Holography and Entanglement Entropy

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Abstract

We review our recent formulation [1, 2] of computing entanglement entropy in a holographic way. The basic examples can be found by applying AdS/CFT correspondence and the holographic formula has successfully been checked in many examples of conformal field theories. We also explain the covariant formulation of holographic entanglement entropy which is closely related to the covariant entropy bound (Bousso bound) in an interesting way.

1 Introduction

In gravitational theories, the degree of freedom which is contained in a given region A is not proportional to the volume, but to the area of its boundary ∂A. This is because if we put a lot of materials inside A, then they eventually make a black hole and this gives the upper bound of the allowed entropy in A. In this way, the property of the gravitational theory is rather different from the familiar systems described by the law of quantum mechanics, where the entropy is extensive. This suggests that the true degree of freedom in a $d+2$ dimensional gravity is actually equally described by that of a $d+1$ dimensional quantum manybody system. This is known as the holographic principle [3]. This idea has been played crucial role in the recent development of the string theory, especially in the context of AdS/CFT correspondence [4]. The AdS/CFT relates the $d+2$ dimensional anti-de Sitter spacetime to a $d+1$ dimensional conformal field theory (CFT).

However, the holography in other spaces such as the de Sitter spacetime has not been studied well. This is because there is no simple way to realize such spaces in string theory, though in principle we can find (slightly complicated) examples for e.g. de-Sitter space [5]. Therefore it is intriguing and helpful to explore a general principle of holography which may allow us to find the holographic dual for any spacetime without relying on explicit examples in string theory. For this purpose, it is a nice idea to find a universal physical observables by which we can formulate the holographic principle generally. Clearly, the correlation functions, which are often quoted and studied in AdS/CFT, are not suitable for this aim, since we need to specify which operators we consider and thus we need to know the precise spectrum or field contents of the dual theory.

The purpose of this talk is to present a candidate of such a useful quantity. We claim that the quantity called entanglement entropy, which can be defined in any quantum mechanical systems, is a universal physical observable in holography. We will explain how the entanglement entropy in quantum field theories (QFTs) is related a certain geometrical quantity in the dual gravity background. In the first half, we assume that the spacetime is static for simplicity, where the entropy is time-independent. In the latter half, we extend the result in the static case to the time-dependent backgrounds by presenting a covariant formulation of holographic entanglement entropy. As will explain later, this construction has an interesting connection to the covariant entropy bound known as the Bousso bound.

This article is organized as follows. In section 2, we will offer an basic definition and properties of entanglement entropy. In section 3, we review the holographic calculation of entanglement entropy in a static spacetimes. In section 4, we consider its generalization to time-dependent spacetimes by looking at the covariant formulation. In section 5, we summarize the conclusions and discuss future problems.

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2 Definition and Properties of Entanglement Entropy

2.1 Definition

In order to define the entanglement entropy, we first divide a given quantum mechanical (manybody) system into two parts (or subsystems) A and B. Accordingly, the total Hilbert space is factorized as

$$H = H_A \otimes H_B. \tag{1}$$

A simple example is a spin chain, which is artificially divided into the left and right part of sites. Next we introduce the reduced density matrix

$$\rho_A = \mathrm{Tr}_B \rho, \tag{2}$$

for the subsystem A by tracing out H_B. ρ is the density matrix of the original system. Indeed, ρ_A is the density matrix when we consider an operator which only depends on the information of H_A. Finally, the entanglement entropy is defined as the von-Neumann entropy for ρ_A i.e.

$$S_A = -\mathrm{Tr}\rho_A \log \rho_A. \tag{3}$$

Notice that even if the total density matrix ρ is that of pure state (i.e. the entropy of ρ is vanishing), still we get a non-vanishing entropy $S_A > 0$ (except that A and B are totally decoupled) since we traced out B and this leads to some ambiguity of information, which is measured by the von-Neumann entropy S_A.

2.2 Basic Properties

Here we summarize the basic properties.

First of all, the entanglement entropy is not an extensive quantity and because of this it has a rather different property than the familiar thermal entropy. But, if we consider the high temperature limit, the entanglement entropy S_A includes a extensive part which is equal to the thermal entropy in A.

Let us assume that the total system is described by a pure state e.g. the system at zero temperature. Then we can show $S_A = S_B$ in a straightforwardly. However, this is violated when ρ is a mixed state.

It is also useful to consider the case where we divide the system in many parts. Especially assume that the Hilbert space is factorized as $H = H_A \otimes H_B \otimes H_C \otimes H_D$. Then we can show the inequality known as the strong subadditivity (see e.g. [6] for a review)

$$S_{A+B+C} + S_B \leq S_{A+B} + S_{B+C}. \tag{4}$$

This has been known to be the most strong constraint which the entanglement entropy should satisfy and it can be derived from the positivity of the norm of Hilbert space. By setting B to zero, (4) is reduced to the subadditivity $S_{A+C} \leq S_A + S_C$. The strong subadditivity represents the concave property of the von-Neumann entropy. For example, in [7], it has been shown that the strong subadditivity applied to 2D CFTs leads to the entropic version of the c-theorem.

2.3 Various Applications

The entanglement entropy has been played important roles in various areas in physics. First of all, it is a crucial quantity in the research of quantum information theory and quantum computation. In this context, the entanglement entropy measures the amount of quantum information [8].

Also recently it has been employed as a quantum order parameter in condense matter systems such as a spin systems, quantum Hall liquid and so on [9, 10]. Especially, it is expected that it can distinguish different quantum vacua such as the presence of anyons when the low energy limit is described by a topological field theory. Notice that in such a topological theory, the correlation functions behave trivial and are not useful. Also in the numerical simulation of quantum many body systems using the density matrix renormalization[2], the entanglement entropy measures the obstruction of the numerical simulation by approximating the degree of freedom by finite size matrices. Thus we expect that it diverges at the quantum phase transition point and this is the reason why the entanglement entropy plays the role of an order parameter.

[2]Roughly speaking this is a quantum version of the method of compressing information.

2.4 Entanglement Entropy in QFT and Area Law

Consider a QFT on a $d+1$ dimensional manifold $R_t \times N$, where R_t and N denote the time direction and the d dimensional space-like manifold, respectively. We define the subsystem by a d dimensional submanifold $A \subset N$ at fixed time $t = t_0$. We call its complement the submanifold B. The boundary of A, which is denoted by ∂A, divides the manifold N into two submanifolds A and B. Then we can define the entanglement entropy S_A by (3). Sometimes this kind of entropy is called geometric entropy as it depends on the geometry of the submanifold A. Since the entanglement entropy is always divergent in a continuum theory, we introduce an ultraviolet cut off a (or a lattice spacing). Then the coefficient in front of the divergence turns out to be proportional to the area of the boundary ∂A of the subsystem A as first pointed out in [11],

$$S_A = \gamma \cdot \frac{\text{Area}(\partial A)}{a^{d-1}} + \text{subleading terms}, \tag{5}$$

where γ is a constant which depends on the system. This behavior can be intuitively understood since the entanglement between A and B occurs at the boundary ∂A most strongly. This result (5) was originally found from numerical computations [11] and checked in many later arguments.

The simple area law (5), however, does not always describe the scaling of the entanglement entropy in generic situations. As we will discuss in details in the next subsections, the entanglement entropy of 1D quantum systems at criticality scales logarithmically with respect to the linear size l of A, $S_A \sim \frac{c}{3}\log l/a$ where c is the central charge of the CFT that describes the critical point.

Before we proceed to further analysis of entanglement entropy, it might be interesting to notice that this area law (5) looks very similar to the Bekenstein-Hawking entropy (BH entropy) of black holes which is proportional to the area of the event horizon

$$S_{BH} = \frac{\text{Area of horizon}}{4G_N}, \tag{6}$$

where G_N is the Newton constant. Intuitively, we can regard S_A as the entropy for an observer who is only accessible to the subsystem A and cannot receive any signals from B. In this sense, the subsystem B is analogous to the inside of a black hole horizon for an observer sitting in A, i.e., outside of the horizon. Indeed, this similarity was an original motivation of considering the entanglement entropy in QFTs [12, 11]. An important motivation of our holographic calculations of the entanglement entropy is actually to explain this similarity from the holographic viewpoint.

2.5 Explicit Computations in 2D CFT

In order to find the entanglement entropy, we first evaluate $\text{tr}_A\, \rho_A^n$, differentiate it with respect to n and finally take the limit $n \to 1$ (remember that ρ_A is normalized such that $\text{tr}_A\, \rho_A = 1$)

$$S_A = \lim_{n\to 1} \frac{\text{tr}_A\, \rho_A^n - 1}{1-n} \tag{7}$$

$$= -\frac{\partial}{\partial n}\text{tr}_A\, \rho_A^n|_{n=1} = -\frac{\partial}{\partial n}\log \text{tr}_A\; \rho_A^n|_{n=1}. \tag{8}$$

This is called the replica trick. Therefore, what we have to do is to evaluate $\text{tr}_A\; \rho_A^n$ in our 2D system. The first line of the above definition (7) without taking the $n \to 1$ limit defines the so-called Tsallis entropy, $S_{n,\text{Tsallis}} = \frac{\text{tr}_A\, \rho_A^n - 1}{1-n}$. [3]

This can be done in the path-integral formalism as follows. We first assume that A is the single interval $x \in [u, v]$ at $t_E = 0$ in the flat Euclidean coordinates $(t_E, x) \in R^2$. The ground state wave function Ψ can be found by path-integrating from $t_E = -\infty$ to $t_E = 0$ in the Euclidean formalism

$$\Psi\left(\phi_0(x)\right) = \int_{t_E=-\infty}^{\phi(t_E=0,x)=\phi_0(x)} D\phi\; e^{-S(\phi)}, \tag{9}$$

[3]The Tsallis entropy is related to the alpha entropy (Rényi entropy) $S_\alpha = \frac{\log \text{tr}_A\, \rho_A^\alpha}{1-\alpha}$ through $S_{\alpha,\text{Tsallis}} = \frac{1}{1-\alpha}[e^{(1-\alpha)S_\alpha} - 1]$. The $\alpha \to 1$ and $\alpha \to \infty$ limits of the alpha entropy give the von Neumann entropy and the single-copy entanglement entropy, respectively.

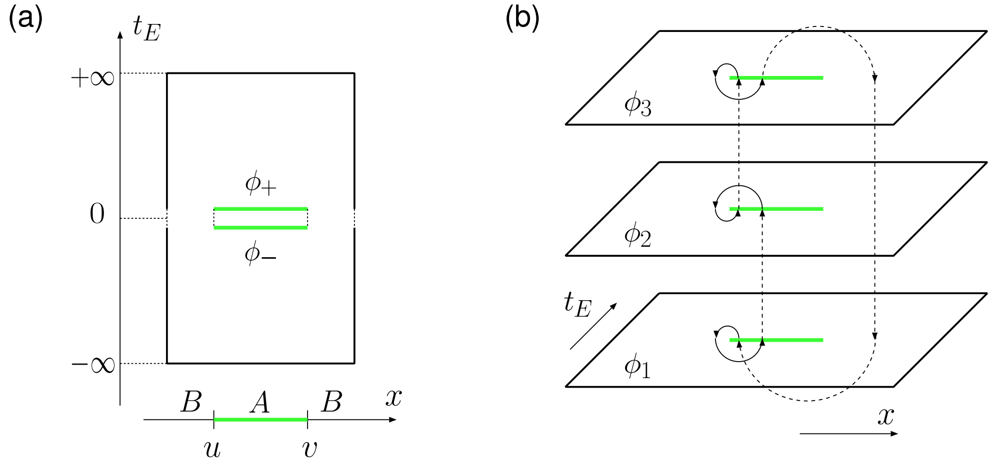

Figure 1: (a) The path integral representation of the reduced density matrix $[\rho_A]_{\phi_+\phi_-}$. (b) The n-sheeted Riemann surface $\mathcal{R}_n$. (Here we take $n=3$ for simplicity.)

where $\phi(t_E, x)$ denotes the field which defines the 2D CFT. The values of the field at the boundary ϕ_0 depends on the spacial coordinate x. The total density matrix ρ is given by two copies of the wave function $[\rho]_{\phi_0\phi_0'} = \Psi(\phi_0)\bar{\Psi}(\phi_0')$. The complex conjugate one $\bar{\Psi}$ can be obtained by path-integrating from $t_E = \infty$ to $t_E = 0$. To obtain the reduced density matrix ρ_A, we need to integrate ϕ_0 on B assuming $\phi_0(x) = \phi_0'(x)$ when $x \in B$.

$$[\rho_A]_{\phi_+\phi_-} = (Z_1)^{-1} \int_{t_E=-\infty}^{t_E=\infty} D\phi\ e^{-S(\phi)} \prod_{x\in A} \delta\left(\phi(+0,x) - \phi_+(x)\right) \cdot \delta\left(\phi(-0,x) - \phi_-(x)\right), \tag{10}$$

where Z_1 is the vacuum partition function on R^2 and we multiply its inverse in order to normalize ρ_A such that $\mathrm{tr}_A\,\rho_A = 1$. This computation is sketched in Fig. 1 (a).

To find $\mathrm{tr}_A\,\rho_A^n$, we can prepare n copies of (10)

$$[\rho_A]_{\phi_{1+}\phi_{1-}}[\rho_A]_{\phi_{2+}\phi_{2-}} \cdots [\rho_A]_{\phi_{n+}\phi_{n-}}, \tag{11}$$

and take the trace successively. In the path-integral formalism this is realized by gluing $\{\phi_{i\pm}(x)\}$ as $\phi_{i-}(x) = \phi_{(i+1)+}(x)$ $(i = 1, 2, \cdots, n)$ and integrating $\phi_{i+}(x)$. In this way, $\mathrm{tr}_A\,\rho_A^n$ is given in terms of the path-integral on an n-sheeted Riemann surface $\mathcal{R}_n$ (see Fig. 1 (b))

$$\mathrm{tr}_A\,\rho_A^n = (Z_1)^{-n} \int_{(t_E,x)\in\mathcal{R}_n} D\phi\ e^{-S(\phi)} \equiv \frac{Z_n}{(Z_1)^n}. \tag{12}$$

To evaluate the path-integral on $\mathcal{R}_n$, it is useful to introduce replica fields. Let us first take n disconnected sheets. The field on each sheet is denoted by $\phi_k(t_E, x)$ $(k = 1, 2, \cdots, n)$. In order to obtain a CFT on the flat complex plane C which is equivalent to the present one on $\mathcal{R}_n$, we impose the twisted boundary conditions

$$\phi_k(e^{2\pi i}(w-u)) = \phi_{k+1}(w-u), \quad \phi_k(e^{2\pi i}(w-v)) = \phi_{k-1}(w-v), \tag{13}$$

where we employed the complex coordinate $w = x + it_E$. Equivalently we can regard the boundary condition (13) as the insertion of two twist operators $\Phi_n^{+(k)}$ and $\Phi_n^{-(k)}$ at $w = u$ and $w = v$ for each ($k-$th) sheet. Thus we find

$$\mathrm{tr}_A\,\rho_A^n = \prod_{k=0}^{n-1} \langle \Phi_n^{+(k)}(u)\Phi_n^{-(k)}(v)\rangle. \tag{14}$$

When ϕ is a real scalar field, this is a non-abelian orbifold. To make the situation simple, assume that ϕ is a complex scalar field. Then we can diagonalize the boundary condition by defining n new fields $\tilde{\phi}_k = \frac{1}{n}\sum_{l=1}^{n} e^{2\pi i lk/n}\phi_l$. They obey the boundary condition

$$\tilde{\phi}_k(e^{2\pi i}(w-u)) = e^{2\pi ik/n}\tilde{\phi}_k(w-u), \quad \tilde{\phi}_k(e^{2\pi i}(w-v)) = e^{-2\pi ik/n}\tilde{\phi}_k(w-v). \tag{15}$$

Thus in this case we can conclude that the system is equivalent to $n-$disconnected sheets with two twist operators $\sigma_{k/n}$ and $\sigma_{-k/n}$ inserted in the $k-$th sheet for each values of k. In the end we find

$$\mathrm{tr}_A\, \rho_A^n = \prod_{k=0}^{n-1}\langle \sigma_{k/n}(u)\sigma_{-k/n}(v)\rangle \sim (u-v)^{-4\sum_{k=0}^{n-1}\Delta_{k/n}} = (u-v)^{-\frac{1}{3}(n-1/n)}, \tag{16}$$

where $\Delta_{k/n} = -\frac{1}{2}\left(\frac{k}{n}\right)^2 + \frac{1}{2}\frac{k}{n}$ is the (chiral) conformal dimension of $\sigma_{k/n}$. When we have m such complex scalar fields we simply obtain

$$\mathrm{tr}_A\, \rho_A^n = \prod_{k=0}^{n-1}\langle \sigma_{k/n}(u)\sigma_{-k/n}(v)\rangle \sim (u-v)^{-\frac{c}{6}(n-1/n)}, \tag{17}$$

setting the central charge $c = 2m$.

To deal with a general CFT with central charge c, we need to go back to the basis (13). The paper [13] showed that the result (17) is generally correct. The argument is roughly as follows. Define the coordinate z as follows

$$z = \left(\frac{w-u}{w-v}\right)^{\frac{1}{n}}. \tag{18}$$

This maps $\mathcal{R}_n$ to the z-plane C. In this simple coordinate system we easily find $\langle T(z)\rangle_C = 0$. Via Schwartz derivative term in the conformal map we obtain a non-vanishing value of $\langle T(w)\rangle_{\mathcal{R}_n}$. From that result, we can learn that twist operators $\Phi_n^{\pm(k)}$ in (14) have conformal dimension $\Delta_n = \frac{c}{24}(1-n^{-2})$. Thus we find the same result (17) for general CFTs as follows from (14).

Applying the formula (8) to (17), we find[4] the famous result [14]

$$S_A = \frac{c}{3}\,\log\frac{l}{a}, \tag{19}$$

where a is the UV cut off (or lattice spacing) and we set $l \equiv v - u$.

By applying appropriate conformal maps, we can compactify a direction of the two dimensional flat space. If we do so in the space direction, after some computations we find the entanglement entropy on a circle with the length L [13] as follows

$$S_A = \frac{c}{3}\log\left(\frac{L}{\pi a}\sin\frac{\pi l}{L}\right), \tag{20}$$

where $l < L$ is the length of the subsystem A.

On the other hand, if we periodically identify the (Euclidean) time direction, we get the result at finite temperature $T = \beta^{-1}$ [13]

$$S_A = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\sinh\frac{\pi l}{\beta}\right). \tag{21}$$

3 Holographic Entanglement Entropy for Static Spacetime

3.1 The Setup of Holography

As we have reviewed we can define the entanglement entropy S_A in any QFTs for each choice of the boundary ∂A. In this sense we always have infinite different quantities for a given QFT. Even though

[4]Here we neglect a constant term which does not depend on l, L and a.

in two dimensional CFT they can be analytically computed by using the conformal map method as in [13], the calculations in higher dimensional QFTs or CFTs are generally complicated and difficult. Nevertheless, we expect that the entanglement entropy play the role of order parameter of quantum phase transitions and it is quite useful if we can compute this quantity explicitly in a strongly coupled theories.

For this purpose, the holographic dual computation, if it exists, will be very useful because we expect that a quantum physical observable in the QFT side corresponds to a certain classical geometrical quantity in the dual gravity theory as is so in the AdS/CFT. Therefore we would like to consider the holographic calculation of the entanglement entropy in QFTs in this section.

The arguments below are not necessarily restricted to the setup of AdS/CFT correspondence, but we consider a rather general setup of the holography.

We will work in the general setup of holography where the (quantum) gravity in the bulk $d+2$ dimensional spacetime M is dual to a QFT on its $(d+1)$ dimensional boundary ∂M. If we stick to the AdS/CFT correspondence, M is the asymptotically AdS spacetime and the gravity on M is dual to a QFT with a UV fixed point defined on the boundary ∂M.

We assume that the spacetime M is static to make the argument simple. We will later discuss general time-dependent cases in the next section. Then we can express M as $M = R_t \times N$, where the d dimensional manifold N represents the time slice and R_t is the time direction. Also on the boundary we have $\partial M = R_t \times \partial N$.

3.2 Holographic Entanglement Entropy

To define the entanglement entropy, we divide the time slice N into A and B as we explained before. Since we are interested in the bulk gravity dual calculation, we would like to somehow extend this division to the bulk spacetime M. Our principle is as follows; as is clear in the area law of entanglement entropy, the boundary ∂A is the most physically important object. So we extend ∂A to a surface γ_A in the entire M such that $\partial\gamma_A = \partial A$. Notice that this is a surface in the time slice N, which is a Euclidean manifold. Of course, there are infinitely many different choices of γ_A. We claim that we have to choose the minimal area surface among them. This is uniquely determined and we call this γ_A below.

We are now in a position to present our holographic formula. We argue that the holographic entanglement entropy is simply given by

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+2)}}, \tag{22}$$

where $G_N^{(d+2)}$ is the Newton constant in the gravity theory on M. The above formula is reminiscent of the Bekenstein-Hawking formula of black hole entropy, though in our case γ_A is no longer than a horizon.

Indeed, we can motivate our formula (22) from the following intuitive argument. The holography relates the bulk gravity to a non-gravitational theory on its boundary. Thus we expect that a part of the bulk corresponds to the information of a certain region in the boundary. In our setup, we relate the information includes B in the boundary theory, whose amount is measure by S_A, to the bulk region defined by the one inside γ_A. The reason why we take the minimal area surface is that we are applying the idea of the entropy bound and we are trying to find the most strict bound. This part will be discussed in detail in the next section.

If we restrict to the AdS/CFT setup, we can formally derive the holographic formula (22) from the bulk to boundary relation (GKPW relation) [15] as shown in [16]. As we have explained in the previous section, the computation of $Tr\rho_A^n$, whose derivative about n in the limit $n \to 1$ leads to the entropy S_A, is equivalent to that of the partition function on the n-copied of the original manifold with the cut along ∂A. In other words, the manifold is defined by putting the negative deficit angle $2\pi(1-n)$ on the original spacetime. Following the AdS/CFT, what we have to do is to extend this geometry on the boundary toward the bulk region. We assume that the deficit angle surfaces extends to the entire the bulk AdS. This is denoted by γ_A. Then the Ricci scalar behaves like a delta function

$$R = 4\pi(1-n)\delta(\gamma_A). \tag{23}$$

Then we plug this in the gravity action

$$S_{AdS} = -\frac{1}{16\pi G_N^{(d+2)}} \int_M dx^{d+2}\sqrt{g}(R+\Lambda) + \cdots, \tag{24}$$

where we only make explicit the bulk Einstein-Hilbert action. Other parts which come from the boundary terms and the other fields contributions do not affect our computation here.

The basic principle of AdS/CFT i.e. the bulk to boundary relation [15] equates the partition function of CFT with the one of AdS gravity. Thus we can holographically calculate the entanglement entropy S_A as follows

$$S_A = -\frac{\partial}{\partial n}\log Tr\rho_A^n|_{n=1} = -\frac{\partial}{\partial n}\left[\frac{(1-n)\mathrm{Area}(\gamma_A)}{4G_N^{d+2}}\right]_{n=1} = \frac{\mathrm{Area}(\gamma_A)}{4G_N^{d+2}}. \tag{25}$$

This reproduces our holographic formula (22). The action principle in the gravity theory requires that γ_A is the minimal area surface.

Finally we would like to point that this holographic formulation assumes the existence of non-trivial minimal surfaces. In the spacetime with a warp factor as in AdS spaces, we expect this property. We think this is an interesting constraint on the spacetime which has a holographic interpretation.

3.3 Many Evidences for the Holographic Formula

Since the above arguments are pretty formal and assume the AdS/CFT correspondence, we need to check explicitly this claim by comparing both sides directly. Indeed, several different checks have been made until now and they have turn out to be all successful. In this subsection we would like to give a very brief overview of these agreements.

- The area law (5) known in QFT can be easily reproduced holographically. The warp factor in the AdS space leads to the UV divergence of the dual CFT [1]. Since the leading contribution to the area of γ_A comes from the region near the boundary, it should be proportional to the area of the boundary i.e. ∂A. This leading divergence of the area clearly scales as $\sim a^{-(d-1)}$ for AdS_{d+2}, which indeed agrees with the area law.
- We find perfect agreements in the lowest dimensional case of the AdS_3/CFT_2 setup [1]. In this case γ_A is a geodesic line which connects the two points which define the division into A and B. It is also possible to show that the entanglement entropy at finite temperature can be reproduced from the geodesics length in the BTZ black holes. These arguments will be reviewed in the next subsection.
- Though in the higher dimension, it is not easy to calculate the entanglement entropy in QFTs analytically, still we can show the semi-qualitative agreements between the CFT and AdS calculations. In particular, for the logarithmic terms of the entropy we can show the precise agreement as its coefficient is proportional to a linear combination of central charges. For details, refer to the second paper of [1].
- In the presence of a horizon, the minimal surface γ_A tends to wrap the (apparent) horizon. Then the wrapped part gives a extensive contribution to the holographic entanglement entropy. This agrees with the fact that the entanglement entropy includes the thermal part and we know that thermal entropy is dual to the black hole entropy which is given by the Bekenstein-Hawking area formula. In other words, our holographic formula generalizes the black hole entropy formula.
- We can holographically derive the strong subadditivity (4) in a very simple way [17] (see also [6]).
- If we apply the holographic formula to the AdS_2/CFT_1 setup, which comes from the near horizon limit of 4D or 5D extremal black holes, then it reproduces the Wald entropy formula in the presence of the higher derivative correction to the Einstein-Hilbert action [18].
- The holographic formula is nontrivially consistent with the covariant entropy bound (Bousso bound). This will be discussed in the next section.

3.4 Holographic Entanglement Entropy in AdS_3/CFT_2

Here we present a detailed analysis of the holographic entanglement entropy in AdS_3/CFT_2. According to AdS/CFT correspondence [4], the gravitational theories on this space are dual to $1+1$ dimensional conformal field theories with the central charge [19]

$$c = \frac{3R}{2G_N^{(3)}}, \tag{26}$$

where $G_N^{(3)}$ is the Newton constant in three dimensional gravity. In the global coordinate, the metric of AdS_3 becomes

$$ds^2 = R^2(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\theta^2). \tag{27}$$

At the boundary $\rho = \infty$ of the AdS_3, the metric is divergent. To regulate relevant physical quantities we need to put a cutoff ρ_0 and restrict the space to the bounded region $\rho \le \rho_0$. This procedure corresponds to the ultra violet (UV) cutoff in the dual conformal field theory. If we define the dimensionless UV cutoff δ ($\propto$ *length*), then we find the relation $e^{\rho_0} \sim \delta^{-1}$. In the example of the previous section, δ should be identified with

$$e^{\rho_0} \sim \delta^{-1} = L/a. \tag{28}$$

Remember that L is the total length of the system and a is the lattice spacing (or UV cutoff). Notice that there is actually an ambiguity about the $\mathcal{O}(1)$ numerical coefficient in this relation[5].

In the global coordinate of AdS_3 (27), the $1+1$ dimensional spacetime, in which the CFT_2 is defined, is identified with the cylinder (t,θ) at the (regularized) boundary $\rho = \rho_0$. Then we consider the AdS dual of the setup of computing the entanglement entropy. The subsystem A corresponds to $0 \le \theta \le 2\pi l/L$ and we can discuss the entanglement entropy by applying our proposal (22). In this lowest dimensional example, the minimal surface γ_A, which plays the role of the holographic screen [3, 20], becomes one dimensional. In other words, it is the geodesic line which connects the two boundary points at $\theta = 0$ and $\theta = 2\pi l/L$ with t fixed (see Fig. 2) .

Then to find the entropy we calculate the length of the geodesic line γ_A. The geodesics in AdS_{d+2} spaces are given by the intersections of two dimensional hyperplanes and the AdS_{d+2} in the ambient $R^{2,d+1}$ space such that the normal vector at the points in the intersections is included in the planes. The explicit form of the geodesic in AdS_3, expressed in the ambient $\vec{X} \in R^{2,2}$ space, is

$$\vec{X} = \frac{R}{\sqrt{\alpha^2-1}}\sinh(\lambda/R)\cdot\vec{x} + R\left[\cosh(\lambda/R) - \frac{\alpha}{\sqrt{\alpha^2-1}}\sinh(\lambda/R)\right]\cdot\vec{y}, \tag{29}$$

where $\alpha = 1 + 2\sinh^2\rho_0\sin^2(\pi l/L)$; x and y are defined by

$$\begin{aligned}\vec{x} &= (\cosh\rho_0\cos t, \cosh\rho_0\sin t, \sinh\rho_0, 0),\\ \vec{y} &= (\cosh\rho_0\cos t, \cosh\rho_0\sin t, \sinh\rho_0\cos(2\pi l/L), \sinh\rho_0\sin(2\pi l/L)).\end{aligned} \tag{30}$$

The length of the geodesic can be found as

$$\text{Length} = \int ds = \int d\lambda = \lambda_*, \tag{31}$$

where λ_* is defined by

$$\cosh(\lambda_*/R) = 1 + 2\sinh^2\rho_0\ \sin^2\frac{\pi l}{L}. \tag{32}$$

Assuming that the UV cutoff energy is large $e^{\rho_0} \gg 1$, we can obtain the entropy (22) as follows (using (26))

$$S_A \simeq \frac{R}{4G_N^{(3)}}\log\left(e^{2\rho_0}\sin^2\frac{\pi l}{L}\right) = \frac{c}{3}\log\left(e^{\rho_0}\sin\frac{\pi l}{L}\right). \tag{33}$$

Indeed, this entropy exactly coincides with the known 2D CFT result (20), including the (universal) coefficients after we remember the relation (28).

[5]However, this ambiguity does not affect universal quantities which do not depend on the cut off a and we will consider such quantities in the later arguments.

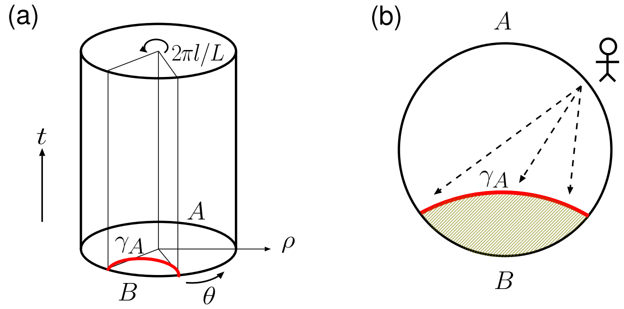

Figure 2: (a) AdS_3 space and CFT_2 living on its boundary and (b) a geodesics γ_A as a holographic screen.

It may be useful to repeat the similar analysis in the Poincare coordinates of AdS_3 $ds^2 = \frac{R^2}{z^2}(-dt^2 + dz^2 + dx^2)$. We pickup the spacial region (again call A) $-l/2 \leq x \leq l/2$ and consider its entanglement entropy. We can find the geodesic line γ_A between $x = -l/2$ and $x = l/2$ for a fixed time t_0

$$(x, z) = \frac{l}{2}(\cos s, \sin s), \quad (\epsilon \leq s \leq \pi - \epsilon). \tag{34}$$

The infinitesimal ϵ is the UV cutoff and leads to the cutoff z_{UV} as $z_{UV} = \frac{l\epsilon}{2}$. Since $e^\rho \sim x^i/z$ near the boundary, we find $z \sim a$. The length of γ_A can be found as

$$\text{Length}(\gamma_A) = 2R \int_\epsilon^{\pi/2} \frac{ds}{\sin s} = -2R \log(\epsilon/2) = 2R \log \frac{l}{a}. \tag{35}$$

Finally the entropy can be obtained as follows

$$S_A = \frac{\text{Length}(\gamma_A)}{4G_N^{(3)}} = \frac{c}{3} \log \frac{l}{a}. \tag{36}$$

This again agrees with the well-known result (19) as expected.

Next we consider how to explain the entanglement entropy at finite temperature $T = \beta^{-1}$ from the viewpoint of AdS/CFT correspondence. Since we assumed that the spacial length of the total system L is infinite, we have $\beta/L \ll 1$. In such a high temperature circumstance, the gravity dual of the conformal field theory is described by the Euclidean BTZ black hole [22]. Its metric looks like

$$ds^2 = (r^2 - r_+^2)d\tau^2 + \frac{R^2}{r^2 - r_+^2}dr^2 + r^2 d\varphi^2. \tag{37}$$

The Euclidean time is compactified as $\tau \sim \tau + \frac{2\pi R}{r_+}$ to obtain a smooth geometry. We also impose the periodicity $\varphi \sim \varphi + 2\pi$. By taking the boundary limit $r \to \infty$, we find the relation between the boundary CFT and the geometry (37)

$$\frac{\beta}{L} = \frac{R}{r_+} \ll 1. \tag{38}$$

The subsystem for which we consider the entanglement entropy is given by $0 \leq \varphi \leq 2\pi l/L$ at the boundary. Then by applying our proposal (22), the entropy can be computed from the length of the space-like geodesic starting from $\varphi = 0$ and ending to $\varphi = 2\pi l/L$ at the boundary $r = \infty$ for a fixed time. To find the geodesic line, it is useful to remember that the Euclidean BTZ black hole at temperature T is equivalent to thermal AdS_3 at temperature $1/T$. If we define the new coordinates

$$r = r_+ \cosh\rho, \quad \tau = \frac{R}{r_+}\theta, \quad \varphi = \frac{R}{r_+}t, \tag{39}$$

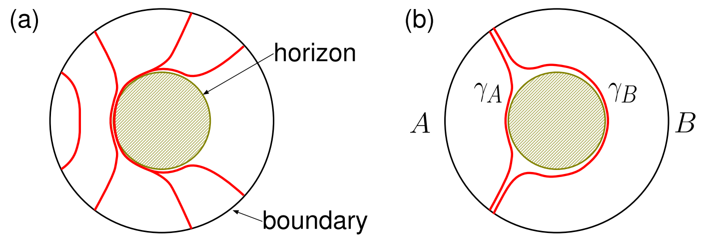

Figure 3: (a) Minimal surfaces γ_A in the BTZ black hole for various sizes of A. (b) γ_A and γ_B wrap the different parts of the horizon.

then the metric (37) indeed becomes the one in the Euclidean Poincare coordinates with t replaced by it. Now the computation of the geodesic line is parallel with what we did just before. We only need to replace $\sinh\rho$ and $\sin t$ with $\cosh\rho$ and $\sinh t$. In the end we find (31) with λ_* is now given by

$$\cosh\left(\frac{\lambda_*}{R}\right) = 1 + 2\cosh^2\rho_0 \sinh^2\left(\frac{\pi l}{\beta}\right), \tag{40}$$

where we took into account the UV cutoff $e^{\rho_0} \sim \beta/a$. Then our area law (22) precisely reproduces the known CFT result (21).

It is also intriguing to understand these calculations geometrically. The geodesic line in the BTZ black hole takes the form shown in Fig. 3(a). When the size of A is small, it is almost the same as the one in the ordinary AdS_3. As the size becomes large, the turning point approaches the horizon and eventually, the geodesic line covers a part of the horizon. This is the reason why we find a thermal behavior of the entropy when $l/\beta \gg 1$ i.e. $S_A \sim \frac{\pi c l}{3\beta}$. The thermal entropy in a conformal field theory is dual to the black hole entropy in its gravity description via the AdS/CFT correspondence. In the presence of a horizon, it is clear that S_A is not equal to S_B (remember B is the complement of A) since the corresponding geodesic lines wrap different parts of the horizon (see Fig. 3(b)). This is a typical property of entanglement entropy at finite temperature as we mentioned in section 2.2.

Also as shown recently in [18], when A is very closed to the total system, γ_A is divided into two pieces, the circle which wraps the horizon and the one localized at the boundary. This leads the precise relation between the entanglement entropy on the circle S_A and the BTZ black hole entropy S_{BH}

$$\lim_{l\to 0} \left(S_A(l) - S_A(L-l)\right) = S_{BH}, \tag{41}$$

where again L is the total length of the boundary.

4 Covariant Holographic Entanglement Entropy and Covariant Entropy Bound

4.1 Covariant Entropy Bound

So far we have only discussed static spacetimes. However, it is much more interesting to consider holography in a time-dependent spacetime as eventually we would like to understand cosmological backgrounds such as the de-Sitter space from a holographic viewpoint. Here we assume that there is a time-like boundary where the metric diverges as is so in the time-dependent asymptotically AdS spaces.

In the previous argument, we assumed a time slice on which we can define minimal surfaces since its signature is Euclidean. However, in our time-dependent case there is no longer a natural choice of the time-slices as we have infinitely many different ways of defining the time slices. Thus we need to consider the entire Lorentzian spacetime. Then we are in a trouble since in Lorentzian geometry there is no minimal area surface as the area vanishing if the surface extends in the light-like direction. In order to resolve this issue, let us remember an analogous problem; the covariant entropy bound so called the Bousso bound.

In general, if we get heavy objects together in a small region and continue to bring another one into the region, this system eventually experiences the gravitational collapse. Therefore we have a upper bound of the mass and entropy which can be included inside of the surface Σ. The bound for the entropy in flat space time is called the Bekenstein bound and it is given by

$$S_\Sigma \leq \frac{\text{Area}(\Sigma)}{4G_N}, \tag{42}$$

where Σ is a codimension two closed surface in the spacetime. It is also more interesting to generalize this bound to any time-dependent backgrounds like the cosmological ones. This requires to find a covariant description. It is obvious that the Bekenstein bound (42) is not covariant since the definition of the entropy included inside Σ is not covariant but depends on the choice of the time slice. The covariant entropy bound was eventually formulated by Bousso [20] and it is given by

$$S_{L(\Sigma)} \leq \frac{\text{Area}(\Sigma)}{4G_N}. \tag{43}$$

The light-like manifold $L(\Sigma)$ is called the light-sheet of Σ. This is defined by the manifold which is generated by the null geodesics starting from the surface Σ. We require that the expansion θ of the null geodesic is non-positive $\theta \leq 0$. In the flat spacetime, this is just a half of light-cone and the same is true for the AdS spacetime as it is conformally flat. Then the quantity $S_{L(\Sigma)}$ means the entropy which pass through the light sheet $L(\Sigma)$, which is covariantly well-defined. One more interesting thing of the Bousso bound is that we can apply the bound even if the surface Σ has boundaries, which is quite useful in the holographic setup as we employ below.

4.2 Covariant Holographic Entanglement Entropy

Now we would like to return to our original question of the covariant holographic entanglement entropy. Our final claim [2] is given by

$$S_A(t) = \frac{\text{Area}(\gamma_A(t))}{4G_N^{d+2}}, \tag{44}$$

where $\gamma_A(t)$ is the extremal surface in the entire Lorentzian spacetime M with the boundary condition $\partial\gamma_A(t) = \partial A(t)$. The time t is the time on the time slice in the boundary ∂M and there is no unique way to extend it to the bulk spacetime M.

This covariant formula (44) has been originally motivated from the Bousso bound (43) in [2]. To see let us remember the fact that the AdS/CFT correspondence with a UV cut off $z > a$ can be regarded as a brane-world setup (RS2 [21]). Since we assume that the cut off is close to the UV $a << R$, the gravity on the $d+1$ dimensional brane theory is very weak as

$$\frac{1}{G_N^{brane}} \sim \frac{R^d}{G_N^{bulk}} \int_a^\infty \frac{dz}{z^d} = \frac{R^d}{(d-1)a^{d-1}} \frac{1}{G_N^{bulk}} >> \frac{R}{G_N^{bulk}}, \tag{45}$$

where we assume the standard metric

$$ds^2 = R^2 \frac{dz^2 + g_{ij}(x)dx^i dx^j}{z^2}, \tag{46}$$

where g_{ij} is the metric on the brane.

Now we would like to ask what is the Bousso bound on the brane gravity theory (see fig.4 in the simplest case of AdS_3/CFT_2). We expect that the brane theory with quantum corrections taken into account is dual to the bulk gravity theory which is classical, based on the standard idea of AdS/CFT correspondence. Therefore we argue that the quantum corrected Bousso bound on the brane can be found as the classical Bousso bound on the brane. First we start with the setup of Bousso bound at the boundary ∂M. We pick up a (closed) surface $\partial\Sigma$ which separates a time slice into the subsystem A and B such that $\partial A = \partial\Sigma$. Now we define the light-sheet for Σ. We consider both the future and past directed ones and call them $\partial L^+(\Sigma)$ and $\partial L^-(\Sigma)$. The reason why we put the symbol ∂ is that we are

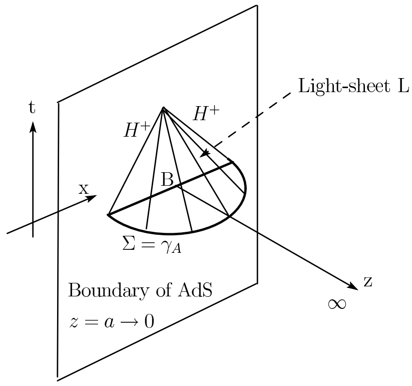

Figure 4: The setup of Bousso bound applied to the AdS_3/CFT_2 in the Poincare coordinate $ds^2 = \frac{R^2}{z^2}(-dt^2+dz^2+dx^2)$. In this simplified case, the future Cauchy horizon H^+ coincides with the future light-sheet $\partial L^+(\Sigma)$. In the above figure we only write the future light-sheet and not the past one just for simplicity.

interested in their bulk extensions $L^{\pm}(\Sigma)$. Again there are infinitely many different ways of extending the boundary light-sheets toward the bulk. We define the surface Σ by the intersection $L^+(\Sigma) \cap L^-(\Sigma)$. For each of such a Σ, we get the Bousso bound (43).

Here the condition of non-positive expansions of the null geodesics on the light-sheets i.e. $\theta^{\pm} \leq 0$ come into play. If there were not this condition we can choose arbitrary Σ and we can take them to be light-like. However, the condition is rather strong enough that the area of allowed Σ takes a non-trivial minimum and therefore we can define an analogue of the minimal surface in this Lorentzian spacetime. The minimum of the area corresponds to the most strict Bousso bound for a given boundary surface $\partial\Sigma$ or equally the choice of the subsystem A.

This minimum of the area occurs when the expansions on the two light-sheets are both vanishing $\theta^{\pm} = 0$. This condition is actually equal to the statement that the surface Σ is an extremal surface again called γ_A, which is defined by the saddle point of the area functional in the Lorentzian spacetime.

The final observation is that the quantum Bousso bound on the brane will be saturated by the entanglement entropy. This is motivated by the fact that the entanglement entropy represents a thermal entropy plus quantum corrections and that it is defined by assuming that the subsystem B is completely smeared, which will be expected to lead to the maximal entropy allowed in the region. If we assume this, then we immediately reach the holographic entanglement entropy formula (44).

Before we conclude, let us discuss an example where we can apply the above covariant formula. We consider the AdS Vaidya solution

$$ds^2 = -(r^2 - m(v))dv^2 + 2dvdr + r^2 d\phi^2. \tag{47}$$

This is the solution to the Einstein equation with the negative cosmological constant in the presence of null matter whose EM tensor looks like $T_{vv} = \frac{1}{2r}\frac{dm(v)}{dv}$. The null energy condition requires $T_{vv} \geq 0$ and thus we find that $m(v)$ is a monotonically increasing function of the (light-cone) time v.

This background is asymptotically AdS_3 and if we assume that $m(v)$ is a constant, then it is equivalent to the static BTZ black hole [22] with the mass m. Thus our background (47) describes an idealized collapse of a radiating star in the presence of negative cosmological constant. The dual theory is expected to be a CFT in a time-dependent background. The time-dependence comes from the time-dependent temperature. We can now apply the covariant entanglement entropy formula (44) and in the end we find

$$S_A(v) = \frac{c}{3}\left[\log\frac{l}{a} + \frac{m(v)l^2}{6} + \cdots\right], \tag{48}$$

as the expansion of small $m(v)$. The null energy condition guarantees that this is a monotonically increasing function of time. This shows that the entanglement entropy in this background is a monotonically increasing function of time as is so in the second law of the thermal entropy. We believe this behavior of entanglement entropy in black hole formation processes is rather general. However, we would like to stress that we are not claiming that the entanglement entropy is always increasing. For example if we start with the system with maximally entangled, the entanglement entropy will decrease after a small perturbation due to the de-coherence phenomenon.

We would also like to mention that if we stay with the brane-world setup we mentioned before and consider the brane-world black hole, then the holographic formula (44) tells us that the quantum corrected entropy of the black hole on the brane is equal to the entanglement entropy in the same theory as pointed out in [23]. This is because the horizon of this black hole is actually an extremal surface.

5 Conclusions and Discussions

In this talk we have presented the holographic formula which computes the entanglement entropy in the dual QFTs. It takes the form of the area law and can be regarded as a generalization of the Bekenstein-Hawking entropy formula. We also gives a covariant formulation which is useful to analyze the holographic dual of the time-dependent background.

There are many interesting future problems. We will mention a few of them here. One thing which would hopefully be clear in near future is the question how much information the entanglement entropy contains. Since we have infinitely many choices of the subsystem A, the entanglement entropy include infinite amount of information. The natural question is whether the information of entanglement entropy in a given QFT is enough to extract the metric of its holographic dual spacetime.

Another intriguing future problem is to understand any implications of holography in cosmological background such as a de-Sitter space from the viewpoint of entanglement entropy. This will be directly related to the understanding of the mysterious horizon entropy of de-Sitter space.

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