blocks of every Lie algebra of rank n are n interconnected $\mathfrak{sl}(2)$ subalgebras.

A slightly more complicated Cartan matrix is

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} . \tag{2.24}$$

Because its rank is equal to two, the resulting Lie algebra now has at least six generators, h_1, h_2, e_1, e_2, f_1, and f_2. But there are two brackets that are not put to zero by the Serre relations, which correspond to additional generators:

$$e_{1+2} \equiv [e_1, e_2], \tag{2.25a}$$

$$f_{1+2} \equiv [f_1, f_2]. \tag{2.25b}$$

If we try to take further Lie brackets, we see that they are killed by the Serre relations:

$$[e_1, e_{1+2}] = [e_1, [e_1, e_2]] = (\mathrm{ad}_{e_1})^{1+1} e_2 = 0, \tag{2.26a}$$

$$[f_1, f_{1+2}] = [f_1, [f_1, f_2]] = (\mathrm{ad}_{f_1})^{1+1} f_2 = 0. \tag{2.26b}$$

The Lie algebra is 8-dimensional, and isomorphic to $\mathfrak{sl}(3)$.

Dynkin diagrams

The data contained in a Cartan matrix can be neatly visualized with the help of Dynkin diagrams. A *Dynkin diagram* consists of a number of nodes that are connected by lines. Given a Cartan matrix, the rules for drawing such a diagram are simple:

- For every row of the Cartan matrix A, draw one node.
- Nodes corresponding to rows i and j $(i \neq j)$ are connected if $A_{ij} \neq 0$.
- The connection consists of $\max(|A_{ij}|, |A_{ji}|)$ lines.

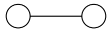
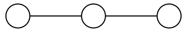
Lie algebra	Dynkin diagram	Cartan matrix
$\mathfrak{sl}(3)$		$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$
$\mathfrak{sl}(4)$		$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

Table 2.1: The Dynkin diagrams and Cartan matrices of $\mathfrak{sl}(3)$ and $\mathfrak{sl}(4)$.

- If $|A_{ij}| > |A_{ji}|$, the connection has an arrow pointing towards node i from node j.
- If the connection has an arrow and if $\min(|A_{ij}|, |A_{ji}|) > 1$, the connection has an additional label indicating $\min(|A_{ij}|, |A_{ji}|)$.

Furthermore, a diagram and its associated Lie algebra are called *simply laced* if the diagram contains no arrows, or equivalently, if the Cartan matrix is symmetric.

Two simple examples of Dynkin diagrams are given in Table 2.1. The complete classification of all finite-dimensional simple Lie algebras [36] is given in Figure 2.1. There are four infinite series: A_n, B_n, C_n, and D_n, where the n denotes the rank of the algebra. The A_n series are isomorphic to the $\mathfrak{sl}(n+1)$ Lie algebras, which we encountered earlier. The four series are infinite in the sense that the rank n can take on any value, while the resulting algebra remains finite. Furthermore there are five exceptional cases: E_6, E_7, E_8, F_4, and G_2.

Besides these five isolated cases and the four infinite series, there are no other finite-dimensional simple Lie algebras. However, if we relax the conditions (2.13d) and (2.13e) and allow for generalized Cartan matrices, the range of options increases dramatically. Lie algebras associated to generalized Cartan matrices are called *Kac-Moody algebras*. When the Cartan matrix is not positive definite, they are infinite-dimensional (see also chapter 4).

2.1.3 Roots

Thus far we have seen that a Lie algebra can be characterized either by all its Lie brackets (or equivalently its structure constants), or by its associated Cartan matrix. There is a third way of describing Lie algebras that is closely linked to both previously described ways. The key object in this case is the *root system*.

A *root* α of a generator $x \in \mathfrak{g}$ is the eigenvalue of x under the adjoint operation

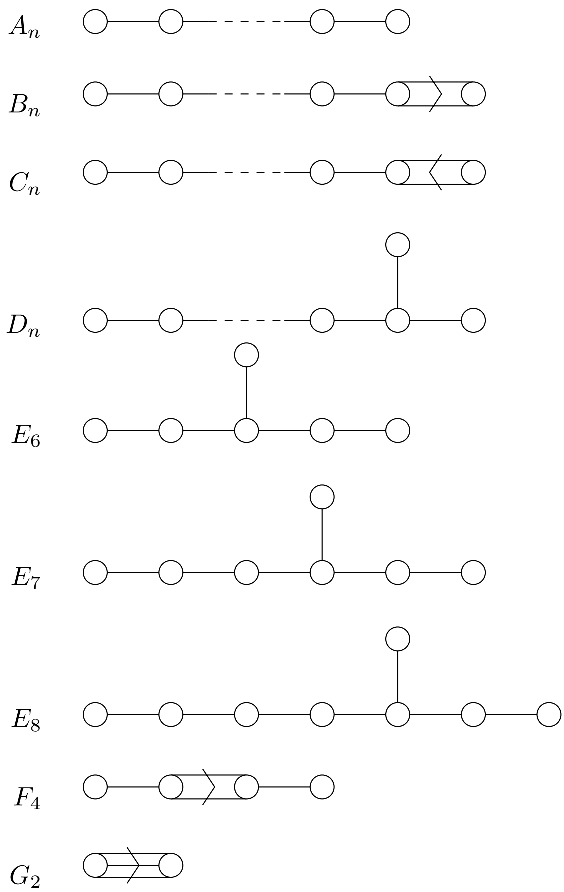


Figure 2.1: Dynkin diagrams of all finite dimensional simple Lie algebras. The subscript denotes the rank of the Lie algebra, or equivalently, the number of nodes.

of a generic element $h \in \mathfrak{h}$:

$$[h, x] = \alpha_x(h)x. \tag{2.27}$$

The root α is just a number. For a fixed element $x \in \mathfrak{g}$ it also acts as a linear function as $\alpha_x : \mathfrak{h} \mapsto F$, sending an element of the Cartan subalgebra to F, the base field over which $\mathfrak{g}$ is a vector space. Thus roots are elements of the space dual to $\mathfrak{h}$,

$$\alpha \in \mathfrak{h}^* \equiv \Phi. \tag{2.28}$$

The space Φ will be called the *root space.* The root space is a vector space with the same dimension as $\mathfrak{h}$, namely $\dim \Phi = n$. As a side note, the function α is only called a root if it is non-zero. Thus the generators in the Cartan subalgebra do not have a root α_h associated to them.

If we have a specific basis h_i of $\mathfrak{h}$, the *simple roots* α_i are defined as

$$\alpha_i(h_j) = A_{ij}, \tag{2.29}$$

where A is again the Cartan matrix. Inspection of (2.16) reveals that the simple roots are the eigenvalues of the Chevalley generators e_i under the adjoint action of h_j, that is, $\alpha_i = \alpha_{e_i}$. The simple roots also form a basis of the root space Φ. To be a bit more precise, we can consider the root of a generic Lie bracket:

$$[h, [x, y]] = \big(\alpha_x(h) + \alpha_y(h)\big)[x, y]. \tag{2.30}$$

This follows straightforwardly from the Jacobi identity (2.3). The roots add up: $\alpha_{[x,y]} = \alpha_x + \alpha_y$. But because all generators are constructed of multiple brackets of the Chevalley generators (see (2.19)), this means that all roots are linear integral combinations of the simple roots:

$$Q = \sum_{i=1}^{n} \mathbb{Z}\alpha_i. \tag{2.31}$$

The lattice Q is called the *root lattice.* However, by courtesy of the Serre relations (2.17), not all points on the root lattice are actual roots. The set of all proper roots is called the *root system* and denoted by $\Delta\,(\subset Q \subset \Phi)$.

A generic root α can be expanded in terms of the simple roots as

$$\alpha = \sum_{i=1}^{n} m^i \alpha_i. \tag{2.32}$$

The numbers m^i are collectively called the *root vector.* Because of the triangular decomposition (2.20) they are either all non-negative ($m^i \geq 0$) or all non-positive

($m^i \leq 0$). In the former case the root is said to be *positive*, and in the latter it is *negative*. Thus the root system Δ splits into two disjoint sets,

$$\Delta = \Delta_+ \cup \Delta_-, \tag{2.33}$$

where Δ_+ contains all the positive roots, and Δ_- contains all the negative roots. Because of the Chevalley involution (2.21) it holds that $\Delta_- = -\Delta_+$. It is therefore enough to study either Δ_+ or Δ_-.

The *root space* $\mathfrak{g}_\alpha$ of a root α is the set of generators of $\mathfrak{g}$ that has α as an eigenvalue under $\mathfrak{h}$. Thus:

$$\mathfrak{g}_\alpha = \big\{x \in \mathfrak{g} \,|\, [h, x] = \alpha(h)x \;\forall\, h \in \mathfrak{h}\big\}. \tag{2.34}$$

The dimension of this space is known as the *multiplicity* of the root, $\mathrm{mult}(\alpha)$. It is the same as the number of generators that share the same root α. For finite Lie algebras the multiplicity is always equal to one, whereas for infinite Lie algebras the multiplicity can degenerate. In section 2.3 we will deal with how to calculate the root multiplicities.

By (2.30) the Lie algebra is graded by means of its roots. In particular, we have for the root spaces

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}. \tag{2.35}$$

The complete Lie algebra $\mathfrak{g}$ can be decomposed into its root spaces as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha. \tag{2.36}$$

The reason why the Cartan subalgebra appears separately in the direct sum is that roots were defined to be non-zero.

The Cartan matrix not only specifies what the simple roots are, it also defines an inner product between them. Because the simple roots are the basis vectors of the root space Φ, the inner product acts on the whole root space:

$$(\cdot|\cdot) : \Phi \times \Phi \mapsto F. \tag{2.37}$$

It is given by the following definition,

$$A_{ij} \equiv 2\frac{(\alpha_i|\alpha_j)}{(\alpha_j|\alpha_j)}, \tag{2.38}$$

from which we can deduce the actual root space metric $B = (B_{ij})$:

$$B_{ij} \equiv (\alpha_i|\alpha_j) = \frac{A_{ij}}{\epsilon_j}. \tag{2.39}$$

Here we have set $\epsilon_i = \frac{2}{(\alpha_i|\alpha_i)}$ (compare equation (2.15)). From equation (2.22) it is clear that the inner product on the root space corresponds to the restriction of the Cartan-Killing form to the CSA. The inner product is unique, but only up to normalization. This means we have the freedom to choose a norm $\alpha_i^2 = (\alpha_i|\alpha_i)$ of one of the simple roots, after which all the others are fixed by

$$\frac{(\alpha_i|\alpha_i)}{(\alpha_j|\alpha_j)} = \frac{A_{ij}}{A_{ji}}. \tag{2.40}$$

If the Cartan matrix is decomposable, we have to fix the normalization once for every indecomposable subpart.

In the mathematical literature it is common to fix the normalization such that the simple roots have at most norm equal to 2, i.e. $\alpha_i^2 \leq 2$. However, I will adhere to the convention that the simple roots have *at least* norm equal to 2, i.e. $\alpha_i^2 \geq 2$. The reason is that in the latter case $\frac{1}{\epsilon_i}$, and thus also the metric on the root space, has only integer values. Note that if the Cartan matrix is symmetric to start with, then it coincides with the root space metric. In that case the norms of all the simple roots are equal, and the algebra is called simply laced.

The inner product between two roots α and β can easily be expanded in their respective roots vectors m^i and n^i as

$$(\alpha|\beta) = \sum_{i,j=1}^{n} B_{ij} m^i n^j. \tag{2.41}$$

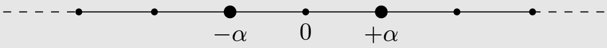
For finite Lie algebras the Cartan matrix A and the root space metric B are positive definite. This entails that root norms are always positive, $\alpha^2 > 0$. However, when conditions (2.13d) and (2.13e) are dropped, A may become indefinite, allowing for null and negative directions. This prompts us to distinguish between the following cases:

$$(\alpha|\alpha) = \begin{cases} > 0 & \text{real root,} \\ = 0 & \text{imaginary (null) root,} \\ < 0 & \text{imaginary root.} \end{cases} \tag{2.42}$$

Thus finite Lie algebras have only real roots, whereas infinite algebras also have imaginary roots.

Example 2.3: Root system of A_1 and A_2

The Lie algebra A_1 is of rank one. This means it has only one simple root α which is given by $[h, e] = \alpha e$. The root system of A_1 consists of just two roots, namely α and $-\alpha$, the latter belonging to the generator f. The root lattice is the line of integers, as depicted in the following image.

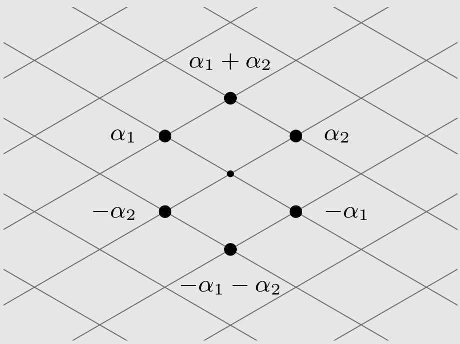


The roots are indicated with big dots, whereas other points on the root lattice that do not correspond to roots are indicated with smaller dots.

The root system of A_2 is a bit more interesting. Being of rank two, it has two simple roots α_1 and α_2. The root space is thus a two-dimensional vector space, spanned by α_1 and α_2. There are six roots in total: the two simple roots and their negatives, and the root of e_{1+2} and its negative (see Example 2.2). By (2.30), the root of e_{1+2} is given by

$$[h, e_{1+2}] = [h, [e_1, e_2]] = (\alpha_1 + \alpha_2)e_{1+2}. \tag{2.43}$$

The picture of the root lattice is as follows:



The roots are again indicated with the big dots, while the other vertices correspond to points on the lattice that are not roots. The small dot in the middle is the origin.

You may have noticed that the angle between the simple roots is not 90°, but 120°. The reason is that the angle is fixed by the inner product $(\cdot|\cdot)$ in the usual way: $\cos\theta = \frac{(\alpha_1|\alpha_2)}{\sqrt{\alpha_1^2\alpha_2^2}} = -\frac{1}{2}$.

2.1.4 Weights

In this section the so-called *weights* will be introduced. Their importance lies in the fact they can be used to describe representations of Lie algebras (see section 2.2). But before weights are discussed, it is convienent to associate to any root α a *coroot* $\alpha^\vee$ by

$$\alpha^\vee = \frac{2\alpha}{(\alpha|\alpha)}. \tag{2.44}$$

Amongst others, this simplifies the expression of the Cartan matrix in terms of the inner product (2.38) a bit:

$$A_{ij} = \big(\alpha_i|\alpha_j^\vee\big). \tag{2.45}$$

We are now in the position to define the *fundamental weights* Λ^i as the duals of the simple coroots,

$$\big(\Lambda^i|\alpha_j^\vee\big) = \delta^i_j. \tag{2.46}$$

The fundamental weights span the weight space dual to the root space Φ, and its elements are called weights. A generic weight λ can be expressed in terms of the fundamentals weights as

$$\lambda = \sum_{i=1}^n p_i \Lambda^i. \tag{2.47}$$

The coefficients p_i are called the *Dynkin labels* of the weight. The lattice P on which the Dynkin labels are strictly integers, i.e.

$$P = \sum_{i=1}^n \mathbb{Z}\Lambda^i, \tag{2.48}$$

is called the *weight lattice* and is dual to the root lattice, $P = Q^*$.

For convenience we can also introduce *cofundamental weights* $\Lambda^{\vee i}$ as the duals of the simple roots:

$$\big(\Lambda^{\vee i}|\alpha_j\big) = \delta^i_j, \tag{2.49a}$$

$$\Lambda^{\vee i} = \frac{2}{(\alpha_i|\alpha_i)}\Lambda^i. \tag{2.49b}$$

The cofundamental weights are not as important as the fundamental weights; their main use is to simplify certain notation.

When we compare equations (2.45) and (2.46) we see that the fundamental weights and the simple roots can be expressed in terms of each other as

$$\Lambda^i = \sum_{j=1}^{n} \left(A^{-1}\right)^{ij} \alpha_j, \tag{2.50a}$$

$$\alpha_i = \sum_{j=1}^{n} A_{ij}\Lambda^j. \tag{2.50b}$$

It is often of interest to know what the Dynkin labels of a particular root are. To that end, we can simply expand a root in both the simple root and the fundamental weight bases, and equate the coefficients. Upon doing so, we see that the root vector m^i and Dynkin labels p_i of a root α are related by

$$p_i = (\alpha|\alpha_i^\vee) = \sum_{j=1}^{n} A_{ji} m^j, \tag{2.51a}$$

$$m^i = \left(\alpha|\Lambda^{\vee i}\right) = \sum_{j=1}^{n} (A^{-1})^{ji} p_j. \tag{2.51b}$$

Following [53], the root vector will be written as $(m^1, \ldots, m^n)$ and the Dynkin labels as $[p_1, \ldots, p_n]$ in order to distinguish between the different bases. Note that points on the root lattice always lie on the weight lattice as well, because the Cartan matrix contains only integers. However, the converse is not necessarily true, as the inverse of the Cartan matrix may contain fractional entries.

The inner product on the root space can be extended by linearity to an inner product on the weight space. Given two weights λ and μ with respective Dynkin labels p_i and q_i, it can be computed to be

$$(\lambda|\mu) = \sum_{i,j=1}^{n} G^{ij} p_i q_j. \tag{2.52}$$

Here G is the metric on the weight space, and is also called the *quadratic form matrix.* A short calculation reveals that is related to the Cartan matrix by

$$G^{ij} \equiv \left(\Lambda^i|\Lambda^j\right) = \tfrac{1}{2}(A^{-1})^{ij} \left(\alpha_j|\alpha_j\right). \tag{2.53}$$

Similarly to the metric on the root space, this expression simplifies when the algebra is simply laced. Then the quadratic form matrix is just the inverse of the Cartan matrix.

Example 2.4: Weight lattice of A_2

The Cartan matrix of the Lie algebra A_2 is

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \tag{2.54}$$

which has an inverse given by

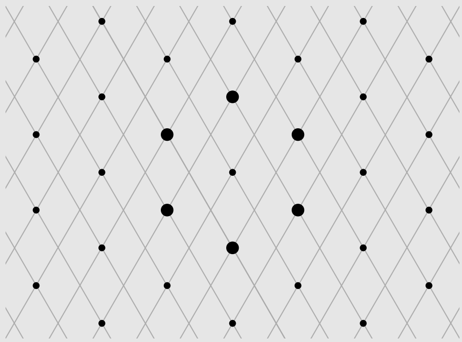
$$A^{-1} = \tfrac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}. \tag{2.55}$$

By equation (2.50a), the fundamental weights are

$$\Lambda^1 = \tfrac{1}{3}\left(2\alpha_1 + \alpha_2\right) \tag{2.56a}$$

$$\Lambda^2 = \tfrac{1}{3}\left(\alpha_1 + 2\alpha_2\right) \tag{2.56b}$$

These two fundamental weights form the basis of the weight lattice, which is drawn below. All the intersections of lines correspond to points on the weight lattice, i.e. weights with integer Dynkin labels. Superposed on the weight lattice are the roots of A_2 (indicated with big dots) and the other points on the root lattice (indicated with small dots). See also Example 2.3.



Note that not all points on the weight lattice coincide with points on the root lattice. This is due to the fact that the inverse Cartan matrix has a factor $\frac{1}{3}$ in front, stemming from its determinant.

2.1.5 The Weyl vectors and the highest root

Two distinguished elements of the weight space are the *Weyl vector* ρ and the *dual Weyl vector* $\rho^\vee$. They can be defined as

$$(\rho|\alpha_i^\vee) = 1, \tag{2.57a}$$

$$(\rho^\vee|\alpha_i) = 1. \tag{2.57b}$$

Expanded in the basis of fundamental weights, both read

$$\rho = \sum_{i=1}^n \Lambda^i, \tag{2.58a}$$

$$\rho^\vee = \sum_{i=1}^n \Lambda^{\vee i}. \tag{2.58b}$$

The Dynkin labels of ρ are thus all equal to one. The Weyl vector will play an important role in the analysis of representations. The dual Weyl vector on the other hand has fractional Dynkin labels when the algebra is not simply laced. Its main usage is to calculate the *height* of a root, which is simply the sum of the components of its root vector m^i:

$$\mathrm{ht}(\alpha) = (\alpha|\rho^\vee) = \sum_{i=1}^n m^i. \tag{2.59}$$

Simple finite Lie algebras have a unique root θ that is the highest of all roots in the root system Δ. This root is called the *highest root*. Similarly, $-\theta$ is the *lowest root* of Δ. Infinite Kac-Moody algebras on the contrary do not have a highest root; their root system 'just goes on forever'.

The components of root vector of the (dual) highest root are called the *(dual) Coxeter labels*, and are denoted by a^i and $a^{\vee i}$, respectively:

$$\theta = \sum_{i=1}^n a^i \alpha_i, \tag{2.60a}$$

$$\theta^\vee = \sum_{i=1}^n a^{\vee i} \alpha_i^\vee, \tag{2.60b}$$

where $\theta^\vee = \frac{2\theta}{(\theta|\theta)}$. Their respective sums plus one are known as the *(dual) Coxeter number* g of the Lie algebra,

$$g = 1 + (\theta|\rho^\vee) = 1 + \sum_{i=1}^n a^i, \tag{2.61a}$$

$$g^\vee = 1 + (\theta^\vee|\rho) = 1 + \sum_{i=1}^n a^{\vee i}. \tag{2.61b}$$

2.1.6 The Weyl group

As is apparent from Example 2.3, the root systems of some of the simplest of Lie algebras have already quite some symmetry. Not surprisingly, the amount of symmetry of the root system tends to increase with its size. This symmetry is captured and described in what is known as the *Weyl group* $W(\Delta)$ of the root system.

The Weyl group is a reflection group [52], generated by so-called *Weyl reflections* w_α. They are are defined as

$$w_\alpha(\beta) = \beta - (\beta|\alpha^\vee)\,\alpha. \tag{2.62}$$

They are proper reflections in the sense that they square to the identity:

$$w_\alpha(w_\alpha(\beta)) = \beta, \tag{2.63}$$

or more succinctly, $w_\alpha^2 = \mathbb{1}$.

A Weyl reflection w_α is a reflection with respect to a hyperplane perpendicular to a fixed root α (see also Figure 2.2). For example, when a root gets reflected with respect to itself, the result is the negative of that root: $w_\alpha(\alpha) = -\alpha$. By virtue of the triangular decomposition (2.20) this is also a root. The same holds for *any* Weyl reflection: the result always lies in the root system. Thus the *orbit* $W(\alpha)$ of a root,

$$W(\alpha) = \bigcup_{w\in W} w(\alpha), \tag{2.64}$$

i.e. is its image under all the elements of the Weyl group, lies in the root system. Moreover, any points on the root lattice between β and $w_\alpha(\beta)$ are also roots:

$$\beta - q\alpha \in \Delta, \quad q \in \{0, \ldots, (\beta|\alpha^\vee)\}. \tag{2.65}$$

This fact yields an iterative procedure to construct the whole root system from just the simple roots. First, consider the orbit of the simple roots. Next, consider the orbits of the 'gaps' of the first orbit, and so on and so forth. This procedure truncates at some point for the finite Lie algebras, but it does not for infinite algebras. In the latter case the best one can do is to calculate the root system up to some given height.

The size of the Weyl group can become rather large. For instance, the Weyl group of A_n has $(n+1)!$ elements, which makes for difficult bookkeeping of every single reflection when n increases. Luckily, any element $w \in W$ can be written as a successive combination of *fundamental reflections* w_i,

$$w = w_{i_1} w_{i_2} \cdots w_{i_k}. \tag{2.66}$$

The *length* $l(w)$ of a Weyl reflection is the minimal number of fundamental reflections needed to write w in the above form. The fundamental reflections are Weyl reflections in the simple roots,

$$w_i \equiv w_{\alpha_i}. \tag{2.67}$$

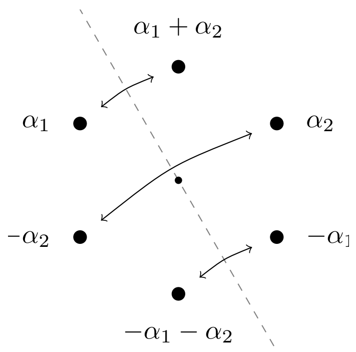


Figure 2.2: The action of $w_2 = w_{\alpha_2}$ on the root system of A_2. The dashed line indicates the hyperplane through the origin perpendicular to α_2. The solid lines with arrows depict the action of w_2.

The whole Weyl group W is thus generated by just the fundamental reflections, of which there are always the same number as n, the rank of the algebra. When we let the fundamental reflections act on the simple roots, equation (2.62) becomes

$$w_i(\alpha_j) = \alpha_j - A_{ji}\alpha_i. \tag{2.68}$$

This means that for a generic root with root vector m^j, the fundamental Weyl reflections act as

$$m^j \xrightarrow{w_i} m^j - p_i\delta_i^j, \tag{2.69}$$

where p_i are the Dynkin labels of the root. Thus a Weyl reflection w_i increases the height of a root if its corresponding Dynkin label p_i is negative, decreases its height if it is positive, and leaves it invariant if it is zero.

We can therefore introduce the set P_-, which contains all roots and weights that have non-positive Dynkin labels:

$$P_- = \left\{ \lambda = \sum_{i=1}^{n} p_i\Lambda^i \;\middle|\; p_i \leq 0 \right\}. \tag{2.70}$$

The subset $P_- \subset P$ of the weight lattice is called the *fundamental Weyl chamber*. By equation (2.69), fundamental Weyl reflections on elements of P_- never decrease their height. Furthermore, the Weyl group acts transitively on P_-. This means that by acting with the Weyl group on all the roots in the fundamental chamber, one obtains the full root system. P_- is also a fundamental domain of W; thus on the Weyl orbit of a weight, exactly one point lies in P_-.

The images of the fundamental chamber under the Weyl group are also called Weyl chambers, albeit not fundamental. Their boundaries are hyperplanes perpen-

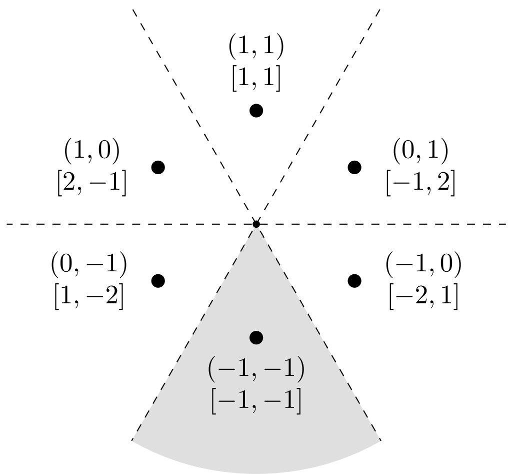


Figure 2.3: The Weyl chambers of the root system of A_2. The fundamental chamber is shaded in gray. The numbers (m^1, m^2) form the root vector, and $[p_1, p_2]$ are the Dynkin labels. The dashed lines are the hyperplanes perpendicular to roots.

dicular to roots through the origin. Figure 2.3 shows the Weyl chambers of the Lie algebra A_2.

As already might be apparent from the discussion above, the action of the Weyl group can be extended by linearity from roots to weights. The Weyl reflections then act on weights in a similar manner as on roots,

$$w_\alpha(\lambda) = \lambda - (\lambda|\alpha^\vee)\,\alpha. \tag{2.71}$$

Here λ is a generic weight. The action of the Weyl group on weights will be used in the analysis of representations.

2.1.7 A bound on root norms

In subsection 2.1.6 it was argued that the whole root system Δ can be constructed from the Weyl orbit of the simple roots, the orbits of the 'gaps' in the orbits of the simple roots, and so on and so forth. This, together with that fact that all Weyl reflections can be written as combinations of fundamental reflections, can be used to derive a bound on the norm of all roots in the root system.

First note that the inner product is associative with respect to Weyl reflections, i.e. $(w(\alpha)|\beta) = (\alpha|w(\beta))$. From this it follows that Weyl reflections preserve the norm of roots:

$$(w(\beta)|w\beta)) = (\beta|\beta)\,. \tag{2.72}$$

So we only need to worry about roots that lie on gaps of fundamental reflections. Consider a root γ that lies in a gap between the root β and its i^{th} fundamental reflection:

$$\gamma = \beta - q\alpha_i, \quad 0 < |q| < |p_i|, \tag{2.73}$$

where $p_i = (\beta|\alpha_i^\vee)$ is the i^{th} Dynkin label of β, and q has the same sign as p_i. Calculating the norm of γ, we find

$$\frac{(\gamma|\gamma) - (\beta|\beta)}{(\alpha_i|\alpha_i)} = q^2 - qp_i < 0. \tag{2.74}$$

As the norm of simple roots is always positive, $(\alpha_i|\alpha_i) > 0$, the norm of γ must be smaller than that of β. Therefore all roots $\alpha \in \Delta$ satisfy

$$(\alpha|\alpha) \leq \alpha_{\max}^2 \tag{2.75}$$

where $\alpha_{\max}^2 = \max\big((\alpha_i|\alpha_i)\big)$ is the norm of the longest simple root.

2.2 Representations

So far Lie algebras have been described as objects on their own, which describe some form of symmetry. But the usefulness of describing this symmetry lies often in letting it act on objects that are symmetric. The mathematical concept of acting with a Lie algebra on another object is called a *representation* of the Lie algebra.

The object in question will be a vector space V. The fact that it carries symmetry is captured in the set $\mathfrak{gl}(V)$, which contains all linear transformations that send V to itself:

$$\mathfrak{gl}(V) : V \mapsto V. \tag{2.76}$$

Then the precise definition of ‘acting on’ is that there exists a map ψ from the Lie algebra $\mathfrak{g}$ to $\mathfrak{gl}(V)$,

$$\psi : \mathfrak{g} \mapsto \mathfrak{gl}(V), \tag{2.77}$$

which preserves the Lie bracket structure on V,

$$\psi_x\psi_y - \psi_y\psi_x = \psi_{[x,y]}, \tag{2.78}$$

for all $x, y \in \mathfrak{g}$. Strictly speaking, the map ψ is called the representation, and the space V the representation space or $\mathfrak{g}$-module. But as is common in the literature, the term ‘representation’ will refer to both in this thesis.

Concretely, a representation is a set of vectors $\{v_\lambda, v_\mu, \ldots\}$ which by the action of ψ get mapped onto each other:

$$\psi_x v_\lambda = v_\mu. \tag{2.79}$$

The module V is the span of these vectors. Now, because the Cartan subalgebra $\mathfrak{h}$ is abelian, it is possible to find a particular basis for V on which the representations of $\mathfrak{h}$ act diagonally:

$$\psi_h v_\lambda = \lambda(h) v_\lambda. \tag{2.80}$$

Here $\lambda(h)$ is nothing more than a number, and is called the weight of the vector v_λ. For the elements in the basis of $\mathfrak{h}$, its specific value is given by

$$\lambda(h_i) = (\lambda|\alpha_i^\vee) = p_i. \tag{2.81}$$

Here p_i are the Dynkin labels of the weight λ (compare equation (2.47)). Similar to how the Lie algebra decomposes into a direct sum of root spaces (see equation (2.36)), the module V splits up in a direct sum of weight spaces,

$$V = \bigoplus_{\lambda \in P(V)} V_\lambda. \tag{2.82}$$

The weight spaces V_λ are the sets of vectors in V that have λ as a weight under the action of $\mathfrak{h}$. The object $P(V)$ is the collection of all weights of V, and is called the *weight diagram*. It must not be confused with the weight lattice P (2.48). The question of which weights are part of the weight diagram will be addressed in the next section.

Lastly, the *multiplicity* of a weight is the dimension of its weight space, that is, $\mathrm{mult}_V(\lambda) = \dim(V_\lambda)$. It follows straightforwardly that the dimension of V is given by the sum of weight multiplicities,

$$\dim(V) = \sum_{\lambda \in P(V)} \mathrm{mult}_V(\lambda). \tag{2.83}$$

In section 2.3 we will see how to calculate weight multiplicities.

Example 2.5: Two common representations

One particularly simple representation is the one where all elements $x \in \mathfrak{g}$ get mapped onto zero:

$$\psi_x = 0. \tag{2.84}$$

The underlying module is one-dimensional. This representation is the trivial or *singlet* representation.

Another example is the representation where the module is taken to be $\mathfrak{g}$ itself, and the map ψ is the adjoint action:

$$\psi_x = \mathrm{ad}_x\,. \tag{2.85}$$

This representation is called the *adjoint* representation.

2.2.1 Integrable lowest weight representations

All irreducible finite-dimensional modules of finite Lie algebras fall into the class of so-called *highest weight representations*. Here instead I will discuss *lowest* weight representations, which are identical in structure. The main difference from highest weight representations is that they are not classified by their highest weight, but *lowest weight vector* v_Λ that satisfies

$$\psi_h v_\Lambda = \Lambda(h) v_\Lambda, \tag{2.86a}$$
$$\psi_{f_i} v_\Lambda = 0, \tag{2.86b}$$

for all $h \in \mathfrak{h}$ and all $i = 1, \dots n$. The lowest weight vector is thus annihilated by the action of the negative Chevalley generators. It follows that any element of $\mathfrak{n}_-$ annihilates it, $\psi(\mathfrak{n}_-)v_\Lambda = 0$. Because the representation map preserves the Lie bracket, the weight increases with a simple root when we act on it with a positive Chevalley generator:

$$\psi_h(\psi_{e_i} v_\Lambda) \propto \big(\Lambda(h) + \alpha_i(h)\big) v_\Lambda \tag{2.87}$$

Hence Λ is indeed the *lowest weight* of the module. It uniquely characterizes the module, which is therefore denoted by $V(\Lambda)$. The whole module is generated by the action of $\mathfrak{n}_+$ on the lowest weight vector,

$$\psi_{\mathfrak{n}_+} v_\Lambda = V(\Lambda). \tag{2.88}$$

When the underlying Lie algebra is finite, the module V itself is finite. This implies that besides a lowest weight it has also highest weight vector $v_{\Lambda'}$ which is annihilated by the positive Chevalley generators,

$$\psi_{e_i} v_{\Lambda'} = 0. \tag{2.89}$$

Thus for finite Lie algebras every lowest weight representation is simultaneously a highest weight representation. This is not the case for infinite Lie algebras, as their lowest weight representations are not bounded from above.

A representation $V(\Lambda)$ is *integrable* when the action of all positive and negative Chevalley generators is locally nilpotent. That is, the repeated action of the same step operator must yield zero at some point:

$$\big(\psi_{e_i}\big)^p v_\lambda = 0, \tag{2.90a}$$
$$\big(\psi_{f_i}\big)^q v_\lambda = 0, \tag{2.90b}$$

for all $v_\lambda \in V(\Lambda)$ and for some finite positive integers p and q. A necessary and sufficient condition for this is that the weight diagram of the representation lies on the weight lattice P,

$$P(V(\Lambda)) \subset P. \tag{2.91}$$

By equation (2.87) this condition simplifies to the requirement that the lowest weight Λ lies on the weight lattice, because the roots do too. In the rest of this thesis the prefix 'integrable' will often be dropped, because all lowest weight representations discussed here will be integrable.

Let us now turn to the question of determining the weight diagram $P(V(\Lambda))$ of a module $V(\Lambda)$. It can be proven that $P(V(\Lambda))$ is invariant under the action of the Weyl group, which provides us with sufficient tools to determine the actual structure of $P(V(\Lambda))$.

First, it is convenient to define the *height* of a weight λ by the amount of simple roots it differs from the lowest weight Λ:

$$\mathrm{ht}_\Lambda(\lambda) = (\lambda - \Lambda|\rho^\vee)\,. \tag{2.92}$$

By this definition, the lowest weight has height zero. But in order for it to be truly a lowest weight, it has to lie in the fundamental Weyl chamber: then all Weyl reflections increase its height. This is most easily seen when we look at the action of a fundamental Weyl reflection on a generic weight:

$$w_i(\lambda) = \lambda - (\lambda|\alpha_i^\vee)\,\alpha_i = \lambda - p_i\alpha_i, \tag{2.93}$$

where p_i are the Dynkin labels of λ. So indeed, when the Dynkin labels are all non-positive, the height of the weight can only increase under Weyl reflections. Weights that lie in the fundamental Weyl chamber are called *dominant* weights.

Having established that Λ is dominant, the full weight diagram can now be constructed from Λ by considering its orbit under the Weyl group. Weights that lie on this orbit belong to $P(V(\Lambda))$,

$$W(\Lambda) \subseteq P(V(\Lambda)). \tag{2.94}$$

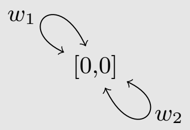
Moreover, weights that lie 'in between' reflections also belong to the weight diagram, although not to the same orbit. Thus for a weight $\lambda \in P(V(\Lambda))$, the points on the line between λ and $w_i(\lambda)$ that differ by a single simple root α_i are part of the weight diagram:

$$\lambda - q\alpha_i \in P(V(\Lambda)), \quad q \in \{0, \dots, (\lambda|\alpha_i^\vee)\}. \tag{2.95}$$

This is enough information to construct $P(V(\Lambda))$. First, consider the orbit of the lowest weight. Next, consider the orbits of the 'gaps' in the first orbit, and so on and so forth until the weight diagram closes. For infinite Lie algebras, one cannot calculate the full weight diagram. There one has to be content to calculate it up to a given height.

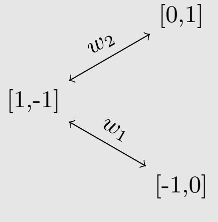
Example 2.6: Weight diagrams of A_2

Let's have a look at some weight diagram of representations of A_2. The smallest diagram is that of the trivial or *singlet* representation. This is the representation with zero lowest weight (i.e. all Dynkin labels are zero). This is also its only weight. The weight diagram is thus as follows:



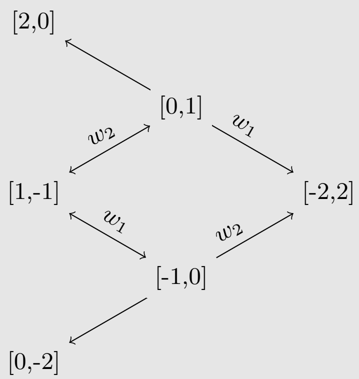
The lines with arrows depict the fundamental Weyl reflections, and the numbers $[p_1, p_2]$ are the Dynkin labels of a weight. w_1 and w_2 map the $[0, 0]$ weight onto itself, because all its Dynkin labels are zero. In the following, Weyl reflections that leave weights invariant will not be drawn.

The next simplest weight diagram is that of the $\Lambda = -1\Lambda^1 + 0\Lambda^2$ lowest weight representation. The Weyl orbit of the lowest weight consists of three weights in total:



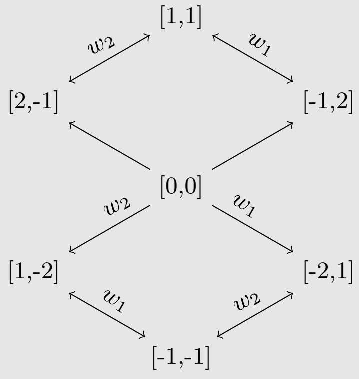
There are no gaps in the reflections, which means the weight diagram consists of one single orbit. Note that reflection w_i flips the sign of the ith Dynkin label.

A more involved weight diagram is that of the $\Lambda = 0\Lambda^1 - 2\Lambda^2$ lowest weight representation:



Here the w_2 reflection brings the lowest weight $[0, -2]$ to $[-2, 2]$. This is indicated by the missing arrows on the $[-1, 0]$ weight; the w_2 reflection 'passes through' it. By equation (2.95), $[-1, 0]$ is then also a weight of the diagram, although it doesn't lie on the same orbit. Hence this diagram consists of two Weyl orbits, one of which is the $[-1, 0]$ weight diagram.

Yet another example is weight diagram of the lowest weight representation associated to the lowest root of A_2, which is $\Lambda = -\theta = -\Lambda^1 - \Lambda^2$.



This weight diagram is identical to the root system of A_2. This not a coincidence of A_2, but is actually valid for all finite Lie algebras if the lowest weight is taken to be the lowest root. The resulting representation is then the adjoint.

2.3 Multiplicities

So far we have studied the question which points on the root and weight lattice are elements of the root system and weight diagram, respectively. What remains to be done is to calculate their multiplicity, that is, the degeneracy of the root and weight spaces.

A convenient way to store the final answer for lowest weight representations is its *character* $\mathcal{X}$,

$$\mathcal{X}_V = \sum_{\lambda \in P(V)} \mathrm{mult}_V(\lambda)\, e^{\lambda}. \tag{2.96}$$

This is a sum over the formal exponents of all the weights in the weight diagram. It can be shown [51] that the character satisfies the *Weyl-Kac character formula*

$$\mathcal{X}_{V(\Lambda)} = \frac{\sum_{w \in W} \epsilon(w) e^{w(\Lambda+\rho)-\rho}}{\prod_{\alpha \in \Delta_+} \left(1 - e^{-\alpha}\right)^{\mathrm{mult}\,\alpha}}. \tag{2.97}$$

Here the function ϵ is given by $\epsilon(w) = (-1)^{l(w)}$, with l being the length of the Weyl reflection. Not only do weight multiplicities follow from the Weyl-Kac character formula, it can also be used to calculate root multiplicities. One then has to evaluate it for the singlet representation, $\Lambda = 0$, whose character is equal to one. The result is

$$\prod_{\alpha \in \Delta_+} \left(1 - e^{-\alpha}\right)^{\mathrm{mult}\,\alpha} = \sum_{w \in W} \epsilon(w) e^{w(\rho)-\rho}. \tag{2.98}$$

This is known as the *denominator identity.*

Although the denominator identity can in principle be used to calculate root multiplicites, a shorter and faster way is to use the *Peterson recursion formula* [71], which is derived from it. It reads

$$(\alpha|\alpha - 2\rho)\, c_\alpha = \sum_{\substack{\beta,\gamma \in Q_+ \\ \beta+\gamma=\alpha}} (\beta|\gamma)\, c_\beta c_\gamma. \tag{2.99}$$

The coefficient c_α is given by

$$c_\alpha = \sum_{k \geq 1} \frac{1}{k}\, \mathrm{mult}\left(\frac{\alpha}{k}\right), \tag{2.100}$$

and will be called the *co-multiplicity* of the root α. The factors in the co-multiplicity for which α/k is not a root do not contribute to the sum, as their multiplicity is zero. As indicated in (2.99), the factors β and γ of α do not have to be roots. But in order for them to contribute to the sum, they do need to be integer multiples of roots. Otherwise their co-multiplicity would vanish.

Using the Peterson recursion formula, it is possible to calculate the multiplicities of all roots. Starting from the simple roots, which have multiplicity one, you can inductively work your way up in the root system. For higher and higher roots the calculation will get more involved, as the number of contributions to the sum will increase.

For the weight multiplicities a similar recursion formula can be deduced, this time from the Weyl-Kac character formula [71]. It reads

$$\operatorname{mult}_\Lambda(\lambda) = \frac{2\sum_{\alpha\in\Delta_+} \operatorname{mult}(\alpha) \sum_{k\geq 1} (-\alpha|\lambda - k\alpha) \operatorname{mult}_\Lambda(\lambda - k\alpha)}{(\Lambda - \rho|\Lambda - \rho) - (\lambda - \rho|\lambda - \rho)} . \tag{2.101}$$

This is a generalization of the Freudenthal recursion formula for finite Lie algebras, in which case the root multiplicities are all equal to one. Starting from the lowest weight Λ, which has multiplicity one, all other weight multiplicities can be calculated by induction on height.

Luckily it is not necessary to calculate the multiplicities for all roots and weights. It can namely be shown that Weyl reflections preserve multiplicities:

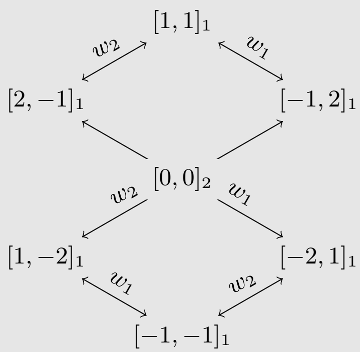
$$\operatorname{mult}(w(\alpha)) = \operatorname{mult}(\alpha), \tag{2.102a}$$

$$\operatorname{mult}_\Lambda(w(\lambda)) = \operatorname{mult}_\Lambda(\lambda). \tag{2.102b}$$

It therefore suffices to calculate the multiplicity once for every Weyl orbit.

Example 2.7: Weight multiplicities of the adjoint A_2 representation

Recall that the weight diagram of the adjoint representation of A_2 was as follows (see Example 2.6):



The subscripts on the Dynkin labels are the multiplicities of the weight. By definition the multiplicity of the lowest weight $[-1,-1]$ is one. As Weyl reflections preserve the multiplicity, all other weights on its orbit also have multiplicity one.

There is just one weight that does not lie on the orbit of $[-1,-1]$, namely $[0,0]$. Since this weight corresponds to the Cartan subalgebra of A_2, its multiplicity is bound to be equal to two. But let us invoke the Freudenthal recursion formula (2.101) to prove it. First we need to determine for which roots and k the weight $\lambda - k\alpha$ for $\lambda = 0$ still lies in the weight diagram. They are

$$0 - \alpha_2 = \Lambda^1 - 2\Lambda^2, \tag{2.103a}$$

$$0 - \alpha_1 = -2\Lambda^1 + \Lambda^2, \tag{2.103b}$$

$$0 - \alpha_1 - \alpha_2 = -\Lambda^1 - \Lambda^2. \tag{2.103c}$$

The integer k is one in all cases, as otherwise the resulting weight would lie outside the diagram. The double sum in (2.101) simplifies to one that runs over the roots α_1, α_2, and $\alpha_1 + \alpha_2$. The multiplicity of $\lambda = 0$ can then be evaluated to give

$$\begin{aligned} \mathrm{mult}_{-\Lambda^1-\Lambda^2}(0) &= 2\frac{(\alpha_1|\alpha_1) + (\alpha_2|\alpha_2) + (\alpha_1+\alpha_2|\alpha_1+\alpha_2)}{(\Lambda^1+\Lambda^2|\Lambda^1+\Lambda^2) + 2\,(\Lambda^1+\Lambda^2|\rho)} \\ &= \frac{2(2+2+2)}{2+4} = 2. \end{aligned} \tag{2.104}$$

2.4 Real forms

Up to this point the field F over which $\mathfrak{g}$ is a vector space has not been specified. In the classification of Lie algebras one assumes that F is algebraically closed, and one usually takes the complex numbers, $F = \mathbb{C}$. In that case, the Cartan matrix A uniquely characterizes the Lie algebra $\mathfrak{g}(A)$. However, the Lie algebras that pop up in the context of supergravities (see chapter 5) are vector spaces over the real numbers, $\mathbb{R}$. As $\mathbb{R}$ is not algebraically closed, there can be multiple non-isomorphic Lie algebras associated to one single Cartan matrix. These different real Lie algebras are called the various *real forms* of the complex algebra.

A real form can be characterized by the signature of its Cartan-Killing form. Note that this wouldn't make sense for an algebra over the complex numbers, as rescaling generators with a factor of i would effectively change the sign of their norm. The generators of real forms can be classified by the sign of their norm: *compact* generators have negative norm, and *non-compact* generators have positive norm. Real forms of Lie algebras are often denoted by $X_{n(m)}$, where X is the type of algebra (A, B, ...), n is its rank, and m the difference between the number of non-compact and compact generators.

It can be shown that the Cartan-Killing form pairs the root spaces of a root and its negative in a non-degenerate manner:

$$\langle \mathfrak{g}_\alpha | \mathfrak{g}_\beta \rangle = 0 \;\text{ if }\; \alpha + \beta \neq 0. \tag{2.105}$$

The Cartan-Killing form on the full algebra in the triangular decomposition (2.20) can therefore be written as

$$\langle \cdot | \cdot \rangle = \begin{pmatrix} DBD & 0 & 0 \\ 0 & 0 & C \\ 0 & C & 0 \end{pmatrix} \begin{matrix} \mathfrak{h} \\ \mathfrak{n}_+ \\ \mathfrak{n}_- \end{matrix} . \tag{2.106}$$

Here DBD is the Cartan-Killing form on $\mathfrak{h}$ (see equation (2.22)), and C is a positive diagonal matrix:

$$C = \operatorname{diag}\left(\langle y | y^T \rangle, \dots, \langle z | z^T \rangle \right) . \tag{2.107}$$

The elements $\{y, \dots, z\}$ are a basis of $\mathfrak{n}_+$, and their transpose $\{y^T, \dots, z^T\}$ a basis of $\mathfrak{n}_-$. The (generalized) transpose is defined by means of the Chevalley involution (2.21),

$$x^T = -\omega(x). \tag{2.108}$$

As the Cartan-Killing form on $\mathfrak{h}$ has the same signature as the Cartan matrix A, we only need to calculate the signature on the subspace $\mathfrak{n}_+ \oplus \mathfrak{n}_-$. We can diagonalize that part of the Cartan-Killing form by taking the combinations $x + x^T$ and $x - x^T$, which results in

$$\langle \cdot | \cdot \rangle = \begin{pmatrix} DBD & 0 & 0 \\ 0 & 2C & 0 \\ 0 & 0 & -2C \end{pmatrix} \begin{matrix} \mathfrak{h} \\ \mathfrak{p} \ominus \mathfrak{h} \\ \mathfrak{l} \end{matrix} . \tag{2.109}$$

The subspaces $\mathfrak{p}$ and $\mathfrak{l}$ are respectively the odd and even eigenspaces of $\mathfrak{g}$ under the Chevalley involution:

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{l}, \tag{2.110}$$

where

$$\mathfrak{p} = \{x \in \mathfrak{g} \,|\, \omega(x) = -x\}, \tag{2.111a}$$
$$\mathfrak{l} = \{x \in \mathfrak{g} \,|\, \omega(x) = x\}. \tag{2.111b}$$

The space $\mathfrak{p}$ is spanned by elements of the form $x + x^T$, and $\mathfrak{l}$ by elements of the form $x - x^T$.

From (2.109) we can easily read of the signature of the Cartan-Killing form. For finite Lie algebras the Cartan matrix is always positive definite. Thus if we simply restrict the field F to be $\mathbb{R}$, then the signature is

$$(\#\text{non-compact}, \#\text{compact}) = \left(\frac{\dim \mathfrak{g} + n}{2}, \frac{\dim \mathfrak{g} - n}{2}\right), \tag{2.112}$$

where n is the rank of the finite algebra. This particular real form is called the *split real form* or the *maximal non-compact* real form. The difference between the number of non-compact and compact generators is always equal to the rank of the algebra. The split real form of an algebra X_n can thus be denoted by $X_{n(+n)}$.

The split real form is not the only real form of a Lie algebra. This can be seen by inspecting the decomposition (2.110). The Lie brackets of the subspaces read

$$[\mathfrak{l}, \mathfrak{l}] \subseteq \mathfrak{l}, \tag{2.113a}$$
$$[\mathfrak{l}, \mathfrak{p}] \subseteq \mathfrak{p}, \tag{2.113b}$$
$$[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{l}. \tag{2.113c}$$

Thus $\mathfrak{l}$ is not just a subspace of $\mathfrak{g}$, but also a subalgebra. From (2.109) it is clear that the Cartan-Killing form on $\mathfrak{l}$ is negative definite. Because all the generators of $\mathfrak{l}$ are compact, it is called the *maximal compact subalgebra* of $\mathfrak{g}$. Real forms in which all the generators are compact, are not surprisingly called *compact* real forms.

The split and compact real forms are the two 'extreme' real forms of a Lie algebra. The former has the maximal number of non-compact generators, while the latter has the maximal number of compact generators (namely all). In between these two options there usually lies an array of other possibilities. However, in this thesis we will only encounter the split real form and the compact real form.

Example 2.8: Split real form of A_2

Recall from Example 2.2 that the 8 generators of A_2 are

$$h_1, h_2 \in \mathfrak{h}, \tag{2.114a}$$
$$e_1, e_2, e_{1+2} \in \mathfrak{n}_+, \tag{2.114b}$$
$$f_1, f_2, f_{2+1} \in \mathfrak{n}_-, \tag{2.114c}$$

where $e_{1+2} = [e_1, e_2]$ and $f_{2+1} = [f_2, f_1]$. To work out the Cartan-Killing form, we can either explicitly calculate the adjoint action of all generators and take traces (equation (2.5)), or extend it by invariance from $\mathfrak{h}$ to the whole of $\mathfrak{g}$ (equation (2.22)). In either case, we get

$$\langle\cdot|\cdot\rangle = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} h_1 \\ h_2 \\ e_1 \\ e_2 \\ e_{1+2} \\ f_1 \\ f_2 \\ f_{2+1}. \end{matrix} \tag{2.115}$$

The generalized transpose of the generators are

$$\begin{aligned} h_1^T &= h_1, & e_1^T &= f_1, & e_{1+2}^T &= -[\omega(e_1), \omega(e_2)] \\ h_2^T &= h_2, & e_2^T &= f_2, & &= +f_{2+1}. \end{aligned} \tag{2.116a}$$

Going to the basis (2.111) of the $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{l}$ decomposition, the Cartan-Killing form becomes

$$\langle\cdot|\cdot\rangle = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix} \begin{matrix} h_1 \\ h_2 \\ e_1 + f_1 \\ e_2 + f_2 \\ e_{1+2} + f_{2+1} \\ e_1 - f_1 \\ e_2 - f_2 \\ e_{1+2} - f_{2+1}. \end{matrix} \tag{2.117}$$

The split real form $A_{2(+2)}$ of A_2 thus has 5 non-compact and 3 compact generators. The subspace $\mathfrak{l}$ is indeed a subalgebra, because its commutators close:

$$[e_1 - f_1, e_2 - f_2] = e_{1+2} - f_{2+1}, \tag{2.118a}$$
$$[e_{1+2} - f_{2+1}, e_1 - f_1] = e_2 - f_2, \tag{2.118b}$$
$$[e_{1+2} - f_{2+1}, e_2 - f_2] = e_1 - f_1. \tag{2.118c}$$

2.5 Cosets and non-linear sigma models

Consider a real Lie group G and its maximal compact subgroup $K(G) \in G$. The (left) *coset* is then defined as the set of equivalence classes of elements $g \in G$ [30]

$$g \sim g' \quad \text{iff} \quad g' = gk, \tag{2.119}$$

with $k \in K(G)$. That is, elements g and g' of G are identified if they are related by an element of the subgroup $K(G)$. Thus the coset is a quotient space, and is denoted by $G/K(G)$. Its dimension is given by

$$\dim G/K(G) = \dim G - \dim K(G). \tag{2.120}$$

We can build a dynamical theory on the coset space as follows. Let ϕ^α be local coordinates on the coset, and $G_{\alpha\beta}(\phi)$ the metric. In addition we have a 'space-time' manifold M with coordinates x^μ and metric $\gamma_{\mu\nu}$. If we let $\phi^\alpha = \phi^\alpha(x)$ be functions of x^μ, we can view them as maps from the space-time M to the target space $G/K(G)$. See also Figure 2.4. The dynamics of these mappings are governed by an action of the form

$$S = -\int_M dx \sqrt{\gamma}\gamma^{\mu\nu} G_{\alpha\beta}(\phi)\partial_\mu\phi^\alpha\partial_\nu\phi^\beta. \tag{2.121}$$

For historical reasons, this particular dynamical realization is called a *non-linear sigma model*. In the analysis below we will consider the case when M is one-dimensional and parameterized by the coordinate t. The action then simplifies to

$$S = -\int dt\, n(t)^{-1} G_{\alpha\beta}(\phi)\partial\phi^\alpha\partial\phi^\beta, \tag{2.122}$$

where $\partial \equiv \partial_t$. The function $n(t) = \sqrt{\gamma}$ ensures reparameterization invariance in the coordinate t.

The action (2.122) can be constructed explicitly by introducing a t dependent group element $V(t) \in G$ that transforms as

$$V(t) \longrightarrow V'(t) = gV(t)k(t), \tag{2.123}$$

where $g \in G$ and $k(t) \in K(G)$. Dropping the explicit coordinate dependence, the Maurer-Cartan form of the group element V reads

$$J = V^{-1}\partial V \in \mathfrak{g}. \tag{2.124}$$

The Maurer-Cartan form is Lie algebra-valued. If $\mathfrak{g}$ is the algebra of G, then the algebra of $K(G)$ is $\mathfrak{l}$, the maximal compact subalgebra of $\mathfrak{g}$. Hence J can be decomposed in a part that belongs to $\mathfrak{l}$, and a part that belongs to its complement, $\mathfrak{p}$ (see

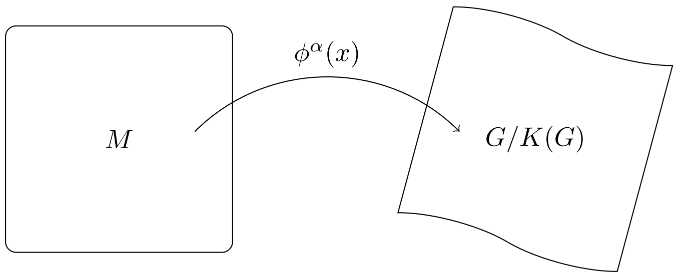


Figure 2.4: The coordinates ϕ^α of the coset as maps from the space-time M to the coset.

equation (2.110)):

$$P = \tfrac{1}{2}\left(J + J^T\right) \in \mathfrak{p}, \tag{2.125a}$$
$$Q = \tfrac{1}{2}\left(J - J^T\right) \in \mathfrak{l}. \tag{2.125b}$$

Here the transpose $()^T$ is the generalized transpose (2.108). The subspace $\mathfrak{p}$ can be interpreted as the 'coset-part' of the algebra $\mathfrak{g}$. The action (2.122) can then be written as

$$S = -\int dt\, n(t)^{-1)} \langle P|P\rangle, \tag{2.126}$$

where $\langle\cdot|\cdot\rangle$ is the usual Cartan-Killing form on $\mathfrak{g}$. This action has a global (rigid) G invariance and a local $K(G)$ gauge invariance. The gauge invariance allows us to parameterize $V(t)$ as

$$V(t) = e^{\phi^\alpha(t) t_\alpha}, \tag{2.127}$$

where the generators t_α only take values in the so-called *Borel gauge* (compare (2.20)),

$$t_\alpha \in \mathfrak{h} \oplus \mathfrak{n}_+. \tag{2.128}$$

The Borel gauge is particularly convenient if we want to calculate the Maurer-Cartan form (2.124) explicitly. This can be done with the help of some Baker-Campbell-Hausdorff formulas,

$$e^{-A}\partial e^A = \partial A + \frac{1}{2}[\partial A, A] + \frac{1}{3!}[[\partial A, A], A] + \cdots, \tag{2.129a}$$
$$e^{-A} B e^A = B + [B, A] + \frac{1}{2}[B, A] + \cdots. \tag{2.129b}$$

To illustrate the above analysis, we will conclude with two simple examples.

Example 2.9: Non-linear realization of A_1

The three generators of A_1 were h, e, and f. Thanks to the Borel gauge, we only need two of them to write down our group element:

$$V(t) = e^{\psi(t)e} e^{\phi(t)h}. \tag{2.130}$$

This group element is related via a coordinate transformation to a group element of the form $e^{\psi' e + \phi' h}$. We will use the former because in that case the Maurer-Cartan form is easier to calculate. It reads

$$J = V^{-1}\partial V = h\partial\phi + e\exp\left(-2\phi\right)\partial\psi. \tag{2.131}$$

The coset element P becomes

$$P = \tfrac{1}{2}(J + J^T) = h\partial\phi + \tfrac{1}{2}(e + f)\exp\left(-2\phi\right)\partial\psi. \tag{2.132}$$

Finally, the action is

$$S = \int dt\, n(t)^{-1}\left(-2\partial\phi\partial\phi - \tfrac{1}{2}e^{-4\phi}\partial\psi\partial\psi\right). \tag{2.133}$$

In the next example we will treat the non-linear realization of A_2. In that case we could in principle use the same approach as in the previous example. However, for future use it will be more convenient to tackle this problem in a slightly different way.

Example 2.10: Non-linear realization of A_2

In the adjoint representation the generators of A_2 can be written as traceless 3×3 matrices. Specifically,

$$h_1 = K^1{}_1 - K^2{}_2, \qquad e_1 = K^1{}_2, \qquad f_1 = K^2{}_1, \qquad [e_1, e_2] = K^1{}_3, \tag{2.134a}$$

$$h_2 = K^2{}_2 - K^3{}_3, \qquad e_2 = K^2{}_3, \qquad f_2 = K^3{}_2, \qquad [f_2, f_1] = K^3{}_1, \tag{2.134b}$$

where the 3×3 matrices $K^a{}_b$ are given by

$$(K^a{}_b)^i{}_j = \delta^{ai}\delta_{bj} - \tfrac{1}{3}\delta^a_b\delta^i_j. \tag{2.135}$$

The indices a, b label the different matrices, and the i, j indices indicate the rows and columns of the matrices. Both sets run from one to three. The Lie bracket then reads

$$[K^a{}_b, K^c{}_d] = \delta^c_b K^a{}_d - \delta^a_d K^c{}_b, \tag{2.136}$$

and the Cartan-Killing form becomes

$$\langle K^a{}_b | K^c{}_d \rangle = \mathrm{Tr}\,(K^a{}_b \cdot K^c{}_d) = \delta^a_d \delta^c_b - \tfrac{1}{3}\delta^a_b \delta^c_d. \tag{2.137}$$

The group element V can then be written as

$$V(t) = \exp\left(h_a{}^b(t) K^a{}_b\right) = e_a{}^b(t) K^a{}_b, \tag{2.138}$$

where $h_a{}^b$ is a generic traceless matrix, and $e_a{}^b = \exp\left(h_a{}^b\right)$ is its standard matrix exponential. Hence, the latter has unit determinant. Because V transforms under global G transformations from the left, and local $K(G)$ transformations from the right, the upper and lower indices of $e_a{}^b$ transform differently. To indicate this, we write

$$V = e_m{}^a K^m{}_a. \tag{2.139}$$

The index m transforms under global G transformations, and the index a under local $K(G)$ transformations. Hence $e_m{}^a$ behaves as a vielbein on the coset space. With this distinction in place, we proceed to calculate the Maurer-Cartan form and the coset element:

$$J = e_a{}^m \partial e_m{}^b K^a{}_b, \tag{2.140a}$$

$$P = e_a{}^m \partial e_m{}^b S^a{}_b, \tag{2.140b}$$

where $e_a{}^m$ is the inverse vielbein, and $S^a{}_b$ the basis elements of the coset:

$$S^a{}_b = \tfrac{1}{2}\left(K^a{}_b + K^b{}_a\right). \tag{2.141}$$

Note that here the generalized transposed is equal to the ordinary matrix transpose. The inner product of the coset basis is easily evaluated to give

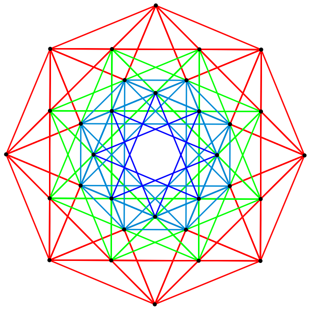
$$\langle S^a{}_b | S^c{}_d \rangle = \delta^a_d \delta^c_b - \tfrac{1}{3}\delta^a_b \delta^c_d. \tag{2.142}$$

If we introduce the metric $g_{mn} = \delta_{ab} e_m{}^a e_n{}^b$, the action reads

$$\begin{aligned} S &= -\tfrac{1}{4}\int dt\, n(t)^{-1}\left(g^{mp}g^{nq} - \tfrac{1}{3}g^{mn}g^{pq}\right)\partial g_{mn}\partial g_{pq} \\ &= -\tfrac{1}{4}\int dt\, n(t)^{-1} g^{mp}g^{nq}\partial g_{mn}\partial g_{pq}. \end{aligned} \tag{2.143}$$

Here g^{mn} is the inverse metric. The last step follows from the fact that, like the vielbein, both the metric and its inverse have unit determinant.

3



Visualizations

From time to time, it is convenient to draw a picture of a Lie algebra or one of its representations. Besides from looking nice, a picture can display their structure at a glance. For instance, both the symmetry of the root system and the height of the roots of A_2 are immediately clear from Figure 3.1, which displays the projection we already encountered in chapter 2. In fact, Figure 3.1 preserves the structure of A_2 exactly. This is possible because the root space of A_2 is two-dimensional. When the rank of the algebra is bigger than two, projections onto two dimensions of the root or weight space loose some of the information. It is then no longer possible to capture both the ordering in height and the full symmetry into one image. What one can do, however, is do one projection that preserves the ordering in height, and another that preserves (some part of) the symmetry. The former can be achieved with a *Hasse diagram*, and the latter with a *Coxeter projection*.

3.1 Hasse diagrams

A *Hasse diagram* is a graph that displays the ordering between the different elements of a set [12, 33], which in our case are the roots of a root system. An example of

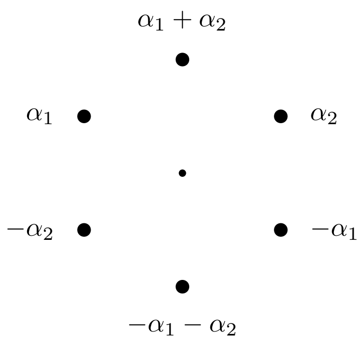


Figure 3.1: The root system of A_2.

a Hasse diagram is given in Figure 3.2. Below I will give the precise definition of a Hasse diagram, and a procedure for drawing them.

The root system Δ can be promoted to an *ordered set* $(\Delta, \geq)$ if we introduce the following partial ordering. A root α is said to be bigger than β if their difference is positive:

$$\alpha \geq \beta \quad \text{if} \quad \alpha - \beta \in Q_+. \tag{3.1}$$

Thus $\alpha - \beta$ has to be a non-negative combination of simple roots. If it is not, the two roots are incomparable. In addition to the partial ordering we need to introduce a so-called *cover relation*. A root α is said to *cover* β if there is no root γ smaller than α and bigger than β:

$$\alpha \succ \beta \quad \text{if} \quad \nexists\, \gamma \;:\; \alpha \geq \gamma \geq \beta. \tag{3.2}$$

For roots this means that one root covers the other only if their difference is one single simple root. With these two relations, a Hasse diagram of Δ can now be drawn according to the following rules:

- If $\alpha \geq \beta$ the vertical coordinate for β is less than that for α.
- If $\alpha \succ \beta$ there is a straight line connecting α and β.

Because $\alpha \geq \beta$ implies $\mathrm{ht}(\alpha) > \mathrm{ht}(\beta)$, the first criterion is satisfied if we assign the vertical coordinate according to the height of the roots. The second criterion is equivalent to drawing straight lines for every fundamental Weyl reflections, as these are used to construct the root system in the first place (see subsection 2.1.6).

What remains to be done is to determine the horizontal coordinate for each root. Although there are various algorithms with varying degree of complexity available (see for example [33]), the following simple recipe works fairly well for root systems.

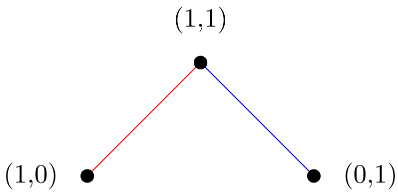


Figure 3.2: Hasse diagram of the positive roots of A_2. The numbers (m^1, m^2) denote the root vector.

The first step is to distribute the simple roots evenly on a horizontal line around the origin. This is achieved by the following horizontal projection $\mathbb{P}_x$ of the simple roots α_i:

$$\mathbb{P}_x(\alpha_i) = \frac{i-1}{n-1} - \frac{1}{2} \equiv x_i, \tag{3.3}$$

where n is the rank of the Lie algebra. The horizontal position of a generic root $\alpha = m^i \alpha_i$ can now be defined as

$$\mathbb{P}_x\left(\alpha\right) = m^i\,\mathbb{P}_x\left(\alpha_i\right) = m^i x_i. \tag{3.4}$$

Note that the explicit summation of the index i has been dropped. From now on, any contracted index will be summed over. We can formalize the above a bit by introducing a *projection vector* φ that satisfies

$$(\alpha_i|\varphi) = x_i. \tag{3.5}$$

Expanded in the basis of simple co-roots, the projection vector φ explicitly reads

$$\varphi = (A^{-1})^{ij} x_j\, \alpha_i^\vee. \tag{3.6}$$

When we take its inner product with a generic root α, we see that it indeed gives us the desired projection (3.4): $(\alpha|\varphi) = m^i x_i$. The complete projection $\mathbb{P} = (\mathbb{P}_x, \mathbb{P}_y)$ can then be written as

$$\mathbb{P}_x(\alpha) = (\alpha|\varphi)\,, \tag{3.7a}$$
$$\mathbb{P}_y(\alpha) = (\alpha|\rho^\vee)\,, \tag{3.7b}$$

where the projection in the vertical coordinate y is just the height of the root.

Note that the horizontal coordinate (3.3) of a simple root α_i strongly depends on its number i. If the order of the simple roots is changed, the Hasse diagram changes shape too. The best looking diagrams are produced when the ordering of nodes in the Dynkin diagram (and thus the ordering of simple roots) matches the connections between the nodes. See also Figure 3.3.

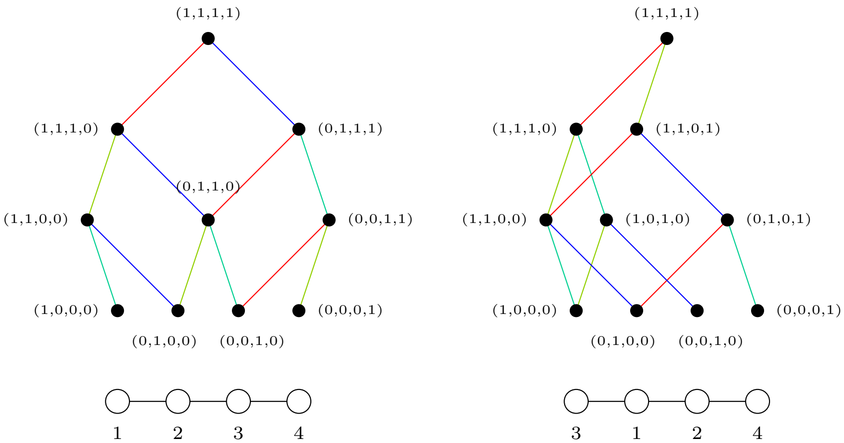


Figure 3.3: Two Dynkin diagrams (below) and Hasse diagrams (above) of the same Lie algebra, A_4. The ordering of nodes in the left Dynkin diagram, indicated with numbers below the nodes, is canonical. The ordering of nodes in the right Dynkin diagram does not match the connections between them, resulting in a Hasse diagram with crossing lines.

The lines drawn in a Hasse diagram represent the Weyl reflections in the simple roots. Say there is a root α projected to the point (x, y). Then the root $\alpha + \alpha_i$ connected to it by the line of the fundamental Weyl reflection w_i gets projected to the point $(x + x_i, y + 1)$. The line of a fundamental reflection is therefore drawn at an angle ϕ given by

$$\phi_{w_i} = \tan^{-1} \frac{1}{x_i}. \tag{3.8}$$

Because x_i is unique for all i, the n distinct fundamental reflections w_i all are drawn at different angles, and reflections in the same simple root are drawn parallel. To distinguish between them even further they will get drawn in different colors, ranging from blue (the first fundamental reflection) to red (the n^{th}).

The Hasse diagram of the full root system is symmetric around the origin, because of the Chevalley involution (2.21). It is therefore customary to draw only the positive roots in a Hasse diagram.

Following the above procedure it is straightforward, though sometimes tedious, to draw a Hasse diagrams of any root system. Figure 3.4 displays for example the Hasse diagrams of various root systems.

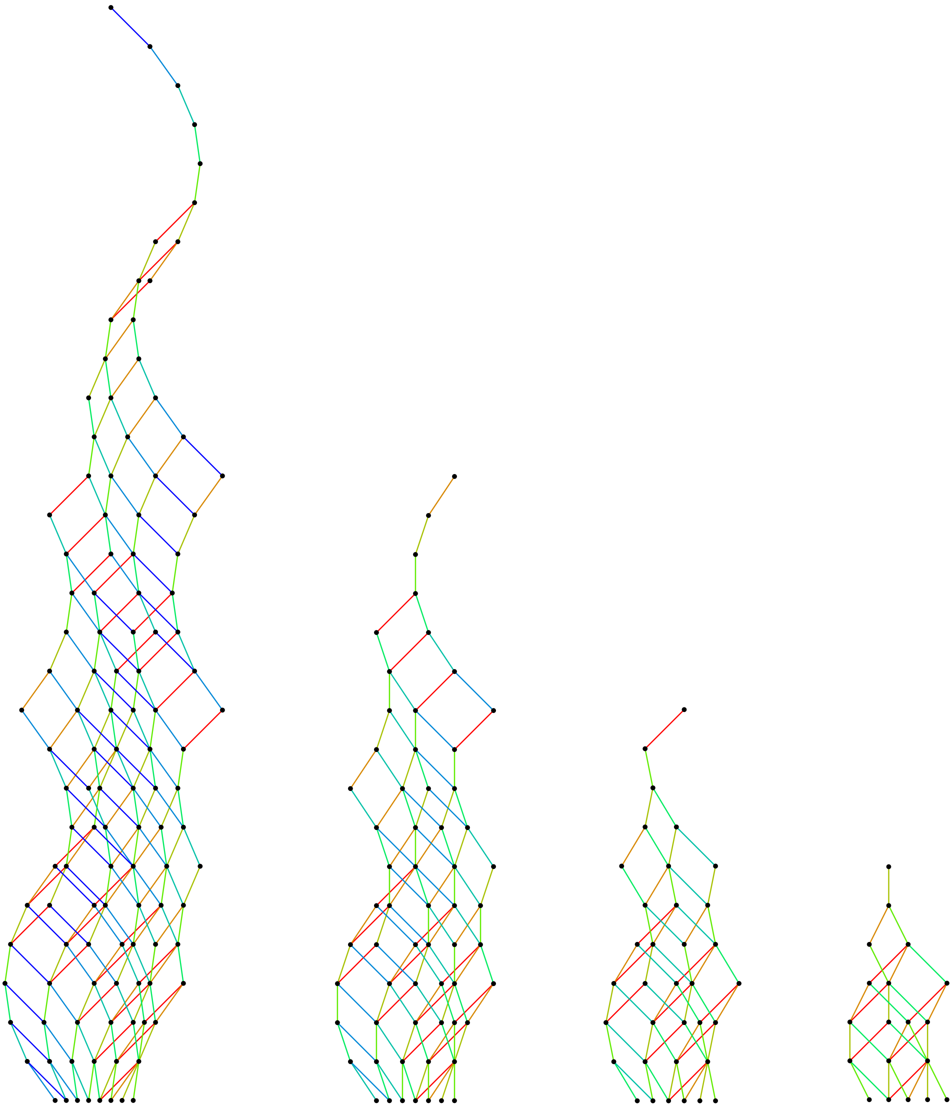


Figure 3.4: Hasse diagram of the positive roots of (a) E_8, (b) E_7, (c) E_6, and (d) D_5. The last three are subdiagrams of the E_8 diagram. The colors of the Weyl reflections are chosen such that they match their embedding within E_8.

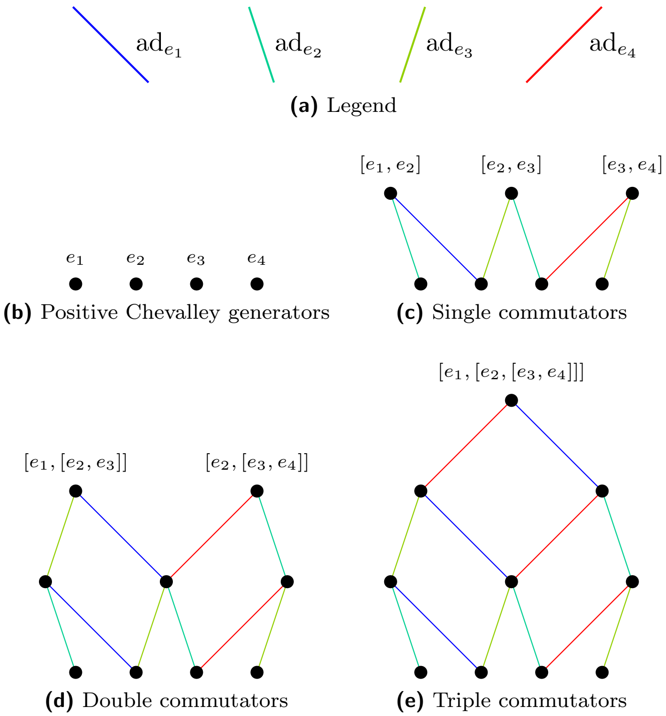


Figure 3.5: The Serre construction for A_4.

3.1.1 Visualizing the Serre construction

Hasse diagrams can serve as a neat tool to visualize the results of the Serre construction, the step-by-step construction of the full algebra from the Cartan matrix (see Example 2.2). One then has to interpret the points in the diagram not as roots, but as the generator they belong to. Furthermore, the lines can then be interpreted as the adjoint action of the respective positive Chevalley generators. Starting at the bottom, the vertical steps in the diagram then represent the steps of the Serre construction.

Figure 3.5 displays the Serre construction for the Lie algebra A_4. One starts out with just the positive Chevalley generators (Figure 3.5b). The first step is to take all (single) commutators $[e_i, e_j]$ of the positive Chevalley generators that are consistent with the Chevalley relations (2.16), the Serre relations (2.17), and the Jacobi identity (2.3), which results in Figure 3.5c. This procedure is then iterated (Figure 3.5d and 3.5e) until it no longer yields new generators.

The analogy presented above is only valid up to a certain point. The Serre construction can give you all the Lie brackets of the algebra, whereas the Hasse diagram does not contain this information. Also, if the multiplicity of a root α is greater than one, Hasse diagrams do not distinguish between the different generators of the root space $\mathfrak{g}_\alpha$.

3.2 Coxeter projections

Where Hasse diagrams try to visualize the ordering of the root system, *Coxeter projections* try to visualize its symmetry. The problem is that the full symmetry of a root system is only revealed in a space of dimension n, which is the rank of the algebra. What one can do for finite-dimensional Lie algebras, however, is project this n-dimensional space onto a carefully chosen 2-dimensional hyperplane such that the projection preserves a part of the full symmetry. The hyperplane in question is known as a *Coxeter plane* [47, 84, 16].

3.2.1 The Coxeter plane

The Coxeter plane can only be defined for finite-dimensional Lie algebras. In order to introduce it, we must first define a distinguished element of the Weyl group, known as the *Coxeter element*, w_c. It is given by the product of all fundamental Weyl reflections:

$$w_c = \prod_{i=1}^{n} w_i. \tag{3.9}$$

The Coxeter element is not unique, but depends on the choice of basis of the root system and the ordering of the above product. However, all Coxeter elements are conjugate to each other in the Weyl group, which implies they share the same properties. In particular, the order of the Coxeter element is always equal to the Coxeter number g of the Lie algebra (2.61a). Thus g is the smallest possible integer such that

$$(w_c)^g = \mathbb{1}. \tag{3.10}$$

Furthermore, it can be shown that w_c has exactly one eigenvalue equal to $e^{\frac{2\pi i}{g}}$ [16]. The corresponding (complex) eigenvector will be denoted by z:

$$w_c(z) = e^{\frac{2\pi i}{g}} z \tag{3.11}$$

Recall that the inner product is associative with respect to Weyl reflections, that is, $(w(\alpha)|\beta) = (\alpha|w(\beta))$. So upon considering the inner product between z and the action of w_c on a generic root, it follows that

$$(w_c(\alpha)|z) = (\alpha|w_c(z)) = e^{\frac{2\pi i}{g}} (\alpha|z) \,. \tag{3.12}$$

Thus when projected onto z, the Coxeter element acts as $(\frac{1}{g})^{\text{th}}$ of a rotation on all roots. This leads us to the concept of a Coxeter plane C, which is spanned by the real and imaginary parts of z:

$$C = \mathbb{R}x_c + \mathbb{R}y_c, \tag{3.13}$$

where

$$x_c = \operatorname{Re} z, \tag{3.14a}$$
$$y_c = \operatorname{Im} z. \tag{3.14b}$$

A Coxeter projection is the projection of a root system onto its Coxeter plane. Its horizontal and vertical components are respectively given by

$$\mathbb{P}_x(\alpha) = (\alpha|x_c)\,, \tag{3.15a}$$
$$\mathbb{P}_y(\alpha) = (\alpha|y_c)\,. \tag{3.15b}$$

Following [84] we will draw lines between roots that are nearest neighbors. That is, we will draw a line between roots α and β if their distance $(\alpha - \beta|\alpha - \beta)$ is minimal. The coloring of the lines depends only on their maximal distance from the origin in the projected graph.

Coxeter projections preserve the g-fold rotational symmetry of the root system, which is generically only a small part of its complete symmetry. Nonetheless the resulting graph can display a rich structure, as for example the E_8 Coxeter plane does (Figure 3.6). Note that the Coxeter projection is always mirror symmetric in the origin, because the negative roots project as $\mathbb{P}(-\alpha) = -\mathbb{P}(\alpha)$. This effectively doubles the rotational symmetry from g-fold to $2g$-fold for Lie algebras whose Coxeter number is odd. For more Coxeter projections, see Appendix B.

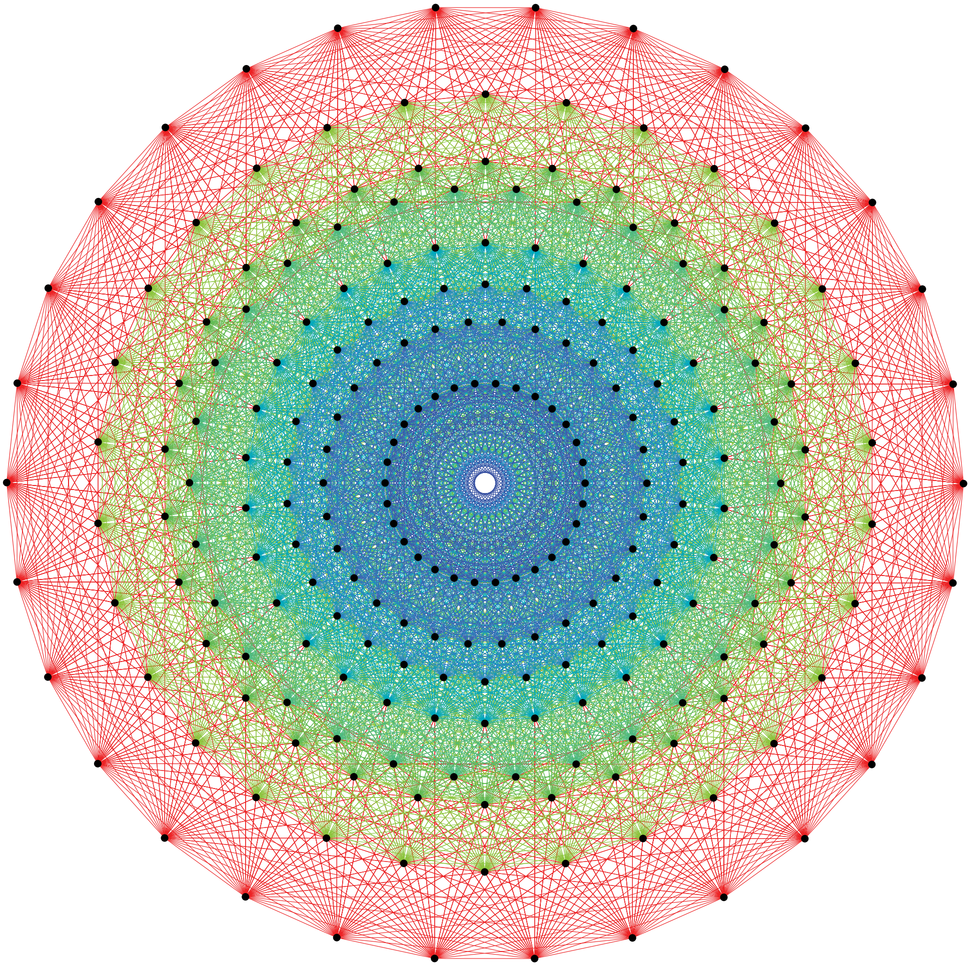


Figure 3.6: Coxeter projection of the roots of E_8.

Example 3.1: Coxeter projection of A_2

When acting on a root vector m^i, the Weyl reflections can be written as $n \times n$ matrices. By equation (2.69), the two fundamental Weyl reflections of A_2 are

$$w_1 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{3.16a}$$

$$w_2 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}. \tag{3.16b}$$

The Coxeter element w_c is the product of the two,

$$w_c = w_1 w_2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}. \tag{3.17}$$

We're looking for an eigenvector of w_c that has an eigenvalue of $e^{\frac{2\pi i}{3}}$, since the Coxeter number of A_2 is $g = 1 + (\rho^\vee|\alpha_1 + \alpha_2) = 3$. The eigenvector z is

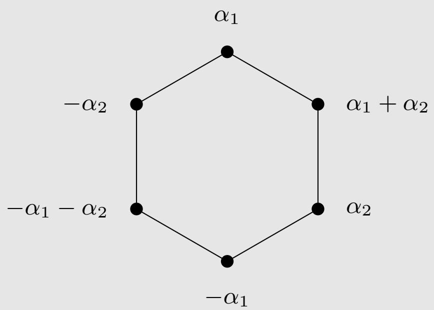
$$z = \begin{pmatrix} 1 + i\sqrt{3} \\ 2 \end{pmatrix}. \tag{3.18}$$

Expanded in terms of components, the horizontal and vertical projections of a root α with root vector m^i then respectively read

$$\mathbb{P}_x(\alpha) = A_{ij} m^i x_c^j = 3m^2, \tag{3.19a}$$

$$\mathbb{P}_y(\alpha) = A_{ij} m^i y_c^j = \sqrt{3}(2m^1 - m^2). \tag{3.19b}$$

Doing this projection for all roots of A_2 results in the following picture:



Not surprisingly, this is the same old picture of the root system we have seen before, but now rotated over an angle of 60 degrees. The lines between the roots indicate the nearest neighbour pairs.

3.2.2 Projections to subalgebras

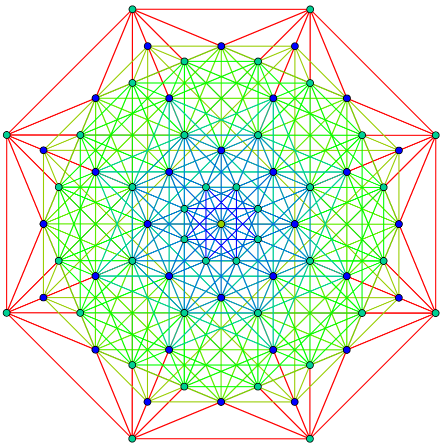
The discussion in subsection 3.2.1 is only valid for finite-dimensional Lie algebras. Although the Coxeter element can be defined for infinite-dimensional Lie algebras in a similar way, it no longer has the nice properties its finite-dimensional counterpart has. For instance, it does not have an eigenvalue of $e^{\frac{2\pi i}{g}}$. One reason for this is that the Coxeter number is ill-defined, because infinite Lie algebras do not have a highest root. A notable exception are the affine Lie algebras: despite the fact that they are infinite, one is still able to define a Coxeter number and do a Coxeter projection. More on this in section 4.1.

However, it is possible to project the root system of an infinite-dimensional Lie algebra onto the Coxeter plane of a *finite* subalgebra $\mathfrak{s}$. The finite subalgebra can be specified by picking a subset α_s $(s = 1, \dots, n-m)$ of the simple roots α_i such that α_s generate a finite root system. The Coxeter element of $\mathfrak{s}$, denoted by w_c^{sub}, is then

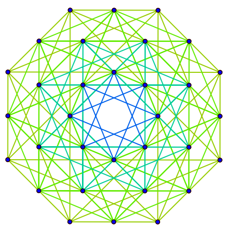
$$w_c^{\text{sub}} = \prod_{s=1}^{n-m} w_s. \tag{3.20}$$

Its order is equal to g_{sub}, the Coxeter number of $\mathfrak{s}$. The rest of analysis follows the same lines as that of subsection 3.2.1.

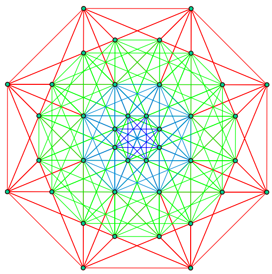
A projection onto a Coxeter plane of a subalgebra can be viewed as a level decomposition of the whole Coxeter projection. For more on level decompositions, see section 4.3. The resulting graph consists of Coxeter projections of representations of the subalgebra, stacked on top of each other. This procedure is of course not limited to infinite Lie algebras, but can also be done for finite cases. For example, in Figure 3.7 the root system of E_8 is projected onto the Coxeter plane of an A_7 subalgebra. Subalgebra projections display the g_{sub}-fold rotational symmetry of the subalgebra. As $g > g_{\text{sub}}$, the resulting picture is less symmetric than the full projection.



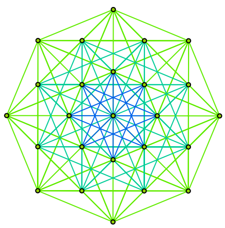
(a) The full subalgebra projection.



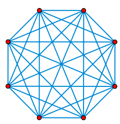
(b) Level 0



(c) Level 1



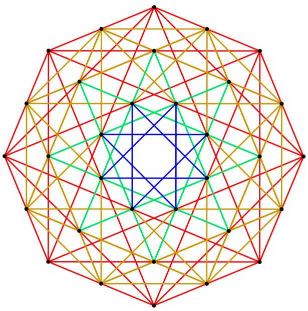
(d) Level 2



(e) Level 3

Figure 3.7: Projection of E_8 onto the Coxeter plane of an A_7 subalgebra, split into the contributions of A_7 representations at different levels.

4



Kac-Moody algebras

A Kac-Moody algebra is a Lie algebra whose Cartan matrix A is generalized. That is, A is not necessarily positive definite (see equation (2.13)). The algebra is finite-dimensional when A is positive definite, and infinite-dimensional otherwise. This chapter will treat special kinds of the latter case. Namely, we will consider the Kac-Moody algebras that can be obtained by enlarging the Cartan matrix (or equivalently, the Dynkin diagram) of finite Lie algebras in a particular way.

4.1 Affine algebras

An *affine Kac-Moody algebra* is a Kac-Moody algebra whose Cartan matrix A has a vanishing determinant [34],

$$\det A = 0. \tag{4.1}$$

This a loosening of condition (2.13d), but condition (2.13e) remains intact: all its principal minors should remain positive. Thus if we remove any node from the associated Dynkin diagram, the remaining diagram should correspond to a finite Lie algebra.

If A is an $(n+1)\times(n+1)$ matrix, its rank is n; it has one eigenvector whose eigenvalue is zero, and is therefore positive semidefinite. The root space Φ thus has

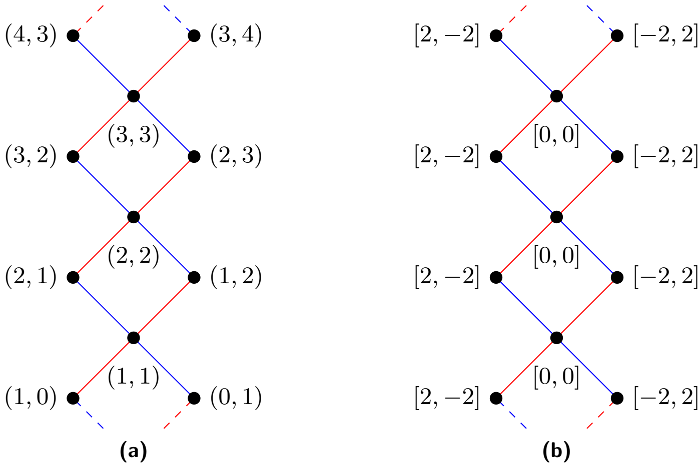


Figure 4.1: Hasse diagrams of the root system of A_1^+ up to height 7. In Figure (a) the labels are the root vectors, in Figure (b) the Dynkin labels.

one null direction, which is the reason the root system Δ becomes infinite. Another point of view is that an affine Lie algebra is infinite-dimensional because the Serre construction does not terminate when A is not positive definite.

Besides the fact root systems of affine algebras are infinite, they are also highly structured. Take for example the simplest affine algebra, known as A_1^+. Its Cartan matrix is given by

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{4.2}$$

The root system Δ of A_1^+ up to a given height is given in Figure 4.1. One feature that is immediately clear is that it is repetitive, which is in fact common to all affine Kac-Moody algebras. See for example also Figure 4.2. To be precise, there is one element $\delta \in \Delta$ for which the following holds:

$$\alpha + m\delta \in \Delta \quad \forall\, \alpha \in \Delta. \tag{4.3}$$

Here m is the smallest possible non-negative integer for which the above statement is true, and δ is the so-called *null root*, which will be defined below. Algebras for which m is equal to one are known as *untwisted* affine algebras. Otherwise they are called *twisted*. In the following we will assume that the affine algebra under consideration is always untwisted.

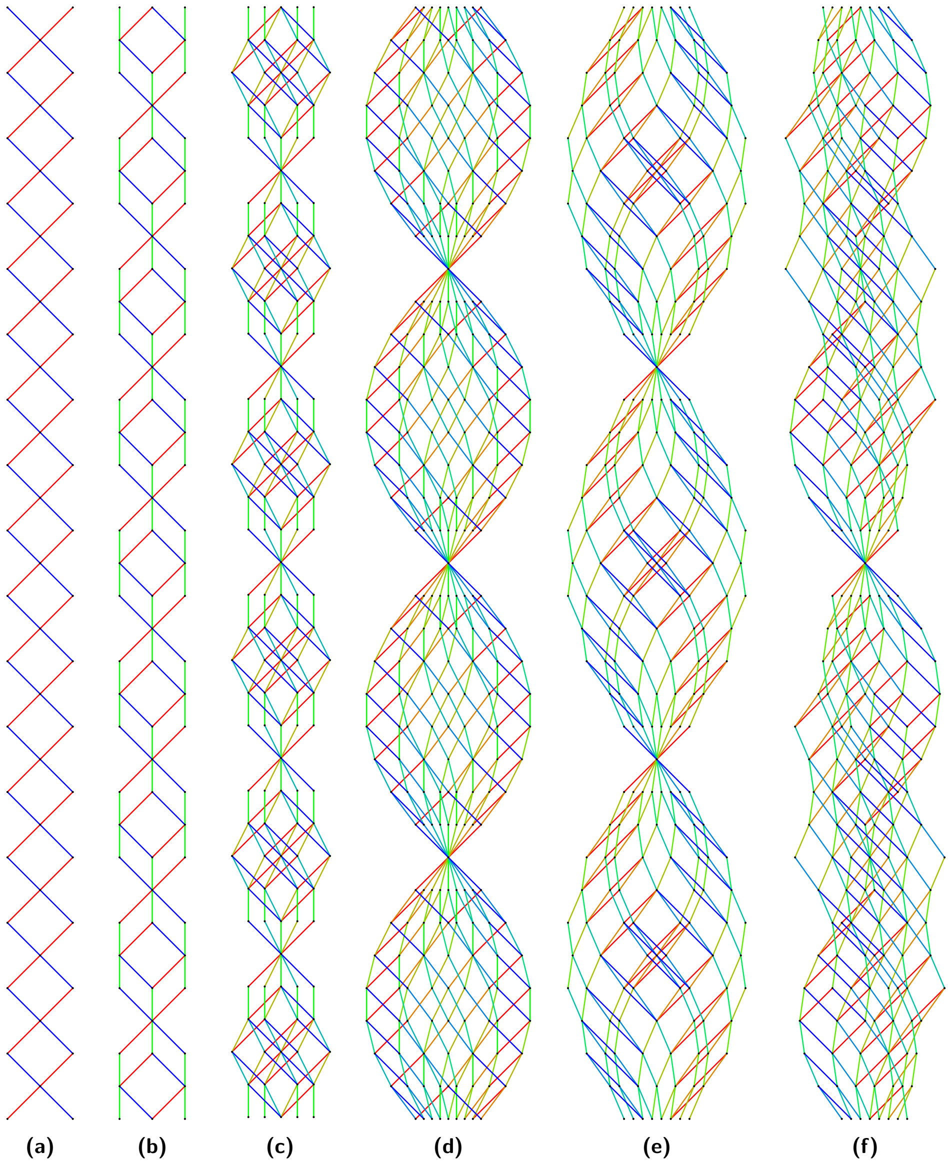


Figure 4.2: Hasse diagrams of root systems of affine Kac-Moody algebras of increasing complexity. From left to right we have: (a) A_1^+, (b) C_2^+, (c) D_4^+, (d) A_8^+, (e) D_7^+, and (f) E_7^+.

The null root δ and its dual $\delta^\vee$ are given by

$$\delta = a^i \alpha_i, \tag{4.4a}$$

$$\delta^\vee = a^{\vee i} \alpha_i^\vee. \tag{4.4b}$$

The coefficients $a^{\vee i}$ and a^i are the (dual) Coxeter labels of the affine Kac-Moody algebra. In contrast to finite algebras, they are defined as the left and right null eigenvectors of the Cartan matrix. They are normalized such that their minimum component is equal to one:

$$a^j A_{ji} = 0 \qquad \min(a^i) = 1 = a^0, \tag{4.5a}$$

$$A_{ij} a^{\vee j} = 0 \qquad \min(a^{\vee i}) = 1 = a^{\vee 0}. \tag{4.5b}$$

This particular normalization ensures that δ lies on the root lattice. For convenience, the index for which both a^i and $a^{\vee i}$ are minimal has been labeled 0. The full index i then runs over $i = 0, 1, \ldots, n$. From their definition it follows that the Coxeter labels a^i and $a^{\vee i}$ satisfy

$$a^{\vee i} = \frac{(\alpha_i|\alpha_i)}{(\alpha_0|\alpha_0)} a^i. \tag{4.6}$$

It follows straightforwardly that the null root is related to its dual by

$$\delta^\vee = \frac{2}{(\alpha_0|\alpha_0)} \delta. \tag{4.7}$$

This is not the usual definition $\delta^\vee = 2\delta/(\delta|\delta)$, as that would have been ill-defined. δ is namely truly a null root in the sense of equation (2.42); its norm $(\delta|\delta)$ is zero.

The null root of A_1^+ is $\delta = \alpha_0 + \alpha_1$. Studying Figure 4.1 in more detail, you may notice that roots differing by δ share the same Dynkin labels p_i,

$$p_i = (\alpha|\alpha_i^\vee) = A_{ji} m^j = (\alpha + \delta|\alpha_i^\vee). \tag{4.8}$$

The reason this happens is that the Dynkin labels of δ are all zero, because its root vector is the null vector of the Cartan matrix. In order to lift this degeneracy of the roots, we can add an additional Dynkin label p_{-1} that distinguishes between α and $\alpha + \delta$. For an arbitrary root α it is defined as

$$p_{-1} = (\alpha|\gamma^\vee), \tag{4.9}$$

where $\gamma^\vee = 2\frac{\gamma}{(\gamma|\gamma)}$. We will call γ the *root of derivation*. It is an element of the root space Φ that satisfies

$$(\delta|\gamma^\vee) = -1. \tag{4.10}$$

This does the trick, as $(\alpha+\delta|\gamma^\vee) = (\alpha|\gamma^\vee) - 1$. Therefore the Dynkin label p_{-1} of $\alpha+\delta$ differs from that of α by -1. Because the zeroth Coxeter label a^0 is always equal to one, the inner product of $\gamma^\vee$ with the simple roots can be defined as

$$(\alpha_i|\gamma^\vee) = -\delta_i^0, \tag{4.11}$$

With this definition γ satisfies (4.10). Note that γ cannot lie in the span of the simple roots. If that were the case, its inner product with δ would be zero. We therefore need to manually add γ to the root space,

$$\Phi = \mathrm{span}\{\alpha_0, \ldots, \alpha_n\} \oplus \gamma. \tag{4.12}$$

Note that the root of derivation is not a member of the root system Δ, as all of the roots in Δ can still be expanded entirely in the basis of simple roots.

In principle the norm of the root of derivation is not fixed by the above analysis. However, things simplify a bit when we choose $(\gamma|\gamma) = (\alpha_0|\alpha_0)$. The above equation then can be written as

$$(\alpha_i^\vee|\gamma) = -\delta_i^0 = \left(-\Lambda^0|\alpha_i^\vee\right). \tag{4.13}$$

Thus γ can be identified with minus the zeroth fundamental weight, Λ^0. When the Cartan matrix is non-degenerate, the fundamental weights can be given by means of its inverse (see equation (2.50)). But when the Cartan matrix is degenerate, as is the case for affine algebras, this is not possible. The fundamental weights can be introduced for affine algebras only because the introduction of γ has lifted the degeneracy of the Cartan matrix.

The Cartan matrix A can namely be extended to a bigger matrix, A'. The extended Cartan matrix A' has one extra row and column that correspond to γ:

$$A_{(-1)i} = (\gamma|\alpha_i^\vee) = (\alpha_i^\vee|\gamma) = A_{i(-1)}. \tag{4.14}$$

The introduction of γ has effectively added a generator h_{-1} to the Cartan subalgebra, on which the action of the simple roots is (compare equation (2.29))

$$\alpha_i(h_{-1}) = A_{(-1)i}. \tag{4.15}$$

The extended Cartan matrix reads in full

$$A' = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & A_{00} & A_{10} & \cdots & A_{0n} \\ 0 & A_{10} & A_{11} & \cdots & A_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & A_{n0} & A_{n1} & \cdots & A_{nn} \end{pmatrix}. \tag{4.16}$$

This matrix is indeed non-degenerate, as its rank is $n+2$. The extended Cartan matrix can be taken as the starting point in the Serre construction of affine Kac-Moody algebras. However, one must then take some additional rules into account.

Namely, there are no Chevalley generators e_{-1} and f_{-1} associated to the extension, but only an extra generator h_{-1} for the Cartan subalgebra. The extra simple root α_{-1} is also absent. If it were present it would correspond to γ, which is, as noted above, not part of the root system. Lastly, the Weyl group is generated by the $n+1$ fundamental reflections $w_0, \ldots, w_n$. The reflection w_{-1} does not exist.

Example 4.1: A_1^+

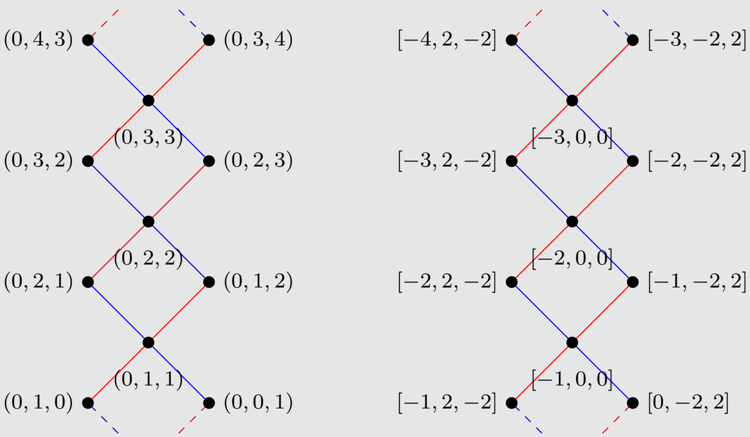
Recall that the Cartan matrix of A_1^+ is equal to

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{4.17}$$

The normalized Coxeter labels are then $a^i = (1,1)$, as $a^j A_{ji} = 0$ and $\min(a^i) = 1$. Because both Coxeter labels are equal to one, we have a freedom to choose which index will be the zeroth. If we pick the first one, the extended Cartan matrix is

$$A' = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}. \tag{4.18}$$

If we use the extended Cartan matrix to calculate the Dynkin labels of the roots via $p_i = A'_{ji} m^j$, Figure 4.1 becomes



The addition of the derivation γ to the root space, and its corresponding row and column to the Cartan matrix, has achieved what it was supposed to achieve: the Dynkin labels are no longer degenerate.

4.1.1 Affine algebras as extensions of finite algebras

Besides being defined by means of Cartan matrices with vanishing determinant, an affine Kac-Moody algebra $\mathfrak{g}$ can also be obtained from a finite semi-simple Lie algebra $\overline{\mathfrak{g}}$. The Lie algebra $\overline{\mathfrak{g}}$ then features as a particular subalgebra of $\mathfrak{g}$ whose roots $\overline{\alpha}$ all satisfy

$$(\overline{\alpha}|\gamma^{\vee}) = 0. \tag{4.19}$$

The subalgebra $\overline{\mathfrak{g}}$ is called the *horizontal* subalgebra of $\mathfrak{g}$. The way $\mathfrak{g}$ can be obtained from $\overline{\mathfrak{g}}$ is as follows. If the Lie bracket of $\overline{\mathfrak{g}}$ reads

$$[T^a, T^b] = f_{ab}{}^c T^c \tag{4.20}$$

a loop algebra $\overline{\mathfrak{g}}_{\text{loop}}$ can be formed by introducing a grading over $\mathbb{Z}$:

$$[T^a_m, T^b_n] = f_{ab}{}^c T^c_{m+n}. \tag{4.21}$$

Here the additional indices run over the integers: $m, n \in \mathbb{Z}$. In order to turn the loop algebra into an affine algebra we have to add two additional generators, the *central element* K and the *derivation* D. With the inclusion of K and D, the complete Lie bracket reads

$$[T^a_m, T^b_n] = f_{ab}{}^c T^c_{m+n} + m\delta_{m,-n}\eta^{ab}K, \tag{4.22a}$$
$$[D, T^a_m] = -mT^a_m, \tag{4.22b}$$
$$[K, D] = [K, T^a_m] = 0. \tag{4.22c}$$

The algebra $\mathfrak{g}$ specified by the above bracket is said to be the *affine extension* of $\overline{\mathfrak{g}}$, and is often denoted by $\overline{\mathfrak{g}}^+$. The original Lie algebra $\overline{\mathfrak{g}}$ can be identified with generators T^a_m whose bracket with D vanishes. This is the same identification as in equation (4.19).

The central element is unique up to normalization, and corresponds to the null direction of the Cartan subalgebra:

$$K = a^{\vee i} h_i. \tag{4.23}$$

Inspection then shows that K indeed commutes with all Chevalley generators e_i and f_i (see equation (2.16)), and thus also with all generators of $\mathfrak{g}^+$.

The derivation D is identical to the extra generator h_{-1} of the previous section. Like γ does for the roots, the derivation 'measures' the $\mathbb{Z}$ grading of the loop generators T^a_m. It also crucially lifts the degeneracy of the Cartan-Killing form, which is given by

$$\langle T^a_m | T^b_n \rangle = \langle T^a | T^b \rangle \delta_{m,-n}, \tag{4.24a}$$
$$\langle K | D \rangle = -1, \tag{4.24b}$$

with all other combinations vanishing. As a consequence, the derivation also lifts the degeneracy of the inner product on the root space. The root space is again defined as the dual of the Cartan subalgebra, $\Phi = \mathfrak{h}^*$, which can be given by

$$\mathfrak{h} = \overline{\mathfrak{h}} \oplus \operatorname{span}\{K, D\}. \tag{4.25}$$

The root system Δ then has the same structure as described in section 4.1. In particular, the simple roots are given by

$$\alpha_s = \overline{\alpha}_s, \tag{4.26a}$$

$$\alpha_0 = \delta - \overline{\theta}, \tag{4.26b}$$

where $\overline{\theta}$ is the highest root and $\overline{\alpha}_s$ are the simple roots of $\overline{\mathfrak{g}}$. The index s runs over the rank of $\overline{\mathfrak{g}}$, that is $i = \{0, 1, \ldots, n\} = \{0, s\}$. Note that the norm of α_0 is the same as that of $\overline{\theta}$, $(\alpha_0|\alpha_0) = (\overline{\theta}|\overline{\theta})$.

By equation (4.26), the Cartan matrix $\overline{A}$ of the horizontal subalgebra $\overline{\mathfrak{g}}$ can be recovered from that of $\mathfrak{g}$ by deleting the zeroth row and column. Equivalently, one can delete the zeroth node of the affine Dynkin diagram to retrieve the diagram of the horizontal subalgebra. Conversely, the Cartan matrix of the affine algebra can be obtained from $\overline{A}$ by adjoining a row and column whose entries correspond to the Dynkin labels of the lowest root of $\overline{\mathfrak{g}}$. Specifically, the row and columns to add are

$$A_{0s} = \left(\delta - \overline{\theta}|\alpha_s^\vee\right) = -a^t A_{ts}, \tag{4.27a}$$

$$A_{s0} = \frac{(\alpha_s|\alpha_s)}{(\overline{\theta}|\overline{\theta})} A_{0s}. \tag{4.27b}$$

4.2 Over- and very-extended algebras

In section 4.1 we argued that the extended Cartan matrix (4.16) of affine Kac-Moody algebras can serve as the starting point in their construction if one takes certain rules into account. When those rules are forgotten, the resulting algebra is not affine. Using the 'ordinary' Serre construction, the result is a so-called *over-extended* Kac-Moody algebra. The nomenclature stems from the fact that an over-extended algebra is an extension of an affine algebra $\mathfrak{g}^+$, who in turn is an extension of its horizontal finite subalgebra $\mathfrak{g}$. Over-extended algebras are denoted by $\mathfrak{g}^{++}$.

By (4.16), their Dynkin diagram is obtained from that of affine algebras by adding a node to the zeroth node of the affine diagram. In Figure 4.4 the over-extended node is denoted by -1. As noted in section 4.1, the over-extended Cartan matrix is non-degenerate. If its rank is $n+2$, it has one negative and $n+1$ positive eigenvalues. The root space of over-extended Kac-Moody algebras has therefore a Lorentzian signature, allowing for real, null, and imaginary roots.

Example 4.2: A_1^+ as the affine extension of A_1

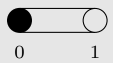
Recall from Example 2.2 and Example 2.3 that the Cartan matrix of the Lie algebra A_1 is given by

$$\overline{A} = \begin{pmatrix} 2 \end{pmatrix} . \tag{4.28}$$

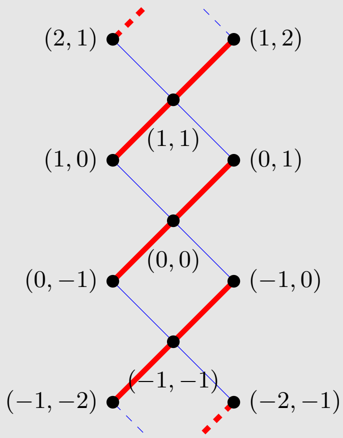
The highest root of A_1 is its only positive root, $\overline{\theta} = \overline{\alpha}_1$, whose Dynkin label is equal to 2. By equation (4.27) the Cartan matrix A of the affine extension of A_1 is thus given by

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} . \tag{4.29}$$

This is, not surprisingly, the same Cartan matrix as in equation (4.2). A_1^+ is thus truly the affine extension of A_1. Its Dynkin diagram is given by



The node labels indicate their ordering. The black zeroth node has been 'deleted': the remaining undeleted Dynkin diagram is that of the horizontal subalgebra A_1. If we again take a look at the root system of A_1^+, we can distinguish infinitely many copies of the root system of A_1 within it:



The red lines have been thickened to indicate the A_1 subalgebras. The middle roots $(0, -1)$, $(0, 0)$, and $(0, 1)$ form the horizontal root subsystem, as they all satisfy equation (4.19).

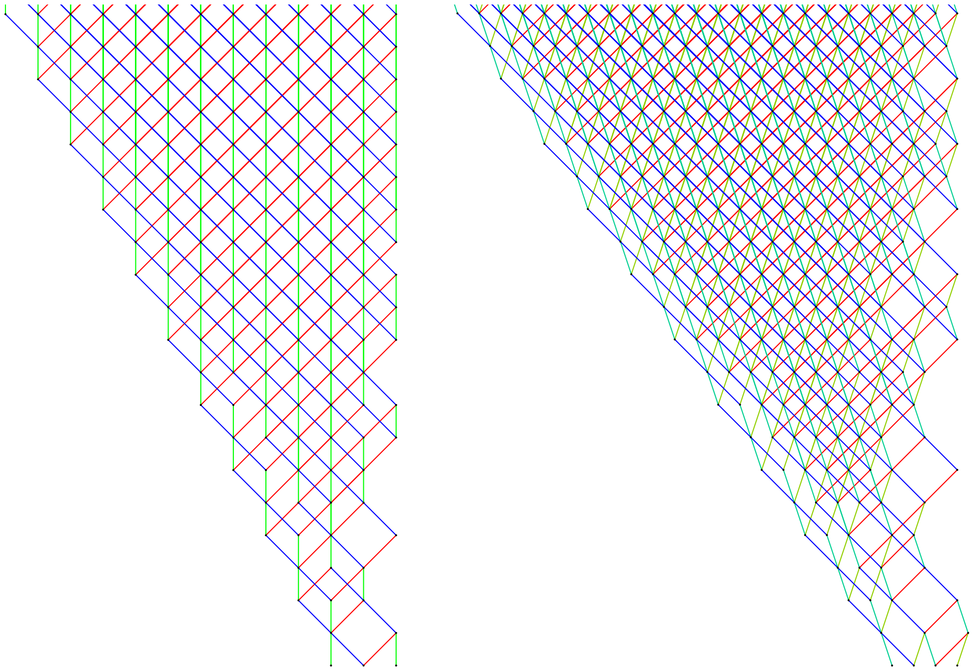


Figure 4.3: Hasse diagrams of the root systems of (a) A_1^{++} and (b) A_1^{+++} up to height 22. The number of roots of A_1^{+++} grows faster in height than that of A_1^{++}.

If we repeat the procedure and attach yet another node to the over-extended node, the resulting diagram is that of a *very-extended* Kac-Moody algebra $\mathfrak{g}^{+++}$. Figure 4.4 lists all the Kac-Moody algebras that are very-extensions of simple Lie algebras. In Figure 4.4 the very-extended node has been labeled -2. Similarly to over-extended algebras, the Cartan matrix of very-extended algebras is also Lorentzian. As opposed to affine algebras, their root systems do not grow linearly in height; both over- and very-extended root systems grow faster. Figure 4.3 shows the root systems of A_1^{++} and A_1^{+++}. Because of the extra very-extended simple root, the root system of very-extended algebras grows faster than that of the over-extended algebras.

In contrast to affine algebras, the root systems of over- and very-extended Kac-Moody algebras cannot (yet) be described fully in closed form. However, it is worth noting that there is a class of Lorentzian Kac-Moody algebras for which one can make general statements about the root system. These Kac-Moody algebras are called *hyperbolic*, and they are characterized by the fact their Cartan matrix contains

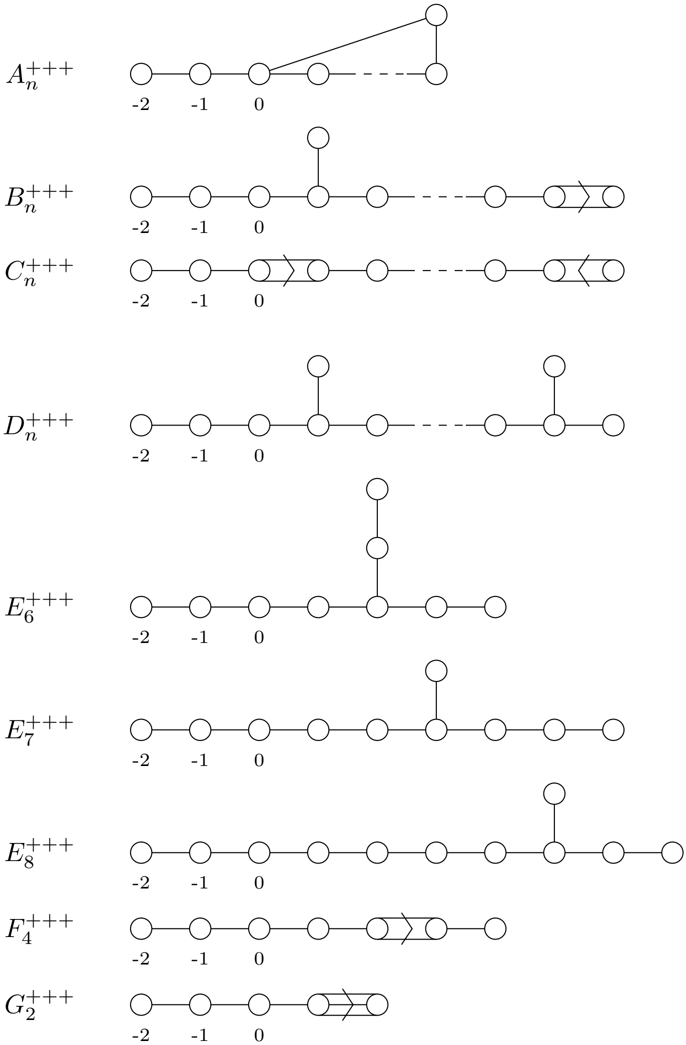


Figure 4.4: Dynkin diagrams of all Kac-Moody algebras that are very-extensions of finite simple Lie algebras. Deleting node -2 results in all of the over-extended Kac-Moody algebras g^{++}, and also deleting node -1 in all of the extended (affine) algebras g^{+}. In all cases the subscript denotes the rank of the unextended algebra.

only positive definite or positive semidefinite submatrices. Thus upon removing any node from their Dynkin diagram, we are left with a diagram of either a finite or an affine algebra. Very-extended Kac-Moody algebras are not hyperbolic; removing the very-extended node yields a Lorentzian over-extended algebra. In contrast, over-extended algebras can be hyperbolic. The root system Δ of hyperbolic algebras consists of all the points on the root lattice whose norm is smaller than norm of the longest simple root, $\alpha^2_{\max}$ [50]:

$$\Delta = \big\{\alpha \in \Delta \,\big|\, (\alpha|\alpha) \leq \alpha^2_{\max}\big\}. \tag{4.30}$$

Unfortunately, this tells us nothing about the root multiplicities. Besides, we would also like to analyze the root system of very-extended Kac-Moody algebras. So for the general case one has to resort to methods that analyze the root system part by part, such as the level decomposition.

4.3 Level decomposition

A *level decomposition* is a way to chop up a Lie algebra $\mathfrak{g}$ in terms of one of its subalgebras $\mathfrak{s}$ [23, 70, 53, 57]. More precisely, under a level decomposition the adjoint representation of $\mathfrak{g}$ branches into a number of representations of $\mathfrak{s}$. This is particularly useful for over- and very-extended Kac-Moody algebras, as their infinite adjoint representation can be described in terms of finite representations of a finite subalgebra. One can of course also perform a level decomposition of finite algebras; see Figure 4.5 for a few examples.

The subalgebra $\mathfrak{s}$ will always be chosen to be *regular*. That is, all simple roots of $\mathfrak{s}$ are also simple roots of $\mathfrak{g}$. Put differently, the Cartan matrix of $\mathfrak{s}$ is obtained from that of $\mathfrak{g}$ by simultaneously deleting certain rows and columns. Equivalently, the Dynkin diagram of $\mathfrak{s}$ is a subdiagram of that of $\mathfrak{g}$ and can be recovered by deleting one or more nodes. The number of nodes that are deleted will be denoted by m. In order to indicate which of the simple roots belong to $\mathfrak{s}$ we will split the indices as $i = (s, a)$. They respectively run over

$$\begin{aligned} i, j, \ldots &: \text{Rank of full algebra, } n, \\ s, t, \ldots &: \text{Rank of regular subalgebra, } n - m, \\ a, b, \ldots &: \text{Number of deleted nodes, } m. \end{aligned} \tag{4.31}$$

Any root $\alpha \in \mathfrak{g}$ can then be written as

$$\alpha = m^i\alpha_i = m^s\alpha_s + l^a\alpha_a, \tag{4.32}$$

where $l^a \equiv m^a$ are called the *levels* of a root. The algebra $\mathfrak{g}$ then splits up into *level*

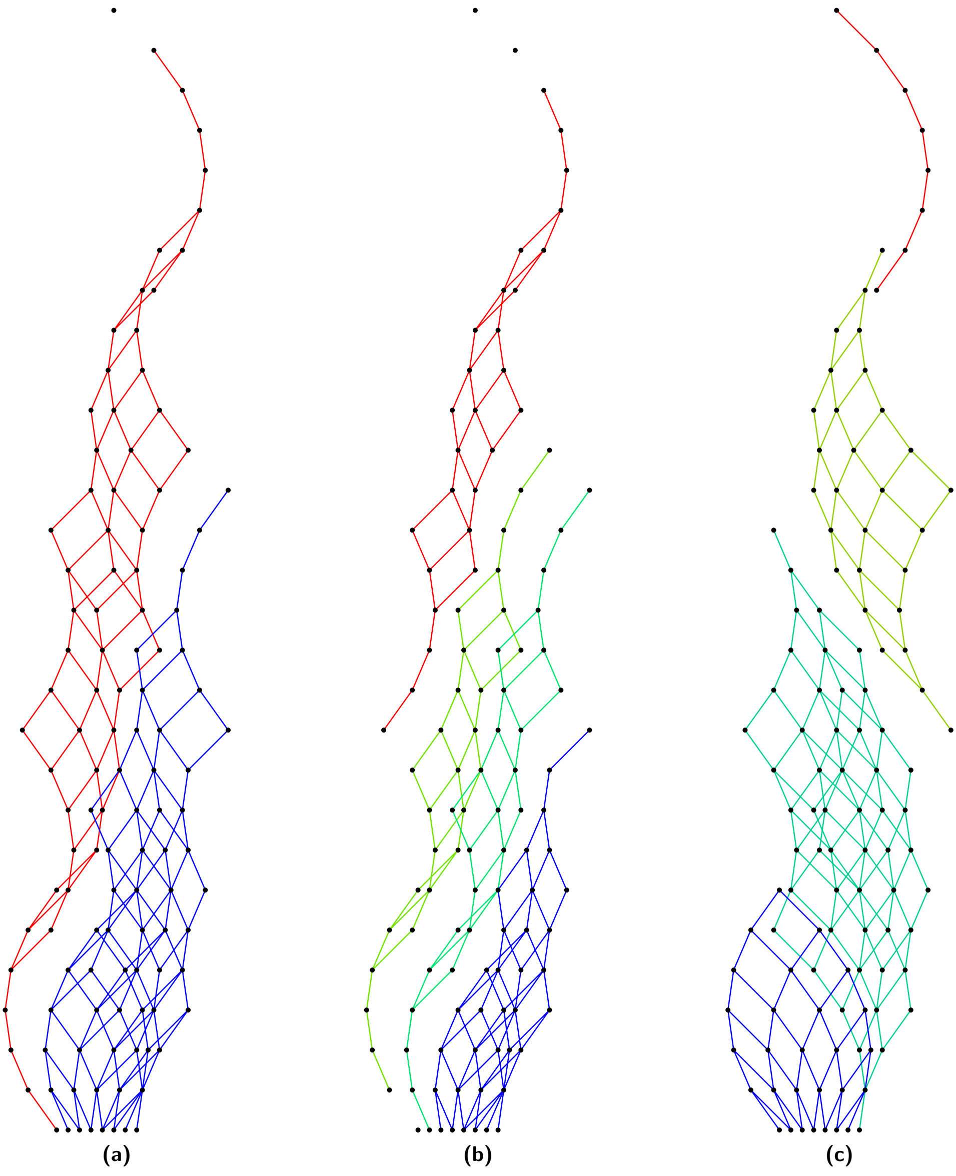


Figure 4.5: Hasse diagrams of the root system of E_8 branched with respect to different regular subalgebras. The subalgebras are: (a) E_7, (b) E_6, and (c) A_7. Compare also Figure 3.4.

spaces $\mathfrak{g}_{l^a}$. They are direct sums of root spaces $\mathfrak{g}^\alpha$ whose root is of level l^a:

$$\mathfrak{g}_{l^a} = \bigoplus_{(\alpha|\Lambda^{\vee a})=l^a} \mathfrak{g}_\alpha. \tag{4.33}$$

The whole algebra $\mathfrak{g}$ can then be written as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{l^a \in \mathbb{Z}^m} \mathfrak{g}_{l^a}. \tag{4.34}$$

Because $\mathfrak{g}$ is graded by means of its roots (see equation (2.35)), the levels also induce a grading on $\mathfrak{g}$:

$$[\mathfrak{g}_{l_1^a}, \mathfrak{g}_{l_2^a}] \subseteq \mathfrak{g}_{l_1^a+l_2^a}. \tag{4.35}$$

The roots of the regular subalgebra $\mathfrak{s}$ all have level zero by construction. This implies that the adjoint action of $\mathfrak{s}$ on a level space cannot change the level,

$$[\mathfrak{s}, \mathfrak{g}_{l^a}] \subseteq \mathfrak{g}_{l^a}. \tag{4.36}$$

Owing to the triangular decomposition (2.20), a level space $\mathfrak{g}_{l^a} \neq \mathfrak{s}$ sits either in the positive part ($\mathfrak{g}_{l^a} \in \mathfrak{n}_+$) or in the negative part ($\mathfrak{g}_{l^a} \in \mathfrak{n}_-$) of $\mathfrak{g}$. In the former case the adjoint action of $\mathfrak{s}$ on $\mathfrak{g}_{l^a}$ can never reach $\mathfrak{n}_-$. The level space $\mathfrak{g}_{l^a}$ is therefore bounded from below. This in turn implies there exist generators $x \in \mathfrak{g}_{l^a}$ that are annihilated by the adjoint action of the negative Chevalley generators of $\mathfrak{s}$:

$$\mathrm{ad}_{f_s}\, x = 0. \tag{4.37}$$

Comparing with equation (2.86), we see that the generators x are lowest weight vectors in lowest weight representations of $\mathfrak{s}$. The level space is thus completely reducible in terms of lowest weight representations. On the other hand, if $\mathfrak{g}_{l^a} \in \mathfrak{n}_-$ it is reducible in terms of highest weight representations. In the final case that $\mathfrak{g}_{l^a} = \mathfrak{s}$ the level space is given by the adjoint representation of $\mathfrak{s}$. We will focus only on the decomposition of $\mathfrak{n}_+$, as the decomposition of $\mathfrak{n}_-$ follows from it using the Chevalley involution (2.21).

As we saw in subsection 2.2.1, if x is a lowest weight vector of $\mathfrak{s}$ its Dynkin labels with respect to $\mathfrak{s}$ are non-positive. By equation (2.51a) it can be easily checked if this is the case. The Dynkin labels of any positive root $\alpha = m^i \alpha_i$ are

$$p_s = A_{ts} m^t + A_{as} l^a. \tag{4.38}$$

Using the above equation, we can check every root to see if it is a lowest weight of $\mathfrak{s}$. It can also be inverted to give

$$m^s = \left(A_{\mathrm{sub}}^{-1}\right)^{ts} \left(p_t - l^a A_{at}\right), \tag{4.39}$$

where A_{sub} is the Cartan matrix of $\mathfrak{s}$. This, together with the requirement that the root norm is bounded from above (2.75), gives us an algorithm to scan for potential lowest weights. The bound (2.75) can be written as

$$\alpha^2 = G^{st}_{\text{sub}} \left(p_s p_t - A_{as} A_{bt} l^a l^b \right) + B_{ab} l^a l^b \leq \alpha^2_{\text{max}}. \tag{4.40}$$

Note that for this formula to be valid, we have to make sure that a long (or short) root in the full algebra is also a long (short) root in the subalgebra, which in general is not automatically the case. Luckily we are always free to choose a normalization such that the root lengths match.

Assuming $\mathfrak{s}$ is finite, the quadratic form matrix G^{st}_{sub} of $\mathfrak{s}$ only has positive entries. Therefore α^2 is a monotonically increasing function of $|p_s|$ at fixed levels l^a. The number of distinct Dynkin labels p_s that respect the bound is thus finite. Furthermore, the root vector associated to p_s has to lie on the root lattice, implying that m^s should be non-negative integers. Hence equation (4.39) restricts the possible values of p_s even more. All in all, there are a finite number of subalgebra representations at a given level if $\mathfrak{s}$ is finite.

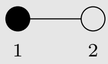
After scanning for possible valid Dynkin labels at fixed levels using equations (4.39) and (4.40), we still have to determine whether the representation $V(\Lambda)$ with lowest weight $\Lambda = p_s \Lambda^s$ actually occurs at level l^a. Let n_{l^a} be the number of distinct Dynkin labels found at level l^a using the above scanning technique. In principle there could be n_{l^a} distinct representations V_i at level l^a. Say a root α occurs as a weight within the weight diagrams of some of the representations V_i. Then the sum of its multiplicity as a weight in the different representations has to add up to its multiplicity as a root:

$$\text{mult}(\alpha) = \sum_{i=1}^{n_{l^a}} \mu_{l^a}(V_i) \, \text{mult}_{V_i}(\alpha). \tag{4.41}$$

The number $\mu_{l^a}(V_i)$ counts how often the representation V_i occurs at level l^a, and is called the *outer multiplicity* of the representation. If one calculates $\text{mult}(\alpha)$ by means of the Peterson recursion formula (2.99) and $\text{mult}_{V_i}(\alpha)$ by means of the Freudenthal recursion formula (2.101), it is relatively straightforward to compute the outer multiplicities.

Example 4.3: Level decomposition of A_2

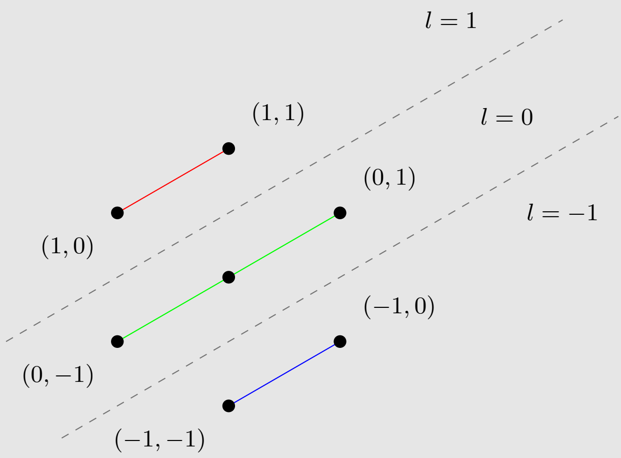
Let us see how the root system of A_2 splits up under the level decomposition with respect to an A_1 regular subalgebra. If we delete the first node, the Dynkin diagram looks like



The simple root of the deleted (black) node will count the level. The root system splits up according to

$$\alpha = l\alpha_1 + m\alpha_2. \tag{4.42}$$

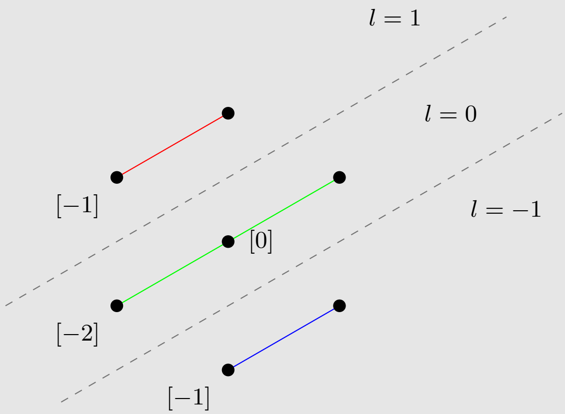
Graphically, it looks like the following:



The colored lines are Weyl reflections of the A_1 subalgebra. The different colors indicate the three levels -1, 0, and 1. From the above picture it is fairly obvious which roots are lowest or highest weights of A_1, but we can also employ the scanning method to find them. Equations (4.39) and (4.40) simplify to

$$m = \tfrac{1}{2}(p + l) \tag{4.43a}$$
$$\alpha^2 = \tfrac{1}{2}\left(p^2 + 3l^2\right) \leq 2. \tag{4.43b}$$

The last equation has no solution for $l \geq 2$, so, as expected, we only have to look at level -1,0, and 1. For $l = \pm 1$, the only allowed values are $p = \pm 1$. For $l = 0$ the options are $p = \{-2, 0, 2\}$. Since we're looking for lowest weights, the positive values of p can be discarded. All in all, the different Dynkin labels we have found are:



Each representation occurs with outer multiplicity one, including the singlet [0] representation. The singlet corresponds to the Cartan subalgebra generator of the deleted node, h_1. Generically, for every deleted node there is one singlet representation. The results of the decomposition can be collected in a table:

l	$-p$	m^i	α^2	dim	mult(α)	μ
-1	1	-1 -1	2	2	1	1
0	2	0 -1	2	3	1	1
0	0	0 0	0	1	2	1
1	1	1 0	2	2	1	1

The rows in the columns are the various representations of the subalgebra A_1 at the different levels. Here l denotes the level, p the Dynkin label, m^i the root vector of the lowest weight, dim the dimension of the representation, mult(α) the multiplicity of the root of the lowest weight, and μ the outer multiplicity of the representation.

To summarize, the level decomposition splits the adjoint A_2 representations into four A_1 representations over three different levels. In terms of the dimensions of the representations, we have made the following branching: $\mathbf{8} \to \mathbf{3} + \mathbf{1} + \mathbf{2} + \mathbf{2}$.

4.4 Non-linear realizations

For infinite-dimensional Kac-Moody algebras one can in principle construct a non-linear realization. But as there are an infinite amount of generators, the parameterization of the group element $V(t)$ (2.127) will contain an infinite amount of terms. This makes it impossible to compute the Maurer-Cartan form (2.124) in full generality. What is possible, however, is to make a consistent truncation of the formally defined action (2.126) for which the solutions of the equations of motion are also solutions to the full model [24, 42]. This can be done by truncating not the group element, but the Maurer-Cartan form either at a given root height or at a given level of a level decomposition. These two methods are similar in spirit, and we will discuss only the latter.

Recall that under a level decomposition the Lie algebra $\mathfrak{g}$ splits into level $\mathfrak{g}_{l^a}$ according to (4.34). The group element (2.127) can thus be written as

$$V(t) = \prod_{l^a \geq 0} V_{l^a}(t) = \prod_{l^a \geq 0} \exp\big(\phi_{l^a}(t) \cdot E_{l^a}\big). \tag{4.44}$$

Here E_{l^a} denotes all the representations of the subalgebra that occur at level l^a. The negative levels do not enter, as we can use the Borel gauge (2.128) to parameterize $V(t)$. Consequently, the Maurer-Cartan form splits as

$$J = V^{-1}\partial V = \sum_{l^a \geq 0} J_{l^a}, \tag{4.45}$$

where the summand schematically reads

$$J_{l^a} = R_{l^a} \cdot E_{l^a}. \tag{4.46}$$

The coefficients R_{l^a} contracting E_{l^a} can depend on scalar fields $\phi_{l'^a}$ whose level is at most l^a, i.e. $l'^a \leq l^a$. The exact form of R_{l^a} depends strongly on the graded structure of $\mathfrak{g}$. Before truncating, the sum in the expansion of J is infinite for Kac-Moody algebras. The truncated version can simply be cut off at any given level t^a,

$$\tilde{J} = \sum_{0 \leq l^a \leq t^a} J_{l^a}. \tag{4.47}$$

This truncated Maurer-Cartan form $\tilde{J}$ can then be used to calculate a (truncated) coset element $\tilde{P}$ and its corresponding action.

Example 4.4: Non-linear realization of $A_1 \subset A_2$

In Example 4.3 we decomposed A_2 with respect to A_1. In this example we will show how to do a non-linear realization of the fields recovered in the level decomposition. Recall that the results of the level decomposition were

l	$-p$	α^2	dim	mult(α)	μ	representation
-1	1	2	2	1	1	F_a
0	2	2	3	1	1	${K^a}_b$
0	0	0	1	2	1	K
1	1	2	2	1	1	E^a

The representations introduced in the last column correspond to the various generators as follows:

$$l = 0: \qquad h_1 = {K^1}_1 + 2{K^2}_2, \qquad e_2 = {K^1}_2, \tag{4.48a}$$

$$h_2 = {K^1}_1 - {K^2}_2, \qquad f_2 = {K^2}_1, \tag{4.48b}$$

$$l = 1: \qquad e_1 = E^2, \qquad [e_1, e_2] = E^1, \tag{4.48c}$$

$$l = -1: \qquad f_1 = F_2, \qquad [f_2, f_1] = F_1. \tag{4.48d}$$

Like in Example 2.10, ${K^a}_b$ are 2×2 matrices, and E^a and F_b are vectors,

$$({K^a}_b)^i{}_j = \delta^{ai}\delta_{bj}, \tag{4.49a}$$

$$(E^a)^i = \delta^{ai}, \tag{4.49b}$$

$$(F_a)_i = \delta_{ai}. \tag{4.49c}$$

All indices run from one to two. As we already know the Cartan-Killing norm on the generators from Example 2.8, we can straightforwardly cast it in the form

$$\langle {K^a}_b | {K^c}_d \rangle = \delta^a_d \delta^c_b - \tfrac{1}{3}\delta^a_b \delta^c_d, \tag{4.50a}$$

$$\langle E^a | F_b \rangle = \delta^a_b. \tag{4.50b}$$

All other combinations vanish. A convenient parameterization of the group element V is

$$V(t) = V_1(t)V_0(t), \tag{4.51a}$$

$$V_0(t) = e^{{h_a}^b(t){K^b}_a}, \tag{4.51b}$$

$$V_1(t) = e^{A_a(t)E^a}. \tag{4.51c}$$

The Maurer-Cartan form splits into a level 0 part J_0 and a level 1 part J_1,

$$J = J_0 + J_1, \tag{4.52a}$$
$$J_0 = e_a{}^m \partial e_m{}^b K^a{}_b, \tag{4.52b}$$
$$J_1 = e_a{}^m \partial A_m E^a, \tag{4.52c}$$

where $e_m{}^a$ is again the matrix exponential of $h_m{}^a$ and plays the role of the vielbein, and $e_a{}^m$ is its inverse. The coset element splits in a similar fashion:

$$P = P_0 + P_1, \tag{4.53a}$$
$$P_0 = e_a{}^m \partial e_m{}^b S^a{}_b, \tag{4.53b}$$
$$P_1 = e_a{}^m \partial A_m S^a, \tag{4.53c}$$

where the coset basis elements S are given by

$$S^a{}_b = \tfrac{1}{2}\left(K^a{}_b + K^b{}_a\right), \tag{4.54a}$$
$$S^a = \tfrac{1}{2}\left(E^a + F_a\right). \tag{4.54b}$$

Note that E^a and F_a are each others transposeds. If we evaluate the Cartan-Killing norm on the coset basis, we find

$$\langle S^a | S^b \rangle = \tfrac{1}{2}\delta^{ab}, \tag{4.55a}$$
$$\langle S^a{}_b | S^c{}_d \rangle = \delta^a_d \delta^c_b - \tfrac{1}{3}\delta^a_b \delta^c_d. \tag{4.55b}$$

Upon introducing the metric $g_{mn} = \delta_{ab} e_m{}^a e_n{}^b$ the action finally becomes

$$S = -\tfrac{1}{4}\int dt\, n(t)^{-1}\Big(\left(g^{mp}g^{nq} - \tfrac{1}{3}g^{mn}g^{pq}\right)\partial g_{mn}\partial g_{pq} + 2g^{mn}\partial A_m \partial A_n\Big). \tag{4.56}$$

This action should of course also follow from a direct reduction of the non-linear sigma model action of a non-decomposed A_2. If we recall from Example 2.10 that action was

$$S = -\tfrac{1}{4}\int dt\, n(t)^{-1} G^{MP} G^{NQ} \partial G_{MN} \partial G_{PQ}, \tag{4.57}$$

and the metric G_{MN} was constrained to have unit determinant. The capital indices run from one to three. It turns out that if we use the Ansatz

$$G_{MN} = \frac{1}{\det(g_{mn})^{1/3}} \begin{pmatrix} g_{mn} + A_m A_n & A_m \\ A_n & 1 \end{pmatrix} \tag{4.58}$$

the action (4.57) exactly reduces to (4.56), just as expected.

5

Supergravity

In the previous chapters we have described the generic theory and structure of Kac-Moody algebras. In this chapter we will treat a class of theories known as *supergravities* in which these infinite-dimensional symmetries appear. For an introduction to supergravity, see for example the lecture notes [89] and [31].

Supergravities are constructed by requiring that supersymmetry is a local symmetry of the theory. Very roughly, supersymmetry transforms bosons into fermions and vice-versa:

$$\delta(\text{boson}) = \text{fermion}, \tag{5.1a}$$

$$\delta(\text{fermion}) = \text{translated boson}. \tag{5.1b}$$

Because supersymmetry is required to hold locally, spacetime translations must also be a local symmetry of the theory. We therefore have diffeomorphism invariance, and expect the theory to contain gravity. Hence the name supergravity, a term which was coined even before the first theories were constructed [32].

The underlying symmetry of supergravity is captured in its superalgebra. A *superalgebra* $\mathfrak{g}$ is a $\mathbb{Z}_2$-graded generalization of a Lie algebra. It splits up in two subalgebras, $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0$ is the even (bosonic) subalgebra and $\mathfrak{g}_1$ the odd (fermionic) subalgebra. The Lie bracket of superalgebras is generalized from (2.1), and reads

$$[x,y] = -[y,x] \subseteq \mathfrak{g}_0 \qquad \forall\, x,y \in \mathfrak{g}_0, \tag{5.2a}$$
$$\{x,y\} = +\{y,x\} \subseteq \mathfrak{g}_0 \qquad \forall\, x,y \in \mathfrak{g}_1, \tag{5.2b}$$
$$[x,y\} = -\{y,x] \subseteq \mathfrak{g}_1 \qquad \forall\, x \in \mathfrak{g}_0, y \in \mathfrak{g}_1. \tag{5.2c}$$

The bosonic subalgebra $\mathfrak{g}_0$ thus constitutes an ordinary Lie algebra. Furthermore, by equation (5.2c) the fermionic subalgebra $\mathfrak{g}_1$ carries a representation of $\mathfrak{g}_0$.

The superalgebra of supergravities is an extension of the usual Poincaré algebra. Its bosonic subalgebra thus always contains Lorentz generators $M_{\mu\nu}$ and the generator of translations P_μ. The fermionic subalgebra contains a number of supersymmetry generators Q^i_α, which come in spinor representations of the Lorentz algebra. The index i runs from 1 to $\mathcal{N}$, the number of supersymmetry generators. The number of *supercharges*, Q, is then defined as the total number of components of all supersymmetry generators,

$$Q = \mathcal{N} q \leq 32. \tag{5.3}$$

Here q is the number of components (i.e. the dimension) of a single irreducible spinor representation. If $Q > 32$, acting with supersymmetry generators on the graviton (which has spin 2) will give rise to fields with spin greater than two [65]. It is thought that these fields cannot consistently couple to themselves or other fields, and hence theories with $Q > 32$ are discarded. Supergravities that satisfy the bound in (5.3) are called *maximal* supergravities.

As is clear from Table 5.1, $D = 11$ is the maximum number of space-time dimensions for a supergravity. We will take this theory as a starting point to see how Kac-Moody symmetries emerge in it.

5.1 Maximal supergravity

Supergravity in eleven dimensions consists of the metric $g_{\mu\nu}$, a rank three antisymmetric gauge field $A_{\mu\nu\rho}$ (both bosons), and the gravitino ψ_μ (a fermion) [21]. The latter carries 128 degrees of freedom, the same as the metric and 3-form combined (44 and 84 d.o.f., respectively). In form notation, the bosonic part of the action reads

$$S = \int \star R - \frac{1}{2} \star \mathrm{d}A_3 \wedge \mathrm{d}A_3 - \frac{1}{6} \mathrm{d}A_3 \wedge \mathrm{d}A_3 \wedge A_3, \tag{5.4}$$

where A_3 is the usual form-shorthand for $\frac{1}{3!} A_{\mu\nu\rho} \mathrm{d}x^\mu \mathrm{d}x^\nu \mathrm{d}x^\rho$. This action carries no global symmetry. When it is reduced on an n-torus, one expects the resulting lower-dimensional supergravity theory to have at least a global $GL(n)$ symmetry,

D	q	$\mathcal{N}_{\max}$
2	1	32
3	2	16
4	4	8
5	8	4
6	8	4
7	16	2
8	16	2
9	16	2
10	16	2
11	32	1
12	64	0

Table 5.1: The number of components of an irreducible spinor (q) in Lorentzian space-times of dimension D, and the corresponding maximum number of supersymmetry generators ($\mathcal{N}_{\max}$). Modified from [31].

which is the symmetry group of the torus. In fact, it turns out that the actual global symmetry is enhanced to the exceptional E_n series [19, 20]. The corresponding groups are given in Table 5.2, and their Coxeter projections in Figure 5.1.

The 128 degrees of freedom of the $D = 11$ theory get upon reduction redistributed over the metric, a set of scalars, vectors, and higher-rank p-forms (see also Table 5.3). The generic form of the lower-dimensional Lagrangian is

$$S = \int \star R - \frac{1}{2} G_{\alpha\beta}(\phi) \star \mathrm{d}\phi^\alpha \wedge \mathrm{d}\phi^\beta - \frac{1}{2} M_{MN}(\phi) \star \mathrm{d}A_1^M \wedge \mathrm{d}A_1^N - \cdots , \tag{5.5}$$

with additional contributions of higher-rank p-forms. The scalars ϕ^α are described by a sigma model on the coset space $G/K(G)$, and thus transform non-linearly under the global symmetry group G. The p-forms carry a linear representation of G. For the vector fields this is always the fundamental representation of G, whereas the higher-rank p-forms transform in representations that are formed by taking appropriate tensor products of the fundamental representation.

The full G symmetry is only manifest when all the p-forms have been dualized by means of Hodge duality to their lowest possible rank. Standard Hodge duality relates p-forms to $(D - p - 2)$-forms through their field strengths. For abelian field strengths, it reads for example

$$\star \, \mathrm{d}A_p = \mathrm{d}A_{D-p-2}. \tag{5.6}$$

So after dualizing all p-forms to their lower-rank duals, the action contains p-forms whose rank is at most $\lfloor \frac{D-2}{2} \rfloor$, where the brackets denote the integral part. In even

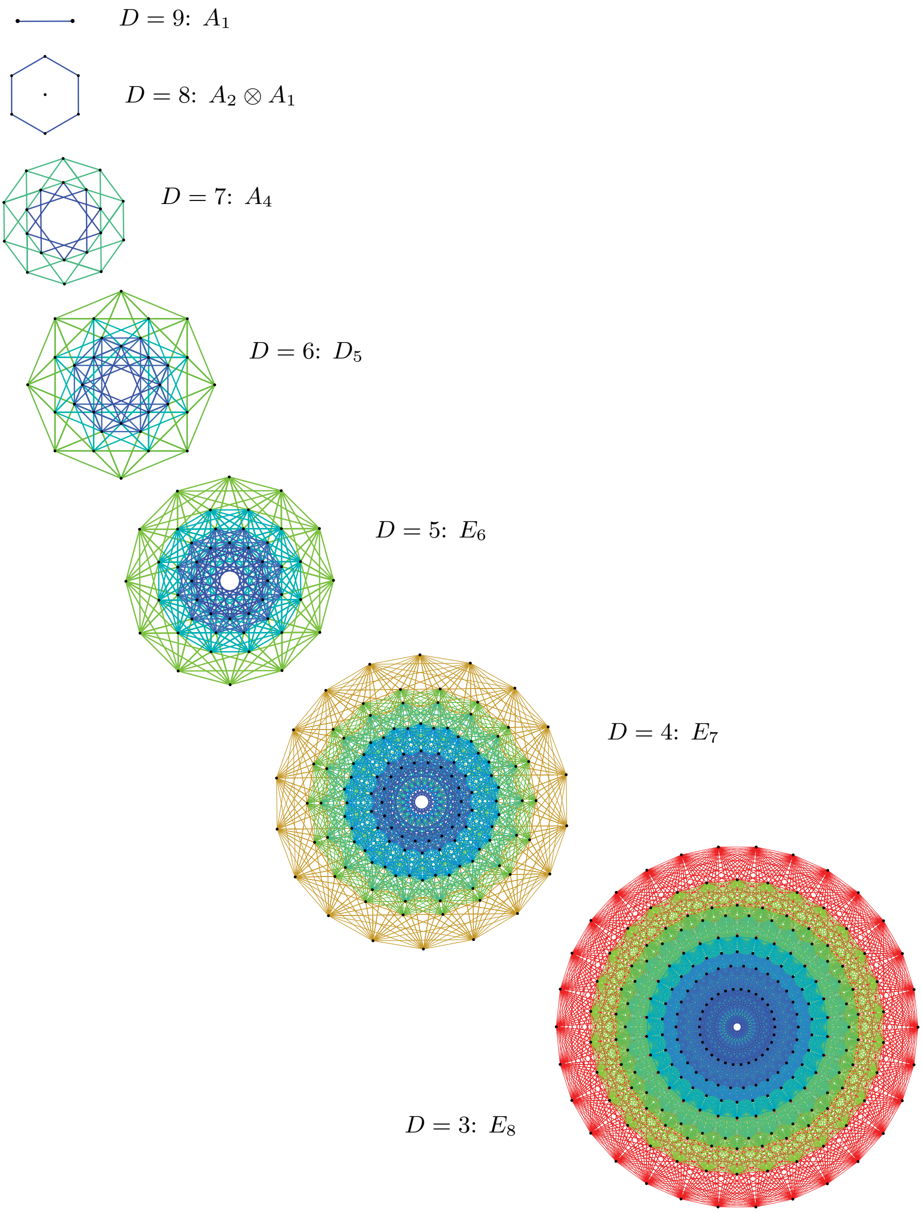


Figure 5.1: Coxeter projections of the hidden duality groups of maximal supergravity in various dimensions.

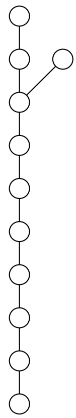
D	G	$K(G)$	$\dim(G,$	$K,$	$G/K)$	diagram
11	1	1	0	0	0	
10	$\mathbb{R}^+$	1	1	0	1	
9	$GL(2)$	$SO(2)$	3	1	2	
8	$SL(3) \times SL(2)$	$SO(3) \times SO(2)$	11	4	7	
7	$SL(5)$	$SO(5)$	24	10	14	
6	$SO(5,5)$	$SO(5) \times SO(5)$	45	20	25	
5	E_6	$USp(8)$	78	36	42	
4	E_7	$SU(8)$	133	63	70	
3	E_8	$SO(16)$	248	120	128	
2	E_9	$K(E_9)$	∞	∞	∞	
1	E_{10} ?	$K(E_{10})$	∞	∞	∞	
	E_{11} ?	$K(E_{11})$	∞	∞	∞	

Table 5.2: The global symmetry group G and its maximal compact subgroup $K(G)$ of maximal supergravity in D dimensions, as obtained from dimensional reduction. The group G is always over the real numbers and a split real form. The Dynkin diagram for a particular dimension contains the node(s) on the same horizontal axis, and all nodes above. Note that the $D = 10$ listed here is the IIA supergravity; in $D = 10$ there is also the IIB supergravity theory that has a global $SL(2)$ symmetry. The IIB theory cannot be obtained from $D = 11$.

dimensions there can be a subtlety involving self-dual p-forms for which a Hodge dualization does not change the rank. In those cases the full G symmetry cannot be realized on the level of the action, but only on the equations of motion.

It is worth noting that there exists a formulation of supergravity in which all the p-forms appear with their duals [9]. This is known as the *democratic formulation* of supergravity. The highest rank forms in that case are the duals of scalars, which are $(D-2)$-forms.

Three is the lowest number of space-time dimensions for which the global symmetry group is still finite. In that case it is E_8 [48, 61], the largest finite exceptional Lie group, and the dimension of the coset space $E_8/SO(16)$ is exactly 128. This matches the fact that all the degrees of freedom are contained in the scalars, as the metric carries no propagating degrees of freedom in three dimensions.

In two dimensions the global symmetry group is enlarged to the affine extension of the three-dimensional group. This was first noted for plain general relativity [38, 37], but it also holds for supergravity. For maximal supergravity the two-dimensional

D	$g_{\mu\nu}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$
11	$\mathbf{44}\times\mathbf{1}$				$\mathbf{84}\times\mathbf{1}$	
IIA	$\mathbf{35}\times\mathbf{1}$	$\mathbf{1}\times 1$	$\mathbf{8}\times\mathbf{1}$	$\mathbf{28}\times\mathbf{1}$	$\mathbf{56}\times\mathbf{1}$	
IIB	$\mathbf{35}\times\mathbf{1}$	$\mathbf{1}\times 2$		$\mathbf{28}\times\mathbf{2}$		$\mathbf{70}\times\frac{1}{2}\ \mathbf{1}$
9	$\mathbf{27}\times\mathbf{1}$	$\mathbf{1}\times 3$	$\mathbf{7}\times(\mathbf{1}+\mathbf{2})$	$\mathbf{21}\times\mathbf{2}$	$\mathbf{35}\times\mathbf{1}$	
8	$\mathbf{20}\times(\mathbf{1{,}1})$	$\mathbf{1}\times 7$	$\mathbf{6}\times(\mathbf{3{,}2})$	$\mathbf{15}\times(\mathbf{3{,}1})$	$\mathbf{20}\times\frac{1}{2}\ (\mathbf{1{,}2})$	
7	$\mathbf{14}\times\mathbf{1}$	$\mathbf{1}\times 14$	$\mathbf{5}\times\mathbf{10}$	$\mathbf{10}\times\mathbf{5}$		
6	$\mathbf{9}\times\mathbf{1}$	$\mathbf{1}\times 25$	$\mathbf{4}\times\mathbf{16}$	$\mathbf{6}\times\frac{1}{2}\ \mathbf{10}$		
5	$\mathbf{5}\times\mathbf{1}$	$\mathbf{1}\times 42$	$\mathbf{3}\times\mathbf{27}$			
4	$\mathbf{2}\times\mathbf{1}$	$\mathbf{1}\times 70$	$\mathbf{2}\times\frac{1}{2}\ \mathbf{56}$			
3	$-\times\mathbf{1}$	$\mathbf{1}\times 128$				

Table 5.3: The physical states of all $3 \leq D \leq 11$ maximal supergravities. The p-columns indicate which p-form potentials are present. All entries except $p = 0$ are of the form 'physical d.o.f. $\times$ G representation,' where G is the duality group. For $p = 0$ the entries read 'physical d.o.f. $\times$ number of scalars.' The factor $\frac{1}{2}$ in $D = 4, 6, 8$ ensures the correct counting of self-dual p-forms.

symmetry becomes $E_9 = E_8^+$ [49, 66, 69], the affine extension of E_8.

Seeing as the rank of the group G increases with every step in the reduction, it is tempting to conjecture that this also happens for the reduction to one space-time dimension. The global symmetry group would then be G^{++}, the over-extension of the $D = 3$ group. In the case of maximal supergravity, this would be $E_{10} = E_8^{++}$ [49, 63]. Taking the conjecture one step further, one can formally reduce to zero dimensions and hope to obtain the very-extension $E_{11} = E_8^{+++}$ [88, 87], which is conjectured to be a symmetry of M-theory.

We will see later in chapter 6 how pieces of the conjectured E_{10} and E_{11} Kac-Moody symmetries nicely tie in with the structure of supergravity. Surprisingly, it turns out the Kac-Moody algebras encode information on possible gauge deformations of the supergravity theory. This is remarkable, as the ungauged theory was the starting point for the conjecture. We will discuss gaugings of supergravity in section 5.3, but first we will briefly touch upon half-maximal supergravity.

5.2 Half-maximal supergravity

Supergravities that have 16 superchargers are called *half-maximal* supergravities. From Table 5.1 one easily deduces that the maximum number of spacetime dimensions is 10 for a half-maximal supergravity. In contrast to maximal supergravity, (ungauged) half-maximal supergravity is not unique in its highest dimension, nor in lower dimensions. Besides the usual graviton multiplet, one can add an arbitrary

D	Mult.	$p=0$	$p=1$	$p=2$
10	GV^n	$\mathbf{1}\times 1$	$\mathbf{8}\times \boldsymbol{n}$	$\mathbf{28}\times \mathbf{1}$
9	GV^{n+1}	$\mathbf{1}\times \Big(1+(1n+1)\Big)$	$\mathbf{7}\times(\boldsymbol{n}+\mathbf{2})$	$\mathbf{21}\times \mathbf{1}$
8	GV^{n+2}	$\mathbf{1}\times \Big(1+(2n+4)\Big)$	$\mathbf{6}\times(\boldsymbol{n}+\mathbf{4})$	$\mathbf{15}\times \mathbf{1}$
7	GV^{n+3}	$\mathbf{1}\times \Big(1+(3n+9)\Big)$	$\mathbf{5}\times(\boldsymbol{n}+\mathbf{6})$	$\mathbf{10}\times \mathbf{1}$
6a	GV^{n+4}	$\mathbf{1}\times \Big(1+(4n+16)\Big)$	$\mathbf{4}\times(\boldsymbol{n}+\mathbf{8})$	$\mathbf{6}\times \mathbf{1}$
6b	GT^{n+4}	$\mathbf{1}\times (5n+25)$		$\mathbf{6}\times \frac{1}{2}(\mathbf{10}+\boldsymbol{n})$
5	GV^{n+5}	$\mathbf{1}\times \Big(1+(5n+25)\Big)$	$\mathbf{3}\times(\boldsymbol{n}+\mathbf{11})$	
4	GV^{n+6}	$\mathbf{1}\times \Big((1,2)+(6n+36,1)\Big)$	$\mathbf{2}\times(\boldsymbol{n}+\mathbf{12})$	
3	GV^{n+7}	$\mathbf{1}\times (8n+64)$		

Table 5.4: The physical states apart from the graviton of all $D=10-m$ half-maximal supergravities coupled to $m+n$ vector multiplets. The multiplet structures are also given: G is the graviton multiplet, V the vector multiplet and T the self-dual tensor multiplet. The physical states of the graviton can be found in Table 5.3.

number of vector multiplets to the theory. We will describe both these multiplets in turn.

The bosonic part of the graviton multiplet of D-dimensional half-maximal supergravity consists of a metric, m vector gauge fields (with $m=10-D$), a two-form gauge field and a single scalar which is the dilaton. It has a global $SO(m)$ symmetry, under which the vectors transform in the fundamental representation.

The other possible multiplet in generic dimensions is the vector multiplet, which contains a vector and m scalars. The effect of adding $m+n$ vector multiplets is to enlarge the symmetry group from $SO(m)$ to $SO(m,m+n)$. The scalars parameterize the corresponding scalar coset while the vectors transform in the fundamental representation. In four dimensions the symmetry becomes $SL(2,\mathbb{R})\times SO(6,6+n)$ while in three dimensions it is given by $SO(8,8+n)$. In the latter case there is symmetry enhancement due to the equivalence between scalars and vectors. The entire theory can be described in terms of the corresponding scalar coset (coupled to gravity).

The above multiplets belong to non-chiral half-maximal supergravity and are the correct and complete story in generic dimensions. In six dimensions, however, the half-maximal theory can be chiral or non-chiral, similar to the maximal theory in ten dimensions. The non-chiral theory is denoted by $D=6a$ and follows the above pattern. The chiral theory, $D=6b$, instead has different multiplets. In particular, the graviton multiplet contains the graviton, five scalars and five self-dual plus one anti-self-dual two-form gauge fields. The global symmetry is given by $SO(5,1)$. The other possible multiplet is that of the tensor, which contains an anti-self-dual two-form and five scalars. Adding $4+n$ of such tensor multiplets to the graviton multiplet enhances the symmetry to $SO(5,5+n)$.

Upon dimensional reduction over a circle, the graviton multiplet splits up into a graviton multiplet plus a vector multiplet. A vector (or tensor) multiplet reduces to a vector multiplet in the lower dimensions. This was the reason for adding $m+n$ instead of n vector or tensor multiplets in any dimension; it can easily be seen that n remains invariant under dimensional reduction. That is, a theory with a certain value of n reduces to a theory with the same value of n in lower dimensions.

The bosonic physical states of the various half-maximal supergravities have been listed in Table 5.4.

5.3 Gaugings

In this section we will quickly review gauge deformations of supergravity theories. For a more in-depth review, see [80]. Besides the global symmetry G described in the previous section, ungauged supergravities have a local symmetry $U(1)^n$. Here n is the number of vector fields of the supergravity. This local symmetry corresponds to the abelian gauge symmetry of the vector fields:

$$\delta A_1^M = \mathrm{d}\Lambda_0^M, \tag{5.7}$$

where $\Lambda_0^M = \Lambda_0^M(x)$ is a coordinate-dependent 0-form parameter.

A *gauging* or *gauge deformation* of the supergravity turns this abelian local symmetry into a non-abelian local symmetry. Another point of view is that it promotes a subgroup $G_0 \in G$ to a local symmetry. This is achieved by introducing covariant derivatives

$$\partial_\mu \longrightarrow D_\mu = \partial_\mu - gA_\mu^M X_M, \tag{5.8}$$

where g is the gauge coupling constant, and X_M are the generators of the subgroup G_0. All the possible choices of a consistent gauge group G_0 are best described in an object that is known as the embedding tensor [68, 67, 94, 95, 91].

5.3.1 The embedding tensor

Because the gauge group G_0 is embedded within G, its generators are a linear combination of the generators t_α of G:

$$X_M \equiv \Theta_M{}^\alpha t_\alpha. \tag{5.9}$$

The constant object $\Theta_M{}^\alpha$ is called the *embedding tensor*, and encodes the gauging completely.

D	Linear constraint	Quadratic constraint
7	$\mathbf{15} \oplus \mathbf{40}$	$\mathbf{5} \oplus \mathbf{45} \oplus \mathbf{70}$
6	$\mathbf{144}$	$\mathbf{10} \oplus \mathbf{126} \oplus \mathbf{320}$
5	$\mathbf{351}$	$\mathbf{27} \oplus \mathbf{1728}$
4	$\mathbf{912}$	$\mathbf{133} \oplus \mathbf{8645}$
3	$\mathbf{1} \oplus \mathbf{3875}$	$\mathbf{3875} \oplus \mathbf{147250}$

Table 5.5: The linear and quadratic constraint representations in various dimensions (D). Adapted from [90].

The embedding tensor is a tensor with two indices that both take values in representations of G. The first index, M, takes values in the fundamental representation $\mathcal{R}_{\text{fund}}$, and the second, α, in the adjoint representation $\mathcal{R}_{\text{adj}}$. The embedding tensor therefore 'lives' in the tensor product of these two representations,

$$\Theta \in \mathcal{R}_{\text{fund}} \otimes \mathcal{R}_{\text{adj}}. \tag{5.10}$$

If we take $D = 4$ maximal supergravity as a concrete example, the embedding tensor a priori can take values in

$$\Theta \in \mathbf{56} \otimes \mathbf{133} = \mathbf{56} \oplus \mathbf{912} \oplus \mathbf{6480}. \tag{5.11}$$

However, there are two sets of constraints that restrict the possible values of the embedding tensor. The first comes from demanding that the gauging is consistent with supersymmetry, although in almost all cases it can also be derived from purely bosonic arguments [93]. Either way, it poses a linear restriction on Θ, and is therefore called the *linear constraint*. Say $\mathbb{P}_{\text{lin}}$ is an operator that projects onto the forbidden components of Θ (i.e. those components that are inconsistent with supersymmetry). Then the linear constraint can be written as

$$\mathbb{P}_{\text{lin}}\, \Theta_M{}^\alpha = 0. \tag{5.12}$$

The linear constraint is usually indicated by the allowed representations of the embedding tensor. By the above equation, they live in the kernel of $\mathbb{P}_{\text{lin}}$. For $D = 4$ maximal supergravity, the linear constraint leaves only the **912** representation:

$$\Theta \in \mathbf{56} \otimes \mathbf{133} = \cancel{\mathbf{56}} \oplus \mathbf{912} \oplus \cancel{\mathbf{6480}}. \tag{5.13}$$

The linear constraint has to be worked out on a case-by-case basis, and some results are collected in Table 5.5.

The second constraint restricting the possible values of the embedding tensor even further comes from demanding that the gauge algebra is self-consistent. In particular Θ must be gauge invariant, or equivalently, the generators X_M must close into an algebra,

$$[X_M, X_N] = -\Theta_M{}^\alpha (t_\alpha)_N{}^P X_P. \tag{5.14}$$

This poses restrictions on the symmetric tensor product of Θ, and is therefore called the *quadratic constraint*. Again, say $\mathbb{P}_{\text{quad}}$ is an operator that projects onto the components of $\Theta_M{}^\alpha \Theta_N{}^\beta$ that are inconsistent with the quadratic constraint. The constraint itself can then be written as

$$\mathbb{P}_{\text{quad}} \left(\Theta_M{}^\alpha \Theta_N{}^\beta \right) = 0. \tag{5.15}$$

Therefore the allowed components of the symmetric tensor product of Θ live in the kernel of $\mathbb{P}_{\text{quad}}$. In contrast to the linear constraint, the quadratic constraint is usually labeled by the representations in the complement of the kernel, i.e. those components that are *not* allowed. For $D = 4$ maximal supergravity, the quadratic constraint evaluates with the help of LiE [60] to

$$\Theta \otimes_s \Theta \in \mathbf{912} \otimes_s \mathbf{912} = \cancel{\mathbf{133}} \oplus \mathbf{1463} \oplus \cancel{\mathbf{8645}} \oplus \mathbf{152152} \oplus \mathbf{253935}, \tag{5.16}$$

where the **133** and **8645** representations constitute the components of the symmetric tensor product that are not allowed. The analysis of the components of the quadratic constraints is a purely group-theoretical problem, and some results are listed in Table 5.5.

Note that the embedding tensor formalism relies on the presence of vectors fields. In $D = 3$ they are absent, as the only physical fields are scalars. In order to proceed, one introduces vectors fields that are not independent but are related to the scalars by a duality relation [68].

5.3.2 The tensor hierarchy

The implementation of the linear and quadratic constraints are necessary for a consistently gauged supergravity. However, they are not sufficient. The introduction of the embedding tensor also forces one to introduce a whole tower of anti-symmetric p-forms that transform in specific representations of the global symmetry group G. This tower is called the *p-form hierarchy* [90, 92], and we will briefly describe it below.

Besides the gauge vectors A_1^M transforming the fundamental representation of G, the embedding tensor also requires the existence of higher-rank p-forms. The whole set of p-forms can be denoted by

$$A_1^M,\ A_2^{\lceil MN \rfloor},\ A_3^{\lceil M \lceil NP \rfloor \rfloor},\ A_4^{\lceil M \lceil N \lceil PQ \rfloor \rfloor \rfloor},\ A_5^{\lceil M \lceil N \lceil P \lceil QR \rfloor \rfloor \rfloor \rfloor}, \ldots \tag{5.17}$$

The tower truncates at D-forms, which are the anti-symmetric forms whose rank is the highest possible in D dimensions. The G representations in which the p-forms live are tensor products of the fundamental representation $\mathcal{R}_{\text{fund}}$. For instance, the 2-form carries indices MN that reside in the first symmetrized tensor product of the fundamental representation:

$$MN : \mathcal{R}_{\text{fund}} \otimes \mathcal{R}_{\text{fund}}. \tag{5.18}$$

The extra brackets $\lceil MN \rfloor$ indicate that the two-form does not take values in the whole tensor product. Rather, it lives in a restricted subspace defined by some projector $\mathbb{P}_2$ whose precise form depends on the embedding tensor,

$$\lceil MN \rfloor : \mathbb{P}_2 \left(\mathcal{R}_{\text{fund}} \otimes \mathcal{R}_{\text{fund}}\right) \equiv \mathcal{R}_{\text{fund}}^{\otimes 2}. \tag{5.19}$$

Here we have defined the representation $\mathcal{R}_{\text{fund}}^{\otimes p}$ to be the representation of the p-form in question. The same holds for the higher-rank p-forms: their G representation can be evaluated by taking the tensor product of the fundamental representation with the $(p-1)$-representation, and projecting with the correct operator:

$$\mathcal{R}_{\text{fund}}^{\otimes p} = \mathbb{P}_p \left(\mathcal{R}_{\text{fund}}^{\otimes (p-1)} \otimes \mathcal{R}_{\text{fund}}\right). \tag{5.20}$$

It turns out that if the various p-forms in the hierarchy are related by Hodge-duality, then their G representations are each other's conjugate. Thus the $(D-2)$-forms transform in the conjugate adjoint representation, the $(D-3)$-forms in the conjugate fundamental representation, and so on and so forth. Furthermore, for these forms one finds exactly the same representation as one would find in a supersymmetry analysis (i.e. Table 5.3 plus their Hodge duals).

The $(D-1)$- and D-forms are special in the sense that they do not have a Hodge dual. Furthermore, they do not carry any propagating degrees of freedom. For those two p-forms at the end of the hierarchy one finds that they transform in the representation conjugate to the embedding tensor and conjugate to its quadratic constraint, respectively. This suggest they may be employed as Lagrange multipliers, enforcing the constancy of Θ and the quadratic constraint [90, 96]:

$$\mathcal{L}_{\text{constraints}} = g A^{M}_{(D-1)\alpha} \wedge Q_M{}^{\alpha} + g^2 A_D^{MN}{}_{\alpha\beta} \wedge Q_{MN}{}^{\alpha\beta}, \tag{5.21}$$

where

$$Q_M{}^{\alpha} = D_\mu \Theta_M{}^{\alpha} \mathrm{d}x^\mu, \tag{5.22a}$$

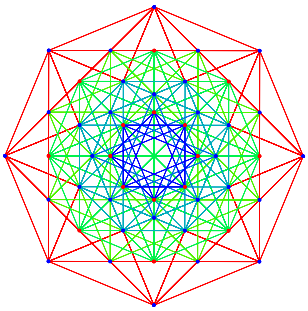
$$Q_{MN}{}^{\alpha\beta} = \mathbb{P}_{\text{quad}} \left(\Theta_M{}^{\alpha} \Theta_N{}^{\beta}\right). \tag{5.22b}$$

Upon adding $\mathcal{L}_{\text{constraints}}$ to the action and varying with respect to $A^{M}_{(D-1)\alpha}$, we find that Θ should be a constant tensor. The linear constraint (5.12) is enforced automatically by only letting those components of Θ enter the action that satisfy it. Varying with respect to $A_D^{MN}{}_{\alpha\beta}$ tells us that Θ should also satisfy the quadratic constraint (5.15).

Because the $(D-1)$-forms couple to the embedding tensor, we will call them *deformation potentials*. And, as the D-forms have the highest rank possible, we will call them *top-form potentials* [1].

The remarkable property of the Kac-Moody algebra E_{11}, and to a lesser extend E_{10}, is that it encodes all the representations of all the p-forms in a unified manner. How this comes about exactly, we will see in the next chapter.

6



The comparison

In the previous chapter we described the spectrum of states of maximal and half-maximal supergravity. In section 6.1 we will see how they can be obtained from the Kac-Moody algebras E_{11} and D_8^{+++}, respectively. In section 6.3 we will try to compare the equations of motion of supergravity to those obtained from a non-linear sigma model of a Kac-Moody algebra. Note that for both the kinematical and dynamical analysis we will only treat bosons; fermions have been discussed in for example [28, 17, 26].

6.1 Kinematics

It was shown in [87, 81, 57] that the spectrum of physical states of the different maximal supergravity theories can be obtained from the very extended Kac-Moody algebra E_{11}. This has been extended to the set of all possible deformation and top-form potentials in [77, 1]. A similar analysis could be done for E_{10} [70, 23, 24] except for the top-form potentials. In addition, non-maximal supergravity and the associated very extended Kac-Moody algebras have been discussed in [57, 82, 2, 73, 56, 44]. We will first review how the the kinematics of maximal supergravity can be obtained, and in subsection 6.1.2 switch to half-maximal supergravity.

6.1.1 Maximal supergravity

The key idea is to decompose E_{11} with respect to subgroups that match the symmetry structure of maximal supergravity in various dimensions. The subalgebra representations resulting from this level decomposition can then be matched with the various supergravity fields. The valid decompositions for E_{11} are always of the type $G \otimes GL(D)$, where G is the duality group in D dimensions and $GL(D)$ refers to the space-time symmetries. Dynkin diagrams are a useful tool to visualize these group decompositions: the decompositions correspond to 'deleting' certain nodes of the diagram in order to obtain two disjoint parts. One of these disjoint parts is the G duality group, and the other is the $SL(D) \otimes \mathbb{R}^+ = GL(D)$ *gravity line*. The extra factor of $\mathbb{R}^+$ for the gravity line comes from the Cartan subalgebra generator of the deleted node. Furthermore, the gravity line must always include the very-extended node. All the valid $3 \leq D \leq 11$ decompositions of E_{11} are listed in Table 6.1. Note that the duality group G contains an extra $\mathbb{R}^+$ factor whenever there is a second disabled node, stemming from the Cartan subalgebra generator of the disabled node. This explains why the duality group of IIA supergravity is $\mathbb{R}^+$ and why those of IIB and $D = 11$ supergravity do not have such a factor.

After the decomposition has been fixed, the generators of E_{11} can be analyzed by means of a level decomposition (see also section 4.3). As the actual level decomposition is quite cumbersome to do by hand, we have written a computer program called SimpLie [5] to do the job. For more on SimpLie, see Appendix A.

The explicit results of the level decompositions of Table 6.1 can be found in section C.1. The spectrum is obtained by associating to each generator a supergravity field in the same representation. This leads to the following fields at each level. At the lowest levels the physical states of the supergravity we started out with (see Table 5.3) appear together with their duals. More precisely: corresponding to any p-form generator we also find a $(D-p-2)$-form. In addition there is a $(D-3,1)$-form with mixed symmetries and possibly $(D-2)$-form generators, which are interpreted as the dual graviton [45, 87, 14, 10] and dual scalars, respectively. The duality relations themselves are not reproduced by the level decomposition: in the absence of dynamics these relations have to be imposed by hand. Beyond the level of the dual graviton we find deformation potentials and top-forms, i.e. $(D-1)$- and D-forms respectively. At the same levels as the deformation potentials and top-forms, and at higher levels still, there are the so-called 'exotic' generators. These exotic generators have a space-time symmetric structure that does not have an obvious counterpart in supergravity. Some of them may be interpreted as infinitely many exotic dual copies of the previously mentioned fields [75, 73]. A schematic representation of the level decomposition and the resulting tower of (physical) states can be found in Figure 6.1.

The $(D-1)$- and D-forms found from E_{11} are listed in Table 6.2. The interesting thing is that they exactly match the representations of the embedding tensor and its quadratic constraint as given in Table 5.5, save for an extra **248** representation in

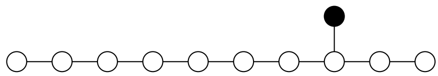
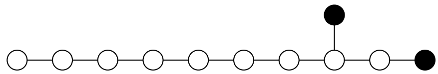
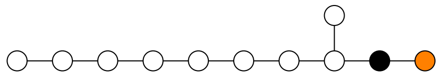
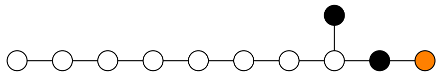
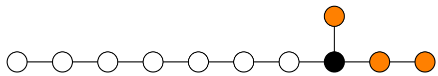
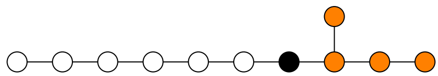
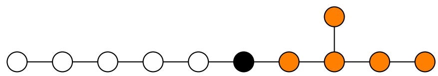
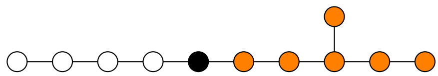
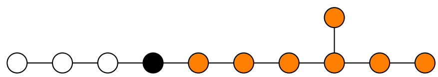
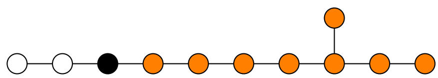
D	G	Grav. line	E_{11} decomposition
11	1	$GL(11)$	
IIA	$\mathbb{R}^+$	$GL(10)$	
IIB	$SL(2)$	$GL(10)$	
9	$GL(2)$	$GL(9)$	
8	$SL(3) \times SL(2)$	$GL(8)$	
7	$SL(5)$	$GL(7)$	
6	$SO(5,5)$	$GL(6)$	
5	E_6	$GL(5)$	
4	E_7	$GL(4)$	
3	E_8	$GL(3)$	

Table 6.1: Global symmetry groups G of all $3 \leq D \leq 11$ maximal supergravities embedded in E_{11}. The groups G can be read off from the decomposition of the Dynkin diagram of E_{11}; they correspond to the orange nodes. The white nodes together with one scaling generator from a deleted node form the gravity line $A_{D-1} \times \mathbb{R}^+ = GL(D)$.

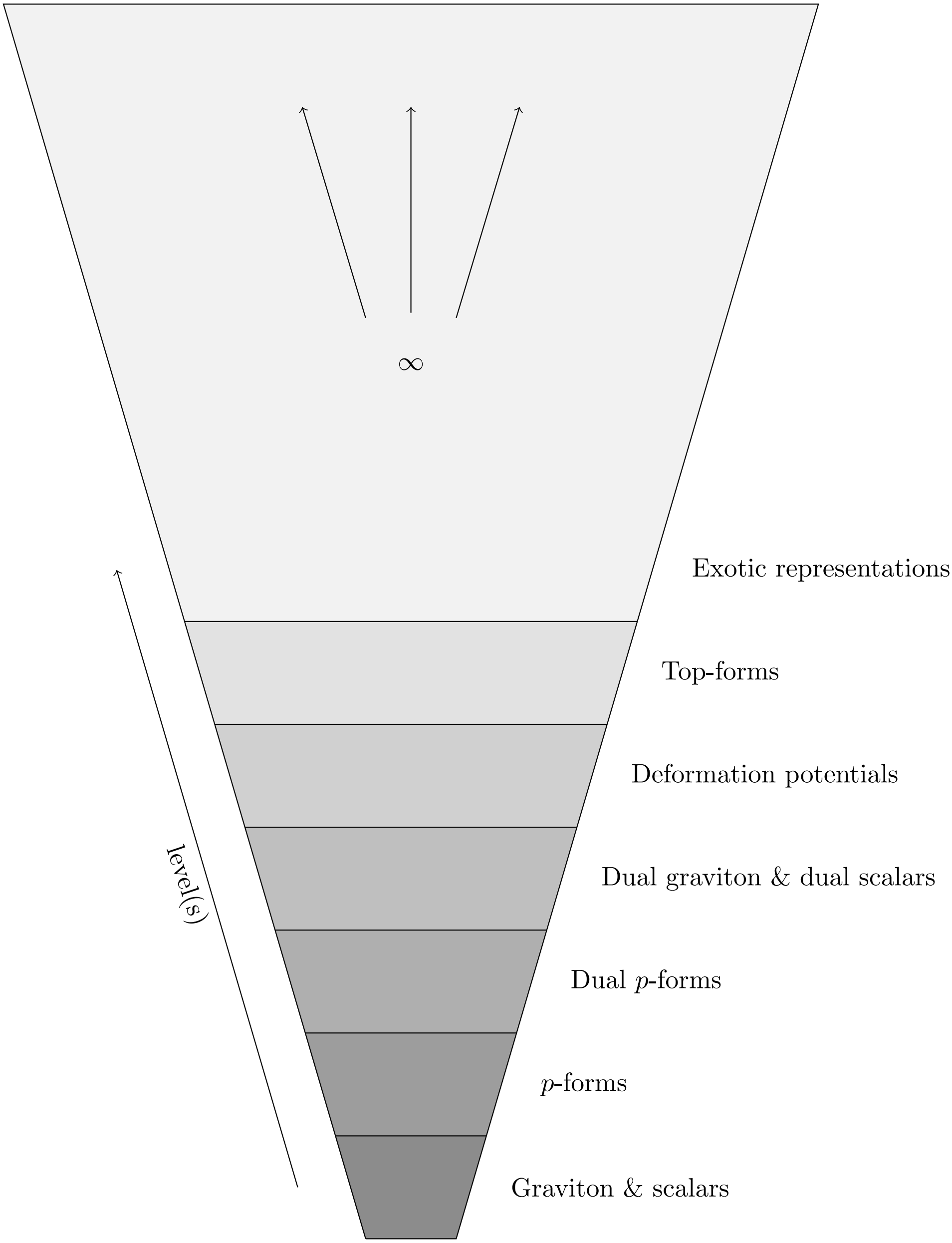


Figure 6.1: Schematic Hasse diagram of E_{11} containing the hierarchy of subalgebra representations common to all decompositions.

D	$(D-1)$-forms $p=1$	$p=2$	D-forms
IIA		$\mathbf{1}$	$\mathbf{1}\oplus\mathbf{1}$
IIB			$\mathbf{2}\oplus\mathbf{4}$
9	$\mathbf{2}\oplus\mathbf{3}$		$\mathbf{2}\oplus\mathbf{2}\oplus\mathbf{4}$
8	$(\mathbf{3},\mathbf{2})\oplus(\mathbf{6},\mathbf{2})$		$(\mathbf{3},\mathbf{1})\oplus(\mathbf{3},\mathbf{1})\oplus(\mathbf{3},\mathbf{3})\oplus(\mathbf{15},\mathbf{1})$
7	$\mathbf{15}\oplus\mathbf{40}$		$\mathbf{5}\oplus\mathbf{45}\oplus\mathbf{70}$
6	$\mathbf{144}$		$\mathbf{10}\oplus\mathbf{126}\oplus\mathbf{320}$
5	$\mathbf{351}$		$\mathbf{27}\oplus\mathbf{1728}$
4	$\mathbf{912}$		$\mathbf{133}\oplus\mathbf{8645}$
3	$\mathbf{1}\oplus\mathbf{3875}$		$\mathbf{248}\oplus\mathbf{3875}\oplus\mathbf{147250}$

Table 6.2: E_{11} predictions for deformation- and top-forms in all $3 \leq D \leq 10$ maximal supergravities. These are representations of the respective duality groups G given in Table 5.2. For the deformation-forms it is also indicated to which type-p deformation they correspond (see section 6.2).

three dimensions. Furthermore, E_{11} predicts top-forms for IIB supergravity, which were recently reconfirmed to exist in [8]. The single deformation potential for IIA supergravity corresponds to Romans' massive deformation thereof [78]. The fact that Romans' theory is not an ordinary gauged supergravity is reflected in the fact that the deformation potential is of 'type 2', which will be explained in section 6.2.

6.1.2 Half-maximal supergravity

The analogy between gauged supergravity and very-extended Kac-Moody algebras is not limited to maximal supergravity and E_{11}. Half-maximal supergravity [2], coupled to $D-10+n$ vector multiplets, reduce to the scalar coset $SO(8,8+n)/SO(8)\times SO(8+n)$ when reduced to three dimensions. In other words, the relevant groups for supergravity theories with 16 supercharges are the B and D series in the above real form. Of these, only three are of split real form, i.e. maximally non-compact, which are given by $n=-1,0,+1$. These correspond to the split forms of B_7, D_8 and B_8, respectively.

We are interested in the decomposition of the very extensions of these algebras with respect to the possible gravity lines. An exhaustive list of the possibilities for the algebras of real split form is given in table Table 6.3. As can be seen from this table, these correspond to the unique D-dimensional supergravity theory with 16 supercharges coupled to $m+n$ vector multiplets with $m=10-D$. The corresponding duality groups G in D dimensions are also given in Table 6.3. Note that there is no second disabled node and therefore no $\mathbb{R}^+$ factor in the duality group for the 6b case and in $D=3,4$.

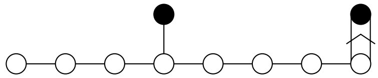
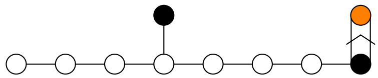
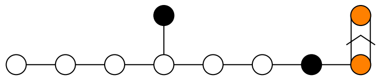
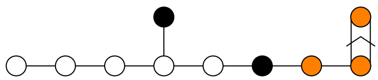
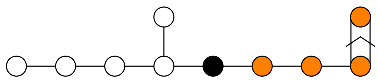
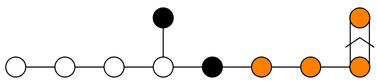
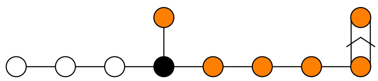
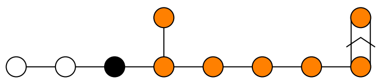
D	G	Multiplets	B_7^{+++} $(n=-1)$
10	$\mathbb{R}^+ \times SO(n)$	GV^n	–
9	$\mathbb{R}^+ \times SO(1,1+n)$	GV^{n+1}	
8	$\mathbb{R}^+ \times SO(2,2+n)$	GV^{n+2}	
7	$\mathbb{R}^+ \times SO(3,3+n)$	GV^{n+3}	
6a	$\mathbb{R}^+ \times SO(4,4+n)$	GV^{n+4}	
6b	$SO(5,5+n)$	GT^{n+4}	
5	$\mathbb{R}^+ \times SO(5,5+n)$	GV^{n+5}	
4	$SL(2) \times SO(6,6+n)$	GV^{n+6}	
3	$SO(8,8+n)$	GV^{n+7}	

Table 6.3: The decompositions of B_7^{+++}, D_8^{+++} and B_8^{+++} with respect to the possible gravity lines. The duality groups G and the multiplet structures (where G is the graviton, V the vector and T the self-dual tensor multiplet) are also given.

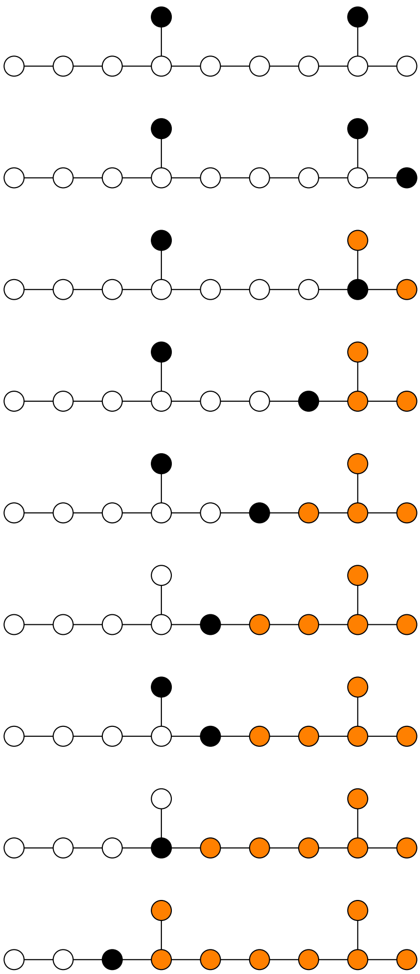
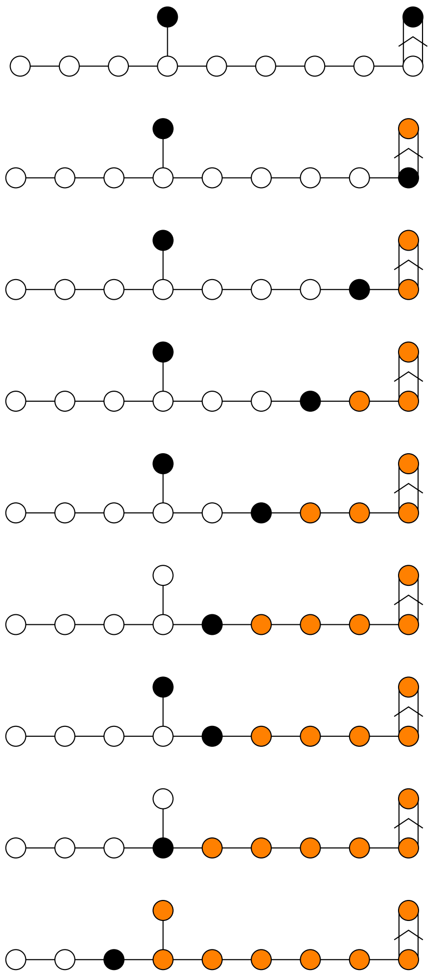
D_8^{+++} $(n = 0)$	B_8^{+++} $(n = 1)$
	

Table 6.3: Continued.

D	$(D-1)$-forms $p=1$,	2,	3.	D-forms constraints on $p=1$	other
10-8	$\ydiagram{1}$ $\ydiagram{1,1,1}$			1 $\ydiagram{1,1}$ $\ydiagram{1,1,1,1}$	
7	$\ydiagram{1}$ $\ydiagram{1,1,1}$		1	1 $\ydiagram{1,1}$ $\ydiagram{1,1,1,1}$	$\ydiagram{1}$
6a	$\ydiagram{1}$ $\ydiagram{1,1,1}$	$\ydiagram{1}$		1 $\ydiagram{1,1}$ $\ydiagram{1,1,1,1}$	1 1 $\ydiagram{2}$ $\ydiagram{1,1}$
6b					$\ydiagram{1}$ $\ydiagram{2,1}$
5	$\ydiagram{1}$ $\ydiagram{1,1}$ $\ydiagram{1,1,1}$			1 $\ydiagram{1}$ $\ydiagram{1}$ $\ydiagram{1,1}$ $\ydiagram{2,1}$ $\ydiagram{1,1,1}$ $\ydiagram{1,1,1,1}$	
4	$(\mathbf{2},\ydiagram{1})$ $(\mathbf{2},\ydiagram{1,1,1})$			$(\mathbf{3},1)$ $(\mathbf{3},\ydiagram{1,1})$ $(\mathbf{3},\ydiagram{1,1,1,1})$ $\ydiagram{1,1}$ $\ydiagram{1,1}$ $\ydiagram{2,1,1}$	
3	1 $\ydiagram{2}$ $\ydiagram{1,1,1,1}$			$\ydiagram{2}$ $\ydiagram{1,1}$ $\ydiagram{1,1,1,1}$ $\ydiagram{2,1,1}$ $\ydiagram{2,1,1,1,1}$	

Table 6.4: The representations of deformation- and top-forms in all half-maximal supergravities. The representations refer to the duality group G given in Table 6.3. We also indicate which type p of deformations they correspond to (see also section 6.2, and to which top-forms one can associate a quadratic constraint on type 1 deformation parameters.

In section C.2 the result of the decomposition of the D_8^{+++} algebras with respect to the different $SL(D)$ subalgebras is given. It can be seen that these decompositions give rise to exactly the physical degrees of freedom [57]. In addition there are the deformation and top-form potentials in the Kac-Moody spectrum. In particular, Table 6.4 summarizes our results for the deformation and top-form potentials for half-maximal supergravity in D dimensions.

Using the embedding tensor approach, an analysis of the linear and quadratic constraints on the possible deformations has been explicitly carried out in $D = 3, 4, 5$ [83]. It turns out that the representations of the quadratic constraints exactly coincide with the representations of the possible top-forms in these dimensions.

6.2 Fundamental p-forms and type-p deformations

Recall from section 4.3 that a level decomposition always induces a grading on the decomposed algebra $\mathfrak{g}$:

$$[\mathfrak{g}_{l_1}, \mathfrak{g}_{l_2}] \subseteq \mathfrak{g}_{l_1+l_2}, \tag{6.1}$$

where, for simplicity, we have assumed there is only one level. This grading implies that all the higher levels can be recovered from the repeated adjoint action of the level one generators on themselves:

$$\underbrace{[\mathfrak{g}_1, \ldots [\mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{g}_1]] \ldots]}_{l \text{ times}} \subseteq \mathfrak{g}_l. \tag{6.2}$$

For the supergravity decompositions of Kac-Moody algebras considered in this chapter, the subalgebra representations at level 1 always correspond to one or more p-forms. We will call these level 1 p-forms *fundamental p-forms.* The fundamental p-forms correspond to the positive simple roots of the disabled nodes in the Kac-Moody algebra. From the decomposed Dynkin diagram one can thus deduce the number and type of these fundamental p-forms: any disabled node connected to the n^{th} node of the gravity line (counting from the very extended node) gives rise to a fundamental $(D-n)$-form. Furthermore, if the disabled node in question is also connected to a node of the duality group the $(D-n)$-form carries a non-trivial representation of the duality group.

The level 1 p-forms are fundamental in the sense that by virtue of (6.2) their commutators generate the other p-forms in the decomposition. Say A_p is a fundamental p-form, and A_q is a p-form that occurs higher in the decomposition. The latter can then be written as

$$\underbrace{[A_p, \ldots [A_p, [A_p, A_p]] \ldots]}_{l \text{ times, } lp=q} \subseteq A_q. \tag{6.3}$$

This allows us to distinguish between deformations potentials that correspond to gauged and massive deformations as follows.

The most familiar class of deformed supergravities are the gauged supergravities. They are special in the sense that the deformations can be seen as the result of gauging a subgroup G_0 of the duality group G (see section 5.3). Not all deformed supergravities can be viewed as gauged supergravities. In the case of maximal supergravity there is one exception: massive IIA supergravity cannot be obtained by gauging the $\mathbb{R}^+$ duality group [78]. The gauged supergravities can be seen as the first in a series of *type-p deformations*. There is a simple criterion that defines to which type of deformation parameter each deformation potential gives rise to. The central observation is that to each $(D-1)$-form A_{D-1} one can associate a unique commutator

$$[A_p, A_{D-p-1}] = A_{D-1}\,, \tag{6.4}$$

where A_p corresponds to a fundamental p-form. The deformation potential corresponding to such a deformation generator gives rise to a type-p deformation parameter.

We observe that each type p deformation is characterized by the fact that a fundamental p-form gauge field becomes massive. For $p = 1$ this leads to gauged supergravities, in which a vector can become massive by absorbing a scalar degree of freedom. Note that other non-fundamental gauge fields may become massive as well. The case $p = 2$ entails a fundamental two-form that becomes massive by 'eating' a vector. The prime example of this is massive IIA supergravity in ten dimensions [78]. Another example is the non-chiral half-maximal supergravity in six dimensions [79]. An example of a $p = 3$ deformation is the half-maximal supergravity theory of [85] where a fundamental three-form potential acquires a topological mass term.

As is evident from Table 6.2 and Table 6.4 the Kac-Moody algebras E_{11} and D_8^{+++} correctly reproduce the type-p deformation potentials for known non-gauged supergravity deformations in ten [78], seven [85], and six [79] dimensions.

6.3 Dynamics

Up to this point we have been concerned with matching the kinematics of (gauged) supergravity and certain Kac-Moody algebras. We will now attempt to take the correspondence one step further, and try to compare the dynamics, i.e. the equations of motion.

6.3.1 E_{10} or E_{11}?

There are two proposals for implementing Kac-Moody symmetries in supergravity theories. In the case of maximal supergravity, one employs the very-extended Kac-Moody algebra E_{11} [88, 87], while the other uses the over-extended E_{10} [23]. Both use a non-linear realization to implement the infinite symmetry. The former approach realizes the E_{11} symmetry directly in the dimension of the corresponding supergravity theory by including an infinite amount of coordinates [76, 74, 72].

In this thesis we will focus on the latter approach, which realizes the E_{10} symmetry by reducing the coordinate dependence to only time. This is inspired by the behavior of gravity near a spacetime singularity, where the spatial points dynamically decouple [7]. The equations of motion in that regime only depend on time, and exhibit chaotic behavior [62] that may be interpreted as taking place in the fundamental Weyl chamber of a hyperbolic Kac-Moody algebra [25, 22]. For maximal supergravity, the hyperbolic Kac-Moody algebra is E_{10}.

In this approach, the idea is to compare a suitably reduced supergravity theory to a dynamic model based on an over-extended Kac-Moody algebra. To compare

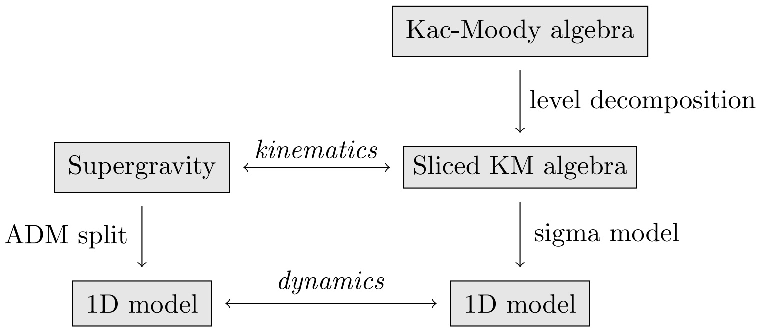


Figure 6.2: Comparing supergravity with a Kac-Moody algebra. The former has to be reduced to one dimension with an ADM-like split, and the latter has to be sliced with a level decomposition and then realized as a non-linear sigma model.

both sides of this correspondence, on the one hand one has to truncate the supergravity fields and break spacetime covariance by choosing an ADM gauge [6] in order to be amenable to a one-dimensional language. On the Kac-Moody side, on the other hand, one has to perform a level decomposition and put the results on a one-dimensional non-linear sigma model (see section 4.4). The correspondence has been schematically depicted in Figure 6.2. Because the non-linear sigma model already provides the time dimension, the Kac-Moody target space has to provide the spatial dimensions. Thus the gravity line to which to decompose the Kac-Moody algebra must be $GL(D-1)$ for the D-dimensional decomposition. In order to get the correct duality group, it is clear that the relevant Kac-Moody algebras are not of the very-extended type, but must be over-extended. In particular, the correct Kac-Moody algebra for a dynamical comparison in this particular way to maximal supergravity is E_{10}, and not E_{11} (see also Figure 6.3).

As one attractive scenario it has been suggested [23, 70, 24] that the higher levels of the over-extended Kac-Moody algebra encode the spatial gradients of the supergravity fields, and so by including all of these states one should finally recover the full unrestricted supergravity in D dimensions (though in an 'unconventional' formulation). This is in contrast to the E_{11} approach, where, as already mentioned, some of the higher level states can be interpreted as dual representations of lower level states [75].

Instead we will interpret part of the higher levels (i.e. the deformation potentials and top-forms) as deformations of pure maximal supergravity. In [55, 41] it has been shown that Romans' massive supergravity in ten dimensions [78], which deforms type IIA supergravity with a mass parameter, is contained in the E_{10} model, upon taking a certain 9-form representation into account. We will try to do the same for $D = 3$ maximal supergravity [3].

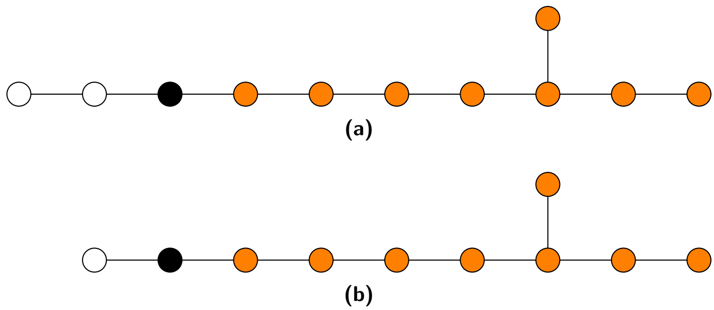


Figure 6.3: Both E_{11} (a) and E_{10} (b) in a $D = 3$ E_8 decomposition. For the very-extended E_{11}, the gravity line has to encode the space-time symmetries, and for the over-extended E_{10} it has to encode the spatial symmetries.

6.3.2 The $E_{10}/K(E_{10})$ coset model

In this section we will focus on gauged supergravity in three dimensions. The advantage of this case is that E_8 is the largest finite-dimensional duality group. As a consequence, the E_{10} equations of motion truncated to level $l = 0$ already match ungauged supergravity reduced to a one-dimensional system. Thus, this model allows a clear distinction between the 'manifest' aspects of the E_{10} conjecture at level $l = 0$ and the more speculative features related to higher levels, such as spatial gradients or gauge couplings.

In order to make contact with three-dimensional supergravity we perform a level decomposition of E_{10} with respect to the subgroup of spatial diffeomorphisms and the duality group:

$$E_{10} \supset SL(2) \otimes E_8. \tag{6.5}$$

This corresponds to deleting the black node in the Dynkin diagram in Figure 6.3b. The representations occurring in this level decomposition can be calculated using the computer program SimpLie [5]. Up to level $l < 3$ we find the $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{e}_8$ representations in Table 6.5, where we indicated the corresponding generators with their symmetries.

We denote by $a, b = 1, 2$ the fundamental indices of $GL(2, \mathbb{R})$ and by $\mathcal{A}, \mathcal{B} = 1, 2 \ldots, 248$ the adjoint indices of E_8. The fields associated to the $l = 0$ generators are the spatial zweibein and the coset scalars. The $l = 1$ fields can be interpreted as gauge vectors. We will take the $l = 2$ fields to be the embedding tensor components θ and $\tilde{\Theta}$. At the negative levels we have the conjugate representations, i.e. the transposed generators of those at the positive levels.

We will now construct the non-linear realization of $E_{10}/K(E_{10})$, in a similar spirit in which the much simpler $A_2/K(A_2)$ has been constructed in Example 4.4 in section 4.4. Much of the details can be found in [3].

Level ℓ	$SL(2) \otimes E_8$ representation	Generator	Interpretation
0	$(\mathbf{1} \oplus \mathbf{3}, \mathbf{1})$	$K^a{}_b$	metric
0	$(\mathbf{1}, \mathbf{248})$	$t^{\mathcal{A}}$	scalars
1	$(\mathbf{2}, \mathbf{248})$	$E^a{}_{\mathcal{A}}$	gauge vectors
2	$(\mathbf{1}, \mathbf{1})$	E	θ
2	$(\mathbf{1}, \mathbf{3875})$	$E_{\mathcal{AB}} = E_{(\mathcal{AB})}$	$\tilde{\Theta}_{\mathcal{MN}}$
2	$(\mathbf{3}, \mathbf{248})$	$E^{ab}{}_{\mathcal{A}} = E^{(ab)}{}_{\mathcal{A}}$	

Table 6.5: $SL(2) \otimes E_8$ representations within E_{10} up to level 2.

The local $K(E_{10})$ invariance allows us to choose a suitable gauge for the E_{10}-valued group element V. In the Borel gauge, we can write V as a product

$$V = V_l V_0 = e^X e^h e^{\mathcal{H}}, \tag{6.6}$$

where V_l and V_0 are group elements corresponding to $l > 0$ and $l = 0$, respectively. Thus we can expand the corresponding algebra elements in the basis of $\mathfrak{e}_{10}$ as

$$X = A_m{}^{\mathcal{M}} E^m{}_{\mathcal{M}} + B_{mn}{}^{\mathcal{M}} E^{mn}{}_{\mathcal{M}} + BE + B^{\mathcal{MN}} E_{\mathcal{MN}} + \cdots , \tag{6.7a}$$

$$h = h_a{}^b K^a{}_b, \tag{6.7b}$$

$$\mathcal{H} = \mathcal{H}_{\mathcal{A}} t^{\mathcal{A}}, \tag{6.7c}$$

where the dots stand for higher-level contributions. Here and in the following, $m, n, \ldots = 1, 2$ and $\mathcal{M}, \mathcal{N} \ldots = 1, 2, \ldots, 248$ denote curved $GL(2)$ and E_8 indices, respectively. This means that they are 'world' indices indicating rigid transformations from the left, while $\mathcal{A}$ and a are flat indices.

The Maurer-Cartan form J can then be computed to give

$$J = V^{-1} \partial V = J_0 + J_1 + J_2 + \cdots . \tag{6.8}$$

The dots stand for the higher-level contributions, which will be truncated. The derivative is with respect to time, i.e. $\partial \equiv \partial_t$. The low-level contributions that will be kept are

$$J_0 = P_{\mathcal{A}} t^{\mathcal{A}} + \tfrac{1}{2} P_a{}^b K^a{}_b, \tag{6.9a}$$

$$J_1 = P_a{}^{\mathcal{A}} E^a{}_{\mathcal{A}}, \tag{6.9b}$$

$$J_2 = PE + P^{\mathcal{AB}} E_{\mathcal{AB}} + P_{ab}{}^{\mathcal{A}} E^{ab}{}_{\mathcal{A}}, \tag{6.9c}$$

where the individual components read

$$P_a{}^b = \tfrac{1}{2}(e_a{}^m \partial e_m{}^b + e_b{}^m \partial e_m{}^a), \tag{6.10a}$$
$$P_{\mathcal{A}} = \tfrac{1}{2}(\mathcal{E}^{-1}\partial\mathcal{E})_{\mathcal{A}}, \tag{6.10b}$$
$$P_a{}^{\mathcal{A}} = \tfrac{1}{2} e_a{}^m \mathcal{E}^{\mathcal{A}}{}_{\mathcal{M}} DA_m{}^{\mathcal{M}}, \tag{6.10c}$$
$$P = \tfrac{1}{2}(\det e)^{-1} DB, \tag{6.10d}$$
$$P^{\mathcal{AB}} = \tfrac{1}{2}(\det e)^{-1}\mathcal{E}^{\mathcal{A}}{}_{\mathcal{M}}\mathcal{E}^{\mathcal{B}}{}_{\mathcal{N}} DB^{\mathcal{MN}}, \tag{6.10e}$$
$$P_{ab}{}^{\mathcal{A}} = \tfrac{1}{2} e_a{}^m e_b{}^n \mathcal{E}^{\mathcal{A}}{}_{\mathcal{M}} DB_{mn}{}^{\mathcal{M}}. \tag{6.10f}$$

Here we have introduced two 'vielbeine' $\exp h$ and $\exp \mathcal{H}$, which are group elements of $\mathfrak{gl}(2)$ and $\mathfrak{e}_8$, respectively. We denote the components of these group elements by $e_m{}^a$ and $\mathcal{E}^{\mathcal{M}}{}_{\mathcal{A}}$, and their inverses by $e_a{}^m$ and $\mathcal{E}^{\mathcal{A}}{}_{\mathcal{M}}$. Furthermore, we have introduced the 'covariant derivatives'

$$DA_m{}^{\mathcal{M}} = \partial A_m{}^{\mathcal{M}}, \tag{6.11a}$$
$$DB_{mn}{}^{\mathcal{P}} = \partial B_{mn}{}^{\mathcal{P}} + \tfrac{1}{2} f_{\mathcal{MN}}{}^{\mathcal{P}} A_{(m}{}^{\mathcal{M}} \partial A_{n)}{}^{\mathcal{N}}, \tag{6.11b}$$
$$DB = \partial B - \tfrac{1}{4}\varepsilon^{ab}\eta_{\mathcal{MN}} A_m{}^{\mathcal{M}} \partial A_n{}^{\mathcal{N}}, \tag{6.11c}$$
$$DB^{\mathcal{MN}} = \partial B^{\mathcal{MN}} - \tfrac{1}{2}\varepsilon^{mn}\mathbb{P}_{\mathcal{PQ}}{}^{\mathcal{MN}} A_m{}^{\mathcal{P}} \partial A_n{}^{\mathcal{Q}}. \tag{6.11d}$$

Here the $\mathbb{P}_{\mathcal{AB}}{}^{\mathcal{CD}}$ projector projects onto the **3875** representation of E_8. It reads [58]

$$\mathbb{P}_{\mathcal{AB}}{}^{\mathcal{CD}} = \tfrac{1}{7}\delta_{(\mathcal{A}}{}^{\mathcal{C}}\delta_{\mathcal{B})}{}^{\mathcal{D}} - \tfrac{1}{56}\eta_{\mathcal{AB}}\eta^{\mathcal{CD}} - \tfrac{1}{14} f^{\mathcal{E}}{}_{\mathcal{A}}{}^{(\mathcal{C}} f_{\mathcal{EB}}{}^{\mathcal{D})}. \tag{6.12}$$

After computing the coset element $\mathcal{P}(t) = \frac{1}{2}(J + J^T)$, it can be plugged into the action to give

$$\begin{aligned} S &= \frac{1}{4}\int dt\, n(t)^{-1}\langle \mathcal{P}(t)|\mathcal{P}(t)\rangle \\ &= \int dt\,(\mathcal{L}_0 + \mathcal{L}_{12})\,. \end{aligned} \tag{6.13}$$

The 'level zero' and 'higher level' Lagrangians $\mathcal{L}_0$ and $\mathcal{L}_{12}$ can be written as

$$\mathcal{L}_0 = n^{-1}P^A P^A + \tfrac{1}{4}n^{-1}\left(P_a{}^b P_a{}^b - P_a{}^a P_b{}^b\right), \tag{6.14a}$$
$$\mathcal{L}_{12} = \tfrac{1}{2}n^{-1}\big(P_a{}^{\mathcal{A}}P_a{}^{\mathcal{A}} + P_{ab}{}^{\mathcal{A}}P_{ab}{}^{\mathcal{A}} + PP + 14P^{\mathcal{AB}}P^{\mathcal{AB}}\big), \tag{6.14b}$$

Note that the index A runs here only over the $E_8/SO(16)$ coset, and not over the whole of E_8. The above Lagrangians can also be written out completely as

$$\mathcal{L}_0 = \tfrac{1}{16} n^{-1} \Big(\partial g_{mn} \partial g_{pq} (g^{mp} g^{nq} - g^{mn} g^{pq}) \, + \tfrac{1}{60} \partial \mathcal{G}^{\mathcal{M}\mathcal{N}} \partial \mathcal{G}^{\mathcal{P}\mathcal{Q}} \mathcal{G}_{\mathcal{M}\mathcal{P}} \mathcal{G}_{\mathcal{N}\mathcal{Q}} \Big),$$
$$\mathcal{L}_{12} = \tfrac{1}{8} n^{-1} \Big(g^{mp} g^{nq} \mathcal{G}_{\mathcal{M}\mathcal{N}} DB_{mn}{}^{\mathcal{M}} DB_{pq}{}^{\mathcal{N}} \, + g^{mn} \mathcal{G}_{\mathcal{M}\mathcal{N}} DA_m{}^{\mathcal{M}} DA_n{}^{\mathcal{N}}$$
$$+ \big(14 \mathcal{G}_{\mathcal{M}\mathcal{P}} \mathcal{G}_{\mathcal{N}\mathcal{Q}} DB^{\mathcal{M}\mathcal{N}} DB^{\mathcal{P}\mathcal{Q}} + DBDB \big) (\det g)^{-1} \Big). \tag{6.15}$$

Here we have introduced the respective $GL(2)$ and E_8 metrics

$$g^{mn} = \delta^{ab} \, e_a{}^m e_b{}^n, \tag{6.16a}$$
$$\mathcal{G}_{\mathcal{M}\mathcal{N}} = \delta_{\mathcal{A}\mathcal{B}} \, \mathcal{E}^{\mathcal{A}}{}_{\mathcal{M}} \mathcal{E}^{\mathcal{B}}{}_{\mathcal{N}}. \tag{6.16b}$$

6.3.3 Gauged supergravity in three dimensions

In this section we review gauged three-dimensional supergravity in a formulation suitable for comparison with the E_{10} analysis of the preceding section.

The bosonic sector of ungauged maximal supergravity in three dimensions contains 128 propagating scalars transforming in the coset $E_8/(Spin(16)/\mathbb{Z}_2)$ and a vielbein $\mathtt{e}_\mu{}^\alpha$ that carries no dynamical degrees of freedom [61]. The scalars can also be described by an (internal) vielbein which we denote by $\mathtt{E}^{\mathcal{M}}{}_{\mathcal{A}}$ (which was denoted $\mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}}$ in [67]). We will use the 'typewriter' font for supergravity variables in order to distinguish them from the corresponding E_{10} quantities. The inverses will be written as $\mathtt{e}_\alpha{}^\mu$ and $\mathtt{E}^{\mathcal{A}}{}_{\mathcal{M}}$. The curved indices are written as Greek indices $\mu, \nu, \ldots = (t, m)$ and the flat indices are $\alpha,\ \beta, \ldots = 0,\ 1,\ 2$.

The bosonic Lagrangian of three-dimensional maximal gauged supergravity is [68, 67]

$$L = L_0 + L_g, \tag{6.17}$$

where the 'ungauged' part of the action L_0 and the 'gauged' part L_g read

$$L_0 = + \, \mathtt{e} \left(\tfrac{1}{4} R - \mathtt{P}_\mu{}^A \mathtt{P}^{\mu A} \right), \tag{6.18a}$$
$$L_g = - \, \mathtt{e} V - \tfrac{1}{4} g \varepsilon^{\mu\nu\rho} \Theta_{\mathcal{M}\mathcal{N}} \mathtt{A}_\mu{}^{\mathcal{M}} \partial_\nu \mathtt{A}_\rho^{\mathcal{N}}$$
$$- \tfrac{1}{12} g^2 \varepsilon^{\mu\nu\rho} \Theta_{\mathcal{M}\mathcal{N}} \Theta_{\mathcal{P}\mathcal{Q}} f^{\mathcal{M}\mathcal{P}}{}_{\mathcal{R}} \mathtt{A}_\mu{}^{\mathcal{N}} \mathtt{A}_\nu{}^{\mathcal{Q}} \mathtt{A}_\rho{}^{\mathcal{R}}, \tag{6.18b}$$

with $\mathtt{e} = \det(\mathtt{e}_\mu{}^\alpha)$. Since there is no kinetic term for them, the gauge fields $\mathtt{A}_\mu{}^{\mathcal{M}}$ do not contain propagating degrees of freedom. The gauging also introduces an indefinite scalar potential, which can be written in the form

$$V = \tfrac{1}{32} g^2 G^{\mathcal{M}\mathcal{N},\mathcal{K}\mathcal{L}} \Theta_{\mathcal{M}\mathcal{N}} \Theta_{\mathcal{K}\mathcal{L}}, \tag{6.19}$$

where [4]

$$G^{\mathcal{M}\mathcal{N},\mathcal{K}\mathcal{L}} \;=\; \tfrac{1}{14} \mathtt{G}^{\mathcal{M}\mathcal{K}} \mathtt{G}^{\mathcal{N}\mathcal{L}} + \mathtt{G}^{\mathcal{M}\mathcal{K}} \eta^{\mathcal{N}\mathcal{L}} - \tfrac{3}{14} \eta^{\mathcal{M}\mathcal{K}} \eta^{\mathcal{N}\mathcal{L}} - \tfrac{4}{6727} \eta^{\mathcal{M}\mathcal{N}} \eta^{\mathcal{K}\mathcal{L}}, \tag{6.20}$$

with the metric $\mathtt{G}^{\mathcal{MN}}$ defined in (6.16), but here with respect to the supergravity E_8 vielbein $\mathtt{E}^{\mathcal{M}}{}_{\mathcal{A}}$.

We now effectively reduce the three-dimensional gauged supergravity theory to a one-dimensional time-like system. For this we perform the ADM-like split of the vielbein

$$\mathtt{e}_\mu{}^\alpha = \begin{pmatrix} N & 0 \\ 0 & \mathtt{e}_m{}^a \end{pmatrix}, \tag{6.21}$$

in which everything depends only on one coordinate $x^0 = t$ and we have split curved indices as $\mu = (t, m)$ and flat ones as $\alpha = (0, a)$ (with signature $(-++)$). Here we have chosen a gauge with vanishing shift N^m, which turns out to be necessary in order to match the E_{10} coset. As stressed before, gauge fixing is crucial for comparing the E_{10} sigma model to supergravity. The field $\mathtt{e}_m{}^a$ denotes the internal 'spatial' vielbein, i.e. an element of $GL(2, \mathbb{R})/SO(2)$.

For the reduction of the gauge fields we choose a temporal gauge

$$\mathtt{A}_t{}^{\mathcal{M}} = 0. \tag{6.22}$$

The reduced Lagrangian (6.17) then reads

$$L^{D=1} = L_0^{D=1} + L_g^{D=1}, \tag{6.23}$$

with

$$L_0^{D=1} = +\,\mathtt{n}^{-1}\mathtt{P}_t{}^A\mathtt{P}_t{}^A + \tfrac{1}{4}\mathtt{n}^{-1}\left(\mathtt{P}_{ab}\mathtt{P}_{ab} - \mathtt{P}_{aa}\mathtt{P}_{bb}\right), \tag{6.24a}$$

$$\begin{aligned} L_g^{D=1} = &-\tfrac{1}{8}g^2\mathtt{e}\mathtt{g}^{mn}(\mathtt{G}^{\mathcal{MN}} + \eta^{\mathcal{MN}})\Theta_{\mathcal{MK}}\Theta_{\mathcal{NL}}\mathtt{A}_m{}^{\mathcal{K}}\mathtt{A}_n{}^{\mathcal{L}} - \mathtt{e}V \\ &+ \tfrac{1}{4}g\Theta_{\mathcal{MN}}\varepsilon^{mn}\mathtt{A}_m{}^{\mathcal{M}}\partial_t\mathtt{A}_n{}^{\mathcal{N}}. \end{aligned} \tag{6.24b}$$

Here we have defined the quantities

$$\mathtt{n} = N(\det(\mathtt{e}_m{}^a))^{-1}, \tag{6.25a}$$

$$\mathtt{P}_{ab} = -N^{-1}\mathtt{e}_{(a}{}^m\partial_t\mathtt{e}_{m|b)}. \tag{6.25b}$$

Furthermore, we have written the Lagrangian entirely in terms of the E_8 'metric' $\mathtt{G}^{\mathcal{MN}}$. For this we have used the identity

$$\mathtt{E}^{\mathcal{M}}{}_A\mathtt{E}^{\mathcal{N}A} = \tfrac{1}{2}\left(\mathtt{G}^{\mathcal{MN}} + \eta^{\mathcal{MN}}\right), \tag{6.26}$$

which follows from the fact that the Cartan-Killing metric $\eta^{\mathcal{MN}}$ differs from $\mathtt{G}^{\mathcal{MN}}$ by a relative sign in the non-compact part.

6.3.4 The correspondence

It is clear that the E_{10} 'level zero' Lagrangian (6.14a) has exactly the same form as its supergravity counterpart (6.24a). We therefore focus on equating the E_{10} 'higher level' Lagrangian $\mathcal{L}_{12}$ (6.15) and the supergravity 'gauged' Lagrangian L_g (6.18b), or rather their equations of motion.

The first step is to eliminate the mixed-symmetry field $B_{mn}{}^{\mathcal{M}}$ on the Kac-Moody side, as it has no obvious counterpart in supergravity. Similar mixed-symmetrical objects have been found to be in a one-to-one correspondence with trombone gaugings [59], but their space-time mixed symmetry lacks a solid understanding. We therefore consistently truncate it by setting its covariant derivate equal to zero,

$$DB_{mn}{}^{\mathcal{M}} = 0. \tag{6.27}$$

If we vary the action (6.13) with respect to the remaining level two fields, we obtain

$$0 = \partial(n^{-1}(\det g)^{-1}DB), \tag{6.28a}$$
$$0 = \partial(n^{-1}(\det g)^{-1}\mathcal{G}_{\mathcal{MP}}\mathcal{G}_{\mathcal{NQ}}DB^{\mathcal{PQ}}). \tag{6.28b}$$

These two equations can be be integrated and identified with the **1** and **3875** components of the embedding tensor:

$$c_1 g\theta = n^{-1}(\det g)^{-1}DB, \tag{6.29a}$$
$$c_2 g\tilde{\Theta}_{\mathcal{MN}} = n^{-1}(\det g)^{-1}\mathcal{G}_{\mathcal{MP}}\mathcal{G}_{\mathcal{NQ}}DB^{\mathcal{PQ}}, \tag{6.29b}$$

where c_1 and c_1 are two arbitrary constants. The above equations can then be subsequently used to write the equation of motion of $A_m{}^{\mathcal{M}}$ from (6.13) as

$$\partial\Big(n^{-1}g^{mn}\mathcal{G}_{\mathcal{MN}}\partial A_n{}^{\mathcal{N}} + g\varepsilon^{mn}\big(\eta_{\mathcal{MN}}\theta + \tilde{\Theta}_{\mathcal{MN}}\big)A_n{}^{\mathcal{N}}\Big) = 0. \tag{6.30}$$

Here we have already chosen the integration constants to be $c_1 = 2$ and $c_2 = \frac{1}{14}$. This allows us to combine the **1** and **3875** components of the embedding tensor into Θ, and integrate the above equation to

$$n^{-1}g^{mn}\mathcal{G}_{\mathcal{MN}}\partial A_n{}^{\mathcal{N}} = g\varepsilon^{mn}\Theta_{\mathcal{MN}}A_n{}^{\mathcal{N}} + \Xi^m{}_{\mathcal{M}}. \tag{6.31}$$

$\Xi^m{}_{\mathcal{M}}$ denotes an integration constant. This integration constant cannot be set to zero without breaking the symmetries. The situation is analogous to the integration leading to the embedding tensor $\Theta_{\mathcal{MN}}$ in (6.29), which generically breaks the global E_8 symmetry once $\Theta_{\mathcal{MN}}$ is constant. Correspondingly, the E_{10} shift symmetry leaves this first-order equation only invariant if the integration constant also transforms as a shift,

$$\delta_\Lambda \Xi^m{}_{\mathcal{M}} = -g\varepsilon^{mn}\Theta_{\mathcal{MN}}\Lambda_n{}^{\mathcal{N}}, \tag{6.32}$$

which is consistent with the time-independence of Ξ. Thus, fixing it to any specific value (as zero) breaks the symmetry, and in this sense supergravity may at best be viewed as a broken phase of E_{10}. After setting $\Xi = 0$ and contracting with $\Theta_{\mathcal{M}\mathcal{N}}$, (6.31) implies

$$g\Theta_{\mathcal{M}\mathcal{N}}\varepsilon^{mn}\partial A_n{}^{\mathcal{N}} + g^2 eN\mathcal{G}^{\mathcal{K}\mathcal{L}}g^{mn}\Theta_{\mathcal{M}\mathcal{K}}\Theta_{\mathcal{N}\mathcal{L}}A_n{}^{\mathcal{N}} = 0. \tag{6.33}$$

This matches the duality relation obtained from the supergravity Lagrangian (6.23) by varying with respect to $\mathtt{A}_m{}^{\mathcal{M}}$, which reads

$$g\Theta_{\mathcal{M}\mathcal{N}}\varepsilon^{mn}\partial\mathtt{A}_n{}^{\mathcal{N}} + \tfrac{1}{2}g^2\mathtt{e}(\mathtt{G}^{\mathcal{K}\mathcal{L}} + \eta^{\mathcal{K}\mathcal{L}})\mathtt{g}^{mn}\Theta_{\mathcal{M}\mathcal{K}}\Theta_{\mathcal{N}\mathcal{L}}\mathtt{A}_n{}^{\mathcal{N}} = 0. \tag{6.34}$$

Finally, we compare the 'Einstein equations' and the equations of motions for the scalars on both side. For supergravity, they read

$$\begin{aligned} 0 = \frac{\delta L_0}{\delta \mathtt{g}^{mn}} &+ \tfrac{1}{2}\mathtt{e}\mathtt{g}_{mn}V \\ &+ \tfrac{1}{16}g^2\mathtt{e}(\mathtt{G}^{\mathcal{M}\mathcal{N}} + \eta^{\mathcal{M}\mathcal{N}})\Theta_{\mathcal{M}\mathcal{K}}\Theta_{\mathcal{N}\mathcal{L}}\left(\mathtt{g}_{mn}\mathtt{g}^{kl}\mathtt{A}_k{}^{\mathcal{K}}\mathtt{A}_l{}^{\mathcal{L}} - 2\mathtt{A}_m{}^{\mathcal{K}}\mathtt{A}_n{}^{\mathcal{L}}\right), \end{aligned} \tag{6.35a}$$

$$\begin{aligned} 0 = \frac{\delta L_0}{\delta \mathtt{G}^{\mathcal{M}\mathcal{N}}} &- \tfrac{1}{8}g^2\mathtt{e}\mathtt{g}^{mn}\Theta_{\mathcal{M}\mathcal{K}}\Theta_{\mathcal{N}\mathcal{L}}\mathtt{A}_m{}^{\mathcal{K}}\mathtt{A}_n{}^{\mathcal{L}} \\ &- \tfrac{1}{7\cdot 32}\mathtt{e}g^2\mathtt{G}^{\mathcal{K}\mathcal{L}}\Theta_{\mathcal{M}\mathcal{K}}\Theta_{\mathcal{N}\mathcal{L}} - \tfrac{1}{16}\mathtt{e}g^2\eta^{\mathcal{K}\mathcal{L}}\Theta_{\mathcal{M}\mathcal{K}}\Theta_{\mathcal{N}\mathcal{L}}, \end{aligned} \tag{6.35b}$$

whereas the equations of motion that follow from the E_{10} coset model are given by

$$\begin{aligned} 0 = \frac{\delta \mathcal{L}_0}{\delta g^{mn}} &+ \tfrac{1}{8}g^2\mathtt{e}G^{\mathcal{M}\mathcal{N}}\Theta_{\mathcal{M}\mathcal{K}}\Theta_{\mathcal{N}\mathcal{L}}\left(g_{mn}g^{kl}A_k{}^{\mathcal{K}}A_l{}^{\mathcal{L}} - A_m{}^{\mathcal{K}}A_n{}^{\mathcal{L}}\right) \\ &+ \tfrac{1}{2}g^2\mathtt{e}g_{mn}\left(\tfrac{1}{56}\mathcal{G}^{\mathcal{M}\mathcal{K}}\mathcal{G}^{\mathcal{N}\mathcal{L}}\tilde{\Theta}_{\mathcal{M}\mathcal{N}}\tilde{\Theta}_{\mathcal{K}\mathcal{L}} + \theta^2\right), \end{aligned} \tag{6.36a}$$

$$0 = \frac{\delta \mathcal{L}_0}{\delta \mathcal{G}^{\mathcal{M}\mathcal{N}}} - \tfrac{1}{8}g^2\mathtt{e}g^{mn}\Theta_{\mathcal{M}\mathcal{K}}\Theta_{\mathcal{N}\mathcal{L}}A_m{}^{\mathcal{K}}A_n{}^{\mathcal{L}} - \tfrac{1}{56}g^2\mathtt{e}\mathcal{G}^{\mathcal{K}\mathcal{L}}\tilde{\Theta}_{\mathcal{M}\mathcal{K}}\tilde{\Theta}_{\mathcal{N}\mathcal{L}}. \tag{6.36b}$$

By comparing (6.35) with (6.36) we observe that the equations are structure-wise the same, but differ in the details. For one thing, on the E_{10} side we generically have just $\mathcal{G}^{\mathcal{M}\mathcal{N}}$ instead of $\frac{1}{2}(\mathcal{G}^{\mathcal{M}\mathcal{N}} + \eta^{\mathcal{M}\mathcal{N}})$. Apart from that, the indefinite contributions to the supergravity potential are not reproduced, but only the leading term quadratic in $\mathcal{G}^{\mathcal{M}\mathcal{N}}$.

Summarizing, we find that the gauging appears exclusively as a consequence of 'switching on' certain higher level degrees of freedom in the level expansion of the Cartan form and the coset equations of motion. The embedding tensor appears naturally in the coset model by integrating the one-dimensional equations of motion. The same holds for duality relation between the scalars and vectors, and the scalar potential. However, the latter is not fully reproduced by E_{10}, but only the positive definite contributions. This is due to the fact that in supergravity the scalar potential is indefinite [90], while the corresponding 2-forms appearing in the E_{10} coset model necessarily enter with a positive definite kinetic term.

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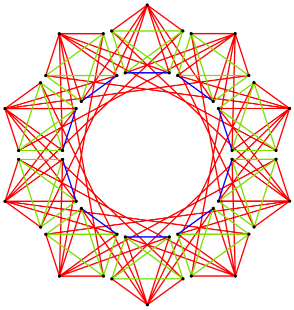


Conclusions

In this thesis we have shown that Kac-Moody symmetries play a unifying role in supergravity. In particular, we have given evidence that certain over- and very-extended Kac-Moody algebras are intimately related to gauged supergravity. Not only do the Kac-Moody algebras correctly predict all the physical degrees of freedom of the supergravities in all dimensions, but they also contain deformation- and top-form potentials that correspond to all the known gaugings of those supergravities. This is a promising unifying picture, although there are a few mismatches. First, the Kac-Moody algebra E_{11} predicts more top-forms than there are quadratic constraints in the corresponding gauged maximal supergravity. Secondly, matching the equations of motion succeeds only up to a certain level. Lastly, the infinite tower of exotic representations in Kac-Moody algebras have no obvious counterpart on the supergravity side.

The deformation potentials, or $(D-1)$-forms, that follow from over- or very-extended Kac-Moody algebras occur in precisely the same representations of the global symmetry group as the embedding tensors of gauged supergravity. Also, the way the $(D-1)$-forms are reached by commutators of the fundamental p-forms tells us whether they correspond to a 'traditional' gauged supergravity, or a massive deformation.

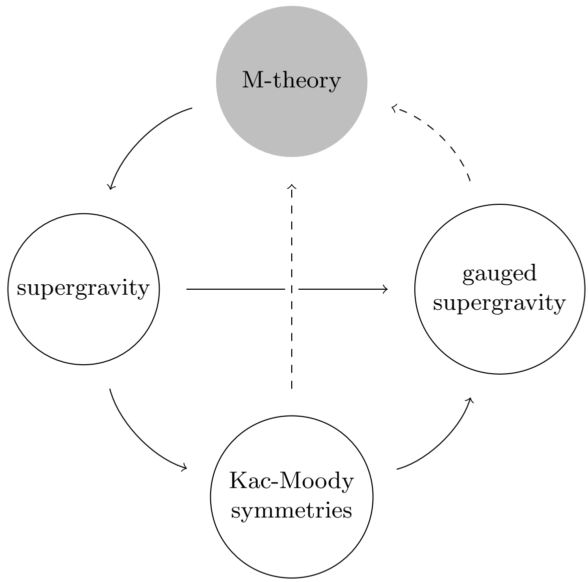


Figure 7.1: The interrelations between M-theory, (gauged) supergravity, and Kac-Moody symmetries.

Furthermore, all the known quadratic constraints of the embedding tensor have a top-form, or D-form, counterpart in very-extended Kac-Moody algebras. However, in certain cases, such as the $D = 3$ or IIB case for E_{11}, there are more top-forms than quadratic constraints. For the IIB case it is known that these extra top-forms can be interpreted to couple to D9-branes of Type IIB string theory [11]. A similar scenario could be possible for the $D = 3$ case, but such an interpretation is currently lacking.

As for the dynamic comparison, we have found that the equations of motion of $D = 3$ gauged maximal supergravity adapted to a one-dimensional language can in part be matched to the E_{10} equations, even though the latter have a priori a rather different form. For one thing, the absence of gauge-covariant derivatives on the E_{10} side agrees with the supergravity expressions, once a gauge-fixing condition which is inevitable for the comparison, has been imposed. Moreover, in spite of the fact that on the E_{10} side all fields appear with a 'kinetic' term, the (truncated) duality relation between vectors and scalars expected from supergravity naturally follows via integrating the one-dimensional equations of motion. Finally, the embedding tensor automatically appears as an integration constant in the right representation. In this sense, none of the essential ingredients of gauged supergravity have to be introduced by hand, but rather they naturally follow from the E_{10} sigma model.

However, a mismatch occurs at higher levels: the scalar potential of gauged supergravity is not fully reproduced by E_{10}. This is somewhat reminiscent to a discrepancy encountered in higher dimensions, once spatial gradients are introduced as the duals of higher-level fields [23].

In total we are led to conclude that further insights, like those of [27, 29], are required in order to understand the precise relation between supergravity theories and the E_{10} sigma model. It would be interesting to see whether modifications and/or extensions of the E_{10} model are possible to compensate for the present mismatches. We note that mismatches already occur before comparing to *gauged* supergravity and so an ultimate resolution of the present discrepancies must await a better understanding of the basic picture.

It is therefore too soon to claim that the Kac-Moody symmetries of E_{11} or E_{10} are truly the symmetries of M-theory, or are the fundamental symmetries of nature. One might conjecture that one of the missing pieces of the puzzle, the infinite tower of exotic representations, might encode some M-theory degrees of freedom. In the same direction it is thought that some of the gauge deformations of supergravity also encode M-theory degrees of freedom [96]. As E_{11} somehow 'knows' of the gauged deformations via deformation and top-form potentials, one might argue that it indirectly contains information on M-theory (see also Figure 7.1).

To conclude, the Kac-Moody symmetries provide an elegant unifying framework for the symmetries of supergravity, although the correspondence is not perfect. Future work might lift the discrepancies, and hopefully allow a sneak peak into the structure of M-theory.

A

SimpLie

In this chapter we will briefly discuss SimpLie [5], a Java program that has specifically been written for the level decompositions of Kac-Moody algebras. However, that is not the only trick it can do. At the moment of writing, it is amongst others capable of the calculation of:

- Root systems
  - Root multiplicties
  - Level decomposition w.r.t regular subalgebras
- Highest weight representations
  - Weight multiplicities
  - Tensor products
  - Branchings w.r.t. regular subalgebras
- Visualizations
  - Hasse diagrams
  - Coxeter projections

There are other computer programs that also perform some of SimpLie's functions, and usually a whole lot more. A list of Lie algebra related programs is for example given in [35], of which perhaps one of the most famous is LiE [60]. However, none of the programs of which the author is aware of are capable of handling

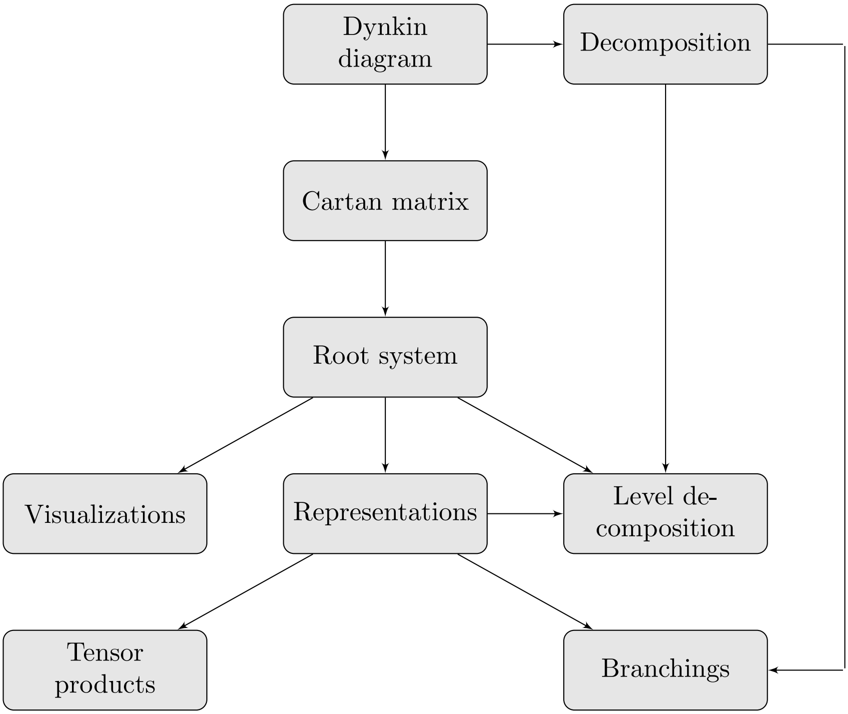


Figure A.1: Schematic work flow of SimpLie

Kac-Moody algebras with an arbitrary Cartan matrix. This is exactly what SimpLie was designed to do from the ground up: the user can enter any Dynkin diagram he or she likes, irrespective of the fact whether the resulting algebra is finite, affine, over- or very-extended, or even more exotic. Based on the Dynkin diagram alone, SimpLie can then calculate any of the quantities given above.

A schematic flowchart of the inner workings of SimpLie is given in Figure A.1. Note that this picture has been simplified rather dramatically from how SimpLie actually works. Interested readers may be redirected to the source code, which is freely available online [5].

Besides being able to perform level decompositions, SimpLie has been written with the idea that it should be:

1. Multi-platform,
2. Easy to extend and adapt,
3. Easy to use.

The first two goals justify Java as a choice of programming language. Java is by design multi-platform, and its object-oriented nature automatically steers one towards a modular design. Whether the third goal is achieved is a more subjective matter, and is best evaluated by trying out SimpLie for one's self.

A.1 User manual

This section contains a short user manual. It is not exhaustive, as SimpLie's features and appearance will most likely change in the future. We will discuss the different windows of SimpLie in turn.

Algebra Setup (Figure A.2) This is the main window of Simplie. Within the white Dynkin diagram panel of the Algebra Setup tab one can interactively 'click' a Dynkin diagram together. The relevant commands are:

- Left mouse-click: Add a node. Hold shift to automatically add a connection to the previous node.
- Middle mouse-click or alt-left: Toggle a node to be deleted / normal.
- Right mouse-click: Bring up the context menu. From the context menu, you can:
    - Add/remove a node if the mouse is on an empty location.
    - Add/remove a connection if the mouse is over a node.
    - Left click on a second node to finish the action.
    - Toggle a node.
    - Remove a node.

A decomposition is specified by toggling one or more nodes to be 'deleted'. The deleted nodes are black and will indicate the levels for the level decomposition. However, when one or more deleted nodes do not have lines connecting them to the undeleted nodes, they will become so-called 'internal nodes'. The internal nodes are orange, and the regular subalgebra nodes (i.e. those that are not deleted) remain white.

By default, the subscripts of the nodes indicate the node ordering. Using the radio buttons on the right it is possible to display to (dual) Coxeter labels. If the Coxeter labels differ from their duals, the Coxeter labels get displayed in bold below the dual. The default ordering on of the nodes is from bottom to top. Using the radio buttons on the right it is possible to change it to from top to bottom.

Algebra Info (Figure A.3) This window gives some basic information about the Lie algebra of the previously entered Dynkin diagram. You can select whether to view the information for the full algebra, the regular subalgebra, or for the internal algebra. You can either view the Cartan matrix, the quadratic form matrix, or the metric on the root space. It also possible to display a list of positive roots up to a given height.

Representations (Figure A.4) This window contains the following three tabs:

- Representation info. Here it it is possible to enter Dynkin labels for a representation and calculate its dimension and weights down to a given depth.
- Tensor products. Here one can enter two sets of Dynkin labels, and calculate their tensor product in terms of irreducible highest weight representations.
- Branching. After entering one set of Dynkin labels, its branching can be calculated with respect to the decomposition entered in the algebra setup window.

Level Decomposition (Figure A.5) In this window a level decomposition of the Dynkin diagram as given in the algebra setup window can be calculated. The column headers of the table containing the decomposition results indicate the following:

- `l` : Level(s).
- `p_r` : Dynkin labels of the regular subalgebra part.
- `p_i` : Dynkin labels of the internal part.
- `vector` : Root vector.
- `a^2` : Root norm.
- `d_r` : Dimension of the regular subalgebra part.
- `d_i` : Dimension of the internal part.
- `mult` : The root multiplicity.
- `mu` : The outer multiplicity of the representation.
- `h` : The height of lowest / highest weight.

Note that these column headers are similar to the ones in Table C.1.

Visualization (Figure A.6) In this window any Lie algebra can be visualized: either a Coxeter plane projection can be made, or a Hasse diagram can be drawn of the root system of the algebra. When choosing the Coxeter plane projection, the full algebra gets projected with respect to its regular subalgebra as specified in the Dynkin diagram.

A.2 List of papers

SimpLie has been used for calculations in the following papers: [1, 2, 3, 73, 56, 13, 44].

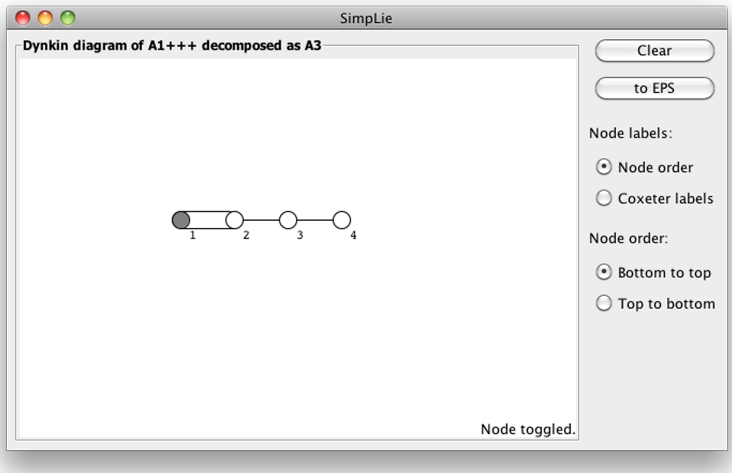


Figure A.2: SimpLie's Algebra Setup window.

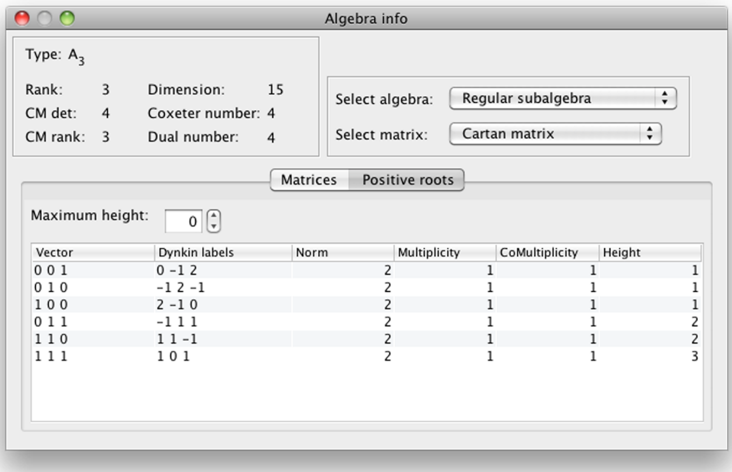


Figure A.3: SimpLie's Algebra Info window.

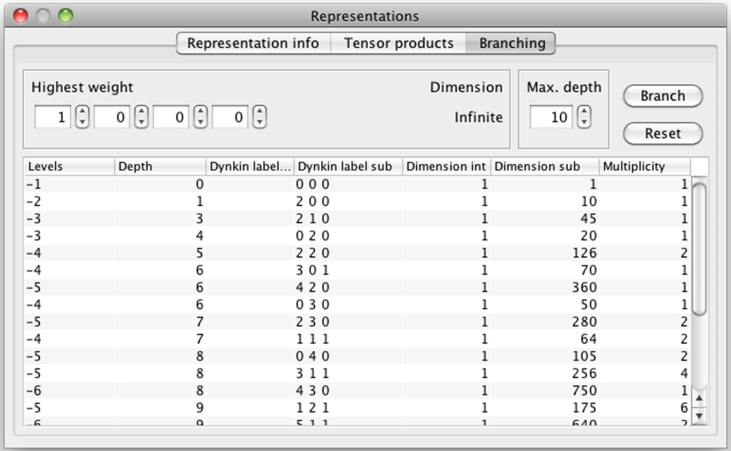


Figure A.4: SimpLie's Representations window.

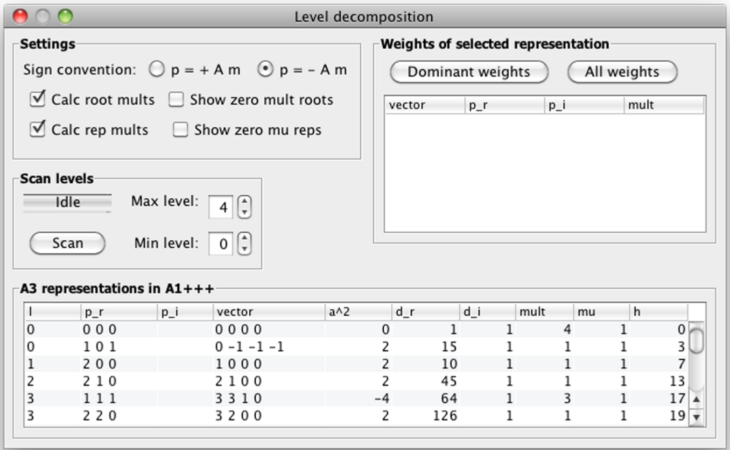


Figure A.5: SimpLie's Level decomposition window.

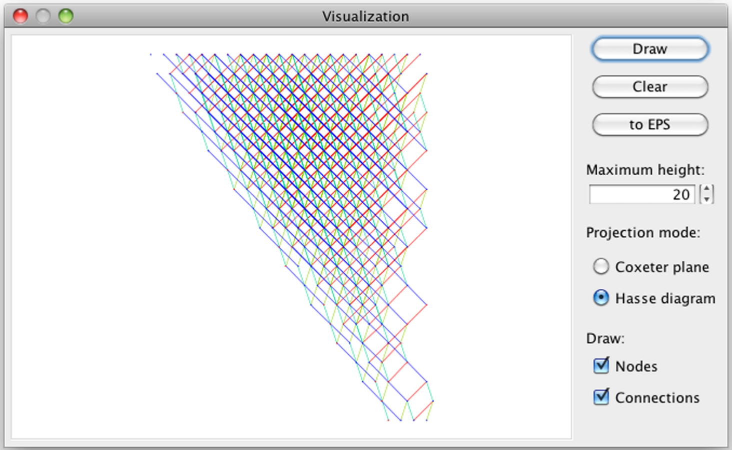


Figure A.6: SimpLie's Visualization window.

B

Hasse diagrams and Coxeter projections

This section contains Hasse diagrams and Coxeter projections of various finite and infinite Lie algebras and representations of finite algebras. All the images are generated with SimpLie [5]. All the Coxeter projections of E_{11} are done up to height 100 in the root system. The subalgebras of E_{11} with respect to which the Coxeter projections have been performed are indicated in Dynkin diagrams below the Coxeter projections, except for the first projection, Figure B.2. For that particular projection, the subalgebra embedding is given in Figure B.1.

In Figures B.9, B.10, B.11, and B.12 Coxeter projections of representations of various finite Lie algebras are given. The Dynkin labels of the representations are indicated in the Dynkin diagram below the Coxeter projection.

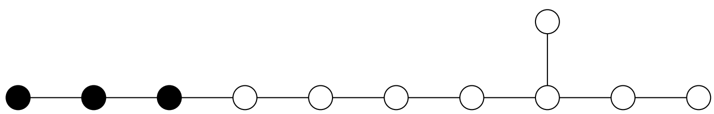
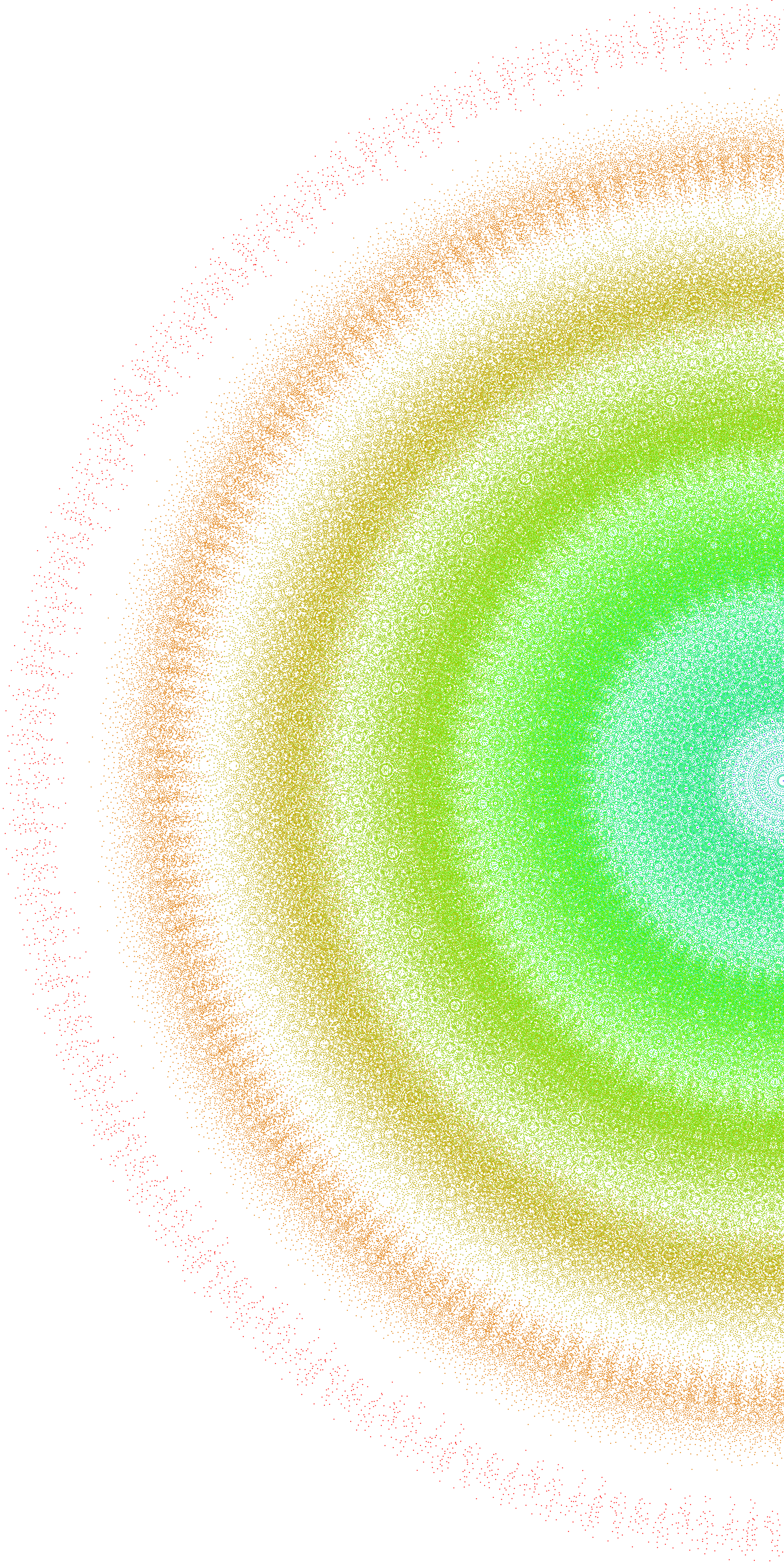


Figure B.1: The $E_8 \subset E_{11}$ embedding for the Coxeter projection of Figure B.2.



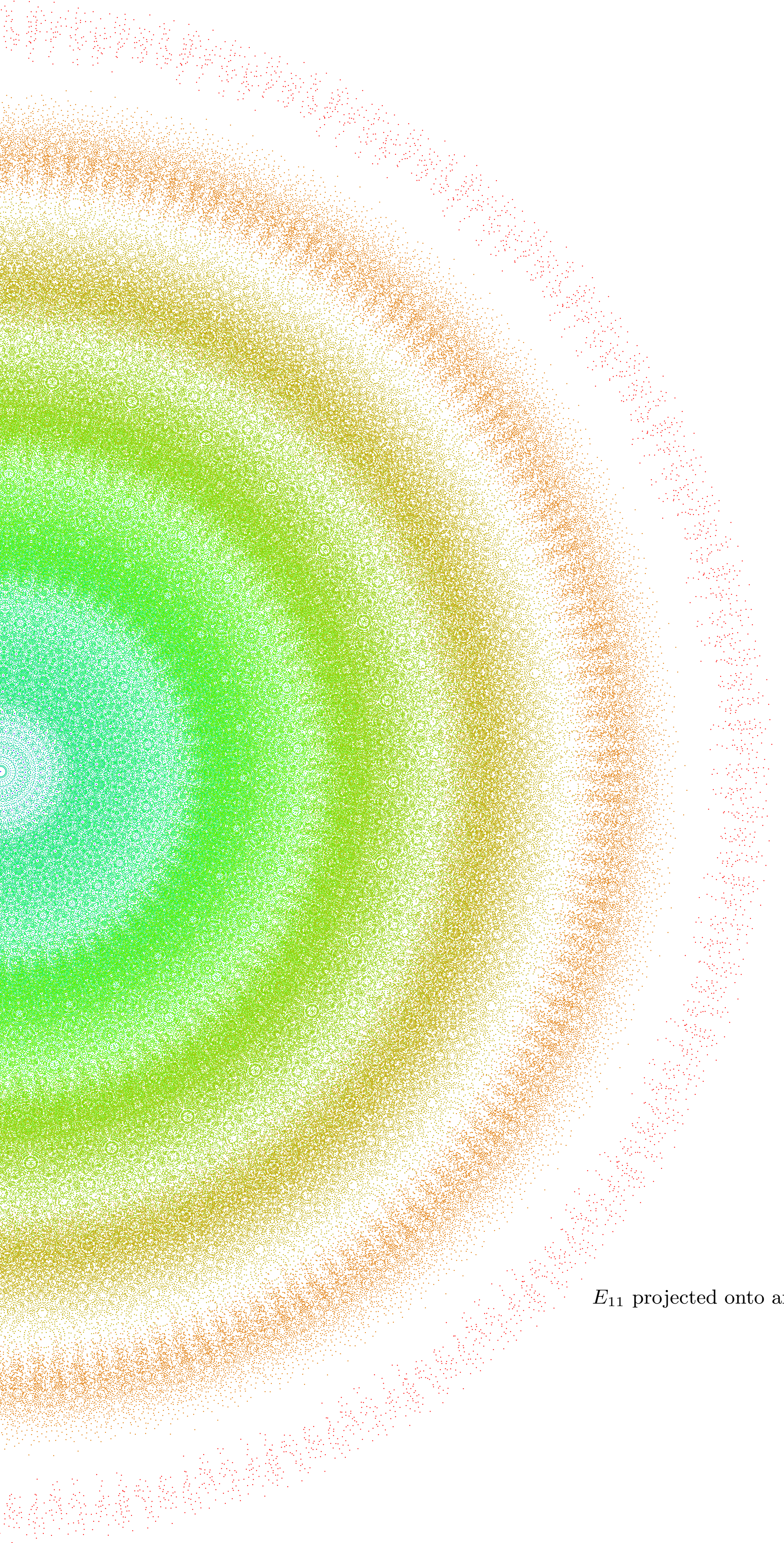


Figure B.2
E_{11} projected onto an E_8 Coxeter plane.

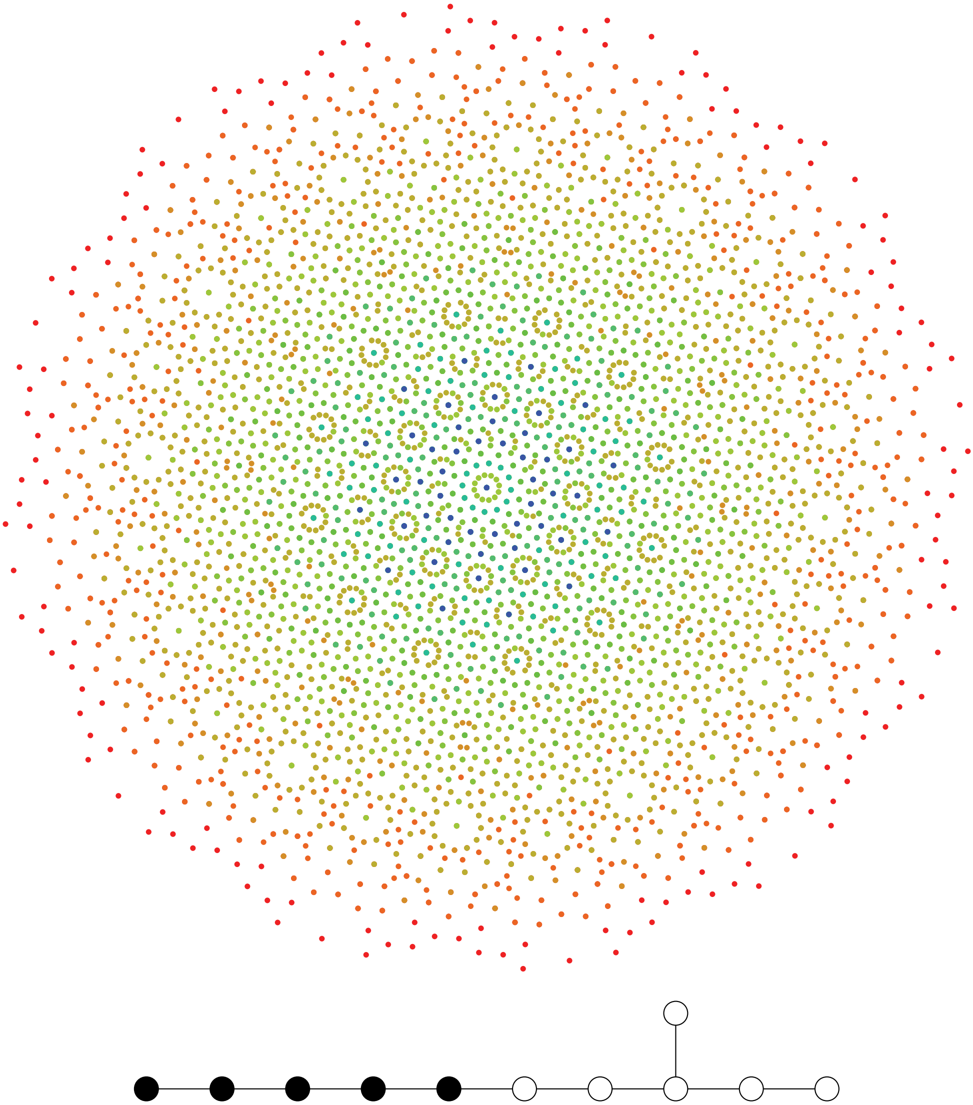


Figure B.3: E_{11} projected onto an E_6 Coxeter plane.

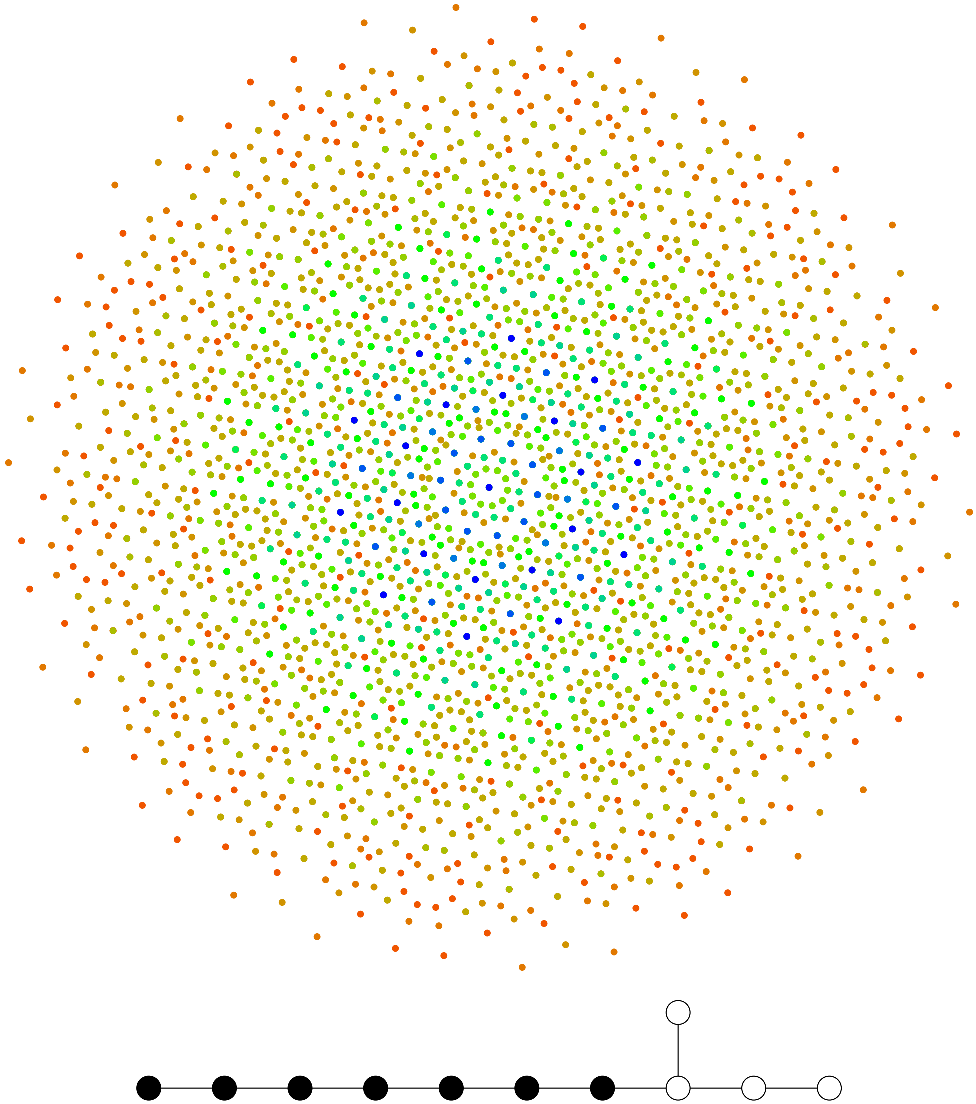


Figure B.4: E_{11} projected onto an A_4 Coxeter plane.

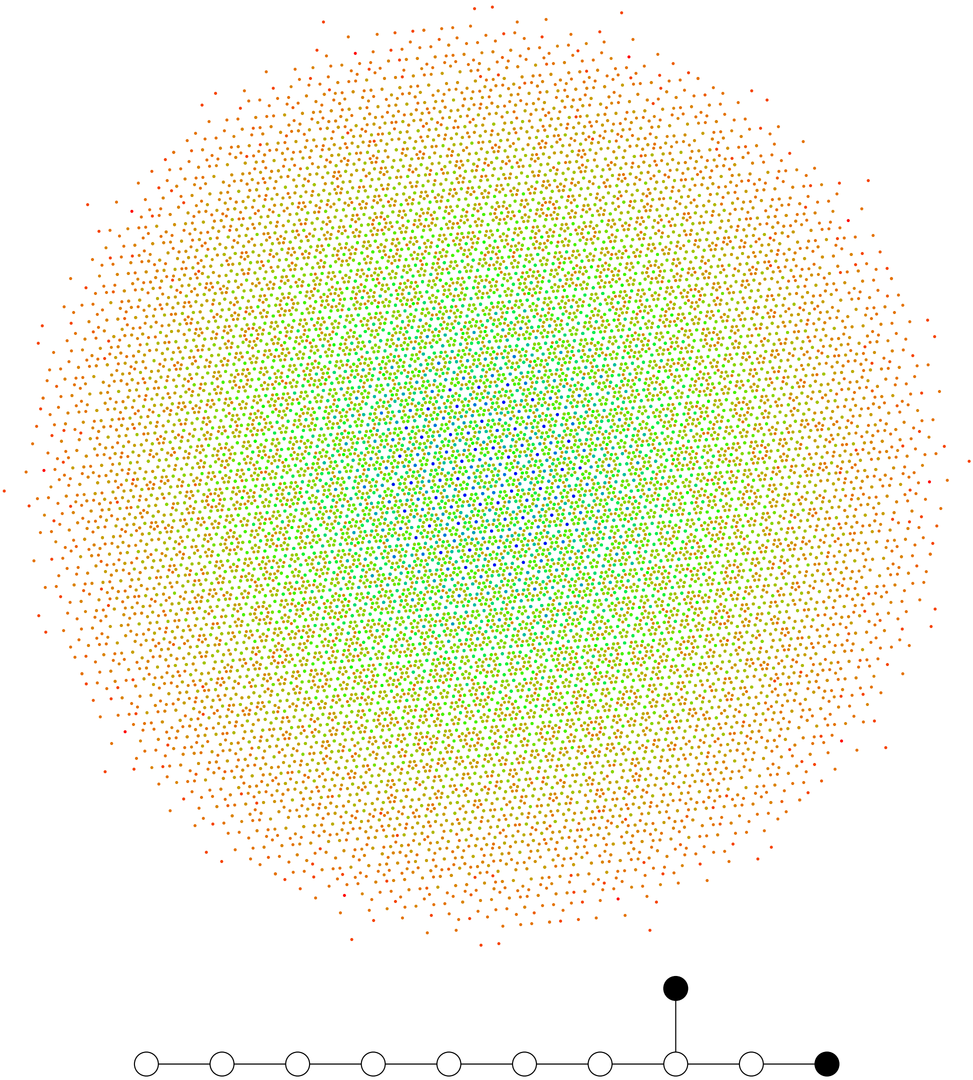


Figure B.5: E_{11} projected onto an A_9 Coxeter plane.

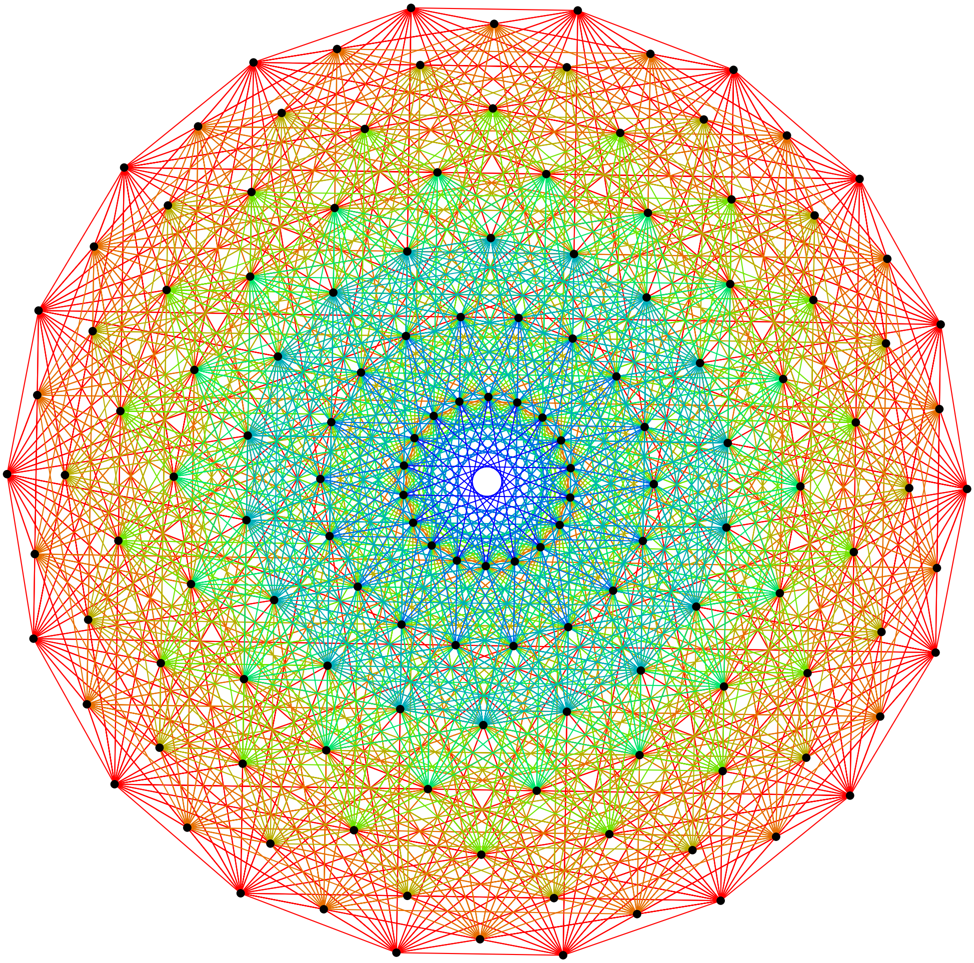


Figure B.6: Coxeter projection of D_{10}.

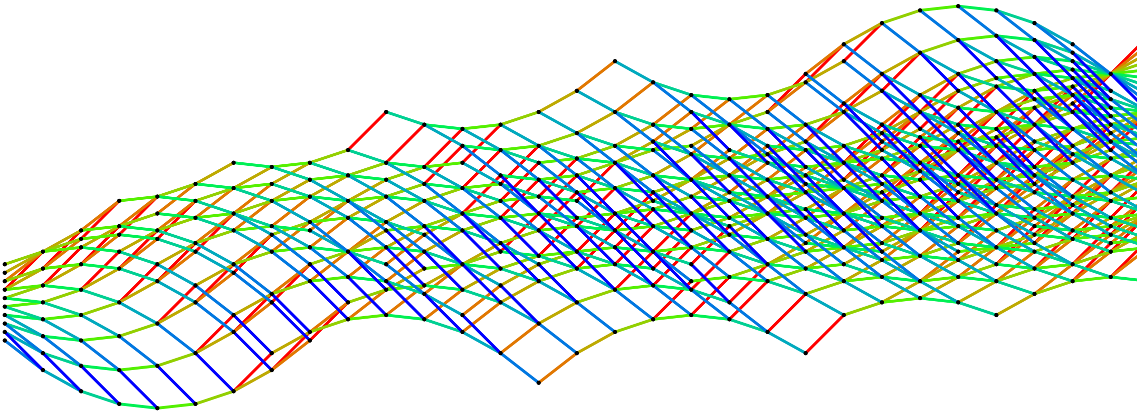


Figure B.7: Hasse diagram of E_{10} up to height 60, rotated over 90°.

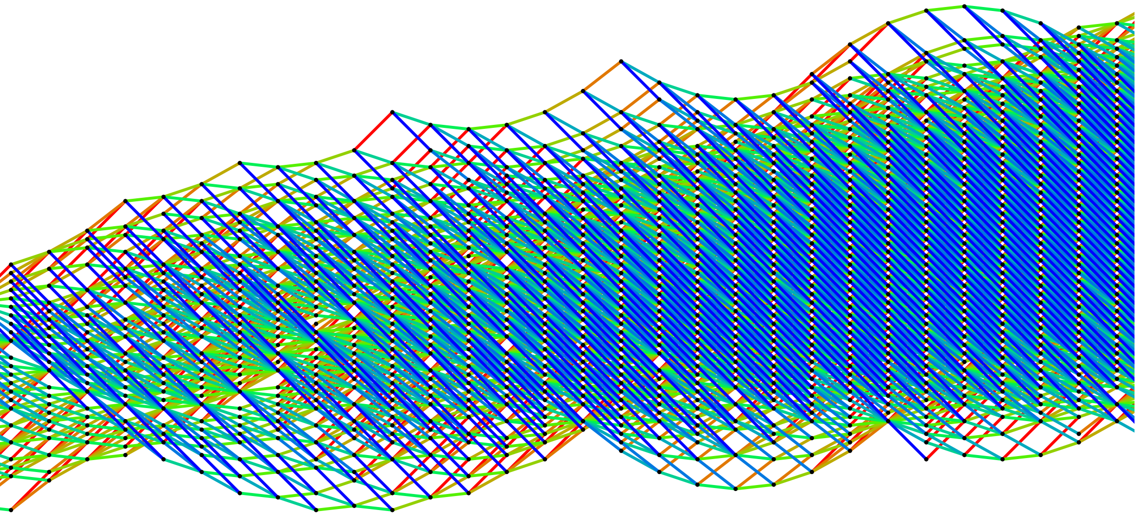


Figure B.7: Continued.

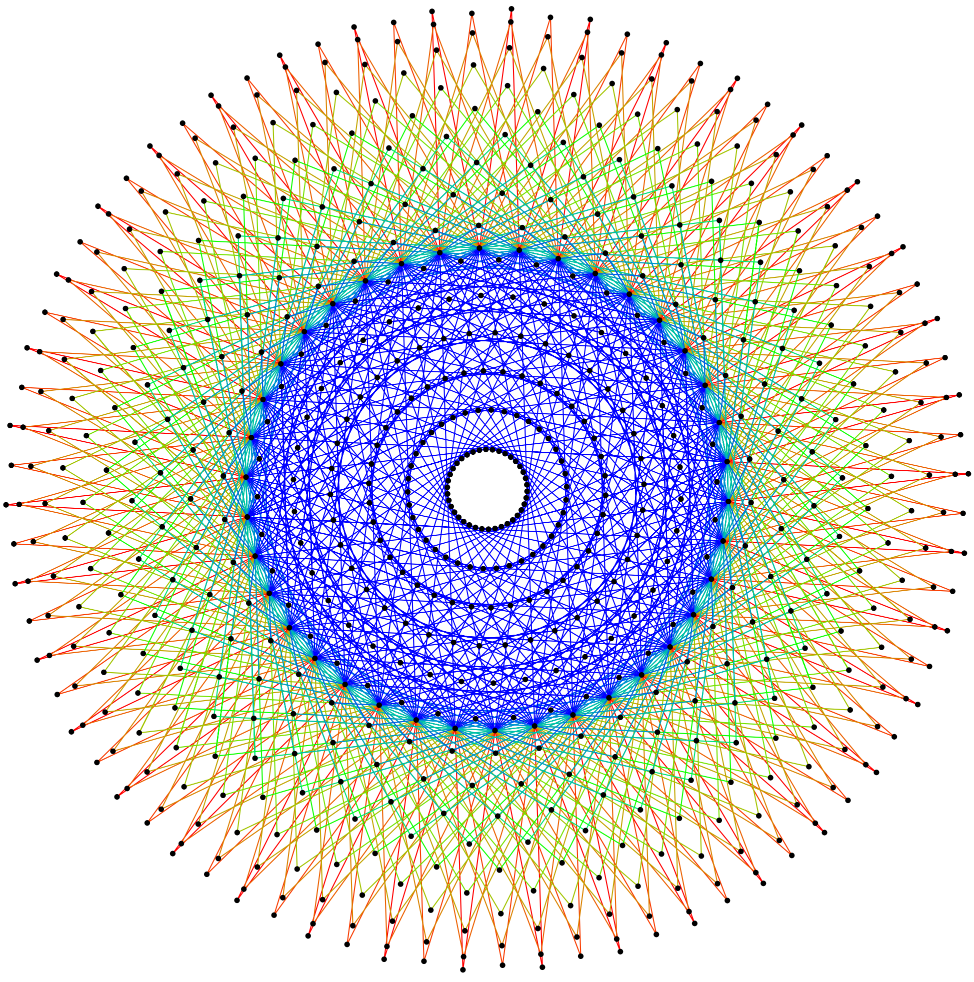


Figure B.8: Coxeter projection of B_{19}.

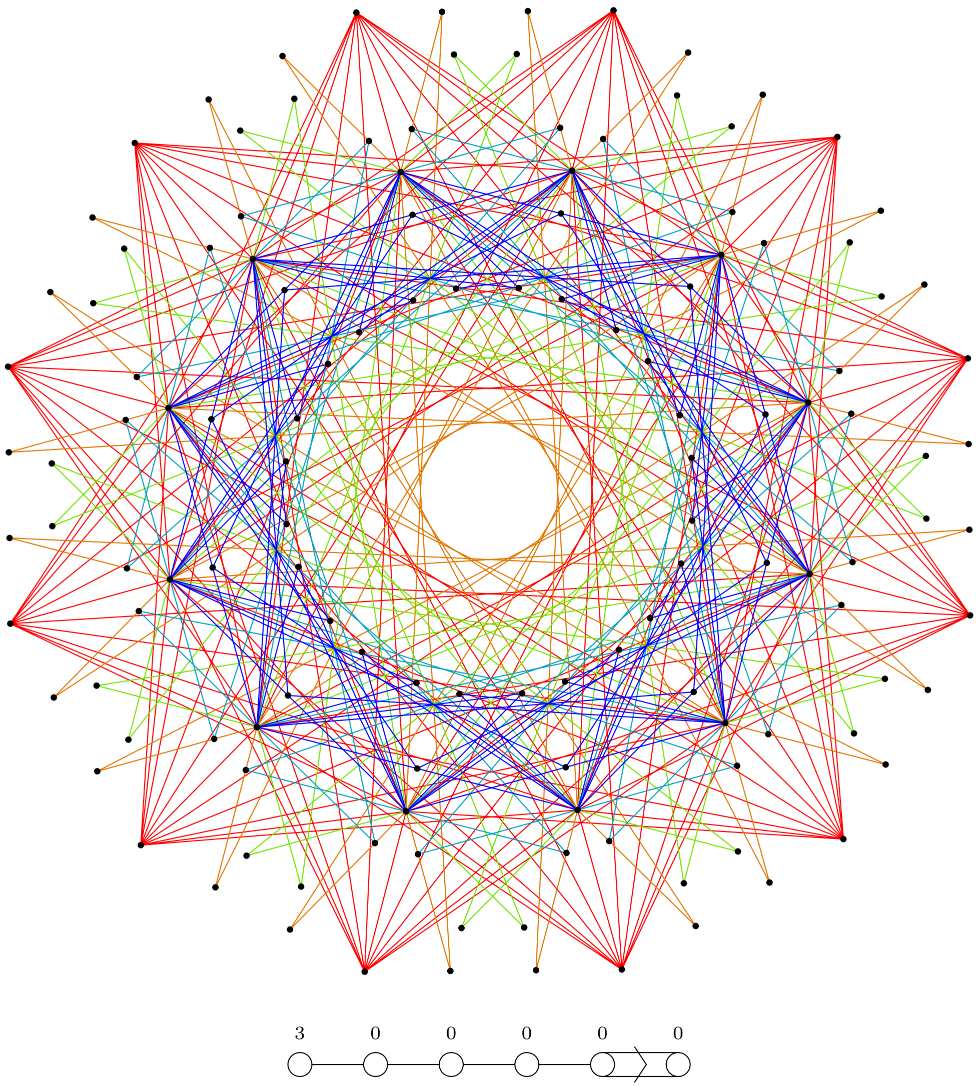


Figure B.9: Coxeter projection of the [300000] representation of B_6.

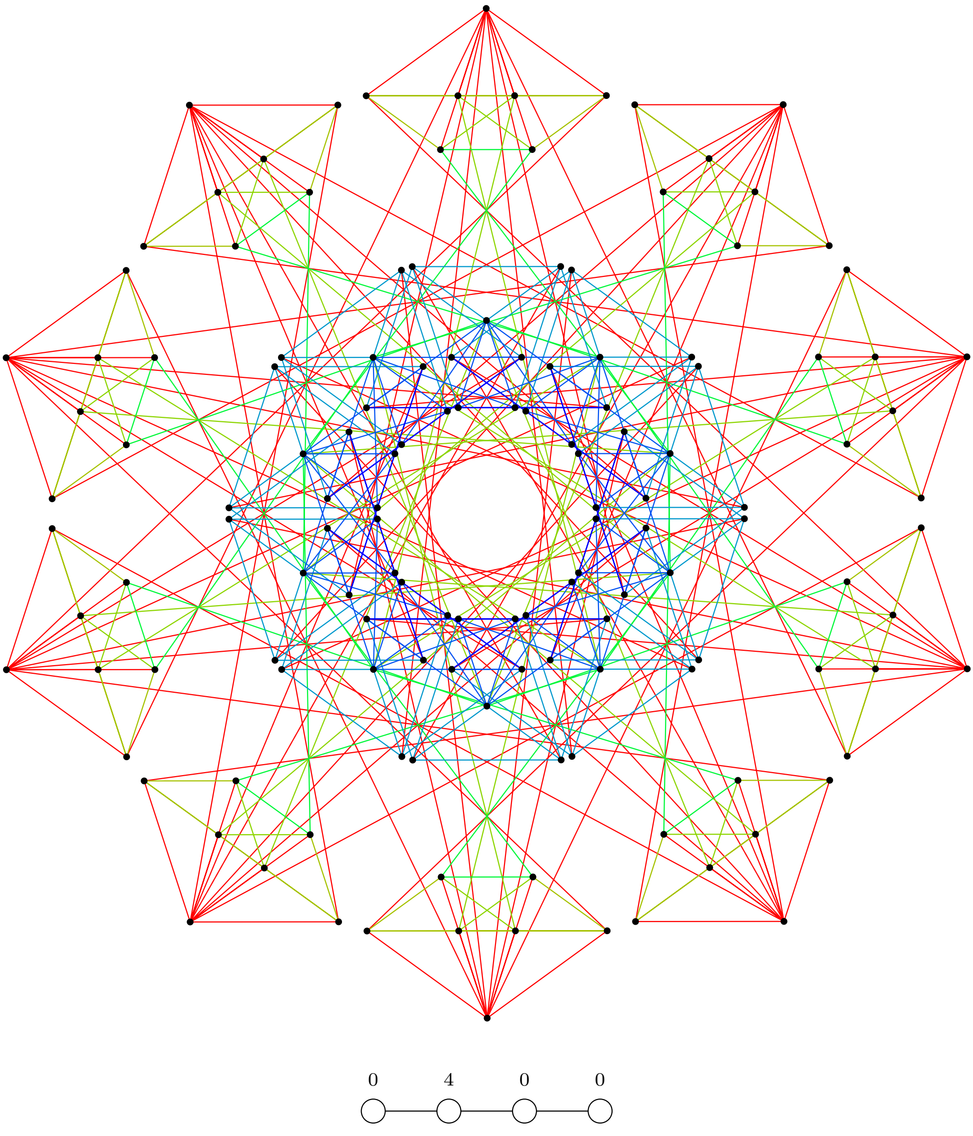


Figure B.10: Coxeter projection of the [0400] representation of A_4.

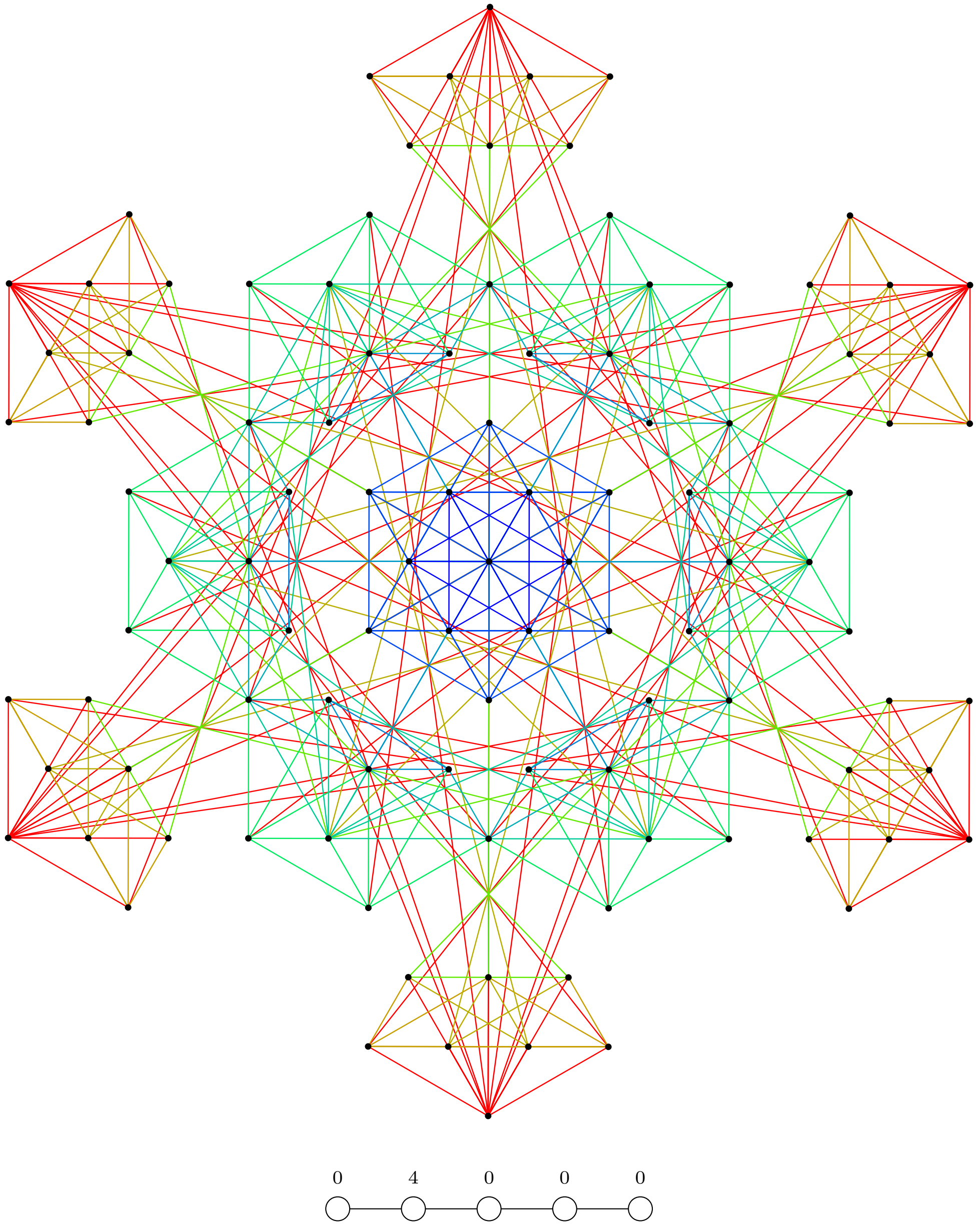


Figure B.11: Coxeter projection of the [04000] representation of A_5.

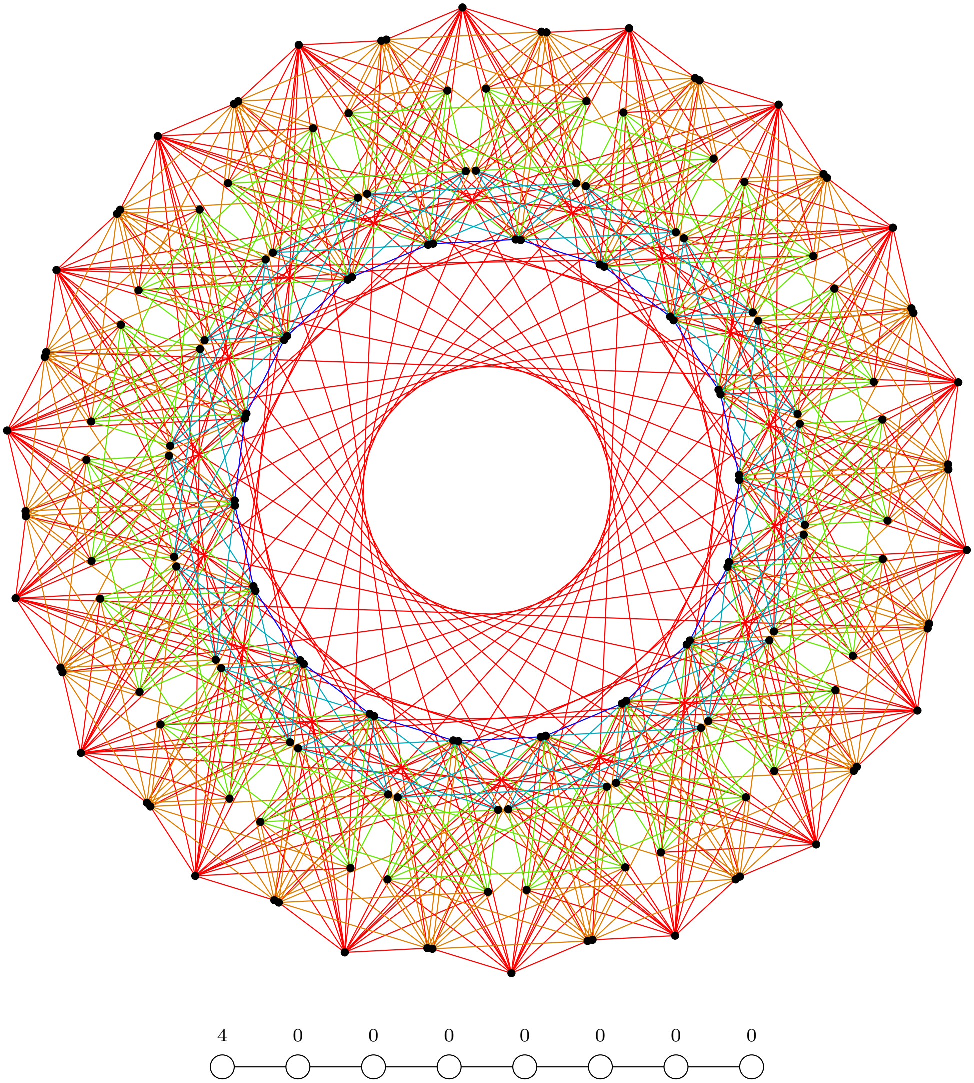


Figure B.12: Coxeter projection of the [40000000] representation of A_8.

C

Decomposition tables

C.1 E_{11} decompositions

Here we list the output of SimpLie [5] at low levels, using the various decompositions of E_{11} as in Table 6.1. The regular subalgebra splits into a part belonging to the gravity line A_n (the white nodes) and a part belonging to the internal duality group G_D (the grey nodes).

In the following tables we respectively list the levels l, the Dynkin labels p_{grav} and p_G of respectively A_n and G_D, the root labels m, the root length α^2, the dimensions d_{grav} and d_G of the representations of respectively A_n and G, the multiplicity of the root mult(α), the outer multiplicity μ, and the interpretation as a physical field. The deformation- and top-form potentials are indicated by 'de' and 'top', respectively. When the internal group does not exist, we do not list the corresponding columns. In all cases the Dynkin labels of the lowest weights of the representations are given. All tables are truncated at the point when the number of indices of the gravity subalgebra representations exceed the dimension. The order of the levels, Dynkin labels, and root labels as they appear in the tables are determined by the order of the node labels on the Dynkin diagram of Figure C.1.

The interpretation of the representations at level zero as the graviton is, unlike the p-forms at higher levels, not quite straightforward. The graviton emerges when one combines the adjoint representation of A_n with a scalar coming from one of the disabled nodes, see [23]. We have indicated these parts of the graviton by $\bar{g}_{\mu\nu}$ and $\hat{g}_{\mu\nu}$, respectively.

Column header	Meaning
l	Level(s) at which the representation occurs.
$-p_{\text{grav}}$	Gravity line part of negative Dynkin labels of lowest weight representation.
$-p_G$	Global symmetry group part of negative Dynkin labels of lowest weight representation.
m	Root vector of the lowest weight.
α^2	Norm of the lowest weight.
d_{grav}	Dimension of the gravity line part of the representation.
d_G	Dimension of the global symmetry group part of the representation.
mult(α)	Root multiplicity of the lowest weight.
μ	Outer multiplicity of the representation.
fields	Interpretation as supergravity fields.

Table C.1: Legend to the decomposition tables.

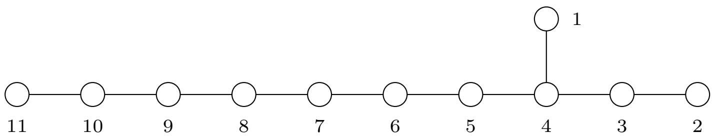


Figure C.1: Dynkin diagram of E_{11} indicating the ordering of Dynkin labels and root vectors in the tables of this section. They are ordered from high to low according to the labels of the nodes.

Table C.2: A_{10} representations in E_{11} $(D = 11)$

l	$-p_{\text{grav}}$	m	α^2	d_{grav}	mult(α)	μ	fields
0	1 0 0 0 0 0 0 0 0 0 1	0 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1	2	120	1	1	$\bar{g}_{\mu\nu}$
0	0 0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	11	1	$\hat{g}_{\mu\nu}$
1	0 0 1 0 0 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0 0 0	2	165	1	1	$p = 3$
2	0 0 0 0 0 1 0 0 0 0 0	2 1 2 3 2 1 0 0 0 0 0	2	462	1	1	$^\star\,(p = 3)$
3	1 0 0 0 0 0 0 0 1 0 0	3 1 3 5 4 3 2 1 0 0 0	2	1760	1	1	$^\star\,g_{\mu\nu}$

Table C.3: A_9 representations in E_{11} (IIA)

l	$-p_{\text{grav}}$	m	α^2	d_{grav}	mult(α)	μ	fields
0 0	1 0 0 0 0 0 0 0 0 1	0 0 -1 -1 -1 -1 -1 -1 -1 -1 -1	2	99	1	1	$\bar{g}_{\mu\nu}$
0 0	0 0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	11	2	$p = 0, \hat{g}_{\mu\nu}$
1 0	0 1 0 0 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0 0 0	2	45	1	1	$p = 2$
0 1	1 0 0 0 0 0 0 0 0 0	0 1 0 0 0 0 0 0 0 0 0	2	10	1	1	$p = 1$
1 1	0 0 1 0 0 0 0 0 0 0	1 1 1 1 0 0 0 0 0 0 0	2	120	1	1	$p = 3$
2 1	0 0 0 0 1 0 0 0 0 0	2 1 2 3 2 1 0 0 0 0 0	2	252	1	1	$^\star\,(p = 3)$
3 1	0 0 0 0 0 0 1 0 0	3 1 3 5 4 3 2 1 0 0 0	2	120	1	1	$^\star\,(p = 1)$
4 1	0 0 0 0 0 0 0 0 0 1	4 1 4 7 6 5 4 3 2 1 0	2	10	1	1	de
2 2	0 0 0 0 0 1 0 0 0	2 2 3 4 3 2 1 0 0 0 0	2	210	1	1	$^\star\,(p = 2)$
3 2	1 0 0 0 0 0 1 0 0	3 2 3 5 4 3 2 1 0 0 0	2	1155	1	1	$^\star\,g_{\mu\nu}$
3 2	0 0 0 0 0 0 0 0 1 0	3 2 4 6 5 4 3 2 1 0 0	0	45	8	1	$^\star\,(p = 0)$
4 2	0 1 0 0 0 0 0 0 1 0	4 2 4 6 5 4 3 2 1 0 0	2	1925	1	1	
4 2	1 0 0 0 0 0 0 0 0 1	4 2 4 7 6 5 4 3 2 1 0	0	99	8	1	
4 2	0 0 0 0 0 0 0 0 0 0	4 2 5 8 7 6 5 4 3 2 1	-2	1	46	2	top
3 3	1 0 0 0 0 0 0 0 1 0	3 3 4 6 5 4 3 2 1 0 0	2	440	1	1	

Table C.4: $A_9 \otimes A_1$ representations in E_{11} (IIB)

l	$-p_{\text{grav}}$	$-p_G$	m	α^2	d_{grav}	d_G	mult(α)	μ	fields
0	1 0 0 0 0 0 0 0 0 1	0	-1 0 0 -1 -1 -1 -1 -1 -1 -1 -1	2	99	1	1	1	$\bar{g}_{\mu\nu}$
0	0 0 0 0 0 0 0 0 0 0	2	0 -1 0 0 0 0 0 0 0 0 0	2	1	3	1	1	$p=0$
0	0 0 0 0 0 0 0 0 0 0	0	0 0 0 0 0 0 0 0 0 0 0	0	1	1	11	1	$\hat{g}_{\mu\nu}$
1	0 1 0 0 0 0 0 0 0 0	1	0 0 1 0 0 0 0 0 0 0 0	2	45	2	1	1	$p=2$
2	0 0 0 1 0 0 0 0 0 0	0	1 1 2 2 1 0 0 0 0 0 0	2	210	1	1	1	$p=4$
3	0 0 0 0 0 1 0 0 0 0	1	2 1 3 4 3 2 1 0 0 0 0	2	210	2	1	1	$^\star\,(p=2)$
4	1 0 0 0 0 0 1 0 0 0	0	2 2 4 5 4 3 2 1 0 0 0	2	1155	1	1	1	$^\star\,g_{\mu\nu}$
4	0 0 0 0 0 0 0 0 1 0	2	3 1 4 6 5 4 3 2 1 0 0	2	45	3	1	1	$^\star\,(p=0)$
5	0 1 0 0 0 0 0 0 1 0	1	3 2 5 6 5 4 3 2 1 0 0	2	1925	2	1	1	
5	1 0 0 0 0 0 0 0 0 1	1	3 2 5 7 6 5 4 3 2 1 0	0	99	2	8	1	
5	0 0 0 0 0 0 0 0 0 0	3	4 1 5 8 7 6 5 4 3 2 1	2	1	4	1	1	top
5	0 0 0 0 0 0 0 0 0 0	1	4 2 5 8 7 6 5 4 3 2 1	-2	1	2	46	1	top

Table C.5: $A_8 \otimes A_1$ representations in E_{11} $(D=9)$

l	$-p_{\text{grav}}$	$-p_G$	m	α^2	d_{grav}	d_G	mult(α)	μ	fields
0 0	1 0 0 0 0 0 0 0 1	0	0 0 0 -1 -1 -1 -1 -1 -1 -1 -1	2	80	1	1	1	$\bar{g}_{\mu\nu}$
0 0	0 0 0 0 0 0 0 0 0	2	0 -1 0 0 0 0 0 0 0 0 0	2	1	3	1	1	$p=0$
0 0	0 0 0 0 0 0 0 0 0	0	0 0 0 0 0 0 0 0 0 0 0	0	1	1	11	2	$p=0, \hat{g}_{\mu\nu}$
1 0	1 0 0 0 0 0 0 0 0	0	1 0 0 0 0 0 0 0 0 0 0	2	9	1	1	1	$p=1$
0 1	1 0 0 0 0 0 0 0 0	1	0 0 1 0 0 0 0 0 0 0 0	2	9	2	1	1	$p=1$
1 1	0 1 0 0 0 0 0 0 0	1	1 0 1 1 0 0 0 0 0 0 0	2	36	2	1	1	$p=2$
1 2	0 0 1 0 0 0 0 0 0	0	1 1 2 2 1 0 0 0 0 0 0	2	84	1	1	1	$p=3$
2 2	0 0 0 1 0 0 0 0 0	0	2 1 2 3 2 1 0 0 0 0 0	2	126	1	1	1	$^\star\,(p=3)$
2 3	0 0 0 0 1 0 0 0 0	1	2 1 3 4 3 2 1 0 0 0 0	2	126	2	1	1	$^\star\,(p=2)$
3 3	0 0 0 0 0 1 0 0 0	1	3 1 3 5 4 3 2 1 0 0 0	2	84	2	1	1	$^\star\,(p=1)$
2 4	0 0 0 0 0 1 0 0 0	0	2 2 4 5 4 3 2 1 0 0 0	2	84	1	1	1	$^\star\,(p=1)$
3 4	1 0 0 0 0 1 0 0 0	0	3 2 4 5 4 3 2 1 0 0 0	2	720	1	1	1	$^\star\,g_{\mu\nu}$
3 4	0 0 0 0 0 0 0 1 0	2	3 1 4 6 5 4 3 2 1 0 0	2	36	3	1	1	$^\star\,(p=0)$

3 4	0 0 0 0 0 0 1 0	0	3 2 4 6 5 4 3 2 1 0 0	0	36	1	8	1	$^\star\,(p=0)$
4 4	1 0 0 0 0 0 1 0	0	4 2 4 6 5 4 3 2 1 0 0	2	315	1	1	1	
4 4	0 0 0 0 0 0 0 1	2	4 1 4 7 6 5 4 3 2 1 0	2	9	3	1	1	de
3 5	1 0 0 0 0 0 1 0	1	3 2 5 6 5 4 3 2 1 0 0	2	315	2	1	1	
3 5	0 0 0 0 0 0 0 1	1	3 2 5 7 6 5 4 3 2 1 0	0	9	2	8	1	de
4 5	0 1 0 0 0 0 1 0	1	4 2 5 7 5 4 3 2 1 0 0	2	1215	2	1	1	
4 5	1 0 0 0 0 0 0 1	1	4 2 5 7 6 5 4 3 2 1 0	0	80	2	8	2	
4 5	0 0 0 0 0 0 0 0	3	4 1 5 8 7 6 5 4 3 2 1	2	1	4	1	1	top
4 5	0 0 0 0 0 0 0 0	1	4 2 5 8 7 6 5 4 3 2 1	-2	1	2	46	2	top
3 6	1 0 0 0 0 0 0 1	0	3 3 6 7 6 5 4 3 2 1 0	2	80	1	1	1	

Table C.6: $A_7 \otimes (A_2 \otimes A_1)$ representations in E_{11} $(D = 8)$

l	$-p_{\text{grav}}$	$-p_G$	m	α^2	d_{grav}	d_G	mult(α)	μ	fields
0	1 0 0 0 0 0 1	0 0 0	0 0 0 0 -1 -1 -1 -1 -1 -1 -1	2	63	1	1	1	$\bar{g}_{\mu\nu}$
0	0 0 0 0 0 0 0	0 1 1	0 -1 -1 0 0 0 0 0 0 0 0	2	1	8	1	1	$p=0$
0	0 0 0 0 0 0 0	2 0 0	-1 0 0 0 0 0 0 0 0 0 0	2	1	3	1	1	$p=0$
0	0 0 0 0 0 0 0	0 0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	1	11	1	$\hat{g}_{\mu\nu}$
1	1 0 0 0 0 0 0	1 0 1	0 0 0 1 0 0 0 0 0 0 0	2	8	6	1	1	$p=1$
2	0 1 0 0 0 0 0	0 1 0	1 0 1 2 1 0 0 0 0 0 0	2	28	3	1	1	$p=2$
3	0 0 1 0 0 0 0	1 0 0	1 1 2 3 2 1 0 0 0 0 0	2	56	2	1	1	$p=3$, $^\star\,(p=3)$
4	0 0 0 1 0 0 0	0 0 1	2 1 2 4 3 2 1 0 0 0 0	2	70	3	1	1	$^\star\,(p=2)$
5	0 0 0 0 1 0 0	1 1 0	2 1 3 5 4 3 2 1 0 0 0	2	56	6	1	1	$^\star\,(p=1)$
6	1 0 0 0 1 0 0	0 0 0	3 2 4 6 4 3 2 1 0 0 0	2	420	1	1	1	$^\star\,g_{\mu\nu}$
6	0 0 0 0 0 1 0	0 1 1	3 1 3 6 5 4 3 2 1 0 0	2	28	8	1	1	$^\star\,(p=0)$
6	0 0 0 0 0 1 0	2 0 0	2 2 4 6 5 4 3 2 1 0 0	2	28	3	1	1	$^\star\,(p=0)$
7	1 0 0 0 0 1 0	1 0 1	3 2 4 7 5 4 3 2 1 0 0	2	216	6	1	1	
7	0 0 0 0 0 0 1	1 2 0	3 1 4 7 6 5 4 3 2 1 0	2	8	12	1	1	de
7	0 0 0 0 0 0 1	1 0 1	3 2 4 7 6 5 4 3 2 1 0	0	8	6	8	1	de
8	0 1 0 0 0 1 0	0 1 0	4 2 5 8 6 4 3 2 1 0 0	2	720	3	1	1	
8	1 0 0 0 0 0 1	0 0 2	4 2 4 8 6 5 4 3 2 1 0	2	63	6	1	1	

8	1 0 0 0 0 0 0 1	2 1 0	3 2 5 8 6 5 4 3 2 1 0	2	63	9	1	1	
8	1 0 0 0 0 0 0 1	0 1 0	4 2 5 8 6 5 4 3 2 1 0	0	63	3	8	1	
8	0 0 0 0 0 0 0 0	0 2 1	4 1 4 8 7 6 5 4 3 2 1	2	1	15	1	1	top
8	0 0 0 0 0 0 0 0	2 1 0	3 2 5 8 7 6 5 4 3 2 1	0	1	9	8	1	top
8	0 0 0 0 0 0 0 0	0 1 0	4 2 5 8 7 6 5 4 3 2 1	-2	1	3	46	2	top

Table C.7: $A_6 \otimes A_4$ representations in E_{11} $(D = 7)$

l	$-p_{\text{grav}}$	$-p_G$	m	α^2	d_{grav}	d_G	mult(α)	μ	fields
0	1 0 0 0 0 1	0 0 0 0	0 0 0 0 0 -1 -1 -1 -1 -1 -1	2	48	1	1	1	$\bar{g}_{\mu\nu}$
0	0 0 0 0 0 0	1 1 0 0	-1 -1 -1 -1 0 0 0 0 0 0 0	2	1	24	1	1	$p = 0$
0	0 0 0 0 0 0	0 0 0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	1	11	1	$\hat{g}_{\mu\nu}$
1	1 0 0 0 0 0	0 0 0 1	0 0 0 0 1 0 0 0 0 0 0	2	7	10	1	1	$p = 1$
2	0 1 0 0 0 0	0 1 0 0	1 0 1 2 2 1 0 0 0 0 0	2	21	5	1	1	$p = 2$
3	0 0 1 0 0 0	1 0 0 0	1 1 2 3 3 2 1 0 0 0 0	2	35	5	1	1	$^\star\ (p = 2)$
4	0 0 0 1 0 0	0 0 1 0	2 1 2 4 4 3 2 1 0 0 0	2	35	10	1	1	$^\star\ (p = 1)$
5	1 0 0 1 0 0	0 0 0 0	3 2 4 6 5 3 2 1 0 0 0	2	224	1	1	1	$^\star\ g_{\mu\nu}$
5	0 0 0 0 1 0	1 1 0 0	2 1 3 5 5 4 3 2 1 0 0	2	21	24	1	1	$^\star\ (p = 0)$
6	1 0 0 0 1 0	0 0 0 1	3 2 4 6 6 4 3 2 1 0 0	2	140	10	1	1	
6	0 0 0 0 0 1	0 1 1 0	3 1 3 6 6 5 4 3 2 1 0	2	7	40	1	1	de
6	0 0 0 0 0 1	2 0 0 0	2 2 4 6 6 5 4 3 2 1 0	2	7	15	1	1	de
7	0 1 0 0 1 0	0 1 0 0	4 2 5 8 7 5 3 2 1 0 0	2	392	5	1	1	
7	1 0 0 0 0 1	1 0 1 0	3 2 4 7 7 5 4 3 2 1 0	2	48	45	1	1	
7	1 0 0 0 0 1	0 1 0 0	4 2 5 8 7 5 4 3 2 1 0	0	48	5	8	1	
7	0 0 0 0 0 0	1 2 0 0	3 1 4 7 7 6 5 4 3 2 1	2	1	70	1	1	top
7	0 0 0 0 0 0	1 0 1 0	3 2 4 7 7 6 5 4 3 2 1	0	1	45	8	1	top
7	0 0 0 0 0 0	0 1 0 0	4 2 5 8 7 6 5 4 3 2 1	-2	1	5	46	1	top

Table C.8: $A_5 \otimes E_5$ representations in E_{11} $(D = 6)$

l	$-p_{\text{grav}}$	$-p_G$	m	α^2	d_{grav}	d_G	mult(α)	μ	fields
0	0 0 0 0 0	0 0 1 0 0	-1 -1 -2 -2 -1 0 0 0 0 0 0	2	1	45	1	1	$p = 0$
0	1 0 0 0 1	0 0 0 0 0	0 0 0 0 0 0 -1 -1 -1 -1 -1	2	35	1	1	1	$\bar{g}_{\mu\nu}$
0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	1	11	1	$\hat{g}_{\mu\nu}$
1	1 0 0 0 0	0 0 0 0 1	0 0 0 0 0 1 0 0 0 0 0	2	6	16	1	1	$p = 1$
2	0 1 0 0 0	0 1 0 0 0	1 0 1 2 2 2 1 0 0 0 0	2	15	10	1	1	$p = 2$, $^\star(p = 2)$
3	0 0 1 0 0	1 0 0 0 0	1 1 2 3 3 3 2 1 0 0 0	2	20	16	1	1	$^\star(p = 1)$
4	0 0 0 1 0	0 0 1 0 0	2 1 2 4 4 4 3 2 1 0 0	2	15	45	1	1	$^\star(p = 0)$
4	1 0 1 0 0	0 0 0 0 0	3 2 4 6 5 4 2 1 0 0 0	2	105	1	1	1	$^\star g_{\mu\nu}$
5	1 0 0 1 0	0 0 0 0 1	3 2 4 6 5 5 3 2 1 0 0	2	84	16	1	1	
5	0 0 0 0 1	1 1 0 0 0	2 1 3 5 5 5 4 3 2 1 0	2	6	144	1	1	de
6	1 0 0 0 1	0 0 0 1 0	3 2 4 6 6 6 4 3 2 1 0	2	35	120	1	1	
6	0 1 0 1 0	0 1 0 0 0	4 2 5 8 7 6 4 2 1 0 0	2	189	10	1	1	
6	0 0 0 0 0	0 1 1 0 0	3 1 3 6 6 6 5 4 3 2 1	2	1	320	1	1	top
6	0 0 0 0 0	2 0 0 0 0	2 2 4 6 6 6 5 4 3 2 1	2	1	126	1	1	top
6	1 0 0 0 1	0 1 0 0 0	4 2 5 8 7 6 4 3 2 1 0	0	35	10	8	1	
6	0 0 0 0 0	0 1 0 0 0	4 2 5 8 7 6 5 4 3 2 1	-2	1	10	46	1	top

Table C.9: $A_4 \otimes E_6$ representations in E_{11} $(D = 5)$

l	$-p_{\text{grav}}$	$-p_G$	m	α^2	d_{grav}	d_G	mult(α)	μ	fields
0	0 0 0 0	1 0 0 0 0 0	-2 -1 -2 -3 -2 -1 0 0 0 0 0	2	1	78	1	1	$p = 0$
0	1 0 0 1	0 0 0 0 0 0	0 0 0 0 0 0 0 -1 -1 -1 -1	2	24	1	1	1	$\bar{g}_{\mu\nu}$
0	0 0 0 0	0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	1	11	1	$\hat{g}_{\mu\nu}$
1	1 0 0 0	0 0 0 0 0 1	0 0 0 0 0 0 1 0 0 0 0	2	5	27	1	1	$p = 1$
2	0 1 0 0	0 1 0 0 0 0	1 0 1 2 2 2 2 1 0 0 0	2	10	27	1	1	$^\star(p = 1)$
3	0 0 1 0	1 0 0 0 0 0	1 1 2 3 3 3 3 2 1 0 0	2	10	78	1	1	$^\star(p = 0)$
3	1 1 0 0	0 0 0 0 0 0	3 2 4 6 5 4 3 1 0 0 0	2	40	1	1	1	$^\star g_{\mu\nu}$
4	0 0 0 1	0 0 1 0 0 0	2 1 2 4 4 4 4 3 2 1 0	2	5	351	1	1	de

4	1 0 1 0	0 0 0 0 0 1	3 2 4 6 5 4 4 2 1 0 0	2	45	27	1	1	
5	1 0 0 1	0 0 0 0 1 0	3 2 4 6 5 5 5 3 2 1 0	2	24	351	1	1	
5	0 0 0 0	1 1 0 0 0 0	2 1 3 5 5 5 5 4 3 2 1	2	1	1728	1	1	top
5	0 1 1 0	0 1 0 0 0 0	4 2 5 8 7 6 5 3 1 0 0	2	75	27	1	1	
5	1 0 0 1	0 1 0 0 0 0	4 2 5 8 7 6 5 3 2 1 0	0	24	27	8	1	
5	0 0 0 0	0 1 0 0 0 0	4 2 5 8 7 6 5 4 3 2 1	-2	1	27	46	1	top

Table C.10: $A_3 \otimes E_7$ representations in E_{11} $(D = 4)$

l	$-p_{\text{grav}}$	$-p_G$	m	α^2	d_{grav}	d_G	mult(α)	μ	fields
0	0 0 0	0 1 0 0 0 0 0	-2 -2 -3 -4 -3 -2 -1 0 0 0 0	2	1	133	1	1	$p = 0$
0	1 0 1	0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 -1 -1 -1	2	15	1	1	1	$\bar{g}_{\mu\nu}$
0	0 0 0	0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	1	11	1	$\hat{g}_{\mu\nu}$
1	1 0 0	0 0 0 0 0 0 1	0 0 0 0 0 0 0 1 0 0 0	2	4	56	1	1	$p = 1$, $^\star\,(p = 1)$
2	0 1 0	0 1 0 0 0 0 0	1 0 1 2 2 2 2 2 1 0 0	2	6	133	1	1	$^\star\,(p = 0)$
2	2 0 0	0 0 0 0 0 0 0	3 2 4 6 5 4 3 2 0 0 0	2	10	1	1	1	$^\star\,g_{\mu\nu}$
3	0 0 1	1 0 0 0 0 0 0	1 1 2 3 3 3 3 3 2 1 0	2	4	912	1	1	de
3	1 1 0	0 0 0 0 0 0 1	3 2 4 6 5 4 3 3 1 0 0	2	20	56	1	1	
4	0 0 0	0 0 1 0 0 0 0	2 1 2 4 4 4 4 4 3 2 1	2	1	8645	1	1	top
4	1 0 1	0 0 0 0 0 1 0	3 2 4 6 5 4 4 4 2 1 0	2	15	1539	1	1	
4	0 2 0	0 1 0 0 0 0 0	4 2 5 8 7 6 5 4 2 0 0	2	20	133	1	1	
4	1 0 1	0 1 0 0 0 0 0	4 2 5 8 7 6 5 4 2 1 0	0	15	133	8	1	
4	0 0 0	0 1 0 0 0 0 0	4 2 5 8 7 6 5 4 3 2 1	-2	1	133	46	1	top
4	2 1 0	0 0 0 0 0 0 0	6 4 8 12 10 8 6 4 1 0 0	2	45	1	1	1	
4	1 0 1	0 0 0 0 0 0 0	6 4 8 12 10 8 6 4 2 1 0	-2	15	1	44	1	

Table C.11: $A_2 \otimes E_8$ representations in E_{11} $(D = 3)$

l	$-p_{\text{grav}}$	$-p_G$	m	α^2	d_{grav}	d_G	mult(α)	μ	fields
0	0 0	0 0 0 0 0 0 0 1	-3 -2 -4 -6 -5 -4 -3 -2 0 0 0	2	1	248	1	1	$p = 0$
0	1 1	0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 -1 -1	2	8	1	1	1	$\bar{g}_{\mu\nu}$
0	0 0	0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	1	11	1	$\hat{g}_{\mu\nu}$
1	1 0	0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1 0 0	2	3	248	1	1	* $(p = 0)$
2	0 1	0 1 0 0 0 0 0 0	1 0 1 2 2 2 2 2 2 1 0	2	3	3875	1	1	de
2	2 0	0 0 0 0 0 0 0 1	3 2 4 6 5 4 3 2 2 0 0	2	6	248	1	1	
2	0 1	0 0 0 0 0 0 0 0	6 4 8 12 10 8 6 4 2 1 0	-2	3	1	44	1	de
3	0 0	1 0 0 0 0 0 0 0	1 1 2 3 3 3 3 3 3 2 1	2	1	147250	1	1	top
3	1 1	0 0 0 0 0 0 1 0	3 2 4 6 5 4 3 3 3 1 0	2	8	30380	1	1	
3	1 1	0 1 0 0 0 0 0 0	4 2 5 8 7 6 5 4 3 1 0	0	8	3875	8	1	
3	0 0	0 1 0 0 0 0 0 0	4 2 5 8 7 6 5 4 3 2 1	-2	1	3875	46	1	top
3	3 0	0 0 0 0 0 0 0 1	6 4 8 12 10 8 6 4 3 0 0	2	10	248	1	1	
3	1 1	0 0 0 0 0 0 0 1	6 4 8 12 10 8 6 4 3 1 0	-2	8	248	44	1	
3	0 0	0 0 0 0 0 0 0 1	6 4 8 12 10 8 6 4 3 2 1	-4	1	248	206	1	top
3	1 1	0 0 0 0 0 0 0 0	9 6 12 18 15 12 9 6 3 1 0	-4	8	1	192	1	

C.2 D_8^{+++} decompositions

In this section we list similar tables as in section C.1, only now for the Kac-Moody algebra D_8^{+++}. The corresponding Dynkin diagram decompositions can be found in Table 6.3. The column-layout is the same as in the preceding section (see also Table C.1). The Dynkin diagram of Figure C.2 indicates the ordering of the Dynkin labels and root vectors.

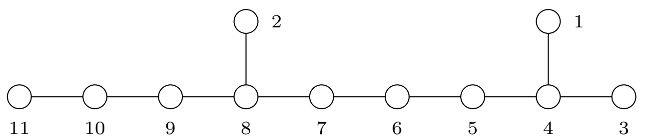


Figure C.2: Dynkin diagram of D_8^{+++} indicating the ordering of Dynkin labels and root vectors in the tables of this section. They are ordered from high to low according to the labels of the nodes.

Table C.12: A_9 representations in D_8^{+++} $(D = 10)$

l	$-p_{\text{grav}}$	m	α^2	d_{grav}	mult(α)	μ	fields
0 0	1 0 0 0 0 0 0 0 1	0 0 -1 -1 -1 -1 -1 -1 -1 -1 -1	2	99	1	1	$\bar{g}_{\mu\nu}$
0 0	0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	11	2	$p = 0, \hat{g}_{\mu\nu}$
1 0	0 1 0 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0 0 0	2	45	1	1	$p = 2$
0 1	0 0 0 0 0 1 0 0 0	0 1 0 0 0 0 0 0 0 0 0	2	210	1	1	$^\star\,(p = 2)$
1 1	1 0 0 0 0 0 1 0 0	1 1 0 1 1 1 1 1 0 0 0	2	1155	1	1	$^\star\, g_{\mu\nu}$
1 1	0 0 0 0 0 0 0 1 0	1 1 1 2 2 2 2 2 1 0 0	0	45	8	1	$^\star\,(p = 0)$
2 1	0 1 0 0 0 0 0 1 0	2 1 1 2 2 2 2 2 1 0 0	2	1925	1	1	
2 1	1 0 0 0 0 0 0 0 1	2 1 1 3 3 3 3 3 2 1 0	0	99	8	1	
2 1	0 0 0 0 0 0 0 0 0	2 1 2 4 4 4 4 4 3 2 1	-2	1	45	1	top

Table C.13: A_8 representations in D_8^{+++} $(D = 9)$

l	$-p_{\text{grav}}$	m	α^2	d_{grav}	mult(α)	μ	fields
0 0 0	1 0 0 0 0 0 0 1	0 0 0 -1 -1 -1 -1 -1 -1 -1 -1	2	80	1	1	$\bar{g}_{\mu\nu}$
0 0 0	0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	11	3	$p = 0, \hat{g}_{\mu\nu}$
1 0 0	1 0 0 0 0 0 0 0	1 0 0 0 0 0 0 0 0 0 0	2	9	1	1	$p = 1$
0 1 0	0 0 0 0 1 0 0 0	0 1 0 0 0 0 0 0 0 0 0	2	126	1	1	$^\star\,(p = 2)$
0 0 1	1 0 0 0 0 0 0 0	0 0 1 0 0 0 0 0 0 0 0	2	9	1	1	$p = 1$
1 1 0	0 0 0 0 0 1 0 0	1 1 0 1 1 1 1 1 0 0 0	2	84	1	1	$^\star\,(p = 1)$
1 0 1	0 1 0 0 0 0 0 0	1 0 1 1 0 0 0 0 0 0 0	2	36	1	1	$p = 2$
0 1 1	0 0 0 0 0 1 0 0	0 1 1 1 1 1 1 1 0 0 0	2	84	1	1	$^\star\,(p = 1)$
1 1 1	1 0 0 0 0 1 0 0	1 1 1 1 1 1 1 1 0 0 0	2	720	1	1	$^\star\, g_{\mu\nu}$
1 1 1	0 0 0 0 0 0 1 0	1 1 1 2 2 2 2 2 1 0 0	0	36	8	2	$^\star\,(p = 0)$
2 1 1	1 0 0 0 0 0 1 0	2 1 1 2 2 2 2 2 1 0 0	2	315	1	1	
2 1 1	0 0 0 0 0 0 0 1	2 1 1 3 3 3 3 3 2 1 0	0	9	8	1	de
1 1 2	1 0 0 0 0 0 1 0	1 1 2 2 2 2 2 2 1 0 0	2	315	1	1	
1 1 2	0 0 0 0 0 0 0 1	1 1 2 3 3 3 3 3 2 1 0	0	9	8	1	de
2 1 2	0 1 0 0 0 0 1 0	2 1 2 3 2 2 2 2 1 0 0	2	1215	1	1	
2 1 2	1 0 0 0 0 0 0 1	2 1 2 3 3 3 3 3 2 1 0	0	80	8	2	
2 1 2	0 0 0 0 0 0 0 0	2 1 2 4 4 4 4 4 3 2 1	-2	1	45	2	top

Table C.14: $A_1 \otimes A_1 \otimes A_7$ representations in D_8^{+++} $(D = 8)$

l	$-p_{\text{grav}}$	$-p_G$	m	α^2	d_{grav}	d_G	mult(α)	μ	fields
0 0	1 0 0 0 0 0 1	0 0	0 0 0 0 -1 -1 -1 -1 -1 -1 -1	2	63	1	1	1	$\bar{g}_{\mu\nu}$
0 0	0 0 0 0 0 0 0	0 2	0 0 -1 0 0 0 0 0 0 0 0	2	1	3	1	1	$p = 0$
0 0	0 0 0 0 0 0 0	2 0	-1 0 0 0 0 0 0 0 0 0 0	2	1	3	1	1	$p = 0$
0 0	0 0 0 0 0 0 0	0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	1	11	2	$p = 0, \hat{g}_{\mu\nu}$
1 0	0 0 0 1 0 0 0	0 0	0 1 0 0 0 0 0 0 0 0 0	2	70	1	1	1	* $(p = 2)$
0 1	1 0 0 0 0 0 0	1 1	0 0 0 1 0 0 0 0 0 0 0	2	8	4	1	1	$p = 1$
1 1	0 0 0 0 1 0 0	1 1	0 1 0 1 1 1 1 1 0 0 0	2	56	4	1	1	* $(p = 1)$
2 0	1 0 0 0 0 0 1	0 0	0 2 0 0 0 1 2 3 2 1 0	2	63	1	1	1	
0 2	0 1 0 0 0 0 0	0 0	1 0 1 2 1 0 0 0 0 0 0	2	28	1	1	1	$p = 2$
1 2	1 0 0 0 1 0 0	0 0	1 1 1 2 1 1 1 1 0 0 0	2	420	1	1	1	* $g_{\mu\nu}$
1 2	0 0 0 0 0 1 0	0 2	1 1 0 2 2 2 2 2 1 0 0	2	28	3	1	1	* $(p = 0)$
1 2	0 0 0 0 0 1 0	2 0	0 1 1 2 2 2 2 2 1 0 0	2	28	3	1	1	* $(p = 0)$
1 2	0 0 0 0 0 1 0	0 0	1 1 1 2 2 2 2 2 1 0 0	0	28	1	8	1	* $(p = 0)$
1 3	1 0 0 0 0 1 0	1 1	1 1 1 3 2 2 2 2 1 0 0	2	216	4	1	1	
1 3	0 0 0 0 0 0 1	1 1	1 1 1 3 3 3 3 3 2 1 0	0	8	4	8	2	de
1 4	0 1 0 0 0 1 0	0 0	2 1 2 4 3 2 2 2 1 0 0	2	720	1	1	1	
1 4	1 0 0 0 0 0 1	0 2	2 1 1 4 3 3 3 3 2 1 0	2	63	3	1	1	
1 4	1 0 0 0 0 0 1	2 0	1 1 2 4 3 3 3 3 2 1 0	2	63	3	1	1	
1 4	1 0 0 0 0 0 1	0 0	2 1 2 4 3 3 3 3 2 1 0	0	63	1	8	1	
1 4	0 0 0 0 0 0 0	0 2	2 1 1 4 4 4 4 4 3 2 1	0	1	3	8	1	top
1 4	0 0 0 0 0 0 0	2 0	1 1 2 4 4 4 4 4 3 2 1	0	1	3	8	1	top
1 4	0 0 0 0 0 0 0	0 0	2 1 2 4 4 4 4 4 3 2 1	-2	1	1	45	2	top

Table C.15: $A_3 \otimes A_6$ representations in D_8^{+++} $(D = 7)$

l	$-p_{\text{grav}}$	$-p_G$	m	α^2	d_{grav}	d_G	mult(α)	μ	fields
0 0	1 0 0 0 0 1	0 0 0	0 0 0 0 0 -1 -1 -1 -1 -1 -1	2	48	1	1	1	$\bar{g}_{\mu\nu}$
0 0	0 0 0 0 0 0	1 1 0	-1 0 -1 -1 0 0 0 0 0 0 0	2	1	15	1	1	$p = 0$
0 0	0 0 0 0 0 0	0 0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	1	11	2	$p = 0, \hat{g}_{\mu\nu}$
1 0	0 0 1 0 0 0	0 0 0	0 1 0 0 0 0 0 0 0 0 0	2	35	1	1	1	$^\star\ (p = 2)$
0 1	1 0 0 0 0 0	0 0 1	0 0 0 0 1 0 0 0 0 0 0	2	7	6	1	1	$p = 1$
1 1	0 0 0 1 0 0	0 0 1	0 1 0 0 1 1 1 1 0 0 0	2	35	6	1	1	$^\star\ (p = 1)$
2 0	0 0 0 0 0 1	0 0 0	0 2 0 0 0 1 2 3 2 1 0	2	7	1	1	1	de
0 2	0 1 0 0 0 0	0 0 0	1 0 1 2 2 1 0 0 0 0 0	2	21	1	1	1	$p = 2$
2 1	1 0 0 0 0 1	0 0 1	0 2 0 0 1 1 2 3 2 1 0	2	48	6	1	1	
2 1	0 0 0 0 0 0	0 0 1	0 2 0 0 1 2 3 4 3 2 1	0	1	6	7	1	top
1 2	1 0 0 1 0 0	0 0 0	1 1 1 2 2 1 1 1 0 0 0	2	224	1	1	1	$^\star\ g_{\mu\nu}$
1 2	0 0 0 0 1 0	1 1 0	0 1 0 1 2 2 2 2 1 0 0	2	21	15	1	1	$^\star\ (p = 0)$
1 2	0 0 0 0 1 0	0 0 0	1 1 1 2 2 2 2 2 1 0 0	0	21	1	8	1	$^\star\ (p = 0)$
1 3	1 0 0 0 1 0	0 0 1	1 1 1 2 3 2 2 2 1 0 0	2	140	6	1	1	
1 3	0 0 0 0 0 1	0 2 0	1 1 0 2 3 3 3 3 2 1 0	2	7	10	1	1	de
1 3	0 0 0 0 0 1	2 0 0	0 1 1 2 3 3 3 3 2 1 0	2	7	10	1	1	de
1 3	0 0 0 0 0 1	0 0 1	1 1 1 2 3 3 3 3 2 1 0	0	7	6	8	1	de
1 4	0 1 0 0 1 0	0 0 0	2 1 2 4 4 3 2 2 1 0 0	2	392	1	1	1	
1 4	1 0 0 0 0 1	1 1 0	1 1 1 3 4 3 3 3 2 1 0	2	48	15	1	1	
1 4	1 0 0 0 0 1	0 0 0	2 1 2 4 4 3 3 3 2 1 0	0	48	1	8	1	
1 4	0 0 0 0 0 0	1 1 0	1 1 1 3 4 4 4 4 3 2 1	0	1	15	8	2	top
1 4	0 0 0 0 0 0	0 0 0	2 1 2 4 4 4 4 4 3 2 1	-2	1	1	45	1	top

Table C.16: $D_4 \otimes A_5$ representations in D_8^{+++} $(D = 6a)$

l	$-p_{\text{grav}}$	$-p_G$	m	α^2	d_{grav}	d_G	mult(α)	μ	fields
0 0	1 0 0 0 1	0 0 0 0	0 0 0 0 0 0 -1 -1 -1 -1 -1	2	35	1	1	1	$\bar{g}_{\mu\nu}$
0 0	0 0 0 0 0	0 0 1 0	-1 0 -1 -2 -1 0 0 0 0 0 0	2	1	28	1	1	$p = 0$
0 0	0 0 0 0 0	0 0 0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	1	11	2	$p = 0, \bar{g}_{\mu\nu}$
1 0	0 1 0 0 0	0 0 0 0	0 1 0 0 0 0 0 0 0 0 0	2	15	1	1	1	$^\star\,(p = 2)$
0 1	1 0 0 0 0	0 0 0 1	0 0 0 0 0 1 0 0 0 0 0	2	6	8	1	1	$p = 1$
1 1	0 0 1 0 0	0 0 0 1	0 1 0 0 0 1 1 1 0 0 0	2	20	8	1	1	$^\star\,(p = 1)$
0 2	0 1 0 0 0	0 0 0 0	1 0 1 2 2 2 1 0 0 0 0	2	15	1	1	1	$p = 2$
2 1	0 0 0 0 1	0 0 0 1	0 2 0 0 0 1 2 3 2 1 0	2	6	8	1	1	de
1 2	0 0 0 1 0	0 0 1 0	0 1 0 0 1 2 2 2 1 0 0	2	15	28	1	1	$^\star\,(p = 0)$
1 2	1 0 1 0 0	0 0 0 0	1 1 1 2 2 2 1 1 0 0 0	2	105	1	1	1	$^\star\,g_{\mu\nu}$
1 2	0 0 0 1 0	0 0 0 0	1 1 1 2 2 2 2 2 1 0 0	0	15	1	8	1	$^\star\,(p = 0)$
2 2	1 0 0 0 1	0 0 1 0	0 2 0 0 1 2 2 3 2 1 0	2	35	28	1	1	
2 2	0 1 0 1 0	0 0 0 0	1 2 1 2 2 2 2 2 1 0 0	2	189	1	1	1	
2 2	0 0 0 0 0	0 0 0 2	0 2 0 0 0 2 3 4 3 2 1	2	1	35	1	1	top
2 2	1 0 0 0 1	0 0 0 0	1 2 1 2 2 2 2 3 2 1 0	0	35	1	8	1	
2 2	0 0 0 0 0	0 0 1 0	0 2 0 0 1 2 3 4 3 2 1	0	1	28	7	1	top
2 2	0 0 0 0 0	0 0 0 0	1 2 1 2 2 2 3 4 3 2 1	-2	1	1	43	2	top
1 3	1 0 0 1 0	0 0 0 1	1 1 1 2 2 3 2 2 1 0 0	2	84	8	1	1	
1 3	0 0 0 0 1	1 1 0 0	0 1 0 1 2 3 3 3 2 1 0	2	6	56	1	1	de
1 3	0 0 0 0 1	0 0 0 1	1 1 1 2 2 3 3 3 2 1 0	0	6	8	8	1	de
1 4	1 0 0 0 1	0 0 1 0	1 1 1 2 3 4 3 3 2 1 0	2	35	28	1	1	
1 4	0 1 0 1 0	0 0 0 0	2 1 2 4 4 4 3 2 1 0 0	2	189	1	1	1	
1 4	0 0 0 0 0	0 2 0 0	1 1 0 2 3 4 4 4 3 2 1	2	1	35	1	1	top
1 4	0 0 0 0 0	2 0 0 0	0 1 1 2 3 4 4 4 3 2 1	2	1	35	1	1	top
1 4	1 0 0 0 1	0 0 0 0	2 1 2 4 4 4 3 3 2 1 0	0	35	1	8	1	
1 4	0 0 0 0 0	0 0 1 0	1 1 1 2 3 4 4 4 3 2 1	0	1	28	8	1	top
1 4	0 0 0 0 0	0 0 0 0	2 1 2 4 4 4 4 4 3 2 1	-2	1	1	45	1	top

Table C.17: $D_5 \otimes A_5$ representations in D_8^{+++} $(D = 6b)$

l	$-p_{\text{grav}}$	$-p_G$	m	α^2	d_{grav}	d_G	mult(α)	μ	fields
0	0 0 0 0 0	0 0 0 1 0	-1 0 -1 -2 -2 -1 0 0 0 0 0	2	1	45	1	1	$p = 0$
0	1 0 0 0 1	0 0 0 0 0	0 -1 0 0 0 0 0 -1 -1 -1 -1	2	35	1	1	1	$\bar{g}_{\mu\nu}$
0	0 0 0 0 0	0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	1	11	1	$\hat{g}_{\mu\nu}$
1	0 1 0 0 0	0 0 0 0 1	0 0 0 0 0 0 1 0 0 0 0	2	15	10	1	1	$p = 2, ^\star(p = 2)$
2	0 0 0 1 0	0 0 0 1 0	0 1 0 0 0 1 2 2 1 0 0	2	15	45	1	1	$^\star(p = 0)$
2	1 0 1 0 0	0 0 0 0 0	1 0 1 2 2 2 2 1 0 0 0	2	105	1	1	1	$^\star g_{\mu\nu}$
3	1 0 0 0 1	0 0 1 0 0	0 1 0 0 1 2 3 3 2 1 0	2	35	120	1	1	
3	0 1 0 1 0	0 0 0 0 1	1 1 1 2 2 2 3 2 1 0 0	2	189	10	1	1	
3	0 0 0 0 0	0 0 0 1 1	0 2 0 0 0 1 3 4 3 2 1	2	1	320	1	1	top
3	1 0 0 0 1	0 0 0 0 1	1 1 1 2 2 2 3 3 2 1 0	0	35	10	8	1	
3	0 0 0 0 0	0 0 0 0 1	1 2 1 2 2 2 3 4 3 2 1	-2	1	10	43	1	top

Table C.18: $D_5 \otimes A_4$ representations in D_8^{+++} $(D = 5)$

l	$-p_{\text{grav}}$	$-p_G$	m	α^2	d_{grav}	d_G	mult(α)	μ	fields
0 0	0 0 0 0	0 0 0 1 0	-1 0 -1 -2 -2 -1 0 0 0 0 0	2	1	45	1	1	$p = 0$
0 0	1 0 0 1	0 0 0 0 0	0 0 0 0 0 0 0 -1 -1 -1 -1	2	24	1	1	1	$\bar{g}_{\mu\nu}$
0 0	0 0 0 0	0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	1	11	2	$p = 0, \hat{g}_{\mu\nu}$
1 0	1 0 0 0	0 0 0 0 0	0 1 0 0 0 0 0 0 0 0 0	2	5	1	1	1	$p = 1$
0 1	1 0 0 0	0 0 0 0 1	0 0 0 0 0 0 1 0 0 0 0	2	5	10	1	1	$p = 1$
1 1	0 1 0 0	0 0 0 0 1	0 1 0 0 0 0 1 1 0 0 0	2	10	10	1	1	$^\star(p = 1)$
0 2	0 1 0 0	0 0 0 0 0	1 0 1 2 2 2 2 1 0 0 0	2	10	1	1	1	$^\star(p = 1)$
1 2	0 0 1 0	0 0 0 1 0	0 1 0 0 0 1 2 2 1 0 0	2	10	45	1	1	$^\star(p = 0)$
1 2	1 1 0 0	0 0 0 0 0	1 1 1 2 2 2 2 1 0 0 0	2	40	1	1	1	$^\star g_{\mu\nu}$
1 2	0 0 1 0	0 0 0 0 0	1 1 1 2 2 2 2 2 1 0 0	0	10	1	8	1	$^\star(p = 0)$
2 2	0 0 0 1	0 0 0 1 0	0 2 0 0 0 1 2 3 2 1 0	2	5	45	1	1	de
2 2	1 0 1 0	0 0 0 0 0	1 2 1 2 2 2 2 2 1 0 0	2	45	1	1	1	
1 3	0 0 0 1	0 0 1 0 0	0 1 0 0 1 2 3 3 2 1 0	2	5	120	1	1	de
1 3	1 0 1 0	0 0 0 0 1	1 1 1 2 2 2 3 2 1 0 0	2	45	10	1	1	
1 3	0 0 0 1	0 0 0 0 1	1 1 1 2 2 2 3 3 2 1 0	0	5	10	8	1	de

2 3	1 0 0 1	0 0 1 0 0	0 2 0 0 1 2 3 3 2 1 0	2	24	120	1	1	
2 3	0 0 0 0	0 0 0 1 1	0 2 0 0 0 1 3 4 3 2 1	2	1	320	1	1	top
2 3	0 1 1 0	0 0 0 0 1	1 2 1 2 2 2 3 3 1 0 0	2	75	10	1	1	
2 3	0 0 0 0	0 0 1 0 0	0 2 0 0 1 2 3 4 3 2 1	0	1	120	7	1	top
2 3	1 0 0 1	0 0 0 0 1	1 2 1 2 2 2 3 3 2 1 0	0	24	10	8	2	
2 3	0 0 0 0	0 0 0 0 1	1 2 1 2 2 2 3 4 3 2 1	-2	1	10	43	2	top
1 4	1 0 0 1	0 0 0 1 0	1 1 1 2 2 3 4 3 2 1 0	2	24	45	1	1	
1 4	0 0 0 0	1 1 0 0 0	0 1 0 1 2 3 4 4 3 2 1	2	1	210	1	1	top
1 4	0 0 0 0	0 0 0 1 0	1 1 1 2 2 3 4 4 3 2 1	0	1	45	8	1	top
1 4	0 1 1 0	0 0 0 0 0	2 1 2 4 4 4 4 3 1 0 0	2	75	1	1	1	
1 4	1 0 0 1	0 0 0 0 0	2 1 2 4 4 4 4 3 2 1 0	0	24	1	8	1	
1 4	0 0 0 0	0 0 0 0 0	2 1 2 4 4 4 4 4 3 2 1	-2	1	1	45	1	top

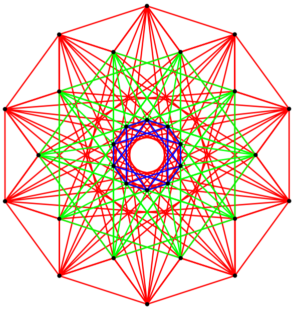
Table C.19: $D_6 \otimes A_1 \otimes A_3$ representations in D_8^{+++} $(D = 4)$

l	$-p_{\text{grav}}$	$-p_G$	m	α^2	d_{grav}	d_G	mult(α)	μ	fields
0	0 0 0	0 0 0 0 0 1 0	-1 0 -1 -2 -2 -2 -1 0 0 0 0	2	1	66	1	1	$p = 0$
0	1 0 1	0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 -1 -1 -1	2	15	1	1	1	$\bar{g}_{\mu\nu}$
0	0 0 0	0 2 0 0 0 0 0	0 -1 0 0 0 0 0 0 0 0 0	2	1	3	1	1	$p = 0$
0	0 0 0	0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	1	11	1	$\hat{g}_{\mu\nu}$
1	1 0 0	0 1 0 0 0 0 1	0 0 0 0 0 0 0 1 0 0 0	2	4	24	1	1	$p = 1, ^\star(p = 1$
2	0 1 0	0 0 0 0 0 1 0	0 1 0 0 0 0 1 2 1 0 0	2	6	66	1	1	$^\star(p = 0)$
2	2 0 0	0 0 0 0 0 0 0	1 1 1 2 2 2 2 2 0 0 0	2	10	1	1	1	$^\star g_{\mu\nu}$
2	0 1 0	0 2 0 0 0 0 0	1 0 1 2 2 2 2 2 1 0 0	2	6	3	1	1	$^\star(p = 0)$
3	0 0 1	0 1 0 0 1 0 0	0 1 0 0 0 1 2 3 2 1 0	2	4	440	1	1	de
3	1 1 0	0 1 0 0 0 0 1	1 1 1 2 2 2 2 3 1 0 0	2	20	24	1	1	
3	0 0 1	0 1 0 0 0 0 1	1 1 1 2 2 2 2 3 2 1 0	0	4	24	8	1	de
4	0 0 0	0 0 0 0 1 0 1	0 2 0 0 0 1 2 4 3 2 1	2	1	2079	1	1	top
4	1 0 1	0 0 0 1 0 0 0	0 2 0 0 1 2 3 4 2 1 0	2	15	495	1	1	
4	0 0 0	0 2 0 1 0 0 0	0 1 0 0 1 2 3 4 3 2 1	2	1	1485	1	1	top
4	1 0 1	0 0 0 0 0 0 2	1 2 1 2 2 2 2 4 2 1 0	2	15	77	1	1	
4	0 2 0	0 0 0 0 0 1 0	1 2 1 2 2 2 3 4 2 0 0	2	20	66	1	1	
4	1 0 1	0 2 0 0 0 1 0	1 1 1 2 2 2 3 4 2 1 0	2	15	198	1	1	

4	1 0 1	0 0 0 0 0 0 1 0	1 2 1 2 2 2 3 4 2 1 0	0	15	66	8	1	
4	0 0 0	0 2 0 0 0 0 1 0	1 1 1 2 2 2 3 4 3 2 1	0	1	198	8	1	top
4	0 0 0	0 0 0 0 0 0 1 0	1 2 1 2 2 2 3 4 3 2 1	-2	1	66	43	2	top
4	2 1 0	0 0 0 0 0 0 0 0	2 2 2 4 4 4 4 4 1 0 0	2	45	1	1	1	
4	0 2 0	0 2 0 0 0 0 0 0	2 1 2 4 4 4 4 4 2 0 0	2	20	3	1	1	
4	1 0 1	0 2 0 0 0 0 0 0	2 1 2 4 4 4 4 4 2 1 0	0	15	3	8	1	
4	1 0 1	0 0 0 0 0 0 0 0	2 2 2 4 4 4 4 4 2 1 0	-2	15	1	44	2	
4	0 0 0	0 2 0 0 0 0 0 0	2 1 2 4 4 4 4 4 3 2 1	-2	1	3	45	1	top

Table C.20: $D_8 \otimes A_2$ representations in D_8^{+++} $(D = 3)$

l	$-p_{\text{grav}}$	$-p_G$	m	α^2	d_{grav}	d_G	mult(α)	μ	fields
0	0 0	0 0 0 0 0 0 0 0 1	-1 -1 -1 -2 -2 -2 -2 -2 0 0 0	2	1	120	1	1	$p = 0$
0	1 1	0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 -1 -1	2	8	1	1	1	$\bar{g}_{\mu\nu}$
0	0 0	0 0 0 0 0 0 0 0 0	0 0 0 0 0 0 0 0 0 0 0	0	1	1	11	1	$\hat{g}_{\mu\nu}$
1	1 0	0 0 0 0 0 0 0 0 1	0 0 0 0 0 0 0 0 1 0 0	2	3	120	1	1	* $(p = 0)$
2	0 1	0 0 0 0 0 0 1 0 0	0 1 0 0 0 0 1 2 2 1 0	2	3	1820	1	1	de
2	2 0	0 0 0 0 0 0 0 0 1	1 1 1 2 2 2 2 2 2 0 0	2	6	120	1	1	
2	0 1	0 2 0 0 0 0 0 0 0	1 0 1 2 2 2 2 2 2 1 0	2	3	135	1	1	de
2	0 1	0 0 0 0 0 0 0 0 0	2 2 2 4 4 4 4 4 2 1 0	-2	3	1	44	1	de
3	0 0	0 1 0 0 1 0 0 0 0	0 1 0 0 0 1 2 3 3 2 1	2	1	60060	1	1	top
3	1 1	0 0 0 1 0 0 0 0 0	0 2 0 0 1 2 3 4 3 1 0	2	8	8008	1	1	
3	1 1	0 1 0 0 0 0 0 1 0	1 1 1 2 2 2 2 3 3 1 0	2	8	7020	1	1	
3	0 0	0 1 0 0 0 0 0 1 0	1 1 1 2 2 2 2 3 3 2 1	0	1	7020	8	1	top
3	1 1	0 0 0 0 0 0 1 0 0	1 2 1 2 2 2 3 4 3 1 0	0	8	1820	8	1	
3	0 0	0 0 0 0 0 0 1 0 0	1 2 1 2 2 2 3 4 3 2 1	-2	1	1820	43	1	top
3	3 0	0 0 0 0 0 0 0 0 1	2 2 2 4 4 4 4 4 3 0 0	2	10	120	1	1	
3	1 1	0 2 0 0 0 0 0 0 0	2 1 2 4 4 4 4 4 3 1 0	0	8	135	8	1	
3	1 1	0 0 0 0 0 0 0 0 1	2 2 2 4 4 4 4 4 3 1 0	-2	8	120	44	2	
3	0 0	0 2 0 0 0 0 0 0 0	2 1 2 4 4 4 4 4 3 2 1	-2	1	135	45	1	top
3	0 0	0 0 0 0 0 0 0 0 1	2 2 2 4 4 4 4 4 3 2 1	-4	1	120	195	1	top
3	1 1	0 0 0 0 0 0 0 0 0	3 3 3 6 6 6 6 6 3 1 0	-4	8	1	192	1	



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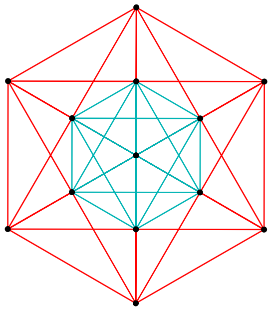

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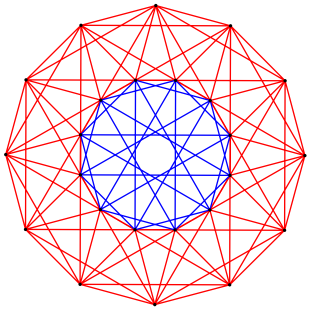

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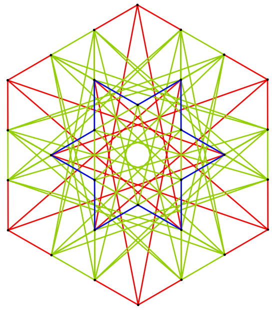


Nederlandse samenvatting

De twee grote mijlpalen van de natuurkunde in de vorige eeuw zijn Einsteins ontdekking van de Algemene Relativiteitstheorie en de constructie van het Standaard Model van de deeltjesfysica. Voor beide gevallen geldt dat ze tot op zekere hoogte geleid zijn door wat men het principe van symmetrie zou kunnen noemen. Voor Einstein leidde dit tot het inzicht dat de wetten van de natuur hetzelfde zouden moeten zijn voor alle waarnemers, of ze nou rechtop staan of ondersteboven hangen, of stil staan of juist versnellen. Voor het Standaard Model houdt het in dat zijn voorspellingen onveranderd blijven onder de symmetrietransformaties van $SU(3) \times SU(2) \times U(1)$.

Het grote falen van de natuurkunde in de vorige eeuw was wellicht het onvermogen om de Algemene Relativiteitstheorie en het Standaard Model te combineren. Deze gecombineerde theorie zou idealiter alle vier krachten in de natuur (te weten de sterke en zwakke kernkracht, het elektromagnetisme en de zwaartekracht) samen brengen onder één noemer. Een van de meest veelbelovende kandidaten voor deze allesomvattende theorie is, alhoewel het tot op heden niet in dit doel slaagt, de snaartheorie. Het uitgangspunt van de snaartheorie is om de puntdeeltjes van het Standaard Model te vervangen door eendimensionale objecten, de zogenoemde snaren. Het idee is dat alle bekende elementaire deeltjes overeen komen met verschillende vibraties van de snaren. De kracht van de snaartheorie, naast het feit dat het de oneindigheden van het Standaard Model 'gladwrijft', is dat één van deze vibraties het zwaartekrachtsdeeltje is. De zwaartekracht komt dus op natuurlijke wijze voort uit de snaartheorie.

Men zou misschien verwachten dat er slechts één enkele unieke theorie is die alle krachten van de natuur tegelijk beschrijft. Helaas voor de snaartheorie zijn er maar liefst vijf verschillende consistente snaartheorieën. Deze vijf theorieën worden Type I, Type IIA, Type IIB, Heterotische $E_8 \times E_8$ en Heterotische $SO(32)$ snaartheorie genoemd. Ze hebben allemaal een zekere mate van supersymmetrie (een symmetrie tussen krachtdeeltjes en materie), en 'leven' allemaal in tien ruimte-tijd dimensies. Dit zag er niet goed uit voor de snaartheorie, totdat de ontdekking werd gedaan dat de vijf verschillende theorieën verbonden zijn door zogenaamde dualiteitssymmetrieën. Daarnaast werd het vermoeden geopperd dat ze allemaal een limiet zijn van een tot op heden onbekende theorie in elf dimensies, M-theorie genaamd. Over M-theorie is vrij weinig bekend. Een van de weinige aanwijzingen die er zijn is de veronderstelde lage energie limiet: de unieke elfdimensionale supersymmetrische zwaartekracht (superzwaartekracht) theorie.

Net als de theorie zelf zijn de mogelijke symmetrieën van M-theorie een raadsel. Maar als ze bekend zouden zijn, dan kan het principe van symmetrie ons helpen in de zoektocht naar een duidelijke omschrijving van M-theorie. Een van de eisen van de symmetrieën van M-theorie is dat ze op zijn minst de dualiteiten tussen de vijf verschillende snaartheorieën moeten bevatten. Een bijkomende aanduiding volgt uit de superzwaartekracht: in bepaalde lage-dimensionale limieten laat de superzwaartekracht een oneindige symmetrie zien. Deze oneindige symmetrieën worden collectief Kac-Moody symmetrieën genoemd. Aangezien ze ook de dualiteitssymmetrieën bevatten, kan men het vermoeden poneren dat ze daadwerkelijk de symmetrie van M-theorie zijn. Echter, dit proefschrift zal geen van dergelijke sterke uitspraken bevatten. Het zal zich toeleggen op de unificerende rol van de Kac-Moody symmetrieën voor de verschillende superzwaartekracht theorieën.

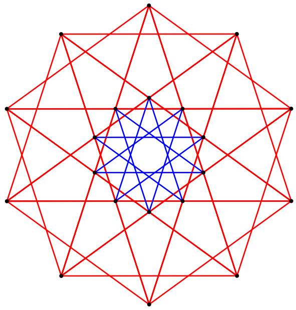
De Kac-Moody symmetrieën laten zich wiskundig beschrijven in een zogenaamde Lie algebra. Het mooie van Lie algebras is dat ze zich kort en bondig laten samenvatten in een object dat hun structuur volledig vastlegt. Dit object wordt de Cartan matrix genoemd, en is niets meer dan een simpele matrix. In de vergelijking met een computerbestand is de Cartan matrix de ingepakte versie van het volledig uitgepakte bestand. De vergelijking loopt alleen wel scheef doordat de 'uitgepakte' Kac-Moody algebra daadwerkelijk oneindig is en op geen enkele harde schijf zou passen, terwijl de Cartan matrix ruim genoeg heeft aan een enkele kilobyte.

Alhoewel het goed begrepen is hoe het 'uitpakken' van de Kac-Moody algebra in zijn werk gaat, is de procedure omslachtig en in de praktijk niet tot in het oneindige door te voeren. Bovendien geeft het weinig inzicht in de volledige structuur van de Kac-Moody algebra, die tot op heden dan ook in nevelen gehuld is. Het beste wat we op dit moment kunnen doen is een klein deel van het oneindige object construeren, en dat vervolgens in stukjes hakken die we wel begrijpen. Dit proces is enigszins analoog aan hoe men een vierdimensionale hyperkubus kan begrijpen door hem te beschrijven in termen van driedimensionale kubussen, net zoals de kubus op zijn beurt is opgebouwd uit een aantal vierkanten. De Kac-Moody algebras kunnen op vele verschillende manieren in stukjes worden gehakt, maar het interessantst zijn de gevallen waarin de resulterende stukjes overeenkomen met structuren die ook voorkomen in de superzwaartekracht theorieën. Eén enkele Kac-Moody algebra, E_{11} genaamd, beschrijft op deze manier de symmetrieën van de superzwaartekrachten met de meeste supersymmetrie in drie tot en met elf dimensies.

Bovendien, zo blijkt uit dit proefschrift, lijkt E_{11} daar bovenop ook nog eens alle mogelijke ijkdeformaties van de superzwaartekrachten te bevatten. Deze ijkdeformaties zijn van femenologisch belang doordat ze een deel van de supersymmetrie breken, en ze een effectieve kosmologische constante invoeren. Maar afgezien daarvan introduceren ze vrijheidsgraden die niet af te leiden zijn uit de unieke elfdimensionale superzwaartekracht theorie. Deze vrijheidsgraden zou men kunnen interpreteren als restanten van M-theorie, en in die hoedanigheid lijkt E_{11} informatie over M-theorie te bevatten.

Alhoewel de Kac-Moody algebras op elegante wijze de verschillende superzwaartekracht theorieën unificeren, zijn er ook een paar punten die niet geheel blijken te werken. Ten eerste voorspellen ze meer beperkingen op de ijkdeformaties dan dat er uit de analyse aan de kant van de superzwaartekracht volgen. Ten tweede komen de bewegingsvergelijkingen die volgen uit de Kac-Moody algebra slechts tot op een bepaalde hoogte overeen met die van de superzwaartekrachten. En tenslotte zijn de Kac-Moody algebras oneindig, wat inhoudt dat ze naast de bekende structuren uit de superzwaartekracht oneindig veel meer exotische structuren bevatten, die niet te relateren zijn aan aan ons bekende fysische theorieën.

Daarom is het dan ook te voorbarig om te stellen dat E_{11} daadwerkelijk de symmetrie van M-theorie is, laat staan de fundamentele symmetrie van de natuur. Veel stukjes van de puzzel passen perfect, maar het zijn de resterende niet-passende stukjes die ons doen vermoeden dat de Kac-Moody symmetrie E_{11} wellicht niet het hele verhaal is.



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