An infinite family of relativistic magnetized finite thin disks.

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Abstract. An infinite family of relativistic finite thin disk model with magnetic field is presented. The model is obtained for solving the Einstein-Maxwell equations for static spacetimes by means of the Horský-Mitskievitch generating conjecture. The vacuum limit of these obtained solutions is the well known Morgan and Morgan solution. The obtained expressions are simply written in terms of oblate spheroidal coordinates. The mass of the disks are finite and the energy-momentum tensor agrees with all the energy conditions. The magnetic field and the circular velocity are evaluated explicitly. All the physical quantities obtained shown an acceptable behavior

1. Einstein-Maxwell Equations and Thin Disks

The vacuum Einstein-Maxwell equations, in geometrized units such that $c = 8\pi G = \mu_0 = \epsilon_0 = 1$, can be written as

$$G_{ab} \;=\; T_{ab}, \quad F^{ab}{}_{;b} = 0, \tag{1}$$

with the electromagnetic energy-momentum tensor given by

$$T_{ab} = F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F_{cd} F^{cd}, \quad F_{ab} = A_{b,a} - A_{a,b}, \tag{2}$$

where A_a is the electromagnetic four potential.

By using the distributional approach [1, 2] and by choosing the electromagnetical potential as $A_a = A(r,z)\delta_a^0$ we can write the surface energy-momentum tensor and the current density of a source like a disk in the canonical form as

$$S^{ab} \;=\; \epsilon V^a V^b + p W^a W^b, \quad j^a = j W^a. \tag{3}$$

Here ϵ and p are the energy density and the azimuthal pressure of the disk, respectively, and we have used the orthonormal tetrad $e_{(b)}^{\ a} = \{V^a, W^a, X^a, Y^a\}$, with

$$V^a = e^{-\Phi}\delta_0^a, \qquad W^a = \frac{e^{\Phi}}{r}\delta_1^a, \qquad X^a = e^{\Phi-\Lambda}\delta_2^a, \qquad Y^a = e^{\Phi-\Lambda}\delta_3^a, \tag{4}$$

and we have defined $\mu = \epsilon + p$ as the mass density on the surface of the disk. The energy-momentum tensor S^{ab} and the current density j^a can be interpreted as the superposition of two counterrotating fluids (see [3] and [4]). In order to do this we cast

$$S^{ab} \;=\; \epsilon_+ U_+^a U_+^b + \epsilon_- U_-^a U_-^b, \qquad j^a = \sigma_+ U_+^a + \sigma_- U_-^a, \tag{5}$$

where

$$\epsilon_+ = \epsilon_- = (\epsilon - p)/2, \qquad \sigma_+ = -\sigma_- = \frac{j e^{\Phi}}{2r}\sqrt{\frac{\epsilon}{p} - 1}, \tag{6a}$$

are the energy densities and charge densities of the two counterrotating fluids. The counterrotating velocity vectors are given by

$$U_\pm^a = \frac{V^a \pm U W^a}{\sqrt{1 - U^2}}, \qquad U^2 = \frac{p}{\epsilon} \leq 1, \tag{7}$$

with U the counterrotating tangential velocity. Therefore we have two counterrotating charged fluids with equal energy densities and equal but opposite charge densities.

2. The Horský-Mitskievitch conjecture and the WLP metric
The line element of a static vacuum spacetime can be written in the Weyl-Lewis-Papapetrou form

$$ds^2 = -f(r,z)^2\, e^{2\Phi_s} dt^2 + \frac{e^{-2\Phi_s} r^2}{f(r,z)^2} d\varphi^2 + f(r,z)^2 e^{2(\Lambda_s - \Phi_s)}(dr^2 + dz^2), \tag{8}$$

which admits two Killing vectors

$$\boldsymbol{\xi_t} = -f(r,z)^2 e^{2\Phi_s} \boldsymbol{dt}, \quad \boldsymbol{\xi_\varphi} = \frac{r^2 e^{-2\Phi_s}}{f(r,z)^2} \boldsymbol{d\varphi}, \tag{9}$$

with $f(r,z)$ an arbitrary function and $\Phi_s = \Phi_s(r,z)$ and $\Lambda_s = \Lambda_s(r,z)$, being (r,z,φ) the usual cylindrical coordinates. The Killing vectors $\boldsymbol{\xi}$ and the electromagnetic four-potential $\boldsymbol{A}$ satisfy the equations

$$\star \boldsymbol{d} \star \boldsymbol{d\xi} = 0, \quad \star \boldsymbol{d} \star \boldsymbol{dA} = 0. \tag{10}$$

Then, Killing vector $\boldsymbol{\xi_\varphi}$ induces the four-potential

$$\boldsymbol{A} = \frac{q r^2}{f e^{2\Phi_s}} \boldsymbol{d\varphi}. \tag{11}$$

It can be verified through standard calculations that the sourceless Einstein-Maxwell equations are fulfilled if

$$f(r,z) = 1 + c_1 f_1(r,z), \tag{12}$$

where the function f_1 must be a solution of the differential equation

$$G^{(a)}{}_{(a)} = -R = 0, \tag{13}$$

arising from the fact that for electrovacuum spacetimes the Einstein tensor, $G_{(a)(b)}$, is traceless. Naturally, if $c_1 = 0$ we arrive to a solution of the vacuum Einstein equation.

As in the reference [5] we can choose $c_1 = q^2$ and $f_1 = r^2 e^{-2\Phi_s}$, then we have the four-potential

$$\mathbf{A} = \frac{qr^2}{q^2 r^2 + e^{2\Phi_s}} d\boldsymbol{\varphi}. \tag{14}$$

Using equations (3) and (4) we found the surface energy density and the azimuthal pressure

$$\epsilon = \frac{4E^2(r)e^{3\Phi_s - \Lambda_s}(1 - r\Phi_{s,r})\Phi_{s,z}}{(e^{2\Phi} + q^2 r^2)^3}, \qquad p = \frac{4rE(r)e^{3\Phi_s - \Lambda_s}[2q^2 r + E(r)\Phi_{s,r}]\Phi_{s,z}}{(e^{2\Phi_s} + q^2 r^2)^3}, \tag{15}$$

the surface mass density and the surface current density

$$\mu = \frac{4E(r)e^{3\Phi_s - \Lambda_s}\Phi_{s,z}}{(e^{2\Phi_s} + q^2 r^2)^2}, \qquad j = \frac{4qre^{4\Phi_s - \Lambda_s}\Phi_{s,z}}{(e^{2\Phi_s} + q^2 r^2)^2}, \tag{16}$$

and the circular velocity

$$U^2 = U_s^2 \left[1 + \frac{2q^2 r}{E(r)\Phi_{s,r}}\right], \tag{17}$$

where $E(r) = e^{2\Phi_s} - q^2 r^2$, and

$$U_s^2 = \frac{r\Phi_{s,r}}{1 - r\Phi_{s,r}} \tag{18}$$

is the circular velocity of the unmagnetized source, i.e. $q = 0$.

Let's restrict the previous general model to obtain a solution of the Einstein-Maxwell equations describing an infinite family of finite static magnetized thin disks. In order to do this, we use as a seed solution (Φ_s, Λ_s), the well known Morgan and Morgan metric disk (ϕ, λ) [6],

$$\phi(x, y) = -\sum_{n=0}^{\infty} C_{2n} q_{2n}(x) P_{2n}(y),$$

where $P_n(y)$ are the usual Legendre polynomials and $q_n(x) = i^{n+1}Q_n(ix)$. $Q_n(z)$ being the Legendre functions of second kind (see [7, 8]). The x and y are the oblate spheroidal coordinates related with the cylindrical coordinates by the relations [8]

$$r^2 = a^2(1 + x^2)(1 - y^2), \qquad z = axy,, \tag{19a}$$

where $0 \le x < \infty$ and $-1 \le y < 1$. The disk has coordinates $x = 0$, $0 \le y^2 < 1$ and, on crossing the disk, y changes sign but does not change in the absolute value. We use the constants C_{2n} as determined by [9]. Then, these solutions correspond to the magnetized version of the well known Morgan-Morgan disks [6]. On the other hand, from the Eq.(15) the energy density of the magnetized disk can be written as

$$\epsilon_n = \frac{4E^2 e^{3\phi_n - \lambda_n}[y + (1 - y^2)\phi_{n,y}]\phi_{n,x}}{ay^2[e^{2\phi_n} + q^2 a^2(1 - y^2)]^3}, \qquad E(y) = e^{2\phi_n} - q^2 a^2(1 - y^2), \tag{20}$$

whereas from the equations (15) and (16) the pressure, p, and mass density, μ, in the surface of the disk are

$$p_n = \frac{4E(1 - y^2)e^{3\phi_n - \lambda_n}\phi_{n,x}[2q^2 a^2 y - \phi_{n,y}E]}{ay^2[e^{2\phi_n} + q^2 a^2(1 - y^2)]^3}, \qquad \mu_n = \frac{4Ee^{3\phi_n - \lambda_n}\phi_{n,x}}{ay[e^{2\phi_n} + q^2 a^2(1 - y^2)]^2}, \tag{21}$$

respectively. Whereas, from the equations (16) and (17) we have for the surface current density and circular velocity

$$j = \frac{4q(1-y^2)^{1/2}e^{4\phi_n - \lambda_n}\phi_{n,x}}{y[e^{2\phi_n} + q^2 a^2 (1-y^2)]^2}, \tag{22}$$

and

$$U_n^2 = U_s^2 \left[1 - \frac{2q^2 a^2 y}{E\phi_{n,y}} \right], \qquad U_s^2 = -\frac{(1-y^2)\phi_{n,y}}{y + (1-y^2)\phi_{n,y}}, \tag{23}$$

being U_n^2 the circular velocity of the unmagnetized source, i.e. $q = 0$.

The non zero components of the magnetic field in the surface of the disk are

$$B_z = -\frac{2qa(1-y^2)^{1/2}e^{2\phi_n}[y + (1-y^2)\phi_{n,y}]}{y[e^{2\phi_n} + q^2 a^2 (1-y^2)]^2}, \qquad B_r = -\frac{2qa(1-y^2)e^{2\phi_n}\phi_{n,x}}{y[e^{2\phi_n} + q^2 a^2 (1-y^2)]^2}, \tag{24}$$

here we have used the magnetic potential of the equation (11) and all the quantities are evaluated on the disk.

We have presented an infinite family of new exact solutions of the vacuum Einstein-Maxwell equations for static and axially symmetric spacetimes. The solutions describe an infinite family of magnetized finite thin disks, the magnetized version of the family of Morgan and Morgan relativistic thin disks [6]. Unlike the Morgan and Morgan seed solution, generated spacetimes are not asymptotically flat. Nevertheless, their energy densities and mass densities are positive everywhere and well behaved. In such way their energy-momentum tensor are in fully agreement with all the energy conditions. The first member, $n = 1$, of the family of the magnetized Morgan-Morgan disk has a singularity at the rim of the disk where the Kretschmann invariant becomes infinite, although its mass density is finite everywhere. Whereas all the $n > 1$ disks are regular. This property in the curvature is, of course, inherited from the seed Morgan and Morgan metric disk. On the other hand, as all the metric functions of the solutions were explicitly computed, these are the first fully integrated exact solutions for such kind of magnetized thin disk sources. Moreover, the method outlined may serve as a guideline to find more physical acceptable solutions in future works. Furthermore their importance as a new family of exact solutions of the Einstein-Maxwell vacuum equations.

References
[1] A. Lichnerowicz, C.R. 1971 *Acad. Sci.* **273**, 528
[2] A. H. Taub 1980 *J. Math. Phys.* **21**, 1423
[3] G. A. González and O. A. Espitia 2003 *Phys. Rev.* D **68**, 104028
[4] G. García-Reyes and G. A. González 2004 *Phys. Rev.* D **70**, 104005
[5] L. Richtereck, J. Novotný and J. Horský 2000 *Czech. J. Phys.* **50**, 925
[6] T. Morgan and L. Morgan 1969 *Phys. Rev.* **183**, 1097
[7] Arfken G. and Weber H 2005 *Mathematical Methods for Physicists* (Academic Press, New York
[8] P. M. Morse and H. Fesbach 1953 *Methods of Theoretical Physics* (McGraw-Hill)
[9] G. A. González and J. I. Reina 2006 *Mon. Not. R. Astron. Soc.* **371**, 1873