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To cite this article: Atirat Meunson and Tanapat Deesuwan 2025 *J. Phys.: Conf. Ser.* **3168** 012014

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# A preliminary study on some properties of the cumulant-based quantum relative Rényi functional

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**Abstract.** In quantum information theory, the quantum relative Rényi functional is a generalisation of quantum relative entropy that quantifies the difference between two quantum states. We study a new type of quantum relative Rényi functional, motivated by the concepts of the cumulant-generating function and quantum surprisal difference, and propose an information-theoretic interpretation of the Rényi parameter. In addition, we demonstrate that this functional can be naturally expressed in a path-integral-like formulation, providing a distinct perspective compared to existing approaches. To support the viability of our proposal, we conduct a preliminary numerical study of its key properties for qubits and qutrits. These properties include the monotonicity in the Rényi parameter ( $\alpha$ ) and the quantum data-processing inequality (QDPI).

## 1 Introduction

Many physical systems inherently exhibit non-deterministic behavior, particularly in quantum mechanics and stochastic processes. From an information-theory perspective, a measure of uncertainty or statistical surprise in an outcome of an observable of interest is called surprisal [1], also known as the information content. When averaged over all possible outcomes of a stochastic system, this yields Shannon entropy [2–4], which serves as a foundational measure in classical information theory.

A closely related quantity is the relative entropy or Kullback–Leibler divergence, which quantifies the average difference in information content between two probability distributions. Shannon entropy and relative entropy are particularly effective for analyzing systems in asymptotic regimes, where information arises from repeated and independent trials. However, in non-asymptotic contexts, such as those relevant to quantum cryptography, one-shot information theory, or finite resource scenarios, Shannon entropy proves insufficient due to its lack of a tunable sensitivity parameter. Specifically, it does not account well for rare events or extreme behavior, which motivates the use of min-entropy and max-entropy [5–7] in these settings.

To address such limitations systematically, Alfréd Rényi introduced a family of entropy measures derived from a generalized set of axioms parameterized by  $\alpha$ . The resulting Rényi entropy [8], and relative Rényi entropy (or Rényi divergence) [9] interpolate between various operational regimes, including Shannon, minimum, and maximum entropy, depending on the choice of  $\alpha$ . These measures have found broad applications in thermodynamics, statistical mechanics [10, 11], black hole thermodynamics [12], and fuzzy clustering [13].



The Rényi entropy framework also extends naturally to the quantum domain. By replacing classical probability distributions with density operators, one obtains the quantum Rényi entropy and the quantum relative Rényi functional, generalizing the von Neumann entropy and the Umegaki quantum relative entropy, respectively. The Petz-type quantum relative Rényi functional [14], which is based on operator convex functions and algebraic structure, was one of the first formulations of this kind. However, it fails to satisfy the quantum data processing inequality (QDPI) for  $\alpha > 2$  and lacks a strong operational interpretation in some non-commutative scenarios.

To address these issues, Müller-Lennert et al. and Wilde et al. independently proposed the sandwiched quantum Rényi divergence [15, 16], which satisfies QDPI for a broader range of  $\alpha$ , particularly for  $\alpha > 2$ , and has a clearer interpretation in contexts such as quantum hypothesis testing [17]. Nevertheless, both Petz and sandwiched versions were not derived from a fundamental statistical approach. Rather, their derivations were based on axiomatic operator-algebraic generalization and operational approaches, respectively.

In this work, we propose a new formulation of the quantum relative Rényi functional grounded in the cumulant-generating function (CGF) of the quantum surprisal difference. This approach provides a conceptually motivated foundation by connecting quantum information measures with statistical cumulants. Furthermore, we demonstrate that this new functional admits a path-integral-like representation reminiscent of the Feynman formalism [18, 19]. This formulation leads to new insights into the role of the Rényi parameter and the structure of quantum information.

To evaluate the validity of this formulation, we numerically investigate two key properties: (i) monotonicity in the Rényi parameter  $\alpha$ , and (ii) the quantum data processing inequality (QDPI). Simulations are performed using randomly generated qubit and qutrit states processed under representative quantum channels. The results indicate that the proposed functional satisfies these properties within specific parameter ranges, motivating further theoretical and numerical study.

## 2 Classical and Quantum Information Measures

We begin by reviewing both classical and quantum information measures that form the foundation of this work. These measures are rooted in the concept of surprisal, which quantifies the information content associated with a classical event or, in the quantum setting, a measurement outcome. Throughout this paper, we adopt the natural logarithm (base  $e$ ) in accordance with conventions in statistical mechanics and quantum physics.

**Definition 1 (Classical Surprisal and Shannon Entropy).** Let  $X$  be a discrete random variable taking the value  $x$  in a finite alphabet  $\mathcal{X}$ , with its distribution described by a probability mass function  $p(X)$  over  $\mathcal{X}$ . The surprisal or information content  $\Lambda(X)$  of the random variable  $X$  is defined as:

$$\Lambda(X) \equiv -\ln(p(X)). \quad (1)$$

The average surprisal over the distribution yields the Shannon entropy:

$$S(X) \equiv \mathbb{E}_X[-\ln(p(X))] = -\sum_{x \in \mathcal{X}} p(x) \ln[p(x)] \quad (2)$$

**Definition 2 (Classical Surprisal Difference and Relative Entropy).** Given two probability distributions over the same random variable  $X$ , denoted as  $p(X)$  and  $q(X)$ , the surprisal difference between them is defined as:

$$\Delta\Lambda(X) \equiv \ln(p(X)) - \ln(q(X)) \quad (3)$$

This quantity captures the difference in information content when describing a random variable  $X$  using distribution  $p(X)$  versus  $q(X)$ . The average surprisal difference is the relative entropy:

$$D(p(X) \parallel q(X)) = \begin{cases} \sum_{x \in \mathcal{X}} p(x) \ln \left( \frac{p(x)}{q(x)} \right) & \text{if } \text{supp}(p(x)) \subseteq \text{supp}(q(x)), \\ +\infty & \text{else.} \end{cases} \quad (4)$$

It quantifies the inefficiency of assuming  $q(X)$  when the true distribution is  $p(X)$ .

**Definition 3 (Classical Rényi Entropy).** The classical Rényi entropy of order  $\alpha$  with  $\alpha \in (0, +\infty) \setminus \{1\}$  is defined as:

$$S_\alpha(X) = \frac{1}{1-\alpha} \ln \left( \sum_{x \in \mathcal{X}} p^\alpha(x) \right) \quad (5)$$

This expression reduces to Shannon entropy when  $\lim \alpha \rightarrow 1$ .

**Definition 4 (Classical Relative Rényi Entropy).** The Rényi divergence of order  $\alpha$  with  $\alpha \in (0, +\infty) \setminus \{1\}$  from  $p(X)$  to  $q(X)$  is defined as:

$$D_\alpha(p(X) \parallel q(X)) = \begin{cases} \frac{1}{\alpha-1} \ln \left( \sum_{x \in \mathcal{X}} p(x)^\alpha q(x)^{1-\alpha} \right), & \text{if } \text{supp}(p(x)) \subseteq \text{supp}(q(x)), \\ +\infty, & \text{else.} \end{cases} \quad (6)$$

This reduces to the relative entropy when  $\lim \alpha \rightarrow 1$ .

**Definition 5 (Quantum Shannon Entropy).** Let  $\rho = \sum_i p(\varphi_i) |\varphi_i\rangle\langle\varphi_i| \in D(\mathcal{H})$  be a quantum state in a Hilbert space  $\mathcal{H}$ , and  $\{|\varphi_i\rangle\}$  be a set of eigenvectors that forms an orthonormal basis in the Hilbert space. The quantum Shannon entropy, also known as the von Neumann entropy, is defined as:

$$S^{\text{Neumann}}(\rho) = -\text{Tr}(\rho \ln(\rho)) \quad (7)$$

The operator  $\Xi \equiv -\ln \rho$  is referred to as the quantum surprisal, capturing the self-information content of the state.

**Definition 6 (Quantum Surprisal Difference and Quantum Relative Entropy).** Let  $\rho = \sum_i p(\varphi_i) |\varphi_i\rangle\langle\varphi_i|$  and  $\sigma = \sum_j q(\psi_j) |\psi_j\rangle\langle\psi_j|$ , where  $\rho, \sigma \in D(\mathcal{H})$ , and  $\varphi_i, \psi_j$  are elements of an orthonormal basis of the Hilbert space. The quantum surprisal difference [20] is defined as:

$$\Delta\Xi = \ln(\rho) - \ln(\sigma) \quad (8)$$

This quantity is the key to our new form of quantum relative Rényi entropy, discussed in the next section. The average of quantum surprisal differences is the quantum relative Rényi entropy:

$$S(\rho) = -\text{Tr}[\rho(\ln \rho - \ln \sigma)]. \quad (9)$$

**Definition 7 (Petz Quantum Relative Rényi Functional).** Let  $\rho = \sum_i p(\varphi_i) |\varphi_i\rangle\langle\varphi_i|$  and  $\sigma = \sum_j q(\psi_j) |\psi_j\rangle\langle\psi_j|$ , where  $\rho, \sigma \in D(\mathcal{H})$ , and  $\varphi_i, \psi_j$  are elements of an orthonormal basis of the Hilbert space, the Petz quantum relative Rényi functional of order  $\alpha$  with  $\alpha \in (0, +\infty) \setminus \{1\}$  is defined as [14]:

$$S_\alpha^{\text{Petz}}(\rho \parallel \sigma) = \begin{cases} \frac{1}{\alpha-1} \ln (\text{Tr}(\rho^\alpha \sigma^{1-\alpha})) & \text{if } \rho \not\perp \sigma \wedge (\text{supp}(\rho) \subseteq \text{supp}(\sigma) \vee \alpha < 1), \\ +\infty & \text{else.} \end{cases} \quad (10)$$

This form reduces to quantum relative entropy as  $\alpha$  approaches 1 and matches the classical relative Rényi entropy when  $\rho$  and  $\sigma$  commute, i.e.,  $[\rho, \sigma] = 0$ . It is well known that this functional satisfies QDPI in the range  $\alpha \in [0, 2]$ .

**Definition 8 (Sandwiched Quantum Relative Rényi Functional)** Let  $\rho = \sum_i p(\varphi_i) |\varphi_i\rangle\langle\varphi_i|$  and  $\sigma = \sum_j q(\psi_j) |\psi_j\rangle\langle\psi_j|$ , where  $\rho, \sigma \in D(\mathcal{H})$ , the sandwiched quantum relative Rényi functional of order  $\alpha$  with  $\alpha \in (0, +\infty) \setminus \{1\}$  is defined as [15, 16]:

$$S_\alpha^s(\rho\|\sigma) = \begin{cases} \frac{1}{\alpha-1} \ln \text{Tr} \left[ \left( \sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}} \right)^\alpha \right], & \text{if } \rho \not\perp \sigma \wedge (\text{supp}(\rho) \subseteq \text{supp}(\sigma) \vee \alpha < 1), \\ +\infty & \text{else.} \end{cases} \quad (11)$$

As well as the Petz quantum relative Rényi functional, the sandwiched form reduces to quantum relative entropy as  $\lim_{\alpha \rightarrow 1}$  and becomes equivalent to its classical counterpart when  $[\rho, \sigma] = 0$ . This kind of relative Rényi entropy satisfies QDPI for all  $\alpha \geq \frac{1}{2}$  and is especially relevant for quantum hypothesis testing.

### 3 Cumulant-based Quantum Relative Rényi Functional

In this section, we discuss how the concept of classical relative Rényi entropy can be viewed as equivalent to the cumulant-generating function of the corresponding surprisal differences. By extending this idea to quantum settings, using the concept of quantum surprisal difference, we show how our new formulation of quantum relative Rényi functional arises. We begin by recalling the definition of a cumulant-generating function (CGF) which offers an alternative equivalent representation of a probability distribution  $p(Y)$ .

$$C_Y(t) = \ln \mathbb{E}[e^{tY}], \quad (12)$$

where  $t \in \mathbb{R}$  is the conjugate variable of  $Y$ . If we choose the random variable of interest to be the surprisal  $\Lambda(X) = -\ln p(X)$ , we will obtain

$$C_{\Lambda(X)}(t) = \ln \mathbb{E} \left[ e^{t\Lambda(X)} \right] = \ln \mathbb{E} \left[ e^{-t \ln p(X)} \right]. \quad (13)$$

By the definition of the CGF, we also have:

$$\mathbb{E} \left[ e^{-t \ln p(X)} \right] = e^{C_{\Lambda(X)}(t)}. \quad (14)$$

Recall the Rényi entropy of order  $\alpha > 0, \alpha \neq 1$ , defined as:

$$S_\alpha(X) = \frac{1}{1-\alpha} \ln \left( \sum_{x \in \mathcal{X}} (p(x))^\alpha \right). \quad (15)$$

The summation inside the logarithm can be expressed as an expectation value of with respect to the probability  $p(X)$  itself:

$$\sum_{x \in \mathcal{X}} (p(x))^\alpha = \mathbb{E} \left[ e^{(\alpha-1)(-\Lambda(X))} \right] = e^{C_{\Lambda(X)}(1-\alpha)}. \quad (16)$$

Substituting Eq. (16) into Eq. (15), we find:

$$\begin{aligned} S_\alpha(X) &= \frac{1}{1-\alpha} \ln \left( e^{C_{\Lambda(X)}(1-\alpha)} \right) = \frac{1}{1-\alpha} C_{\Lambda(X)}(1-\alpha) \\ &= \frac{1}{1-\alpha} \ln \left( \sum_{x \in \mathcal{X}} p(x) e^{-(1-\alpha) \ln p(x)} \right). \end{aligned} \quad (17)$$

By applying the same approach, we obtain the following relationship between the cumulant-generating function of the classical surprisal difference and the classical relative Rényi entropy:

$$\begin{aligned} \frac{1}{\alpha-1} C_{\Delta\Lambda(X)}(\alpha-1) &= \frac{1}{\alpha-1} \ln \mathbb{E} \left[ e^{(\alpha-1)[\ln P(X) - \ln Q(X)]} \right] \\ &= S_\alpha(P(X) \parallel Q(X)). \end{aligned} \quad (18)$$

where  $\alpha-1 \in \mathbb{R}$  serves as the conjugate variable to the surprisal difference. By rewriting the standard formulation of a classical relative Rényi entropy in terms of surprisal differences this way, one can clearly see that the parameter  $\alpha$  can be interpreted as a logarithmic base modifier, effectively altering the scale at which information is measured via a surprisal. In particular, when the factor  $\alpha-1$  is absorbed into the surprisal difference, it effectively changes the logarithmic base, thus rescaling the measure of information.

$$\begin{aligned} (\alpha-1) [\ln(P(X)) - \ln(Q(X))] &= \text{sgn}(\alpha-1) |\alpha-1| [\ln(P(X)) - \ln(Q(X))] \\ &= \text{sgn}(\alpha-1) (\log_{e^{1/|\alpha-1|}}(P(X)) - \log_{e^{1/|\alpha-1|}}(Q(X))) \end{aligned} \quad (19)$$

The classical relative Rényi entropy can be expressed as a cumulant-generating function of the relative surprisal. This observation is foundational: since a cumulant-generating function uniquely determines a probability distribution, it encapsulates all the statistical properties of the underlying variable. In this light, the relative Rényi entropy encodes the full distributional structure of the surprisal difference, extending far beyond the average behavior captured by the Shannon relative entropy. Thus, Rényi's framework provides a more complete characterisation of statistical divergence between distributions.

We can extend this concept to quantum settings by replacing probability distributions with density matrices. The cumulant-generating function in this case is the following:

$$C_{\Delta\Xi}(\alpha-1) = \ln \mathbb{E}[e^{(\alpha-1)[\ln \rho - \ln \sigma]}] \quad (20)$$

Hence, we can define our new quantum relative Rényi functional based on the concept of the cumulant-generating function of the quantum surprisal difference as follows.

**Definition 5:** Let  $\rho = \sum_i p(\varphi_i) |\varphi_i\rangle\langle\varphi_i|$  and  $\sigma = \sum_j q(\psi_j) |\psi_j\rangle\langle\psi_j|$ , where  $\rho, \sigma \in D(H)$ , and  $\{\varphi_i\}$  and  $\{\psi_j\}$  individually form an orthonormal basis in a Hilbert space, the cumulant-based quantum relative Rényi functional of order  $\alpha$  with  $\alpha \in (0, +\infty) \setminus \{1\}$  is defined as:

$$S_\alpha^Q(\rho \parallel \sigma) = \begin{cases} \frac{1}{\alpha-1} \ln \text{Tr} (\rho e^{(\alpha-1)(\ln \rho - \ln \sigma)}) & \text{if } \rho \not\perp \sigma \wedge (\text{supp}(\rho) \subseteq \text{supp}(\sigma) \vee \alpha < 1), \\ +\infty & \text{else.} \end{cases} \quad (21)$$

The cumulant-based quantum relative Rényi functional reduces to the quantum relative entropy when  $\alpha \rightarrow 1$ , and it also recovers the classical relative Rényi functional if  $[\rho, \sigma] = 0$ .

### 3.1 Comparison with Existing Quantum Rényi Functionals

Our cumulant-based quantum relative Rényi functional, as defined in Eq. (21), differs fundamentally from the established Petz and sandwiched quantum relative Rényi functionals in its motivation, structure, and interpretive foundation.

First, while both forms were developed from operator-algebraic and operational frameworks, respectively, our formulation arises directly from the cumulant-generating function (CGF) of the quantum surprisal difference, which is based on solid statistical principles. This origin provides a clear interpretation: the Rényi parameter  $\alpha$  plays the role of a conjugate variable that controls the scale of fluctuations in surprisal statistics.

Second, unlike the Petz form which takes the algebraic form  $\text{Tr}[\rho^\alpha \sigma^{1-\alpha}]$  and the sandwiched form which involves symmetric operator orderings such as  $\text{Tr}[(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}})^\alpha]$ , our cumulant-based form structurally mirrors an expectation over the exponential of a non-commuting observable ( $\log \rho - \log \sigma$ ). This leads naturally to a path-integral-like representation, as shown in Section 4, which is not readily available for other formulations.

Third, our functional retains reduction to the classical relative Rényi entropy and quantum relative entropy in the commutative and  $\alpha \rightarrow 1$  limits, respectively, similarly to the Petz and sandwiched versions. However, the cumulant structure provides additional insights into the higher-order fluctuations of information content, going beyond average-case behavior. While a full analytical characterization, particularly regarding the quantum data processing inequality (QDPI), remains ongoing, our numerical study in Section 5 shows that the proposed functional satisfies monotonicity in  $\alpha$  and complies with QDPI over significant parameter ranges.

In summary, the cumulant-based formulation offers a distinct statistical foundation, an intuitive interpretation via fluctuation theory, and a natural bridge to path-integral formalism, which are not directly present in the existing Petz and sandwiched forms. We acknowledge that desirable properties such as positivity, additivity, and full QDPI compliance have not yet been analytically established in this manuscript. However, numerical evidence supports monotonicity in  $\alpha$  and partial QDPI compliance. As this work is intended as a preliminary study, we have deliberately focused on introducing the formulation, outlining its statistical and structural foundations, and providing initial empirical validation. A rigorous analytic investigation of these properties will be addressed in future work.

#### 4 Path-Integral-Like Formulation of Cumulant-Based Quantum Relative Rényi functional

Originally proposed by R. Feynman, the path integral formalism [18, 19] describes the dynamics of a quantum system through a quantity called the action. Instead of focusing on a single path, as in the classical approach, the path integral sums over all possible paths, with each path weighted by a phase factor determined by its corresponding action.

Let us consider the cumulant-based quantum relative Rényi functional in Eq.(21). We can rewrite the matrix  $\rho$  in front of the exponential term with spectral decomposition as

$$\begin{aligned} S_\alpha^Q(\rho||\sigma) &= \frac{1}{\alpha-1} \ln \text{Tr} \left( \sum_i p(\varphi_i) |\varphi_i\rangle \langle \varphi_i| e^{(\alpha-1)(\ln \rho - \ln \sigma)} \right) \\ &= \frac{1}{\alpha-1} \ln \left( \sum_i p(\varphi_i) \langle \varphi_i| e^{(\alpha-1)(\ln \rho - \ln \sigma)} |\varphi_i\rangle \right) \end{aligned} \quad (22)$$

Consider  $f_{\rho,\sigma}(\alpha-1) = \sum_i p(\varphi_i) \langle \varphi_i| e^{(\alpha-1)(\ln \rho - \ln \sigma)} |\varphi_i\rangle$ . Since the two density matrices  $\rho$  and  $\sigma$  do not commute in general, we cannot reduce the exponential term to a simple product. To properly account for the non-commutativity, we apply the Lie–Trotter formula to the exponential term. Thus, we obtain

$$f_{\rho,\sigma}(\alpha-1) = \lim_{n \rightarrow \infty} \sum_i p(\varphi_i) \langle \varphi_i| \left( e^{\frac{(\alpha-1)}{n} \ln \rho} e^{-\frac{(\alpha-1)}{n} \ln \sigma} \right)^n |\varphi_i\rangle \quad (23)$$

where  $n \rightarrow +\infty$ .

One can see that this structure is similar to a propagator in the Feynman path integral formulation, where the path in this case is like “a loop” where the start and the end are at the same point. However, there is no time parameter in this system, and thus we refer to it as a

propagator-like functional. This analogy becomes even clearer when we insert the completeness relation, allowing us to express the functional as a sum over cyclic, path-integral-like trajectories in the corresponding Hilbert space,

$$\begin{aligned}
 f_{\rho,\sigma}(\alpha - 1) &= \lim_{n \rightarrow \infty} \left( \sum_i p(\varphi_i) \langle \varphi_i | e^{\frac{(\alpha-1)}{n} \ln \rho} e^{-\frac{(\alpha-1)}{n} \ln \sigma} \sum_j |\varphi_j\rangle \langle \varphi_j | e^{\frac{(\alpha-1)}{n} \ln \rho} e^{-\frac{(\alpha-1)}{n} \ln \sigma} \times \dots \right. \\
 &\quad \left. \times \sum_k |\varphi_k\rangle \langle \varphi_k | e^{\frac{(\alpha-1)}{n} \ln \rho} e^{-\frac{(\alpha-1)}{n} \ln \sigma} | \varphi_i \rangle \right) \\
 &= \lim_{n \rightarrow \infty} \left( \prod_{\beta=1}^n \sum_{i'_\beta} \sum_{j'_{\beta-1}} \langle \varphi_{i'_\beta} | e^{\frac{(\alpha-1)}{n} \ln \rho} e^{-\frac{(\alpha-1)}{n} \ln \sigma} | \varphi_{j'_{\beta-1}} \rangle \delta(\varphi_{i'_{\beta-1}}, \varphi_{j'_{\beta-1}}) p(\varphi_{i'_n}) \delta(\varphi_{i'_n}, \varphi_{j'_0}) \right)
 \end{aligned} \tag{24}$$

To proceed with the decomposition and facilitate the path-integral-like representation, we insert the identity operator between the exponential and the outer operators in the trace. Specifically, we use the completeness relation  $\sum_{m'_\beta} |\psi_{m'_\beta}\rangle \langle \psi_{m'_\beta}| = \mathbb{I}$ , where  $\{\psi_{m'_\beta}\}$  denotes the eigenbasis of the reference operator  $\hat{\sigma}$ . This choice enables the diagonalization of  $\log \hat{\sigma}$ , and thereby simplifies the expression within the exponential. The resulting decomposition leads to the following form:

$$\begin{aligned}
 f_{\rho,\sigma}(\alpha - 1) &= \lim_{n \rightarrow \infty} \left( \prod_{\beta=1}^n \sum_{i'_\beta} \sum_{j'_{\beta-1}} \sum_{m'_\beta} \langle \varphi_{i'_\beta} | \psi_{m'_\beta} \rangle \langle \psi_{m'_\beta} | \varphi_{j'_{\beta-1}} \rangle e^{\frac{(\alpha-1)}{n} \ln p(\varphi_{i'_\beta})} e^{-\frac{(\alpha-1)}{n} \ln q(\psi_{m'_\beta})} \delta(\varphi_{i'_{\beta-1}}, \varphi_{j'_{\beta-1}}) \right. \\
 &\quad \left. \times p(\varphi_{i'_n}) \delta(\varphi_{i'_n}, \varphi_{i'_0}) \right)
 \end{aligned} \tag{25}$$

The term of inner product between  $\langle \varphi_{i'_\beta} | \psi_{m'_\beta} \rangle$  and  $\langle \psi_{m'_\beta} | \varphi_{j'_{\beta-1}} \rangle$  provides the multiple product of phase factor and conditional probability amplitude in each inner product.

$$\begin{aligned}
 f_{\rho,\sigma}(\alpha - 1) &= \lim_{n \rightarrow \infty} \left( \prod_{\beta=1}^n \sum_{i'_\beta} \sum_{j'_{\beta-1}} \sum_{m'_\beta} e^{i[\Theta(\psi_{i',m',\beta}) - \Theta(\psi_{m',j',\beta-1})]} \cdot e^{\frac{1}{2} \ln(\mathbb{P}(\varphi_{i'_\beta} | \psi_{m'_\beta}) \mathbb{P}(\psi_{m'_\beta} | \varphi_{j'_{\beta-1}}))} \right. \\
 &\quad \left. \times e^{\frac{(\alpha-1)}{n} \ln \left( \frac{p(\varphi_{i',\beta})}{q(\psi_{m',\beta})} \right)} \delta(\varphi_{i'_{\beta-1}}, \varphi_{j'_{\beta-1}}) p(\varphi_{i'_n}) \delta(\varphi_{i'_n}, \varphi_{i'_0}) \right)
 \end{aligned} \tag{26}$$

We note that the phase factor arising from the two inner products behaves similarly to an action term in the path integral formalism. However, in our construction, the "action" consists of both real and imaginary parts. Therefore, we can define a complex-action-like functional as follows:

$$\mathcal{A}_{\rho,\sigma}[\{\varphi, \psi\}] = i [\Theta(\psi_{i',m',\beta}) - \Theta(\psi_{m',j',\beta-1})] + \frac{1}{2} \ln \left[ \mathbb{P}(\varphi_{i'_\beta} | \psi_{m'_\beta}) \mathbb{P}(\psi_{m'_\beta} | \varphi_{j'_{\beta-1}}) \right]. \tag{27}$$

Therefore, the cumulant-based quantum relative Rényi functional can be naturally expressed

in the path-integral-like formulation as follows:

$$S_{\alpha}^Q(\rho||\sigma) = \frac{1}{\alpha - 1} \ln \left( \lim_{n \rightarrow \infty} \left( \prod_{\beta=1}^n \sum_{i'_{\beta}} \sum_{j'_{\beta-1}} \sum_{m'_{\beta}} e^{A_{\rho,\sigma}[\{\varphi,\psi\}]} e^{\frac{\alpha-1}{n} \ln \left( \frac{p(\varphi_{i'_{\beta}})}{q(\psi_{m'_{\beta}})} \right)} \delta(\varphi_{i'_{\beta-1}}, \varphi_{j'_{\beta-1}}) \right. \right. \\ \left. \left. \times p(\varphi_{i'_n}) \delta(\varphi_{i'_n}, \varphi_{i'_0}) \right) \right) \quad (28)$$

where  $n$  is the time-step-like factor and  $\alpha - 1$  is the logarithmic scaling factor.

## 5 Numerical Investigation of Standard Properties of the Proposed Formulation

In this section, we present a numerical investigation of two key properties expected of a quantum information measure: (i) monotonicity with respect to the Rényi parameter  $\alpha$  and (ii) the quantum data processing inequality (QDPI), within the framework of the proposed cumulant-based quantum relative Rényi functional.

### 5.1 Monotonicity in the Rényi parameter $\alpha$

In both classical and quantum settings, the Rényi entropy and relative Rényi entropy often exhibit monotonic behavior in  $\alpha$ . Specifically, the sandwiched quantum relative Rényi functional is known to be monotonically increasing for all  $\alpha \in (0, +\infty)$ , the Petz-type quantum relative Rényi entropy also satisfies monotonicity with respect to  $\alpha$  over the same range.

We numerically evaluate whether the cumulant-based quantum relative Rényi functional defined in Eq.(21) satisfies monotonicity in  $\alpha$ . To do so, we generate 10,000 random pairs of density matrices (for qubit and qutrit systems) and evaluate the functional across varying  $\alpha$  values while keeping the matrices fixed. The result is shown in Figure 1.

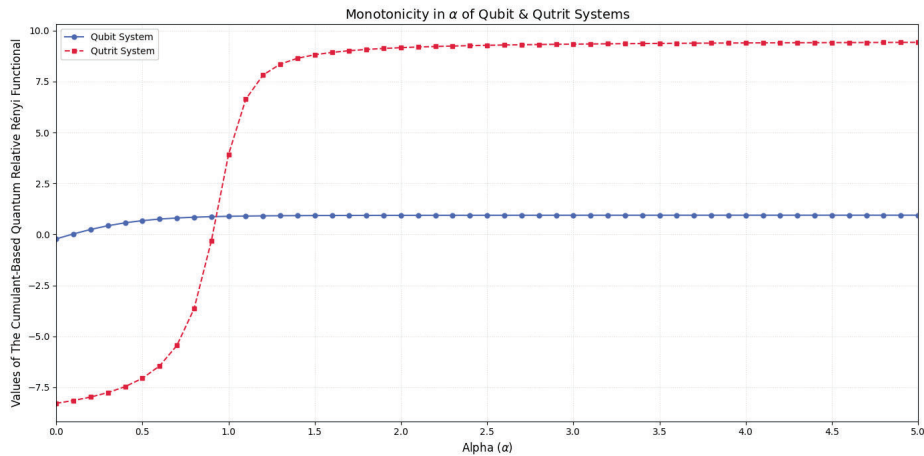


Figure 1: Numerical verification of the monotonicity in the Rényi parameter  $\alpha$  for the cumulant-based quantum relative Rényi functional, evaluated on representative superposition states of qubit and qutrit systems.

This result is only one example of the testing procedure. However, when we test with other states, the tendency toward monotonicity remains valid. The results demonstrate that the cumulant-based functional exhibits strictly increasing monotonic behavior for both qubit

and qutrit systems. This confirms that the proposed our functional satisfies the monotonically increasing in  $\alpha$  as follows:

$$S_{\alpha_1}^Q(\rho \parallel \sigma) \leq S_{\alpha_2}^Q(\rho \parallel \sigma), \quad (29)$$

where  $\alpha_1 \leq \alpha_2$ .

### 5.2 Quantum Data Processing Inequality (QDPI).

QDPI [21] is a desired property of relative entropy and its Rényi extension, both in classical and quantum versions, which states that the distinguishability between two quantum states cannot increase under the action of any completely positive trace-preserving (CPTP) map  $\Phi$ . This can be written as:

$$S_{\alpha}^Q(\Phi(\rho) \parallel \Phi(\sigma)) \leq S_{\alpha}^Q(\rho \parallel \sigma). \quad (30)$$

We test this inequality numerically using several representative quantum channels, namely, the **quantum depolarizing channel** [22], which is commonly used to model how noise affects a quantum state, especially when quantum information is transmitted through an imperfect channel. In other words, it refers to a map where an arbitrary quantum state may change to a maximally mixed state with probability  $p \in [0, 1]$  after it applies, which is defined as

$$\Phi_{\text{Dep}}(\rho) = (1-p)\rho + p\frac{\mathbb{I}}{d}, \quad (31)$$

where  $d$  is the dimension of the system. Moreover, we use a **generalized noisy identity channel**, which is a modification of the depolarizing channel,

$$\Phi_{\text{G}}(\rho) = (1-p)\rho + p\sigma, \quad (32)$$

where  $\sigma$  is an arbitrary quantum state. We note that the depolarizing channel is a special case of this generalized channel when  $\sigma = \frac{\mathbb{I}}{d}$ . Another noisy model is the **dephasing channel** [21], which describes a type of noise that alters the phase of a quantum state without affecting the information encoded in its eigenvalues. For a single qubit, this channel can be expressed as

$$\Phi_{\text{Deph}}(\rho) = (1-p)\rho + pZ\rho Z, \quad (33)$$

where  $Z$  is the Pauli- $Z$  operator. The final noise model used in our numerical experiment is the **amplitude damping channel** [23], which describes energy dissipation processes in a quantum system. Its action on a single qubit can be represented using the following Kraus operators:

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad K_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}, \quad (34)$$

To assess compliance with the quantum data processing inequality (QDPI), we define the success rate as the proportion of sampled state pairs  $(\rho, \sigma)$  that satisfy the inequality  $S_{\alpha}^Q(\Phi(\rho) \parallel \Phi(\sigma)) \leq S_{\alpha}^Q(\rho \parallel \sigma) + \varepsilon$ , for a small numerical tolerance  $\varepsilon = 10^{-4}$ . For each quantum channel and each noise parameter  $p$ , we generate 10 independent sets of 10,000 random pairs of qubit states. For each pair, we evaluate the cumulant-based relative Rényi functional across a range of Rényi parameters  $\alpha \in [0, 10]$  with a step size of  $\Delta\alpha = 0.1$ . The success rate reported in Figure 2 corresponds to the compliance rate across all 10 runs for each  $\alpha$  and  $p$  combination.

Key observations from the numerical results can be summarized as follows. For the Generalized Noisy Identity channel, all values of the noise parameter  $p$  satisfy the QDPI within the channel-specific range  $\alpha \in (0.9, 1.4)$ , while the  $p$ -dependent ranges extend up to  $\alpha \in (0.8, 4.1)$

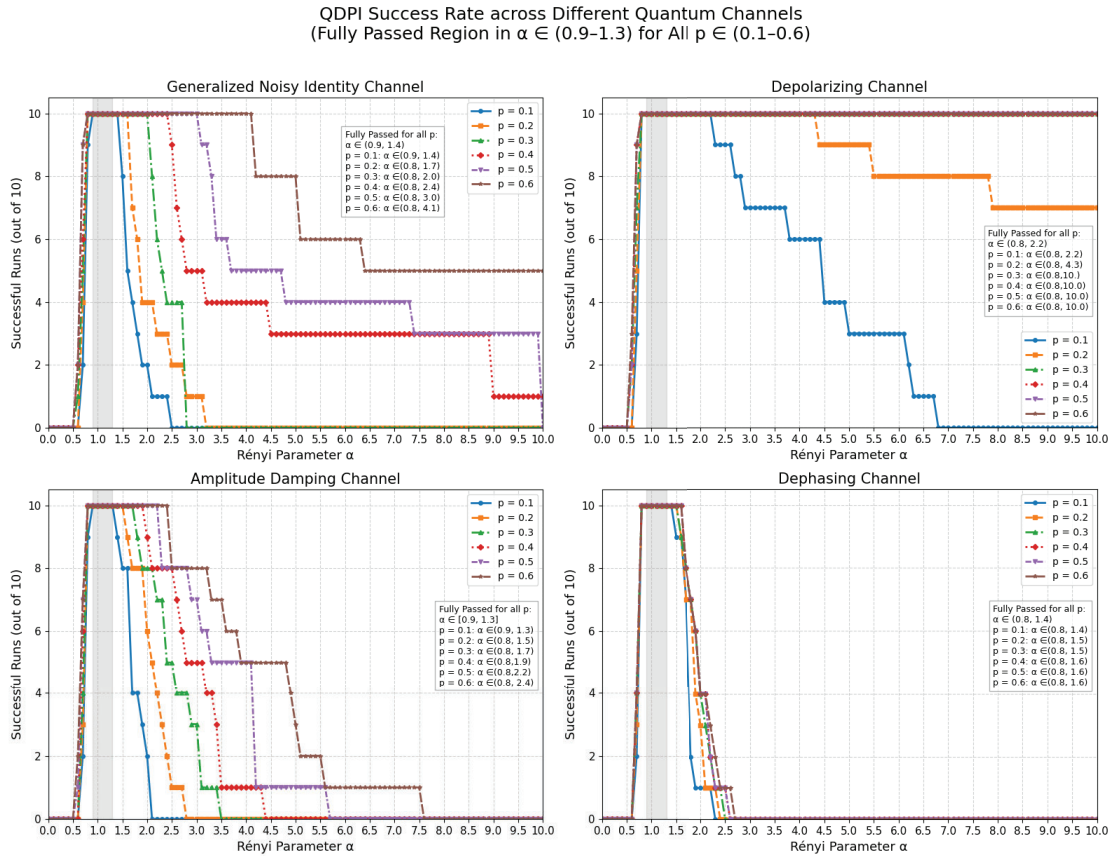


Figure 2: Number of successful runs (out of 10) in which all 10,000 randomly generated state pairs satisfy the QDPI condition for each value of the Rényi parameter  $\alpha$ . Each subplot corresponds to a different quantum channel with varying noise parameters  $p$ . The gray shaded region denotes the universal range  $\alpha \in (0.9, 1.3)$  where compliance is consistently observed across all tested channels. Distinct line styles and markers indicate different values of  $p$ .

for  $p = 0.6$  and remain within  $\alpha \in (0.9, 1.4)$  for  $p = 0.1$ . For the Depolarizing channel, all  $p$  values satisfy the QDPI for  $\alpha \in (0.8, 2.2)$ , with full compliance extending to  $\alpha \in (0.8, 10.0)$  when  $p \geq 0.3$ . The Amplitude Damping channel exhibits the most stringent requirements with  $\alpha \in (0.9, 1.3)$  for all  $p$  values. For the Dephasing channel, the QDPI is satisfied for all  $p$  in the range  $\alpha \in (0.8, 1.4)$ , with a slight extension to  $\alpha \in (0.8, 1.6)$  observed at  $p = 0.6$ . These results lead to the conclusion that the Rényi parameter  $\alpha$  satisfies QDPI for all  $p$  (where  $p \in \{0.1, 0.2, \dots, 0.6\}$ ) and all channels when  $\alpha \in (0.9, 1.3)$ .

## 6 Conclusion and Discussion

This study explores a new formulation of the quantum relative Rényi functional based on cumulant-generating function, providing an alternative perspective to existing approaches like the Petz and sandwiched versions. The proposed functional connects statistical concepts with quantum information theory and admits a path-integral-like representation, while numerical simulations suggest that it maintains key properties such as monotonicity in  $\alpha$  and partial compliance with the quantum data processing inequality for certain parameter ranges. This

range-dependent compliance with QDPI is particularly intriguing: our functional satisfies the inequality only within specific intervals of the Rényi parameter  $\alpha$ , and these intervals vary across different quantum channels. One possible explanation lies in the nature of our functional structure itself. The exponential of  $(\log \rho - \log \sigma)$  does not generally preserve operator monotonicity or convexity outside certain ranges of  $\alpha$ , especially in non-commuting cases. Furthermore, the absence of symmetric operator ordering, as in the sandwiched form, may limit the generality of QDPI compliance. The consistent success observed across different channels in the narrow range  $\alpha \in (0.9, 1.3)$  may suggest a natural stability region associated with near-classical behavior. In this regime, the functional approximates the quantum relative entropy and is less sensitive to non-commutativity.

In addition to its theoretical construction, the cumulant-based functional may have future relevance in quantum resource theories where fluctuations and higher-order information are operationally significant. In particular, resource frameworks involving thermodynamic non-equilibrium, coherence under stochastic control could benefit from a functional or divergence that reflects not just average distinguishability but also the statistical dispersion encoded in the cumulant structure. These possibilities merit further exploration, both in terms of operational interpretation and application in quantifying resource monotones.

### Acknowledgments

This work was supported by the National Science, Research and Innovation Fund (NSRF) through the Program Management Unit for Human Resources and Institutional Development, Research and Innovation under grant number B39G680007.

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