$O(d, d)$ covariant formulation of Type II supergravity and Scherk-Schwarz reduction

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Abstract. T-duality is a stringy symmetry which relates string backgrounds with different space-time geometries. In the low energy limit, it manifests itself as a continuous $O(d, d)$ symmetry acting on supergravity fields, after dimensional reduction on a d dimensional torus. Double Field Theory (DFT) is a T-duality covariant extension of string theory which aims to realize $O(d, d)$ as a manifest symmetry for the low energy effective space-time actions of string theory without dimensional reduction. The mathematical framework needed to construct DFT goes beyond Riemannian geometry and is related to Hitchin's generalized geometry program. On the other hand, Scherk-Schwarz reduction of DFT of Type II strings with a duality twist in $O(d, d)$ yields Gauged Double Field Theory (GDFT), that can be regarded as an $O(d, d)$ covariant extension of gauged supergravity. The purpose of this contribution is to give a short review on Scherk-Schwarz reductions of DFT and its intriguing connections to integrable deformations of string sigma models.

1. Introduction

T-duality is a stringy symmetry which relates string backgrounds with different space-time geometries [1]. It is natural to expect that T-duality (along with other stringy dualities) should also be a symmetry of the corresponding supergravity theory, the effective theory which describes string theory in the low energy limit. Indeed, compactified supergravity theories possess non-compact global symmetries [2] as had been understood as early as in 70s [3, 4, 5, 6], and T-duality is realized as a discrete subgroup of these symmetries. Historically, these non-compact global symmetries are called hidden symmetries, as they arise after dimensional reduction on a torus and only after certain dualizations [7, 8]. Compactifying eleven dimensional supergravity on a $d + 1$ dimensional torus and dualizing certain fields one ends up with a $10 - d$ dimensional supergravity theory, with a global symmetry group which includes $O(d, d, R)$[1]. After quantization, the discrete subgroup $O(d, d, Z)$ is promoted to the T-duality symmetry group of the toroidally compactified type II string theories.

Double Field Theory (DFT) of Type II strings is an extension of massless Type II string theories, in which the T-duality symmetry is already manifest in higher dimensions without the requirement of dimensional reduction. In the low energy limit, this provides manifestly $O(d, d)$ invariant actions for the corresponding supergravity theories. This is achieved by extending the space-time by introducing dual, winding type coordinates [9, 10, 11, 12, 13]. In [14], a manifestly $O(d, d)$ invariant action was constructed on such a doubled space, based on two dynamical

[1] In the manuscript, $O(d, d, R)$ is simply referred to as $O(d, d)$.

fields called the generalized metric and the generalized dilaton field. The action comes with an $O(d,d)$ covariant constraint, called the strong constraint, imposition of which effectively halves the number of coordinates the fields and the gauge parameters can depend on. A trivial solution of this constraint is obtained when none of the fields or gauge parameters in the theory depend on the winding type coordinates. In this case, the fields are said to be in supergravity frame. As the name suggests, when this solution of the constraint is imposed, the DFT action constructed in [12] reduces to the standard NS-NS action for the massless fields of string theory. Later in [15], Zwiebach and Hohm constructed a manifestly $Spin^+(d,d)$ invariant action, which reduces in the supergravity frame to the Ramond-Ramond sector of the democratic formulation of Type II supergravity theory [16]. In this action, the dynamical fields are two spinor fields: χ and $\mathbb{S}$. The spinor χ encodes all the p-form gauge potentials and their Hodge duals that live in the RR sector of Type II theory. The field $\mathbb{S}$ is the spinor field which projects onto the generalized metric under the double covering homomorphism between $O(d,d)$ and $Pin(d,d)$. These actions will be given in Section 2.

As mentioned above, both the NS-NS and the RR sectors of the DFT action are invariant under the global symmetry group $Pin(d,d)$.[2] It is well known that theories with a global symmetry group can be dimensionally reduced by using a Scherk-Schwarz (SS) type ansatz [18, 19]. This provides a generalization of the Kaluza-Klein (KK) dimensional reduction scheme, which allows the higher dimensional fields to have a dependence on the coordinates of the internal space. The coordinate dependence is dictated by how the fields transform under the global symmetry group. This yields in lower dimensions gaugings, mass terms and a scalar potential, all determined by the duality twist matrix, which is an element of the global symmetry group. SS reduction is known to yield a consistent dimensional reduction in the sense that solutions of the lower dimensional field equations are also solutions of the field equations derived from the higher dimensional parent theory. This consistency is guaranteed by the fact that the reduction ansatz is determined by the global symmetry.

SS reduction of DFT was studied in [20, 21, 22, 23, 24] for the NS-NS sector and in [17] for the RR sector. The resulting theory dubbed Gauged Double Field Theory (GDFT) reduces in the supergravity frame to gauged supergravity. Interestingly, it is possible to obtain also the *non-geometric gaugings* this way [25]. This means that these gaugings cannot be obtained as a result of standard geometric compactification of supergravity or cannot be T-dualized to one of those. The origin of such gaugings had been previously related with the existence of the so-called *non-geometrix fluxes* in higher dimensions [26, 27, 28]. It is remarkable that GDFT provides a framework in which non-geometric fluxes can be treated on the same footing with the geometric fluxes [21].

Another remarkable feature of GDFT is that it sets a convenient stage for understanding certain solution generating duality transformations and integrable deformations of string sigma models. It has been understood recently that non-Abelian T-duality, Yang-Baxter deformations and Poisson Lie T-duality can be understood in terms of local $O(d,d)$ transformations [29, 30, 31, 32, 33, 34, 35, 36, 37, 38]. This makes DFT and GDFT a very convenient framework for describing these dualities and deformations.

The purpose of this article is to give a short review of SS reductions of DFT and its intriguing connections to integrable deformations of string sigma models. The structure of the review is as follows. In the next section, we start by introducing the basic ingredients in the construction of DFT. The $O(d,d)$ invariant actions of both the NS-NS and the RR sectors will be given in that section. In Section 3, we will describe the SS reduction of DFT, again for both sectors. We will introduce the fluxes in this section, which describe GDFT as a deformation of DFT. Section 4 explains how non-Abelian T-duality and Yang-Baxter deformations can be described within the

[2] In the RR sector, this is broken to $Spin(d,d)$ due to the chirality condition on χ and is further reduced to $Spin^+(d,d)$ due to the existence of a self-duality constraint. See [15, 17] for more details.

framework of GDFT. We end in Section 5 with a discussion of results.

2. A brief review of double field theory

In closed string field theory, all fields depend on two types of coordinates: the usual space-time coordinates x and the dual winding type coordinates $\tilde{x}$. The dual coordinates are conjugate to the winding degrees of freedom of strings, in the same way space coordinates and momenta are conjugate variables in classical field theory. The aim of DFT is to realize this in the sector of massless fields in order to construct a manifestly T-duality invariant action describing this sector. In DFT, the space-time and winding type coordinates combine to form an $O(d,d)$ vector transforming as:

$$X'^M = h^M{}_N X^N, \quad X^M = \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix}. \tag{1}$$

Here $\tilde{x}_i$ are the dual coordinates, and $h^M{}_N$ is a general $O(d,d)$ matrix. Remember that an $O(d,d)$ matrix h is of the form

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2}$$

where a, b, c and d are $d \times d$ matrices satisfying

$$a^t c + c^t a = 0, \quad b^t d + d^t b = 0, \quad a^t d + c^t b = I. \tag{3}$$

In what follows we will always decompose the indices M labelling the $O(d,d)$ representation as ${}^M = ({}_i, {}^i)$, where ^{i} and ${}_i$ label representations of the $GL(d)$ subgroup of $O(d,d)$. We will raise and lower indices by the $O(d,d)$ invariant metric η (presented in equation (4) below), so that $X_M = \eta_{MN} X^N$. The doubling of the coordinates is formal in the sense that the consistency of the DFT action that we will present shortly relies on imposition of a constraint, which effectively halves the number of coordinates that the fields and the gauge parameters can depend on. The constraint is $O(d,d)$ invariant and is given below:

$$\partial^M \partial_M A \;=\; \eta^{MN} \partial_M \partial_N A \;=\; 0\,, \qquad \partial^M A\, \partial_M B \;=\; 0\,, \qquad \eta^{MN} \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4}$$

Here, A and B represent any fields or parameters of the theory. The first of the above constraints is called the *weak constraint* and follows from the level matching constraint in closed string theory. The second constraint is stronger and is called the *strong constraint.* A trivial solution of the strong constraint is obtained when all the fields and gauge parameters in the theory depend only on the standard coordinates x and have no dependence on the dual coordinates $\tilde{x}$. In this case, the theory is said to be in the supergravity frame, since the DFT action reduces to the standard supergravity action in this frame. It can be shown that for any other solution of the strong constraint, there is a choice of coordinates $(x', \tilde{x}')$ related to $(x, \tilde{x})$ by an $O(d,d)$ transformation in such a way that the fields and gauge parameters have no dependence on the coordinates $\tilde{x}'$ [13]. This means that the theory is restricted to lie in a maximally isotropic subspace (with respect to the metric preserved by $O(d,d)$) of the total $2d$ dimensional doubled space [12].[3]

[3] Another important solution of the strong constraint is obtained by allowing the dilaton field to have a linear dependence on one of the winding type coordinates, say $\tilde{x}_i$. In order to satisfy (4), none of the fields (including the dilaton field) may have a dependence on the corresponding space coordinate x^i, or equivalently the background supports $U(1)$ isometry with the corresponding Killing vector field $\frac{\partial}{\partial x^i}$. In this case, the theory is said to be in the generalized supergravity frame, since the DFT equations are known to reduce to generalized supergravity equations of [39, 40] for this solution [41, 42].

The DFT action can be presented in terms of a generalized vielbein, as was first done in [9] (see [43] for later developments). Below we present the generalized metric formulation, which was first constructed by Hohm, Hull and Zwiebach for the NS-NS sector [14], and then by Hohm, Kwak and Zwiebach for the RR sector [15]:

$$\mathcal{S} = \int \; dx d\tilde{x} \left(\mathcal{L}_{\text{NS}-\text{NS}} + \mathcal{L}_{\text{RR}} \right), \tag{5}$$

where

$$\mathcal{L}_{\text{NS}-\text{NS}} = e^{-2d} \; \mathcal{R}(\mathcal{H}, d), \tag{6}$$

and

$$\mathcal{L}_{\text{RR}} = \frac{1}{4} (\partial\!\!\!/ \chi)^\dagger \; \mathbb{S} \; \partial\!\!\!/ \chi = \frac{1}{4} \langle \; \partial\!\!\!/ \chi, \; C^{-1} \mathbb{S} \; \partial\!\!\!/ \chi \rangle. \tag{7}$$

This action has to be complemented by the following self-duality constraint

$$\partial\!\!\!/ \chi = -\mathcal{K} \; \partial\!\!\!/ \chi, \quad \mathcal{K} \equiv C^{-1} \mathbb{S}. \tag{8}$$

Let us explain the ingredients in the equations (6) and (7). The field $\mathcal{H}$ is the generalized metric and it is constructed from the semi-Riemannian metric g and the B-field b, which is an antisymmetric 2-form field:

$$\mathcal{H}_{MN} = \begin{pmatrix} \mathcal{H}^{ij} & \mathcal{H}^{i}{}_{j} \\ \mathcal{H}_{i}{}^{j} & \mathcal{H}_{ij} \end{pmatrix} = \begin{pmatrix} g^{ij} & -g^{ik} b_{kj} \\ b_{ik} g^{kj} & g_{ij} - b_{ik} g^{kl} b_{lj} \end{pmatrix}. \tag{9}$$

$\mathcal{H}$ is a symmetric $O(d,d)$ matrix and as such it satisfies $\mathcal{H}_{MP} \eta^{PQ} \mathcal{H}_{QR} = \eta^{MR}$. The term $\mathcal{R}(\mathcal{H}, d)$ is the generalized Ricci scalar and its explicit form is:

$$\begin{aligned} \mathcal{R}(\mathcal{H}, d) \;\; = \;\; & 4\mathcal{H}^{MN} \partial_M \partial_N d - \partial_M \partial_N \mathcal{H}^{MN} - 4\mathcal{H}^{MN} \partial_M d \partial_N d + 4\partial_M \mathcal{H}^{MN} \partial_N d \\ & + \frac{1}{8} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_N \mathcal{H}_{KL} - \frac{1}{2} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_K \mathcal{H}_{NL} \\ & + \frac{1}{2} \partial_M \varepsilon^a{}_P \partial^M \varepsilon^b{}_Q S_{ab} \eta^{PQ}. \end{aligned} \tag{10}$$

Here, $\varepsilon^a{}_P$ is the generalized vielbein with $\mathcal{H}_{MN} = \varepsilon^a{}_M S_{ab} \varepsilon^b{}_N$, where $S_{ab} = diag(-1, 1, \cdots, 1; -1, 1, \cdots, 1)$ is the planar metric. The term in the last line is not in the original generalized metric formulation of DFT and vanishes when the strong constraint is imposed [22]. The field d in (6) is the generalized dilaton field and it is defined as

$$e^{-2d} = \sqrt{g} e^{-2\phi}. \tag{11}$$

The field χ in the action (7) is a $Clif(d,d)$ spinor field that packages the p-form gauge fields in the RR sector of Type II supergravity. For Type IIA one has odd degree forms, whereas for Type IIB χ packages even degree forms. The spinor field is differentiated with the Dirac operator

$$\partial\!\!\!/ \equiv \Gamma^M \partial_M = \Gamma^i \partial_i + \Gamma_i \tilde{\partial}^i, \tag{12}$$

where $\tilde{\partial}^i = \frac{\partial}{\partial \tilde{x}_i}$. Here the Gamma matrices $\Gamma^M = (\Gamma_i, \Gamma^i)$ are the matrix representations of the generators of the Clifford algebra $Clif(d,d) \equiv Clif(R^{2d}, \eta)$:

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}, \tag{13}$$

where the $O(d,d)$ invariant metric η is as in (4). The labelling is such that the first d elements span a maximally isotropic subsace with respect to the metric η, and the the remaining elements span the orthogonal complement. Then one has

$$\{\Gamma_i, \Gamma^j\} \ = \ 2\delta_i{}^j \ , \qquad \{\Gamma_i, \Gamma_j\} \ = \ 0 \ , \qquad \{\Gamma^i, \Gamma^j\} \ = \ 0 \ . \tag{14}$$

It is also useful to define $\psi^M \equiv \frac{1}{\sqrt{2}}\Gamma^M$. The spinorial action of the Clifford algebra elements Γ^M on spinor fields χ is best understood when one identifies the spinor fields with (not necessarily homogenous) differential forms (or with polyforms, as is commonly called in physics literature) of the exterior algebra $\bigwedge^\bullet R^d$ [44, 45, 46, 15, 17]. A generic spinor field φ

$$\varphi(x, \tilde{x}) = \sum_p \frac{1}{p!} C_{i_1 \ldots i_p}(x, \tilde{x}) \psi^{i_1} \ldots \psi^{i_p} \tag{15}$$

is identified with the differential form

$$\varphi(x, \tilde{x}) = \sum_p \frac{1}{p!} C_{i_1 \ldots i_p}(x, \tilde{x}) dx^{i_1} \wedge \ldots \wedge dx^{i_p}. \tag{16}$$

Then, the action of ψ^i and ψ_i on χ is by wedge product and by contraction, respectively. More precisely, one has

$$\psi^i.\chi = \psi^i \wedge \chi, \quad \psi_i.\chi = i_{\psi_i}\chi, \tag{17}$$

where one defines $i_{\psi_i}\psi^j = \delta_i{}^j$. More details can be found in [15, 17].

The Ramond-Ramond sector couples to the NS-NS sector via the spinor field $\mathbb{S}$. Here $\mathbb{S}$ is the spinor field which projects to the generalized metric $\mathcal{H}$ under the double covering homomorphism ρ between $O(d,d)$ and $Pin(d,d)$, that is, $\rho(\mathbb{S}) = \mathcal{H}$. In Lorentzian signature, the generalized metric $\mathcal{H}$ is in the coset $SO^-(d,d)$[4], and there are subtleties in lifting this to an element $\mathbb{S}$ of $Spin^-(d,d)$ (for a detailed discussion, see [15]). So, in [15] the following viewpoint was adopted: it is the spin field $\mathbb{S} \in Spin^-(d,d)$, rather than the generalized metric, which has to be regarded as the fundamental gravitational field. The generalized metric $\mathcal{H}$ is then constructed by projecting onto the corresponding unique element in $SO^-(d,d)$. Finally, the matrix C is called the charge conjugation matrix and is defined as

$$C = \Lambda_1^{\pm} \ldots \Lambda_d^{\pm} \tag{18}$$

with

$$\Lambda_i^{\pm} = (\psi^i \mp \psi_i), \tag{19}$$

where the upper plus sign is for even dimensions and the lower minus sign is for odd dimensions. It can be shown that the following useful relation to Hodge duality * (defined with respect to the semi-Riemannian metric g) holds:

$$*\lambda(A) = -C^{-1} S_g^{-1} A. \tag{20}$$

Here,

$$\lambda(A_n) \equiv (-1)^{Int[n/2]} A_n = (-1)^{n(n-1)/2} A_n \tag{21}$$

[4] When the space-time metric g is positive definite, so is the generalized metric $\mathcal{H}$, and hence its components form a matrix that lies in $SO^+(d,d)$. In this case, the corresponding spin group element $\mathbb{S}$ is in $Spin^+(d,d)$. However, when the semi-Riemannian metric g, has Lorentzian signature, then $\mathcal{H}$ is in $SO^-(d,d)$ and correspondingly $\mathbb{S}$ lives in $Spin^-(d,d)$. Here, $SO^+(d,d)$ is the component of $SO(d,d)$ connected to the identity. It is also a subgroup, whereas its complement, $SO^-(d,d)$ is a coset of $SO^+(d,d)$.

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for an n-form A_n, and action of λ on a polyform A is defined by linear extension. Also, $S_g^{-1} = S_{g^{-1}}$ is the $Spin(d,d)$ element that projects onto the $SO(d,d)$ element

$$h_{g^{-1}} \equiv \begin{pmatrix} g^{-1} & 0 \\ 0 & g \end{pmatrix} \tag{22}$$

under the double covering homomorphism ρ that is, $\rho(S_{g^{-1}}) = h_{g^{-1}}$.

The inner product $\langle\ ,\ \rangle$ that determines the action in (7) is the Mukai pairing, which is a $Spin(d,d)$ invariant bilinear form on the space of spinors [47, 44]. It is defined as $\langle\ ,\ \rangle : S \otimes S \to \wedge^n T^*M$:

$$\langle \chi_1, \chi_2 \rangle = (\tau(\chi_1) \wedge \chi_2)_{\text{top}} = \sum_j (-1)^j (\chi_1^{2j} \wedge \chi_2^{n-2j} + \chi_1^{2j+1} \wedge \chi_2^{n-2j-1}), \quad \chi_1, \chi_2 \in \wedge^\bullet T^*M. \tag{23}$$

Here, $()_{\text{top}}$ denotes the top degree component of the form, and the superscript k denotes the k-form component of the form.

2.1. Gauge symmetries of the action

It can be shown that the DFT actions (6) and (7) are invariant under the following gauge transformations [13]

$$\begin{aligned}
\delta_\xi \mathcal{H}_{MN} &= \widehat{\mathcal{L}}_\xi \mathcal{H}_{MN} \\
&\equiv \xi^P \partial_P \mathcal{H}_{MN} + \big(\partial_M \xi^P - \partial^P \xi_M\big)\, \mathcal{H}_{PN} + \big(\partial_N \xi^P - \partial^P \xi_N\big)\, \mathcal{H}_{MP}\ , \\
\delta d &= \xi^M \partial_M d - \frac{1}{2} \partial_M \xi^M\ , \\
\delta_\xi \chi &= \widehat{\mathcal{L}}_\xi \chi \ \equiv\ \xi^M \partial_M \chi \ +\ \frac{1}{\sqrt{2}} \partial\!\!\!/ \xi^M \Gamma_M\, \chi \qquad (24) \\
&= \xi^M \partial_M \chi \ +\ \frac{1}{2} \partial_N \xi_M \Gamma^N \Gamma^M \chi\ , \\
\delta_\xi \mathcal{K} &= \xi^M \partial_M \mathcal{K} + \frac{1}{2} \big[\Gamma^{PQ},\, \mathcal{K}\big] \partial_P \xi_Q\ , \\
\delta_\lambda \chi &= \partial\!\!\!/ \lambda = \frac{1}{\sqrt{2}} \Gamma^M \partial_M \lambda, \qquad (25)
\end{aligned}$$

where λ is a space-time dependent spinor. Here, the generator $\xi^M = (\tilde{\xi}_i, \xi^i)$ is an $O(d,d)$ vector and Γ^{PQ} is defined as $\Gamma^{PQ} \equiv \frac{1}{2}[\Gamma^P, \Gamma^Q]$. In the frame $\tilde{\partial}^i = 0$, the gauge parameter $\xi^M = (\tilde{\xi}_i, \xi^i)$ combines the diffeomorphism parameter $\xi^i(x)$ and the Kalb-Ramond gauge parameter $\tilde{\xi}_i(x)$. The parameter λ generates the double field theory version of the abelian gauge symmetry of p-form gauge fields.

It can be shown that the gauge transformations close to form a gauge algebra:

$$\begin{aligned}
\big[\delta_{\xi_1}, \delta_{\xi_2}\big] &= -\delta_{[\xi_1,\xi_2]_C}, \\
\big[\delta_\lambda, \delta_\xi\big] &= \delta_{\widehat{\mathcal{L}}_\xi},
\end{aligned} \tag{26}$$

provided that the strong constraint (4) is satisfied. Here, $[,]_C$ is the C-bracket, which is defined as

$$\big[\Sigma_1,\ \Sigma_2\big]_C^M = 2\Sigma_{[1}^N \partial_N \Sigma_{2]}^M - \Sigma_{[1}^P \partial^M \Sigma_{2]P}. \tag{27}$$

If we formally write $\Sigma = \xi + \tilde{\xi}$ and expand out the bracket in (27), we obtain

$$\begin{aligned} [\xi_1 + \tilde{\xi}_1, \xi_2 + \tilde{\xi}_2]_C &= [\xi_1, \xi_2] + \mathcal{L}_{\tilde{\xi}_1}\xi_2 - \mathcal{L}_{\tilde{\xi}_2}\xi_1 - \frac{1}{2}\tilde{d}(\tilde{i}_{\tilde{\xi}_1}\xi_2 - \tilde{i}_{\tilde{\xi}_2}\xi_1) \\ &+ [\tilde{\xi}_1, \tilde{\xi}_2] + \mathcal{L}_{\xi_1}\tilde{\xi}_2 - \mathcal{L}_{\xi_2}\tilde{\xi}_1 - \frac{1}{2}d(i_{\xi_1}\tilde{\xi}_2 - i_{\xi_2}\tilde{\xi}_1). \end{aligned} \tag{28}$$

Here, $\tilde{d}$ is defined with respect to the dual coordinates so that acting on a function f, one has $(\tilde{d}f)^i = \tilde{\partial}^i f$ and $(\tilde{d}f)_i = 0$. $\tilde{i}$ is defined so that $i_{\tilde{\xi}_1}\xi_2 = \tilde{i}_{\xi_2}\tilde{\xi}_1 = (\tilde{\xi}_1)_i \xi_2^i$ and $\tilde{\mathcal{L}} = \tilde{d}\tilde{i} + \tilde{i}\tilde{d}$.

We digress here in order to discuss Lie bialgebroids and Courant algebroids, where the bracket (28) above plays a fundamental role. A Lie algebroid A is a vector bundle over a base manifold M which comes with a Lie bracket $[,]_A$ on its sections that satisfies the Leibniz rule and the Jacobi identity and an *anchor*, which is a Lie algebra homomorphism $\rho : A \to TM$ so that

$$\rho([X, Y]_A) = [\rho(X), \rho(Y)],$$

where X, Y are sections of A and $[,]$ is the standard Lie bracket of vector fields. Now, suppose that the dual vector bundle A^* also carries a Lie algebroid structure with a Lie bracket $[,]_{A^*}$ on its sections and an anchor ρ^*. If the two algebroid structures on A and A^* are compatible (the technical definition of compatibility is given in [48]), then the pair (A, A^*) forms a Lie bialgebroid. In this case, it is possible to construct a Courant algebroid structure on the direct sum bundle $A \oplus A^*$ as was first shown in [48] [5]. A Courant algebroid is a vector bundle E over a manifold M equipped with a non-degenerate symmetric bilinear form (,) on the bundle, a skew-symmetric bracket $[,]_{Cour}$ on sections of E and a bundle map (anchor) $\rho : E \to TM$ with certain properties (a modifed Jacobi identity, a modified Leibniz rule for the bracket, the bracket and the bilinear form must be compatible in a certain sense and ρ is a Lie algebra homomorphism [49]). When (A, A^*) forms a Lie bialgebroid, one takes as the inner product the natural inner product $(X + \eta, Y + \xi) = \frac{1}{2}(\xi(X) + \eta(Y))$, with $X, Y \in A$ and $\xi, \eta \in A^*$ and as the anchor $\rho = \rho_A + \rho_{A^*}$, where ρ_A and ρ_{A^*} are the anchors on the Lie algebroids A and A^*, respectively. The skew symmetric bracket on $E = A \oplus A^*$ can be written in terms of the brackets on A and A^* [48]. Just like the Lie bracket $[,]_A$ induces a differential d_A on sections of $\bigwedge^\bullet A^*$, the bracket $[,]_{A^*}$ induces a differential d_{A^*} on sections of $\bigwedge^\bullet A$. Then $\mathcal{D} \equiv d_A + d_{A^*}$ is used to differentiate the sections of the exterior bundle of E. The compatibility between the Lie algebroid structures on A and A^* makes sure that the inner product, the bracket, the anchor and the differential constructed on E define a Courant algebroid. The bracket that appears in (28) is exactly of the same form as the bracket on a Courant algebroid $A \oplus A^*$ if we call the differentials on A and A^* d and $\tilde{d}$, respectively. An interesting case is when $A = TM$ with the standard Lie bracket and $\rho = Id$ (which obviously constitute a Lie algebroid structure) and $A^* = T^*M$ is the cotangent bundle with zero Lie bracket (also a Lie algebroid structure trivially compatible with the one on TM). In this case, the bracket that defines a Courant algebroid structure on the direct sum bundle $TM \oplus T^*M$ is the famous Courant bracket

$$[X + \xi, Y + \eta] = [X, Y] + L_X\eta - L_Y\xi + \frac{1}{2}d\left(i_X\eta - i_Y\xi\right), \tag{29}$$

which was first studied in [50], where Courant used it to define a new geometrical structure called Dirac structure, which unifies Poisson geometry and presymplectic geometry. The same

[5] A special case of this construction is when the base manifold M is point, in which case the Lie algebroid becomes a Lie algebra g. In this particular case, one can construct a Lie algebra structure on the direct sum of the underlying vector space of g and its dual g^* provided that they form a Lie bialgebra and the doubled structure which replaces the Courant algebroid is a Drinfeld double. The Drinfeld double construction plays an essential role in the discussion of YB deformations, as we will discuss in subsection 4.2.

bracket plays a prominent role in generalized geometry program of Nigel Hitchin in defining a generalized complex structure on the complexification of the bundle $TM \oplus T^*M$, which is a geometric structure that unifies complex structure and symplectic structure [45, **?**, 44].

The Courant bracket (29) is exactly what the bracket (27) reduces to when the trivial solution $\tilde{\partial}^i = 0$ of the strong constraint is imposed. Automorphisms of the Courant bracket are the diffeomorphisms and 2-form gauge transformations, which are the symmetries of string theory [44]. Correspondingly, the gauge transformations (25) and (24) reduce in the supergravity frame to diffeomorphisms of the metric g and the B-field b and the gauge transformations of the B-field, with ξ and $\tilde{\xi}$ being the parameters for these gauge transformations. On the other hand, when all the fields and gauge parameters are in the supergravity frame, that is when $\tilde{\partial}^i = 0$, the action given in (6) reduces to the standard NS-NS action for the massless fields of string theory and the action (7) reduces to the RR sector of the democratic formulation of Type II supergravity theories. The duality constraint (8) reduces to the conventional duality relation for p-form fields in the democratic formulation

$$F^{(p)} = (-1)^{(d-p)(d-p-1)/2} * F^{(d-p)}. \tag{30}$$

The field equations in the democratic formulation for the higher degree fields which are absent in the standard formulation become, after applying (30), the Bianchi identities for the lower degree fields in the standard formulation.

3. Scherk-Schwarz reduction of DFT

Twisted toroidal compactification or Scherk-Schwarz reduction is a useful way of introducing masses into supergravity and string compactifications [18, 19]. Standard toroidal compactification on a torus T^d involves expanding all the fields in the theory in terms of the harmonics of the torus and then truncating away all the massive fields (which are very heavy in the limit of vanishing volume for the torus). This is equivalent to imposing a Kaluza-Klein (KK) dimensional reduction anzats, that is, taking all the fields independent of the internal toroidal coordinates. If the higher dimensional theory possesses a global symmetry group, then it is possible to keep some of the massive modes in the truncation without violating the consistency of the compactification. This is equivalent to imposing a duality twisted reduction anzats which allows the fields to depend on the internal coordinates in a way dictated by the global symmetry group. In their seminal papers [18, 19], Scherk and Schwarz showed that this also yields a consistent compactification, meaning that the solutions of the lower dimensional field equations can be uplifted to solutions of the higher dimensional parent theory. If a higher dimensional field, call it ψ, transforms in a certain representation of G, the SS dimensional anzats imposed on ψ is

$$\psi(x, y) = U(y)\,[\psi(x)], \quad U(y) \in G, \tag{31}$$

where $U(y)$ acts on G in that particular representation. Here, x and y correspond to the external and internal coordinates, respectively. Requirement of consistency imposes certain restrictions on $U(y)$, as we will discuss shortly in the context of DFT. In compacitifications of supergravity, the global symmetry group exploited in the SS dimensional reduction usually comes from a previous toroidal reduction. For example, consider a theory compactified on a two-torus T^2. The lower dimensional theory has an $SL(2, R)$ symmetry as part of its global symmetry group, as $SL(2, R)$ is the large diffeomorphism group of T^2. In a further compactification on a circle S^1 one can introduce a duality-twisted ansatz for these fields as in (31), where $U(y)$ is in $SL(2, R)$ [51, 52, 53, 54, 55]. The total 3 dimensional internal space is usually called a twisted torus in the literature. In contrast, DFT possesses a global $O(d, d)$ symmetry, which is already manifest in higher dimensions, without requiring toroidal dimensional reduction. This makes it possible to introduce a duality twisted anzats for all the fields in the higher dimensional theory, yielding mass terms and gaugings that cannot be obtained by standard compactification schemes. For example, when DFT is reduced to 4 dimensions and the fields are restricted to the supergravity frame, one obtains the whole electric bosonic sector of four dimensional gauged N = 4 supergravity [56] as was shown in [21, 20, 22]. Prior to their work, only a certain class of gaugings present in the theory had been known to have a higher dimensional origin. Below we will give a quick description of the duality twisted reduction of DFT, not only in the NS-NS sector, but also in the RR sector, whose SS reduction was first studied in [17].

The DFT action presented in (7) is invariant under the following transformations:

$$\mathbb{S}(X) \quad \longrightarrow \mathbb{S}'(X') \;=\; (S^{-1})^{\dagger}\, \mathbb{S}(X)\, S^{-1}\;, \quad \chi(X) \longrightarrow \chi(X') = S\chi(X). \tag{32}$$

Here $S \in Spin^{+}(d,d)$ and $X' = hX$, where $h = \rho(S) \in SO(d,d)$. The generalized dilaton field is invariant. The duality group is broken to $Spin(d,d)$ because the spinor field χ in (7) must have a fixed chirality, depending on whether it is related to Type IIA or Type IIB string theory, and the full $Pin(d,d)$ does not preserve the fixed chirality. Moreover, a general $Spin(d,d)$ transformation does not preserve the self-duality constraint (8) and hence the duality group is further reduced to the subgroup $Spin^{+}(d,d)$. The transformation of $\mathbb{S}$ implies the following transformation rule for the generalized metric $\mathcal{H} = \rho(\mathbb{S})$:

$$\mathcal{H}(X) \quad \longrightarrow \mathcal{H}'(X') \;=\; (h^{-1})^{T}\, \mathcal{H}(X)\, h^{-1}\;. \tag{33}$$

These transformation rules dictate the anzats for the duality twisted reduction of the DFT action:

$$\mathbb{S}(X,Y) \;=\; (S^{-1})^{\dagger}(Y)\mathbb{S}(X)S^{-1}(Y), \tag{34}$$

$$\chi(X,Y) \;=\; S(Y)\chi(X). \tag{35}$$

Here, X denotes collectively the coordinates of the reduced theory, whereas Y denotes the internal coordinates, and the whole dependence on the internal coordinates is encoded in the duality twist matrix $S(Y)$. Note that the internal and external coordinates can be further decomposed as $Y = (\tilde{y}, y)$ and $X = (\tilde{x}, x)$, where the coordinates with a tilde are the dual coordinates associated with winding excitations. The anzats (34) implies the following anzats in the NS-NS sector:

$$\mathcal{H}_{MN}(X,Y) = U^{A}_{\;M}(Y)\mathcal{H}_{AB}(X)U^{B}_{\;N}(Y). \tag{36}$$

Although the generalized dilaton field is invariant under $O(d,d)$ transformations, it is possible to introduce the following anzats for it, due to its invariance under constant shifts:

$$d(X,Y) = d(X) + \rho(Y). \tag{37}$$

Imposing this non-trivial ansatz on the generalized dilaton field gives rise to an overall conformal scaling in the NS-NS sector and in order to induce this factor also in the RR sector, one needs to modify the anzats on the field χ as follows:

$$\chi(X,Y) = e^{-\rho(Y)}S(Y)\chi(X). \tag{38}$$

On the other hand, we have for the gauge parameters

$$\xi^{M}(X,Y) = (U^{-1})^{M}_{\;A}\hat{\xi}^{A}(X) \tag{39}$$

and

$$\lambda(X,Y) = e^{-\rho(Y)}S(Y)\hat{\lambda}(X). \tag{40}$$

When we plug in the ansatze (34, 36, 37, 38, 39) and (40) in the DFT action and the gauge transformation rules, we end up with a deformation of DFT, determined by the twist matrices $U(Y), S(Y)$ and the function $\rho(Y)$. For consistency, we need the following conditions to be satisfied: first of all, the Y dependence should drop both in the reduced action and the gauge algebra so that we have a truly dimensionally reduced theory. Also, we need the reduced action to be invariant under the deformed gauge transformation rules and also the gauge algebra must close. One finds that all these conditions are satisfied provided that the following conditions on the twist matrices hold:

Firstly, the external coordinates X must remain untwisted, that is, we should have

$$(U^{-1})^M_A \partial_M g(X) = \partial_A g(X). \tag{41}$$

The second condition is

$$\partial^P (U^{-1})^M_A \partial_P g(X) = 0. \tag{42}$$

If a given coordinate and its dual are either both external or both internal, then this condition is trivially satisfied. A similar condition must be imposed on ρ, too:

$$\partial^P \rho \partial_P g(X) = 0. \tag{43}$$

In addition, the weak and the strong constraint has to be imposed on the external space so that

$$\partial_A \partial^A V(X) = 0, \quad \partial_A V(X) \partial^A W(X) = 0, \tag{44}$$

for any fields or gauge parameters V, W that have a dependence on the coordinates of the external space only. Under these conditions, it can be checked that all the Y dependence of the resulting theory is encoded in the entities called *flux*, which are defined in terms of twisted matrices as in below:

$$f_{ABC} = 3\Omega_{[ABC]}, \quad \eta_A = \partial_M (U^{-1})^M_A - 2(U^{-1})^M_A \partial_M \rho, \tag{45}$$

where ρ is as in (37) and

$$\Omega_{ABC} = -(U^{-1})^M_A \partial_M (U^{-1})^N_B U^D_N \eta_{CD}. \tag{46}$$

Note that Ω_{ABC} are antisymmetric in the last two indices: $\Omega_{ABC} = -\Omega_{ACB}$. We also make the following definition

$$f_A = -\partial_M (U^{-1})^M_A = \Omega^C{}_{AC}. \tag{47}$$

Note that, due to complete antisymmetry of f_{ABC} in its indices, the only independent blocks of f_{ABC} out of the 8 possible combinations are $f^I{}_{JK}, f_{IJK}, f^{IJ}{}_K$ and f^{IJK}, $I = 1, \cdots, d$. It is customary to call these the geometric flux, the H-flux, the Q-flux and the R-flux, respectively [57].

It can be shown that the conditions (41) and (42) imply the following conditions on fluxes:

$$f^A{}_{BC} \partial_A g(X) = 0, \quad f^A \partial_A g(X) = 0. \tag{48}$$

Note that the second condition in (48) and (43) imply together that η_A should also satisfy

$$\eta^A \partial_A g(X) = 0. \tag{49}$$

In order for the Y dependence to drop from the reduced theory, the fluxes must be constant. Also, the following Jacobi identity and the orthogonality condition should be satisfied for the closure of the gauge algebra:

$$f_{E[AB} f_{C]D}{}^E = 0, \tag{50}$$

and

$$\eta^A f_{ABC} = 0. \tag{51}$$

These conditions are sufficient to ensure that the SS reduction of the DFT action of the NS-NS sector of Type II string theory is consistent. It was shown in [17] that the consistency of the SS reduction of the RR sector requires one more condition, which is given below:

$$f_{ABC} f^{ABC} = 0. \tag{52}$$

The theory obtained under these conditions is called Gauged Double Field Theory (GDFT). The reduction is consistent and the action and the gauge transformation rules of the reduced theory are determined completely by the constant fluxes. This leads to an important principle that will play a crucial role in section 4: *reductions with inequivalent twist matrices yield the same dimensionally reduced theory, if the fluxes associated with these twist matrices are equivalent.* As mentioned above, in the supergravity frame GDFT reduces to gauged supergravity with fluxes corresponding to gauge parameters. Indeed, it can be shown that the constraints to be obeyed by the fluxes determining the GDFT (of the NS-NS sector) are in one-to-one correspondence with the constraints of half-maximal gauged supergravities. The extra condition that comes from the requirement of the consistency of the reduction of the RR sector implies that the gauged theory in hand corresponds to a truncation of maximal supergavity [58].

We would like to note that it is not necessary to impose the strong constraint in the internal space, that is, one does not need to impose

$$\partial^P U^A_{\ M} \partial_P U^B_{\ N}. \tag{53}$$

Therefore, the duality twisted anzats (34)-(37) allow for a relaxation of the strong constraint on the total space. This idea was exploited in [59] to find massive deformations of Type IIA theory. It was shown there that the massive Type IIA theory can be obtained from a duality twisted reduction of DFT with a twist matrix that violates both the weak and the strong constraint explicitly, where the mass deformation is induced through the reduction of the RR sector. This is interesting because massive Type IIA theory cannot be obtained from a conventional dimensional reduction of eleven dimensional supergravity.

The GDFT action resulting from the duality twisted reduction of DFT is as below:

$$S_{GDFT} = v \int dx\, d\tilde{x} \left(e^{-2d}(\mathcal{R} + \mathcal{R}_f) + \frac{1}{4}\langle \nabla\!\!\!/\chi, C^{-1}\mathbb{S}\, \nabla\!\!\!/\chi\rangle \right), \tag{54}$$

where v is defined as

$$v = \int d^d Y e^{-2\rho(Y)}, \tag{55}$$

and

$$\mathcal{R} \to \mathcal{R}_{\rm def} = \mathcal{R} + \mathcal{R}_f, \tag{56}$$

with

$$\begin{aligned} \mathcal{R}_f \;=\; & -\frac{1}{2} f^A_{\ BC} \mathcal{H}^{BD} \mathcal{H}^{CE} \partial_D \mathcal{H}_{AE} - \frac{1}{12} f^A_{\ BC} f^D_{\ EF} \mathcal{H}_{AD} \mathcal{H}^{BE} \mathcal{H}^{CF} \\ & -\frac{1}{4} f^A_{\ BC} f^B_{\ AD} \mathcal{H}^{CD} - 2\eta_A \partial_B \mathcal{H}^{AB} + 4\eta_A \mathcal{H}^{AB} \partial_B d - \eta_A \eta_B \mathcal{H}^{AB}. \end{aligned} \tag{57}$$

The Dirac operator $\nabla\!\!\!/$ in (54) is defined as

$$\begin{aligned} \nabla\!\!\!/ \;\equiv\; & \partial\!\!\!/ + \frac{1}{12\sqrt{2}} f_{ABC} \Gamma^A\, \Gamma^B\, \Gamma^C + \frac{1}{2\sqrt{2}} \eta_B \Gamma^B \\ =\; & \partial\!\!\!/ + \frac{1}{6} f_{ABC} \psi^A\, \psi^B\, \psi^C + \frac{1}{2} \eta_B \psi^B. \end{aligned} \tag{58}$$

The constraint (52) implies that it is a nilpotent operator: $\not\nabla^2 = 0$. The fact that it is determined by the fluxes f_{ABC} and f_A follows from the fact that the spinorial twist matrix $S(Y)$ projects under the double covering homomorphism ρ between $Pin(d,d)$ and $O(d,d)$ onto the twist matrix $U(Y)$, and hence these elements are generated by the same Lie algebra element in different representations [17].

3.1. More on fluxes

In the next section, where we will discuss string dualities and integrable deformations of string sigma models in the context of GDFT, we will see that invariance of the fluxes under the relevant $O(d,d)$ transformation will be of crucial importance. Due to its important role in that section, we would like to give a more detailed picture of their algebraic structure in this subsection.

A generic element T of $O(d,d)$ has the following form

$$T = \begin{pmatrix} e^A & e^A B \\ \beta e^A & \beta e^A B + (e^{-A})^T \end{pmatrix} . \tag{59}$$

Any such matrix can be generated as a product of the following $O(d,d)$ matrices:

$$h_B = \begin{pmatrix} 1 & -B \\ 0 & 1 \end{pmatrix} \quad B^T = -B , \tag{60}$$

$$h_\beta = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}, \quad \beta^T = -\beta \tag{61}$$

$$h_A = \begin{pmatrix} e^A & 0 \\ 0 & (e^{-A})^T \end{pmatrix} . \tag{62}$$

If the corresponding $Spin(d,d)$ elements are denoted by S_B, S_β and S_A, then their spinorial action on a spinor field α (regarded as a polyform as discussed in previous sections) is

$$S_B : \quad \alpha \longmapsto e^{-B} \wedge \alpha = (1 - B + \frac{1}{2} B \wedge B - \ldots) \wedge \alpha, \tag{63}$$

$$S_\beta : \quad \alpha \longmapsto e^{\beta} \alpha = (1 + i_\beta + \frac{1}{2} i_\beta^2 + \cdots)\alpha, \tag{64}$$

$$S_A : \quad \alpha \longmapsto \frac{1}{\sqrt{detR}} (e^A)^* \alpha. \tag{65}$$

Here $B = \frac{1}{2} B_{kl} \psi^k \wedge \psi^l$, and $\beta = \frac{1}{2} \beta^{kl} \psi_k \wedge \psi_l$. Also, $i_\beta \alpha = \frac{1}{2} \beta^{ij} i_{\psi_i} (i_{\psi_j} \alpha)$ and $R^* \alpha = R_j{}^i \psi^j \wedge i_{\psi_i} \alpha$, with $R = e^A$, which is the usual action of $GL^+(d)$ on forms, where $GL^+(d)$ is the space of (orientation preserving) linear transformations of strictly positive determinant.

The flux associated with $U(Y)$ in (59) can be computed from (45). We do not present the result here, but rather refer the reader to [57] (see their equation (5.16) for the general expressions and equation (5.18) for the form of the fluxes in the supergravity frame, that is, when $\tilde{\partial}^i = 0$.) It can be checked that in the supergravity frame, the fluxes become the structure functions for the Roytenberg bracket $[,]_{Royt}$ [49, 60, 61, 62], which is obtained from the Courant bracket by twisting it with a 2-form field B and a bivector field β. Equivalently, one can take (e_i, e^i) as a non-holonomic basis for the direct sum bundle $TM \oplus T^*M$ satisfying

$$[e_i, e_j] = f_{ij}{}^k e_k \tag{66}$$

$$de^k = \frac{1}{2} f_{ij}{}^k e^i \wedge e^j, \tag{67}$$

and twist them as

$$e_i \to \hat{e}_i = e_i + B_{ij}e^j, \qquad e^i \to \hat{e}^i = e^i + \beta^{ij}e_j.$$

Calculating the Courant bracket of the resulting basis elements gives

$$\begin{aligned} [e_i, e_j]_{Royt} &= [\hat{e}_i, \hat{e}_j]_{Cour} = f_{ij}{}^k \hat{e}_k + H_{ijk}\hat{e}^k, \\ \left[e_i, e^j\right]_{Royt} &= [\hat{e}_i, \hat{e}^j]_{Cour} = f_{ik}^{\;j} \hat{e}^k + Q_i^{jk}\hat{e}_k, \\ \left[e^i, e^j\right]_{Royt} &= \left[\hat{e}^i, \hat{e}^j\right]_{Cour} = Q_k^{ij}\hat{e}^k + R^{ijk}\hat{e}_k. \end{aligned} \tag{68}$$

Explicit expressions or $f_{ij}{}^k, H_{ijk}, Q_i{}^{jk}$ and R^{ijk} are given in equation (5.18) of [57]. When $H = R = 0$ and f, Q are constant, (68) becomes the Lie algebra on a Drinfeld double with f and Q being the structure constants of the Lie algebra on g and dual of g (see footnote 5). Imposing the generalized Jacobi identity to be obeyed by the Courant bracket, one ends up with a set of constraints to be obeyed by the fluxes. When these conditions are satisfied, the bracket is a Courant bracket and $(\hat{e}_i, \hat{e}^i)$ forms a basis for the sections of a Courant algebroid. These non-constant fluxes play a fundamental role in the so-called flux formulation of DFT [57, 43]. In this formulation, the twist matrix $U(y)$ plays the role of a generalized vielbein used to define flat derivatives: $\mathcal{D}_A = U_A{}^M\partial_M$ and the action and the field equations are written in terms of the fluxes (see 2.18, 2.58 and 2.59 of [57]). For the consistency of the theory (e.g. for gauge invariance of the action, closure of the gauge algebra, covariance of generalized tensors), the fluxes must obey certain Bianchi identities (see 1.5 of [57] for the identities coming from the NS sector). When the strong constraint is imposed, these Bianchi identities are exactly the same as the constraints fluxes should obey so that they form the structure functions of the Roytenberg algebra (68) [61]. Therefore, we see that the underlying mathematical structure best suited to describe the non-constant fluxes is a Courant algebroid. Indeed, it was shown in [63] that the axioms for a Courant algebroid can be written in terms of three equations, which take the following form in local coordinates [64, 65]:

$$\eta^{IJ}\rho_I^i\rho_J^j = 0, \tag{69}$$

$$\rho_I^i\partial_i\rho_J^j - \rho_J^i\partial_i\rho_I^j - \eta^{KL}\rho_K^jF_{LIJ} = 0, \tag{70}$$

$$4\rho^i_{[L}\partial_iF_{IJK]} + 3\eta^{MN}F_{M[IJ}F_{KL]N} = 0. \tag{71}$$

Here e^I is a local basis for the sections of the Courant algebroid, F_{IJK} are the structure functions of the Courant bracket:

$$\left[e^I, e^J\right] = \eta^{IK}\eta^{JL}F_{KLM}e^M, \tag{72}$$

and ρ is the anchor with components $\rho = (\rho^i{}_j, \rho^{ij})$. If we identify the components of ρ with the twist matrix $U(y)$ in (45), (70) becomes the definition for the DFT fluxes (in the supergavity frame) and (71) is just a compact way of writing the Bianchi identities for fluxes [57, 64]. For constant fluxes, the Bianchi identities reduce to the constraints to be obeyed by the fluxes for the consistency of the SS reduction, which we discussed in the previous section. As an interesting remark, we note that generalizing the set of axioms (69-71) in such a way that they yield the expressions and Bianchi identities for the DFT fluxes (without imposing the constraint), an algebroid structure for DFT was proposed in [64].

4. Integrable deformations of string sigma models and GDFT

Recently, it has been understood that certain interesting integrable deformations of string sigma models can be described as coordinate dependent $O(d, d)$ transformations [29, 30, 31, 32, 33, 34, 35, 36]. Moreover, it was shown in general terms that $O(d, d)$ transformations preserve classical integrability [37]. This makes DFT and GDFT a very convenient setting for discussing such deformations. In this final section of this review paper, we will discuss non-Abelian T-duality and Yang-Baxter deformations, and describe how regarding them as transformations in DFT makes the analysis of the deformed theories easier, by following closely the papers [34] and [36].

4.1. Non-Abelian T-duality

Non-Abelian T-duality (NATD) is a generalization of T-duality for strings on backgrounds with non-Abelian isometries [66, 67, 68, 69, 70, 71, 72]. NAT dual background of a given string background can be found by using the standard tools of the Buscher method. For a generic non-linear sigma model with isometry group G, one starts with gauging the symmetry group (or a subgroup of it) and introduces Lagrange multiplier terms which constrain the gauge field to be pure gauge. Integrating out the Lagrange multipliers, one obtains the original model. Integrating out the gauge field gives the NATD model, for which the Lagrange multiplier terms play the role of coordinates on the dual manifold. Recently, it was shown that NATD can be described as a coordinate dependent $O(d,d)$ transformation [29, 30, 31, 34, 35, 73]. The transformation under NATD of the metric, the B-field and the dilaton field are determined by the coordinate dependent $O(d,d)$ matrix (which we also call T_{NATD}) obtained by embedding the following $O(n,n)$ matrix in $O(d,d)$ in the usual way (see 4.2.28-4.2.29 of [74]):

$$T_{\text{NATD}} = \begin{pmatrix} 0 & 1_n \\ 1_n & \theta_{IJ} \end{pmatrix}, \qquad \theta_{IJ} = \nu_K C_{IJ}{}^K. \tag{73}$$

Here C_{IJ}^K are the structure constants of the Lie algebra of the isometry group G with respect to a fixed basis T_I. We assume that G is n dimensional and acts without isotropy so that $I, J = 1, \cdots, n$. The corresponding $Spin(d,d)$ element is

$$S_{\text{NATD}} = C_n S_\theta = S_\beta C_n. \tag{74}$$

The factors S_θ and S_β in S_{NATD} are as in (63), (64) with $\theta_{ij} = \nu_k C_{ij}{}^k$ and $\beta_{ij} = \nu_k C_{ij}{}^k$, respectively. C_n is the charge conjugation matrix

$$C_n = \Lambda_1 \cdots \Lambda_n, \tag{75}$$

where

$$\Lambda_i = (\psi^i - \psi_i). \tag{76}$$

Now suppose that a supergravity background with non-Abelian isometry G is given and we would like to find the NAT dual of it by dualizing with respect to G by transforming it with T_{NATD} and S_{NATD}. For this, DFT is a very convenient framework since all the DFT fields transform naturally under $O(d,d)$ and its spinorial counterpart. So, instead of working with the standard supergravity fields, we work with the corresponding DFT fields: the generalized metric, the generalized dilaton field and the spinorial field χ that encodes the RR gauge potentials and their Hodge duals. To be more precise, the spinorial field that transforms is F (that encodes the field strengths, not the gauge potentials), which is related to χ as $F = e^{-B}\not{\partial}\chi$. Due to the isometry respected by the background, these fields can be written in the following form:

$$\begin{aligned} \mathcal{H}^{MN}(x,\theta) &= L^M{}_A(\theta)\mathcal{H}^{AB}(x)L^N{}_B(\theta), && (77)\\ \mathcal{K}(x,\theta) &= S_L(\theta)\mathcal{K}(x)S_L^{-1}(\theta), && (78)\\ F(x,\theta) &= e^{-B'}S_L(\theta)e^B F(x) = S_L(\theta)F(x), && (79) \end{aligned}$$

where

$$L = \begin{pmatrix} l^T & 0 \\ 0 & l^{-1} \end{pmatrix}. \tag{80}$$

Here, $\theta = \{\theta^i, i = 1, \cdots, n\}$ are coordinates for G and components of l are the coefficients of the left invariant 1-forms $\sigma^I = l^I{}_i d\theta^i$ on G defined from the Maurer-Cartan form: $g^{-1}dg = \sigma^I T_I$. The second equality in equation (79) is due to the special diagonal form of L.

Note that the isometry of the background has made it possible to write all the fields in a separated form as in a SS reduction, where the matrix $L(\theta)$ plays the role of a duality twist. For this reason, we will refer to the fields $\mathcal{H}(x), \mathcal{K}(x), F(x)$ on the right hand side that depend only on the x coordinates *untwisted fields*, and in this context the background fields will be referred to as *twisted fields*. These twisted fields satisfy the field equations of DFT in the supergravity frame since they form a supergravity background. What about the untwisted fields? As discussed in detail in [34], it can be shown that these fields satisfy a set of equations which is a deformation of the DFT equations and the deformation is determined solely by the fluxes associated with the twist matrix.[6] When the twist matrix is as in (80) one can easily compute that the associated flux is the geometric flux with $f_{ij}{}^k = C_{ij}{}^k$. The untwisted fields can now be twisted by a a different twist matrix $U(Y)$, which satisfies the constraints (41), (42) to obtain DFT fields $\mathcal{H}(x, Y), \mathcal{K}(x, Y)$ and $F(x, Y)$. These new twisted fields (different from the DFT fields associated with the seed background since $U(Y) \neq L(\theta)$) satisfy the field equations of DFT (and hence of supergravity if we identify the new coordinates Y with space-time coordinates) if and only if $U(Y)$ produces the same constant flux as with L, that is geometric flux with $f_{ij}{}^k = C_{ij}{}^k$ (remember our discussion in section 3, on page 12). Now comes the key point: our matrix T_{NATD} which generates the NAT dual backgrounds is exactly of this form, that is, the flux associated with it is just the geometric flux with the same components! (In fact with a generic matrix of the form (73) one would also have H-flux, but it vanishes in our case due to the identity $C_{L[I}{}^H C_{JK]}{}^L = 0$ which follows from the Jacobi identity for the Lie algebra of G.) As a result, one immediately concludes that the fields of the NAT dual background must also satisfy the supergravity equations, that is, NATD is a solution generating transformation. The dual supergravity fields can be read off from the transformed DFT fields given below:

$$\mathcal{H}'(x,\nu) = T_{\text{NATD}}(\nu)\mathcal{H}(x)(T_{\text{NATD}})^t(\nu), \quad (81)$$
$$\mathcal{K}'(x,\nu) = S_{\text{NATD}}(\nu)\mathcal{K}(x)(S_{\text{NATD}})^{-1}(\nu), \quad (82)$$
$$F'(x,\nu) = e^{-\sigma(\nu)}e^{-B'(x,\nu)}S_{\text{NATD}}(\nu)e^{B(x)}F(x), \quad (83)$$
$$d'(x,\nu) = d(x) + \sigma(\nu). \quad (84)$$

$B'(x, \nu)$ that appears in (83) is read off from $\mathcal{H}'(x, \nu)$ in (81). The field $\sigma(\nu)$ in (84) and (83) is non-vanishing only when the isometry group is non-unimodular.

As we mentioned in section 3, the invariance of fluxes under the NATD transformation plays a crucial role in showing that NATD is a solution generating transformation. Interestingly, this idea is also sufficient to prove that NATD also preserves supersymmetry of the background, at least for backgrounds of a certain type. Indeed, very recently in [75] transformation of supersymmetry equations under NATD was discussed also using the framework GDFT for backgrounds which are of the form $Mink_{3,1} \times M$, where M is six dimensional. Demanding that the four dimensional solution preserves at least $\mathcal{N} = 1$ supersymmetry implies that the structure group of the generalized tangent bundle $TM \oplus T^*M$ of the six dimensional internal manifold M is reduced from $SO(6, 6)$ to $SU(3) \times SU(3)$. This topological condition on the internal manifold implies the existence of two globally defined compatible pure spinors Φ_1 and Φ_2 of non-vanishing norm. These $Clif(6, 6)$ spinors can be constructed from the internal spinors arising from the 10 dimensonal Killing spinors generating the supersymmetry transformations in

[6] These are almost the same as the field equations of GDFT, except for the equations coming from variations of the field χ. There are also some subtleties when the twist matrix is not in $Spin^+(d, d)$. See [34] for a detailed discussion.

10 dimensions. For preservation of supersymmetry, these pure spinors (polyforms) must satisfy the following set of first order differential equations, as was first shown in [76]:

$$d(e^{2A-\phi}e^B \wedge \Phi_1) = 0, \tag{85}$$
$$d(e^{2A-\phi}e^B \wedge \Phi_2) = e^{2A-\phi}dA \wedge e^B \wedge \bar{\Phi}_2 + \frac{i}{8}e^{3A}e^B \wedge \lambda(*_6 F). \tag{86}$$

For detailed information and derivation of these equations, see Appendix A of [77] and Appendix B of [78]. Under NATD, the pure spinors transform as

$$\Phi \to \Phi' = \sqrt{G}\ S_{\text{NATD}}.\Phi = \sqrt{G}\ e^{-B'}S_{\text{NATD}}e^B\Phi, \tag{87}$$

where

$$G = det((g+B)+\theta)^{-1}. \tag{88}$$

In [75] these equations were embedded in DFT and it was shown that (87) maps solutions to solutions by using the framework of GDFT, the key idea again being the preservation of fluxes.

4.2. Yang-Baxter deformations

Yang-Baxter (YB) sigma model was introduced by Klimcik in [79] as a particular type of deformation for Principal Chiral Models (PCM) with simple compact Lie group G as the target manifold. The deformation is based on solutions of modified classical Yang-Baxter equation (mCYBE) associated with the Lie algebra of G. The integrability of the YB sigma model was shown again by Klimcik later in [80]. The applicability of YB deformations was extended to symmetric coset spaces in [81], which in turn was applied to $AdS_5 \times S^5$ in [82]. The NS-NS sector of the corresponding deformed supergravity background was found in [83]. When applied to AdS backgrounds, such deformations are usually called η-deformations. It is also possible to consider similar deformations based on the classical Yang-Baxter equation (CYBE), as opposed to mCYBE and the resulting models are called homogeneous Yang-Baxter models. Such deformations of the $AdS_5 \times S^5$ string were first studied in [84]. These deformations are particularly interesting, as they provide a natural generalization of the Lunin-Maldacena (LM) deformations [85], obtained by T-duality-shift-T-duality (TsT) transformations [86, 87, 88]. Recently it has been understood that YB deformations, like NATD, can also be described as a coordinate dependent $O(d,d)$ transformation. In fact, this is not surprising, since it is by now well understood that YB deformation is related to NATD [89, 90, 91, 30]. The first work in this direction was [32], where it was proposed that the homogeneous YB deformation could be obtained by the action of a β-transformation. This proposal is based on observation and is supported by working out the example of the $AdS_5 \times S^5$ background. Later in [33], a proof of the proposal of [32] was given for deformations of the $AdS_5 \times S^5$ background. Then in [36], it was shown that the deformed sigma model constructed in [92] as a deformation of a generic string sigma model can also be obtained by the action of a coordinate dependent $O(d,d)$ transformation. Before we present the details, let us note that the approach adopted in the papers [93, 94, 95, 96], which regards YB deformation as an open-closed string map [97] also contributes to the understanding of the $O(d,d)$ structure of YB deformations, as the open-closed string map can be regarded as a special case of β-transformation [36].

The $O(d,d)$ transformation that generates the YB deformation is as follows[7]:

$$T_{\rm YB} = L.\begin{pmatrix} 1 & 0 \\ \eta R_g^{IJ} & 1 \end{pmatrix}. \tag{89}$$

Here, the first factor L is as in (80) and the second factor is a β-transformation (61) with the bivector field R_g constructed from an R-matrix on the Lie algebra of the isometry group. An R-matrix on a Lie algebra g with a non-degenerate inner product is a skew-symmetric endomorphism on g, which satisfies the CYBE [98]:

$$[RX, RY] - R([RX, Y] + [X, RY]) = 0, \quad \forall X, Y \in g. \tag{90}$$

When this equation is satisfied, the skew-symmetric bracket

$$[X, Y]_R = [RX, Y] + [X, RY] \tag{91}$$

satisfies the Jacobi identity and hence also defines a Lie bracket. The dressed R-matrix R_g is constructed by extending R to the whole group manifold by the adjoint action:

$$R_g = Ad_{g^{-1}} R Ad_g. \tag{92}$$

Obviously, R_g satisfies CYBE if R does, since the adjoint action is an automorphism of the Lie bracket. An interesting fact is that the brackets $[,]$ and $[,]_R$ are compatible in the sense that it is possible to construct a Lie bracket (from the brackets $[,]$ and $[,]_R$) on the direct sum $g \oplus g^*$, that is (g, g^*) is a Lie bialgebra and $g \oplus g^*$ has a Drinfeld double structure (see footnote 5). Now, one can construct a bivector field from the dressed R-matrix R_g (which we will also call R_g) in the following way: first one defines a non-degenerate 2-form ω

$$\kappa(RX, Y) = \omega(X, Y), \quad X, Y \in g. \tag{93}$$

Here κ is the non-degenerate inner product on g. ω is indeed a 2-form as the skew-symmetry property of R implies that $\omega(X, Y) = -\omega(Y, X)$. Also, it is non-degenerate, at least on a subalgebra of g called the Frobenius subalgebra[8]. On this subalgebra one can define a bivector field β

$$\beta = \frac{1}{2} R_g^{IJ} T_I \wedge T_J, \tag{94}$$

with

$$\omega(T_I, T_J) = (R_g^{-1})_{IJ}. \tag{95}$$

In the above expressions, T_I, T_J are the generators of the Frobenius subalgebra. The fact that R_g satisfies the CYBE is then equivalent to the condition that the Schouten bracket (a natural

[7] The matrix (89) differs slightly both from the one in [32] and the one in [36]. This is because in [32] the $O(d,d)$ matrix (equal to $T_{\rm YB}L^{-1}$) acts on the seed background directly, whereas here our formalism requires that the YB matrix should act on untwisted fields. On the other hand, [36] is interested in how untwisted fields are mapped to untwisted fields under YB deformation, and this mapping is generated by $L^{-1}T_{\rm YB}$. This is the reason why the flux associated with the YB matrix found in [36] is Q-flux, whereas here we will find that the Q-flux is zero and the geometric flux is preserved. See Appendix B of [36] for more details.

[8] It is known that skew-symmetric solutions for CYBE on a finite dimensional Lie algebra g are in one-to-one correspondence with quasi-Frobenius subalgebras of g. On such a subalgebra the R-matrix acts as a non-degenerate operator. If this subalgebra is Abelian, the R-matrix is called Abelian; if this subalgebra is unimodular, the R-matrix is called unimodular. The rank of the R-matrix is equal to the dimension of the quasi-Frobenius subalgebra. For more detail, see [99] and Proposition 22.6 and Proposition 3.1.6 in the book [100].

extension of the Lie bracket to multivector fields) of β with itself (which is normally a trivector) vanishes:

$$[\beta, \beta]_S = 0. \tag{96}$$

Equivalently, the bivector field β defines a Poisson structure.

In order to find the supergravity background corresponding to the YB deformed sigma model, one has to form the untwisted DFT fields as we did in the previous section and transform them as in (81)-(83), where now the $O(d,d)$ and $Spin(d,d)$ matrices involved are $T_{\rm YB}$ and $S_{\rm YB}$. The resulting fields solve equations of DFT (and hence of Type II supergravity since one remains in the supergravity frame) because the flux associated with the YB transformation is just geometric flux with $f_{ij}{}^k = C_{ij}{}^k$. In fact for an arbitrary bivector field, a twist matrix of the form (89) would also have R-flux. However, it is identically zero in our case, resulting directly from the fact that β defines a Poisson structure.

5. Outlook

Symmetry plays a fundamental role in physics. The dynamics of some of the most successful theories in physics is *dictated* by symmetry principles. This is the case with the standard model of particle physics, where the theory is based on gauge symmetries described by Lie algebras and also is the case with general relativity, where the structure of the theory is determined by demanding invariance under diffeomorphisms and Lorentz transformations. String theory possesses duality symmetries, which do not exist in conventional field theories. In recent years, remarkable progress has been made in constructing field theories which are *dictated* by these duality symmetries. This review paper has focused on some aspects of one of these theories, Double Field Theory, which is based on T-duality.

As we have seen in parts of the paper, construction of DFT requires using a mathematical framework that goes beyond semi-Riemannian geometry. Most of the mathematical structures are constructed (at least in the supergravity frame) on the direct sum bundle $TM \oplus T^*M$, a prototypical Courant algebroid. This Courant algebroid also plays a fundamental role for the generalized geometry program of Nigel Hitchin [45, 46], and indeed many constructions in DFT are closely related to those in generalized geometry.

Instead of discussing these interesting mathematical structures in detail, we have preferred in this paper to focus on a rather practical feature of DFT: its role in generating new, interesting supergravity solutions. Recently, it has been understood that certain (integrable) deformations of string sigma models can be described as coordinate dependent $O(d,d)$ transformations. On the other hand, when embedded in DFT, transformations of the supergravity fields under $O(d,d)$ (and its spinorial counterpart) can be described very easily. This makes DFT a very convenient setting for studying such deformations. We studied in detail two such (coordinate dependent) $O(d,d)$ transformations. The first one generates the non-Abelian T-dual of a given supergravity background with non-Abelian isometries, and the other one generates the supergravity fields in the target space of the Yang-Baxter deformation of a string sigma model. In both cases, the transformed fields can be expressed in a separated form as in a SS reduction, so the transformed theory can be easily analyzed by using tools from GDFT. This is a deformation of DFT determined by fluxes, and we have seen that invariance of fluxes under the associated $O(d,d)$ transformation is the key idea in showing that NATD and YB deformations are both solution generating transformations. It is also the key idea in showing that NATD preserves supersymmetry.

The approach we have discussed here makes it clear that classification of solution generating transformations for supergravity can be turned into a less difficult, algebraic problem: classification of fluxes. Indeed, this approach has been used very recently in the interesting papers [101, 102, 38]. An interesting future direction would be to try and understand solution generating transformations in terms of morphisms of Courant algebroids. Indeed, we have seen that fluxes, under the conditions imposed by the requirement of consistency of DFT, form structure functions of a Courant algebroid. Hence, understanding morphisms of Courant algebroids preserving structure functions might be a fruitful way to explore solution generating transformations.

T-duality covariant formulation of string theory has many remarkable features. It leads to new, interesting mathematical structures and also offers practical solutions to some interesting problems. We believe both these directions are worth further exploration.

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