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Integrable Systems

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Integrable Systems and Knot Theory

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Abstract

A general theory is presented to construct link polynomials, topological invariants for knots and links, from exactly solvable (integrable) models. Representations of the braid group and the Markov traces on the representations are made through the general theory which is based on fundamental properties of the models. In addition, the equivalence of algebraic and graphical formulation is proved. Various examples including Alexander, Jones, Kauffman and new link polynomials are explicitly shown. In a word, the soliton theory contains an essence of the knot theory.

1 Introduction

In 1965, Zabusky and Kruskal [1] introduced a new concept, soliton, in the study of nonlinear waves. The soliton system has an infinite number of conserved quantities and is proved to be a completely integrable system. When we extend the soliton theory to quantum completely integrable systems, there emerges a unified viewpoint on various exactly solvable models in 1+1 dimensional field theory and in 2-dimensional classical statistical mechanics. To each model we can associate a family of commuting transfer matrices which are generators of an infinite number of conserved quantities. The condition for the commutability is the Yang-Baxter relation. [2,3,4,5]

Recently, the Yang-Baxter relation has been found to be a key to several fields in mathematical physics. In particular, Y. Akutsu and the authors found a general method to obtain various link polynomials [6,7,8,9] and their extensions from exactly solvable models. [10,11, 12,13,14,15,16,17,19,20,21,22,23] The purpose of this paper is to summarize the general theory for construction of known and new link polynomials from exactly solvable models.

Several problems in physics such as path integrals, fractional statistics and quantum gravity theory are related to the braid group. [24,25,26,27,28,29] In particular, it is interesting that solvable models and conformal field theories share many mathematical features in common. [30,31,32,33,34]

The outline of this paper is given in the following. In §2, the Yang-Baxter relations for S-matrices, vertex models and IRF models are introduced. Then, the mothod for construction of the representations of the braid group is given. In §3, link polynomials are constructed by algebraic and graphical approaches. The crossing symmetry is used for the graphical calculation of the link polynomials. In §4, appplications to several models are shown. Link polynomials obtained include Alexander, Jones, Kauffman and new link polynomials. In §5, link polynomials are constructed from solvable models with graded symmetry. The last section is devoted to concluding remarks.

2 Exactly solvable models and braids

Let us first introduce factorized S-matrices. We write the amplitude of the scattring process: $i \rightarrow k,\ j \rightarrow \ell$ as $S_{j\ell}^{ik}(u)$, where u is the rapidity difference of incoming (outgoing) particles. In general, the "charge" variables i, j, k and ℓ of $S_{j\ell}^{ik}(u)$ take vector values (weight vectors). The factorized S-matrices represent the elastic scattering of particles where only the exchanges of momenta and the phase shifts occur. The rapidity difference of the scattering particles can be depicted by the angle in the diagram. When $S_{j\ell}^{ik}(u)$ is non-zero only for the case $i+j=k+\ell$, we say that the model has "charge conservation" property. [10,19,20]

The Yang-Baxter relation for the S-matrices reads as

$$\begin{aligned} &\sum_{abc} S_{cr}^{bq}(u) S_{kc}^{ap}(u+v) S_{jb}^{ia}(v) \\ = \quad &\sum_{abc} S_{bq}^{ap}(v) S_{cr}^{ia}(u+v) S_{kc}^{jb}(u). \qquad (1) \end{aligned}$$

This relation is often referred to as the factorization equation [2,5].

In two-dimensional statistical mechanics, [3, 20] there are two types of solvable models, vertex models and IRF models. We introduce vertex models. The Boltzmann weight (statistical weight) $w(i,j,k,\ell;u)$ of a vertex model is defined for a configuration $\{i,j,k,\ell\}$ round a vertex. Here the parameter u is called spectral parameter which controls the anisotropy (and strength) of the interactions for the model.

For vertex models the Yang-Baxter relation is given by

$$\begin{aligned} &\sum_{abc} w(b,c,q,r;u) w(a,k,p,c;u+v) \times \\ &\qquad w(i,j,a,b;v) \\ = \quad &\sum_{abc} w(a,b,p,q;v) w(i,c,a,r;u+v) \times \\ &\qquad w(j,k,b,c;u). \qquad (2) \end{aligned}$$

It is known that factorized S-matrices are mathematically equivalent to the corresponding solvable vertex models.

We consider IRF models. The Boltzmann weight $w(a,b,c,d;u)$ of an IRF model is defined on a configuration $\{a,b,c,d\}$ round a face. IRF models have constraints on the configurations. By $b \sim a$ we denote that the "spin" b is admissible to the "spin" a under the constraint of the model. If the conditions $b \sim a, a \sim d, b \sim c$ and $c \sim d$ are all satisfied, then the configuration $\{a,b,c,d\}$ is called to be allowed. The Boltzmann weights for not-allowed configurations are set to be 0. For IRF models the Yang-Baxter relation is written as

$$\begin{aligned} &\sum_{c} w(b,d,c,a;u) w(d,e,f,c;u+v) \times \\ &\qquad w(c,f,g,a;v) \\ &= \sum_{c} w(d,e,c,b;v) w(b,c,g,a;u+v) \times \\ &\qquad w(c,e,f,g;u). \qquad (3) \end{aligned}$$

The IRF configuration a,b,c,d corresponds to the vertex configuration by $i=a-d, j=b-a, k=b-c$ and $\ell=c-d$.

Factorized S-matrices satisfy the following basic relations in addition to the Yang-Baxter relation. [10,14,15,19,20]

1) standard initial condition

$$S_{j\ell}^{ik}(u=0) = \delta_{i\ell}\delta_{jk}, \qquad (4)$$

where δ_{ij} is the Kronecker's symbol.

2) inversion relation (unitarity condition)

$$\sum_{mp} S_{p\ell}^{mk}(u) S_{jm}^{ip}(-u) = \rho(u)\rho(-u)\delta_{i\ell}\delta_{jk}, \quad (5)$$

where $\rho(u)$ is a model-dependent function.

3) second inversion relation (second unitarity condition)

$$\begin{aligned} &\sum_{pm} S_{p\ell}^{im}(\lambda-u) S_{mj}^{kp}(\lambda+u) \times \\ &\quad \left(\frac{r(m)r(p)}{r(i)r(j)r(k)r(\ell)} \right)^{1/2} \\ &= \rho(u)\rho(-u)\delta_{ij}\delta_{k\ell}. \qquad (6) \end{aligned}$$

We call the parameter λ crossing parameter and $\{r(i)\}$ crossing multipliers.

4) crossing symmetry

$$S_{j\ell}^{ik}(u) = S_{ki}^{j\ell}(\lambda-u) \left(\frac{r(i)r(\ell)}{r(j)r(k)} \right)^{\frac{1}{2}}. \qquad (7)$$

Here, we have introduced the notation $\bar{k}$ for the "antiparticle" of k. We assume that $r(\bar{k}) = 1/r(k)$. Note that the second inversion relation and the crossing symmetry define the crossing multipliers.

The Boltzmann weights for most of IRF models satisfy the basic relations corresponding to (4)-(7). For example, the crossing symmetry is

$$\begin{aligned} &w(a,b,c,d;u) \\ &= w(b,c,d,a;\lambda-u)\left(\frac{\psi(a)\psi(c)}{\psi(b)\psi(d)}\right)^{1/2} \end{aligned} \quad (8)$$

where $\{\psi(\ell)\}$ are the crossing multipliers for the IRF model. Crossing multipliers $\{\psi(\ell)\}$ for an IRF model are related to those for the corresponding vertex model by $r^2(j) = \psi(b)/\psi(a)$, when $j = b - a$ and $b \sim a$. We shall see that the basic relations and the Yang-Baxter relation are related intimately to the local moves on link diagrams, known as the Reidemeister moves in knot theory.

In order to relate exactly solvable models with the braid group we introduce Yang-Baxter operator $X_i(u)$. [10,14,19,20] For factorized S-matrices, we define Yang-Baxter operator by

$$\begin{aligned} X_i(u) &= \sum_{abcd} S^{cb}_{da}(u) I^{(1)} \otimes \cdots \otimes e^{(i)}_{ac} \otimes e^{(i+1)}_{bd} \\ &\quad \otimes I^{(i+2)} \otimes \cdots \otimes I^{(n)}. \end{aligned} \quad (9)$$

Here $I^{(i)}$denotes the identity matrix and e_{ab} a matrix such that $(e_{ab})_{jk} = \delta_{ja}\delta_{kb}$. The Yang-Baxter operators $\{X_i(u)\}$ satisfy the following relations (Yang-Baxter algebra),

$$\begin{aligned} &X_i(u)X_{i+1}(u+v)X_i(v) \\ &= X_{i+1}(v)X_i(u+v)X_{i+1}(u), \quad (10) \\ &X_i(u)X_j(v) = X_j(v)X_i(u), \\ &\quad |i-j| \geq 2. \quad (11) \end{aligned}$$

In terms of the Yang-Baxter operators, the Yang-Baxter relations for factorized S-matrices, solvable vertex and IRF models are in the same form.

The braid group B_n [35] is defined by a set of generators, $b_1, \cdots, b_{n-1}$ which satisfy

$$\begin{aligned} b_i b_{i+1} b_i &= b_{i+1} b_i b_{i+1}, \\ b_i b_j &= b_j b_i, \qquad |i-j| \geq 2. \quad (12) \end{aligned}$$

The operation b_i makes $(i+1)$-th string cross above i-th string.

Braid is a fundamental object in knot theory since any oriented link can be expressed by a closed braid. The equivalent braids expressing the same link are mutually transformed by a finite sequence of two types of operations, Markov moves I and II. The Markov trace $\phi(\cdot)$ is a linear functional on the representation of the braid group which have the following properties (the Markov properties):

$$\begin{aligned} I.\ \phi(AB) &= \phi(BA), \quad A, B \epsilon B_n, \quad (13) \\ II.\ \phi(Ab_n) &= \tau\phi(A), \\ \phi(Ab_n^{-1}) &= \bar{\tau}\phi(A), \\ &\quad A \epsilon B_n, b_n \epsilon B_{n+1}, \quad (14) \end{aligned}$$

where

$$\tau = \phi(b_i), \bar{\tau} = \phi(b_i^{-1}) \quad \text{for all } i. \quad (15)$$

From the Markov trace we obtain a link polynomial $\alpha(\cdot)$ as [10,19,20]

$$\alpha(A) = (\tau\bar{\tau})^{-\frac{n-1}{2}}\left(\frac{\bar{\tau}}{\tau}\right)^{\frac{1}{2}e(A)}\phi(A), \qquad A \epsilon B_n. \quad (16)$$

Here $e(A)$ is the exponent sum of b_i's in the braid A, which is equivalent to the writhe of the link diagram (cf.(30)). It is easy to show that $\alpha(\cdot)$ defined by (16) is indeed invariant under the Markov moves.

The braid operator $G_i(+)$, the inverse operator $G_i(-)$ and the identity I are given by [10]

$$\begin{aligned} G_i(\pm) &= \lim_{u\to\infty} X_i(\pm u)/\rho(\pm u), \quad (17) \\ I &= X_i(0). \quad (18) \end{aligned}$$

The limit $u \to \infty$ (more precisely, an infinity in a certain direction in the complex u-plane) requires that factorized S-matrices (the Boltzmann weights) be parametrized by hyperbolic or trigonometric functions. In statistical mechanics, it implies that the model is at the criticality. Hereafter we shall write the matrix elements of the braid operator as

$$\begin{aligned} G^{ab}_{cd}(\pm) &= \lim_{u\to\infty} X^{ab}_{cd}(\pm u)/\rho(\pm u) \\ &= \lim_{u\to\infty} S^{cb}_{da}(\pm u)/\rho(\pm u). \quad (19) \end{aligned}$$

Then we can express the braid operator (17) constructed from the Yang-Baxter operator as

$$G_i(\pm) = \sum_{abcd} G^{ab}_{cd}(\pm) I^{(1)} \otimes \cdots \otimes$$

$$e_{ac}^{(i)} \otimes e_{bd}^{(i+1)} \otimes I^{(i+2)} \otimes \cdots \otimes I^{(n)} \quad (20)$$

To summarize, corresponding to an exactly solvable model, we obtain a representation of the braid group by using the formula (17).

3 Construction of link polynomials

3.1 The Markov trace

We shall obtain link polynomials by constructing the Markov trace on the representations of the braid group derived from the solvable models. For factorized S-matrices and vertex models, the Markov trace takes the following form [10,19,20]

$$\phi(A) = \frac{\hat{T}r(H(n)A)}{\hat{T}r(H(n))}, \qquad A \epsilon B_n,$$

$$[H(n)]_{b_1 b_2 \cdots b_n}^{a_1 a_2 \cdots a_n} = \prod_{j=1}^{n} r^2(a_j)\delta_{b_j}^{a_j}, \quad (21)$$

where $\delta_b^a = \delta_{ab}$ is the Kronecker's symbol. For the models with the crossing symmetry (and the second inversion relation), $r(p)$ is nothing but the crossing multiplier of the model. The trace $\phi(\cdot)$ defined in (21) is the Markov trace since we can prove the Markov property I by the "charge conservation" property and the Markov property II by the following conditions:

$$\Sigma_b G_{ab}^{ab}(\pm) r^2(b) = \chi(\pm) \quad \text{(independent of } a). \quad (22)$$

The r-factors are related to $\chi(\pm)$ as $\bar{r}/r = \chi(-)/\chi(+)$.

We can prove the extended Markov property [14,16,19,20] :

$$\sum_b X_{ab}^{ab}(u) h(b) = H(u;\eta)\rho(u) \quad \text{(independent of } a), \quad (23)$$

where the function $H(u;\eta)$ is called characteristic function. This relation is an extension of (22) into the case of finite spectral parameter.

For IRF models we introduce a "constrained trace" $\tilde{T}r(A)$ by [14,16,19,20]:

$$\tilde{T}r(A) = \tilde{\sum_{\ell_1 \ell_2 \cdots \ell_n}} A_{\ell_0 \ell_1 \cdots \ell_n}^{\ell_0 \ell_1 \cdots \ell_n} \frac{\psi(\ell_n)}{\psi(\ell_0)}, \quad (\ell_0 : \text{fixed}) \quad (24)$$

where the symbol Σ with ˜ represents the summataion over admissible multi-indices $\{\ell_i : \ell_{i+1} \sim \ell_i\}$ for $i = 0, \cdots, n-1$ with ℓ_0 being fixed. Then the Markov trace $\phi(\cdot)$ is written as

$$\phi(A) = \frac{\tilde{T}r(A)}{\tilde{T}r(I(n))}, \qquad A \epsilon B_n, \quad (25)$$

where $I(n)$ is the "identity" operator for n strings. We can prove the extended Markov property also for IRF models. [14,16,19,20] In conclusion, the extended Markov property (and the charge conservation condition for vertex models) is sufficient for the existence of the Markov trace. This completes the algebraic construction of link polynomials from exactly solvable models.

3.2 Graphical calculation

The crossing symmetry is significant in algebraic and graphical aspects of the knot theory. For solvable models with the crossing symmetry, the Yang-Baxter operator becomes the Temperley-Lieb operator at the point $u = \lambda$. [15] In fact, setting

$$E_i = X_i(\lambda), \quad (26)$$

we find that the operators $\{E_i\}$ satisfy the following relations (the Temperley-Lieb algebra) [36] :

$$\begin{aligned} E_i E_{i\pm 1} E_i &= E_i, \\ E_i^2 &= q^{\frac{1}{2}} E_i, \\ E_i E_j &= E_j E_i, \quad |i-j| \geq 2, \quad (27) \end{aligned}$$

where the quantity $q^{1/2}$ is related to the crossing multipliers $r(a)$ (or $\psi(i)$) by [10,14,15]

$$\begin{aligned} q^{\frac{1}{2}} &= \sum_j r^2(j), \quad \text{for S-matrix} \\ &\quad \text{(for vertex model)}, \\ &= \sum_{b \sim a} \frac{\psi(b)}{\psi(a)}, \quad \text{for IRF model.} \quad (28) \end{aligned}$$

In (28) the summation is over all states b allowable to a.

Let us consider the graphical meaning of the relations (27). From the crossing symmetry and the standard initial condition we have [15, 19]

$$S_{da}^{cb}(\lambda) = \left(\frac{r(a)r(c)}{r(b)r(d)}\right)^{\frac{1}{2}} S_{\bar{b}\bar{c}}^{da}(0) = r(a)\delta(a,\bar{b}) \cdot r(c)\delta(c,\bar{d}), \quad (29)$$

where $\delta(a,c) = \delta_{ac}$ is the Kronecker's symbol.

We can regard the elements $r(c)\delta(c,\bar{d})$ and $r(a)\delta(a,\bar{b})$ as the weights for the pair-annihilation diagram and the pair-creation diagram, respectively. Then, the Yang-Baxter operator at $u = \lambda$ is depicted as the monoid diagram, by which the Temperley-Lieb algebra is explained. This interpretation is consistent with a fact that the energy at the point $u = \lambda$ is related to the pair-creation energy.

For IRF models, the weights $\{\psi(a)/\psi(b)\}^{1/2}$ and $\{\psi(c)/\psi(b)\}^{1/2}$ correspond to the pair-annihilation and pair-creation diagrams, respectively.

We can formulate link polynomials with the crossing symmetry directly on link diagrams. Link diagram $\hat{L}$ is a 2-dimensional projection of a link L. The writhe $w(\hat{L})$ is the sum of signs for all crossings C_i in the link diagram :

$$w(\hat{L}) = \Sigma_{C_i} \epsilon(C_i). \quad (30)$$

We calculate "statistical sum" $Tr(\hat{L})$ on the diagram $\hat{L}$ by summing over braid diagrams, pair-creation diagrams and pair-annihilation diagrams. The link polynomial for the link L is given by

$$\alpha(L) = c^{-w(\hat{L})} \frac{Tr(\hat{L})}{Tr(\hat{K}_0)}. \quad (31)$$

Here $\hat{K}_0$ is the trivial knot diagram (a loop) and the constant c is defined by a relation

$$G_i E_i = c E_i, \quad (32)$$

or by (cf.(22))

$$c = \left(\frac{\chi(-)}{\chi(+)}\right)^{\frac{1}{2}}. \quad (33)$$

It is easy to see that $\alpha(L)$ is invariant under the Reidemeister moves, and therefore $\alpha(L)$ is a topological invariant of the link L. Thus we have shown that the link polynomials constructed from solvable models with the crossing symmetry are also graphically formulated. The monoid diagram and the weights for the creation and annihilation diagrams were used by L.H. Kauffman [37] for the Bracket polynomial which gives a graphical calculation of the Jones polynomial. We have derived monoid operators from the crossing symmetry of solvable models by a general formula (26).

We have an important conclusion. The graphical formulation applied to closed braids yields the Markov trace. For the link polynomials with the crossing symmetry, the formulation based on the Markov trace is thus equivalent to the graphical formulation.

It is interesting that link diagrams are considered as the Feynman diagrams for the high energy processes of charged particles and the link polynomials as the scattering amplitudes. At the lowest point in the diagram there occur a pair creation and at the highest point a pair annihilation. It is also interesting that, if we regard the link diagrams as distorted 2-dimensional lattices, the link polynomials are considered as the partition functions.

4 Various examples

4.1 N-state vertex model

From the N-state vertex models, a hierarchy of link polynomials is obtained by the general method presented in §3 and §4.[10] The model corresponds to the factorized S-matrices with spin s particles, where $N = 2s + 1$. [38,39]

Using the N-state vertex model (asymmetrized by the symmetry breaking transformation), we get the braid operator which satisfies an N-th order relation: [10]

$$(G_i - C_1)(G_i - C_2) \cdots (G_i - C_N) = 0 \quad (34)$$

where for $j = 1, 2, \cdots, N$

$$C_j = (-1)^{j+N} t^{\frac{1}{2}N(N-1) - \frac{1}{2}j(j-1)}, \quad t = e^{2\lambda}. \quad (35)$$

We call a relation for G_i such as (34) reduction relation of the braid operator. The crossing multiplier for the asymmetrized N-state vertex model is

$$r(k) = e^{-\lambda k} = t^{-k/2}, \quad k = -s, -s+1, \cdots, s, \quad (36)$$

where $s = (N-1)/2$.

The extended Markov property [14,20] is satisfied with the characteristic function given as

$$H(u;\lambda) = \frac{\sinh(N\lambda - u)}{\sinh(\lambda - u)}. \qquad (37)$$

The constants τ and $\bar{\tau}$ are

$$\tau = 1/(1 + t + \cdots + t^{N-1}), \qquad (38)$$
$$\bar{\tau} = t^{N-1}/(1 + t + \cdots + t^{N-1}). \qquad (39)$$

It is remarkable that there exists an infinite sequence of link polynomials corresponding to the N-state vertex models ($N = 2, 3, 4, 5, \cdots$). [10,20] The $N = 2$ case corresponds to the Jones polynomial.[7] In the $N \geq 3$ cases we have new link polynomials. From the reduction relation, we obtain the skein relations (the Alexander-Conway relations) for the link polynomials:

$$\alpha(L_+) = (1-t)t^{\frac{1}{2}}\alpha(L_0) + t^2\alpha(L_-), \quad (N = 2) \qquad (40)$$

$$\alpha(L_{2+}) = t(1 - t^2 + t^3)\alpha(L_+) + (t^4 - t^5 + t^7)\alpha(L_0) - t^8\alpha(L_-), \qquad (N = 3) \qquad (41)$$

$$\alpha(L_{3+}) = t^{3/2}(1 - t^3 + t^5 - t^6)\alpha(L_{2+}) + t^6(1 - t^2 + t^3 + t^5 - t^6 + t^8)\alpha(L_+) + t^{25/2}(-1 + t - t^3 + t^6)\alpha(L_0) - t^{20}\alpha(L_-), \qquad (N = 4). \qquad (42)$$

In (40), by L_+, L_0 and L_- we have denoted links which have the same configuration except b_i, b_i^0 and b_i^{-1} at an intersection. Similarly, L_{2+}, L_+, L_0 and L_- in (41) and L_{3+}, L_{2+}, L_+, L_0 and L_- in (42) should be understood. For general N, the skein relation is of N-th degree relating links $L_{(N-1)+}, \cdots, L_0, L_-$.

4.2 Graph state IRF model

We can construct solvable IRF models corresponding to arbitrary graphs in any dimensions.[40,14] We call them graph state IRF models. We may express the constraint of the model by a graph. In the graph each point represents the spin state. When a spin c is admissible to d then the point c is connected by a line to the point d. For ADE type graphs of Dynkin diagrams, the models are called ADE models.[41] There also exist solvable models with elliptic parametrization for extended Dynkin diagrams [40,42].

Let us construct the graph state IRF models.[14] We solve the eigenvalue equation for the graph ;

$$\sum_{b\sim a}\psi(b) = \Lambda\psi(a), \qquad (43)$$

where the summation is over all spin state b admissible to a. Constructing the Temperley-Lieb operator

$$[E_i]_{k_1\cdots k_n}^{p_1\cdots p_n} = \prod_{j=0}^{i-1}\delta_{k_{i-1}}^{k_{i+1}}\frac{\psi(p_i)\psi(k_i)}{\psi(p_{i-1})}\prod_{j=i+1}^{n}\delta_{k_j}^{p_j}, \qquad (44)$$

we have the Yang-Baxter operartor

$$X_i(u) = \frac{\sinh(\lambda - u)}{\sinh(\lambda)}\left(I + \frac{\sinh u}{\sinh(\lambda - u)}E_i\right). \qquad (45)$$

From the model we get the braid operator by taking the limit $u \to \infty$ and the Markov trace on the braid group representation by using the crossing multipliers. The link polynomial satisfies the second degree skein relation.

We can consider vertex models corresponding to the graph state IRF models under the Wu-Kadanoff-Wegner transformation and the base-point-infinity limit .[21] From these vertex and IRF models we have multi-variable braid matrices.[21]

4.3 ABCD IRF models

The IRF model corresponding to affine Lie algebra $A_{m-1}^{(1)}$ ($B_m^{(1)}$, $C_m^{(1)}$, $D_m^{(1)}$) is called $A_{m-1}^{(1)}$ ($B_m^{(1)}$, $C_m^{(1)}$, $D_m^{(1)}$) model.[43] The crossing parameter λ and the sign factor σ are defined as

$$\begin{aligned} \lambda = &\quad m\omega/2, \quad \sigma = 1 \quad \text{for } A_{m-1}^{(1)}, \\ \lambda = &\quad (2m-1)\omega/2, \quad \sigma = 1 \quad \text{for } B_m^{(1)}, \\ \lambda = &\quad (m+1)\omega, \quad \sigma = -1 \quad \text{for } C_m^{(1)}, \\ \lambda = &\quad (m-1)\omega, \quad \sigma = 1 \quad \text{for } D_m^{(1)}, \end{aligned} \qquad (46)$$

where ω is a parameter. The reduction relations are

$$(G_i - 1)(G_i + \gamma^2) = 0 \quad \text{for } A_{m-1}^{(1)}, \qquad (47)$$

$$(G_i - 1)(G_i - \beta)(G_i + \gamma^2) = 0 \quad \text{for } B_m^{(1)}, C_m^{(1)} \text{ and } D_m^{(1)}, \qquad (48)$$

with

$$\begin{aligned}\gamma &= e^{-i\omega} \\ &\qquad \text{for } A_{m-1}^{(1)},\ B_m^{(1)},\ C_m^{(1)} \text{ and } D_m^{(1)}, \\ \beta &= \sigma e^{-i[2\lambda+\omega(1+\sigma)]} \\ &\qquad \text{for } B_m^{(1)},\ C_m^{(1)} \text{ and } D_m^{(1)}. \end{aligned} \tag{49}$$

The extended Markov property is proved and the characteristic functions are calculated as

$$\begin{aligned}H(u) &= \frac{\sin(m\omega-u)}{\sin(\omega-u)}, \quad \text{for } A_{m-1}^{(1)}, \\ H(u) &= \frac{\sigma\sin(2\lambda-u)\sin(\sigma\omega+\lambda-u)}{\sin(\lambda-u)\sin(\omega-u)}, \\ &\qquad \text{for } B_m^{(1)},\ C_m^{(1)} \text{ and } D_m^{(1)}. \end{aligned} \tag{50}$$

(The explicit forms of the crossing multipliers are given in [16]). Using the reduction relations and the Markov traces, we obtain the generalized skein relations:

$$\begin{aligned}&\alpha(L_+) = (1-t)t^{(m-1)/2}\alpha(L_0) + t^m\alpha(L_-) \\ &\qquad \text{for } A_{m-1}^{(1)}, \\ &\alpha(L_{2+}) = \\ &\qquad = (1-t+\beta)e^{-i(2\lambda+\omega(\sigma-1))}\cdot\alpha(L_+) \\ &+(t+\beta t-\beta)e^{-2i(2\lambda+\omega(\sigma-1))}\cdot\alpha(L_0) \\ &-t\beta e^{-3i(2\lambda+\omega(\sigma-1))}\cdot\alpha(L_-), \\ &\text{for } B_m^{(1)},\ C_m^{(1)} \text{ and } D_m^{(1)}, \end{aligned} \tag{51}$$

where

$$t = e^{-2i\omega}. \tag{52}$$

For $A_{m-1}^{(1)}$ model, the Alexander polynomail is obtained by the limit $m \to 0$, while $m = 2$ corresponds to the Jones polynomial.

Link polynomials thus obtained are one-variable invariants for each fixed m. It is noted that m is independent of t. We now have two variables t and m. The link polynomial constructed from $A_{m-1}^{(1)}$ model corresponds to the two-variable extension [8] of the Jones polynomial. The link polynomials from $B_m^{(1)}$, $C_m^{(1)}$, $D_m^{(1)}$ models correspond to the Kauffman polynomial [9]. We thus have explicit realizations of the Kauffman polynomial and the two-variable extension of the Jones polynomial (HOMFLY polynomial). The braid matrices given by Turaev [44,45] correspond to the vertex-model analog of the braid matrices constructed from $A_{m-1}^{(1)}$, $B_m^{(1)}$, $C_m^{(1)}$, $D_m^{(1)}$ IRF models. From the IRF models we can construct braid matrices and the Markov trace for the vertex models by the Wu-Kadanoff-Wegner transformation and the base-point-infinity limit. [14] For example, from A-type IRF models we obtain the multi-state vertex models [46] related to SU(n). In the limit, the Markov trace [16] for the IRF model leads to that [44] for the vertex model.

5 Super Vertex models

5.1 gl(M|N) Vertex models

We
shall explain construction of link polynomials from vertex models with graded symmetry.[22] We consider a family of solvable vertex models associated with gl(M|N).[22,46] We prepare a set of signs $\{\epsilon_i\}$

$$\epsilon_i = 1 \text{ or} -1, \quad \text{for } i = 1,\cdots,M+N. \tag{53}$$

The sign ϵ_i represents the 'parity' of the edge state i. We also introduce 'grade' $p(i) \in \{0,1\}$ of the edge state i as $\epsilon_i = (-1)^{p(i)}$. The number of positive (resp. negative) signs is given by M (resp. N). In this way we have introduced the graded symmetry. For any set of signs $\{\epsilon_i\}$ we have a solution of the Yang-Baxter relation. Non-zero elements of the Boltzmann weights are given as follows:

$$\begin{aligned}w(a,a,a,a;u) &= \sinh(\eta-\epsilon_a u)/\sinh\eta, \\ w(a,b,b,a;u) &= \begin{cases}\exp(-u) & \text{for } a<b, \\ \exp(u) & \text{for } a>b,\end{cases} \\ w(a,b,a,b;u) &= \pm\sinh u/\sinh\eta \\ &\qquad \text{for } a \neq b, \end{aligned} \tag{54}$$

where η is a parameter and the edge variables a and b take values $1,2,\cdots,M+N$. The models have the charge conservation property: $w(a,b,c,d;u) = 0$ unless $a+b = c+d$.

The elements of the braid matrices are derived from (54) and (19) with $\rho(u) = \sinh(\eta-u)/\sinh\eta$:

$$\begin{aligned}G_{aa}^{aa}(+) &= \begin{cases}1 & \text{for } \epsilon_a = 1, \\ -t & \text{for } \epsilon_a = -1\end{cases} \\ G_{ab}^{ab}(+) &= \begin{cases}0 & \text{for } a<b, \\ 1-t & \text{for } a>b,\end{cases} \\ G_{ba}^{ab}(+) &= \mp t^{1/2} \qquad \text{for } a \neq b. \end{aligned} \tag{55}$$

Here a variable t is defined by $t = \exp(2\eta)$. Depending on the choice of the signs $\{\epsilon_a\}$ [22], we obtain 2^{M+N} different representations. Note that by replacing t with t^{-1} and multiplying the braid matrix by $-t$, we have an equivalent representation.

Each representation has only two eigenvalues 1 and $-t$. The braid matrices satisfy the Hecke algebra relations. Thus, to summarize, the Hecke algebra appears in the braid matrices associated with the Lie superalgebra gl(M|N). [22]

By taking the limit $\eta \to 0$ we get the graded permutation operator from the representation of the braid group (55). In this sense, the braid operator is a q-analogue of the graded permutation operator.

5.2 Link polynomials

Through the general theory we construct the Markov trace on the representations derived in the previous subsection. For any grading $\{\epsilon_i\}$, the Markov trace is given by

$$\phi(A) = \frac{\mathrm{Tr}(H(n)A)}{\mathrm{Tr}(H(n))}, \quad A \epsilon B_n,$$
$$[H(n)]^{a_1 a_2 \cdots a_n}_{b_1 b_2 \cdots b_n} = \prod_{j=1}^{n} h(j)\delta^{a_j}_{b_j}. \quad (56)$$

Here the diagonal matrix h is

$$h(j) = \epsilon_j \exp\{\eta(\sum_{k=1}^{j-1} 2\epsilon_k + \epsilon_j - M + N)\},$$
$$\text{for } j = 1, \cdots M + N. \quad (57)$$

In the limit $\eta \to 0$, the trace with matrix h reduces to the supertrace $strA = \sum_i \epsilon_i A_{ii}$. We can prove the extended Markov property, [14,16,19,20]

$$\sum_b X^{ab}_{ab}(u)h(b) = H(u;\eta)\rho(u)$$
$$(\text{ independent of } a), \quad (58)$$

where the characteristic function $H(u;\eta)$ is given by

$$H(u;\eta) = \frac{\sinh((M-N)\eta - u)}{\sinh(\eta - u)}. \quad (59)$$

This is a generalization of the characteristic function for the A_{M-1} (sl(M)) model given in (50).[16,19,20]

The link polynomial obtained from the vertex model associated with gl(M|N) satisfies the skein relation:

$$\alpha(L_+) = t^{p/2}(1-t)\alpha(L_0) + t^{p+1}\alpha(L_-). \quad (60)$$

where

$$p = M - N - 1. \quad (61)$$

Since the skein relation is of second degree, the link polynomial is calculable only by the relation. We now have a hierarchy of link polynomials which depends on the number $p = M - N - 1$. It is interesting that as far as p is common we have the same link polynomial [22]. To repeat, from different models related to gl(M|N) with $p = M - N - 1$ we obtain the same link polynomial. Note that the hierarchy includes the case $p = 0$ where $\bar{\tau}/\tau = 1$.

The HOMFLY polynomial [8] is characterized by the second degree skein relation:

$$\alpha(L_+) = \omega^{1/2}(1-t)\alpha(L_0) + \omega t \alpha(L_-). \quad (62)$$

Here t and ω are independent (continuous) variables. We see that the link polynomials constructed from the gl(M|N) type vertex models correspond to the cases $\omega = t^p$, $p \in \mathbf{Z}$ of the HOMFLY polynomial. Based on the Markov traces we thus obtain a hierarchy of link polynomials corresponding to the HOMFLY polynomial.[22]

The link polynomial for $p = -1$ is the Alexander polynomial.[6] The case $p = 1$ corresponds to the Jones polynomial. [7] Therefore we have a number of braid matrices with different sizes which lead to the Alexander polynomial and the Jones polynomial [22].

6 Concluding remarks

We have shown that various link polynomials are systematically constructed from exactly solvable (integrable) models. The Yang-Baxter relation, which is a sufficient condition of the solvability of the models, plays a central role in the theory.

The existence and properties of the link polynomials [10] constructed from the N-state vertex model [39] can be proved also by the construction of composite models (fusion method) in terms of the Temperley-Lieb algebra and by the graphical formulation derived from the crossing symmetry.[15] Note that the combination of the crossing symmetry and the Temperley-Lieb algebra characterizes the link polynomials.

Due to the limited space we have omitted a discussion on construction of two-variable link invariants [12,13,19,20]. Those invariants may be regarded as two-variable extension of the link polynomials constructed from A type composite vertex and IRF models. In papers [12,13], an algorithm for calculation of the two-variable link invariants for any link has been established, and some examples have been given.

For any combinations of braid matrices which have the Markov traces, multivariable link polynomials with higher skein relations have been constructed. Using braid matrices with the Markov traces, we obtain a composite (hybrid-type) braid matrix and a composite Markov trace from them, and therefore a link polynomial. [23] Thus we have a variety of link polynomials with multivariables.

It is now established that there exists a list of link polinomials. This fact is significant not only in mathematics but also in other areas of sciences, since there are many interestig problems concerning applications of link polynomials. We believe that the various link polynomials exhibited in this paper will be helpfull for studying those applications in physics, chemistry and biology.

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References

[1] N.J.Zabusky and M.D.Kruskal, Phys.Rev.Lett. **15** (1965) 240.

[2] C.N. Yang, Phys. Rev. Lett. **19** (1967) 1312.

[3] R.J. Baxter, Ann. of Phys. **70** (1972) 323; R.J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, 1982).

[4] L.A. Takhtajan and L.D. Faddeev, Russian Math. Surveys **34** (1979) 11.

[5] A.B. Zamolodchikov and A.B. Zamolodchikov, Ann. of Phys. **120** (1979) 253. M. Karowski, H.J. Thun, T.T. Truong and P.H. Weisz, Phys. Lett. **67B** (1977) 321. K. Sogo, M. Uchinami, A. Nakamura and M. Wadati, Prog. Theor. Phys. **66** (1981) 1284.

[6] J.W. Alexander, Trans. Amer. Math. Soc. **30** (1928) 275.

[7] V.F.R. Jones, Bull. Amer. Math. Soc. **12** (1985) 103.

[8] P. Freyd, D. Yetter, J. Hoste, W.B.R. Lickorish, K. Millett and A. Ocneanu, Bull. Amer. Math. Soc. **12** (1985) 239. J.H. Przytycki and K.P. Traczyk, Kobe J. Math. **4** (1987) 115.

[9] L.H. Kauffman, *On Knots* (Princeton University Press, 1987).

[10] Y. Akutsu and M. Wadati, J. Phys. Soc. Jpn. **56** (1987) 839, 3039.

[11] Y. Akutsu, T. Deguchi and M. Wadati, J. Phys. Soc. Jpn. **56** (1987) 3464.

[12] Y. Akutsu and M. Wadati, Commun. Math. Phys. **117** (1988) 243.

[13] T. Deguchi, Y. Akutsu and M. Wadati, J. Phys. Soc. Jpn. **57** (1988) 757.

[14] Y. Akutsu, T. Deguchi and M. Wadati, J. Phys. Soc. Jpn. **57** (1988) 1173.

[15] T. Deguchi, M. Wadati and Y. Akutsu, J. Phys. Soc. Jpn. **57** (1988) 1905.

[16] T. Deguchi, M. Wadati and Y. Akutsu, J. Phys. Soc. Jpn. **57** (1988) 2921.

[17] M. Wadati and Y. Akutsu, Prog. Theor. Phys. Suppl. **94** (1988) 1.

[18] M. Wadati, T. Deguchi and Y. Akutsu, in *Nonlinear Evolution Equations, Integrability and Spectral Methods*, ed. A. Fordy (Manchester University Press, 1990).

[19] Y. Akutsu, T. Deguchi and M. Wadati, in *Braid Group, Knot Theory and Statistical Mechanics*, ed. C.N. Yang and M.L. Ge (World Scientific Pub., 1989) p. 151.

[20] M. Wadati, T. Deguchi and Y. Akutsu, Phys. Reports **180** (1989) 427. T. Deguchi, M. Wadati and Y. Akutsu, Adv. Stud. in pure Math. **19** (1989), Kinokuniya-Academic Press, p. 193. M. Wadati, Y. Akutsu and T. Deguchi, in *Nonlinear Physics* ed. by Gu Chaohao et. al. (Springer-Verlag, Berlin, Heidelberg, 1990), p. 111. T. Deguchi, Link Polynomials, Linking Number and Exactly Solvable Models, in KEK Report 89-22 (1990), p. 45.

[21] T. Deguchi , Int. J. Mod. Phys. **A5** (1990) 2195.

[22] T. Deguchi, J. Phys. Soc. Jpn. **58** (1990) 3441. T. Deguchi and Y. Akutsu, J. Phys. A : Math. Gen. **23**(1990) 1861.

[23] T. Deguchi, J. Phys. Soc. Jpn. **59** (1990) 1119.

[24] A. Tsuchiya and Y. Kanie, Adv. Stud. Pure Math. **16** (1988) 297.

[25] T. Kohno, Ann. Inst. Fourier, Grenoble **37** 4, (1987) 139.

[26] Y.S. Wu, Phys. Rev. Lett. **52** (1984) 2103. G.W. Semenoff, Phys. Rev. Lett. **61** (1988) 517. A.M. Polyakov, Mod. Phys. Lett. **3A** (1988) 325.

[27] C. Rovelli and L. Smolin, Phys. Rev. Lett. 61 (1988) 1155.

[28] J. Fröhlich, in *Nonperturbative Quantum Field Theory*, ed. G. 't Hooft et. al (Plenum Pub.,1988) p. 71.

[29] E. Witten, Commun. Math. Phys. **121** (1989) 351.

[30] A. Kuniba, Y. Akutsu and M. Wadati, J. Phys. Soc. Jpn. **55** (1986) 3285.

[31] E. Verlinde, Nucl. Phys. **B300** [FS22] (1988) 360.

[32] G. Moore and N. Seiberg, Commun. Math. Phys. **123** (1989).

[33] K.H. Rehren and B. Schroer, Nucl. Phys. **B312** (1989) 715.

[34] M. Wadati, Y. Yamada and T. Deguchi, J. Phys. Soc. Jpn. **58** (1989) 1153.

[35] J.S. Birman, *Braids, Links and Mapping Class Groups* (Princeton University Press, 1974).

[36] H.N.V. Temperley and E.H. Lieb, Proc. Roy. Soc. London **A322** (1971) 251.

[37] L. H. Kauffman, Contemporary Mathematics AMS **7** (1988) 283.

[38] A.B. Zamolodchikov and V.A. Fateev, Sov. J. Nucl. Phys. **32** (1980) 293.

[39] K. Sogo, Y. Akutsu and T. Abe, Prog. Theor. Phys. **70** (1983) 730,739;

[40] Y. Akutsu, A. Kuniba and M. Wadati, J. Phys. Soc. Jpn. **55** (1986) 1466.

[41] V. Pasquier, J. Phys. A: Math. Gen. **20** (1987): L217, L221.

[42] A. Kuniba and T. Yajima, J. Phys. A: Math. Gen. **21** (1988) 519; J. Stat. Phys. **50** Nos. 3/4, (1988) 829.

[43] M. Jimbo, T. Miwa and M. Okado, Commun. Math. Phys. **116** (1988) 353.

[44] V.G. Turaev, Invent. Math. **92** (1988) 527.

[45] N. Yu. Reshetikhin, LOMI preprint E-4-87,E-17-87, Leningrad 1988.

[46] Cherie L. Schultz, Phys. Rev. Lett. **46** (1981) 629. I.V. Cherednik, Theor. Math. Phys. **43** (1980) 356. O. Babelon, H. J. de Vega and C. M. Viallet, Nucl. Phys. **B190** (1981) 542. J.H.H. Perk and C.L. Schultz, Phys. Lett. **84A** (1981) 407.

DISCUSSION

Q. **M. Bowick** *(Syracuse Univ.)*: Can one obtain link invariants from conformal field theories perturbed away from criticality but remaining integrable?

A. **M. Wadati**: I agree that it is a challenging problem. But is seems to me that we cannot obtain link invariants from models at off-criticality.

QUANTIZATION OF $g\ell(N,\mathbb{C})/U(1)$ AT ROOTS OF UNITY AND PARAFERMIONS

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ABSTRACT

The equivalence between the quantum group $(g\ell(N,\mathbb{C};L)/U(1))_{q,s} \equiv g\ell(N;L)_{q,s}$ at $q^2=-1$ over a non-Grassmannian field and $s\ell(L|N-L)_s$ over a Grassmannian field is discussed. The equivalence extends to $g\ell(N;L)_{q,1}(q^2=-1) \sim s\ell(L|N-L)$. This suggests a generalization of $s\ell(L|N-L)$ to Z_m-grading via $g\ell(N;L)_{q,s}$ at $q^2=m^{th}$ root of unity, m>2. Specifically, representations $g\ell(2;1)_{q,s}$ at m^{th} root of unity are shown, via their fusion and braiding properties, to transform as s-deformed parafermions, or spin-1/m anyons. They contrast sharply with corresponding representations of $s\ell(2)_s$.

Recently representations of quantum groups, especially $s\ell(N,\mathbb{C})_q$ (henceforth $s\ell(N)_q$) at roots of unity have attracted a great deal of attention.[1] Here we discuss representations at roots of unity of another quantum group $(g\ell(N,\mathbb{C};L)/U(1))_{q,s}$ (henceforth $g\ell(N;L)_{q,s}$ and called twisted quantum group of A_{N-1} in [2,3]). Some of the especially interesting properties of these representations are already known: *(a)* For N=2, s generic and $q^2=-1$, the representation gives the Alexander-Conway link polynomial, whose counterpart is the Jones polynomial derivable from the fundamental representation of $s\ell(2)_q$, q generic.[2] *(b)* The state model associated with the Alexander-Conway polynomial is the free fermion model.[3,4] *(c)* There is a hierarchy of Alexander-Conway link polynomials corresponding N=2, s generic and $q^2=m^{th}$ root of unity.[2,5] *(d)* The representations of $g\ell(N;L)_{q,s}$ at $q^2=-1$ coincide with those of $s\ell(L|N-L)_s$, whose associated link polynomials are just Witten's Wilson-lines for the 3D topological Chern-Simons theory with gauge group SU(L|N-L).[3,6]

In this report (where $\omega_m \equiv \exp(2\pi i/m)$, $qm \equiv \omega_m^{-1/2}$) we give a summary of properties of the representations of $g\ell(2;1)_{qm,s}$, s generic ($g\ell(2;L)_{q,s}$ reduces to $s\ell(2)_s$ unless L=1). They are parafermionic and unlike the representations of $s\ell(2)_s$, s generic, which have a one-to-one correspondence to the representations of $s\ell(2)$. We show that $g\ell(2;1)_{qm,s}$ provides a generalization of the Z_2-grading of $s\ell(2)$ to Z_m-grading.

The generators of the Hopf algebra[2] of $g\ell(2;1)_{q,s}$, denoted by $\mathcal{A}$, are I, H and $X^{\pm}$, where in the classical limit I generates the U(1) factor in $g\ell(2) \sim s\ell(2)\times U(1)$ and the other three generate $s\ell(2)$. In the quantized case, I is still central to $\mathcal{A}$, and $[H,X^{\pm}]= \pm 2X^{\pm}$ and $[X^+,X^-]=(k^2-k^{-2})/(q-q^{-1})$ as in $s\ell(2)_q$, except that

$$k = q^{(H-I)/2}\, s^{I/2} \tag{1}$$

instead of $k=q^{H/2}$ in $s\ell(2)_q$. It is possible to absorb the effect of I on k in (1) into H by a redefinition of the latter, which will no longer be traceless.[7] For reasons that will become transparent we use the expression (1) in which the role of I and that of the second parameter s is made explicit from the outset (in which case the respective numbers of generators in the Cartan subalgebra and deformation parameters still match). For convenience we write p≡s/q. First note the trivial special case of (1) at $p^2=1$, whence the U(1) factor in $g\ell(N;L)_{q,s}$ is modded out and $\mathcal{A}$ is reduced to $s\ell(2)_q$, whose properties are well known.

We consider only the nontrivial case $p^2 \neq 1$. Then $\mathcal{A}$ has a finite representation over the vector field V only when q^2 is a root of unity:

$$q^2=\omega^{-1}=e^{-2\pi i/m}, \quad m = \text{positive integer} \tag{2}$$

The same result obtains when one chooses, instead of (2), $q^2=\omega^{m'}$, provided m′ is prime to m. Given (2), the elements $(X^{\pm})^m$ are central in $\mathcal{A}$, and a fundamental m-dimensional matrix representation π: $\mathcal{A}\to End(V)$ is obtained when the relations

$$\pi((X^{\pm})^m) = 0 \tag{3}$$

are imposed. In what follows, it will be understood that all expressions given for elements in $\mathcal{A}$ are those under the homomorphism π, and that $[\rho]=\{|i\rangle;\ i=1 \text{ to } m\}$ is a basis for V, with the highest (lowest) state with respect to X^+ being $|1\rangle$ ($|m\rangle$). Then $|m\rangle$ ($|1\rangle$) are the highest (lowest) state with respect to X^-. With the aid of the derived relation[8] (meant to hold when acted on a state $\in \mathrm{Ker}X^+ \backslash \mathrm{Im}(X^+)^{m-1}$)

$$[(X^+)^u,(X^-)^v] =$$

$$(X^-)^{v-u}\ \frac{[v]_q!}{[v-u]_q!}\ \prod_{j=1}^{v}\ \frac{k^2q^{-u+j}-k^{-2}q^{u-j}}{q-q^{-1}} \tag{4}$$

one obtains from standard methods:

$$I|i\rangle = (m-1)|i\rangle, \qquad H|i\rangle = (m+1-2i)|i\rangle \tag{5}$$

$$X^-|i\rangle = \left([i]_q(s^{m-1}q^{-I+1}-s^{-m+1}q^{I-1})/(q-q^{-1})\right)^{1/2}|i+1\rangle,$$

$$\langle i|X^+|i+1\rangle = \langle i+1|X^-|i\rangle \tag{6}$$

The R-matrix may be calculated from the method either of Drinfeld[9] or of [2]. Here we only give its m^2 eigenvalues, whose degeneracies determine the *fusion rule* of the direct product $[\rho]\otimes[\rho]$ and whose values characterize the *braiding* of the irreducible representations in the direct product, as expressed in the following two relations

$$[\rho]\otimes[\rho]= \bigoplus_{j=1}^{m} [\sigma_j],\ \text{(dimensionality of } [\sigma_j]=n_j) \tag{7}$$

$$R[\sigma_j] = r_j[\sigma_j] \tag{8}$$

That is, the degeneracy of r_j is n_j, and $\Sigma_j n_j=m^2$. For the R-matrix under study, r_j and n_j are given by

$$\{r_j,\ n_j;\ j=1 \text{ to } m\} =$$
$$\{-(-1)^j\omega^{(1-j)(j-2)/2}s^{(m-1)(3-2j)},\ m;\ j=1 \text{ to } m\} \tag{9}$$

There are m distinct eigenvalues, all with degeneracy m. This contrasts sharply with the R-matrix, denoted by R′, of the m-dimensional representation $[\rho']$ of $s\ell(2)_s$, whose eigenvalues r'_j and degeneracies n'_j for generic s are given by

$$\{r'_j,\ n'_j;\ j=1 \text{ to } m\} =$$
$$\{-(-1)^j s^{(m-1)[1-2m+(2m+1)j-j^2]},\ 2m-2j+1;\ j=1 \text{ to } m\} \tag{10}$$

For m=2, the link polynomials corresponding to $[\rho]$ and $[\rho']$ are respectively just the Alexander-Conway and Jones polynomials[2,5]. It follows from the fact $\mathcal{A}$ coincides with the Hopf algebra $\mathcal{A}'$ of $s\ell(2)_s$ in the limit $s^2=q^2=\omega^{-1}$ that $R(s^2=\omega^{-1})=R'(s^2=\omega^{-1})$. On the other hand (9) and (10) are discretely distinct. Therefore at least one of the relations cannot be continuous in that limit. It turns out that both are not; for a detailed discussion see [10].

To have a better understanding of the difference between (9) and (10) we return to $\mathcal{A}$ (instead of the homomorphism π) and consider, instead of $X^\pm$, the generators

$$Y^+ = q^{H/2}X^+, \qquad Y^- = X^-q^{H/2} \tag{11}$$

Define an x-commutator to be $[A,B]_x \equiv AB-xBA$. Then, instead of having a commutation relation like $X^\pm$ do, $Y^\pm$ satisfy

$$[Y^+,Y^-]_\omega = A\omega^{H/2}(\omega^{(I-H)/2}s^I - \omega^{-(I-H)/2}s^{-I}) \tag{12}$$

where $\omega=q^{-2}$ and A is a nonessential normalization constant so long as $q^2\neq 1$. The coproduct on $Y^\pm$ now has a nonstandard appearance[2]: $\Delta(Y^\pm) = Y^\pm\otimes q^H p^{I/2} + p^{-I/2}\otimes Y^\pm$.

The left-hand side of (12) is an ω-commutator. In particular, when $\omega=-1$, it is an *anti-commutator*. In this case, under the homomorphism π of (5) for m=2, the right-hand side of (12) is proportional to (s^I-s^{-I}), which vanishes in the limit $s\to 1$. If one replaces the normalization constant A by $(s-s^{-1})^{-1}$, then (12) is exactly the commutation relation satisfied by the raising and lowering generators of $s\ell(1|1)_s$ (note that the fundamental representation of H in $s\ell(1|1)_s$ is proportional to the unit matrix, just as that of I is). In this sense $g\ell(2;1)_{q2,s}$ is equivalent to $s\ell(1|1)_s$.

To understand this notion further, consider (9) and (10) for the case m=2, and write the two states $|1\rangle$ and $|2\rangle$ as $|+\rangle$ and $|-\rangle$, the representations $[\sigma_j]$ for j=1 and 2 (see (7)) as [b] and [f], and $[\sigma'_j]$ as [s] and [a], respectively. For reason that will be clear presently, b, f, s and a stand for boson, fermion, symmetric and antisymmetric, respectively. We have

$$R[b]=s[b], \quad R[f]=-s^{-1}[f] \quad (\text{for } g\ell(2;1)_{q2,s}) \tag{13}$$

$$R'[s]=s[s], \quad R'[a]=-s^{-1}[a] \quad (\text{for } s\ell(2)_s) \tag{14}$$

The two sets of equations appear identical, but they carry quite different meanings. It suffices to point out that whereas both the symmetric states $|+\rangle|+\rangle$ and $|-\rangle|-\rangle$ lie in the three dimensional [s] in the case of $s\ell(2)_s$, in the case of $\mathcal{gl}(2;1)_{q2}$ $|+>|+>$ lies in the two dimensional [b] while $|->|->$ lies in the two dimensional [f]. Thus, in the limit $s\to 1$, [f] changes sign under braiding not because it is antisymmetric, like [a] is, but because its constituents are fermionic.

It is important to distinguish how [f] is given a fermionic exchange property (here, because $(Y^{\pm})^2$ are central, there is no difference between braiding and transposition) in (the unquantized) $\mathfrak{sl}(1|1)$ and in $\mathcal{gl}(2;1)_{q2,1}$. In the former, which has a trivial coproduct, the task is achieved by making the vector space explicitly contain a Grassmann variable, namely the state $|->$. In the latter the fermionic property of $|->$ is encoded in the braiding property of R in a Hopf algebra with a nontrivial coproduct, while the vector space is *nonGrassmannian*.

The analysis above can be transplanted onto $\mathcal{gl}(N;L)_{q2,s}$ [3] to demonstrate its equivalence to $\mathfrak{sl}(L|N-L)_s$. This explains why, for the fundamental representations of the two quantum groups, the link polynomials, which are actually eigenvalues of invariants of the quantum group, are identical, as are their associated graded vector models, and why the latter are nonquasiclassical.[3] The equivalence carries over to the limit $s\to 1$ to establish the equivalence between the Hopf algebra $\mathcal{gl}(N;L)_{q2,1}$ and the graded Lie algebra $\mathfrak{sl}(L|N-L)$. For $\mathcal{gl}(N;L)_{q2,s}$ the formula (13) still applies, except that the dimensionality of [b] is N(N-1)/2+L and that of [f] is N(N+1)/2-L. These are to be contrasted with the dimensionalities of [s] and [a] in $\mathfrak{sl}(N)_s$, being respectively N(N+1)/2 and N(N-1)/2.

The Z_2-grading of $\mathfrak{sl}(2)$ into $\mathfrak{sl}(1|1)$ does not lend itself to a direct generalization to higher gradings. However, the discussion above shows that a Z_m-grading can be achieved by way of the Hopf algebra of $\mathcal{gl}(2;1)_{q,s}$ at $q^2 = \omega_m^{-1}$, which in the following we call $\mathcal{A}_m$. Recall that the configuration space for a system of states having the property of higher than Z_2 grading is nonsimply connected, so that, instead of transposition, one must speak of braiding of two states. This explains why a quantum group is necessary for higher gradings. That $\mathcal{A}_m$ has the property of a Z_m-graded algebra is already clear from (9) and (12), especially when the latter is recast into the form

$$[Y^+,Y^-]_{\omega_m} = \alpha(s)(P_m-\beta(s)) \qquad (m>2) \qquad (15)$$

where P_m is idempotent of order m, and α and β are central elements depending on s and I. The right-hand side of (15) does not vanish in the limit $s\to 1$ for m>2, so it is not necessary to have a factor $(s-s^{-1})^{-1}$.

From (9), the fusion states $[\sigma_j]$ defined in (8) for $\mathcal{A}_m$ at s=1 braid as

$$R[\sigma_j] = (-1)^{j+1}\omega^{v_j}[\sigma_j];$$
$$v_j = -(j-1)(j-2)/2 \quad (\mathrm{mod}\ m) \qquad (16)$$

In particular $[\sigma_1]$=[b] is bosonic, $[\sigma_2]$=[f] is fermionic, while the other states are such that $R^m[\sigma_j]=\pm[\sigma_j]$. These latter states may be interpreted as anyonic states with "spin" 1/m; they are direct generalizations of a fermionic state, which has spin 1/2. The dimensionality of $[\sigma_j]$ is m, independent of j. Thus the representation $[\rho]$ of $\mathcal{A}_m$ is parafermionic. (Since the link polynomial for $\mathcal{A}_2$, is just the Wilson line for the supersymmetric Chern-Simons theory with SU(1|1) gauge symmetry,[3,6] one is intrigued with the possibility of the link polynomials for $\mathcal{A}_m$, m>2, being related to the Wilson lines for fractionally supersymmetric[11] Chern-Simons theories.) In comparison, for $\mathfrak{sl}(2)$, the corresponding fusion states $[\sigma'_j]$ are just normal spin m-j states: they have respective dimensionalities 2(m-j)+1 and are either symmetric (j odd) or antisymmetric (j even) under R'. Since $\mathfrak{sl}(2)_s$ is a continuous deformation of $\mathfrak{sl}(2)$, the eigenstates of R' for generic s cannot be anyonic even as they have unusual braiding properties. They are just normal spin states deformed. For a discussion of the situation at $s^2=\omega_m^{-1}$, when $\mathcal{A}_m$ coincides with $\mathfrak{sl}(2)_s$, see [10].

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References

[1] L. Alvarez-Gaume, C. Gomez & G. Siserra, Nucl. Phys. B330(1990)347; V. Pasquier & H. Saleur, Nucl. Phys. B330(1990)523; C. DeConcini & V.G. Kac, *Representations of quantum groups at roots of 1*, Pisa preprint no. 75 (1990); E. Date, M. Jimbo, M. Miki & T. Miwa, *R matrix for cyclic representations of $U_q(\mathfrak{sl}(3,\mathbb{C})$ at $q^3=1$*, Kyoto preprint RIMS-696 (1990) and *Cyclic representations of $U_q(\mathfrak{sl}(n+1,\mathbb{C}))$ at $q^N=1$*, Kyoto preprint RIMS-703 (1990); V.V. Bazhanov & R.M. Kashaev, *Cyclic L-operators related with a 3-state $\mathcal{R}$-matrix*, Kyoto preprint RIMS-702 (1990).

[2] H.C. Lee, *Q-deformation of $s\ell(2,\mathbb{C})\times Z_n$ and link invariants*, to appear in "Physics & Geometry" (Proc. NATO ARW, Lake Tahoe, 1989 July), Eds. L.L. Chau and W. Nahm (Plenum 1990); *Twisted quantum groups of A_n and the Alexander-Conway polynomial*, Chalk River preprint TP-90-0220 *(to appear in Pac. J. Math.)*.

[3] H.C. Lee & M. Couture, *Twisted quantum groups of A_n (II): Ribbon links, SU(M|L) Chern-Simons theory and graded vertex models*, Chalk River preprint TP-90-0505 *(submitted to Nucl. Phys. B)*.

[4] T. Deguchi & Y. Akutsu, *Graded solution to the Yang-Baxter relation and link polynomials*, U. Tokyo-Komaba preprint (1989).

[5] H.C. Lee, M. Couture & N.C. Schmeing, *Connected link polynomials*, Chalk River preprint TP-88-1124R (1988, unpublished); M. Couture, H.C. Lee & N.C. Schmeing, *A new family of N-state representations of the braid group*, in "Physics, Geometry and Topology", Ed. H.C. Lee (Plenum, 1990).

[6] J. Horne, Nucl. Phys. B334(1990)669.

[7] M. Couture, *On some quantum R-matrices associated with representations of $U_q s\ell(2,\mathbb{C})$ when q is a root of unity*, Chalk River preprint *(1990, submitted to J. Phys. A)*

[8] V. Pasquier & H. Saleur, see [1].

[9] V. Drinfeld, *Quantum groups*, in Proc. 1986 Int. Cong. Math. (Berkeley, 1987), Vol. 1, 798.

[10] H.C. Lee, in preparation.

[11] C. Ahn, D. Bernard & A. LeClair, *Fractional supersymmetries in perturbed coset CFT's and integrable soliton theory*, CLNS preprint 90/987 (1990).

DISCUSSION

Q. **A. LeClair** *(Cornell Univ.)*: Why do you call your symmetries fractional supersymmetries if you don't have the Poincaré generators in the algebra? I don't think the name is justified.

A. **H. C. Lee**: The representations are those for $1/m$—statistics anyons. I mention fractional supersymmetry because I think the representations are characteristic of those of fractional supersymmetric systems, plus the fact that the link invariants for $(sl(n/n)\times U(1))_{q^2=-1,s}$ are exactly the link invariants of Wilson lines in the three-dimensional supersymmetric topological field theory with $SU(n/n)$ gauge group.

THE TRANCATED SOLUTIONS AND BÄCKLUND TRANSFORMATION FOR THE THREE-WAVE EQUATIONS

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ABSRTACT

We prove explicitly that 3-wave equations exhibit the Painlevé properties. The trancated solutions automatically provide the Bäcklund transformation.

A powerful generalization of Painlevé test for ordinary differential equation was applied to the system of partial differential equations[1]. It was proven that when a partial differential equation (PDE) is soluable by inverse scattering transform and a system of ordinary differential equations (ODE) is obtained from this PDE by an exact similarity reduction, then the solution associated with the Gel'fand-Levitan-Marchenko equation will possess the Painlevé property, namely the general solution can have no movable singular points other than poles. Furthermore it was proposed[2] that, without recoursing to the reduction to an ODE, a PDE has also the Painlevé property when the solution of the PDE are single-valued about the movable singularity manifolds.

In this note, we shall investigate the Painlevé property of 3-wave equations. It has been known that 3-wave equations, commonly refered to as equations of exact resonance in non-linear integrable system, have soliton, multi-solitons solutions as well as Lax pair. Yet the analytical test of Painlevé property has been lacking. We shall start with the 3-wave interaction equations of 1+1 dimension, and expand the solutions in terms of Laurent series about a singular manifold, or pole manifold.

Set us denote ϕ as a singularity manifold of 2N-2 real dimension determined by the condition

$$\phi(Z_1, Z_2, \cdots Z_N) = 0. \tag{1}$$

The Painlevé properties state that for a solution $u(Z_1, Z_2, \cdots Z_N)$ of a PDE, u is of the simple poles about the movable singularties. Therefore it can be expressed as

$$u = \phi^\alpha \sum_i u_i \phi^i, \tag{2}$$

where u_i and ϕ are functions of $Z_1, ... Z_N$ and α is some integers. The values of α as well as the u_i can be determined by substituting eq.(2) into PDE. The integrability condition for a nonlinear PDE can be tested if the consistent recursion relations of u_i exist.

Consider the 3-wave equations under the parametric interactions of the wave packets in the following expression,

$$\frac{\partial u_1}{\partial t} + v_1 \frac{\partial u_1}{\partial x} - \eta u_2 u_3 = 0, \tag{3a}$$

$$\frac{\partial u_2}{\partial t} + v_2 \frac{\partial u_2}{\partial x} - \eta u_3 u_1 = 0, \tag{3b}$$

$$\frac{\partial u_3}{\partial t} + v_3 \frac{\partial u_3}{\partial x} + \eta u_1 u_2 = 0, \tag{3c}$$

where v_i are the group velocities of the 3-wave packet u_i respectively. The sign in front of η in eq.(3c) is positive, while those in eq.(3a) and (3b) are negative. This implies that the relative velocity of u_1 to u_3 is opposit to that of u_2 to u_3.

Set us expand the solutions of 3-wave $u_i(x,t)$ in terms of $\phi(x,t)$, namely

$$u_i(x,t) = \sum_{j=0} u_{ij}(x,t)\phi(x,t)^{j-\alpha_i} \tag{4}$$

where i takes the values from 1 to 3. Substituting eq.(4) into eq.(3) and analyzing each order in power series of $\phi(x,t)$. The requirement that solutions u_i contain no terms other than single pole in $\phi(x,t)$ forces us to put

$$\alpha_1 = \alpha_2 = \alpha_3 = 1. \tag{5}$$

The lowest expansion coefficients u_{i0} are related by the equations similar to those of exact resonance,

$$u_{10}\phi_t + v_1 u_{10}\phi_x + \eta u_{20}u_{30} = 0, \tag{6a}$$

$$u_{20}\phi_t + v_2 u_{20}\phi_x + \eta u_{30}u_{10} = 0, \tag{6b}$$

$$u_{30}\phi_t + v_3 u_{30}\phi_x - \eta u_{10}u_{20} = 0. \tag{6c}$$

For the sake of conciseness in calculating the coefficients u_{in}, we use the following abbriviation

$$D_i = \partial_t + v_i\partial_x, \tag{7}$$

then the recursion relations for u_{1n}, u_{2n} and u_{3n} $(n \geq 0)$ can be obtained from the coefficients in the series expansion of the terms with (n-2) power in ϕ, i.e.

$$\begin{pmatrix} (n-1)D_1\phi & -\eta u_{30} & -\eta u_{20} \\ -\eta u_{30} & (n-1)D_2\phi & \eta u_{10} \\ \eta u_{20} & \eta u_{10} & (n-1)D_3\phi \end{pmatrix} \begin{pmatrix} u_{1n} \\ u_{2n} \\ u_{3n} \end{pmatrix} = \begin{pmatrix} A_{n-1} \\ B_{n-1} \\ C_{n-1} \end{pmatrix}, \tag{8}$$

where A_{n-1}, B_{n-1} and C_{n-1} are given in terms of ϕ_t, ϕ_x, and u_{i0} up to u_{in-1}. The coefficients u_{in} exist only if the determinant of the 3×3 matrix of the last equation does not vanish.

The detailed evaluation of the determinant leads to

$$det \begin{pmatrix} (n-1)D_1\phi & .. & .. \\ & .. & \\ & .. & \end{pmatrix} = -\eta^3(n+1)(n-2)^2 u_{10}u_{20}u_{30}. \tag{9}$$

Therefore u_{in} for $n \geq 3$ can be solved uniquely if the left hand side of eq.(9) does not equal zero, and hence the recursion relations among u_{in} and u_{in-1} can be established. The conditions fail for $n = -1$ and $n = 2$. Obviously, the case for $n = -1$ will correspond to the Painlevé expansions containing terms of movable singularities with double poles. For the case $n = 2$, eq.(8) reduces to

$$\begin{pmatrix} D_1\phi & -\eta u_{30} & -\eta u_{20} \\ -\eta u_{30} & D_2\phi & -\eta u_{10} \\ \eta u_{20} & \eta u_{10} & D_3\phi \end{pmatrix} \begin{pmatrix} u_{12} \\ u_{22} \\ u_{32} \end{pmatrix} = \begin{pmatrix} -D_1 u_{11} + \eta u_{21}u_{31} \\ -D_2 u_{21} + \eta u_{11}u_{31} \\ -D_3 u_{31} + \eta u_{11}u_{21} \end{pmatrix}. \tag{10}$$

The vanishing of the determinant in the 3×3 matrix on the left hand side of last equation implies that there exists a complete arbitrary choice of u_{12}, u_{22} and u_{32} only if they are subject to a compatibity condition that eq.(10) becomes only two independent equations instead of three.

To do this, let us solve u_{i0} in terms of $D_i\phi$ from the nonlinear system of eq.(6). Simple algebraic manipulation provides us with

$$\begin{aligned} u_{10}^2 &= -\frac{1}{\eta^2}(D_2\phi D_3\phi), \\ u_{20}^2 &= -\frac{1}{\eta^2}(D_3\phi D_1\phi), \\ u_{30}^2 &= +\frac{1}{\eta^2}(D_1\phi D_2\phi). \end{aligned} \tag{11}$$

But on the other hand, eq.(8) for $n = 1$ can be calculated explicitly as

$$\begin{pmatrix} 0 & -\eta u_{30} & -\eta u_{20} \\ -\eta u_{30} & 0 & -\eta u_{10} \\ \eta u_{30} & \eta u_{10} & 0 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{21} \\ u_{31} \end{pmatrix} = \begin{pmatrix} -D_1 u_{10} \\ -D_2 u_{20} \\ -D_3 u_{30} \end{pmatrix}. \tag{12}$$

Combining eq.(11) and eq.(12), u_{i1} are readily obtained as follows,

$$u_{11} = \frac{1}{2\eta^2}\frac{(D_2 D_3\phi)}{u_{10}},$$

$$u_{21} = \frac{1}{2\eta^2}\frac{(D_3 D_1\phi)}{u_{20}}, \tag{13}$$

$$u_{31} = \frac{-1}{2\eta^2}\frac{(D_1 D_2 \phi)}{u_{30}}.$$

One can verify that

$$\begin{aligned} & u_{10}(-D_1 u_{11} + \eta u_{21} u_{31}) \\ & = u_{20}(-D_2 u_{21} + \eta u_{11} u_{31}) \\ & = u_{30}(+D_3 u_{31} - \eta u_{11} u_{21}). \end{aligned} \tag{15}$$

Since the condition $n = 2$ allows us to choose arbitrarily the expansion coefficients u_{12}, u_{22} and u_{32} in the general solutions of eq.(4). A set of trancated solutions can be achieved if we set u_{i2} equal to zero because all the coefficient u_{in} for $n \geq 3$ will automatically vanish according to eq. (8). The solution can then be simplied as

$$\begin{aligned} u_1 &= \frac{1}{\phi} u_{10} + u_{11}, \\ u_2 &= \frac{1}{\phi} u_{20} + u_{21}, \\ u_3 &= \frac{1}{\phi} u_{30} + u_{31}, \end{aligned} \tag{16}$$

where u_{i1} are solutions to the 3-wave equations, i.e.

$$\begin{aligned} D_1 u_{11} + \eta u_{21} u_{31} &= 0, \\ D_2 u_{21} + \eta u_{31} u_{11} &= 0, \\ D_3 u_{31} - \eta u_{11} u_{21} &= 0. \end{aligned} \tag{17}$$

By means of eq.(11) and eq.(17), the Bäcklund transformation for 3-wave equations can be expressed as follows,

$$\begin{aligned} u_1(x,t) &= \frac{1}{\eta\phi(x,t)}[-(D_2\phi)(D_3\phi)]^{\frac{1}{2}} + u_{11}(x,t), \\ u_2(x,t) &= \frac{1}{\eta\phi(x,t)}[-(D_3\phi)(D_1\phi)]^{\frac{1}{2}} + u_{21}(x,t), \\ u_3(x,t) &= \frac{1}{\eta\phi(x,t)}[-(D_1\phi)(D_2\phi)]^{\frac{1}{2}} + u_{31}(x,t), \end{aligned} \tag{18}$$

namely, a particular form of and arbitrary function of $\phi(x,t)$ can be added to the solutions u_{i1} to achieve a new set of solutions.

References

(1) M.J. Abolowitz, A. Ramani and H. Segur, J. Math. Phys.21, (1980)715;
ibid 21, (1980)1006.

(2) John Weiss, M. Tabor and George Carnevale, J. Math. Phys. 24, (1983)522;
M. Jimbo, M.D. Kruskal and T. Miwa, Phys. Lett. 92A, (1982)59;
John Weiss, J. Math. Phys. 24, (1983)1405;
A.Roy Chowdhury and Minati Naskar, J. Math Phys. 28, (1987)1809;
W.H. Steeb and N.Euler, lett. Math. Phys. 14, (1987)99;
A.C. Newell, M. Tabor and Y.B. Zeng, Physica 29D, (1987)1.

Exotic Solutions of Yang-Baxter Equations and Yang-Baxterization Approach

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ABSTRACT

The new solutions of Yang-Baxter equations associated with the fundamental representations of B_n, C_n and D_n are derived through the braid group representations and the trigonometric Yang - Baxterization.

Remarkable progress has been made in the derivation of trigonometric solutions of Yang-Baxter equations (YBE) associated with simple Lie algebras(1,2). The standard approach is to make q-deformation of classical Lie algebras, namely, based on the current formulation of quantum group including its loop extension(3). We call this type of solutions "standard" one. However, on the basis of the same Lie algebraic structure it allows to generate new family of solutions of YBE, which is different from the standard one and is called "exotic" family of solutions of YBE.

Our strategy is stated in the following.

(I) In order to solve YBE

$$\check{R}_{12}(x)\check{R}_{23}(xy)\check{R}_{12}(y) = \check{R}_{23}(y)\check{R}_{12}(xy)\check{R}_{23}(x)$$

where $x=e^{-u}$ is the spectral parameter relating with the rapidity for two-particle collision, the asymptotic behavior $T=R(x)_{u=\infty}$ satisfying

$$T_{12}T_{23}T_{12}=T_{23}T_{12}T_{23}$$

is firstly solved for given Lie algebraic structure. T is reffred to braid group representation(BGR).

(II) By using the trigonometric Yang-Baxterization (4,5) prescription (T-YB) $\check{R}(x)$ can be generated for a given BGR. So far the T-YB has been established for those BGR's which possess distinct eigenvalues being three and four(two , needless to say).

In this talk we only discuss the new solutions of YBE associated with the fundamental representations of Lie algebras B_n, C_n and D_n. As was known that the corresponding BGR's possess three distinct eigenvalues.

First we calculate the BGR's which are given by $(w=q-q^{-1})$

$$T=\sum_{k\neq 0} u_k\, e_{kk}\otimes e_{kk} + w\sum_{\substack{k<m\\ k+m\neq 0}} e_{kk}\otimes e_{mm} + \sum_{\substack{k\neq m\\ k+m\neq 0}} e_{km}\otimes e_{mk} + \sum_{k,m} a_{km}\, e_{k-m}\otimes e_{-km}$$

where $u_k=q$ or $-q^{-1}$ for $k=0$ and $u_{-k}=u_k$. $k,m\in[(N-1)/2,\ldots,-(N-1)/2]$ where $N=2n+1, 2n$ and $2n$ for B_n, C_n and

D_n, respectively. The a_{km} are given by

$$a_{km}=\begin{cases} 1 & (k=m=0) \\ u_k^{-1} & (k=m\neq 0) \\ w\left[1-u_m^{-1}\left(\prod_{j=1}^{m-1} u_j^{-2}\right)\right] & (k=-m<0) \\ (-1)^{k+m+1} w u_{k+m}^{-\frac{1}{2}} \prod_{j=1}^{|k+m|-1} u_j^{-1} & (k=0<m,\text{ or } k<m=0) \\ (-1)^{k+m+1} w u_m^{-\frac{1}{2}} u_k^{-\frac{1}{2}} \left(\prod_{j=|k|+1}^{|m|-1} u_j^{-1}\right) & (0<k<m,\text{ Or } k<m<0) \\ (-1)^{k+m+1} w u_m^{-\frac{1}{2}} u_k^{-\frac{1}{2}} \left(\prod_{j=|k|}^{m-1} u_j^{-1}\right)\cdot\left(\prod_{i=1}^{|k|-1} u_i^{-2}\right) & (k<0, m>0, k+m\neq 0) \end{cases}$$

for $B_n^{(1)}$.

$$a_{km}=\begin{cases} u_k^{-1} & (k=m) \\ w\left[1-\varepsilon u_m^{-1}\left(\prod_{j=1}^{m-\frac{1}{2}} u_{j-\frac{1}{2}}^{-2}\right) u_{\frac{1}{2}}\right] & (k=-m\ \ 0) \\ -w u_m^{-\frac{1}{2}} u_k^{-\frac{1}{2}} \left(\prod_{j=1}^{m-\frac{1}{2}} u_{j-\frac{1}{2}}^{-1}\right) u_k^{-\frac{1}{2}} & (0<k<m,\text{ or } k<m<0\ ) \\ -\varepsilon w u_m^{-\frac{1}{2}} \left(\prod_{j=|k|+\frac{1}{2}}^{m-\frac{1}{2}} u_{j-\frac{1}{2}}^{-1}\right)\cdot\left(\prod_{i=1}^{|k|+\frac{1}{2}} u_{i-\frac{1}{2}}^{-2}\right) u_k^{\frac{1}{2}} u_{\frac{1}{2}}^{\varepsilon} & (k\ \ 0, m\ \ 0, k+m=0) \end{cases}$$

with $-\varepsilon=1$ for $C_n^{(1)}$ and $\varepsilon=1$ for $D_n^{(1)}$.

The distinct eigenvalues are given by

$$(T-\lambda_1)(T-\lambda_2)(T-\lambda_3)=0$$

where

	λ_1	λ_2	λ_3
B_n	q	$-q^{-1}$	$\left(\prod_{j=1}^{n} u_j^{-2}\right)$
C_n	$-q^{-1}$	q	$-\left(\prod_{j=1}^{n} u_{j-\frac{1}{2}}^{-2}\right) u_{\frac{1}{2}}^{-1}$
D_n	q	$-q^{-1}$	$\left(\prod_{j=1}^{n} u_{j-\frac{1}{2}}^{-2}\right) u_{\frac{1}{2}}$.

Next we Yang-Baxterize the solutions derived above to give the corresponding $\check{R}(x)$s. It has been proved that if BGR T satisfies the relation (5)

$$\lambda_3^{-1}(T_{12}T_{23}^{-1}T_{12}-T_{23}T_{12}^{-1}T_{23})-\lambda_1(T_{12}^{-1}T_{23}T_{12}^{-1}-T_{23}^{-1}T_{12}T_{23}^{-1})-\left(1+\frac{\lambda_1}{\lambda_2}+\frac{\lambda_1}{\lambda_3}+\frac{\lambda_2}{\lambda_3}\right)$$

$$-\lambda_2^{-1}(T_{12}-T_{23})+\lambda_2(T_{12}^{-1}-T_{23}^{-1})=0,$$

then it can be T-YB to

$$R(x)=\lambda_1 x(x-1)T^{-1}+\left(1+\frac{\lambda_1}{\lambda_2}+\frac{\lambda_1}{\lambda_3}+\frac{\lambda_2}{\lambda_3}\right)\cdot xI-\lambda_3^{-1}(x-1)T.$$

Now the calculations convince the validity (for the derived T) of the statement. Hence for the considered cases we obtain the general solutions of YBE.

The result is shown in the following form.

$$R(x)=\sum_{k\neq 0} u_k\, e_{kk}\otimes e_{kk}-(q^2-1)(x-\xi)$$

$$\cdot\Big(\sum_{\substack{k<m\\ k+m\neq 0}} +x\sum_{\substack{k>m\\ k+m\neq 0}}\Big)e_{kk}\otimes e_{mm}$$

$$+q(x-1)(x-\xi)\sum_{\substack{k\neq m\\ k+m\neq 0}} e_{km}\otimes e_{mk}$$

$$+\sum_{k,m} a_{km}(x)\, e_{k-m}\otimes e_{-km}$$

where

$$u_k(x)=\begin{cases}(x-q^2)(x-\xi) & \text{when } u_k=q\\ -q^2(x-q^{-2})(x-\xi) & \text{when } u_k=-q^{-1}\end{cases}$$

and

$$a_{km}=q(x-1)(x\tilde{a}_{km}-\xi a_{km})+$$

$$(\xi-1)(q^2-1)x\,\delta_{k-m}$$

$$\xi=\begin{cases}q^{-1}\lambda_3^{-1} & \text{for } B_n^{(1)} \text{ and } D_n^{(1)}\\ -q\ \ \lambda_3^{-1} & \text{for } C_n^{(1)},\end{cases}$$

$$\tilde{a}_{km}(u_k)=a_{mk}(u_k^{-1}).$$

The other permitted solution is obtained by the interchange $q \leftrightarrow -q^{-1}$ and keeping λ_3 unchanged. Such a solution corresponds to the "twisted" one. For instance, it gives rise to $A_{2n}^{(2)}$ or $A_{2n-1}^{(2)}$ corresponding to $B_n^{(1)}$ or $D_n^{(1)}$, respectively . As for the correspondence of $C_n^{(1)}$ it deserves to be understood.

We would like to make comments to the above discussion.

(a) Taking $u_k=q$ for all k our solutions go back to those derived by Jimbo (6). The other choice leads to new solutions which are called exotic ones. Nothing is surprise to appear such a new type of solutions of YBE because the usual classical limit is not required here. Actually the new solution for A_n had been found by Gervais et al (7) in terms of different approach.

(b) In contrast with the standard solutions the exotic ones possess very different properties such as without the usual classical limits, different Hopf algebraic structure due to Faddeev-Reshetikhin-Takhtajan approach (8), some of them even cannot be diagonalized and so on.

(c) We can prove that the exotic solutions still satisfy Birman-Wenzl algebra (9). This fact is determined by the fundamental representations of B_n, C_n and D_n. In a sense our discussion provides another explicit example for Jones' theory (4).

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References

(1) M.Jimbo(ed), Yang-Baxter Equation in Integrable Systems. Word Scientific
(2) C.N.Yang and M.L.Ge(eds), Braid Group, Knot Theory and Statistical Mechanics, World Scientific.
(3) M.Jimbo, pp111 in ref.(2).
(4) V.Jones, Commn Math Phys125(89)459
(5) M.L.Ge, Y.S.Wu, K.Xue, ITP-SB-90-02
(6) M.Jimbo, Commn Math Phys102(86)537
(7) E.Cremmer, Gervais, LPTEN 89/19
(8) L.D.Faddeev et al, Alg.Anal, 1(89)
(9) Y.Cheng, M.L.Ge and K.Xue, ITP-SB, 90-24.
(10) About the calculation method for deriving the BGR, see, M.L.Ge et al, Inter J.Mod Phys, 4(89)3351, J.Phys.23A 605(90), 23A 2273(90).

FIELD THEORY FROM INTEGRABLE-SYSTEM POINT OF VIEW

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ABSTRACT

We survey the Geometrical Integrability Properties: linear equations, conservation laws, Riemann-Hilbert transformations, Bianchi-Bäcklund transformations, Ricatti Equations, and Kac-Moody Algebra are discussed for many nonlinear systems: various chiral models in two dimensions, SL(2C) systems of Sine-Gordon, KdV, and Liouville equations; self-dual Yang-Mills equations in four dimensions; extended supersymmetric Yang Mills, and supergravity equations in four and ten dimensions. Physical applications of these properties are also commented on.

INTRODUCTION

The attempt of this line of research is to treat Yang–Mills and gravitational fields as nonlinear systems, and to see how much they possess the geometrical integrability properties, which have been the guiding force in many two–dimension nonlinear systems. Though the study so far has been quite formal and mathematical, the ultimate goal is for particle physics: to solve the full Yang–Mills and gravitational fields, and to formulate new ways to quantize the fields.

Recently, linear systems and conservation laws have been constructed for the extended conformal supergravity theories,[1,2] which have been shown to be the consequences of light–like integrability in curved extended superspace.[3,4] This gives a general picture of a unifying description of equations of motion of classical fields from the point of view of geometrical integrability, which had its origin in the study of many two–dimensional nonlinear systems,[5–8] and in the study of self–dual Yang–Mills equations.[8–12] Such a view that equations of motion of classical fields, nonlinear in four dimensional space, become linear in extended superspace[13–19] helps to find classical solutions, and points to new ways of quantizing the theory.

The generic structure of geometrical integrability properties can be summarized in Figure 1.

The heart of the matter is first to find linear systems with parameters. The linear systems are usually of the form

$$\nabla_X \psi(X,Y) = 0, \qquad \nabla_Y \psi(X,Y) = 0,$$

where ∇_X, ∇_Y are some generalized covariant derivatives in some generalized geometrical spaces; e.g. ordinary space–time space plus complex parameters; ordinary space–time with superspace extensions plus complex parameters, loop spaces, noncommutative geometrical spaces, etc. The integrability of ψ requires $[\nabla_X, \nabla_Y\} = 0$, i.e. curvatureless. The equations of motion or the original nonlinear systems then follow from this generalized curvatureless condition. It is from these linear systems with parameters that powerful methods can be used to generate new solutions.

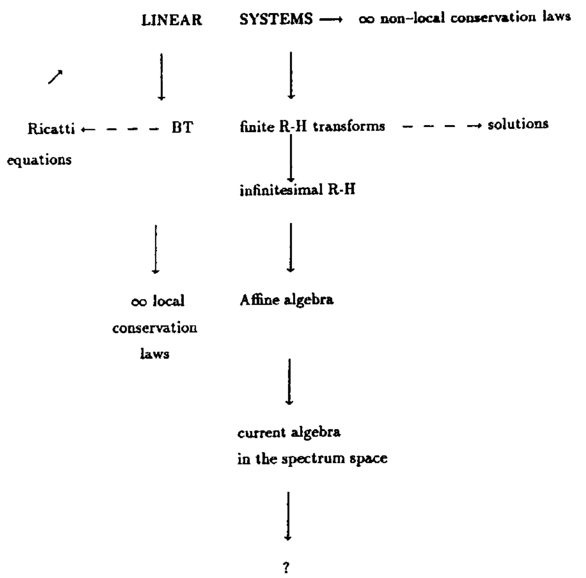

Fig. 1 : Some Generic Structures of integrable Non–linear systems

The general theme of these methods are gauge transformations of ψ, and it is through the analyticity properties in the complex parameters that the nonlocal conservation laws can be derived and that the original nonlinear differential equations are converted into integral equations in the complex-parameter space, i.e., the finite Riemann–Hilbert (RH) transformations. The RH transforms satisfy group properties and their infinitesimal elements form the affine Kac–Moody algebra, which can be viewed as the result of the current algebra in the complex parameter space. Another branch of development is to derive parametric Bianchi–Bäcklund transformation (BT) from a special kind of finite RH transformation. Such BT's historically were found by guesswork and now we have a more general understanding of their origin, i.e., gauge transformation with linear dependence in the complex parameter. From the BT's with parameters, local conservation laws can be derived, and another nonlinear system, the Ricatti equations can be constructed. The Ricatti equations then can be shown to share the same linear systems.[7]

The beautiful and surprising thing is that so many equations of motions in physics possess these geometrical integrability properties when the proper formulations and proper extended spaces are found. The two dimensional systems include chiral models: principal; symmetric–space; superized; with Wess–Zumino term and its superized version; the Ernst equations (which are reduced systems of stationary axially symmetric Einstein equations and also static axially symmetric self–dual Yang–Mills equations); Sine–Gordon equations, KdV equations, Liouville equations in the SL(2C) formulations. All the developments in Fig. (1) have been done for these two-dimensional systems. The four dimensional models includes self–dual Yang–Mills; supersymmetric Yang–Mills; and the conformal supergravity equations. All of them have been shown to have infinite nonlocal conservation laws and linear systems, from which integrability properties of varied degrees as listed in Figure 1 have been derived.

The important extended space that has emerged from such studies is the extended superspace. In respect to whether Nature has manifesting representation of supersymmetry as supersymmetrical particle states or actual superspaces, superspace has already demonstrated its richness as a useful framework to look at our physical equations and may turn out to be an essential part of physical description as complex number is.

RECENT RESEARCH RESULTS

I. Supergravity Theories

I.a. $\underline{D=4}$: It has been shown that light–like integrability conditions for $n \geq 5, 6, 7, 8$ lead to conformal supergravity equations of motion.[3]

I.b. $\underline{D=4}$: Linear systems have been constructed from all ($n = 1, ..., 8$) the light–like integrability conditions.[1] These linear systems help to solve the light–like constraints and thus equations of motion for $n = 5, 6, 7, 8$; and helps to solve the light–like constraints for $n = 1, 2, 3, 4$ for off–shell formulation.

I.c. $\underline{D=10, n=1}$: It has been shown that light–like integrability constraints lead to equations (Poincare) of motion only if an additional algebraic constraint is imposed.[4] Thus the light–like integrability constraints can allow an off–shell formulation of the theory.

I.d. $\underline{D=10, n=1}$: Linear systems and conservation laws can be constructed for the light–like integrability conditions,[2] and thus useful for the off–shell formulation of the $D = 10$, $n = 1$ supergravity theory. In the construction of the linear systems and conservation laws, it is essential to use the bi–spinor representation for the light–like vectors.

II. Supersymmetric Yang–Mills Theories

In addition to the similar developments[15–18] as mentioned in section I for $D = 4$, supergravity theories, our recent new addition is the construction of linear systems, and an infinite number of nonlocal conservation laws using the bi–spinor representation[20] for any light–like vector in $D = 6$ and 10. These will be certainly useful for constructing new solutions in $D = 6$ and 10, and then in $D = 4$ by dimensional reduction.

III. Progress Made For The $D = 4$, Self–Dual Yang–Mills Equation

III.a. Permutability property has been shown to be true for the Chau–Prasad–Sinha Bäcklund transformations (BT).[21]

III.b. The sequence, Parametric BT → Riccati → linear systems, has been constructed for the self–dual Yang–Mills equations.[22]

III.c. A generalized Bäcklund transformation, which is capable of generating instanton solutions has been constructed for the (supersymmetric) self-dual Yang–Mills equations.[23]

IV. The $D = 2$ Theories

IV.a. The Ernst equations which are reduced non-linear systems of static and axially symmetric Einstein, or Yang–Mills equations: linear systems, infinite-nonlocal conservation laws, finite Riemann–Hilbert transforms, and infinitesimal RH transform $\Rightarrow$ Kac–Moody algebra; Bäcklund transformations, etc. have been thoroughly discussed.[24]

IV.b. All the integrability properties as listed in IV.a. have been constructed for the super–chiral equations with Wess–Zumino term.[25]

IV.c. A general gauge covariant formulation, as well as all the integrability properties have been constructed for general symmetric–space chiral fields.[26]

V. General Integrability Discussions

V.a. A unifying derivation of BT has been given from the point of view of finite Riemann-Hilbert transformation.[27]

V.b. A general discussion of Kac-Moody algebra has been made from the point of view of infinitesimal Riemann–Hilbert transformation.[28]

V.c. Using the special Riemann-problem technique of Zakharov et al., we derive an explicit N-step Bäcklund transformation for a class of 1+1 dimensional nonlinear evolution equations.[29]

Now we are ready to move forward in two fronts: first, finding solutions to the full Yang–Mills equations. The essential new feature in the search for classical solutions for the full Yang–Mills and supergravity equations is the use of superspace, and to develop two–complex–variable Riemann–Hilbert transforms, contrasting to the one–complex–variable Riemann–Hilbert transform used in two–dimensional systems and the self-dual Yang–Mills systems. And second, quantizing the super–Yang–Mills and supergravity fields from these new points of view.[30]

VI. Approach to Quantization To approach quantum field theory from this geometrical–integrability point of view, the following work has been done:

VI. a. We have studied the light–cone Hamiltonian formalism of the nonabelian chiral model with Wess–Zumino term in arbitrary coupling constant. The monodromy matrices and their bracket structure are derived explicitly and discussed.[31]

VI. b. From an action for the self–dual Yang–Mills (SDYM) system, we have constructed a higher dimensional version of the Kac–Moody–Virasoro algebra which appears as the symmetry of this system. We have also constructed a SDYM hierarchy with using these algebras.[32]

VI. c. We have studied a conformally invariant theory which consists of scalar fields and a gauge field. The presence of gauge fields introduces interesting phase factors given by the line integral of gauge potential in the correlation functions.[33]

FUTURE PLANS

Developing quantum field theories from this geometrical-integrability point of view will be the emphasis for the near future. Work done as given in VI.1, VI.2, and VI.3 are just the beginning. Certainly, we do not expect that Yang–Mills and general relativity field equations are integrable in the way integrability has revealed in two dimensional models. However the linear systems have given us a very strong hint that these realistic four–dimensional field theories are "partially" integrable. How to make best use of these "relics" of integrability and get as much information out as possible will be the challenge. The hope is that after the integrable part is extracted out, the nonintegrable part will be much easier to deal with. The approach may also provide a non-perturbative approach to these highly and complex nonlinear dynamical system.

On the classical side, there is still much to be done, e.g., I would like to construct classical solutions to the full Yang–Mills equations and the full conformal gravity equations via the geometrical–integrability properties so far we have found. It is conceivable that we may eventually write down a general form of solutions to these equations as Penrose had done to all linear free massless field equations.

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I would like to thank the organizers for the interesting Conference, especially to the organizers of this session, Dr. M.L. Ge and M. Wadati for inviting me to present this talk. This work is partly supported by U.S. Department of Energy.

REFERENCES

1. Chau, L.–L., "Linear Systems and Conservation Laws of Gravitational Fields in Four Plus Extended Superspace", Phys. Lett. B<u>202</u> (1988) 238.

2. Chau, L.-L. and B. Milewski, Phys. Lett. B. 216 (1989) 330.

3. Chau, L.-L. and Lim, C.-S., Phys. Rev. Lett. 56, 294 (1986).

4. Chau, L.-L. and Milewski, B., "Light-Like Integrability and Supergravity Equations of Motion in $D = 10$, $N = 1$ Superspace", UCD-87-05-R*, Chau, L.-L. and Lim, C.S., Mod. Phys. A, 4 (1989) 3819.

5. For general discussions on such equations, see, for example, G.L. Lamb, Jr., "Elements of Soliton Theory" (Wiley, New York, 1980); M.J. Ablowitz and H. Segur, "Solitons and the Inverse Scattering Transform" (SIAM, Philadelphia, 1981); F. Calogero and A. Degasperis, "Spectral Transform and Solitons, I" (North-Holland, Amsterdam, 1982).

6. Faddeev, L.D. and N. Yu. Reshetikhin, Annals of Physics 167 (1986) 227; E.K. Sklyanin, L.A. Takhtadzhyan, and L.D. Faddeev, Teoreticheskaya i Matematcheskaya Fiziks, Vol. 40 (1979) 194.

7. Novikov, S., S.V. Manakov, L.P. Pitaevskii and V.E. Zakharov, "Theory of Solitons: The Inverse Scattering Method" (Consultants Bureau, New York, 1984).

8. Chau, L.-L., Proceedings of the 1980 Guangzhou Conference on Theoretical Particle Physics, Van Notstrand Reinhold, New York 1981 and Science Press of PRC; and in Proceedings of Nonlinear Phenomena, Mexico, 1982, *Lecture Notes in Physics* **189** (Springer-Verlag, New York, 1983).

9. Yang, C.N., Phys. Rev. Lett. 38 (1977) 1377. S. Ward, Phys. Lett. 61A (1977) 81. Y. Brihaye, D.B. Fairlie, J. Nuyts, R.F. Yates, JMP 19 (1978) 2528.

10. Atiyah, M.F. and R.S. Ward, Commun. Math. Phys. 55 (1977) 117.

11. Atiyah, M.F., V.G. Drinfeld, N.J. Hitchin and Yu.I. Manin, Phys. Lett. A 65 (1978) 185.

12. Corrigan, E.F., D.B. Fairlie, R.G. Yates and P. Goddard, Commun. Math. Phys. 58 (1978) 223.

13. Salam, A. and J. Strathdee, Phys. Lett. 51B, 353 (1974).

14. For review see J. Scherk, in *Recent Development in Gravitation* (Plenum, New York, 1979); B. Zumino, in *Proceedings of the NATO Advanced Study Institute on Recent Developments in Gravity, Cargése, 1978* (Plenum, New York, 1979), p. 405; J. Wess and J. Bagger, *Supersymmetry and Supergravity* (Princeton Univ. Press, Princeton, N.J., 1983).

15. Ferber, A., Nucl. Phys. 77B, 394 (1978); Witten, E., Phys. Lett. 77B, 394 (1978); M. Sohnuis, Nucl. Phys. B136, 461 (1978).

16. Volovich, I.V., Phys. Lett. 129B, 429 (1983).

17. Devchan, C., Nucl. Phys. B238, 333 (1984).

18. Chau, L.-L., Ge, M.-L., Popowicz, Z., Phys. Rev. Lett. 52, 1940 (1984); Chau, L.-L., Ge, M.-L., Lim, C.-S., Phys. Rev. D33, 1056 (1986).

19. Chau, L.-L., Proceedings of the 1983 Berkeley Workshop, "Vertex Operators in Mathematics and Physics", Eds. J. Lepowsky, S. Mandelstam, I.M. Singer.

20. Chau, L.-L. and Milewski, B., Phys. Lett. 198 (1987) 356.

21. Chau, L.-L. and Chinea, F.J., Lett. in Math. Phys. 12, 189 (1986); Chau, L.-L., Yen, H.-C., Chen, H.-H., Chinea, F.J., Lee, C-R., Shaw, J.-C., Mod. Phys. Lett. 1, 285 (1986).

22. Chau, L.-L. and Yen, H.-C., J. Math. Phys. 28, 1167 (1986).

23. Chau, L.-L., J.C. Shaw, and H.C. Yen, J. of Mod. Phys. A, 4 (1989) 2715.

24. Chau, L.-L., Chou, K.-C., Hou, B.-Y., Song, X.-C., Phys. Rev. 34D, 1814 (1986).

25. Chau, L.-L. and Yen, H.C., Phys. Lett. B 177, 368 (1986).

26. Chau, L.-L., Hou, B.-Y., Phys. Lett. 145B, 347 (1984); Chau, L.-L., Hou, B.-Y., Song, X.-C., Phys. Lett. 151B, 421 (1985).

27. Chau, L.-L., Ge, M.-L., Yen H.-C., "Finite Riemann-Hilbert Transformation and A Unifying Derivation of Bäcklund Transformation", UCD-87- , (1987).

28. Chau, L.-L., Ge, M.-L., J. of Math. Phys. 30 (1989) 166.

29. Chau, L.-L., Shaw, J.C., Yen, H.C., "An Alternative Explicit Construction of N-Solutions in 1+1 Dimensions", UCD-90-5, to appear in J. Math. Phys. (1990).

30. For a recent review on the field: see Chau, L.-L., "Geometrical Integrability and Equations of Motion in Physics: A Unifying View," talk given at the Workshop held at Nankai Insti-

tute of Mathematics, Nankai University Tianjin, China, August 1987; Appeared as Nankai Lectures on Mathematical Physics, "Integrable Systems", World Scientific.

31. Chau, L.-L., Yamanaka, I., "Quantization of Chiral Model with Wess-Zumino Term in the Light-Cone Coordinate," UCD-90-4, (1990).

32. Chau, L.-L., Yamanaka, I., "A Virasoro Algebra in Self-Dual Yang-Mills System," UCD-90-7, (1990).

33. Chau, L.-L., Yu, Y., "Chern-Simons Gauged Conformal Field Theories" UCD-90-6 (1990).

DISCUSSION

Q. **Yong-Shi Wu** *(Univ. Utah)*: What are the boundary conditions you impose at $x_- = 0$ and $x_- = L$?

A. **L. L. Chau**: We have tried both the vanishing condition and the periodic boundary condition. For both cases the troubling terms violating the Yang-Baxter equations are present.

SKYRMIONS in (2+1) DIMENSIONS

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Abstract: We consider instanton and anti-instanton solutions of the $O(3)$ σ-model in two Euclidean dimensions modified by the addition of apropriate potential and skyrme-like terms as static solitons (and anti-solitons) - skyrmions of the same model in (2+1) dimensions. We find that in contradistinction to the pure $O(3)$ σ-model the addition of the potential and skyrme terms stabilises the skyrmions and that the force between them is repulsive. In the scattering process initiated at low relative velocities the skyrmions bounce back while at larger velocities they scatter at right angles. The scattering is quasi-elastic and the skyrmions preserve their shape after the collision. On the other hand a skyrmion and an antiskyrmion attract each other and annihilate into pure radiation.

1. Introduction

Over the last few years sigma models in low dimensions have become an increasingly important area of research. Although the σ-models are integrable in two dimensions [1-3] it appears that only very special models are integrable[4] in (2+1) dimensions. In particle physics we are interested primarily in Lorentz invariant models. But all such σ-models in (2+1) dimensions appear to be nonintegrable, and so it is natural to consider numerical evolutions in these cases.

The simplest Lorentz invariant (2+1) dimensional σ model is the O(3) model, which contains three real scalar fields, $\vec{\phi} \equiv (\phi^1, \phi^2, \phi^3)$. In (2+1) dimensions $\vec{\phi}$ is a function of the space-time coordinates (t, x, y) which we also write as (x^0, x^1, x^2). The model is defined by the Lagrangian density

$$\mathcal{L} = \tfrac{1}{4}(\partial^\mu \vec{\phi}) \cdot (\partial_\mu \vec{\phi}) , \qquad (1.1)$$

together with the constraint $\vec{\phi} \cdot \vec{\phi} = 1$, *i.e.* $\vec{\phi}$ lies on a unit sphere S^2_ϕ. In (1.1) the Greek indices take values $0, 1, 2$ and label space-time coordinates, and ∂_μ denotes partial differentiation with respect to x^μ. Note that we have set the velocity of light, c, equal to unity, so that in all our calculations we can use dimensionless quantities. The Euler-Lagrange equations derived from (1.1) are

$$\partial^\mu \partial_\mu \vec{\phi} + (\partial^\mu \vec{\phi} \cdot \partial_\mu \vec{\phi}) \vec{\phi} = \vec{0} . \qquad (1.2)$$

For boundary conditions we take

$$\vec{\phi}(r, \theta, t) \to \vec{\phi}_0(t) \quad \text{as} \quad r \to \infty , \qquad (1.3)$$

where (r, θ) are polar coordinates and where $\vec{\phi}_0$ is independent of the polar angle θ. In two Euclidean dimensions (*i.e.* taking $\vec{\phi}$ to be independent of time) this condition ensures finiteness of the action, which is precisely the requirement for quantisation in terms of path integrals. In (2+1) dimensions it leads to a finite potential energy. The boundary condition (1.3) introduces nontrivial topological aspects into the theory and it allows us to intoduce an integer-valued topological charge given by

$$N = \frac{1}{8\pi} \int \epsilon_{ij} \vec{\phi} \cdot (\partial_i \vec{\phi} \times \partial_j \vec{\phi}) \, d^2x \ , \qquad (1.4)$$

where ϵ_{ij} is the antisymmetric symbol on two indices such that $\epsilon_{12} = -\epsilon_{21} = 1$.

It is convenient to express the $\vec{\phi}$ fields in terms of their stereographic projection onto the complex plane W

$$\phi^1 = \frac{W + W^\star}{1 + |W|^2}, \quad \phi^2 = i \frac{W - W^\star}{1 + |W|^2}, \qquad \phi^3 = \frac{1 - |W|^2}{1 + |W|^2}. \qquad (1.5)$$

The W formulation is very useful, because it is in this formulation that the static solutions take the simplest form; namely, as originally shown by Belavin and Polyakov[5] and Woo,[6] they are given by W being any rational function of either $x + iy$ or of $x - iy$. It is easy to see that the topological charge of these solutions is a positive or a negative integer respectively. By convention the first case corresponds to instantons and the other to anti-instantons.

Can we consider the instanton solutions as static solutions of the same model in (2+1) dimensions? Can we have any nonstatic solitons? Of course, the static solutions can be made to move with arbitrary velocity, simply by Lorentz boosting. Being extended structures with a localised energy, they resemble the familiar examples of solitons in (1+1) dimensions. But are they solitons in the strict sense? In particular, one may wonder whether they are stable under small perturbations and also whether they preserve their shape and velocity in scattering processes.

These are the problems that will be discussed in this talk. The talk is based on the work some parts of which have been performed in collaboration with R.A. Leese and which has been the subject of a series of papers.[7] [8] [9] [10]

Looking at the problem of stability we observe that the model has no intrinsic scale and so admits the existence of solitons of arbitrary size. Hence under small perturbations the solitons can either expand indefinitely or shrink to become infinitely tall spikes of zero width. Our simulations have shown that this is exactly what happens in this model. In fact, as soon as the solitons are purturbed, *e.g.* start moving, they start shrinking.

We have analysed this problem in some detail and have found[7] that the solitons of the $O(3)$ σ model are unstable. This is true not only in the full simulation of the model but also[11] in the approximation to the full simulation provided by the so-called "collective coordinate" approach in which the evolution is approximated by geodesic motion on the manifold of static solutions. Such an approximation is clearly very reliable at small velocities; however, all our studies have shown[11] [7] that it is also reliable even at unexpectedly high velocities (~ 0.5 of the velocity of light).

A few words about our numerical procedures. Most of our simulations were performed in Los Alamos using a 4th order Runge-Kutta method of simulating time evolution. We used the Los Alamos Connection Machine working in double precision and also some Los Alamos Crays. We also performed some calculations, using double precision, on the Floating Point System Machine and on Multiflow Trace. Almost all our simulations were performed on fixed lattices which varied from 201×201 to 512×512, with lattice spacing $\delta x = \delta y = 0.02$. The time step was 0.01.

So far as the boundary conditions are concerned most of our simulations were performed with fixed boundary conditions as all the effects associated with the variation of the fields at the boundaries are very small. However, even though small, they are nonzero and so we tested their effects by introducing some absorption or by extrapolating the fields at the boundaries. We have found that the waves coming from the boundaries or the waves reflected from the boundaries can effect our results quite significantly. In particular, some preliminary results obtained on smaller size lattices, were not confirmed in our bigger lattice simulations. Having tested our results by changing the lattice size and varying the boundary conditions we are reasonably confident of our results; although we believe some more work would be required to be absolutely certain.

2. Skyrme Model

To stabilise the $O(3)$ model we introduced a scale into the model that would prevent the instantons from both shrinking and expanding. Guided by the ideas of Skyrme[12] [13] we chose to add to our Lagrangian density the following extra terms

$$L_e = -\frac{1}{4}\Big(\theta_1\Big((\partial^\mu\vec{\phi}\,.\,\partial_\mu\vec{\phi})^2 - (\partial^\mu\vec{\phi}\,.\,\partial^\nu\vec{\phi})(\partial_\mu\vec{\phi}\,.\,\partial_\nu\vec{\phi})\Big) + \theta_2(1+\phi^3)^4\Big), \tag{2.1}$$

where θ_1 and θ_2 are two new (real) parameters of the model. It is clear that the model based on the Lagrangian with these terms is still Lorentz invariant and for positive values of θ's its Hamiltonian is positive definite. Moreover, despite the appearance to the contrary, the Lagrangian does not contain time derivatives higher than two and so its equation of motion takes the conventional form.

$$\begin{aligned}&\partial_\mu\partial^\mu\phi^i - (\vec{\phi}\,.\,\partial_\mu\partial^\mu\vec{\phi})\phi^i\\&- 2\theta_1\Big[\partial_\mu\partial^\mu\phi^i(\partial_\nu\vec{\phi}\,.\,\partial^\nu\vec{\phi}) + \partial_\nu\phi^i(\partial_\mu\partial^\nu\vec{\phi}\,.\,\partial^\mu\vec{\phi})\\&\partial_\nu\partial_\mu\phi^i(\partial^\nu\vec{\phi}\,.\,\partial^\mu\vec{\phi}) - \partial_\mu\phi^i(\partial^\nu\partial_\nu\vec{\phi}\,.\,\partial^\mu\vec{\phi})\\&+ (\partial_\mu\vec{\phi}\,.\,\partial^\mu\vec{\phi})(\partial_\nu\vec{\phi}\,.\,\partial^\nu\vec{\phi})\phi^i - (\partial_\nu\vec{\phi}\,.\,\partial_\mu\vec{\phi})\\&(\partial^\nu\vec{\phi}\,.\,\partial^\mu\vec{\phi})\phi^i\Big] + 2\theta_2(1+\phi^3)^3(\delta_{i3} - \phi^i\phi^3) = 0.\end{aligned} \tag{2.2}$$

The equation (2.2) is rather difficult to solve, but if we restrict ourselves to looking for static solutions and then consider $\vec{\phi}$, which corresponds to W being analytical (*i.e.* $W = W(x+iy)$) then it is easy to check that

$$W = \lambda(x+iy) \tag{2.3}$$

is a static solution if

$$\lambda = \sqrt[4]{\frac{\theta_2}{2\theta_1}}. \tag{2.4}$$

This is a particular case of the one instanton solution of the $O(3)$ σ model, but with the fixed "size" (determined by λ).

It is easy to show that this solution is stable with respect to any perturbations. In fact, if we try to evolve it with (2.4) different from λ we find that the system has an excess of energy which it uses to bring its size up or down to the correct value and at the same time it sends out a wave of radiation.

So what are the scattering properties of our skyrmions? First we looked at the behaviour of two static skyrmions. Thus we considered the field configuration described by

$$W = \frac{(x+iy-a)(x+iy+a)}{2\mu a} \tag{2.5}$$

This configuration describes two skyrmions (located at $\pm a$); their widths are the same and are given by $\lambda = \frac{1}{\mu}$. As (2.5) is not a solution of (2.2) it evolves and as it evolves the system develops some kinetic energy. Looking at this evolution we have found that the forces acting on the skyrmions are quite complicated; they have both repulsive and attractive components. At first the attractive forces win and the skyrmions approach each other. However, this does not last long; very soon the process is reversed, the skyrmions repel and move away from each other.

During this reversal the system performs some internal oscillations. In fact, in our simulations we observed two internal oscillations - which involved the kinetic energy flowing twice in the original direction of motion followed by its flow at 90° before the system stabilised and the skyrmions started moving away from each other. As they moved away they accellerated. Moreover, our studies have shown that the qualitative behaviour of the interaction does not depend on the values of θ's; as we increase their values all effects are the same but become more pronounced.

Next we looked at the scattering properties of two skyrmions sent towards each other at some velocity v. We implemented this idea by starting with the field configuration

$$W(x,y,t) = \frac{(x+iy-a+vt)(x+iy+a-vt)}{2\mu(a-vt)} \tag{2.6}$$

and calculating from it $W(x,y,0)$ and $\partial_t W(x,y,0)$. We chose $a = 1.0$ and considered the dependence of the evolution on the values of v. Again, we found that for all values of θ's the qualitative properties of the scattering were the same. At small values of velocity the skyrmions scattered back to back. When we increased the initial velocity the skyrmions came closer and closer together before scattering back to back, then they spent longer and longer in a quasi-trapped state before bouncing back, and finally above a certain critical valueof the velocity v_{cr} they scattered at 90° to the original direction of motion in their centre of mass.

When the initial velocity is critical or higher the skyrmions manage to come very close together before scattering; at their closest they form a ring from which the outgoing skyrmions emerge. When the ring is formed the skyrmions loose their identity - hence it does not make sense to enquire which skyrmion goes where.

We also looked at the dependence of the critical velocity on the values of θ's. We observed an increase of v_{cr} with the increase of θ's and a sort of levelling off (or even a small decrease) at larger values of θ's.

We have also observed that as the skyrmions move towards each other with their velocities approaching their critical value, the time during which the skyrmions stayed close together increades, implying the trapped nature of the quasi bound-state formed by the skyrmions.

When two skyrmions are put initially on top of each other, the energy density describes a ring very reminiscent of the ring formed during the scattering process. The observed evolution showed small oscillations in the value of the size of the ring, followed, quite suddenly, by the separation of skyrmions and their motion away from each other with some finite velocity very close to the effective critical velocity (for the corresponding value of θ's).

Hence we see that the mechanism of the formation of the ring and the possibility of having two skyrmions on top of each other have the same origin.

3. More General Systems

We also looked at systems consisting of one skyrmion and one antiskyrmion. In this case we considered as our initial configuration

$$W = \frac{(x+iy-a)(x-iy+a)}{2\mu a}. \tag{3.1}$$

It is easy to check that (3.1) is not a solution of the equation of motion. Moreover, the system is clearly unstable and when started at rest the two extended structures approached each other and annihilated into pure radiation. During their approach the skyrmion and antiskyrmion attracted each other and so accellerated while moving towards each other. We found that before their annihilation the skyrmion and antiskyrmion preserved their identities very well. After the interaction the system represented just pure waves; what is interesting is that their maxima flowed at 90° to the original direction of motion. It is easy to check that the outgoing structures really represented pure radiation waves and not skyrmions and antiskyrmions; it is enough to observe that they moved with the velocity of light and that their topological charge gradually decreased.

We performed several simulations varying the initial value of v and changing the values of θ_2. We found no significant dependence on θ's showing that for a system of skyrmions and antiskyrmions, which is characterised by strong attractive forces already at the $O(3)$ level, the additional forces generated by the potential and skyrme terms in the Lagrangian have little effect on the main features of the scattering. We performed several others simulations including the interesting case of a skyrmion and an antiskyrmion rotating around each other. In this last case we found that the skyrmion and the antiskyrmion had almost got trapped in an orbit around each other; however, due to their interaction, at a certain time they slowed down their circular motion, moved towards each other and then annihilated into pure radiation. The maximum of the radiation was again sent out at 90° to the direction motion just before the annihilation.

4. Some Comments

We have seen that the skyrmions behave very much like real solitons. In the scattering involving only skyrmions they preserve their shape and although during the scattering some radiation effects are present, these effetcs are always very small. The situation is different, however, for systems involving skyrmions and antiskyrmions; they interact with each other very strongly and annihilate into pure radiation. The outgoing radiation follows the scattering of skyrmions above their critical velocity; namely the peaks of radiation are sent out at 90° to the original direction of motion in the centre of mass. This phenomenon has also been observed in many scatterings of other extended structures, such as monopoles or vortices[14] and so we believe that it is probably very typical of all scattering of extended structures in (2+1) dimensions.

In conclusion we see that the modified $O(3)$ model, although non-integrable, is almost integrable in that it has many features in common with many integrable models. Most differences or deviations are rather small. As most physically relevant models are not integrable our results suggest that the results found in some integrable models should not be dissmissed as not relevant; it is quite likely that some of these results may also hold in models which, strictly speaking, are not integrable but whose deviations from integrability are rather small.

Most of our results agree with the results obtained in the collective coordinate approximation. This suggests that this approximation is much better than could be first thought of on purely general grounds. Finally, our results suggest that the modified $O(3)$ σ model is a good candidate for being a toy model of solitons in (2+1) dimensions.

REFERENCES

1. Y.Y. Goldschmidt and E. Witten, *Phys. Lett*, **91 B**, 392 (1980)
2. V.E. Zakharov and A.V. Mikhailov, *Sov. Phys. JETP* **47**, 1017 (1979)
3. J. Harnad, Y. Saint-Aubin and S. Shnider, *Comm. Math. Phys.*, **92**, 329 (1984)
4. R.S. Ward, *Nonlinearity*, **1**, 671 (1988)
5. A.A. Belavin and A.M. Polyakov, *JETP Lett*, **22**, 245 (1975)
6. G. Woo*J. Math. Phys.*, **18**, 1264 (1977)
7. R.A. Leese, M. Peyrard and W.J. Zakrzewski - *Nonlinearity*, **3**, 387 (1990)
8. R.A. Leese, M. Peyrard and W.J. Zakrzewski - *Nonlinearity*, **3**, 773 (1990)
9. W.J. Zakrzewski - Soliton-like Scattering in the $O(3)$ σ model in (2+1) Dimensions - *Nonlinearity* to appear (1990)
10. M. Peyrard, B. Piette and W.J. Zakrzewski - Soliton Scattering in the Skyrme Model in (2+1) Dimensions 1. and 2., Durham University preprints DTP-90/37 and 39 (1990)
11. R.A. Leese, Low-energy Scattering of solitons in the CP^1 model, *Nucl. Phys.* **B**, to appear (1990)
12. T.H.R. Skyrme - *Proc. Roy. Soc.***A 260**, 127 (1961)
13. E. Witten - *Nucl. Phys.* **B 223**, 433 (1983), G. Adkins, C. Nappi and E. Witten - *Nucl. Phys.* **B228**, 552 (1983)
14. for an extended discussion see ref. 11