Institute of Physics Publishing
Journal of Physics: Conference Series **24** (2005) 213–218
doi:10.1088/1742-6596/24/1/024
VI Mexican School on Gravitation and Mathematical Physics

S^3 Group-manifold Reduction of Gravity

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Abstract. In this contribution we exhibit a new consistent group-manifold reduction of pure Einstein gravity in the vielbein formulation when the compactification group manifold is S^3. The novel feature in the reduction is to consider the two 3-dimensional Lie algebras that S^3 admits. We discuss the characteristics of the lower-dimensional theory and we emphasize the results generated by the new group-manifold reduction. As an application we show that the lower-dimensional theory admits a domain wall solution which upon uplifting to the higher-dimension results to be the self-dual (in both curvature and spin connection) Kaluza-Klein monopole.

1. Introduction

Dimensional reductions of theories such as gravity, supergravities and extended objects are topics subject to an intensive research activity. A well known type of consistent dimensional reductions are the *group-manifold reductions* [1, 2]. These reductions are based on the fact that the parametrization for the metric and the other higher-dimensional fields are invariant under a transitively-acting group of isometries in the internal space.

In this contribution we are interested in dimensional reductions on Bianchi IX group manifolds, which are defined to be manifolds with an $SO(3)$ or $SU(2)$ isometry group acting on 3-surfaces. It turns out that both isometry groups are characterized locally by the same 3-dimensional Lie algebra [3]. However these groups are topologically different, $SU(2)$ is the double covering of $SO(3)$ and they correspond to S^3 and RP^3 respectively (RP^3 is S^3 with antipodal points identified).

If one performs the group-manifold reduction of pure Einstein gravity considering to the metric as the basic field, the space-time symmetry is the only symmetry that can be used in the reduction. The lower-dimensional theory obtained in this way is an Einstein-Maxwell-scalars gauged theory where the isometry group of the internal space becomes the gauge group of both the Maxwell fields and the scalars of the coset space. In the case of Bianchi IX group manifolds the gauged group of the lower-dimensional theory is either $SO(3)$ or $SU(2)$. For this reason in the literature this dimensional reduction is called $S^3 = SU(2)$ group-manifold reduction.

Although the metric formulation is appropriate for pure gravity, the presence of spinors requires the introduction of a longer set of variables. These are the vielbein fields which describe local orthonormal Lorentz frames at each space-time point and with respect to which the spinors are defined. In order to treat the group-manifold reduction in the general case it is therefore important to perform the reduction of the gravitational sector using the vielbein fields as basic variables. In this formulation gravity has two different local symmetries, the space-time symmetry and the tangent Lorentz symmetry. The standard group-manifold reduction of gravity in the vielbein formulation considers in the same way as in the metric formulation only the space-time symmetry [2]. The main point of this contribution is to show that in the case of the group manifold S^3, there does exists a special transformation in the internal tangent space that introduces non trivial differences into the group-manifold reduction of gravity [4]. We shall show that these differences are: a) a new term in the components of the spin connection with two internal indices and b) an additional gauge coupling for the scalars of the coset space. As an application of these results, we shall discuss the domain wall type solutions of the lower-dimensional theory at the level of the first order differential equations that emerge from the self-duality condition of the spin connection.

2. Parametrization of the vielbein

In this section we exhibit a parametrization of the vielbein in terms of lower-dimensional fields that considers besides the usual 3-dimensional Lie algebra associated to the general coordinate symmetry of the internal space another 3-dimensional Lie algebra associated to the tangent Lorentz symmetry [5]. In the following discussion we assume a $(D+3)$ split of the $(D+3)$ space-time coordinates $x^{\hat{\mu}} = (x^\mu, z^\alpha)$ where $\mu = \{0, 1, \ldots, D-4\}$ are the indices of the D-dimensional space-time and $\alpha = \{1, 2, 3\}$ are the indices of the internal coordinates. The corresponding flat indices of the tangent space are denoted by $\hat{a} = (a, m)$. The group indices are also denoted with the letters $m, n, \ldots,$.

The parametrization of the vielbein is

$$\hat{e}_{\hat{\mu}}{}^{\hat{a}}(x,z) = \begin{pmatrix} e^{c_1\varphi(x)} e_\mu{}^a(x) & e^{c_2\varphi(x)} A_\mu{}^\alpha(x,z)\, L_\alpha{}^p(x,z) \\ \\ 0 & e^{c_2\varphi(x)} L_\alpha{}^p(x,z) \end{pmatrix}, \tag{1}$$

where c_1 and c_2 are constants whose values are $c_1 = -\sqrt{3}/\sqrt{2(D+1)(D-2)}$ and $c_2 = -c_1(D-2)/3$ [1]. The A_μ's are gauge fields and $L_\alpha{}^p(x,z)$ is a 3×3 scalar matrix whose internal coordinate dependence are given by

$$A_\mu{}^\alpha(x,z) = A_\mu{}^m(x)\,(U^{-1}(z))_m{}^\alpha\,, \tag{2}$$

$$L_\alpha{}^p(x,z) = U_\alpha{}^m(z)\, L_m{}^n(x)\, \Lambda_n{}^p(z)\,. \tag{3}$$

The internal coordinate dependence related to the general coordinate symmetry appears via the matrix $U_\alpha{}^m(z)$, which is defined in terms of the left-invariant Maurer-Cartan 1-forms $\sigma^m \equiv dz^\alpha U_\alpha{}^m$, of a 3-dimensional Lie group. By definition these 1-forms satisfy the Maurer-Cartan equations $2d\sigma^m = -f_{np}{}^m \sigma^n \wedge \sigma^p$, where the $f_{mn}{}^p$ are independent of z^m and form the structure constants of the group-manifold

$$f_{mn}{}^p = -2(U^{-1}(z))_m{}^\alpha (U^{-1}(z))_n{}^\beta\, \partial_{[\alpha} U_{\beta]}{}^p(z)\,. \tag{4}$$

The corresponding Lie algebra is given in terms of the Killing vectors which represent the dual base to the Maurer-Cartan 1-forms ($\sigma^m \mathbf{K_n} = \delta^m{}_n$)

$$[\mathbf{K}_m, \mathbf{K}_n] = f_{mn}{}^p \mathbf{K}_p\,. \tag{5}$$

[1] The values of c_1 and c_2 ensure that the reduction of the Einstein-Hilbert action yields a pure Einstein-Hilbert term in D-dimensions, with no pre-factor involving the scalar φ, and that φ has a canonically normalized kinetic term in D-dimensions.

The vielbein parametrization in the standard group-manifold reduction considers internal coordinate dependence only via the matrix U [2]. The novel ingredient in the parametrization (3) is the introduction of the orthogonal matrix $\Lambda(z)$ which is taken in the *adjoint representation* of the 3-dimensional Lie algebra (5) [6]. The property of orthogonality indicates that $\Lambda(z)$ can be introduced as a transformation in the internal tangent space. Explicitly Λ is defined as

$$\Lambda(z) = e^{z^1 R_1} e^{z^2 R_2} e^{z^3 R_3}, \tag{6}$$

where the constant matrices R_m are the generators of $gl(3,\Re)$ and are given by the *adjoint representation* of the parameters of the internal transformations, $R_m = f_{mn}{}^p \mathbf{e}_p{}^n = ad_{\mathbf{K}}(\mathbf{K}_m)$. They satisfy the $SO(3)$ Lie algebra

$$[R_m, R_n] = f_{mn}{}^p R_p. \tag{7}$$

Additionally the quantities depending on the internal coordinates are related by the equation

$$(R_m)_n{}^p = (U^{-1}(z))_m{}^\alpha (\Lambda^{-1}(z))_n{}^q \partial_\alpha \Lambda_q{}^p(z)\,. \tag{8}$$

The group-manifold reduction works out because the internal coordinate dependence can be factored out in any geometrical quantity due to the fact that it always appears in one of the two possible combinations (4) or (8).

Using the vielbein parametrization (1)-(3) the $(D+3)$-dimensional interval is

$$ds^2_{D+3} = e^{2c_1\varphi} g_{\mu\nu} dx^\mu dx^\nu - e^{2c_2\varphi} \mathcal{M}_{mn} (dx^\mu A_\mu{}^m + \sigma^m)(dx^\nu A_\nu{}^n + \sigma^n), \tag{9}$$

where

$$\mathcal{M}_{mn}(x) \equiv -L_m{}^p(x) L_n{}^q(x)\, \eta_{pq}\,. \tag{10}$$

In general $L_m{}^n(x)$ describes the five dimensional $SL(3,\Re)/SO(3)$ scalar coset space. It transforms under a global $SL(3,\Re)$ acting from the left and a local $SO(3)$ symmetry acting from the right. By a gauge fixing of the SO(3) symmetry, it is possible to find an explicit representative of it [7, 8]. The matrix $\mathcal{M}_{mn}(x)$ represents the local $SO(3)$ invariant metric of the internal manifold. In particular for a Bianchi IX group manifold the Lie algebra (5) corresponds to the algebra of the maximal compact subgroup of $SL(3,\Re)$.

3. The D-dimensional action

The important quantity in the group-manifold reduction of the $(D+3)$-dimensional Einstein-Hilbert action is the spin connection $\hat\omega$. By using the parametrization (1)-(3) of the vielbein, the components of the $(D+3)$-dimensional spin connection are

$$\begin{aligned}
\hat\omega_{ab} &= \omega_{ab} - 2c_1 e^{-c_1\varphi} \hat e_{[a}\partial_{b]}\varphi + \frac{1}{2} e^{(c_2-2c_1)\varphi} F_{ab}{}^m L_{mn} e^n, \qquad (11)\\
\hat\omega_{am} &= (\Lambda^{-1})_m{}^n \left[e^{-c_1\varphi} e^p \left(c_2 \partial_a\varphi \eta_{pn} + (L^{-1})_{(p}{}^q \mathcal{D}_a L_{q|n)} \right) + \frac{1}{2} e^{(c_2-2c_1)\varphi} F_{ab}{}^p L_{pn} \hat e^b \right],\\
\hat\omega_{mn} &= (\Lambda^{-1})_m{}^p (\Lambda^{-1})_n{}^q \left[-\hat e^a e^{-c_1\varphi} (L^{-1})_{[p}{}^r \mathcal{D}_a L_{r|q]} \right.\\
&\qquad \left. + e^r e^{-c_2\varphi} \left(\mathcal{F}_{r[pq]} - \frac{1}{2}\mathcal{F}_{pqr} + (\mathcal{R}_r)_{pq} \right) \right].
\end{aligned}$$

In these expressions $\hat e^a = e^{c_1\varphi} e^a$ and $\hat e^m = e^{c_2\varphi}(A^n + \sigma^n) L_n{}^p(x) \Lambda_p{}^m \equiv e^p \Lambda_p{}^m$ are the $D+3$ components of the vielbein, $F^m = 2\partial A^m - f_{np}{}^m A^n A^p$ is the $SU(2)$ gauge vector field strength. The scalar functions $\mathcal{F}_{mnp}$ and $(\mathcal{R}_p)_{mn}$ are defined as

$$\mathcal{F}_{mnp} \equiv (L^{-1})_m{}^q (L^{-1})_n{}^r L_{sp} f_{qr}{}^s \quad \text{and} \quad (\mathcal{R}_p)_{mn} \equiv (L^{-1})_p{}^r (R_r)_{mn}\,, \tag{12}$$

whilst the covariant derivative of the scalar coset is given by

$$\mathcal{D}_\mu L_m{}^n = \partial_\mu L_m{}^n - A_\mu{}^p L_q{}^n f_{mp}{}^q + A_\mu{}^p L_m{}^q f_{qp}{}^n . \tag{13}$$

Notice that the covariant derivative of the scalar coset reflects the gauging of the two Lie algebras under consideration. The second term corresponds to the standard $SU(2)$ gauging of the internal coordinate symmetry whereas the third one corresponds to the $SO(3)$ gauging of the adjoint representation of the $SU(2)$ Lie algebra.

Using (11) the group-manifold reduction of the $(D+3)$-dimensional Einstein-Hilbert action leads to

$$S = C \int d^D x \sqrt{|g|}\, [\mathcal{R} + \tfrac{1}{4}\mathrm{Tr}\left(\mathcal{D}\mathcal{M}\mathcal{M}^{-1}\right)^2 + \tfrac{1}{2}(\partial\varphi)^2 - \tfrac{1}{4} e^{-\frac{2c_1}{3}(D+1)\varphi} F^m \mathcal{M}_{mn} F^n - \mathcal{V}] , \tag{14}$$

where $\mathcal{V}$ is the scalar potential

$$\mathcal{V} = \tfrac{1}{4} e^{\frac{2c_1}{3}(D+1)\varphi}\; [2\mathcal{M}^{mn} f_{mp}{}^q f_{nq}{}^p + \mathcal{M}^{mn}\mathcal{M}^{pq}\mathcal{M}_{rs} f_{mp}{}^r f_{nq}{}^s] , \tag{15}$$

and C the $SU(2)$ group volume. From the covariant derivative of the scalar coset (13), it is direct to compute the covariant derivative of the internal metric $\mathcal{M}$.

In conclusion, the two differences produced by apply the new group-manifold reduction with respect to the standard one [2] are reflected in the terms $(\mathcal{R}_p)_{mn}$ of $\hat{\omega}_{mn}$ and in the additional $SO(3)$ gauging of $L_m{}^n$. These differences are not manifest in the reduced action and therefore in the equations of motion neither. The reduced Lagrangian has the same functional form independently of the dimensional reduction used (either the standard group-manifold or the new group-manifold reduction). This conclusion is expected because the difference in the parametrization of the vielbein is a transformation in the tangent space. However the new group-manifold reduction has leaved its imprint in the internal components of the spin connection.

4. Domain wall solutions

After dimensional reduction the D-dimensional field content is $\{e_\mu{}^a, L_m{}^n, \varphi, A^m\}$. The 5-dimensional scalar coset $L_m{}^n$ contains two dilatons and three axions. An explicit representation of $L_m{}^n$ in terms of the five scalars can be found in [7, 8]. In order to simplify the discussion is convenient to consider the following consistent truncated parametrization of the scalar coset

$$L_m{}^n(x) = \mathrm{diag}(a e^{-c_2\varphi}, b e^{-c_2\varphi}, c e^{-c_2\varphi}), \tag{16}$$

where we have set the axions to zero and the relation of the three scalar functions a, b, c with the two dilatons of the scalar coset and the dilaton φ is

$$a \equiv e^{c_2\varphi - \frac{\sigma}{\sqrt{3}}}, \qquad b \equiv e^{c_2\varphi + \frac{\sigma}{2\sqrt{3}} - \frac{\phi}{2}}, \qquad c \equiv e^{c_2\varphi + \frac{\sigma}{2\sqrt{3}} + \frac{\phi}{2}}. \tag{17}$$

We are interested in solutions of cohomogeneity one also known as domain wall solutions. These are solutions of the theory in the truncation $A_\mu = 0$ that depend only on one spatial coordinate y orthogonal to the compactification manifold, hence we take the following ansatz

$$ds_D^2 = f^2(y) dx_{(D-1)}^2 - g^2(y) dy^2, \qquad a = a(y),\; b = b(y),\; c = c(y). \tag{18}$$

It turns out that by taking

$$f(y) = (abc)^{-c_1/3c_2} \quad \text{and} \quad g(y) = (abc)^{(3c_2 - c_1)/3c_2} , \tag{19}$$

the $D+3$ non-trivial equations of motions of the higher-dimensional action are reduced to three equations of motion and one constraint [4].

In the variables (17), the D-dimensional interval (18) can be rewritten as

$$ds_D^2 = (abc)^{-2c_1/3c_2} dx_{(D-1)}^2 - (abc)^{(6c_2-2c_1)/3c_2} dy^2, \tag{20}$$

and upon uplifting, the $(D+3)$-dimensional space-time is of the form $\Re^{D-2,1} \times M_4$, explicitly

$$ds_{D+3}^2 = dx_{D-1}^2 - ((abc)^2 dy^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2). \tag{21}$$

The D-dimensional domain wall solutions (20) and the Bianchi IX metrics M_4 are completely given by the three positive scalar functions $a(y)$, $b(y)$ and $c(y)$. The solutions describe cohomogeneity one self-dual solutions (in the curvature R_{IJ}) to the 4-dimensional Euclidean Einstein gravity in empty space. It was found that the self-duality condition of the curvature gives origin to second order differential equations of motion that accept two different sets of first integrals [9]. Each set consists of the three equations

$$2\frac{\partial_y a}{a} = -a^2 + b^2 + c^2 - 2\lambda bc\,, \qquad \text{and cyclic.} \tag{22}$$

If $\lambda = 0$ the set of equations is known as the BGPP system [10], whereas if $\lambda = 1$, the set of equations is known as the Atiyah-Hitchin system [11]. The BGPP system can be obtained directly without integration by demanding that the spin connection 1-forms of the metric M_4 in the basis $(abcdy, a\sigma^1, b\sigma^2, c\sigma^3)$ be self-dual [10]. This parametrization is the one used in the standard group-manifold reduction. When the three invariant directions are different, i.e. $a \neq b \neq c$ the BGPP system admits the BGPP metrics as solutions [10] whilst when two of them are equal i.e. $(a = b \neq c)$ admit the Eguchi-Hanson metrics as solutions [12, 13].

If we apply the new S^3 group-manifold reduction we have six independent non-vanishing components of the spin connection $(\hat{\omega}_{ym},\ \hat{\omega}_{mn})$. By require self-duality in these components of the spin connection (11) we get the Atiyah-Hitchin first order system. When two of the tree invariant directions are equal i.e. $(a = b \neq c)$ this system admits the Taub-NUT family of metrics as solutions [9].

We have a clear picture of the relation between the two different Bianchi type IX group-manifold reductions and the domain wall type solutions of the reduced theory. Because the equations of motion are the same in both cases, the domain wall solutions coincide as well. However from the first order differential equations point of view, the solutions are divided into two disjoint sets. One of these sets is given by the metrics that solve the BGPP system ($\lambda = 0$ in (22)) and the another one by the metrics that solve the Atiyah-Hitchin system ($\lambda = 1$ in (22)). If we reduce applying the standard group-manifold reduction the domain walls that solve the BGPP system are self-dual in both the curvature and the spin connection whereas that the metrics in the another set of solutions are self-dual only in the curvature. If instead we reduce applying the new S^3 group-manifold reduction the conclusion is the opposite. The possibility of relate the different first-order systems with the inclusion (or not) of the matrix Λ was already noticed in [5, 14].

It is well known that one of the Eguchi-Hanson metrics and one of the Taub-NUT metrics are the only complete non-singular $SO(3)$ hyper-Kähler metrics in four dimensions [9], both of them are obtained in the case in which two of the invariant directions are equal. From the $(D+3)$-dimensional point of view these two solutions correspond to $\Re^{D-2,1} \times M_4$ with either M_4 the Eguchi-Hanson metric [12] whose generic orbits are RP^3 [10] or the self-dual Taub-NUT solution whose generic orbits are S^3 [15]. In the latter case, the complete $(D+3)$-metric is known as the Kaluza-Klein monopole [16, 17].

5. Discussion

A possible application of the results of this paper is in the context of 8-dimensional supergravities. The $SO(3)$ 8-dimensional gauged supergravity obtained by apply the standard group-manifold reduction to the 11-dimensional supergravity has 1/2 BPS domain wall solutions that satisfy the BGPP system [8, 7]. This happens because the equations that are obtained by require a self-dual spin connection of M_4 are exactly the same that the ones obtained by require a 1/2 BPS solution to the supersymmetry transformation rules. The uplifted solutions are 1/2 BPS except for an especial case which uplift to 11-dimensional flat space and hence becomes fully supersymmetric (it corresponds to have equal invariant directions $a = b = c$). A disturbing fact is that the Kaluza-Klein monopole is also 1/2 BPS in 11-dimensions, however if one reduce it applying the standard group-manifold reduction the supersymmetry in 8-dimensions becomes fully broken. This happens because in the vielbein parametrization of the standard group-manifold reduction this solution does not has self-dual spin connection in 8-dimensions. We believe the results of this paper open the possibility to construct a 8-dimensional gauged supergravity by apply the new S^3 group-manifold reduction to the 11-dimensional supergravity and the hope is that this gauged supergravity owns a 1/2 BPS domain-wall solution which upon uplifting becomes the 11-dimensional Kaluza-Klein monopole. This possibility is currently under research.

Acknowledgments

We wish to thank the organizers of the Sixth Mexican School on Gravitation and Mathematical Physics in 2004 for the organization of the very nice school and the opportunity to give a presentation. We are also grateful to Eric Bergshoeff, Ulf Gran, Hugo Morales, Mikkel Nielsen, Tomás Ortín and Diederik Roest for useful discussions at different stages of this work.

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