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Thermodynamics and phase transition of Bardeen black hole via Rényi statistics in grand canonical ensemble and canonical ensemble

Zefeng Wang, Haizhen Ren, Juhua Chen[a], Yongjiu Wang

Department of Physics, Key Laboratory of Low Dimensional Quantum Structures and Quantum Control of Ministry of Education, and Synergetic Innovation Center for Quantum Effects and Applications, Hunan Normal University, Changsha 410081, Hunan, People's Republic of China

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Abstract The thermodynamics of the Bardeen black hole in asymptotically flat space is investigated with the corrected first law of thermodynamics via Rényi statistics. The nonextensive parameter λ gives the possibility to the thermal stability of Bardeen black hole, and there is a Hawking–Page phase transition in the grand canonical ensemble (fixed the potential), which is similar to the cases of Bardeen black hole and corrected Bardeen black hole in asymptotically anti-de Sitter (AdS) space via standard Gibbs–Boltzmann (GB) statistics. By introducing the general Smarr formula via Rényi statistics, the thermodynamic pressure P is defined with the parameter λ and its conjugate quantity V is the thermodynamic volume (not a geometric spherical volume with horizon radius r_h). The thermodynamics of the asymptotically flat Bardeen black hole via Rényi statistics in the canonical ensemble (fixed the charge q) behaves like the van der Waals system, which is also same as the asymptotically Bardeen-AdS black hole via GB statistics. The analogy between the thermodynamics of the asymptotically flat Bardeen black hole from Rényi statistics and the Bardeen-AdS black hole from GB statistics makes us to consider what is the relation between the nonextensive parameter λ and the cosmological constant Λ.

1 Introduction

Black hole is excepted by general relativity inside whose event horizon nothing can escape. At first, the black hole is considered to only have mass, angular momentum and charge (if exists) and no temperature or entropy [1–3], but the quantum field theory in curved space changes this view. Then the black hole thermodynamics was founded by Bekenstein and Hawking et al. [4–8], suggested that a black hole can have entropy and nonzero temperature. According to the laws of black hole thermodynamics, the entropy of the black hole is proportional to its area and the surface gravity of black hole plays the role of its temperature. The black hole thermodynamics pushes us to consider the black hole as a thermal system [6,9].

A zero-charged black hole in asymptotically flat space can only have negative heat capacity with standard Gibbs-Boltzmann (GB) statistics, which means the black hole cannot be in thermal equilibrium [10]. But Hawking and Page proved that there are two branches of uncharged black hole in asymptotically anti-de Sitter space, i.e. the small black hole branch is unstable with negative heat capacity and the large black hole branch can be stable with positive heat capacity. Moreover, there is a minimum temperature T_{min} below which the two branches of black hole cannot occur, and there is another certain temperature T_{HP} which is the point of Hawking–Page phase transition [9,10]. With the development of AdS/CFT correspondence, other researches on the black hole thermodynamics were done for the charged AdS black holes [11–13].

Some researches [14,15] suggested that the standard GB statistics will be violated for the long-range interactions (black hole system and etc.). Recently, some authors [9,16–18] also pointed out that the standard stability analysis may not be applicable for the black hole system because it depends strongly on the additivity of entropy function, which fails to hold on the Bekenstein–Hawking entropy. The mentioned problems all suggested that applying GB statistics to a self-gravitation system may bring an uncomplete result. It is well known that the black hole entropy is proportional to its area rather than its volume, which indicates that the black hole

[a] e-mail: jhchen@hunnu.edu.cn (corresponding author)

entropy is nonextensive. The difficulties force us to consider the black hole thermodynamics with nonextensive statistics and nonextensive entropy. Fortunately, A weaker composition rule [19] which is the famous Abe's composition rule can be taken to apply for the nonextensive approach,

$$H_\lambda(S(A,B)) = H_\lambda(S(A)) + H_\lambda(S(B)) + \lambda H_\lambda(S(A))H_\lambda(S(B)), \quad (1)$$

where H is a differentiable function of entropy S, A, B are two independent systems, and $\lambda \in \mathbb{R}$ is a constant parameter.

One of nonextensive entropy, Tsallis entropy [20–23] takes the form

$$S_q = \frac{1 - \sum_{i=1}^{W} p_i^q}{q-1}, \; q \in \mathbb{R}, \quad (2)$$

where $W \in \mathbb{N}$ is the total number of possible (microscopic) configurations and p_i is the associated possibilities. When $q \to 1$, the standard GB entropy $S_{BG} = -\Sigma_{i=1}^{W} p_i ln p_i$ will be recovered. The composition rule of nonextensive Tsallis entropy can be written as

$$S_q(A,B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B), \quad (3)$$

which satisfies the Abe's pseudoadditive rule by setting $H_\lambda(S) = S_q$ and $\lambda = 1 - q$.

The empirical temperature cannot be well defined from the pseudoadditivity of Tsallis entropy, so Biró and Ván [15] considered the logarithmic formal of Tsallis entropy

$$L(S_q) = \frac{1}{1-q}[ln(1 + (1-q)S_q)] = S_R. \quad (4)$$

Interestingly, it is just the well-known Rényi entropy [24,25], which is additive under Abe's composition rule. By defining a nonextensive parameter $\lambda = 1 - q$, we have

$$S_R = \frac{1}{\lambda} ln(1 + \lambda S_q). \quad (5)$$

The Rényi entropy is compatible with the zeroth law of thermodynamics and the empirical temperature can be defined as

$$\frac{1}{T_R} = \frac{\partial S_R(E)}{\partial E}, \quad (6)$$

where E is the energy of system.

In Ref. [9] the nonextensive parameter λ plays the role of pressure just like the cosmological constant Λ in anti-de Sitter background. The Rényi statistics has also been applied into several different black holes in different background [14, 16–18,26,27], but still not been used to consider the regular black holes.

Regular black hole is a set of black hole solutions which origin from preventing the singularity difficulty. Since the first regular black hole solution was found by Bardeen [28], some deeper researches about the regular black hole were done [29–40]. Particularly, Eloy Ayó-beato and Alberto García [41–44] proved that the regular black holes are actually a set of solutions of Einstein equation by considering the nonlinear electromagnetical sources (more references seen [45–50]).

When we consider the thermodynamics of regular black holes, the different temperatures will be obtained by the surface gravity approach and the thermodynamic approach, which may lead to giving up one of the first law of thermodynamics or Bekenstein area law. To avoid the different temperatures obtaining by different approaches, Ma and Zhao [51] corrected the first law of black hole thermodynamics.

The thermodynamics of regular black holes has been investigated with several different approaches, but it still has some difficulties which are nonadditivity of entropy function with GB statistics and different temperatures obtaining with different approaches. To solve these problems, we apply the corrected first law of thermodynamics of black hole and Rényi statistics to the thermodynamics of Bardeen black hole.

This paper is organized as follows. In Sect. 2, we briefly review the Bardeen black hole spacetime and its corrected thermal properties. In Sect. 3, we introduce the general Smarr formula of Bardeen black hole via Rényi statistics and derive the corrected first law of regular black hole of Rényi statistics. In Sect. 4, we investigate the thermodynamics quantities of Bardeen black hole with GB statistics and corrected first law of thermodynamics in the grand canonical ensemble and the canonical ensemble. In Sect. 5, the thermodynamic quantities in the Rényi model will be studied. The brief conclusions and discussions are given in the last section. In this paper, the geometric units $G = \hbar = c = k_B = 1$ will be chosen.

2 The first law of thermodynamics for regular black holes

In this section, we briefly review the first law of thermodynamics of regular black holes. The line element of spherically symmetric Bardeen black hole is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + sin^2\theta d\varphi^2), \quad (7)$$

where the function $f(r)$ is

$$f(r) = 1 - \frac{2m(r)}{r}, \quad (8)$$

the effective mass $m(r)$ in function $f(r)$ is

$$m(r) = \frac{Mr^3}{(r^2+q^2)^{\frac{3}{2}}}, \quad (9)$$

where the parameters M and q are the black hole mass and the magnetic charge, respectively. When $q > \frac{4\sqrt{3}M}{9}$, there are inner horizon (Killing horizon) and outer horizon (event horizon). Two horizons meet when $q = \frac{4\sqrt{3}M}{9}$ and there is no horizon when $q > \frac{4\sqrt{3}M}{9}$. Setting r_h as the radius of event horizon and solving $f(r_h) = 0$ for M, we can get

$$M = \frac{\left(q^2 + r_h^2\right)^{3/2}}{2r_h^2}. \tag{10}$$

The temperature is defined by surface gravity: $T_h = f'(r_h)/(4\pi)$, so

$$T_h = \frac{r_h^2 - 2q^2}{4\pi r_h \left(q^2 + r_h^2\right)}. \tag{11}$$

On the other hand, the temperature can be calculated from the first law of thermodynamics. For Bardeen black hole the first law of black hole thermodynamics takes the form

$$dM = T_H dS + \Phi_h dq, \tag{12}$$

where S is the entropy of black hole which equals to quarter of the area of black hole event horizon and Φ_h is the magnetic potential of black hole. The temperature calculated with the first law is

$$T_H = \left(\frac{\partial M}{\partial S}\right)_q = \frac{\left(r_h^2 - 2q^2\right)\sqrt{q^2 + r_h^2}}{4\pi r_h^4}, \tag{13}$$

which is not equal to the temperature taken from the surface gravity way, i.e. $T_H \neq T_h$. Considering the correspondence $\{S, dE, T_H\} \leftrightarrow \{A/4, C(M, q, r_h)dM, T_h\}$, the first law of thermodynamics is corrected as [51]

$$dE = C(M, q, r_h)dM = T_h dS + \Phi_H dq, \tag{14}$$

where the parameter $C(M, q, r_h) = m(r_h)/M = r_h^3/(r_h^2 + q^2)^{\frac{3}{2}}$, A is the area of black hole horizon, Φ_H is the corrected magnetic potential and E is the internal energy. Thus we have

$$T_h = C(M, q, r_h)\left(\frac{\partial M}{\partial S}\right)_q = C(M, q, r_h)T_H. \tag{15}$$

The magnetic potential of Bardeen black hole can be obtained by the corrected first law

$$\Phi_H = C(M, q, r_h)\left(\frac{\partial M}{\partial q}\right)_S = \frac{3qr_h}{2\left(q^2 + r_h^2\right)}. \tag{16}$$

3 Thermodynamics and general Smarr formula from Rényi statistics

The thermodynamics and general Smarr formula in the Rényi model will be reviewed in this section. As a self-gravitation system, the black hole thermodynamics can be considered under the nonadditive Tsallis statistics which has the general composition rule. Though the Tsallis statistics can be applied for the non-extensity nature of black hole, the empirical temperature through the zeroth law of thermodynamics cannot be well defined. To avoid this difficulty, the formal logarithm of the Tsallis entropy, which is the well known Rényi entropy, will be taken. The Bekenstein–Hawking entropy of black hole can be considered as the Tsallis entropy from which the Rényi entropy function of black hole takes the form

$$S_R = \frac{ln(1 + \lambda S_{BH})}{\lambda}, \tag{17}$$

which is additive with the Abe's composition rule. The Rényi temperature can be written as

$$\begin{aligned} T_R &= \frac{1}{\partial S_R/\partial E} = \frac{1}{\partial S_R/(C(M, q, r_h)\partial M)} \\ &= T_h(1 + \lambda S_{BH}). \end{aligned} \tag{18}$$

The Smarr formula i.e. the mass formula of Bardeen black hole in asymptotically flat space can be written as

$$M = \int_S (2T^\mu_\nu - T\delta^\mu_\nu)\xi^\nu_{(t)} d\Sigma_\mu + 2\Omega J + \frac{\kappa}{4\pi}A, \tag{19}$$

in which the surface gravity term is actually the temperature term $2TS$. From the discussion above, the relation of temperature and entropy between the Gibbs–Boltzmann statistics and the Rényi statistics is

$$T_h = \frac{T_R}{e^{\lambda S_R}},\ S_{BH} = \frac{e^{\lambda S_R}}{\lambda}. \tag{20}$$

From the Ref. [9], the temperature term can be expanded as

$$\begin{aligned} M &= 2T_h S_{BH} + \cdots \\ &= 2T_R S_R - \lambda T_R S_R^2 + \cdots + O(\lambda^2), \end{aligned} \tag{21}$$

where the S_R can be expanded as

$$S_R = \pi r_h^2 - \frac{\lambda\pi^2 r_h^4}{2} + O(\lambda^2). \tag{22}$$

The pressure P and the volume V will be defined by expanding the temperature term of M, which will be shown latter, and the $P - v$ criticality and the van der Waals like phase transition of the canonical ensemble will occur naturally.

Moreover, we can derive the corrected first law of regular black hole of Rényi statistics. Following Ref. [9], substituting Eq. (20) into the corrected first law of regular black hole Eq. (14), we obtain

$$\begin{aligned} dE = C(M, q, r_h)dM &= \frac{T_R}{e^{\lambda S_R}} d\left(\frac{e^{\lambda S_R} - 1}{\lambda}\right) + \Phi_H dq \\ &= T_R dS_R + \frac{\lambda\pi r_h^3(r_h^2 - 2q^2)}{8(r_h^2 + q^2)} d\lambda + \Phi_H dq + O(\lambda^2) \\ &\approx T_R dS_R + V dP + \Phi_H dq, \end{aligned} \tag{23}$$

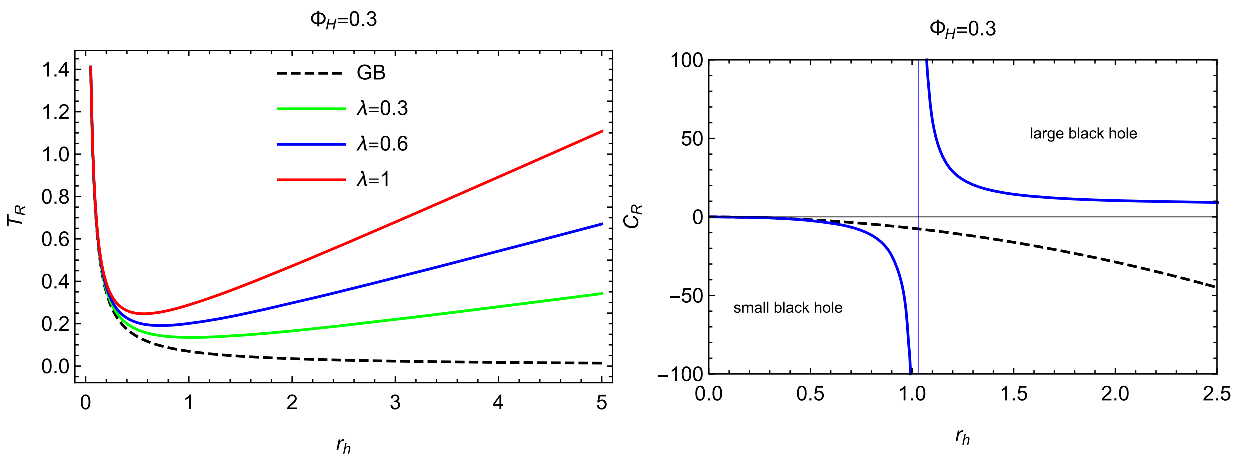

Fig. 1 Left: Rényi temperature of the Bardeen black hole T_R versus event horizon r_h at fixed magnetic potential $\Phi_H = 0.3$ for $\lambda = 0.3$ (blue), $\lambda = 0.6$ (green), and $\lambda = 1$ (red) comparing with the GB case i.e. $\lambda = 0$ (black dashed). Right: the heat capacity C_R of the Bardeen black hole versus r_h at fixed potential $\Phi = 0.3$ and $\lambda = 0.3$. The heat capacity is negative when $r_h < r_c$ and is positive when $r_h > r_c$ while it cannot be positive in the GB case i.e. $\lambda = 0$ (black dashed)

where $P = \frac{3\lambda(r_h^2-2q^2)}{32(r_h^2+q^2)}$, $V = \frac{4\pi r_h^3}{3}$. From the discussion above and comparing with Eq. (14), the E cannot be interpreted as the internal energy, but reinterpreted as the enthalpy H in the Rényi extended phase space. From the Legendre transformation $E_{in} = H - PV$, the corrected first law of regular black hole in Rényi statistics is

$$dE_{in} = T_R dS_R - PdV + \Phi_H dq, \tag{24}$$

where the E_{in} is the internal energy in Rényi statistics. The internal energy now be defined as $E_{in} = E - PV$.

4 Thermodynamics of Bardeen black hole from standard Gibbs–Boltzmann statistics with corrected first law of thermodynamics

The physical quantities of a black hole can be treated as thermodynamic quantities because of the analogy between black hole mechanics and the laws of thermodynamics. The thermodynamics of Bardeen black hole can be considered in the grand canonical ensemble and the canonical ensemble.

4.1 Grand canonical ensemble

In the grand canonical ensemble, the magnetic charge q of black hole changes with the surrounding heat bath, and the magnetic potential Φ_H can be set to be fixed. As mentioned in Sect. 2, the magnetic potential of Bardeen black hole can be written as

$$\Phi_H = \frac{3qr_h}{2\left(q^2 + r_h^2\right)}. \tag{25}$$

Consequently, the internal energy of black hole can be considered as

$$E = C(M, q, r_h)M = \frac{r_h}{2}, \tag{26}$$

and the Bekenstein–Hawking entropy is

$$S_{BH} = \frac{A}{4} = \pi r_h^2. \tag{27}$$

Other quantities can be calculated as

$$T_h = \frac{\sqrt{9 - 16\Phi_H^2} - 1}{8\pi r_h}, \tag{28}$$

$$C_{\Phi_H} = T_h\left(\frac{\partial S_H}{\partial T_h}\right)_{\Phi_H} = \left(\frac{\partial E}{\partial T_h}\right)_{\Phi_H} = -\frac{4\pi r_h^2}{\sqrt{9 - 16\Phi_H^2} - 1}, \tag{29}$$

$$G_H = E - T_h S_{BH} - q\Phi_H = \frac{1}{8} r_h \left(\sqrt{9 - 16\Phi_H^2} - 1\right), \tag{30}$$

where T_h is the Hawking temperature, C_{Φ_H} is the heat capacity at fixed magnetic potential Φ_H, and G_H is the Gibbs free energy. The black hole cannot be in stable because the heat capacity C_{Φ_H} is negative at arbitrary r_h. This can be seen from the dashed black line in Fig. 1.

4.2 Canonical ensemble

In the canonical ensemble we fixed the magnetic charge q of black hole. Considering two horizons meet, the mass and the magnetic charge of extremal black hole can be written as $M_e = 3\sqrt{3}q/4$, and the radius of the extremal black hole is

$r_e = \sqrt{2}q$. Consequently, the internal energy of black hole is

$$E = C(M, q, r_h)(M - M_e) = \frac{1}{4}r\left(2 - \frac{3\sqrt{3}qr^2}{\left(q^2 + r^2\right)^{3/2}}\right), \tag{31}$$

the Bekenstein–Hawking entropy is still $S_{BH} = \pi r_h^2$ and other quantities can be calculated as

$$T_h = \frac{r_h^2 - 2q^2}{4\pi r_h\left(q^2 + r_h^2\right)}, \tag{32}$$

$$C_q = T_h\left(\frac{\partial S_H}{\partial T_h}\right)_q = \left(\frac{\partial E}{\partial T_h}\right)_q = -\frac{2\pi r_h^2\left(r_h^2 + q^2\right)^2}{-7q^2r_h^2 + r_h^4 - 2q^4}, \tag{33}$$

$$F_H = E - T_hS_{BH} = \frac{q^2r}{q^2 + r^2} + \frac{r^3}{4\left(q^2 + r^2\right)} - \frac{3\sqrt{3}qr^3}{4\left(q^2 + r^2\right)^{3/2}}, \tag{34}$$

where the C_q is the heat capacity of black hole with fixed the magnetic charge q and F_H is the Helmoholtz free energy. From the heat capacity C_q, two branches of black hole can be founded, i.e., the one with positive heat capacity and the other with negative heat capacity. From $C_q > 0$, r_h can be solved as

$$\sqrt{2}q < r_h < \frac{\sqrt{\sqrt{57} + 7}q}{\sqrt{2}}. \tag{35}$$

The heat capacity will be negative when $r_h > \frac{\sqrt{\sqrt{57}+7}q}{\sqrt{2}}$, which means that at $r_h = \frac{\sqrt{\sqrt{57}+7}q}{\sqrt{2}}$ the capacity diverges and temperature gets maximum

$$T_{max} = \frac{1}{\sqrt{19\sqrt{57} + 143}\pi q}. \tag{36}$$

It is found that the phase transition of Bardeen black hole depends on the choice of ensemble, which may be resulted from the long-range gravitation and electromagnetic interactions of the system.

5 Thermodynamics of Bardeen black hole from Rényi statistics with corrected first law of thermodynamics

In this section, the thermodynamics of Bardeen black hole will be studied with the Rényi approach in the grand canonical ensemble (fixed potential) and the canonical ensemble (fixed charge).

5.1 Grand canonical ensemble

The thermodynamic quantities of the grand canonical ensemble will be calculated in the Rényi model firstly, then the state equation $P - v$ will be discussed.

As mentioned in Sect. 3, the Rényi temperature in the grand canonical ensemble can be obtained

$$T_R = \frac{\left(\sqrt{9 - 16\Phi_H^2} - 1\right)\left(1 + \lambda\pi r_h^2\right)}{8\pi r_h}, \tag{37}$$

with the above temperature, the heat capacity can be calculated

$$C_R = T_R\left(\frac{\partial S_R}{\partial T_R}\right)_{\Phi_H} = \left(\frac{\partial E}{\partial T_R}\right)_{\Phi_H} = \frac{4\pi r_h^2}{\left(\sqrt{9 - 16\Phi_H^2} - 1\right)\left(\pi\lambda r_h^2 - 1\right)}. \tag{38}$$

The heat capacity is negative when $r_h < r_c$ and is positive when $r_h > r_c$. For the critical point $r_c = \sqrt{1/\lambda\pi}$, C_R is divergent. So there are two possible black hole branches, which are the small black hole branch with negative heat capacity and the large black hole branch with positive heat capacity. Substituting r_c into Eq. (10), the critical mass can be obtained

$$M_c = \frac{3}{32}\sqrt{\frac{3}{2\pi}}\lambda\left(\frac{3 - \sqrt{9 - 16\Phi_H^2}}{\lambda\Phi_H^2}\right)^{3/2}, \tag{39}$$

which means that the Bardeen black hole in Rényi statistics is in the small black hole branch when $M < M_c$ and in the large black hole branch when $M > M_c$.

The heat capacity C_R is proportional to the inverse of the slope of the temperature T_R, we have

$$C_R = \frac{1}{2}\left(\frac{1}{\partial T_R/\partial r_h}\right). \tag{40}$$

The critical radius r_c is the extremum point of T_R i.e. $T_R'(r_c) = 0$. With this relation, the Bardeen black hole can be stable, which has positive heat capacity when $r_h > r_c$ and negative heat capacity when $r_h < r_c$. While there is no extremum point in the case of GB statistics.

We can also solve Eq. (37) for $r_h(T_R)$ function of temperature T_R and the minimum temperature can be obtained by solving $r_h(T_R) = r_c = 1/\sqrt{\lambda\pi}$

$$T_{R,min} = \frac{1}{4}\left(\frac{\sqrt{9\lambda - 16\lambda\Phi_H^2}}{\sqrt{\pi}} - \frac{\sqrt{\lambda}}{\sqrt{\pi}}\right), \tag{41}$$

and the minimum temperature $T_{R,min} = \frac{1}{2}\sqrt{\frac{\lambda}{\pi}}$ corresponds to the Schwarzschild case when $\Phi_H = 0$. It is also interesting

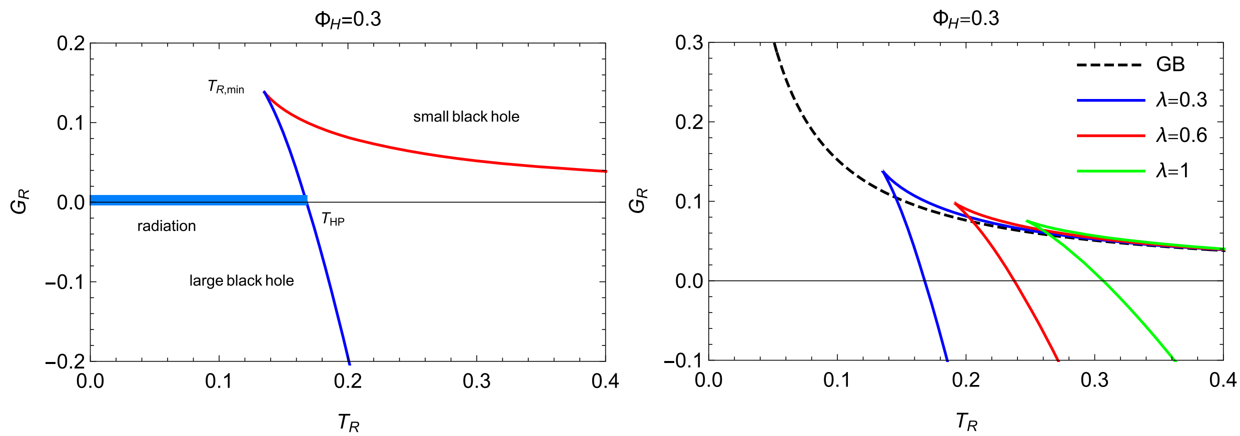

Fig. 2 Left: Gibbs free energy versus temperature for fixed Φ_H in the Rényi model. Right: Gibbs free energy vs. temperature with different nonextensive parameter $\lambda = 0.3$ (blue),0.6 (red),1 (green). The GB case is plotted with dashed black line

that the minimum temperature becomes larger with higher level of nonextensivity i.e. higher value of λ.

The Rényi statistics, in which black hole can be locally stable in asymptotically flat space with certain level of nonextensivity, gives different conclusions from GB statistics. There is no black hole phase when $T_R < T_{R,min}$, because the event horizon cannot exist due to the imaginary value $r_h(T_R)$. The hot thermal radiation in asymptotically flat space will collapse to a black hole with higher temperature than a minimum temperature i.e. $T_R > T_{R,min}$, which also occurs in the case of black hole in the AdS background from GB statistics.

In the grand canonical ensemble, the Gibbs free energy will be considered in discussing the global stability of the black hole. So in the Rényi model, the corresponding Gibbs free energy function should be written with Rényi entropy and temperature as

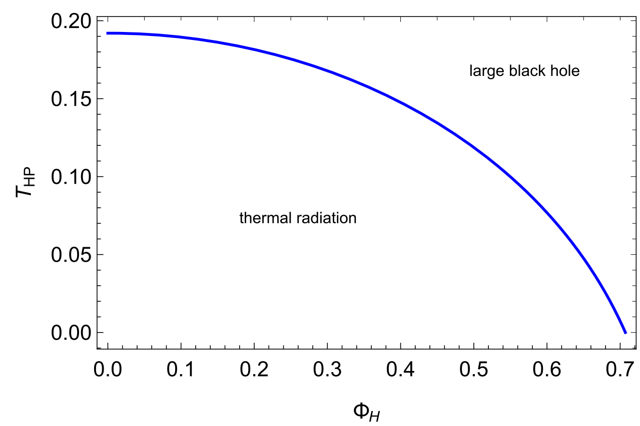

Fig. 3 The Hawking–Page temperature T_{HP} versus the magnetic potential Φ_H. The line corresponds to the Hawking–Page phase transition line which divides the phase space into two regions, the thermal radiation phase and the large black hole phase

$$G_R = E - T_R S_R - q\Phi_H = \frac{\left(\sqrt{r_h^2\left(9-16\Phi_H^2\right)} + r_h\left(4\Phi_H^2-3\right)\right)\left(\left(\pi\lambda r_h^2+1\right)\ln\left(\pi\lambda r_h^2+1\right) - 2\pi\lambda r_h^2\right)}{2\pi\lambda r_h\left(\sqrt{r_h^2\left(9-16\Phi_H^2\right)} - 3r_h\right)}, \tag{42}$$

The Fig. 2 shows that the Bardeen black hole is globally unstable in the GB case as its free energy is always positive. From Fig. 2, we can see the free energy of black hole will be negative when the temperature is high enough, so the black hole is global stable with the Rényi parameter λ. The Hawking–Page temperature T_{HP} can be obtained from $G_R = 0$

$$T_{HP} \approx \sqrt{\lambda}\left(0.17527\sqrt{9-16\Phi_H^2} - 0.17527\right), \tag{43}$$

which is about 1.24 times $T_{R,min}$.

The Hawking–Page temperature T_{HP} versus magnetic potential Φ_H is plotted in Fig. 3. The range of magnetic potential can be obtained from the horizon structure of Bardeen black hole mentioned in Sect. 2 and the extreme value of magnetic potential obtained when $q = 4\sqrt{3}M/9$. From Fig. 3, the Hawking-Page transition line is similar to the $T - \mu$ phase diagram of confining/deconfining phase transition at finite chemical potential. This shows that the properties of asymptotically flat black hole using Rényi statistics is similar to the asymptotically AdS black hole using GB statistics [9,11–13].

Now we turn to the cusp of G_R versus T_R, which occurs at $T_{R,min}$. The cusp corresponds to the small/large black hole phase transition. This cusp corresponds to a second-order phase transition because the heat capacity, which is the second order derivative of Gibbs free energy respect to temperature $C_R = -T_R(\frac{\partial^2 G_R}{\partial T_R^2})_{\Phi_H}$, is divergent. At the cusp, the

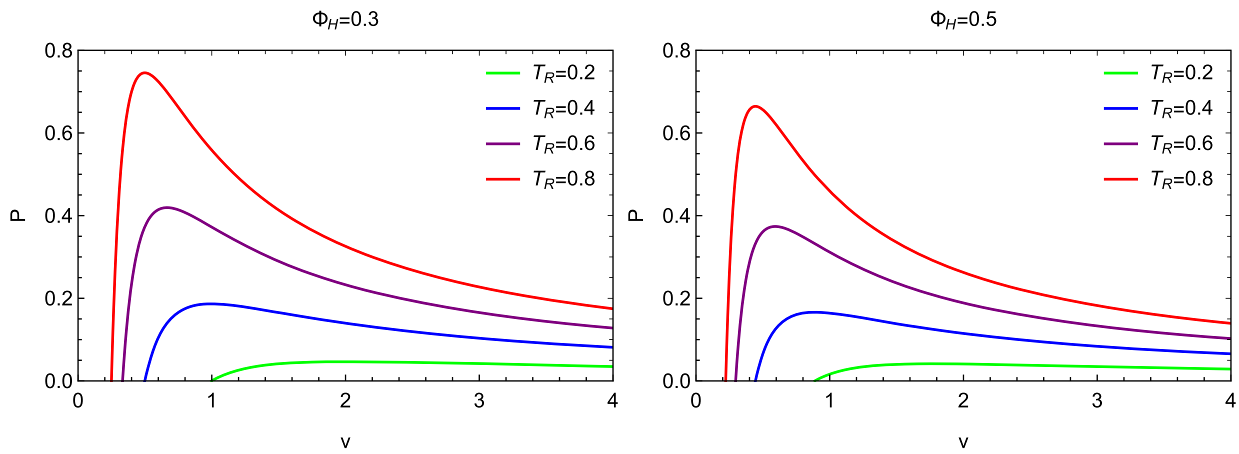

Fig. 4 $P-v$ diagram of the Bardeen black hole for fixed magnetic potential $\Phi_H = 0.3$ (left) and $\Phi_H = 0.5$ (right). There are two branches of the black hole for a giving temperature, one of which is small black hole phase (P increases with v) and another is the large black hole phase (P decreases with v)

critical Rényi entropy S_{Rc} and the critical Gibbs free energy G_{Rc} is

$$S_{Rc} = \frac{ln2}{\lambda},\ G_{Rc} = -\frac{(ln2-1)\left(\sqrt{9-16\Phi_H^2}-1\right)}{4\sqrt{\pi}\sqrt{\lambda}}. \tag{44}$$

Gibbs free energy has the maximum value G_{Rc} and the Rényi entropy is a constant with a fixed λ at the cusp point.

Then we focus on the $P-v$ phase. As mentioned in Sect. 3, the general Smarr formula with Rényi statistics can be written as

$$M = 2T_R S_R - \lambda T_R S_R^2 + \cdots + O(\lambda^2). \tag{45}$$

For the grand canonical ensemble, substituting the T_R and S_R into Eq. (45), we obtain

$$M = 2T_R S_R - \left(-\frac{\pi\lambda r_h^3\left(-4\Phi_H^2-\sqrt{9-16\Phi_H^2}+3\right)}{6-2\sqrt{9-16\Phi_H^2}}\right) + \cdots + O(\lambda^2), \tag{46}$$

with which the pressure P and the volume V can be defined as

$$P = -\frac{3\lambda\left(-4\Phi_H^2-\sqrt{9-16\Phi_H^2}+3\right)}{8\left(6-2\sqrt{9-16\Phi_H^2}\right)},\quad V = \frac{4\pi r_h^3}{3}, \tag{47}$$

and the general Smarr formula can be written as

$$M = 2T_R S_R - 2PV + \cdots. \tag{48}$$

It needs to be emphasized that the $V = 4\pi r_h^3/3$ is not the geometric spherical volume with the horizon radius r_h, but the thermodynamic volume which is the conjugate quantity of the pressure. The specific volume is $v = 8r_h/3$, not $v = 2r_h$ in AdS case, which allows us to get a better ratio Pv/T in Rényi statistics approaching to the standard van der Waals system [9]. With the specific volume, the equation of state can be written as

$$P = \frac{6\sqrt{6}\left(4\Phi_H^2+\sqrt{9-16\Phi_H^2}-3\right)T_R}{v\left(\sqrt{9-16\Phi_H^2}-3\right)\left(\sqrt{16\Phi_H^2+9}-6\right)\sqrt{\sqrt{16\Phi_H^2+9}+3}} + \frac{4\left(4\Phi_H^2+\sqrt{9-16\Phi_H^2}-3\right)}{3\pi v^2\left(\sqrt{9-16\Phi_H^2}-3\right)}. \tag{49}$$

The pressure P versus the specific volume v is plotted in Fig. 4 which shows that there is a maximum value of P. The maximum value P_{max} can be obtained by solving the condition $(\frac{\partial P}{\partial v})_{T_R,\Phi_H} = 0$. The corresponding critical volume for P_{max} is v_c. For a given temperature, there are two branches of black hole when $P < P_{max}$ which are small black hole branch with $v < v_c$ and the large black hole branch with $v > v_c$. For the large black hole branch, the pressure P decreases with the specific volume v, which means the thermodynamic stability for the large black hole branch. In contrary to the large black hole branch, the small black hole branch is unstable. The second derivative $(\frac{\partial^2 P}{\partial v^2})_{T_R,\Phi_H}$ at v_c is nonzero, which means that there is no critical behavior for Bardeen black hole in the grand canonical ensemble.

5.2 Canonical ensemble

The thermal properties of Bardeen black hole in the canonical ensemble will be investigated in this section. From Eqs. (18)

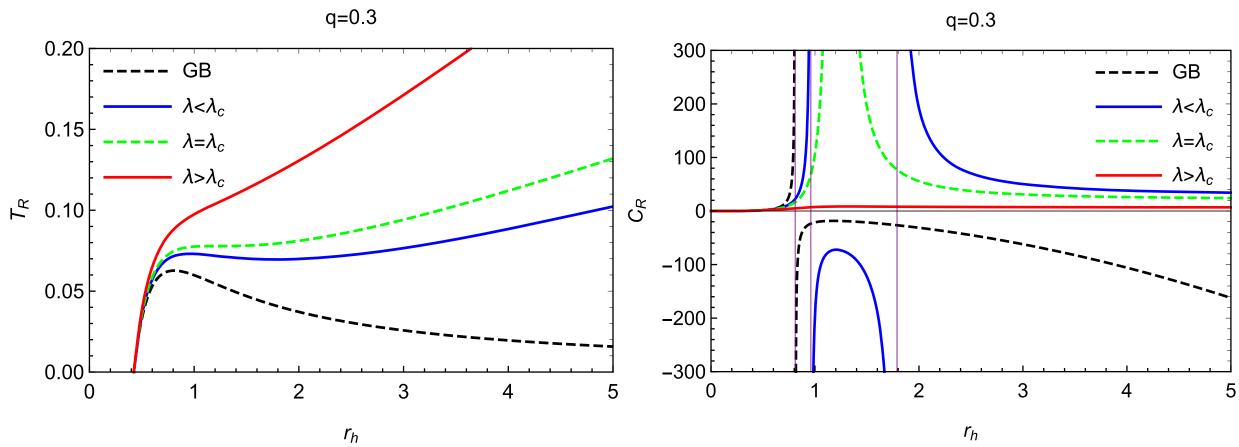

Fig. 5 Left: the Rényi temperature of the Bardeen black hole T_R versus event horizon r_h at fixed magnetic charge $q = 0.3$ with different values of the nonextensitive parameter λ. There are two turning points for $\lambda < \lambda_c$. Right: the heat capacity C_R of Bardeen black hole versus r_h at fixed $q = 0.3$ with different values of λ

and (32), the Hawking temperature via Rényi model is

$$T_R = \frac{\left(r_h^2 - 2q^2\right)\left(\pi\lambda r_h^2 + 1\right)}{4\pi r_h\left(r_h^2 + q^2\right)}, \tag{50}$$

as the function of event horizon r_h is plotted with fixed $q = 0.3$ and different values of λ in Fig. 5.

These temperature curves of Bardeen black hole in the Rényi model are similar to the uncharged black holes in AdS background. Interestingly, the critical phase transition will occur at different values of λ, which can be obtained with the condition

$$\left(\frac{\partial T_R}{\partial r_+}\right)_q = \left(\frac{\partial^2 T_R}{\partial r_+^2}\right)_q = 0. \tag{51}$$

For this condition, the critical horizon r_c, critical nonextensitive parameter λ_c and the critical Rényi temperature T_c can be numerically calculated

$$\begin{aligned} r_c &\approx 4.0552q, \\ \lambda_c &\approx \frac{0.00846318}{q^2}, \\ T_c &\approx \frac{0.0233533}{q}. \end{aligned} \tag{52}$$

When $\lambda < \lambda_c$, there are local maximum and local minimum in the temperature curve. When $\lambda = \lambda_c$, these two extremum points meet at the critical event horizon r_c.

The Rényi heat capacity is

$$\begin{aligned} C_R &= T_R\left(\frac{\partial S_R}{\partial T_R}\right)_q \\ &= \left(\frac{\partial E}{\partial T_R}\right)_q \\ &= \frac{2\pi r_h^2\left(r_h^2 + q^2\right)^2}{q^4\left(2 - 2\pi\lambda r_h^2\right) + q^2 r_h^2\left(5\pi\lambda r_h^2 + 7\right) + r_h^4\left(\pi\lambda r_h^2 - 1\right)}. \end{aligned} \tag{53}$$

Unlike the GB case, three branches of black hole are allowed in the canonical ensemble via Rényi statistics, two positive branches and one negative branch as shown in Fig. 5. It shows that when horizon r_h equals to the extremum points mentioned above, the heat capacity is divergent, which means there is a phase transition. The temperature at these two extremum points are $T_1 = 0.0695703$ and $T_2 = 0.0730847$ for the blue curve in Fig. 5.

In the canonical ensemble, it is better to consider the global stability of Bardeen black hole with Helmholtz free energy. The Helmholtz free energy in the Rényi statistics is

$$\begin{aligned} F_R &= E - T_R S_R \\ &= \frac{1}{4} r_h\left(2 - \frac{3\sqrt{3}qr_h^2}{\left(r_h^2 + q^2\right)^{3/2}}\right) \\ &\quad - \frac{\left(r_h^2 - 2q^2\right)\left(\pi\lambda r_h^2 + 1\right)\ln\left(\pi\lambda r_h^2 + 1\right)}{4\pi\lambda r_h\left(r_h^2 + q^2\right)}, \end{aligned} \tag{54}$$

where $E = C(M, q, r_h)(M - M_e)$.

Figure 6 shows the Helmholtz free energy versus the Rényi temperature of Bardeen black hole. The swallowtail behavior occurs only at $\lambda < \lambda_c$, which is similar to the van der Waals liquid/gas system.

The general Smarr formula in the Rényi model is

$$M = 2T_R S_R - \lambda T_R S_R^2 + \cdots + O(\lambda^2). \tag{55}$$

Substituting Eqs. (17) and (50) into Eq. (55) and holding the first order of λ, we have

$$M = 2T_R S_R - \frac{\pi\lambda r_h^3\left(r_h^2 - 2q^2\right)}{4\left(r_h^2 + q^2\right)} + \cdots \tag{56}$$

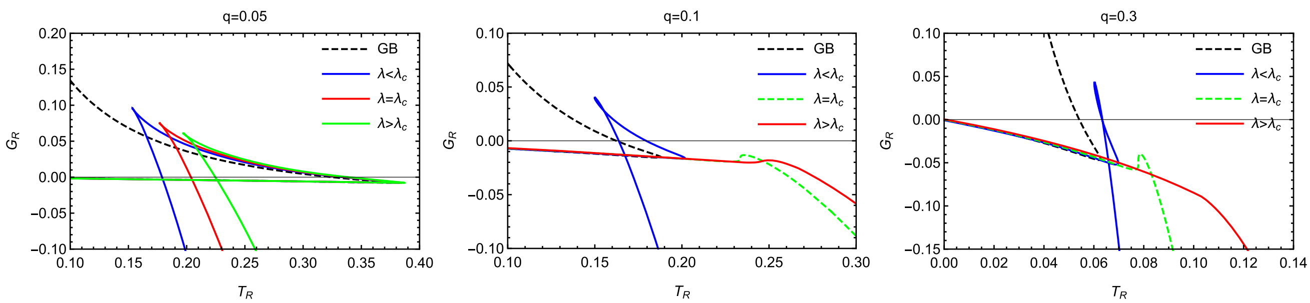

Fig. 6 The Rényi Helmholtz free energy F_R versus the Rényi temperature T_R with different magnetic charge q and different λ. The swallow-tail-shape occurs only at $\lambda < \lambda_c$. As the charge q increases, the swallow-tail-shape deviates from the standard shape, which may be resulted from the nonlinear magnetic charge of Bardeen black hole

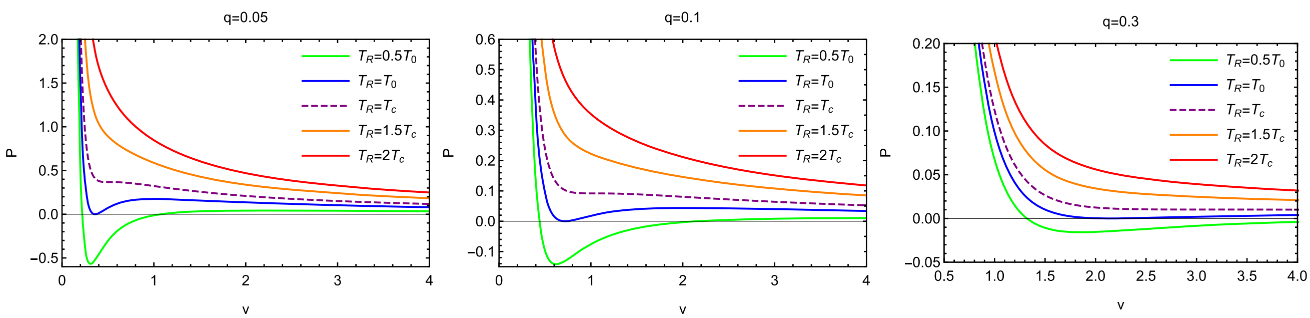

Fig. 7 The pressure P versus v is plotted with different temperatures. The dashed line represents the critical curve $T_R = T_c$. The solid curves below to the critical curve are the case of $T_R < T_c$, in which there are three branches of black hole, one unstable (P increases with v) and two stable (P decreases with v) branches. The solid curves above the dashed line are the case of $T_R > T_c$, which are analogous to the idea gas phase of van der Waals system

Defining the pressure P and the volume V as

$$P = \frac{3\lambda\left(r_h^2 - 2q^2\right)}{32\left(r_h^2 + q^2\right)}, \quad V = \frac{4\pi r_h^3}{3}, \tag{57}$$

the general Smarr formula can be written as

$$M = 2T_R S_R - 2PV + \cdots \tag{58}$$

Using the relation $v = 8r_h/3$, the equation of state can be obtained

$$P = \frac{T_R}{v} - \frac{18}{64\pi q^2 + 9\pi v^2} + \frac{4}{3\pi v^2}, \tag{59}$$

which the $P-v$ diagram will behavior like the van der Waals system as shown in Fig. 7. The critical point can be obtained with the conditions

$$\left(\frac{\partial P}{\partial v}\right)_{T_R,q} = 0, \ \left(\frac{\partial^2 P}{\partial v^2}\right)_{T_R,q} = 0. \tag{60}$$

The critical pressure P_c, specific volume v_c and temperature T_c can be numerically calculated

$$P_c \approx \frac{0.000917584}{q^2}, \quad v_c \approx 10.4641q, \quad T_c \approx \frac{0.0261711}{q}. \tag{61}$$

So, the critical compressibility factor is

$$\frac{P_c v_c}{T_c} \approx 0.366882, \tag{62}$$

which is independent on the black hole's mass and magnetic charge. Interestingly, this result is similar to the ratio of standard van der Waals fluid $\frac{3}{8}$.

The relation P versus v in Fig. 7 is same as van der Waals system. The critical isothermal curve at $T_R = T_c$ is plotted with purple dashed line. When $T_R < T_c$, there is a small black hole phase corresponding to the liquid phase and a large black hole phase corresponding to the gas phase. The small black hole and the large black hole phase are stable with positive compression coefficient (P decreases with v). There is a unstable part between the small black hole and large black hole phase, which has the negative compression coefficient (P increases with v) and stands for the mixture of liquid and gas phase. Solving the following conditions

$$P = 0, \left(\frac{\partial P}{\partial v}\right)_{T_R, q} = 0, \tag{63}$$

the minimum temperature T_0 can be obtained at $v_0 = \frac{4}{3}\sqrt{2\left(\sqrt{57}+7\right)}q$,

$$T_0 = \frac{1}{\sqrt{19\sqrt{57}+143}\pi q}. \tag{64}$$

When $T_0 < T < T_c$, there exists the phase transition similar to van der Waals system. The black hole will be like the ideal gas and there is no phase transition when the temperature is high enough $T > T_c$.

6 Conclusion and discussion

In this paper, to avoid the incompatible temperature of regular black hole obtained with different approaches, we introduced the corrected first law of thermodynamics and studied the thermodynamics of the Bardeen black hole in both the grand canonical ensemble and the canonical ensemble via Rényi statistics. In the grand canonical ensemble (fixed charge potential), the Bardeen black hole can be locally stable and there are small black hole and large black hole branches with the nonextensive parameter $0 < \lambda < 1$, while these two branches will not occur in GB statistics. Moreover, there is a Hawking–Page phase transition between the thermal radiation phase and the large black hole phase depending on the nonextensive parameter λ. In the canonical ensemble (fixed charge q): (1) when $\lambda < \lambda_c$ (critical nonextensive parameter), there is a small/large black hole phase transition (first order); (2) when $\lambda > \lambda_c$, the phase transition above will disappear and the large black hole phase will just be a possibility. These behaviors are similar to the cases of Bardeen black hole and corrected Bardeen black hole in AdS background via GB statistics (see [48–50]). Furthermore, by introducing the general Smarr formula via Rényi statistics, which the parameter λ plays the thermodynamic pressure P, we found that the thermal phase structure in the canonical ensemble just behaves as the van der Waals system. The same phase transition also occurs in the case of Bardeen-AdS black hole via GB statistics.

Both the nonextensive parameter λ in Rényi statistics and the cosmological constant Λ in AdS spacetime play as the thermodynamic pressure P. The analogy between the case of asymptotically flat Bardeen black hole via Rényi statistics and Bardeen-AdS black hole via GB statistics suggests that the nonextensive parameter λ should have some connections with the cosmological constant Λ. This connection may lead us to understand the nonextensive parameter λ more deeply.

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