EFFECTS OF NONLINEAR SYNCHRO-BETATRON COUPLING IN LARGE ELECTRON STORAGE RINGS

Y. Alexahin
Joint Institute for Nuclear Research, Dubna 141980 Russia

Abstract
Effects of synchro-betatron coupling (SBC) specific to large electron storage rings with low emittance lattices and high synchrotron tunes are considered. An adequate mathematical apparatus based on the Lie transform theory is briefly outlined. With HERA-e taken as an example the nonlinear SBC is shown to limit the off-momentum dynamic aperture and produce anomalous vertical emittance.

1 INTRODUCTION

The possible impact of the synchro-betatron coupling (SBC) on the performance of electron/positron storage rings has long been acknowledged [1]. There are common features of large e^+e^- rings used as circular colliders, synchrotron radiation sources and damping rings of linear colliders - low value of the natural emittance, high synchrotron tune, large amount of energy stored in longitudinal oscillations - which make influence of the SBC especially important.

At large amplitudes the SBC can limit particle stability thus reducing the lifetime; at small amplitudes it can transfer energy from longitudinal to transverse oscillations increasing the transverse emittance.

These effects can be studied by tracking with MAD [2], however, an analytical tool is desirable which would enable one to analyze mechanisms of the SBC in particular cases.

2 SOURCES OF NONLINEAR COUPLING

Introducing 6D phase space column vector of coordinates and momenta

$$\underline{z} = (x,\ p_x,\ y,\ p_y,\ \sigma,\ \delta_p)^{\mathrm{T}} \tag{1}$$

and $\theta=s/R$ we have the following equation of motion

$$\dot{\underline{z}} \equiv \frac{d}{d\theta}\underline{z} = \underline{F} = S\cdot\frac{\partial}{\partial \underline{z}}H + \underline{F}^{(\mathrm{rad})},$$
$$S = S_2 \oplus S_2 \oplus S_2,\quad S_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2}$$

which is governed by the radiation reaction force $\underline{F}^{(\mathrm{rad})}$ and the Hamiltonian

$$H = -hRx(1+\delta_p) + (1+hx)\left[\frac{\mathbf{p}_\perp^2 R}{2(1+\delta_p)} - \frac{eR}{cp_0}A_s\right], \tag{3}$$

where A_s takes into account stationary magnetic and RF fields.

The primary source of nonlinear SBC is the kinetic energy term in the square brackets; it contains resonance driving terms $2Q_\alpha \pm mQ_s$, $(Q_\alpha - Q_\alpha) \pm mQ_s$, $\alpha = x, y$. In the quasi-static limit ($|Q_s| \rightarrow 0$) the effect of the former terms is the *off-momentum beta-beating*, whereas the latter produce *chromaticity* of the betatron tunes. When $|Q_s|$ is comparable with fractional betatron tunes these concepts lose their validity, the synchrotron oscillations should be treated on an equal footing.

Due to finite dispersion, when $p_\alpha = p_\alpha^{(\beta)} + D'_\alpha \delta_p$, the kinetic energy term can drive sidebands $Q_\alpha \pm mQ_s$ of the integer betatron resonances as well.

The adverse effects of the kinetic energy nonlinearity are compensated with the help of sextupole families [3], which in the presence of both vertical and horizontal dispersion introduce by themselves 3DoF coupling, $Q_x \pm Q_y \pm Q_s$.

A classical Hamiltonian mechanism driving linear synchro-betatron resonances is the RF field in the presence of finite dispersion [1]. Due to a shift in stable phase angle needed to compensate for the synchrotron radiation losses it can drive odd-order resonances.

The (mean part of) radiation reaction force also contributes to the SBC due to its dependence on the particle transverse position in quadrupoles and/or nonlinear wigglers. This effect called *radiative beta-synchrotron coupling* [4] can even limit stable transverse amplitude at high energies.

With increasing order the resonance strength provided directly by the mentioned mechanisms rapidly falls off, however their cross-talk described by high-order perturbation theory may lead to a strong excitation. To analyze such effects in 3DoF we follow the mathematical approach outlined in Ref. [5].

3 LIE-TRANSFORM METHOD

Let us introduce into the vector field $\underline{F}$ a parameter ε so that at $\varepsilon = 0$ it is integrable, e.g. linear, and try to find a transformation to new dynamical variables

$$\underline{u} = \hat{T}(\underline{z},\theta;\varepsilon)\underline{z} \equiv \underline{Z}(\underline{z},\theta;\varepsilon),\quad \hat{T}(\underline{z},\theta;0) = I, \tag{4}$$

in which the equation of motion

$$\dot{\underline{u}} = \underline{G}(\underline{u},\theta;\varepsilon),\quad \underline{G}(\underline{u},\theta;0) = \underline{F}(\underline{u},\theta;0), \tag{5}$$

is easier to analyze.

Defining the transformation by the equation

$$\frac{\partial}{\partial\varepsilon}\underline{Z}(\underline{z},\theta;\varepsilon) = \underline{V}\big(\underline{Z}(\underline{z},\theta;\varepsilon),\theta;\varepsilon\big), \tag{6}$$

$\underline{V}$ being called a *Lie-dragging field*, we obtain the equation for the inverse operator

$$\frac{\partial}{\partial\varepsilon}\hat{T}^{-1} = -\hat{L}_{\underline{V}}\hat{T}^{-1},\quad \hat{L}_{\underline{V}} \equiv \underline{V}\cdot\frac{\partial}{\partial \underline{u}} \tag{7}$$

which has the formal solution

$$\hat{T}^{-1} = \hat{\Gamma}_\varepsilon \exp[-\int_0^\varepsilon \hat{L}_{\underline{V}} d\varepsilon'], \tag{8}$$

where operator $\hat{\Gamma}_\varepsilon$ orders compositions of the Lie-derivative operators so that their ε-arguments increase from the right to the left. Arbitrary vector fields are transformed with the help of the matrix operator

$$\hat{\mathcal{F}}^{-1} = \hat{T}^{-1}\frac{\partial \underline{Z}}{\partial \underline{z}}, \tag{9}$$

which satisfies the equation

$$\frac{\partial}{\partial \varepsilon}\hat{\mathcal{F}}^{-1} = -\hat{\mathcal{L}}_{\underline{V}}\hat{\mathcal{F}}^{-1}, \quad \hat{\mathcal{L}}_{\underline{V}}\underline{U} \equiv \hat{L}_{\underline{V}}\underline{U} - \hat{L}_{\underline{U}}\underline{V} = -\hat{\mathcal{L}}_{\underline{U}}\underline{V}, \tag{10}$$

and can be presented in a form similar to (8).

The Lie-dragging field $\underline{V}$ is related to the original and new vector fields by the following basic equation

$$\frac{\partial}{\partial \theta}\underline{V} + \hat{\mathcal{L}}_{\underline{G}}\underline{V} = \frac{\partial}{\partial \varepsilon}\underline{G} - \hat{\mathcal{F}}^{-1}\frac{\partial}{\partial \varepsilon}\underline{F} \tag{11}$$

which in principle permits to find $\underline{V}$ for a given $\underline{G}$ or *vice versa.*

Having constructed its solution we may add the fluctuating part to the radiation reaction force, transform it with the help of operator (9) and solve the Fokker-Planck equation in the new variables to find the phase space distribution of radiating particles.

4 NEAR-HAMILTONIAN SYSTEM

If the synchrotron radiation is weak we may exclude it from the normalization process and add afterwards.

In this case, putting in (2) $\underline{F}^{(\mathrm{rad})} = 0$ and introducing a new Hamiltonian, K, and a scalar generating function, w, via the relations

$$\underline{G} = S\cdot\frac{\partial}{\partial \underline{u}}K, \quad \underline{V} = S\cdot\frac{\partial}{\partial \underline{u}}w, \tag{12}$$

we can reduce general equation (11) to Dewar's equation [6] which in turn by expanding everything in power series

$$w = \sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}w_{n+1}, \quad K = \sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}K_n, \quad H = \sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}H_{0n}, \ldots \tag{13}$$

can be reduced to Deprit's equations (see e.g. [5])

$$\hat{D}_0 w_n \equiv \frac{\partial}{\partial \theta}w_n + [w_n, K_0] = K_n - H_{0n} - \Sigma_n, \tag{14}$$

where the Poisson brackets were introduced and

$$\Sigma_n = \sum_{m=1}^{n-1}\left\{\binom{n-1}{m-1}[w_m, K_{n-m}] + \binom{n-1}{m}H_{m,n-m}\right\}, \quad H_{m,n} = \sum_{j=1}^{m}\binom{m-1}{j-1}[w_j, H_{m-j,n}]$$

Deprit's equations permit to build w and K in the same recursive process so as to assure the existence of the (formal) solution: K_n absorbs terms from the r.h.s. of eq.(14) which belongs to the null space of operator $\hat{D}_0$ and can not be relegated to w_n, these being the detuning and the close resonance terms.

As a preliminary step it is convenient to introduce the *linear normal form variables* which are essentially the coefficients of expansion in the eigenvectors (assumed here to be 2π-periodic) of the transfer matrix:

$$\underline{z} = \sum_{\alpha} a_\alpha\, \underline{v}_\alpha(\theta) + c.c., \quad a_\alpha = -i\underline{v}_\alpha^{*}(\theta)\cdot S\cdot\underline{z}, \tag{15}$$

where index α numbers the normal modes, which we still denote as x, y, s for simplicity.

Equations of motion for a_α and its complex conjugate can be cast into the Hamiltonian form

$$\dot{a}_\alpha = \frac{\partial}{\partial a_\alpha^{*}}\tilde{H}, \quad \dot{a}_\alpha^{*} = -\frac{\partial}{\partial a_\alpha}\tilde{H} \tag{16}$$

with an imaginary Hamiltonian

$$\tilde{H} = i\sum_{\alpha} Q_\alpha a_\alpha a_\alpha^{*} + iH^{(\mathrm{h.o.t.})} \tag{17}$$

where $H^{(\mathrm{h.o.t.})}$ stands for the nonlinear part of the original Hamiltonian, $Q_s < 0$ above transition.

Deprit's equations can be easily solved in terms of the eigenfunctions of operator $\hat{D}_0$

$$\Psi_{\mathbf{l},\mathbf{m}} = \prod_{\alpha} a_\alpha^{l_\alpha} a_\alpha^{*m_\alpha}, \quad \Psi_{\mathbf{l},\mathbf{m}}^{*} = \Psi_{\mathbf{m},\mathbf{l}}, \quad l_\alpha, m_\alpha \geq 0, \tag{18}$$

which will be referred to by a 6-tuple of their indices, (l_x, m_x, l_y, m_y, l_s, m_s).

Table 1

$l_x\, m_x\, l_y\, m_y\, l_s\, m_s$	$w_{\mathbf{l},\mathbf{m}}(0)$ [m^{-1}]	$\Delta\varepsilon_y$ [nm]
000102	-0.0220 - 0.1972 i	0.03
000201	0.6440 + 1.8924 i	<0.01
000210	-0.3404 - 2.2283 i	<0.01
001101	1.5418 - 24.873 i	0.10
011001	6.9705 + 1.1064 i	0.07
012000	586.37 + 113.43 i	0.33
020001	-0.7459 + 7.8636 i	-
020010	0.6088 - 8.0303 i	-

Table 1 shows some terms in the first-order generating function at $\theta = 0$ obtained for the HERA-e HE_REV2_NONINT3 lattice with the arc cell phase advances μ_x/μ_y= 90°/60° and the tunes Q_x= 58.2105, Q_y=46.1219, $|Q_s|$=0.0452 in the presence of errors [7].

5 OFF-MOMENTUM DYNAMIC APERTURE

Large values of 020001 and 020010 terms suggest strong modulation of the horizontal amplitude by energy oscillations which may limit a particle stability, both horizontal and vertical due to a large cross-detuning $\partial Q_y/\partial W_x$= -3.3·10^4 m^{-1}, W_x being the Courant-Snyder invariant.

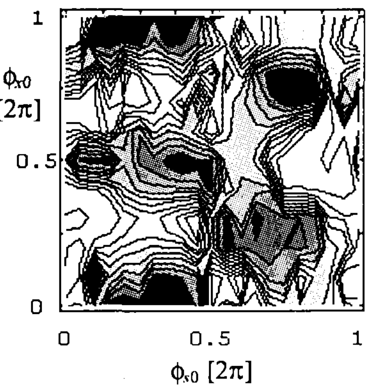

Figure 1. Maximal vertical stable amplitude vs. initial phase angles at W_x = 1 μm, δ_p = 0.5% and ϕ_{y0} = 0.

Results obtained by tracking with radiation at E_0 = 27.5 GeV are presented in Fig.1 as "swamp" plots, with the color changing from white to black as the maximal stable vertical amplitude is reducing from f_y= (W_y[μm])$^{1/2}$ > 1.5 to a vanishing one.

Along with the importance of SBC these findings show necessity to employ nonlinearly normalized variables to eliminate uncertainty arising from the dependence on the initial phase angles.

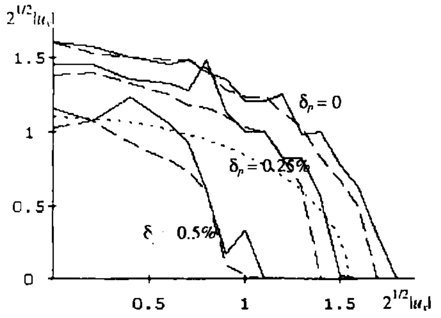

Figure 2. Dynamic aperture in the first order normal form variables with and without chromatic functions correction at different synchrotron amplitudes. The dotted line shows the 10σ ellipse for $\varepsilon_x = 24$ nm, $\varepsilon_y = \varepsilon_x/2$.

The dynamic aperture calculated in these variables gives a really available volume of the phase space. Fig.2 shows effect of the chromatic functions correction (by only some 20% achievable with the sextupole families foreseen in the considered lattice).

6 NONLINEAR VERTICAL EMITTANCE

In the case of a canonical transformation the radiation reaction force transforms according to

$$G_i^{(rad)} = (\hat{\mathcal{T}}^{-1} \underline{F}^{(rad)})_i = \sum_{j,k} F_j^{(rad)}(\hat{T}^{-1}\underline{u})\, S_{ik} \frac{\partial}{\partial u_k}(S \cdot \hat{T}^{-1}\underline{u})_j \qquad (19)$$

Due to the transformation the nonlinear coupling enters the diffusion coefficients modifying the equilibrium distribution. For example in the non-resonant case terms of the type ($h_m a_y a_s^m$ - c.c.) in the original Hamiltonian (17) contribute to the apparent emittance of the (almost) vertical mode as

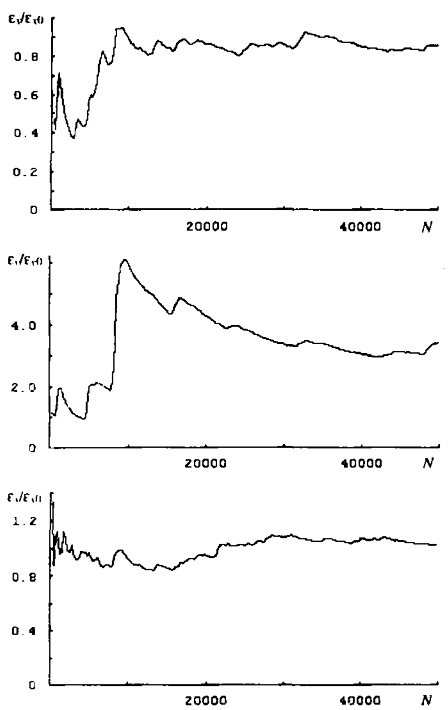

Figure 3. Averaged values of the action variables (in units of the linear emittances) vs. number of turns of averaging.

$$\varepsilon_y = \langle a_y a_y^* \rangle \approx \varepsilon_{y0} + m!\,(|w_1(0)|^2 + \frac{mJ_s}{J_y}\overline{|w_1|^2})\,\varepsilon_{s0}^m$$
$$\approx \varepsilon_{y0} + m!\,(1 + \frac{mJ_s}{J_y})\frac{\overline{|h_m|^2}}{\Delta_m^2}\varepsilon_{s0}^m \qquad (20)$$

where J_α are the damping partition numbers and $\Delta_m = Q_y - m|Q_s| - n$ is the distance from the resonance.

Analytically calculated contributions from several terms are cited in Table 1. Table 2 shows emittances obtained with the MAD EMIT routine (linear) and by tracking for 50000 turns (≈ 80 transverse damping times) with quantum fluctuations (see Fig.3).

Table 2

case	ε_x [nm]	ε_y [nm]	ε_s [μm]
EMIT, ideal lattice	25.66	0.685	8.069
EMIT, misaligned	34.93	2.478	8.023
quantum tracking	29.97	8.495	8.195

The large nonlinear vertical emittance can be attributed to the $Q_y = Q_x - 2|Q_s|$ resonance reached at $W_x/2 \approx 26$nm$\approx 0.7\varepsilon_{x0}$ due to the high cross-anharmonicity. Deprit's algorithm permits to calculate the resonance strength and find the phase space trajectories which for initial values of the action variables $I_\perp \equiv I_{x0} + I_{y0} = \varepsilon_{x0} + \varepsilon_{y0}$, $I_{s0} = \varepsilon_{s0}$ are shown in Fig.4.

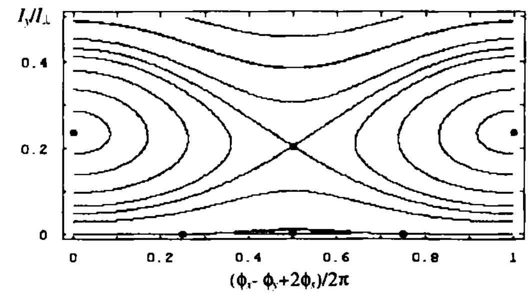

Figure 4. Phase space trajectories on $Q_y = Q_x - 2|Q_s|$ resonance.

For a particle trapped into the large resonance island $< I_y > \approx 0.24\ I_\perp = 9$ nm in a fair agreement with the tracking data. Trapping can also explain the steep rise of the mean vertical action variable seen in Fig.3b.

The largest contribution to the resonance strength comes from the cross-talk of the 011001 and 110001 terms so the main cure in the considered case is a reduction of the vertical dispersion.

References

[1] A.Piwinsky, A.Wrulich, DESY 76-07 (1976)
[2] H.Grote, F.Iselin, "The MAD program. Version 8.21. User's Reference Manual", 1997.
[3] B.W.Montague, *in* Proc CAS Advanced Acc. Physics Course, Oxford, 1985 (ed. S.Turner), CERN 87-03, p.75 (1987)
[4] J.Jowett, *in* Proc. 4th Workshop on LEP Performance, Chamonix, 1994, p.47.
[5] L.Michelotti, Intermediate Classical Dynamics with Applications to Beam Physics. J.Wiley, 1995.
[6] R.L.Dewar, J.Physics, A9, 2043 (1976).
[7] Y.Alexahin, DESY HERA 98-06 (1998)