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MODULATIONAL INSTABILITY OF NONLINEAR WAVES IN A COLD QUARK–GLUON PLASMA

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Using a simple equation of state for the quark–gluon plasma (QGP), we expand the hydrodynamic equations around equilibrium configurations. The resulting differential equations describe the propagation of perturbations in the energy density. We derive in detail the nonlinear Schrödinger equation (NLSE) which governs the modulation instability (MI) in the cold quark–gluon plasma.

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1. Introduction

For the last several years, a tremendous effort has been dedicated to a fairly new branch of high-energy physics, namely, the field of quark–gluon plasma (QGP) [1]. Systems consisting of deconfined quarks and gluons, the fundamental constituents of matter and the mediators of the strong force, are produced in controlled laboratory conditions in reactions of heavy nuclei at ultrarelativistic energies. These so-called "quark–gluon plasmas" exist at very high temperatures and energy densities similar to those found a few microseconds after the Big Bang. The quark–gluon plasma is a special kind of state of matter, which is believed to be formed in ultrarelativistic heavy-ion collisions or existed for a few microseconds after the Big Bang. Quantum

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Chromodynamics (QCD) predicts that deconfined phase transition will take place at high temperature and/or high density. As a result, the nuclear matter will undergo a transition to the quark–gluon plasma which is composed of quarks and gluons. One main goal for relativistic heavy-ion collision experiments is to seek this new state of matter. For the last ten years, the study of QGP properties has attracted intense interest. Recently, we have learned many things about the QGP and one of the most striking is that it is an almost perfect fluid with very small viscosity. Highly localized perturbations can exist and propagate through a fluid. The most famous are the Korteweg–de Vries (KdV) solitons, which are solutions of the KdV equation. Previous studies on nonlinear waves in cold and warm nuclear matter can be found in [2, 3]. In [4], wave propagation has been studied in cold and dense matter, both in a hadron gas phase and in a quark–gluon plasma phase. Works on nonlinear waves in cold QGP in the mean field approach were published in [5] and their extension to three dimensions was published in [6]. Perturbations in fluids with different equations of state (EOS) generate different nonlinear wave equations: the breaking wave equation, KdV, Burgers *etc.* Among these equations, we find the Kadomtsev–Petviashvili (KP) equation [7], which is a nonlinear wave equation in three spatial and one temporal coordinate. It is the generalization of the KdV equation to higher dimensions. This equation has been found with the application of the reductive perturbation method [8] to several different problems such as the propagation of solitons in multicomponent plasmas, dust acoustic waves in hot dust plasmas and dense electron–positron–ion plasma [9]. In [10], Fogaça *et al.* considered hadronic matter at finite temperature and studied the effects of temperature on the KdV soliton. In [11], they started the study of perturbations in the QGP at zero and finite temperature. The conclusion found in that work was that the existence of KdV solitons in a QGP depends on details of the EOS and with a simple MIT Bag Model EOS there is no KdV soliton. A further study of the equation of state, carried out in [12], showed that if nonperturbative effects are included in the EOS through gluon condensates, then new terms appear in the expression of the energy density and pressure. Otherwise, for the last few years, great attention has been paid to the study of the modulation instability (MI) of solitons in the context of the nonlinear Schrödinger equation (NLSE) [13, 14], due to their relevance in wave propagation stability. To complement and provide new insights into what has been already published, we propose here to investigate the modulational instability of waves in a cold quark–gluon plasma.

One of the first equations of state of the quark–gluon plasma was the one derived from the MIT (Massachusetts Institute of Technology) Bag Model [15]. Due to its simplicity, it has been widely used in astrophysics and cosmology. A further study of the equation of state (EOS), carried out in [5], showed that if nonperturbative effects are included in the EOS through gluon condensates, then new terms appear in the expression of the energy density and pressure, and in the present work, we will take into account these new terms to study the modulational instability of waves in a cold quark–gluon plasma. This study might be applied to the deconfined cold quark matter in compact stars and cold quark–gluon plasma formed in heavy-ion collisions at the intermediate energies in the Facility for Antiproton and Ion Research (FAIR) [16] or Nuclotron-based Ion Collider Facility (NICA) [17].

The aim of this paper is therefore to derive the nonlinear Schrödinger equation (NLSE) after the proper treatment of the hydrodynamical equations. In the next section, we present the basic equations of our theoretical model. In Section 3, we introduce the equation of state. A weak nonlinear analysis is carried out in Section 4 to derive the NLSE. Our results are given in Section 5, and a summary of our findings is given in Section 6.

2. Relativistic fluid dynamics

In this part, we review the essential applications of one-dimensional relativistic hydrodynamics. In this work, we use the natural units $c = 1$, $\hbar = 1$ and (Boltzmann's constant) $k_{\mathrm{B}} = 1$. The relativistic version of the Euler equation [11, 12, 18, 19] is given by

$$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla}\right) \vec{v} = -\frac{1}{(\varepsilon + p)\,\gamma^2} \left(\vec{\nabla}_p + \vec{v}\frac{\partial p}{\partial t}\right) , \tag{1}$$

where $\vec{v}$, ε, p and γ are the velocity, energy density, pressure and the Lorentz factor, respectively. Space and time coordinates will be in fm (1 fm $= 10^{-15}$ m). The relativistic version of the continuity equation for the baryon density ρ_B in ideal relativistic hydrodynamics is [18]

$$\partial_\nu j_B^\nu = 0\,. \tag{2}$$

Since $j_B^\nu = u^\nu \rho_B$, the above equation could be rewritten as follows [11, 12]:

$$\frac{\partial \rho_B}{\partial t} + \gamma^2 \vec{v} \rho_B \left(\frac{\partial v}{\partial t} + \vec{v} \cdot \vec{\nabla}\vec{v}\right) + \vec{\nabla} \cdot (\rho_B \vec{v}) = 0\,. \tag{3}$$

In the one-dimensional Cartesian relativistic fluid dynamics, the velocity field is written as $\vec{v} = v(x,t)\widehat{x}$, where $\widehat{x}$ is the unit vector in the x direction. Equations (1) and (3) can be rewritten in the simple form

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{\left(v^2 - 1\right)}{(\varepsilon + p)} \left(\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial t}\right) , \tag{4}$$

and

$$v\rho_B\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}\right)+\left(1-v^2\right)\left(\frac{\partial \rho_B}{\partial t}+\rho_B\frac{\partial v}{\partial x}+v\frac{\partial \rho_B}{\partial x}\right)=0\,. \tag{5}$$

3. The QGP equation of state

In this section, we shall use the equation of state derived from the MIT Bag Model. This equation was obtained and developed for the strongly interacting quark–gluon plasma (sQGP) at zero temperature [5, 12]. It has been applied to the calculation of the structure of compact quark stars [20]. The energy density and the pressure are given by [12]

$$\begin{aligned}\varepsilon \;=\;& \left(\frac{27g^2}{2m_G^2}\right)\rho_B^2+\left(\frac{27g^2}{2m_G^4}\right)\rho_B\frac{\partial^2\rho_B}{\partial x^2}+\left(\frac{27g^2}{2m_G^6}\right)\rho_B\frac{\partial^4\rho_B}{\partial x^4}\\ &+\left(\frac{27g^2}{2m_G^8}\right)\frac{\partial^2\rho_B}{\partial x^2}\frac{\partial^4\rho_B}{\partial x^4}+\boldsymbol{B}_{\mathrm{QCD}}+3\frac{\gamma_Q}{2\pi^2}\frac{k_{\mathrm{F}}^4}{4}\end{aligned} \tag{6}$$

and

$$\begin{aligned}p =& \left(\frac{27g^2}{2m_G^2}\right)\rho_B^2+\left(\frac{18g^2}{m_G^4}\right)\rho_B\frac{\partial^2\rho_B}{\partial x^2}-\left(\frac{9g^2}{m_G^6}\right)\rho_B\frac{\partial^4\rho_B}{\partial x^4}-\left(\frac{9g^2}{2m_G^4}\right)\frac{\partial\rho_B}{\partial x}\frac{\partial\rho_B}{\partial x}\\ &+\left(\frac{9g^2}{2m_G^6}\right)\frac{\partial^2\rho_B}{\partial x^2}\frac{\partial^2\rho_B}{\partial x^2}-\left(\frac{9g^2}{m_G^8}\right)\frac{\partial^2\rho_B}{\partial x^2}\frac{\partial^4\rho_B}{\partial x^4}-\left(\frac{9g^2}{2m_G^8}\right)\frac{\partial^3\rho_B}{\partial x^3}\frac{\partial^3\rho_B}{\partial x^3}\\ &-\left(\frac{9g^2}{m_G^6}\right)\frac{\partial\rho_B}{\partial x}\frac{\partial^3\rho_B}{\partial x^3}-\boldsymbol{B}_{\mathrm{QCD}}+\frac{\gamma_Q}{2\pi^2}\frac{k_{\mathrm{F}}^4}{4}\,.\end{aligned} \tag{7}$$

In Eqs. (6) and (7), γ_Q is the quark degeneracy factor $\gamma_Q = 2(\text{spin}) \times 3(\text{flavor}) = 6$ and k_{F} is the Fermi momentum defined by the baryon number density by $\rho_B = \frac{k_{\mathrm{F}}^3}{\pi^2}$. The other various parameters g, m_G and $\boldsymbol{B}_{\mathrm{QCD}}$ are the coupling of the hard gluons, the dynamical gluon mass and the bag constant in terms of the gluon condensate, respectively.

4. Derivation of the Nonlinear Schrödinger Equation (NLSE)

We first write Eqs. (4) and (5) in terms of the dimensionless variables

$$\widehat{\rho}=\frac{\rho_B}{\rho_0}\,,\qquad \widehat{v}=\frac{v}{c_{\mathrm{s}}}\,, \tag{8}$$

where ρ_0 is an equilibrium (or reference) density, upon which perturbations may be generated, and c_{s} is the speed of sound.

To investigate the modulation of the wave, we employ the standard reductive perturbation technique (RPT) [8] to derive the appropriate NLSE. The independent variables are stretched as $\xi = \sigma\,(x - V_{\rm g}t)$ and $\tau = \sigma^2 t$, where σ is a small parameter and $V_{\rm g}$ is the group velocity of the wave. The dependent variables are then expanded as

$$\begin{aligned} \rho &= 1 + \sum_{n=1}^{\infty}\sigma^{(n)} \sum_{l=-\infty}^{\infty} \rho_l^{(n)}(\xi,\tau)\mathrm{e}^{il(kx-\omega t)}\,, \\ \upsilon &= \sum_{n=1}^{\infty}\sigma^{(n)} \sum_{l=-\infty}^{\infty} \upsilon_l^{(n)}(\xi,\tau)\mathrm{e}^{il(kx-\omega t)}\,, \end{aligned} \tag{9}$$

where ρ and υ satisfy the reality condition $\Phi_{-l}^{(n)} = (\Phi_l^{(n)})^*$ and the asterisk denotes complex conjugate. The derivative operators appearing in the fluid equations are written as

$$\begin{aligned} \frac{\partial}{\partial t} &\rightarrow \frac{\partial}{\partial t} + \sigma^2\frac{\partial}{\partial \tau} - \sigma V_{\rm g}\frac{\partial}{\partial \xi}\,, \\ \frac{\partial}{\partial x} &\rightarrow \frac{\partial}{\partial x} + \sigma\frac{\partial}{\partial \xi}\,. \end{aligned} \tag{10}$$

Substituting the expression of the operators Eq. (10) and the expression of the expanded variables Eq. (9) into Eqs. (4)–(5), we obtain the $n^{\rm th}$-order reduced equations

$$
\begin{aligned}
&\left(\frac{27g^2\rho_0^2}{m_G^2}c_{\mathrm{s}}\right)\Bigg[-il\omega\upsilon_l^{(n)}+\frac{\partial\upsilon_l^{(n-2)}}{\partial\tau}-V_{\mathrm{g}}\frac{\partial\upsilon_l^{(n-1)}}{\partial\xi}+2\sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty}\left(-i\omega\left(l-l'\right)\upsilon_{l-l'}^{(n-n')}\rho_{l'}^{(n')}+\rho_{l'}^{(n'-2)}\frac{\partial\upsilon_{l-l'}^{(n-n')}}{\partial\tau}-V_{\mathrm{g}}\rho_{l'}^{(n'-1)}\frac{\partial\upsilon_{l-l'}^{(n-n')}}{\partial\xi}\right)\\
&+\sum_{n',n''=1}^{\infty}\sum_{l',l''=-\infty}^{\infty}\left(-i\omega\left(l-l'-l''\right)\upsilon_{l-l'-l''}^{(n-n'-n'')}\rho_{l'}^{(n')}\rho_{l''}^{(n'')}+\rho_{l'}^{(n'-2)}\rho_{l''}^{(n'')}\frac{\partial\upsilon_{l-l'-l''}^{(n-n'-n'')}}{\partial\tau}-V_{\mathrm{g}}\rho_{l'}^{(n'-1)}\rho_{l''}^{(n'')}\frac{\partial\upsilon_{l-l'-l''}^{(n-n'-n'')}}{\partial\xi}\right)\Bigg]\\
&+3\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}c_{\mathrm{s}}\Bigg[-il\omega\upsilon_l^{(n)}+\frac{\partial\upsilon_l^{(n-2)}}{\partial\tau}-V_{\mathrm{g}}\frac{\partial\upsilon_l^{(n-1)}}{\partial\xi}+\frac{4}{3}\sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty}\left(-i\omega\left(l-l'\right)\upsilon_{l-l'}^{(n-n')}\rho_{l'}^{(n')}+\rho_{l'}^{(n'-2)}\frac{\partial\upsilon_{l-l'}^{(n-n')}}{\partial\tau}-V_{\mathrm{g}}\rho_{l'}^{(n'-1)}\frac{\partial\upsilon_{l-l'}^{(n-n')}}{\partial\xi}\right)\\
&+\frac{2}{9}\sum_{n',n''=1}^{\infty}\sum_{l',l''=-\infty}^{\infty}\Bigg(-i\omega\left(l-l'-l''\right)\upsilon_{l-l'-l''}^{(n-n'-n'')}\rho_{l'}^{(n')}\rho_{l''}^{(n'')}+\rho_{l'}^{(n'-2)}\rho_{l''}^{(n'')}\frac{\partial\upsilon_{l-l'-l''}^{(n-n'-n'')}}{\partial\tau}\\
&-V_{\mathrm{g}}\rho_{l'}^{(n'-1)}\rho_{l''}^{(n'')}\frac{\partial\upsilon_{l-l'-l''}^{(n-n'-n'')}}{\partial\xi}\Bigg)\Bigg]+\left(\frac{27g^2\rho_0^2}{m_G^2}c_{\mathrm{s}}^2\right)\Bigg[\sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty}\left(ik\left(l-l'\right)\upsilon_{l-l'}^{(n-n')}\upsilon_{l'}^{(n')}+\upsilon_{l'}^{(n'-1)}\frac{\partial\upsilon_{l-l'}^{(n-n')}}{\partial\xi}\right)\\
&+2\sum_{n',n''=1}^{\infty}\sum_{l',l''=-\infty}^{\infty}\left(ik\left(l-l'-l''\right)\upsilon_{l-l'-l''}^{(n-n'-n'')}\upsilon_{l'}^{(n')}\rho_{l''}^{(n'')}+\upsilon_{l'}^{(n'-1)}\rho_{l''}^{(n'')}\frac{\partial\upsilon_{l-l'-l''}^{(n-n'-n'')}}{\partial\xi}\right)\Bigg]\\
&+\left(\frac{63g^2\rho_0^2}{2m_G^4}c_{\mathrm{s}}^2\right)\Bigg\{\sum_{n',n''=1}^{\infty}\sum_{l',l''=-\infty}^{\infty}\Bigg[\left(ik\left(l-l'-l''\right)\right)^2\left(il'k\rho_{l-l'-l''}^{(n-n'-n'')}\upsilon_{l'}^{(n')}\upsilon_{l''}^{(n'')}+\rho_{l-l'-l''}^{(n-n'-n'')}\upsilon_{l''}^{(n'')}\frac{\partial\upsilon_{l'}^{(n'-1)}}{\partial\xi}\right)\\
&+2ik\left(l-l'-l''\right)\left(il'k\upsilon_{l'}^{(n'-1)}\upsilon_{l''}^{(n'')}\frac{\partial\rho_{l-l'-l''}^{(n-n'-n'')}}{\partial\xi}+\upsilon_{l''}^{(n'')}\frac{\partial\rho_{l-l'-l''}^{(n-n'-n'')}}{\partial\xi}\frac{\partial\upsilon_{l'}^{(n'-2)}}{\partial\xi}+il'k\upsilon_{l'}^{(n'-2)}\upsilon_{l''}^{(n'')}\frac{\partial^2\rho_{l-l'-l''}^{(n-n'-n'')}}{\partial\xi^2}\right)\Bigg]\Bigg\}\\
&+\left(\frac{9g^2\rho_0^2}{2m_G^6}c_{\mathrm{s}}^2\right)\Bigg\{\sum_{n',n''=1}^{\infty}\sum_{l',l''=-\infty}^{\infty}\Bigg[\left(ik\left(l-l'-l''\right)\right)^4\left(il'k\rho_{l-l'-l''}^{(n-n'-n'')}\upsilon_{l'}^{(n')}\upsilon_{l''}^{(n'')}+\rho_{l-l'-l''}^{(n-n'-n'')}\upsilon_{l''}^{(n'')}\frac{\partial\upsilon_{l'}^{(n'-1)}}{\partial\xi}\right)
\end{aligned}
$$

$$
+4\left(ik\left(l-l'-l''\right)\right)^3\left(il'k\upsilon_{l'}^{(n'-1)}\upsilon_{l''}^{(n'')}\frac{\partial\rho_{l-l'-l''}^{(n-n'-n'')}}{\partial\xi}+\upsilon_{l''}^{(n'')}\frac{\partial\rho_{l-l'-l''}^{(n-n'-n'')}}{\partial\xi}\frac{\partial\upsilon_{l'}^{(n'-2)}}{\partial\xi}\right.
$$

$$
\left.\left.\left.+6(ik\left(l-l'-l''\right))^2 il'k\upsilon_{l'}^{(n'-2)}\upsilon_{l''}^{(n'')}\frac{\partial^2\rho_{l-l'-l''}^{(n-n'-n'')}}{\partial\xi^2}\right)\right]\right\}+3\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}c_{\mathrm{s}}^2\left[\sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty}\left(ik\left(l-l'\right)\upsilon_{l-l'}^{(n-n')}\upsilon_{l'}^{(n')}+\upsilon_{l'}^{(n'-1)}\frac{\partial\upsilon_{l-l'}^{(n-n')}}{\partial\xi}\right)\right.
$$

$$
\left.+\frac{4}{3}\sum_{n',n''=1}^{\infty}\sum_{l',l''=-\infty}^{\infty}\left(ik\left(l-l'-l''\right)\upsilon_{l-l'-l''}^{(n-n'-n'')}\upsilon_{l'}^{(n')}\rho_{l''}^{(n'')}+\upsilon_{l'}^{(n'-1)}\rho_{l''}^{(n'')}\frac{\partial\upsilon_{l-l'-l''}^{(n-n'-n'')}}{\partial\xi}\right)\right]
$$

$$
=\left(\frac{27g^2\rho_0^2}{m_G^2}c_{\mathrm{s}}^2\right)\left[\sum_{n',n''=1}^{\infty}\sum_{l',l''=-\infty}^{\infty}\left(ik\left(l-l'-l''\right)\rho_{l-l'-l''}^{(n-n'-n'')}\upsilon_{l'}^{(n')}\upsilon_{l''}^{(n'')}+\upsilon_{l'}^{(n'-1)}\upsilon_{l''}^{(n'')}\frac{\partial\rho_{l-l'-l''}^{(n-n'-n'')}}{\partial\xi}\right)\right]
$$

$$
+\left(\frac{18g^2\rho_0^2}{m_G^4}c_{\mathrm{s}}^2\right)\left[\sum_{n',n''=1}^{\infty}\sum_{l',l''=-\infty}^{\infty}\left(\left(ik\left(l-l'-l''\right)\right)^3\rho_{l-l'-l''}^{(n-n'-n'')}\upsilon_{l'}^{(n')}\upsilon_{l''}^{(n'')}+3\left(ik\left(l-l'-l''\right)\right)^2\upsilon_{l'}^{(n'-1)}\upsilon_{l''}^{(n'')}\frac{\partial\rho_{l-l'-l''}^{(n-n'-n'')}}{\partial\xi}\right.\right.
$$

$$
\left.\left.+3ik\left(l-l'-l''\right)\upsilon_{l'}^{(n'-2)}\upsilon_{l''}^{(n'')}\frac{\partial^2\rho_{l-l'-l''}^{(n-n'-n'')}}{\partial\xi^2}\right)\right]-\left(\frac{9g^2\rho_0^2}{m_G^6}c_{\mathrm{s}}^2\right)\left[\sum_{n',n''=1}^{\infty}\sum_{l',l''=-\infty}^{\infty}\left(\left(ik\left(l-l'-l''\right)\right)^5\rho_{l-l'-l''}^{(n-n'-n'')}\upsilon_{l'}^{(n')}\upsilon_{l''}^{(n'')}\right.\right.
$$

$$
\left.\left.+5\left(ik\left(l-l'-l''\right)\right)^4\upsilon_{l'}^{(n'-1)}\upsilon_{l''}^{(n'')}\frac{\partial\rho_{l-l'-l''}^{(n-n'-n'')}}{\partial\xi}+10ik\left(l-l'-l''\right)^3\upsilon_{l'}^{(n'-2)}\upsilon_{l''}^{(n'')}\frac{\partial^2\rho_{l-l'-l''}^{(n-n'-n'')}}{\partial\xi^2}\right)\right]
$$

$$
+\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}c_{\mathrm{s}}^2\left[\sum_{n',n''=1}^{\infty}\sum_{l',l''=-\infty}^{\infty}\left(ik\left(l-l'-l''\right)\rho_{l-l'-l''}^{(n-n'-n'')}\upsilon_{l'}^{(n')}\upsilon_{l''}^{(n'')}+\upsilon_{l'}^{(n'-1)}\upsilon_{l''}^{(n'')}\frac{\partial\rho_{l-l'-l''}^{(n-n'-n'')}}{\partial\xi}\right)\right]
$$

$$
-\left(\frac{27g^2\rho_0^2}{2m_G^2}\right)\left[2ilk\rho_l^{(n)}+2\frac{\partial\rho_l^{(n-1)}}{\partial\xi}+\sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty}\left(ilk\rho_{l-l'}^{(n-n')}\rho_{l'}^{(n')}+\rho_{l'}^{(n'-1)}\frac{\partial\rho_{l-l'}^{(n-n')}}{\partial\xi}+\rho_{l-l'}^{(n-n')}\frac{\partial\rho_{l'}^{(n'-1)}}{\partial\xi}\right)\right]
$$

$$
\begin{aligned}
&-\left(\frac{9g^2\rho_0^2}{m_G^4}\right)\left[\sum_{n',n''=1}^{\infty}\sum_{l',l''=-\infty}^{\infty}\left(\left(ik\left(l-l'\right)\right)^2 il'k\rho_{l-l'}^{(n-n')}\rho_{l'}^{(n')}+il'k\rho_{l'}^{(n'-2)}\frac{\partial^2\rho_{l-l'}^{(n-n')}}{\partial\xi^2}+\left(ik\left(l-l'\right)\right)^2\rho_{l-l'}^{(n-n')}\frac{\partial\rho_{l'}^{(n'-1)}}{\partial\xi}\right.\right.\\
&\left.\left.+2ik\left(l-l'\right)il'k\rho_{l'}^{(n'-1)}\frac{\partial\rho_{l-l'}^{(n-n')}}{\partial\xi}+2ik\left(l-l'\right)\frac{\partial\rho_{l-l'}^{(n-n')}}{\partial\xi}\frac{\partial\rho_{l'}^{(n'-2)}}{\partial\xi}\right)\right]-\left(\frac{18g^2\rho_0^2}{m_G^4}\right)\left[\left(ilk\right)^3\rho_l^{(n)}+3\left(ilk\right)^2\frac{\partial\rho_l^{(n-1)}}{\partial\xi}+3ilk\frac{\partial^2\rho_l^{(n-1)}}{\partial\xi^2}\right.\\
&\left.+\sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty}\left(\left(ik\left(l-l'\right)\right)^3\rho_{l-l'}^{(n-n')}\rho_{l'}^{(n')}+3\left(ik\left(l-l'\right)\right)^2\rho_{l'}^{(n'-1)}\frac{\partial\rho_{l-l'}^{(n-n')}}{\partial\xi}+3ik\left(l-l'\right)\rho_{l'}^{(n'-2)}\frac{\partial^2\rho_{l-l'}^{(n-n')}}{\partial\xi^2}\right)\right]\\
&+\left(\frac{18g^2\rho_0^2}{m_G^6}\right)\left[\sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty}\left(ik\left(l-l'\right)\left(il'k\right)^4\rho_{l-l'}^{(n-n')}\rho_{l'}^{(n')}+4(il'k)^3\left(ik\left(l-l'\right)\rho_{l-l'}^{(n-n')}\frac{\partial\rho_{l'}^{(n'-1)}}{\partial\xi}\right.\right.\right.\\
&\left.\left.\left.+\frac{\partial\rho_{l-l'}^{(n-n')}}{\partial\xi}\frac{\partial\rho_{l'}^{(n'-2)}}{\partial\xi}\right)+6ik\left(l-l'\right)\left(il'k\right)^2\rho_{l-l'}^{(n-n')}\frac{\partial^2\rho_{l'}^{(n'-2)}}{\partial^2\xi}+\left(il'k\right)^4\rho_{l'}^{(n'-1)}\frac{\partial\rho_{l-l'}^{(n-n')}}{\partial\xi}\right)\right]\\
&+\left(\frac{9g^2\rho_0^2}{m_G^6}\right)\left[(ilk)^5\rho_l^{(n)}+5(ilk)^4\frac{\partial\rho_l^{(n-1)}}{\partial\xi}+10(ilk)^3\frac{\partial^2\rho_l^{(n-1)}}{\partial^2\xi}+\sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty}\left(\left(ik\left(l-l'\right)\right)^5\rho_{l-l'}^{(n-n')}\rho_{l'}^{(n')}\right.\right.\\
&\left.\left.+5\left(ik\left(l-l'\right)\right)^4\rho_{l'}^{(n'-1)}\frac{\partial\rho_{l-l'}^{(n-n')}}{\partial\xi}+10\left(ik\left(l-l'\right)\right)^3\rho_{l'}^{(n'-2)}\frac{\partial^2\rho_{l-l'}^{(n-n')}}{\partial^2\xi}\right)\right]\\
&+\left(\frac{18g^2\rho_0^2}{m_G^8}\right)\left[\sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty}\left(\left(ik\left(l-l'\right)\right)^3\left(il'k\right)^4\rho_{l-l'}^{(n-n')}\rho_{l'}^{(n')}+4\left(ik\left(l-l'\right)\right)^3\left(il'k\right)^3\rho_{l-l'}^{(n-n')}\frac{\partial\rho_{l'}^{(n'-1)}}{\partial\xi}\right.\right.\\
&+6\left(ik\left(l-l'\right)\right)^3\left(il'k\right)^2\rho_{l-l'}^{(n-n')}\frac{\partial^2\rho_{l'}^{(n'-2)}}{\partial\xi^2}+3\left(ik\left(l-l'\right)\right)^2\left(il'k\right)^4\rho_{l'}^{(n'-1)}\frac{\partial\rho_{l-l'}^{(n-n')}}{\partial\xi}\\
&\left.\left.+12\left(ik\left(l-l'\right)\right)^2\left(il'k\right)^3\frac{\partial\rho_{l-l'}^{(n-n')}}{\partial\xi}\frac{\partial\rho_{l'}^{(n'-2)}}{\partial\xi}+3ik\left(l-l'\right)\left(il'k\right)^4\rho_{l'}^{(n'-2)}\frac{\partial^2\rho_{l-l'}^{(n-n')}}{\partial^2\xi}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{9g^2\rho_0^2}{m_G^8}\right)\left[\sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty}\left(\left(ik\left(l-l'\right)\right)^2\left(il'k\right)^5\rho_{l-l'}^{(n-n')}\rho_{l'}^{(n')}+5\left(ik\left(l-l'\right)\right)^2\left(il'k\right)^4\rho_{l-l'}^{(n-n')}\frac{\partial\rho_{l'}^{(n'-1)}}{\partial\xi}\right.\right.\\
&+10ik\left(l-l'\right)\left(il'k\right)^4\frac{\partial\rho_{l-l'}^{(n-n')}}{\partial\xi}\frac{\partial\rho_{l'}^{(n'-2)}}{\partial\xi}+10\left(ik\left(l-l'\right)\right)^2\left(il'k\right)^3\rho_{l-l'}^{(n-n')}\frac{\partial^2\rho_{l'}^{(n'-2)}}{\partial\xi^2}+2ik\left(l-l'\right)\left(il'k\right)^5\rho_{l'}^{(n'-1)}\frac{\partial\rho_{l-l'}^{(n-n')}}{\partial\xi}\\
&\left.\left.+\left(il'k\right)^5\rho_{l'}^{(n'-2)}\frac{\partial^2\rho_{l-l'}^{(n-n')}}{\partial^2\xi}\right)\right]-\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}\left[ilk\rho_l^{(n)}+\frac{\partial\rho_l^{(n-1)}}{\partial\xi}+\frac{1}{6}\sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty}\left(ilk\rho_{l-l'}^{(n-n')}\rho_{l'}^{(n')}+\rho_{l'}^{(n'-1)}\frac{\partial\rho_{l-l'}^{(n-n')}}{\partial\xi}+\rho_{l-l'}^{(n-n')}\frac{\partial\rho_{l'}^{(n'-1)}}{\partial\xi}\right)\right]\\
&-\left(\frac{27g^2\rho_0^2}{2m_G^2}c_{\mathrm{s}}\right)\left[2\sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty}\left(-i\omega\left(l-l'\right)\rho_{l-l'}^{(n-n')}\upsilon_{l'}^{(n')}+\upsilon_{l'}^{(n'-2)}\frac{\partial\rho_{l-l'}^{(n-n')}}{\partial\tau}-V_{\mathrm{g}}\upsilon_{l'}^{(n'-1)}\frac{\partial\rho_{l-l'}^{(n-n')}}{\partial\xi}\right)\right.\\
&+\sum_{n',n''=1}^{\infty}\sum_{l',l''=-\infty}^{\infty}\left(-i\omega\left(l-l''\right)\rho_{l-l'-l''}^{(n-n'-n'')}\rho_{l'}^{(n')}\upsilon_{l''}^{(n'')}+\rho_{l'}^{(n'-2)}\upsilon_{l''}^{(n'')}\frac{\partial\rho_{l-l'-l''}^{(n-n'-n'')}}{\partial\tau}\right.\\
&\left.\left.+\rho_{l-l'-l''}^{(n-n'-n'')}\upsilon_{l''}^{(n'')}\frac{\partial\rho_{l'}^{(n'-2)}}{\partial\tau}-V_{\mathrm{g}}\rho_{l'}^{(n'-1)}\upsilon_{l''}^{(n'')}\frac{\partial\rho_{l-l'-l''}^{(n-n'-n'')}}{\partial\xi}-V_{\mathrm{g}}\rho_{l-l'-l''}^{(n-n'-n'')}\upsilon_{l''}^{(n'')}\frac{\partial\rho_{l'}^{(n'-1)}}{\partial\xi}\right)\right]\\
&-\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}c_{\mathrm{s}}\left[\sum_{n'=1}^{\infty}\sum_{l'=-\infty}^{\infty}\left(-i\omega\left(l-l'\right)\rho_{l-l'}^{(n-n')}\upsilon_{l'}^{(n')}+\upsilon_{l'}^{(n'-2)}\frac{\partial\rho_{l-l'}^{(n-n')}}{\partial\tau}-V_{\mathrm{g}}\upsilon_{l'}^{(n'-1)}\frac{\partial\rho_{l-l'}^{(n-n')}}{\partial\xi}\right)\right.\\
&+\frac{1}{6}\sum_{n',n''=1}^{\infty}\sum_{l',l''=-\infty}^{\infty}\left(-i\omega\left(l-l''\right)\rho_{l-l'-l''}^{(n-n'-n'')}\rho_{l'}^{(n')}\upsilon_{l''}^{(n'')}+\rho_{l'}^{(n'-2)}\upsilon_{l''}^{(n'')}\frac{\partial\rho_{l-l'-l''}^{(n-n'-n'')}}{\partial\tau}\right.\\
&\left.\left.+\rho_{l-l'-l''}^{(n-n'-n'')}\upsilon_{l''}^{(n'')}\frac{\partial\rho_{l'}^{(n'-2)}}{\partial\tau}-V_{\mathrm{g}}\rho_{l'}^{(n'-1)}\upsilon_{l''}^{(n'')}\frac{\partial\rho_{l-l'-l''}^{(n-n'-n'')}}{\partial\xi}-V_{\mathrm{g}}\rho_{l-l'-l''}^{(n-n'-n'')}\upsilon_{l''}^{(n'')}\frac{\partial\rho_{l'}^{(n'-1)}}{\partial\xi}\right)\right]\,. \qquad (11)
\end{aligned}
$$

and

$$
\begin{aligned}
&-\frac{i}{c_{\mathrm{s}}} l\omega \rho_l^{(n)} + \frac{1}{c_{\mathrm{s}}} \frac{\partial \rho_l^{(n-2)}}{\partial \tau} - \frac{V_{\mathrm{g}}}{c_{\mathrm{s}}} \frac{\partial \rho_l^{(n-1)}}{\partial \xi} + ilk\upsilon_l^{(n)} + \frac{\partial \upsilon_l^{(n-1)}}{\partial \xi} \\
&+\sum_{n'=1}^{\infty} \sum_{l'=-\infty}^{\infty} \Bigg[ik\left(l-l'\right)\left(\upsilon_{l-l'}^{(n-n')} \rho_{l'}^{(n')} + \rho_{l-l'}^{(n-n')} \upsilon_{l'}^{(n')}\right) \\
&+\rho_{l'}^{(n'-1)} \frac{\partial \upsilon_{l-l'}^{(n-n')}}{\partial \xi} + \upsilon_{l'}^{(n'-1)} \frac{\partial \rho_{l-l'}^{(n-n')}}{\partial \xi} \\
&+c_{\mathrm{s}} \left(-i\omega\left(l-l'\right) \upsilon_{l-l'}^{(n-n')} \upsilon_{l'}^{(n')} + \upsilon_{l'}^{(n'-2)} \frac{\partial \upsilon_{l-l'}^{(n-n')}}{\partial \tau} - V_{\mathrm{g}} \upsilon_{l'}^{(n'-1)} \frac{\partial \upsilon_{l-l'}^{(n-n')}}{\partial \xi} \right) \Bigg] \\
&-\sum_{n',n''=1}^{\infty} \sum_{l',l''=-\infty}^{\infty} c_{\mathrm{s}} \Bigg[i\omega\left(l-l'-l''\right)\left(\upsilon_{l-l'-l''}^{(n-n'-n'')} \rho_{l'}^{(n')} \upsilon_{l''}^{(n'')} - \rho_{l-l'-l''}^{(n-n'-n'')} \upsilon_{l'}^{(n')} \upsilon_{l''}^{(n'')}\right) \\
&-\rho_{l'}^{(n'-2)} \upsilon_{l''}^{(n'')} \frac{\partial \upsilon_{l-l'-l''}^{(n-n'-n'')}}{\partial \tau} + V_{\mathrm{g}} \rho_{l'}^{(n'-1)} \upsilon_{l''}^{(n'')} \frac{\partial \upsilon_{l-l'-l''}^{(n-n'-n'')}}{\partial \xi} \\
&+\upsilon_{l'}^{(n'-2)} \upsilon_{l''}^{(n'')} \frac{\partial \rho_{l-l'-l''}^{(n-n'-n'')}}{\partial \tau} - V_{\mathrm{g}} \upsilon_{l'}^{(n'-1)} \upsilon_{l''}^{(n'')} \frac{\partial \rho_{l-l'-l''}^{(n-n'-n'')}}{\partial \xi} \Bigg] = 0 \,. \qquad (12)
\end{aligned}
$$

For the first-order ($n = 1$) equation with ($l = 1$), we can obtain the first-order quantities in terms of $\rho_1^{(1)}$ as

$$
\begin{aligned}
&-c_{\mathrm{s}}\omega \left[\left(\frac{27g^2\rho_0^2}{m_G^2} \right) + 3\pi^{\frac{2}{3}} \rho_0^{\frac{4}{3}} \right] \upsilon_1^{(1)} + \left[\left(\frac{27g^2\rho_0^2}{m_G^2} \right) k - \left(\frac{18g^2\rho_0^2}{m_G^4} \right) k^3 \right. \\
&\left. - \left(\frac{9g^2\rho_0^2}{m_G^6} \right) k^5 + \pi^{\frac{2}{3}} \rho_0^{\frac{4}{3}} k \right] \rho_1^{(1)} = 0 \,, \qquad (13)
\end{aligned}
$$

$$
-\frac{i\omega}{c_{\mathrm{s}}} \rho_1^{(1)} + ik\upsilon_1^{(1)} = 0 \,. \qquad (14)
$$

The solution for the first harmonics is

$$
\upsilon_1^{(1)} = \frac{\omega}{kc_{\mathrm{s}}} \rho_1^{(1)} \,. \qquad (15)
$$

Thus, we obtain the following dispersion relation:

$$
\frac{\omega^2}{k^2} = c_{\mathrm{s}}^2 \left[1 - \frac{k^2}{A} \left(\frac{18g^2\rho_0^2}{m_G^4} + \frac{9g^2\rho_0^2}{m_G^6} k^2 \right) \right] , \qquad (16)
$$

with

$$c_{\rm s}^2 = \frac{\left(\frac{27g^2\rho_0^2}{m_G^2}\right) + \pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}}{\left(\frac{27g^2\rho_0^2}{m_G^2}\right) + 3\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}} \tag{17}$$

and

$$A = \left(\frac{27g^2\rho_0^2}{m_G^2}\right)c_{\rm s}^2 + 3\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}c_{\rm s}^2 = \left(\frac{27g^2\rho_0^2}{m_G^2}\right) + \pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}\,. \tag{18}$$

For the second-order reduced equation ($n = 2$) with ($l = 1$), the following equations are obtained:

$$\begin{aligned}
&-i\omega c_{\rm s}\left(\frac{27g^2\rho_0^2}{m_G^2} + 3\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}\right)\upsilon_1^{(2)} - V_{\rm g}c_{\rm s}\left(\frac{27g^2\rho_0^2}{m_G^2} + 3\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}\right)\frac{\partial\upsilon_1^{(1)}}{\partial\xi} \\
&+ik\left(\frac{27g^2\rho_0^2}{m_G^2} - \frac{18g^2\rho_0^2}{m_G^4}k^2 - \frac{9g^2\rho_0^2}{m_G^6}k^4 + \pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}\right)\rho_1^{(2)} \\
&+\left(\frac{27g^2\rho_0^2}{m_G^2} - \frac{54g^2\rho_0^2}{m_G^4}k^2 - \frac{45g^2\rho_0^2}{m_G^6}k^4 + \pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}\right)\frac{\partial\rho_1^{(1)}}{\partial\xi} = 0\,,
\end{aligned} \tag{19}$$

$$-\frac{i}{c_{\rm s}}\omega\rho_1^{(2)} - \frac{V_{\rm g}}{c_{\rm s}}\frac{\partial\rho_1^{(1)}}{\partial\xi} + ik\upsilon_1^{(2)} + \frac{\partial\upsilon_1^{(1)}}{\partial\xi} = 0\,, \tag{20}$$

then we obtain the following group velocity:

$$V_{\rm g} = \frac{k}{\omega}c_{\rm s}^2\left[1 - \frac{k^2}{A}\left(\frac{36g^2\rho_0^2}{m_G^4} + \frac{27g^2\rho_0^2}{m_G^6}k^2\right)\right]\,. \tag{21}$$

In order to discuss the issue of causality, we have to study the group velocity which characterize the propagation speed of the fluid [21, 22]. The causality of the theory is determined by the behavior of the real parts of the frequencies. For the small k, the group velocity is given by $V_{\rm g} = \frac{\partial\,{\rm Re}\,\omega}{\partial k} \simeq c_{\rm s}$. This is nothing but the usual sound velocity, so the perturbations studied here seem to be consistent with causality.

From the above equations, we conclude that there is no causality violation since there is no divergence in the group velocity and it converges very quickly. For large k, we have $V_{\rm g} = \frac{\partial\,{\rm Re}\,\omega}{\partial k} \to 0$.

For $(n = 2)$ and $(l = 2)$, we obtain the following set of equations:

$$\begin{aligned}
&-\omega c_{\rm s}\left[\left(\frac{54g^2\rho_0^2}{m_G^2}+6\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}\right)\upsilon_2^{(2)}+\left(\frac{81g^2\rho_0^2}{m_G^2}+5\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}\right)\rho_1^{(1)}\upsilon_1^{(1)}\right]\\
&+k\left[\left(\frac{27g^2\rho_0^2}{m_G^2}c_{\rm s}^2+3\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}c_{\rm s}^2\right)\left(\upsilon_1^{(1)}\right)^2+\left(\frac{54g^2\rho_0^2}{m_G^2}-\frac{144g^2\rho_0^2}{m_G^4}k^2\right.\right.\\
&-\left.\frac{288g^2\rho_0^2}{m_G^6}k^4+2\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}\right)\rho_2^{(2)}+\left(\frac{27g^2\rho_0^2}{m_G^2}-\frac{27g^2\rho_0^2}{m_G^4}k^2-\frac{27g^2\rho_0^2}{m_G^6}k^4\right.\\
&+\left.\left.\frac{27g^2\rho_0^2}{m_G^8}k^6+\frac{1}{3}\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}\right)\left(\rho_1^{(1)}\right)^2\right]=0\,,
\end{aligned} \tag{22}$$

$$-\frac{2}{c_{\rm s}}\omega\rho_2^{(2)}+2k\upsilon_2^{(2)}+2k\rho_1^{(1)}\upsilon_1^{(1)}-\omega c_{\rm s}\left(\upsilon_1^{(1)}\right)^2=0\,, \tag{23}$$

from which we obtain the following expressions of the second-order quantities:

$$\begin{aligned}
\rho_2^{(2)} &= A_\varphi\left(\rho_1^{(1)}\right)^2\,,\\
\upsilon_2^{(2)} &= B_\varphi\left(\rho_1^{(1)}\right)^2\,.
\end{aligned} \tag{24}$$

For $(n, l) = (2, 0)$,

$$\begin{aligned}
&-V_{\rm g}c_{\rm s}\left(\frac{27g^2\rho_0^2}{m_G^2}+3\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}\right)\upsilon_0^{(2)}+\left(\frac{27g^2\rho_0^2}{m_G^2}+\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}\right)\rho_0^{(2)}\\
&+\left[\frac{\omega^2}{k^2}\left(\frac{27g^2\rho_0^2}{m_G^2}+3\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}\right)-\frac{\omega V_{\rm g}}{k}\left(\frac{108g^2\rho_0^2}{m_G^2}+8\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}\right)\right.\\
&\times\left.\left(\frac{27g^2\rho_0^2}{m_G^2}-\frac{45g^2\rho_0^2}{m_G^4}k^2+\frac{9g^2\rho_0^2}{m_G^6}k^4+\frac{9g^2\rho_0^2}{m_G^8}k^6+\frac{1}{3}\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}\right)\right]\left|\rho_1^{(1)}\right|^2=0\,,
\end{aligned} \tag{25}$$

$$-\frac{1}{c_{\rm s}}V_{\rm g}\rho_0^{(2)}+\upsilon_0^{(2)}+\frac{\omega}{kc_{\rm s}}\left(\frac{\omega V_{\rm g}}{k}-2\right)\left|\rho_1^{(1)}\right|^2=0, \tag{26}$$

from which we derive

$$\begin{aligned}
\rho_0^{(2)} &= C\varphi\left|\rho_1^{(1)}\right|^2\,,\\
\upsilon_0^{(2)} &= D_\varphi\left|\rho_1^{(1)}\right|^2\,.
\end{aligned} \tag{27}$$

Proceeding to the third order in ε ($n = 3$) with $l = 1$, and substituting the derived expressions from the cases of ($n = 2, l = 2$) and ($n = 2, l = 0$) into the components for ($n = 3, l = 1$) of the reduced equations, we obtain the following NLS equation:

$$i\frac{\partial\Psi}{\partial\tau} + \frac{1}{2}P\frac{\partial^2\Psi}{\partial^2\xi^2} + Q\,|\Psi|^2\,\Psi = 0\,, \tag{28}$$

where $\Psi \equiv \rho_1^{(1)}$ denotes the amplitude of the first-order density perturbation. In the above equation, the coefficients P and Q are given by

$$P = \frac{1}{\omega}\left[\frac{2\omega}{k}V_{\rm g} - V_{\rm g}^2 - \frac{\omega^2}{k^2} - \frac{c_{\rm s}^2 k^2}{A}\left(\frac{54g^2\rho_0^2}{m_G^4} + \frac{90g^2\rho_0^2}{m_G^6}k^2\right)\right] \tag{29}$$

and

$$\begin{aligned} Q \;=\; & \frac{\omega c_{\rm s}^2}{2}\left(\frac{\omega}{kc_{\rm s}}(B_\varphi + D_\varphi) - \left(\frac{27g^2\rho_0^2}{m_G^2}\right)\frac{(A_\varphi + C\varphi)}{A} + \frac{c_1}{A}\right) \\ & + \frac{kc_{\rm s}^3}{2A}\left(\frac{27g^2\rho_0^2}{m_G^2}(B_\varphi - D_\varphi) + \pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}(B_\varphi - 5D_\varphi)\right) \\ & + \frac{k^2c_{\rm s}^2}{\omega A}\left[-\left(\frac{27g^2\rho_0^2}{2m_G^2} + \frac{1}{6}\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}\right)(A_\varphi + C\varphi) + \left(\frac{9g^2\rho_0^2}{m_G^4}k^2\right)(6A_\varphi + C\varphi)\right. \\ & \left. + \left(\frac{9g^2\rho_0^2}{2m_G^6}k^4\right)(3A_\varphi + C\varphi) - \left(\frac{54g^2\rho_0^2}{m_G^8}k^6\right)A_\varphi\right]\,, \end{aligned} \tag{30}$$

with

$$\begin{aligned} c_1 &= \frac{27g^2\rho_0^2}{m_G^2} + \frac{27g^2\rho_0^2}{2m_G^4}k^2 - \frac{27g^2\rho_0^2}{2m_G^6}k^4 - 2\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}\,, \\ A_\varphi &= \frac{1}{18k^2\left(\frac{2g^2\rho_0^2}{m_G^4} + \frac{5g^2\rho_0^2}{m_G^6}k^2\right)}\left(\frac{\omega^2}{k^2}A_\mu + A_\lambda\right)\,, \\ A_\mu &= \frac{-9g^2\rho_0^2}{m_G^2} + \frac{6g^2\rho_0^2}{m_G^4}k^2 + \frac{3g^2\rho_0^2}{m_G^6}k^4 + \pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}\,, \\ A_\lambda &= \frac{9g^2\rho_0^2}{m_G^2} - \frac{9g^2\rho_0^2}{m_G^4}k^2 - \frac{9g^2\rho_0^2}{m_G^6}k^4 + \frac{9g^2\rho_0^2}{m_G^8}k^6 + \frac{1}{9}\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}\,, \end{aligned}$$

$$\begin{aligned}
B_\varphi &= \frac{\omega}{18k^3 c_\mathrm{s}\left(\frac{2g^2\rho_0^2}{m_G^4}+\frac{5g^2\rho_0^2}{m_G^6}k^2\right)}\left(\frac{\omega^2}{k^2}B_\mu + B_\lambda\right)\,,\\
B_\mu &= \frac{-9g^2\rho_0^2}{m_G^2}+\frac{24g^2\rho_0^2}{m_G^4}k^2+\frac{48g^2\rho_0^2}{m_G^6}k^4+\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}\,,\\
B_\lambda &= \frac{9g^2\rho_0^2}{m_G^2}-\frac{45g^2\rho_0^2}{m_G^4}k^2-\frac{99g^2\rho_0^2}{m_G^6}k^4+\frac{9g^2\rho_0^2}{m_G^8}k^6+\frac{1}{9}\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}\,,\\
C\varphi &= \frac{\omega}{k(V_\mathrm{g}^2-c_\mathrm{s}^2)}\left(C_\mu+\frac{c_\mathrm{s}^2}{A}C_\lambda\right)\,,\\
C_\mu &= \frac{\omega}{k}\left(2-V_\mathrm{g}^2\right)-2c_\mathrm{s}^2V_\mathrm{g}\,,\\
C_\lambda &= \frac{k}{\omega}\left(\frac{27g^2\rho_0^2}{m_G^2}-\frac{45g^2\rho_0^2}{m_G^4}k^2+\frac{9g^2\rho_0^2}{m_G^6}k^4+\frac{9g^2\rho_0^2}{m_G^8}k^6+\frac{1}{3}\pi^{\frac{2}{3}}\rho_0^{\frac{4}{3}}\right)\,,\\
D_\varphi &= \frac{\omega}{kc_\mathrm{s}(V_\mathrm{g}^2-c_\mathrm{s}^2)}\left(D_\mu+\frac{c_\mathrm{s}^2}{A}D_\lambda\right)\,,\\
D_\mu &= \frac{\omega V_\mathrm{g}}{k}\left(2-c_\mathrm{s}^2\right)+2\left(c_\mathrm{s}^2-V_\mathrm{g}^2\left(1+c_\mathrm{s}^2\right)\right)\,,\\
D_\lambda &= V_\mathrm{g}c_\mathrm{s}C_\lambda\,.
\end{aligned} \tag{31}$$

5. Numerical results

Let us now investigate the stability/instability of the modulated wave packets in a quark–gluon plasma model on the basis of the NLS equation (28) that governs the Modulation Instability (MI) of the quark–gluon plasma. Based on the linear stability analysis [23] when modulation on the wave amplitude packet takes place in a direction, which is oblique to the direction of the pump carrier wave propagation, we consider the dynamic solution of the NLSE (28). Accordingly, we separate the amplitude into two parts

$$\Psi = (\Psi_0 + \delta\Psi(\chi))\exp(-i\Delta\tau)\,, \tag{32}$$

where Ψ_0 is the constant (real) amplitude of the pump carrier wave, $\delta\Psi$ is the small amplitude perturbation. Thus, $\delta\Psi \ll \Psi_0$, $\chi = K\xi - \Omega\tau$ is the modulation phase with $K \ll k$ and $\Omega \ll \omega$, respectively, the wave number and the frequency of the modulation and Δ a nonlinear frequency shift. After linearizing Eq. (28) and using Eq. (32), we obtain the governing equation for small perturbation $\delta\Psi$

$$\Delta = -Q\left|\Psi_0\right|^2\,, \tag{33}$$

$$i\frac{\partial\delta\Psi}{\partial\tau}+P\frac{\partial^2\delta\Psi}{\partial\xi^2}+Q\left|\Psi_0\right|^2(\delta\Psi+\delta\Psi^*)=0\,, \tag{34}$$

where we have $\delta\Psi^*$ as the complex conjugate for $\delta\Psi$. Introducing the transformation $\delta\Psi = U + iV$ introducing the latter expression for $\delta\Psi$ in Eq. (34), and separating the real and imaginary parts, we obtain the following equations:

$$\begin{aligned} \frac{\partial V}{\partial \tau} &= P\frac{\partial^2 U}{\partial \xi^2} + 2Q\left|\Psi_0\right|^2 U\,, \\ \frac{\partial U}{\partial \tau} &= -P\frac{\partial^2 V}{\partial \xi^2}\,. \end{aligned} \tag{35}$$

Considering that the amplitude perturbation $\delta\Psi$ varies as $\sim \exp[i(K\xi - \Omega\tau)]$, one can obtain from the system of Eqs. (35) the following nonlinear dispersion relation given by

$$\Omega^2 = P^2K^2\left(K^2 - \frac{2Q\left|\Psi_0\right|^2}{P}\right)\,. \tag{36}$$

From the dispersion relation (36), we can note that the stability of the wave packets depends critically on the sign of the coefficient P/Q. It is clear that if the coefficient $P/Q < 0$, then we have a positive value for Ω^2 so the frequency Ω is real for any values of the wave number K, the wave packet is then modulationally stable in the presence of a small perturbation $\delta\Psi$. On the other hand, when the coefficient $P/Q > 0$, the MI would set in when Ω becomes imaginary. This happens when the modulation wave number K of an external perturbation is smaller than the critical value K_{c}, given by $K_{\mathrm{c}} = \sqrt{2Q\frac{|\Psi_0|^2}{P}}$. In this case, the perturbation grows exponentially in time. Furthermore, the maximum growth rate is given by $Q\left|\Psi_0\right|^2$ and is attained at $K = \sqrt{K_{\mathrm{c}}}/2$. Two types of stationary solutions are possible: *(i)* stable solutions called dark envelope soliton when $PQ < 0$, and *(ii)* unstable solutions called bright envelope soliton when $PQ > 0$. For unstable wave packet ($PQ > 0$), we have envelope soliton given by

$$\Psi = \sqrt{\left|\frac{2\gamma}{Q}\right|}\operatorname{sech}\left(\left|\frac{\gamma}{P}\right|\xi\right)\exp(i\gamma\tau)\,, \tag{37}$$

where γ is a real constant. For stable wave packet ($PQ < 0$), we obtain modulationally stable wave with special solution known as envelope dark soliton given by

$$\Psi = \sqrt{\left|\frac{\gamma}{Q}\right|}\tanh\left(\left|\frac{\gamma}{2P}\right|\xi\right)\exp(i\gamma\tau)\,. \tag{38}$$

It is obvious from Eqs. (37) and (38) that the width and the amplitude of the solitons vary with P and Q, respectively. The soliton width is proportional to $|P|$ and the soliton amplitude is inversely proportional to $|Q|$. As apparent from Eqs. (29) and (30) for P and Q, it is observed that these coefficients depend upon number of parameters such as equilibrium density (or reference) ρ_0, dynamical mass of the hard gluon m_G and coupling constant g. The results obtained from the numerical analysis for various parameters for the modulational instability of a quark–gluon plasma are plotted in the form of graphs. The variation of P/Q as a function of k for different values of ρ_0 is shown in Fig. 1. It is observed that the wave remains stable at small wave numbers $k < k_c$ and MI sets in when $k > k_c$. The dark solitons occur in the former case, *i.e.*, for large wavelength, while bright envelope solitons occur in the latter region. The wave number for which the instability sets in is called the critical wave number. This critical value decreases with increasing ρ_0. Next, the effect of dynamical mass of the hard gluon m_G on the modulational instability is studied in Fig. 2. The latter, in which we have depicted the ratio P/Q as a function of k for different values of m_G, displays such an influence. Although, both the dark and bright excitations are obtained for small and large k, respectively, the trend is in contrast to the earlier observations, *i.e.*, the critical value k_c increases with increasing m_G. Lastly, the influence of the coupling constant g is shown in Fig. 3. Both the stable and unstable regions are formed with varying the coupling constant. It is observed that an increase in the value of coupling constant *i.e.*, as g increases, the critical value decreases.

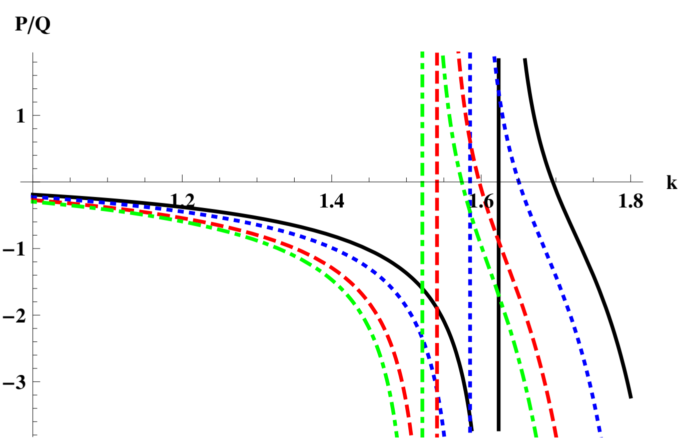

Fig. 1. Variation of the NLSE coefficients P/Q with the carrier wave number k for different values of the equilibrium (or reference) densities $\rho_0 = 1,2$ fm^{-3} (solid curve), $\rho_0 = 1,5$ fm^{-3} (dotted curve), $\rho_0 = 2$ fm^{-3} (dashed curve) and $\rho_0 = 2,3$ fm^{-3} (dot-dashed curve), with fixed values of $g = 0,35$ and $m_G = 290$ MeV.

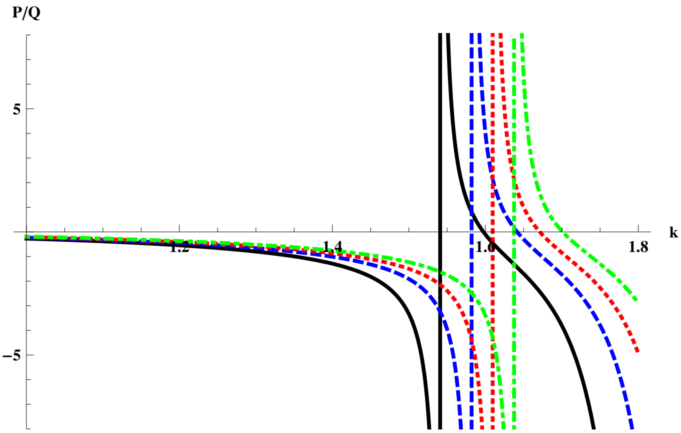

Fig. 2. Variation of the NLSE coefficients P/Q with the carrier wave number k for different values of the dynamical mass of the hard gluon $m_G = 290$ MeV (solid curve), $m_G = 296$ MeV (dotted curve), $m_G = 300$ MeV (dashed curve) and $m_G = 304$ MeV (dot-dashed curve), with fixed values of $\rho_0 = 2$ fm^{-3} and $g = 0, 35$.

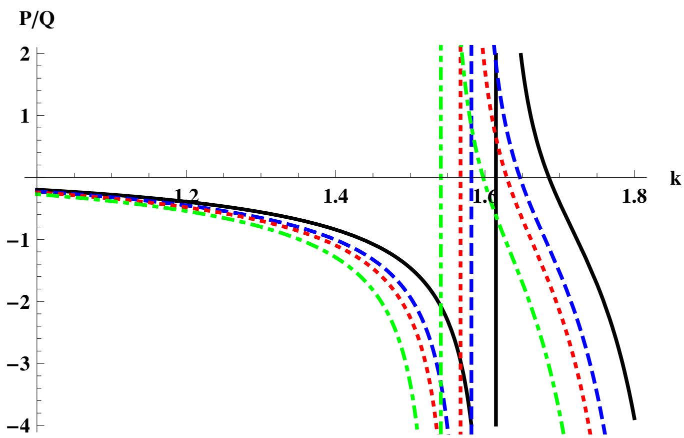

Fig. 3. Variation of the NLSE coefficients P/Q with the carrier wave number k for different values of the coupling constant $g = 0, 30$ (solid curve), $g = 0, 32$ (dotted curve), $g = 0, 33$ (dashed curve) and $g = 0, 35$ (dot-dashed curve), with fixed values of $\rho_0 = 2$ fm^{-3} and $m_G = 290$ MeV.

6. Conclusion

To conclude, we have addressed the problem of the modulational instability of a cold quark–gluon plasma. NLSE is derived making use of the standard reductive perturbation technique. It has been found that the equilibrium density, the dynamical mass, and the coupling constant modify the regions of the stability/instability of the wave. The effect of the equilibrium density (or reference) ρ_0 on the modulational instability is studied. Dark as well as bright excitations occurred in all cases. The critical wave number at which the instability sets in decreases as the equilibrium density increases. Furthermore, the effect of the dynamical mass of the hard gluon m_G on the modulational instability is studied. Although both the dark and bright excitations are obtained for small and large k, respectively, the trend is in contrast to the earlier observations, *i.e.*, the critical value $k_{\rm c}$ increases with increasing m_G. Lastly, the influence of the coupling constant g shows that both stable and unstable regions are formed with varying the coupling constant. It is observed that an increase in the value of coupling constant *i.e.*, as g increases, the critical value decreases.

The findings of this investigation may be helpful in understanding strongly interacting medium phenomena and astrophysical situations. The study of nonlinear waves in hadron physics is an interesting and fast-developing field. This study will help to interpret and understand the data from the Large Hadron Collider at CERN. We hope that our investigation may aid to understand the nonlinear structures that may occur in quark–gluon plasmas.

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REFERENCES

[1] T. Boeckel, S. Schettler, J. Schaffner-Bielich, «The cosmological QCD phase transition revisited», *Prog. Part. Nucl. Phys.* **66**, 266 (2011).

[2] A.Y. Abul-Magd, I. El-Taher, F.M. Khaliel, «Soliton formation in nuclear matter», *Phys. Rev. C* **45**, 448 (1992).

[3] D.A. Fogaça, F.S. Navarra, «Solitons in relativistic mean field models», *Phys. Lett. B* **639**, 629 (2006).

[4] D.A. Fogaça, F.S. Navarra, L.G. Ferreira Filho, «Viscosity, wave damping, and shock-wave formation in cold hadronic matter», *Phys. Rev. C* **88**, 025208 (2013).

[5] D.A. Fogaça, F.S. Navarra, «Gluon condensates in a cold quark–gluon plasma», *Phys. Lett. B* **700**, 236 (2011).

[6] D.A. Fogaça, F.S. Navarra, L.G. Ferreira Filho, «Kadomtsev–Petviashvili equation in relativistic fluid dynamics», *Commun. Nonlinear Sci. Numer. Simul.* **18**, 221 (2013); arXiv:1202.213 [math-ph].

[7] B.B. Kadomtsev, V.I. Petviashivili, «On the stability of solitary waves in weakly dispersing media», *Sov. Phys. Dokl.* **15**, 539 (1970).

[8] H. Washimi, T. Taniuti, «Propagation of Ion-Acoustic Solitary Waves of Small Amplitude», *Phys. Rev. Lett.* **17**, 996 (1966).

[9] W.M. Moslem, R. Sabry, P.K. Shukla, «Three dimensional cylindrical Kadomtsev–Petviashvili equation in a very dense electron–positron–ion plasma», *Phys. Plasmas* **17**, 032305 (2010).

[10] D.A. Fogaça, L.G. Ferreira Filho, F.S. Navarra, «Sound waves and solitons in hot and dense nuclear matter», *Nucl. Phys. A* **819**, 150 (2009).

[11] D.A. Fogaça, L.G. Ferreira Filho, F.S. Navarra, «Nonlinear waves in a quark gluon plasma», *Phys. Rev. C* **81**, 055211 (2010).

[12] D.A. Fogaça, L.G. Ferreira Filho, F.S. Navarra, «Korteveg–de Vries solitons in a cold quark–gluon plasma», *Phys. Rev. D* **84**, 054011 (2011).

[13] O. Bouzit, M. Tribeche, A.S. Bains, «Modulational instability of ion-acoustic waves in plasma with a q-nonextensive nonthermal electron velocity distribution», *Phys. Plasmas* **22**, 084506 (2015).

[14] T.S. Gills, A.S. Bains, C. Bedi, «Modulational instability of dust acoustic solitons in multicomponent plasma with kappa-distributed electrons and ions», *Phys. Plasmas* **17**, 013701 (2010).

[15] A. Chodos *et al.*, «New extended model of hadrons», *Phys. Rev. D* **9**, 3471 (1974).

[16] CBM Collaboration (T. Ablyazimov *et al.*), «Challenges in QCD matter physics — The scientific programme of the Compressed Baryonic Matter experiment at FAIR», *Eur. Phys. J. A* **53**, 60 (2017); B. Friman *et al.* (Eds.) «The CBM Physics Book Compressed Baryonic Matter in Laboratory Experiments. Lectures Notes in Physics Vol. 814», Springer, Berlin, Heidelberg 2011.

[17] `http://theor0.jinr.ru/twiki-cgi/view/NICA` and references therein; see also: K.A. Bugaev, `arXiv:0909.0731 [nucl-th]` and references therein.

[18] S. Weinberg, «Gravitation and Cosmology», *Wiley*, New York 1972.

[19] L. Landau, E. Lifchitz, «Fluid Mechanics», *Pergamon Press*, Oxford 1987.

[20] B. Franzon, D.A. Fogaça, F.S. Navarra, J.E. Horvath, «Self-bound interacting QCD matter in compact stars», *Phys. Rev. D* **86**, 065031 (2012).

[21] H.M. Nussenzveig, «Causality and Dispersion Relations», *Academic Press*, 1972.

[22] D.A. Fogaça, H. Marrochio, F.S. Navarra, J. Noronha, «Nonlinear waves in second order conformal hydrodynamics», *Nucl. Phys. A* **934**, 18 (2015), `arXiv:1402.5548 [nucl-th]`.

[23] M.R. Amin, G.E. Morfill, P.K. Shukla, «Modulational instability of dust-acoustic and dust-ion-acoustic waves», *Phys. Rev. E* **58**, 6517 (1998).