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Louise Dolan, Peter Goddard

Current Algebra on the Torus

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Abstract We derive the N-point one-loop correlation functions for the currents of an arbitrary affine Kac-Moody algebra. The one-loop amplitudes, which are elliptic functions defined on the torus Riemann surface, are specified by group invariant tensors and certain constant tau-dependent functions. We compute the elliptic functions via a generating function, and explicitly construct the invariant tensor functions recursively in terms of Young tableaux. The lowest tensors are related to the character formula of the representation of the affine algebra. These general current algebra loop amplitudes provide a building block for open twistor string theory, among other applications.

1 Introduction

Current algebra conformal field theory is often an important ingredient to supply gauge symmetry in string theory. The tree level N-point correlation functions of the currents (1) of an affine Kac-Moody Lie algebra (2), $\hat{\mathfrak{g}}$,

$$[J^a_m, J^a_n] = f^{ab}{}_c J^c_{m+n} + \kappa^{ab} m \delta_{m,-n}, \tag{1.1}$$

associated with a finite-dimensional algebra, $\mathfrak{g}$, are especially simple, and expressed as a sum over products of differences, with the group tensors given by the level and structure constants of the affine algebra.

The current correlators on the torus have more structure, but turn out to be computable in terms of elliptic functions, and specified by constant but tau-dependent group invariant tensors. Recursion relations for these correlation functions (3) become tedious to evaluate for large numbers of currents. In this paper we calculate the one-loop N-point current correlation functions explicitly for an arbitrary Lie

University of North Carolina, Chapel Hill, NC 27599, USA. ldolan@physics.unc.edu · Institute for Advanced Study, Princeton, NJ 08540, USA

group, and describe their dependence on rather neat combinations of Weierstrass functions and on group tensors given in terms of the character of the representation.

Loop calculations were considered for vertex operator algebras in (4; 5), for particular constructions and levels of current algebras in (6; 7), and for particular Lie groups (8). Our general treatment of the affine current correlators is possible due to the simple holomorphic operator products of the currents. Loop correlation functions for other fields related to current algebras tend to be less completely accessible, although widely studied (9; 10; 11; 12; 13; 14; 15; 16; 17; 18).

Our interest in the current algebra torus correlator was initially motivated by its appearance in the gluon loop amplitude (3) for open twistor string theory (19; 20). The N-point torus current correlator should be helpful to pursue perturbation theory there. The twistor string (21; 19), and efforts to formulate it as a heterotic theory (22), although mixing conformal supergravity with Yang-Mills, also provides an enticing framework for a QCD string. Our analysis of current algebra on the torus provides a fundamental building block that will have general applications.

The plan of this paper is as follows. In Sect. 2.1, we first use the representation of a current algebra as bilinear expressions in (Neveu-Schwarz) fermions to evaluate current algebra tree amplitudes. The expressions obtained involve the tensors formed from the traces of products of the real matrices representing $\mathfrak{g}$ and can be described by a set of graphical rules that will be extended later in the paper to yield loop amplitudes. Although the tensors depend on the representation chosen, this dependence cancels out in the expressions for the tree amplitudes because these are determined by κ^{ab}. For a compact simple algebra, we can take $\kappa^{ab} = k\delta^{ab}$ and we can obtain the general tree amplitude by scaling terms in the result obtained for any given representation by appropriate powers of k.

Notwithstanding this, in Sect. 2.2, we find it useful to give a more generally phrased version of the construction, due to Frenkel and Zhu (FZ) (1). This generalizes the traces of representation matrices to invariant m^{th} order tensors κ_m, satisfying conditions (2.15) and (2.16), which determine κ_m in terms of κ_{m-1} uniquely up to an arbitrary symmetric invariant tensor ω_m. The successive freedoms, represented by the ω_n, have no effect on the tree amplitudes constructed using the κ_m. We isolate a "connected" part of the tree amplitude, which possesses only simple poles and show that, like the full amplitude, this just depends on $\kappa_2 = \kappa$, and so not on the ω_m. In Sect. 2.3 we give a proof that, given a suitable κ_{m-1}, there exists a κ_m satisfying (2.15) and (2.16), and we give explicit formulae for the general κ_3 and κ_4. Our proof of the existence of κ_n does not itself provide a convenient algorithmic construction and we give this in Appendix A using Young tableaux and the representation theory of the permutation group.

In Sect. 3.1, we begin by computing the n-point one loop amplitude using the representation of the current algebra as bilinears in Neveu-Schwarz fermions. The result is given by a modification of the graphical rules used in Sect. 2.1 to describe tree amplitudes. Similar rules describe two other versions of the loop: one in which we use Neveu-Schwarz fermions but also incorporate a factor of $(-1)^{N_b}$, where N_b is the fermion number operator, into the trace defining the loop; and one where we use Ramond rather than Neveu-Schwarz fermions. These rules involve the tensors constructed from traces of representation matices, used in the fermionic construction of tree amplitudes. There nearly all the structure resulting

from varying the representation, reflected in the 'arbitrary' symmetric tensors ω_m coming into the FZ construction, was irrelevant, but this is not so for the loops.

To approach the construction of the general one loop current algebra amplitude, we isolate a connected part of the amplitude in Sect. 3.2, which has only single poles, as we did for the tree amplitudes. The residues of this connected part for the n-point loop are specified in terms of the $(n-1)$-point loop and this means that the n point loop is determined in this way up to a symmetric invariant n^{th} order tensor, $\omega_n(\tau)$, depending only on the torus modulus, τ. In Sect. 3.3, we first obtain general forms for the two- and three-point loops in terms of symmetric invariant tensors $\omega_2(\tau)$ and $\omega_3(\tau)$ and Weierstrass $\mathscr{P}$ and ζ functions. The general form for the n-point loop is given by an adaptation of the rules for tree amplitudes, expressed in terms of Weierstrass σ functions through

$$\nu^{-n}H_n = \prod_{j=1}^{n} \frac{\sigma(\mu_j+\nu,\tau)}{\sigma(\nu,\tau)\sigma(\mu_j,\tau)} = \sum_{m=0}^{\infty} H_{n,m}\nu^{m-n}, \tag{1.2}$$

which is elliptic as a function of ν and $\mu_1,\dots\mu_n$, provided that $\sum_{j=1}^{n}\mu_j = 0$, and in terms of n^{th} order invariant tensor functions of τ, $\kappa_{n,m}(\tau)$, with $n \geq m \geq 2$, defined inductively by (3.55) and (3.56) (which are similar to (2.15) and (2.16)), starting from invariant symmetric tensors $\kappa_{n,0}(\tau) = \omega_n(\tau)$. In Appendix B we discuss properties of the functions $H_{n,m}$ and in Appendix C we show how the general results of this section relate to those previously obtained in (3) for two-, three- and four-point loops.

The symmetric tensors ω_n, irrelevant in the construction of tree amplitudes in Sect. 2, provide the extra structure necessary for the construction of the one-loop amplitudes. They are not arbitrary but can be determined in terms of traces of zero modes of the currents, $\text{tr}\left(J_0^{a_{i_1}}J_0^{a_{i_2}}\dots J_0^{a_{i_n}}w^{L_0}\right)$, symmetrized over the indices a_j. In 4.1, we establish recurrence relations relating the traces over symmetrized products of currents, in terms of which the $\omega_n(\tau)$ are initially defined, to symmetrized traces of their zero modes, showing how this works out in detail for $n = 2,3$ and 4. More precisely, $\omega_n(\tau)$ are defined in terms of the connected parts of the symmetrized traces of currents and, in 4.2, we use the recurrence relations to determine $\omega_n(\tau)$ in terms of the connected part of the symmetrized trace of zero modes.

Then, in Sect. 4.3, we show how the symmetrized traces of zero modes of the currents can be determined in terms of

$$\chi(\theta,\tau) = \text{tr}\left(e^{iH\cdot\theta}w^{L_0}\right), \tag{1.3}$$

the character of the representation of $\hat{\mathfrak{g}}$ provided by the space of states of the theory. While the analysis up to this point has not made any assumptions about the Lie algebra $\mathfrak{g}$, in this section we assume that it is compact and, for ease of exposition, take it to be simple. The method depends on using the Harish-Chandra isomorphism of the center of the enveloping algebra of $\mathfrak{g}$, that is the ring of Casimir operators of $\mathfrak{g}$, onto the polynomials in H invariant under the action of the Weyl group, $W_{\mathfrak{g}}$ of $\mathfrak{g}$.

Section 5 provides a summary of our results.

2 Current Algebra Trees

2.1 Current algebra and the Fermionic tree construction

We consider a conformal field theory containing the affine algebra, $\hat{\mathfrak{g}}$, given by (1.1), where m,n are integers and $f^{ab}{}_c$ are the structure constants of $\mathfrak{g}$ and κ^{ab} is a symmetric tensor invariant with respect to $\mathfrak{g}$. [If the generators of the algebra satisfy the hermiticity condition $J_n^{a\dagger} = J^a_{-n}$, $f^{ab}{}_c$ is pure imaginary and κ^{ab} is real.] For a general introductory review see (23).

We consider evaluating the vacuum expectation value

$$\mathscr{A}^{a_1a_2\ldots a_n}_{\text{tree}}(z_1,z_2,\ldots,z_n) = \langle 0|J^{a_1}(z_1)J^{a_2}(z_2)\ldots J^{a_n}(z_n)|0\rangle, \tag{2.1}$$

where

$$J^a(z) = \sum_n J^a_n z^{-n-1}, \qquad J^a_n|0\rangle = 0, \quad n \geq 0, \qquad (J^a_n)^\dagger = J^a_{-n}. \tag{2.2}$$

The currents $J^a(z)$ satisfy the operator product expansion

$$J^a(z_1)J^b(z_2) \sim \frac{\kappa^{ab}}{(z_1-z_2)^2} + \frac{f^{ab}{}_c J^c(z_2)}{z_1-z_2} \tag{2.3}$$

and the tree amplitudes satisfy the asymptotic condition

$$\mathscr{A}^{a_1a_2\ldots a_n}_{\text{tree}}(z_1,z_2,\ldots,z_n) = \mathscr{O}(z_j^{-2}) \quad \text{as} \quad z_j \to \infty, \tag{2.4}$$

because in this limit $\langle 0|J^a(z) \sim \langle 0|J^a_1 z^{-2}$.

Because of the locality of the currents $J^a(z)$ relative to one another, the tree amplitude (2.1) is symmetric under simultaneous permutations of the z_i and a_i,

$$\mathscr{A}^{a_{\rho(1)}a_{\rho(2)}\cdots a_{\rho(n)}}_{\text{tree}}(z_{\rho(1)},z_{\rho(2)},\ldots,z_{\rho(n)}) = \mathscr{A}^{a_1a_2\ldots a_n}_{\text{tree}}(z_1,z_2,\ldots,z_n), \tag{2.5}$$

where $\rho \in \mathfrak{S}_n$, the group of permutations on n objects

The condition (2.3) gives all the singularities of the n-point function in terms of $(n-1)$- and $(n-2)$-point functions. Thus, given (2.4), using Cauchy's Theorem, we can inductively calculate the n-point function for any n starting from the two-point function,

$$\langle 0|J^a(z_1)J^b(z_2)|0\rangle = \frac{\kappa^{ab}}{(z_1-z_2)^2}, \tag{2.6}$$

i.e. the n-point function is determined by the invariant symmetric tensor κ^{ab}. A general prescription for doing this has been given by Frenkel and Zhu (1), which we shall discuss in Sect. 2.2, but first we shall note the explicit calculation when $J^a(z)$ is given as a bilinear in fermionic oscillators.

Given a representation $J^a_0 \mapsto t^a = iM^a$ of $\mathfrak{g}$, where the M^a are N-dimensional real antisymmetric matrices satisfying

$$[M^a,M^b] = -if^{ab}{}_c M^c, \tag{2.7}$$

we can represent $J^a(z)$ as a bilinear in Neveu-Schwarz fermionic fields,

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1} = \frac{i}{2} M^a_{ij} b^i(z) b^j(z), \tag{2.8}$$

where

$$b^i(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} b^i_r z^{-r-\frac{1}{2}}, \quad \{b^i_r, b^j_s\} = \delta_{r,-s}\delta^{ij}, \quad b^i_r|0\rangle = 0, \quad r>0, \quad 1\le i,j\le d \tag{2.9}$$

Then J^a_n satisfies (1.1) with

$$\kappa^{ab} = -\tfrac{1}{2}\mathrm{tr}(M^a M^b) = \tfrac{1}{2}\mathrm{tr}(t^a t^b). \tag{2.10}$$

Note

$$b^i(z_1)b^j(z_2) =: b^i(z_1)b^j(z_2) : + \frac{\delta^{ij}}{z_1 - z_2}, \tag{2.11}$$

with the usual definition of normal ordering.

Using Wick's theorem, we can evaluate the tree amplitude (2.1) and describe the result as follows. The n-point function can be written as a sum over permutations
$\rho \in \mathfrak{S}_n$ with no fixed point. Each such permutation can be written as a product of cycles, $\rho = \xi_1\xi_2\ldots\xi_r$ and we associate to ρ a product $F_\rho = (-1)^r f_{\xi_1} f_{\xi_2}\ldots f_{\xi_r}$, where the function f_ξ is associated with the cycle $\xi = (i_1, i_2 \ldots i_m)$, defined by

$$f_\xi = \frac{\frac{1}{2}\mathrm{tr}(t^{a_1}t^{a_2}\ldots t^{a_m})}{(z_{i_1}-z_{i_2})(z_{i_2}-z_{i_3})\ldots(z_{i_m}-z_{i_1})}. \tag{2.12}$$

The n-point tree amplitude is then constructed as the sum of these products over the permutations $\rho \in \mathfrak{S}'_n$, the subset of $\mathfrak{S}_n$ with no fixed points,

$$\mathscr{A}^{a_1 a_2\ldots a_n}_{\mathrm{tree}}(z_1, z_2, \ldots, z_n) = \sum_{\rho\in\mathfrak{S}'_n} F^{a_1 a_2\ldots a_n}_\rho(z_1, z_2, \ldots, z_n). \tag{2.13}$$

2.2 The Frenkel-Zhu construction

Frenkel and Zhu have shown how the fermionic construction of the last section can be modified to give the general construction for the tree amplitude (2.1). Again the n-point function (2.1) is written as a sum over permutations with no fixed point, $\rho = \xi_1\xi_2\ldots\xi_r$, written as a product of cycles, with which is associated $F_\rho = (-1)^r f_{\xi_1} f_{\xi_2}\ldots f_{\xi_r}$, where now

$$f_\xi = \frac{\kappa_m^{a_{i_1} a_{i_2}\ldots a_{i_m}}}{(z_{i_1}-z_{i_2})(z_{i_2}-z_{i_3})\ldots(z_{i_m}-z_{i_1})}, \tag{2.14}$$

and the m-order tensors κ_m are defined inductively by the conditions

$$\kappa_m^{a_1 a_2 a_3\ldots a_m} - \kappa_m^{a_2 a_1 a_3\ldots a_m} = f^{a_1 a_2}{}_b \kappa_{m-1}^{b a_3\ldots a_m}, \tag{2.15}$$

and

$$\kappa_m^{a_1a_2a_3\ldots a_m} = \kappa_m^{a_2a_3\ldots a_ma_1}. \tag{2.16}$$

The n-point tree amplitude is then constructed as in (2.13).

A graphical way of describing the Frenkel-Zhu construction (or the fermionic construction) is to say that the n-point tree amplitude is given by summing over all graphs with n vertices where the vertices carry the labels $1,2,\ldots,n$, and each vertex is connected by directed lines to other vertices, one of the lines at each vertex pointing towards it and one away from it. Then each graph consists of a number of directed "loops" or cycles, $\xi = (i_1, i_2 \ldots i_m)$, with which we associate the expression (2.14) and the expression associated with the whole graph is the product of the expressions for the various cycles multiplied by a factor of -1 for each cycle.

As is implied by comparing (2.12) and (2.14), a solution to the conditions (2.15) and (2.16) can be constructed by setting $\kappa_m^{a_1a_2a_3\ldots a_m} = \mathrm{tr}(t^{a_1}t^{a_2}\ldots t^{a_m})$ or, more generally,

$$\kappa_m^{a_1a_2\ldots a_m} = \mathrm{tr}(Kt^{a_1}t^{a_2}\ldots t^{a_m}), \tag{2.17}$$

where t^a is any finite-dimensional representation of $\mathfrak{g}$, *i.e.*

$$[t^a, t^b] = f^{ab}{}_c t^c, \tag{2.18}$$

and K is any matrix commuting with all the t^a, *i.e.* invariant under the action of $\mathfrak{g}$. K is to be chosen so that $\kappa_2^{ab} = \mathrm{tr}(Kt^at^b) = \kappa^{ab}$ as in (1.1), which can be done for any invariant tensor κ^{ab} if $\mathfrak{g}$ is compact and t^a a faithful representation.

It is straightforward to verify that (2.13) has the singularity structure implied by the operator product expansion (2.3), provided that κ_m satisfies (2.15) and (2.16), and satisfies the asymptotic condition (2.4), and thus is inductively determined by Cauchy's Theorem, given the two-point function (2.6). Thus, it does not depend on the choice of κ_m satisfying (2.15) and (2.16), apart from through $\kappa_2 = \kappa$. (In particular, although different choices of representation t^a result in different tensors κ_m, as defined through (2.17), these differences cancel out in (2.14), apart from dependence on κ_2.) In fact, the stronger statement holds that the connected parts, that is the sums of (2.14) over permutations of $(i_1, i_2 \ldots i_m)$, only depend on the κ's through κ_2. This is expressed in the following proposition:

Proposition 1 *If $\mathfrak{g}$ is a Lie algebra and the tensors $\kappa_m^{a_1a_2\ldots a_m}$, where $1 \le a_j \le \dim\mathfrak{g}$, are defined for $m \le N$, and satisfy*

$$\kappa_m^{a_1a_2a_3\ldots a_m} - \kappa_m^{a_2a_1a_3\ldots a_m} = f^{a_1a_2}{}_b \kappa_{m-1}^{ba_3\ldots a_m}, \tag{2.19}$$

and

$$\kappa_m^{a_1a_2a_3\ldots a_m} = \kappa_m^{a_2a_3\ldots a_ma_1}, \tag{2.20}$$

where $f^{ab}{}_c$ are the structure constants of $\mathfrak{g}$, then the tensor functions

$$\mathscr{A}^{a_1a_2\ldots a_m}_{tree,C}(z_1,z_2,\ldots,z_m) = \frac{1}{m}\sum_{\rho\in\mathfrak{S}_m} f_{(\rho(1),\rho(2),\ldots,\rho(m))} \tag{2.21}$$

$$= \frac{1}{m}\sum_{\rho\in\mathfrak{S}_m} \frac{\kappa_m^{a_{\rho(1)}a_{\rho(2)}\cdots a_{\rho(m)}}}{(z_{\rho(1)}-z_{\rho(2)})(z_{\rho(2)}-z_{\rho(3)})\cdots(z_{\rho(m)}-z_{\rho(1)})}$$

$$= \sum_{\rho\in\mathfrak{S}_{m-1}} \frac{\kappa_m^{a_{\rho(1)}a_{\rho(2)}\cdots a_{\rho(m-1)}a_m}}{(z_{\rho(1)}-z_{\rho(2)})(z_{\rho(2)}-z_{\rho(3)})\cdots(z_{\rho(m-1)}-z_m)(z_m-z_{\rho(1)})} \tag{2.22}$$

depend on the κ_m only through κ_2.

Proof of the Proposition The result follows from Cauchy's Theorem because the functions F defined by (2.22) satisfy

$$\mathscr{A}^{a_1a_2\ldots a_m}_{\text{tree},C}(z_1,z_2,\ldots,z_m) = \mathscr{O}(z_1^{-2}), \quad \text{as} \quad z_1\to\infty$$

and

$$\mathscr{A}^{a_1a_2a_3\ldots a_m}_{\text{tree},C}(z_1,z_2,\ldots,z_m) \sim \frac{f^{a_1a_2}{}_b}{z_1-z_2}\mathscr{A}^{ba_3\ldots a_m}_{\text{tree},C}(z_2,\ldots,z_m) \quad \text{as} \quad z_1\to z_2,$$

and so can be calculated inductively from

$$\mathscr{A}^{ab}_{\text{tree},C}(z_1,z_2) = \frac{\kappa^{ab}}{(z_1-z_2)^2}.$$

2.3 The tensors κ_n

The conditions

$$\kappa_n^{a_1a_2a_3\ldots a_n} - \kappa_n^{a_2a_1a_3\ldots a_n} = f^{a_1a_2}{}_b\kappa_{n-1}^{ba_3\ldots a_n}, \tag{2.23}$$

and

$$\kappa_n^{a_1a_2a_3\ldots a_n} = \kappa_n^{a_2a_3\ldots a_na_1} \tag{2.24}$$

are sufficient to ensure that the amplitudes defined by (2.13) with $F_\rho = (-1)^r f_{\xi_1}f_{\xi_2}\ldots f_{\xi_r}$, where f_ξ is given by (2.14), depend on the κ_m only through κ_2. However, κ_2 does not uniquely determine κ_m through (2.23) and (2.24). In this section, we shall discuss the existence and uniqueness of solutions to these equations. Although the arbitrariness in κ_m, given κ_2, is not relevant for tree amplitudes, we shall see in 3 that this freedom is very relevant for the construction of the one-loop amplitudes.

The conditions (2.23) and (2.24) have some immediate consequences. First, if κ_{n-1} satisfies (2.23) for some κ_n which also satisfies (2.24), then κ_{n-1} is invariant because

$$\sum_{j=1}^n f^{ba_j}{}_c\kappa^{a_1\ldots a_{j-1}ca_{j+1}\ldots a_n} = \sum_{j=1}^n(\kappa^{a_1\ldots a_{j-1}ba_ja_{j+1}\ldots a_n} - \kappa^{a_1\ldots a_{j-1}a_jba_{j+1}\ldots a_n}) = 0,$$

using (2.23) and then (2.24). Thus for (2.23) and (2.24) to have a solution for a given κ_{n-1} then this tensor must be invariant.

Second, if $\kappa_n^{a_1a_2a_3\ldots a_n}$ and $\tilde{\kappa}_n^{a_1a_2a_3\ldots a_n}$ both satisfy (2.23) with the same $\kappa_{n-1}^{a_1a_2a_3\ldots a_{n-1}}$ and both satisfy the cyclic property (2.24), then the difference

$$\omega_n^{a_1a_2a_3\ldots a_n} = \tilde{\kappa}_n^{a_1a_2a_3\ldots a_n} - \kappa_n^{a_1a_2a_3\ldots a_n}$$

is cyclically symmetric and satisfies

$$\omega_n^{a_1a_2a_3\ldots a_n} = \omega_n^{a_2a_1a_3\ldots a_n}.$$

These two symmetries generate the whole of $\mathfrak{S}_n$ so that ω_n must be a symmetric tensor. Conversely, if ω_n is symmetric, it follows that $\tilde{\kappa}_n = \kappa_n + \omega_n$ satisfies (2.23) and (2.24) if κ_n does. So κ_{n-1} defines κ_n through (2.23) and (2.24), assuming a solution exists, up to a symmetric tensor ω_n. We establish the existence of the solution in the following proposition:

Proposition 2 *If $\mathfrak{g}$ is a Lie algebra, define inductively the spaces $\mathscr{K}_n$ to consist of the invariant n^{th} order tensors $\kappa_n^{a_1a_2\ldots a_n}$, where $1 \le a_j \le \dim\mathfrak{g}$, such that*

$$\kappa_n^{a_1a_2a_3\ldots a_n} - \kappa_n^{a_2a_1a_3\ldots a_n} = f^{a_1a_2}{}_b\kappa_{n-1}^{ba_3\ldots a_n}, \tag{2.25}$$

for some $\kappa_{n-1} \in \mathscr{K}_{n-1}$, where $f^{ab}{}_c$ are the structure constants of $\mathfrak{g}$, and

$$\kappa_n^{a_1a_2a_3\ldots a_n} = \kappa_n^{a_2a_3\ldots a_na_1}, \tag{2.26}$$

with $\mathscr{K}_0 = \{0\}$. Then, for each $\kappa_{n-1} \in \mathscr{K}_{n-1}$, there exists a κ_n satisfying (2.25) *and* (2.26) *that is unique up to the addition of a symmetric invariant tensor ω_n. The solution can be uniquely specified by requiring that it be orthogonal to all symmetric n^{th} order tensors.*

Proof of the Proposition We define the action of $\rho \in \mathfrak{S}_n$ on n^{th} order tensors τ_n by

$$(\rho\tau_n)^{a_1a_2\ldots a_n} = \tau_n^{a_{\rho^{-1}(1)}a_{\rho^{-1}(2)}\cdots a_{\rho^{-1}(n)}}$$

so that this provides a representation of $\mathfrak{S}_n$ on n^{th} order tensors: $(\rho\sigma)\tau_n = \rho(\sigma\tau_n)$. For any n^{th} order tensor τ_n write

$$\eta(\rho,\tau_n) = \tau_n - \rho\tau_n; \tag{2.27}$$

then we can write

$$\tau_n = \frac{1}{n!}\sum_{\rho\in\mathfrak{S}_n}\eta(\rho,\tau_n) + \omega_n, \tag{2.28}$$

where

$$\omega_n = \frac{1}{n!}\sum_{\rho\in\mathfrak{S}_n}\rho\tau_n, \tag{2.29}$$

is the symmetrization of the tensor τ_n. If τ_n is invariant, $\eta(\rho,\tau_n)$ is also invariant. Then, if $\kappa_n \in \mathscr{K}_n$,

$$\frac{1}{n!}\sum_{\rho\in\mathfrak{S}_n}\eta(\rho,\kappa_n) \tag{2.30}$$

is also in $\mathscr{K}_n$ and satisfies (2.25) for the same $\kappa_{n-1}\in\mathscr{K}_{n-1}$; further, it is orthogonal to any symmetric tensor. It is clear from (2.28) that, taking $\tau_n=\kappa_n\in\mathscr{K}_n$, κ_n is orthogonal to all symmetric tensors only if ω_n, defined as in (2.29), vanishes. Thus, (2.30) is the unique solution to (2.25) and (2.26) for the given κ_{n-1}, with this property.

We now proceed to use the expression (2.30) to show there exists a solution to (2.25) and (2.26) for a given $\kappa_{n-1}\in\mathscr{K}_{n-1}$. From (2.27),

$$\eta(\rho_1\rho_2,\kappa_n)=\kappa_n-\rho_1\rho_2\kappa_n=\eta(\rho_1,\kappa_n)+\rho_1\eta(\rho_2.\kappa_n) \tag{2.31}$$

and, so,

$$\eta(\rho_1\dots\rho_k,\kappa_n)=\sum_{j=1}^{k}\rho_1\dots\rho_{j-1}\eta(\rho_j,\kappa_n). \tag{2.32}$$

We can use this to give a formula for a given κ_n, in terms of $\kappa_{n-1}\in\mathscr{K}_{n-1}$, by expressing each $\rho\in\mathfrak{S}_n$ as a product of transpositions of adjacent indices and then using (2.25) and (2.26). However, such expressions are not unique, so we need to address this by first working in the free group, $\tilde{\mathfrak{S}}_n$, generated by these transpositions, defining a function $\tilde{\phi}:\tilde{\mathfrak{S}}_n\to\mathscr{T}_n$, the space of n^{th} order invariant tensors, for each $\kappa_{n-1}\in\mathscr{K}_{n-1}$, and then checking that we can impose the appropriate relations to obtain a definition for $\rho\in\mathfrak{S}_n$. In this way, we will obtain an n^{th} order tensor $\phi(\rho,\kappa_{n-1})$, $\rho\in\mathfrak{S}_n,\kappa_{n-1}\in\mathscr{K}_{n-1}$, which will provide the desired element $\phi(\kappa_{n-1})\in\mathscr{K}_n$ on averaging over $\rho\in\mathfrak{S}_n$. To show this, we finally show that $\phi(\kappa_{n-1})$ satisfies (2.25) and (2.26).

$\mathfrak{S}_n$ is generated by transpositions $\{\sigma_i : 1\le i\le n\}$, where

$$\sigma_i(i)=i+1,\quad \sigma_i(i+1)=i,\quad \sigma_i(j)=j,\ j\neq i,i+1, \tag{2.33}$$

which satisfy the relations

$$\sigma_i^2=1,\qquad (\sigma_i\sigma_{i+1})^3=1,\qquad (\sigma_i\sigma_j)^2=1,\ |i-j|>1. \tag{2.34}$$

Let $\tilde{\mathfrak{S}}_n$ be the free group on the generators $\{\tilde{\sigma}_i : 1\le i\le n-1\}$ and $\mathfrak{W}_n$ the smallest normal subgroup of $\tilde{\mathfrak{S}}_n$ containing $\{\tilde{\sigma}_i^2,\ 1\le i\le n-1;\quad (\tilde{\sigma}_i\tilde{\sigma}_{i+1})^3,\ 1\le i\le n-2;\quad (\tilde{\sigma}_i\tilde{\sigma}_j)^2\,|i-j|>1\}$. Then $\tilde{\mathfrak{S}}_n/\mathfrak{W}_n\cong\mathfrak{S}_n$ with $\tilde{\sigma}_i\mapsto\sigma_i$ defining an homomorphism $\tilde{\mathfrak{S}}_n\to\mathfrak{S}_n$, which we shall denote by $\tilde{\rho}\mapsto\rho$. (See (24), p. 63.) Each $\tilde{\rho}\in\tilde{\mathfrak{S}}_n$ can be written $\rho=\tilde{\sigma}_{i_1}\tilde{\sigma}_{i_2}\dots\tilde{\sigma}_{i_k}$, where $1\le|i_j|\le n-1$ and $\sigma_i^{-1}=\sigma_{-i}$. We can define a function $\tilde{\phi}:\tilde{\mathfrak{S}}_n\to\mathscr{K}_n$ in terms of $\tilde{\phi}(\tilde{\sigma}_i)$, $1\le i\le n-1$, with $\tilde{\phi}(\tilde{\sigma}_i^{-1})=\tilde{\phi}(\tilde{\sigma}_i)$ and $\tilde{\phi}(1)=0$, by

$$\tilde{\phi}(\tilde{\sigma}_{i_1}\dots\tilde{\sigma}_{i_k})=\sum_{j=1}^{k}\sigma_{i_1}\dots\sigma_{i_{j-1}}\tilde{\phi}(\tilde{\sigma}_{i_j}). \tag{2.35}$$

Then

$$\tilde{\phi}(\tilde{\rho}_1\tilde{\rho}_2) = \rho_1\tilde{\phi}(\tilde{\rho}_2) + \tilde{\phi}(\tilde{\rho}_1). \tag{2.36}$$

We now show that Ker $\tilde{\phi} \cap \mathfrak{W}_n$ is a normal subgroup of $\tilde{\mathfrak{S}}_n$. Suppose $\tilde{\rho} \in$ Ker $\tilde{\phi} \cap \mathfrak{W}_n$, so that $\rho = 1 \in \mathfrak{S}_n$ and $\tilde{\phi}(\tilde{\rho}) = 0$. Then

$$\tilde{\phi}(\tilde{\rho}_1\tilde{\rho}\rho_1^{-1}) = \rho_1\rho\tilde{\phi}(\tilde{\rho}_1^{-1}) + \rho_1\tilde{\phi}(\tilde{\rho}) + \tilde{\phi}(\tilde{\rho}_1) = \rho_1\tilde{\phi}(\tilde{\rho}_1^{-1}) + \tilde{\phi}(\tilde{\rho}_1) = \tilde{\phi}(1) = 0,$$

so that $\tilde{\rho}_1\tilde{\rho}\tilde{\rho}_1^{-1} \in$ Ker $\tilde{\phi} \cap \mathfrak{W}_n$ and this is a normal subgroup of $\tilde{\mathfrak{S}}_n$. So if we can show that

$$\{\tilde{\sigma}_i^2,\, 1 \le i \le n-1;\, (\tilde{\sigma}_i\tilde{\sigma}_{i+1})^3,\, 1 \le i \le n-2;\, (\tilde{\sigma}_i\tilde{\sigma}_j)^2,\, |i-j| > 1\} \subset \text{Ker } \tilde{\phi} \tag{2.37}$$

we must have Ker $\tilde{\phi} \cap \mathfrak{W}_n = \mathfrak{W}_n$, *i.e.* $\mathfrak{W}_n \subset$ Ker $\tilde{\phi}$, because $\mathfrak{W}_n$ is the smallest normal subgroup containing these elements. Then $\tilde{\phi}$ induces a map $\phi : \mathfrak{S}_n \to \mathscr{K}_n$ with $\phi(\rho) = \tilde{\phi}(\tilde{\rho})$, because $\phi(\tilde{\rho}\tilde{w}) = \phi(\tilde{\rho}) + \rho\phi(\tilde{w}) = \phi(\tilde{\rho})$ if $\tilde{w} \in \mathfrak{W}_n$.

Next we show that (2.37) holds if we define

$$\tilde{\phi}(\tilde{\sigma}_i, \kappa_{n-1}) = f^{\alpha_i\alpha_{i+1}}{}_\beta\, \kappa_{n-1}^{\alpha_1\ldots\alpha_{i-1}\beta\alpha_{i+2}\ldots\alpha_n} \in \mathscr{T}_n, \tag{2.38}$$

for $\kappa_{n-1} \in \mathscr{K}_{n-1}$. We will write out the argument for $\tilde{\sigma}_1^2, (\tilde{\sigma}_1\tilde{\sigma}_2)^3, (\tilde{\sigma}_1\tilde{\sigma}_3)^2$ and the arguments for other values of i, j follow by similar arguments. First, writing $\kappa = \kappa_{n-1}$,

$$\tilde{\phi}(\tilde{\sigma}_1^2, \kappa) = \tilde{\phi}(\tilde{\sigma}_1, \kappa) + \sigma_1\tilde{\phi}(\tilde{\sigma}_1, \kappa),$$

implying

$$\tilde{\phi}(\tilde{\sigma}_1^2, \kappa)^{\alpha_1\alpha_2\alpha_3\ldots\alpha_n} = f^{\alpha_1\alpha_2}{}_\beta\, \kappa^{\beta\alpha_3\ldots\alpha_n} + f^{\alpha_2\alpha_1}{}_\beta\, \kappa^{\beta\alpha_3\ldots\alpha_n} = 0.$$

Second

$$\begin{aligned}\tilde{\phi}((\tilde{\sigma}_1\tilde{\sigma}_3)^2, \kappa) &= \tilde{\phi}(\tilde{\sigma}_1, \kappa) + \sigma_1\tilde{\phi}(\tilde{\sigma}_3, \kappa) + \sigma_1\sigma_3\tilde{\phi}(\tilde{\sigma}_1, \kappa) + \sigma_1\sigma_3\sigma_1\tilde{\phi}(\tilde{\sigma}_3, \kappa) \\ &= \tilde{\phi}(\tilde{\sigma}_1, \kappa) + \sigma_1\sigma_3\tilde{\phi}(\tilde{\sigma}_1, \kappa) + \sigma_1\tilde{\phi}(\tilde{\sigma}_3, \kappa) + \sigma_3\tilde{\phi}(\tilde{\sigma}_3, \kappa)\end{aligned} \tag{2.39}$$

implying

$$\begin{aligned}\tilde{\phi}((\tilde{\sigma}_1\tilde{\sigma}_3)^2, \kappa)^{\alpha_1\alpha_2\alpha_3\alpha_4\ldots\alpha_n} &= f^{\alpha_1\alpha_2}{}_\beta\, \kappa^{\beta\alpha_3\alpha_4\ldots\alpha_n} + f^{\alpha_2\alpha_1}{}_\beta\, \kappa^{\beta\alpha_4\alpha_3\ldots\alpha_n} \\ &\quad + f^{\alpha_3\alpha_4}{}_\gamma\, \kappa^{\alpha_2\alpha_1\gamma\ldots\alpha_n} + f^{\alpha_4\alpha_3}{}_\gamma\, \kappa^{\alpha_1\alpha_2\gamma\ldots\alpha_n} \\ &= f^{\alpha_1\alpha_2}{}_\beta f^{\alpha_3\alpha_4}{}_\gamma\, \kappa^{\beta\gamma\alpha_5\ldots\alpha_n} \\ &\quad + f^{\alpha_2\alpha_1}{}_\beta f^{\alpha_3\alpha_4}{}_\gamma\, \kappa^{\beta\gamma\alpha_5\ldots\alpha_n} = 0.\end{aligned} \tag{2.40}$$

Third,

$$\begin{aligned}\tilde{\phi}((\tilde{\sigma}_1\tilde{\sigma}_2)^3, \kappa) &= \tilde{\phi}(\tilde{\sigma}_1, \kappa) + \sigma_1\tilde{\phi}(\tilde{\sigma}_2, \kappa) + \sigma_1\sigma_2\tilde{\phi}(\tilde{\sigma}_1, \kappa) + \sigma_1\sigma_2\sigma_1\tilde{\phi}(\tilde{\sigma}_2, \kappa) \\ &\quad + \sigma_2\sigma_1\tilde{\phi}(\tilde{\sigma}_1, \kappa) + \sigma_2\tilde{\phi}(\tilde{\sigma}_2, \kappa) \\ &= f^{\alpha_1\alpha_2}{}_\beta\, \kappa^{\beta\alpha_3\alpha_4\ldots\alpha_n} + f^{\alpha_1\alpha_3}{}_\beta\, \kappa^{\alpha_2\beta\alpha_4\ldots\alpha_n} \\ &\quad + f^{\alpha_2\alpha_3}{}_\beta\, \kappa^{\beta\alpha_1\alpha_4\ldots\alpha_n} + f^{\alpha_2\alpha_1}{}_\beta\, \kappa^{\alpha_3\beta\alpha_4\ldots\alpha_n}\end{aligned}$$

$$
\begin{aligned}
&+f^{\alpha_3\alpha_1}{}_\beta \kappa^{\beta\alpha_2\alpha_4\ldots\alpha_n} + f^{\alpha_3\alpha_2}{}_\beta \kappa^{\alpha_1\beta\alpha_4\ldots\alpha_n} \\
&= f^{\alpha_1\alpha_2}{}_\beta f^{\beta\alpha_3}{}_\gamma \kappa^{\gamma\alpha_4\ldots\alpha_n} + f^{\alpha_1\alpha_3}{}_\beta f^{\alpha_2\beta}{}_\gamma \kappa^{\gamma\alpha_4\ldots\alpha_n} \\
&+f^{\alpha_2\alpha_3}{}_\beta f^{\beta\alpha_1}{}_\gamma \kappa^{\gamma\alpha_4\ldots\alpha_n} = 0
\end{aligned} \tag{2.41}
$$

by the Jacobi identity. This establishes (2.37).

Thus, for each $\kappa_{n-1} \in \mathscr{K}_{n-1}$, we can define $\phi(\rho, \kappa_{n-1}) = \tilde{\phi}(\tilde{\rho}, \kappa_{n-1})$ and define $\phi : \mathscr{K}_{n-1} \to \mathscr{K}_n$ by

$$
\phi(\kappa_{n-1}) = \frac{1}{n!} \sum_{\rho \in \mathfrak{S}_n} \phi(\rho, \kappa_{n-1}). \tag{2.42}
$$

We shall now show that $\phi(\kappa_{n-1})$ satisfies (2.25) and (2.26). From (2.36),

$$
\phi(\rho_1\rho_2, \kappa_{n-1}) = \rho_1\phi(\rho_2, \kappa_{n-1}) + \phi(\rho_1, \kappa_{n-1}). \tag{2.43}
$$

Then, for any $\sigma \in \mathfrak{S}_n$,

$$
\begin{aligned}
\phi(\kappa_{n-1}) - \sigma\phi(\kappa_{n-1}) &= \frac{1}{n!} \sum_{\rho \in \mathfrak{S}_n} \phi(\rho, \kappa_{n-1}) - \frac{1}{n!} \sum_{\rho \in \mathfrak{S}_n} \sigma\phi(\rho, \kappa_{n-1}) \\
&= \frac{1}{n!} \sum_{\rho \in \mathfrak{S}_n} \phi(\rho, \kappa_{n-1}) - \frac{1}{n!} \sum_{\rho \in \mathfrak{S}_n} \phi(\sigma\rho, \kappa_{n-1}) \\
&\quad + \frac{1}{n!} \sum_{\rho \in \mathfrak{S}_n} \phi(\sigma, \kappa_{n-1}) \\
&= \phi(\sigma, \kappa_{n-1}).
\end{aligned} \tag{2.44}
$$

Taking $\sigma = \sigma_1$ in (2.44), we have

$$
\phi(\kappa_{n-1})^{\alpha_1\alpha_2\alpha_3\ldots\alpha_n} - \phi(\kappa_{n-1})^{\alpha_2\alpha_1\alpha_3\ldots\alpha_n} = f^{\alpha_1\alpha_2}{}_\beta \kappa_{n-1}^{\beta\alpha_3\ldots\alpha_n}
$$

so that (2.25) holds.

If $\sigma = \sigma_1\sigma_2\ldots\sigma_{n-1}$, then $\sigma(j) = j+1$, $1 \le j \le n-1$ and $\sigma(n) = 1$, *i.e.* σ is cyclic permutation of $(1, 2, \ldots, n)$ so that

$$
\begin{aligned}
\phi(\sigma, \kappa_{n-1}) &= \sum_{j=1}^{n-1} \sigma_1 \ldots \sigma_{j-1} \phi(\sigma_j, \kappa_{n-1}) \\
&= \sum_{j=1}^{n-1} \sigma_1 \ldots \sigma_{j-1} f^{\alpha_j\alpha_{j+1}}{}_\beta \kappa_{n-1}^{\alpha_1\ldots\alpha_{j-1}\beta\alpha_{j+2}\ldots\alpha_n} \\
&= \sum_{j=1}^{n-1} f^{\alpha_1\alpha_{j+1}}{}_\beta \kappa_{n-1}^{\alpha_2\ldots\alpha_j\beta\alpha_{j+2}\ldots\alpha_n} = 0
\end{aligned} \tag{2.45}
$$

as $\kappa_{n-1} \in \mathscr{K}_{n-1}$ is invariant. Thus putting $\sigma = \sigma_1\sigma_2\ldots\sigma_{n-1}$ in (2.44) gives

$$
\phi(\kappa_{n-1})^{\alpha_1\alpha_2\ldots\alpha_n} = \phi(\kappa_{n-1})^{\alpha_n\alpha_1\ldots\alpha_{n-1}},
$$

so that (2.26) holds.

By averaging (2.44) over $\sigma \in \mathfrak{S}_n$, we see that

$$\sum_{\sigma\in\mathfrak{S}_n} \sigma\phi(\kappa_{n-1}) = 0,$$

so that it is orthogonal to all symmetric tensors and so the unique solution to (2.25) and (2.26) with this property. This completes the proof of Proposition 2.

Note that Proposition 2 implies that $\mathscr{K}_n/\mathscr{S}_n \cong \mathscr{K}_{n-1}$, where $\mathscr{S}_n \subset \mathscr{K}_n$ is the space of symmetric invariant tensors. As particular instances, we have that if κ^{ab} is an invariant symmetric tensor, the general solution for (2.25) and (2.26) for $n = 3$ is

$$\kappa_3^{abc} = \tfrac{1}{2} f^{ab}{}_e \kappa^{ec} + \omega^{abc}, \tag{2.46}$$

where ω^{abc} is symmetric, and invariant for κ_3 to be invariant. In this case, the general solution to (2.25) and (2.26) for $n = 4$ is

$$\begin{aligned}\kappa_4^{abcd} = {} & \tfrac{1}{6} f^{ab}{}_e f^{cd}{}_g \kappa^{eg} + \tfrac{1}{6} f^{da}{}_e f^{bc}{}_g \kappa^{eg} + \tfrac{1}{2} f^{ab}{}_e \omega^{ecd} + \tfrac{1}{2} f^{bc}{}_e \omega^{ead} \\ & + \tfrac{1}{2} f^{ac}{}_e \omega^{ebd} + \omega^{abcd},\end{aligned} \tag{2.47}$$

where ω^{abcd} is symmetric.

While Proposition 2 proves the existence of a solution to (2.25) and (2.26) for a given $\kappa_{n-1} \in \mathscr{K}_{n-1}$, it does not provide an explicit expression for such a solution unless we have a method of specifying expressions for each element $\rho \in \mathfrak{S}_n$ as a product $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_k}$ of transpositions. In Appendix A, we derive an explicit expression for κ_n in terms of κ_{n-1} using the representation theory of $\mathfrak{S}_n$ and Young tableaux.

3 Current Algebra on the Torus

3.1 Fermionic loop constructions

We consider the loop amplitude

$$\mathscr{A}_{\text{loop}}^{a_1a_2\ldots a_n}(\nu_1,\nu_2,\ldots,\nu_n,\tau) = \operatorname{tr}\left(J^{a_1}(\rho_1)J^{a_2}(\rho_2)\ldots J^{a_n}(\rho_n)w^{L_0}\right)(2\pi i)^n \prod_{j=1}^n \rho_j, \tag{3.1}$$

where

$$\rho_j = e^{2\pi i\nu_j}, \qquad w = e^{2\pi i\tau}, \tag{3.2}$$

and begin by reviewing the explicit expressions for this amplitude when $J^a(\rho)$ is given as a bilinear in fermionic fields. First, we take $J^a(\rho)$ to be given in terms of Neveu-Schwarz fields by (2.8). Defining the partition function

$$\chi_{NS}(\tau) = \prod_{r=\frac{1}{2}}^{\infty}(1+w^r)^N, \tag{3.3}$$

we can write

$$tr(b^i(\rho_1)b^j(\rho_2)w^{L_0}) = \frac{1}{2\pi i(\rho_1\rho_2)^{\frac{1}{2}}}\,\chi_{NS}(\nu_1-\nu_2,\tau)\,\chi_{NS}(\tau)\delta^{ij}, \tag{3.4}$$

where

$$\chi_{NS}(\nu,\tau)=2\pi i\sum_{r=\frac{1}{2}}^{\infty}\frac{e^{-2\pi ir\nu}+w^r e^{2\pi ir\nu}}{1+w^r}=\frac{\theta_1'(0,\tau)\theta_3(\nu,\tau)}{\theta_3(0,\tau)\theta_1(\nu,\tau)}\sim\frac{1}{\nu}\quad\text{as}\quad\nu\to 0 \tag{3.5}$$

With $J^a(\rho)$ given by (2.8), the two-point function is

$$\begin{aligned}\mathrm{tr}\left(J^a(\rho_1)J^b(\rho_2)w^{L_0}\right) &= -\frac{\kappa^{ab}}{4\pi^2\rho_1\rho_2}\chi_{NS}(\nu_1-\nu_2,\tau)^2\chi_{NS}(\tau)\\ &= -\frac{\kappa^{ab}}{4\pi^2\rho_1\rho_2}\mathscr{P}_{NS}(\nu_1-\nu_2,\tau)\chi_{NS}(\tau),\end{aligned} \tag{3.6}$$

where $\kappa^{ab}=-\frac{1}{2}\mathrm{tr}(M^aM^b)=\frac{1}{2}\mathrm{tr}(t^at^b)$, and

$$\begin{aligned}\mathscr{P}_{NS}(\nu,\tau) &= \frac{\theta_1'(0,\tau)^2\theta_3(\nu,\tau)^2}{\theta_3(0,\tau)^2\theta_1(\nu,\tau)^2}\sim\frac{1}{\nu^2},\quad\text{as}\quad\nu\to 0\\ &= \frac{\theta_3''(0,\tau)}{\theta_3(0,\tau)}-\left(\frac{\theta_1'(\nu,\tau)}{\theta_1(\nu,\tau)}\right)'\\ &= \frac{\theta_3''(0,\tau)}{\theta_3(0,\tau)}+2\eta(\tau)+\mathscr{P}(\nu,\tau).\end{aligned} \tag{3.7}$$

Here the Weierstrass $\mathscr{P}$ function,

$$\mathscr{P}(\nu,\tau)=-\left(\frac{\theta_1'(\nu,\tau)}{\theta_1(\nu,\tau)}\right)'-2\eta(\tau), \tag{3.8}$$

with

$$\eta(\tau)=-\frac{1}{6}\frac{\theta_1'''(0,\tau)}{\theta_1'(0,\tau)}. \tag{3.9}$$

(See (25), p. 361.)

The general prescription for the n-point loop amplitude (3.1), with $J^a(\rho)$ given by (2.8), is given by a modification of the Frenkel-Zhu construction of 2.2, by writing (3.1) as a sum over permutations $\rho\in\mathfrak{S}_n$ with no fixed point. If $\rho=\xi_1\xi_2\ldots\xi_r$, a product of disjoint cycles, we associate to ρ a product

$$F_\rho^{NS}=(-1)^r f_{\xi_1}^{NS}f_{\xi_2}^{NS}\ldots f_{\xi_r}^{NS}\chi_{NS}(\tau), \tag{3.10}$$

where the function f_ξ^{NS} associated with the cycle $\xi=(i_1,i_2\ldots i_m)$ is defined by

$$f_\xi^{NS}=\kappa^{a_{i_1}a_{i_2}\ldots a_{i_m}}\chi_{NS}(\nu_{i_1}-\nu_{i_2},\tau)\chi_{NS}(\nu_{i_2}-\nu_{i_3},\tau)\ldots\chi_{NS}(\nu_{i_m}-\nu_{i_1},\tau), \tag{3.11}$$

$$\kappa^{a_1 a_2 \ldots a_n} = \tfrac{1}{2}\mathrm{tr}(t^{a_1} t^{a_2} \ldots t^{a_n}) = \tfrac{1}{2} i^n \mathrm{tr}(M^{a_1} M^{a_2} \ldots M^{a_n}). \tag{3.12}$$

The n-point loop amplitude is then constructed as the sum of these products over the permutations $\rho \in \mathfrak{S}'_n$, the subset of $\mathfrak{S}_n$ with no fixed points,

$$\mathscr{A}^{a_1 a_2 \ldots a_n}_{\mathrm{loop}}(\nu_1, \nu_2, \ldots, \nu_n, \tau) = \sum_{\rho \in \mathfrak{S}'_n} F^{NS\, a_1 a_2 \ldots a_n}_{\rho}(\nu_1, \nu_2, \ldots, \nu_n, \tau). \tag{3.13}$$

Again this construction can be described graphically by summing over all graphs with n vertices where the vertices carry the labels $1, 2, \ldots, n$, and each vertex is connected by directed lines to other vertices, one of the lines at each vertex pointing towards it and one away from it. An expression (3.11) is associated with each cycle, together with factor of -1, and the product of these cycle expressions is associated with the whole graph.

For example, this gives as the expression for the three-point loop

$$\begin{aligned} \mathrm{tr}\left(J^a(\rho_1) J^b(\rho_2) J^c(\rho_3) w^{L_0}\right) = {} & \frac{-ik}{8\pi^3 \rho_1 \rho_2 \rho_3} f^{abc} \chi_{NS}(\tau) \chi_{NS}(\nu_1 - \nu_2, \tau) \\ & \times \chi_{NS}(\nu_2 - \nu_3, \tau) \chi_{NS}(\nu_3 - \nu_1, \tau) \end{aligned} \tag{3.14}$$

if $\mathrm{tr}(M^a M^b) = -2k\delta^{ab}$, so that $\kappa^{ab} = k\delta^{ab}$, and δ^{ab} is used to raise and lower indices.

We can modify the above to give a second fermionic construction by defining the partition function

$$\chi^-_{NS}(\tau) = \prod_{r=\frac{1}{2}}^{\infty} (1 - w^r)^N. \tag{3.15}$$

We can write

$$\mathrm{tr}\left(b^i(\rho_1) b^j(\rho_2) w^{L_0} (-1)^{N_b}\right) = \frac{1}{2\pi i (\rho_1 \rho_2)^{\frac{1}{2}}} \chi^-_{NS}(\nu_1 - \nu_2, \tau) \chi^-_{NS}(\tau) \delta^{ij}, \tag{3.16}$$

where

$$\chi^-_{NS}(\nu, \tau) = 2\pi i \sum_{r=\frac{1}{2}}^{\infty} \frac{e^{-2\pi i r \nu} - w^r e^{2\pi i r \nu}}{1 - w^r} = \frac{\theta'_1(0,\tau)\theta_4(\nu,\tau)}{\theta_4(0,\tau)\theta_1(\nu,\tau)} \sim \frac{1}{\nu} \quad \text{as} \quad \nu \to 0; \tag{3.17}$$

and, if we replace $\chi_{NS}(\nu, \tau)$ by $\chi^-_{NS}(\nu, \tau)$ in (3.11), with $\chi^-_{NS}(\tau)$ replacing $\chi_{NS}(\tau)$ in (3.10), the above construction for the loop amplitudes gives

$$\mathrm{tr}\left(J^{a_1}(\rho_1) J^{a_2}(\rho_2) \ldots J^{a_n}(\rho_n) w^{L_0} (-1)^{N_b}\right), \tag{3.18}$$

where $N_b = \sum_{r>0} b_{-r} b_r$. In particular, the two-point function is

$$\begin{aligned} \mathrm{tr}\left(J^a(\rho_1) J^b(\rho_2) w^{L_0} (-1)^{N_b}\right) &= -\frac{\kappa^{ab}}{4\pi^2 \rho_1 \rho_2} \chi^-_{NS}(\nu_1 - \nu_2, \tau)^2 \chi^-_{NS}(\tau) \\ &= -\frac{\kappa^{ab}}{4\pi^2 \rho_1 \rho_2} \mathscr{P}^-_{NS}(\nu_1 - \nu_2, \tau) \chi^-_{NS}(\tau), \end{aligned} \tag{3.19}$$

where

$$\begin{aligned}\mathscr{P}^{-}_{NS}(\nu,\tau) &= \frac{\theta_1'(0,\tau)^2\theta_4(\nu,\tau)^2}{\theta_4(0,\tau)^2\theta_1(\nu,\tau)^2}\\ &= \frac{\theta_4''(0,\tau)}{\theta_4(0,\tau)} - \left(\frac{\theta_1'(\nu,\tau)}{\theta_1(\nu,\tau)}\right)'\\ &= \frac{\theta_4''(0,\tau)}{\theta_4(0,\tau)} + 2\eta(\tau) + \mathscr{P}(\nu,\tau). \end{aligned} \tag{3.20}$$

A third fermionic construction is given by using the Ramond operators,

$$\begin{aligned} d^i(\rho) = \sum_{m\in\mathbb{Z}} d^i_m \rho^{-m+\frac{1}{2}}, \quad \{d^i_m, d^j_n\} = \delta_{m,-n}\delta^{ij},\\ d^i_m|0\rangle = 0, \quad m>0, \quad 1\le i,j,\le N. \end{aligned} \tag{3.21}$$

Defining the Ramond partition function

$$\chi_R(\tau) = \prod_{n=1}^{\infty}(1+w^n)^N, \tag{3.22}$$

we can write

$$\mathrm{tr}\left(d^i(\rho_1)d^j(\rho_2)w^{L_0}\right) = \frac{1}{2\pi i(\rho_1\rho_2)^{\frac{1}{2}}} chi_R(\nu_1-\nu_2,\tau)\chi_R(\tau)\delta^{ij},$$

where

$$\begin{aligned}\chi_R(\nu,\tau) &= \pi i + 2\pi i\sum_{m=1}^{\infty}\frac{e^{-2\pi i m\nu}+w^m e^{2\pi i m\nu}}{1+w^m} = \frac{\theta_1'(0,\tau)\theta_2(\nu,\tau)}{\theta_2(0,\tau)\theta_1(\nu,\tau)}\\ &\sim \frac{1}{\nu} \quad \text{as} \quad \nu\to 0. \end{aligned} \tag{3.23}$$

If we now replace $\chi_{NS}(\nu,\tau)$ by $\chi_R(\nu,\tau)$ in (3.11), with $\chi_R(\tau)$ replacing $\chi_{NS}(\tau)$ in (3.10), the construction for the loop amplitudes gives

$$\mathrm{tr}\left(J^{a_1}(\rho_1)J^{a_2}(\rho_2)\ldots J^{a_n}(\rho_n)w^{L_0}\right), \tag{3.24}$$

where now, instead of (2.8),

$$J^a(\rho) = \frac{i}{2}M^a_{ij}d^i(\rho)d^j(\rho). \tag{3.25}$$

The two-point function is now

$$\begin{aligned}\mathrm{tr}\left(J^a(\rho_1)J^b(\rho_2)w^{L_0}\right) &= -\frac{\kappa^{ab}}{4\pi^2\rho_1\rho_2}\chi_R(\nu_1-\nu_2,\tau)^2\chi_R(\tau)\\ &= -\frac{\kappa^{ab}}{4\pi^2\rho_1\rho_2}\mathscr{P}_R(\nu_1-\nu_2,\tau)\chi_R(\tau) \end{aligned} \tag{3.26}$$

where

$$\begin{aligned}\mathscr{P}_R(\nu,\tau) &= \frac{\theta_1'(0,\tau)^2\theta_2(\nu,\tau)^2}{\theta_2(0,\tau)\theta_1(\nu,\tau)^2}\\ &= \frac{\theta_2''(0,\tau)}{\theta_2(0,\tau)} - \left(\frac{\theta_1'(\nu,\tau)}{\theta_1(\nu,\tau)}\right)'\\ &= \frac{\theta_2''(0,\tau)}{\theta_2(0,\tau)} + 2\eta(\tau) + \mathscr{P}(\nu,\tau). \end{aligned} \tag{3.27}$$

3.2 General torus amplitudes and connected parts

The loop amplitude

$$\begin{aligned}\mathscr{A}_{\rm loop}^{a_1a_2\ldots a_n}(\nu_1,\nu_2,\ldots,\nu_n,\tau) &= {\rm tr}\left(J^{a_1}(\rho_1)J^{a_2}(\rho_2)\ldots J^{a_n}(\rho_n)w^{L_0}\right)\\ &\quad\times(2\pi i)^n\prod_{j=1}^n\rho_j, \end{aligned} \tag{3.28}$$

is invariant under $\nu_j\to\nu_j+1$ and $\nu_j\to\nu_j+\tau$ for each j individually, so that it is defined on the torus obtained by identifying $\nu\in\mathbb{C}$ with $\nu+1$ and $\nu+\tau$. Because of the locality of the currents $J^{a_j}(\rho_j)$, the amplitude is also symmetric under simultaneous permutations of the ρ_j and the a_j.

From (2.3) we have that

$$J^a(\rho_1)J^b(\rho_2)\sim\frac{\kappa^{ab}}{(2\pi i)^2\rho_1\rho_2(\nu_1-\nu_2)^2}+\frac{f^{ab}{}_cJ^c(\rho_2)}{2\pi i\rho_1(\nu_1-\nu_2)}\quad\text{as}\quad\nu_1\to\nu_2. \tag{3.29}$$

Thus the singularities of the n-point loop amplitude on the torus are determined in terms of the $(n-1)$-point and $(n-2)$-point loop amplitudes. This means that knowledge of the $(n-1)$-point and $(n-2)$-point loop amplitudes determines the n-point loop amplitude up to a constant on the torus, that is a function of τ. (See, *e.g.*, (26), p. 29.) Because of the permutation symmetry of the amplitude (3.28), this leaves the n-point loop determined up to a symmetric invariant tensor function of τ, given the $(n-1)$-point and $(n-2)$-point loops.

The sum over permutations in the expression (3.13) for the loop in the fermionic construction cases can be divided into terms which collect together the same ρ_i in each cycle. Such terms are labeled by the division of the variables $\{\rho_1,\rho_2,\ldots,\rho_n\}$ into subsets, each consisting of at least two elements (corresponding to the restriction to permutations with no fixed points). The full loop amplitude is then the sum over these terms. Such terms are products of "connected parts", each of which involves one of the subsets of $\{\rho_1,\rho_2,\ldots,\rho_n\}$, say $\{\rho_{i_1},\rho_{i_2},\ldots,\rho_{i_m}\}$, given by an expression like (2.21),

$$\mathscr{A}_{{\rm loop},C}^{a_1a_2\ldots a_m}(\nu_1,\nu_2,\ldots,\nu_m,\tau)=-\frac{1}{m}\sum_{\rho\in\mathfrak{S}_m}f^{NS}_{(\rho(1),\rho(2),\ldots,\rho(m))}\chi_{NS}(\tau), \tag{3.30}$$

in the NS case. The amplitudes $\mathscr{A}_{\text{loop},C}$ have a simpler structure than the full amplitudes $\mathscr{A}_{\text{loop}}$ in that they have only single poles for $m > 2$, rather than both single and double poles. For $m > 2$, the connected amplitudes satisfy the conditions

$$\mathscr{A}_{\text{loop},C}^{a_1 a_2 \ldots a_m}(\nu_1, \nu_2, \ldots, \nu_m, \tau) \sim \frac{1}{\nu_m - \nu_j} f^{a_m a_j}{}_{a'_j} \mathscr{A}_{\text{loop},C}^{a_1 a_2 \ldots a_{j-1} a'_j a_{j+1} \ldots a_{m-1}} \times (\nu_1, \nu_2, \ldots, \nu_{m-1}, \tau), \tag{3.31}$$

which are sufficient to specify the m-point connected amplitude $\mathscr{A}_{\text{loop},C}$ in terms of the $(m-1)$-point connected amplitude, again up to a symmetric invariant tensor function of τ.

Motivated by the fermionic constructions, we can give a general definition of the connected part of the loop amplitude in a familiar way. If $A = \{i_1, i_2, \ldots, i_n\}$ is a set of distinct positive integers, define

$$\mathscr{A}^A \equiv \mathscr{A}_{\text{loop}}^{a_{i_1} a_{i_2} \ldots a_{i_n}}(\nu_{i_1}, \nu_{i_2}, \ldots, \nu_{i_n}, \tau) = \operatorname{tr}\left(J^{a_{i_1}}(\rho_{i_1}) J^{a_{i_2}}(\rho_{i_2}) \ldots J^{a_{i_n}}(\rho_{i_n}) w^{L_0}\right) \times (2\pi i)^n \prod_{j=1}^{n} \rho_{i_j}. \tag{3.32}$$

Let $P = (A_1, A_2, \ldots, A_r)$ be a division of the integers $A = \{i_1, i_2, \ldots, i_n\} = A_1 \cup A_2 \cup \ldots \cup A_r$ into a number of disjoint subsets; let $\mathfrak{P}_A$ denote the collections of such divisions; and denote the partition function by

$$\chi(\tau) = \operatorname{tr}\left(w^{L_0}\right). \tag{3.33}$$

Then we can define the connected amplitude $\mathscr{A}_C^A$ inductively by

$$\mathscr{A}^A = \sum_{P \in \mathfrak{P}_A} \chi(\tau)^{1-|P|} \prod_{A_j \in P} \mathscr{A}_C^{A_j}, \tag{3.34}$$

where $|P| = r$, the number of subsets contained in the division P, together with the vanishing of the one point function $\mathscr{A}_C^{\{i\}} = 0$, and, consequently,

$$\mathscr{A}_C^{\{i,j\}} = \mathscr{A}^{\{i,j\}} = \operatorname{tr}\left(J^{a_i}(\rho_i) J^{a_j}(\rho_j) w^{L_0}\right) (2\pi i)^2 \rho_i \rho_j. \tag{3.35}$$

Equation (3.34) is of the form given for the NS case by (3.13) together with (3.10), where

$$\mathscr{A}_C^{\{i_1 \ldots i_m\}} = -\sum_{\rho \in \mathfrak{S}_m} f^{NS}_{i_{\rho(1)} \ldots i_{\rho(m)}} \chi, \qquad \chi(\tau) = \chi_{NS}(\tau).$$

Equation (3.34) defines an inductive procedure because we can write it as

$$\mathscr{A}_C^A = \mathscr{A}^A - \sum_{P \in \mathfrak{P}'_A} \chi(\tau)^{1-|P|} \prod_{A_j \in P} \mathscr{A}_C^{A_j}, \tag{3.36}$$

where $\mathfrak{P}'_A$ denotes the same collection of divisions of A into disjoint subsets but omitting the division of A into the single set consisting of itself. If we single out a point $i \in A$, we can rewrite the inductive definition of $\mathscr{A}_C^A$,

$$\mathscr{A}^A = \mathscr{A}_C^A + \sum_{B \in \mathfrak{R}_A^i} \mathscr{A}_C^B \mathscr{A}^{A \sim B} / \chi, \tag{3.37}$$

where $\mathfrak{R}_A^i$ denotes the proper subsets of $\mathscr{A}$ which contain i.

The point of this definition of

$$\mathscr{A}_C^A \equiv \mathscr{A}_C^{a_{i_1}a_{i_2}\ldots a_{i_n}}(\nu_{i_1},\nu_{i_2},\ldots,\nu_{i_n},\tau) \tag{3.38}$$

is that, for $m>2$, the double poles at $\nu_i=\nu_j$ present in $\mathscr{A}^A$ have been removed and only single poles remain. A double pole remains in $\mathscr{A}_C^{\{i,j\}}$ defined by (3.35),

$$\mathscr{A}_C^{\{i,j\}} \sim \frac{\kappa^{a_ia_j}}{(\nu_i-\nu_j)^2}\chi(\tau) \qquad \text{as } \nu_i\sim\nu_j,$$

and this is its only singularity. To demonstrate the absence of the double pole at $\nu_i=\nu_j$ in $\mathscr{A}_C^A$, $m>2$, we use induction and (3.37). We note that the residue of the double pole on the left hand side is $k\delta^{a_ia_j}\mathscr{A}^{A\sim\{i,j\}}$, and in the sum on the right hand side, assuming inductively that the result is true for smaller amplitudes, the double pole occurs only in the term involving $\mathscr{A}_C^B$ for $B=\{i,j\}$ and the residue for this term is $\delta^{a_ia_j}k\chi$ multiplied by $\mathscr{A}^{A\sim B}/\chi$, *i.e.* the same as on the left hand side, so that these residues cancel and $\mathscr{A}_C^A$ has no double pole at $\nu_i=\nu_j$. A similar argument shows that $\mathscr{A}_C^A$ satisfies the same relations for the residues at single poles as $\mathscr{A}^A$, so that (3.31) holds.

3.3 Structure of torus amplitudes

In general write

$$\mathscr{A}_{\text{loop},C}^{a_1a_2\ldots a_n}(\nu_1,\nu_2,\ldots,\nu_n,\tau) \equiv -\mathscr{F}_n^{a_1a_2\ldots a_n}(\nu_1,\nu_2,\ldots,\nu_n,\tau)\chi(\tau), \tag{3.39}$$

so that, for $n>2$,

$$\begin{aligned}\mathscr{F}_n^{a_1a_2\ldots a_n}(\nu_1,\nu_2,\ldots,\nu_n,\tau) \sim\ & \frac{1}{\nu_n-\nu_j}f^{a_ma_j}{}_{a'_j}\mathscr{F}_{n-1}^{a_1a_2\ldots a_{j-1}a'_ja_{j+1}\ldots a_{n-1}} \\ & \times(\nu_1,\nu_2,\ldots,\nu_{n-1},\tau)\end{aligned} \tag{3.40}$$

as $\nu_n\sim\nu_j$, which specifies $\mathscr{F}_n$ on the torus in terms of $\mathscr{F}_{n-1}$ up to a function of τ,

$$\omega_n^{a_1a_2\ldots a_n}(\tau), \tag{3.41}$$

which, because of the properties of $\mathscr{F}_n$, must be an invariant symmetric tensor. Inductively, this determines $\mathscr{F}_n$ in terms of $\mathscr{F}_2$ and these invariant tensors, ω_m, $2<m\le n$. The 2-point function, $\mathscr{F}_2$, has only a double pole,

$$\mathscr{F}_2^{ab}(\nu_1,\nu_2,\tau) \sim -\frac{\kappa^{ab}}{(\nu_1-\nu_2)^2} \quad \text{as} \quad \nu_1\to\nu_2. \tag{3.42}$$

In general (3.42) implies that the general form of the two-point function is

$$\mathscr{F}^{ab}(\nu_1,\nu_2,\tau) = -\kappa^{ab}\mathscr{P}(\nu_1-\nu_2,\tau)+\omega_2^{ab}(\tau), \tag{3.43}$$

where $\omega_2^{ab}(\tau)$ is a symmetric invariant tensor. In the NS, NS$^-$ and R cases,

$$\omega_2^{ab}(\tau) = -\kappa^{ab}\left[\frac{\theta_s''(0,\tau)}{\theta_s(0,\tau)} + 2\eta(\tau)\right] \tag{3.44}$$

with $s = 3,4,2$, respectively.

We can construct the general three-point loop, $\mathscr{F}_3$, as follows; we start by rewriting (3.43) as

$$\mathscr{F}_2^{ab}(\nu_1,\nu_2,\tau) = -\kappa^{ab}\mathscr{P}_{NS}(\nu_1-\nu_2,\tau) + \tilde{\omega}_2^{ab}(\tau). \tag{3.45}$$

We then have that $\mathscr{F}^{abc}(\nu_1,\nu_2,\nu_3,\tau)$ differs from what it is in the NS case,

$$kf^{abc}\chi_{NS}(\nu_1-\nu_2,\tau)\chi_{NS}(\nu_2-\nu_3,\tau)\chi_{NS}(\nu_3-\nu_1,\tau), \tag{3.46}$$

by a function defined on torus, whose residues at $\nu_1=\nu_2$, $\nu_2=\nu_3$, $\nu_3=\nu_1$ are all $if^{ab}{}_e\tilde{\omega}_2^{ec}(\tau)$. To construct such a function, consider the Weierstrass ζ function (see (27), p. 445),

$$\zeta(\nu,\tau) = \frac{\theta_1'(\nu,\tau)}{\theta_1(\nu,\tau)} + 2\eta(\tau)\nu \tag{3.47}$$

$\zeta(\nu,\tau)$ has the properties:

$$\zeta(\nu+1,\tau) = \zeta(\nu,\tau) + 2\eta(\tau), \qquad \zeta(\nu+\tau,\tau) = \zeta(\nu,\tau) + 2\eta(\tau)\tau - 2\pi i \tag{3.48}$$

$$\begin{aligned} &\zeta'(\nu,\tau) = -\mathscr{P}(\nu,\tau), \quad \zeta(-\nu,\tau) = -\zeta(\nu,\tau), \\ &\zeta(\nu,\tau) = \frac{1}{\nu} + \mathscr{O}(\nu^3), \quad \text{as} \quad \nu \to 0. \end{aligned} \tag{3.49}$$

It follows that

$$\zeta(\nu_1-\nu_2,\tau) + \zeta(\nu_2-\nu_3,\tau) + \zeta(\nu_3-\nu_1,\tau) \tag{3.50}$$

is defined on the torus and has residue 1 at $\nu_1=\nu_2$, $\nu_2=\nu_3$ and $\nu_3=\nu_1$. Thus the general form for

$$\mathscr{F}_3^{abc}(\nu_1,\nu_2,\nu_3,\tau) = kf^{abc}\chi_{NS}(\nu_1-\nu_2,\tau)\chi_{NS}(\nu_2-\nu_3,\tau)\chi_{NS}(\nu_3-\nu_1,\tau) + f^{abe}\tilde{\omega}_2^{ec}(\tau)\,[\zeta(\nu_1-\nu_2,\tau)+\zeta(\nu_2-\nu_3,\tau) + \zeta(\nu_3-\nu_1,\tau)] + 2\omega_3^{abc}(\tau), \tag{3.51}$$

where ω_3 is a symmetric invariant tensor, because this has the residues specified by (3.40). We could proceed to express the n-point connected loop amplitude as the expression in the NS case (3.11) with additional terms, but, instead, we adopt an approach that is more symmetric between all the terms. To this end, we define functions, $H_{n,m}(\mu_1,\ldots,\mu_n,\tau)$, symmetric under the permutations of the μ_i, initially for $0 \le m \le 4$, by

$$\begin{aligned}
H_{n,0}(\mu,\tau) &= 1,\\
H_{n,1}(\mu,\tau) &= \sum_{j=1}^{n}\zeta_j,\\
2H_{n,2}(\mu,\tau) &= \left(\sum_{j=1}^{n}\zeta_j\right)^2 + \sum_{j=1}^{n}\zeta_j',\\
6H_{n,3}(\mu,\tau) &= \left(\sum_{j=1}^{n}\zeta_j\right)^3 + 3\sum_{j=1}^{n}\zeta_j\sum_{j=1}^{n}\zeta_j' + \sum_{j=1}^{n}\zeta_j'',\\
24H_{n,4}(\mu,\tau) &= \left(\sum_{j=1}^{n}\zeta_j\right)^4 + 6\left(\sum_{j=1}^{n}\zeta_j\right)^2\sum_{j=1}^{n}\zeta_j' + 4\sum_{j=1}^{n}\zeta_j\sum_{j=1}^{n}\zeta_j''\\
&\quad + 3\left(\sum_{j=1}^{n}\zeta_j'\right)^2 + \sum_{j=1}^{n}\zeta_j''' + k_4,
\end{aligned} \tag{3.52}$$

where $\mu = (\mu_1,\ldots,\mu_n)$, $\zeta_j = \zeta(\mu_j,\tau)$, and $k_4 = k_4(\tau)$ is a constant on the torus to be determined. Then the singularities in the μ_j of $H_{n,m}(\mu_1,\ldots,\mu_n,\tau)$ are simple poles at $\mu_j = 0$ for $n > 2$, and the residue

$$\operatorname*{Res}_{\mu_n=0} H_{n,m}(\mu_1,\ldots,\mu_n,\tau) = H_{n-1,m-1}(\mu_1,\ldots,\mu_{n-1},\tau), \tag{3.53}$$

for $1 \le m \le 4$ and $n > 2$. This can be verified case by case but we shall give a general argument below.

The $H_{n,m}(\mu,\tau)$ are not single valued for μ_j on the torus but, if we impose the constraint that $\mu_1 + \ldots + \mu_n = 0$, they are. So $H_{n,m}(\nu_{12},\ldots,\nu_{n1},\tau)$, where $\nu_{ij} = \nu_i - \nu_j$ is defined on the torus and, for $n > 2$, just has poles at $\nu_i = \nu_{i+1}$, $1 \le i \le n$, with $\nu_{n+1} \equiv \nu_1$. For $n = 2$,

$$H_{2,1}(\nu_{12},\nu_{21},\tau) = 0, \qquad H_{2,2}(\nu_{12},\nu_{21},\tau) = -\mathscr{P}(\nu_{12},\tau). \tag{3.54}$$

By Proposition 2 of Sect. 2.3, we can define n^{th} order tensors $\kappa_{n,m}(\tau)$, $n \ge m \ge 0$, $n \ge 2$, by the conditions

$$\kappa_{n,m}^{a_1a_2\ldots a_n}(\tau) - \kappa_{n,m}^{a_2a_1\ldots a_n}(\tau) = f^{a_1a_2}{}_b\kappa_{n-1,m-1}^{ba_3\ldots a_n}(\tau), \tag{3.55}$$

$$\kappa_{n,m}^{a_1 a_2 \dots a_n}(\tau) = \kappa_{n,m}^{a_2 \dots a_n a_1}(\tau), \tag{3.56}$$

together with the requirement that $\kappa_{n,m}$ be orthogonal to all symmetric tensors for $m > 0$ and $n > 2$, and the initial condition that $\kappa_{2,2} = \kappa, \kappa_{2,1} = 0$ and $\kappa_{n,0}(\tau) = \omega_n(\tau)$, a symmetric invariant tensor. Then, setting

$$\begin{aligned}\langle \kappa_{n,m} H_{n,m} \rangle^{a_1 a_2 \dots a_n}(\nu_1, \nu_2, \dots, \nu_n, \tau) = \frac{1}{n} \sum_{\rho \in \mathfrak{S}_n} \kappa_{n,m}^{a_{\rho(1)} \dots a_{\rho(n)}} \\ \times H_{n,m}(\nu_{\rho(1)\rho(2)}, \dots, \nu_{\rho(n)\rho(1)}, \tau),\end{aligned} \tag{3.57}$$

$\mathscr{F}_n = \langle \kappa_{n,n-m} H_{n,n-m} \rangle$, $n \geq m$, provides a solution to (3.40) for each m. By the linearity of those equations, we obtain the solution,

$$\mathscr{F}_n = \sum_{m=0}^{n} \langle \kappa_{n,m} H_{n,m} \rangle, \qquad n \geq 2. \tag{3.58}$$

Because so far we only have $H_{n,m}$ for $0 \leq m \leq 4$, (3.58) is only valid for $2 \leq n \leq 4$. Explicitly,

$$\begin{aligned}\mathscr{F}_2 &= \langle \kappa_2 H_{2,2} \rangle + \langle \kappa_{2,0} \rangle, \\ \mathscr{F}_3 &= \langle \kappa_3 H_{3,3} \rangle + \langle \kappa_{3,1} H_{3,1} \rangle + \langle \kappa_{3,0} \rangle, \\ \mathscr{F}_4 &= \langle \kappa_4 H_{4,4} \rangle + \langle \kappa_{4,2} H_{4,2} \rangle + \langle \kappa_{4,1} H_{4,1} \rangle + \langle \kappa_{4,0} \rangle,\end{aligned} \tag{3.59}$$

where we have written $\kappa_n \equiv \kappa_{n,n}$.

To demonstrate that $H_{n,m}$ has the desired properties, $0 \leq m \leq 4$, and to extend its definition to higher values of m, we note we can write

$$H_{n,m}(\mu, \tau) = \frac{1}{m!} \left[\sum_{j=1}^{n} (\partial_j + \zeta_j) \right]^m 1, \qquad \text{for } 1 \leq m \leq 3, \tag{3.60}$$

where $\partial_j = \partial / \partial \mu_j$. (Here, and in what follows, $n \geq 2$.) The ζ function can be written in terms of the Weierstrass σ function (see (27), p. 447)

$$\zeta(\mu, \tau) = \frac{\sigma'(\mu, \tau)}{\sigma(\mu, \tau)}, \qquad \sigma(\mu, \tau) = e^{\eta(\tau)\mu^2} \frac{\theta_1(\mu, \tau)}{\theta_1'(0, \tau)}, \tag{3.61}$$

with $\sigma(-\mu, \tau) = -\sigma(\mu, \tau)$, and

$$\sigma(\mu, \tau) = \sum_{s=0}^{\infty} f_s(\tau) \mu^{2s+1} = \mu + f_2(\tau)\mu^5 + \dots, \qquad \text{because } f_1 = 0. \tag{3.62}$$

Then, defining $\hat{H}_{n,m}(\mu, \tau)$ by the right hand side of (3.60) for all $m \geq 0$, we have as $\mu_n \to 0$,

$$\begin{aligned}\hat{H}_{n,m}(\mu, \tau) &= \frac{1}{m! \sigma(\mu_n, \tau)} \left[\partial_n + \sum_{j=1}^{n-1} (\partial_j + \zeta_j) \right]^m \sigma(\mu_n, \tau) \\ &= \frac{1}{\mu_n m!} \left[\partial_n + \sum_{j=1}^{n-1} (\partial_j + \zeta_j) \right]^m \sum_{s=0}^{\infty} f_s \mu_n^{2s+1} + \mathscr{O}(1) \\ &= \frac{1}{\mu_n} \sum_{s=0}^{[\frac{1}{2}m - \frac{1}{2}]} f_s(\tau) \hat{H}_{n-1, m-2s-1}(\mu', \tau) + \mathscr{O}(1),\end{aligned} \tag{3.63}$$

where $\mu' = (\mu_1, \ldots, \mu_{n-1})$ and $[\frac{1}{2}m - \frac{1}{2}]$ is the greatest integer less than or equal to $\frac{1}{2}m - \frac{1}{2}$. Thus

$$\operatorname*{Res}_{\mu_n=0} \hat{H}_{n,m}(\mu,\tau) = \sum_{s=0}^{[\frac{1}{2}m-\frac{1}{2}]} f_s(\tau)\hat{H}_{n-1,m-2s-1}(\mu,\tau); \tag{3.64}$$

In particular,

$$\operatorname*{Res}_{\mu_n=0} \hat{H}_{n,m}(\mu,\tau) = \hat{H}_{n-1,m-1}(\mu',\tau), \qquad 1 \le m \le 4,$$

so that (3.53) holds for $1 \le m \le 4$ and $n > 2$, but

$$\operatorname*{Res}_{\mu_n=0} \hat{H}_{n,5}(\mu,\tau) = \hat{H}_{n-1,4}(\mu,\tau) + f_2\hat{H}_{n-1,0}(\mu',\tau).$$

To see how to modify $\hat{H}_{n,m}$ to give an $H_{n,m}$ that satisfies (3.53) for all $m \ge 1$, write

$$\hat{H}_n(\mu,\tau;\nu) = \sum_{m=0}^{\infty} \nu^m \hat{H}_{n,m}(\mu,\tau), \tag{3.65}$$

we have from (3.64),

$$\operatorname*{Res}_{\mu_n=0} \hat{H}_n(\mu,\tau;\nu) = \sigma(\nu,\tau)\hat{H}_{n-1}(\mu',\tau;\nu). \tag{3.66}$$

So, if we define

$$\frac{\nu^n}{\sigma(\nu,\tau)^n}\hat{H}_n(\mu,\tau;\nu) = H_n(\mu,\tau;\nu) = \sum_{m=0}^{\infty} H_{n,m}(\mu,\tau)\nu^m, \tag{3.67}$$

the definition of $H_{n,m}$ in (3.52) is unchanged for $1 \le m \le 4$ (except that the constant, $k_4/24$, in the defintion of $H_{n,4}$ is determined to be $-nf_2$), and

$$\operatorname*{Res}_{\mu_n=0} H_n(\mu,\tau;\nu) = \nu H_{n-1}(\mu',\tau;\nu), \qquad n \ge 1, \tag{3.68}$$

i.e. $$\operatorname*{Res}_{\mu_n=0} H_{n,m}(\mu,\tau) = H_{n-1,m-1}(\mu',\tau). \qquad n \ge m \ge 1. \tag{3.69}$$

From (3.60)

$$\begin{aligned}
\hat{H}_n(\mu,\tau;\nu) &= \sum_{m=0}^{\infty} \frac{\nu^m}{m!}\left[\sum_{j=1}^{n}(\partial_j + \zeta_j)\right]^m 1 \\
&= \left[\prod_{j=1}^{n}\frac{1}{\sigma(\mu_j,\tau)}\right]\sum_{m=0}^{\infty}\frac{\nu^m}{m!}\left[\sum_{j=1}^{n}\partial_j\right]^m \prod_{j=1}^{n}\sigma(\mu_j,\tau) \\
&= \prod_{j=1}^{n}\frac{\sigma(\mu_j+\nu,\tau)}{\sigma(\mu_j,\tau)}
\end{aligned} \tag{3.70}$$

and so

$$H_n(\mu,\tau;\nu)=\frac{\nu^n}{\sigma(\nu,\tau)^n}\prod_{j=1}^{n}\frac{\sigma(\mu_j+\nu,\tau)}{\sigma(\mu_j,\tau)}. \tag{3.71}$$

Note that

$$\nu^{-n}H_n(\mu,\tau;\nu)=\prod_{j=1}^{n}\frac{\sigma(\mu_j+\nu,\tau)}{\sigma(\nu,\tau)\sigma(\mu_j,\tau)} \tag{3.72}$$

is elliptic as a function of the μ_j and of ν provided that we impose the constraint that $\mu_1+\mu_2+\ldots+\mu_n=0$. From (3.71) we see directly that

$$\begin{aligned}\operatorname*{Res}_{\mu_n=0}H_n(\mu,\tau;\nu)&=\frac{\nu^n}{\sigma(\nu,\tau)^n}\frac{\sigma(\nu,\tau)}{\sigma'(0,\tau)}\prod_{j=2}^{n}\frac{\sigma(\mu_j+\nu,\tau)}{\sigma(\mu_j,\tau)}\\&=\nu H_{n-1}(\mu',\tau;\nu).\end{aligned} \tag{3.73}$$

Properties of $H_n(\mu,\tau;\nu)$ are discussed in Appendix B. In particular, it is shown that

$$H_{n,n-1}(\nu_{12},\ldots,\nu_{n-1,n},\nu_{n,1},\tau)=0. \tag{3.74}$$

The relation (3.69) shows that (3.58) provides the general form of the n-point connected loop amplitude, with $H_{n,m}$ defined as the moments of (3.71). It specifies $\mathscr{F}_n$ in terms of the invariant symmetric tensors $\kappa_{2,2}=\kappa$ and $\kappa_{n,0}=\omega_n$,

$$\begin{aligned}\mathscr{F}_n^{a_1a_2\ldots a_n}(\nu_1,\nu_2,\ldots,\nu_n,\tau)=\frac{1}{n}\sum_{\rho\in\mathfrak{S}_n}\sum_{m=0}^{n}\kappa_{n,m}^{a_{\rho(1)}\cdots a_{\rho(n)}}\\ H_{n,m}(\nu_{\rho(1)\rho(2)},\ldots,\nu_{\rho(n)\rho(1)},\tau).\end{aligned} \tag{3.75}$$

Since the symmetrization of $\kappa_{n,m}^{a_1a_2\ldots a_n}$ is zero for $m>0$ and $n\geq 3$, we can evaluate $\kappa_{n,0}$ in terms of connected parts of traces of the $J^a(\rho)$ by symmetrizing (3.75) over the group indices only, yielding

$$\mathscr{F}^{a_1a_2\ldots a_n}(\nu_1,\nu_2,\ldots,\nu_n,\tau)_S=(n-1)!\kappa_{n,0}^{a_1\ldots a_n}(\tau),\quad n\geq 3, \tag{3.76}$$

where we define

$$\mathscr{F}^{a_1a_2\ldots a_n}(\nu_1,\nu_2,\ldots,\nu_n,\tau)_S=\frac{1}{n!}\sum_{\rho\in\mathfrak{S}_n}\mathscr{F}_n^{a_{\rho(1)}\cdots a_{\rho(n)}}(\nu_1,\nu_2,\ldots,\nu_n,\tau). \tag{3.77}$$

Equation (3.76) can be written

$$\begin{aligned}\omega_n^{a_1\ldots a_n}(\tau)\equiv\kappa_{n,0}^{a_1\ldots a_n}(\tau)=-\frac{(2\pi i)^n}{(n-1)!\chi(\tau)}\left[\prod_{j=1}^{n}\rho_j\right]\operatorname{tr}\\ \times\left(J^{a_1}(\rho_1)J^{a_2}(\rho_2)\ldots J^{a_n}(\rho_n)w^{L_0}\right)_{C,S},\end{aligned} \tag{3.78}$$

for $n \geq 3$; ω_2 is determined by (3.43). Note that this implies that the symmetrized connected part of the trace

$$\left[\prod_{j=1}^{n}\rho_j\right]\mathrm{tr}\left(J^{a_1}(\rho_1)J^{a_2}(\rho_2)\ldots J^{a_n}(\rho_n)w^{L_0}\right)_{C,S} \tag{3.79}$$

is independent of the ν_j. (This follows directly from symmetrizing (3.40) because this shows that all the residues of this elliptic function vanish, implying that it is a constant on the torus.) We will relate this to the trace of zero modes of the currents in Sect. 4. In Appendix C we show that the formulae given for two-, three- and four-point loops in (3) are equivalent to (3.75) for $n \leq 4$.

4 Zero Modes

4.1 Recurrence relations and traces of zero modes

The symmetric tensor ω_n is given in terms of the symmetrized connected part of the trace of currents by (3.78). We seek to express this in terms of traces of symmetrized products of zero modes, J_0^a. To this end consider

$$\mathrm{tr}\left(J^{a_1}(\rho_1)J^{a_2}(\rho_2)\ldots J^{a_r}(\rho_r)J_0^{a_{r+1}}\ldots J_0^{a_n}w^{L_0}\right)\prod_{j=1}^{r}\rho_j. \tag{4.1}$$

These functions are not elliptic as functions of the ν_j, $1 \leq j \leq r$, if $r < n$; to see this move $J^{a_1}(\rho_1)$ around the trace, through w^{L_0}, to calculate the effect of sending $\nu_1 \to \nu_1 + \tau$, and we find that it is not invariant because terms, proportional to $f^{a_1 a_j}{}_e$ are generated on commuting $J^{a_1}(\rho_1)$ with $J_0^{a_j}$, $j > r$. However, these terms clearly disappear on symmetrizing over all the indices $(a_1, a_2, \ldots, a_n)$, so that

$$\mathrm{tr}\left(J^{a_1}(\rho_1)J^{a_2}(\rho_2)\ldots J^{a_r}(\rho_r)J_0^{a_{r+1}}\ldots J_0^{a_n}w^{L_0}\right)_S\prod_{j=1}^{r}\rho_j, \tag{4.2}$$

is elliptic in ν_j, $1 \leq j \leq r$ and so a suitable function to consider.

Symmetrizing the recurrence relation (3)

$$\begin{aligned}
&\mathrm{tr}\left(J^{a_1}(\rho_1)J^{a_2}(\rho_2)\ldots J^{a_n}(\rho_n)w^{L_0}\right) = \rho_1^{-1}\mathrm{tr}\left(J_0^{a_1}J^{a_2}(\rho_2)\ldots J^{a_n}(\rho_n)w^{L_0}\right)\\
&\quad + i\sum_{j=2}^{n}\frac{\Delta_1(\nu_j-\nu_1,\tau)}{\rho_1}f^{a_1 a_j}{}_{a'_j}\\
&\quad \times\mathrm{tr}\left(J^{a_2}(\rho_2)\ldots J^{a_{j-1}}(\rho_{j-1})J^{a'_j}(\rho_j)J^{a_{j+1}}(\rho_{j+1})\ldots J^{a_n}(\rho_n)w^{L_0}\right)\\
&\quad + k\sum_{j=2}^{n}\frac{\Delta_2(\nu_j-\nu_1,\tau)}{\rho_1\rho_j}\delta^{a_1 a_j}\\
&\quad \times\mathrm{tr}\left(J^{a_2}(\rho_2)\ldots J^{a_{j-1}}(\rho_{j-1})J^{a_{j+1}}(\rho_{j+1})\ldots J^{a_n}(\rho_n)w^{L_0}\right),
\end{aligned} \tag{4.3}$$

where

$$\Delta_1(\nu,\tau) = \frac{i}{2\pi}\frac{\theta_1'(\nu,\tau)}{\theta_1(\nu,\tau)} - \frac{1}{2}, \quad \Delta_2(\nu,\tau) = \frac{1}{4\pi^2}\left(\frac{\theta_1'(\nu,\tau)}{\theta_1(\nu,\tau)}\right)' = -\frac{1}{4\pi^2}\mathscr{P}(\nu,\tau) - \frac{1}{2\pi^2}\eta(\tau), \tag{4.4}$$

we obtain

$$\begin{aligned}&\mathrm{tr}\left(J^{a_1}(\rho_1)J^{a_2}(\rho_2)\dots J^{a_n}(\rho_n)w^{L_0}\right)_S \\ &= \frac{1}{\rho_1}\mathrm{tr}\left(J_0^{a_1}J^{a_2}(\rho_2)\dots J^{a_n}(\rho_n)w^{L_0}\right)_S \\ &\quad + k\sum_{j=2}^{n}\frac{\Delta_2(\nu_j-\nu_1,\tau)}{\rho_1\rho_j} \\ &\quad \times\left[\delta^{a_1a_j}\mathrm{tr}\left(J^{a_2}(\rho_2)\dots J^{a_{j-1}}(\rho_{j-1})J^{a_{j+1}}(\rho_{j+1})\dots J^{a_n}(\rho_n)w^{L_0}\right)\right]_S.\end{aligned} \tag{4.5}$$

This generalizes to

$$\begin{aligned}&\mathrm{tr}\left(J^{a_1}(\rho_1)J^{a_2}(\rho_2)\dots J^{a_r}(\rho_r)J_0^{a_{r+1}}\dots J_0^{a_n}w^{L_0}\right)_S \\ &= \frac{1}{\rho_1}\mathrm{tr}\left(J^{a_2}(\rho_2)\dots J^{a_r}(\rho_r)J_0^{a_1}J_0^{a_{r+1}}\dots J_0^{a_n}w^{L_0}\right)_S + k\sum_{j=2}^{r}\frac{\Delta_2(\nu_j-\nu_1,\tau)}{\rho_1\rho_j} \\ &\times\left[\delta^{a_1a_j}\mathrm{tr}\left(J^{a_2}(\rho_2)\dots J^{a_{j-1}}(\rho_{j-1})J^{a_{j+1}}(\rho_{j+1})\dots J^{a_r}(\rho_r)J_0^{a_{r+1}}\dots J_0^{a_n}w^{L_0}\right)\right]_S.\end{aligned} \tag{4.6}$$

Applying this for $r=n=2$,

$$\begin{aligned}\mathrm{tr}\left(J^{a_1}(\rho_1)J^{a_2}(\rho_2)w^{L_0}\right)\rho_1\rho_2 &= \mathrm{tr}\left(J_0^{a_1}J^{a_2}(\rho_2)w^{L_0}\right)\rho_2 + k\Delta_2(\nu_2-\nu_1,\tau)\delta^{a_1a_2}\chi(\tau) \\ &= \mathrm{tr}\left(J_0^{a_1}J_0^{a_2}w^{L_0}\right) + k\Delta_2(\nu_2-\nu_1,\tau)\delta^{a_1a_2}\chi(\tau);\end{aligned} \tag{4.7}$$

using (3.43) and (4.4), we have

$$\omega_2^{ab} = \frac{4\pi^2}{\chi(\tau)}\mathrm{tr}\left(J_0^aJ_0^bw^{L_0}\right) - 2\delta^{ab}k\eta(\tau). \tag{4.8}$$

Taking $r=n=3$,

$$\begin{aligned}\mathrm{tr}\left(J^{a_1}(\rho_1)J^{a_2}(\rho_2)J^{a_3}(\rho_3)w^{L_0}\right)_S\rho_1\rho_2\rho_3 &= \mathrm{tr}\left(J_0^{a_1}J^{a_2}(\rho_2)J^{a_3}(\rho_3)w^{L_0}\right)_S\rho_2\rho_3 \\ &= \mathrm{tr}\left(J_0^{a_1}J_0^{a_2}J^{a_3}(\rho_3)w^{L_0}\right)_S\rho_3 \\ &= \mathrm{tr}\left(J_0^{a_1}J_0^{a_2}J_0^{a_3}w^{L_0}\right)_S\end{aligned} \tag{4.9}$$

because $\mathrm{tr}(J^{a_3}(\rho_3)w^{L_0}) = \mathrm{tr}(J_0^{a_3}w^{L_0}) = 0$, so that

$$\omega_3^{abc} = \frac{4\pi^3 i}{\chi(\tau)}\mathrm{tr}\left(J_0^a J_0^b J_0^c w^{L_0}\right)_S. \tag{4.10}$$

For $r = n = 4$,

$$\begin{aligned}
&\mathrm{tr}\left(J^{a_1}(\rho_1)J^{a_2}(\rho_2)J^{a_3}(\rho_3)J^{a_4}(\rho_4)w^{L_0}\right)_S\rho_1\rho_2\rho_3\rho_4\\
&= \mathrm{tr}\left(J_0^{a_1}J^{a_2}(\rho_2)J^{a_3}(\rho_3)J^{a_4}(\rho_4)w^{L_0}\right)_S\rho_2\rho_3\rho_4\\
&\quad + k\Delta_2(\nu_2-\nu_1)\left[\delta^{a_1a_2}\mathrm{tr}\left(J^{a_3}(\rho_3)J^{a_4}(\rho_4)w^{L_0}\right)\right]_S\rho_3\rho_4\\
&\quad\quad + k\Delta_2(\nu_3-\nu_1)\left[\delta^{a_1a_3}\mathrm{tr}\left(J^{a_2}(\rho_2)J^{a_4}(\rho_4)w^{L_0}\right)\right]_S\rho_2\rho_4\\
&\quad\quad\quad + k\Delta_2(\nu_4-\nu_1)\left[\delta^{a_1a_4}\mathrm{tr}\left(J^{a_2}(\rho_2)J^{a_3}(\rho_3)w^{L_0}\right)\right]_S\rho_2\rho_3\\
&= \mathrm{tr}\left(J_0^{a_1}J_0^{a_2}J^{a_3}(\rho_3)J^{a_4}(\rho_4)w^{L_0}\right)_S\rho_3\rho_4\\
&\quad + k\left(\Delta_2(\nu_2-\nu_1)+\Delta_2(\nu_3-\nu_1)+\Delta_2(\nu_4-\nu_1)\right)\left[\delta^{a_1a_2}\mathrm{tr}\left(J_0^{a_3}J_0^{a_4}w^{L_0}\right)\right]_S\\
&\quad\quad + k^2\left(\Delta_2(\nu_2-\nu_1)\Delta_2(\nu_3-\nu_4)+\Delta_2(\nu_3-\nu_1)\Delta_2(\nu_2-\nu_4)\right.\\
&\quad\quad\quad \left.+\Delta_2(\nu_4-\nu_1)\Delta_2(\nu_2-\nu_3)\right)\left[\delta^{a_1a_2}\delta^{a_3a_4}\right]_S\chi(\tau)\\
&\quad\quad\quad + k\Delta_2(\nu_3-\nu_2)\left[\delta^{a_2a_3}\mathrm{tr}\left(J_0^{a_1}J^{a_4}(\rho_4)w^{L_0}\right)\right]_S\rho_4\\
&\quad\quad\quad\quad + k\Delta_2(\nu_4-\nu_2)\left[\delta^{a_2a_4}\mathrm{tr}\left(J_0^{a_1}J^{a_3}(\rho_3)w^{L_0}\right)\right]_S\rho_3\\
&= \mathrm{tr}\left(J_0^{a_1}J_0^{a_2}J_0^{a_3}J_0^{a_4}w^{L_0}\right)_S\\
&\quad + k\left[\delta^{a_1a_2}\mathrm{tr}\left(J_0^{a_3}J_0^{a_4}w^{L_0}\right)\right]_S\sum_{i<j}\Delta_2(\nu_i-\nu_j)\\
&\quad\quad + k^2\left(\Delta_2(\nu_2-\nu_1)\Delta_2(\nu_3-\nu_4)+\Delta_2(\nu_3-\nu_1)\Delta_2(\nu_2-\nu_4)\right.\\
&\quad\quad\quad \left.+\Delta_2(\nu_4-\nu_1)\Delta_2(\nu_2-\nu_3)\right)\left[\delta^{a_1a_2}\delta^{a_3a_4}\right]_S\chi(\tau).
\end{aligned}\tag{4.11}$$

Then, since

$$\begin{aligned}
&\mathrm{tr}\left(J^{a_1}(\rho_1)J^{a_2}(\rho_2)J^{a_3}(\rho_3)J^{a_4}(\rho_4)w^{L_0}\right)_C = \mathrm{tr}\left(J^{a_1}(\rho_1)J^{a_2}(\rho_2)J^{a_3}(\rho_3)J^{a_4}(\rho_4)w^{L_0}\right)\\
&- \left[\mathrm{tr}\left(J^{a_1}(\rho_1)J^{a_2}(\rho_2)w^{L_0}\right)\mathrm{tr}\left(J^{a_3}(\rho_3)J^{a_4}(\rho_4)w^{L_0}\right)\right.\\
&\quad + \mathrm{tr}\left(J^{a_1}(\rho_1)J^{a_3}(\rho_3)w^{L_0}\right)\mathrm{tr}\left(J^{a_2}(\rho_2)J^{a_4}(\rho_4)w^{L_0}\right)\\
&\quad\quad \left. + \mathrm{tr}\left(J^{a_1}(\rho_1)J^{a_4}(\rho_4)w^{L_0}\right)\mathrm{tr}\left(J^{a_2}(\rho_2)J^{a_3}(\rho_3)w^{L_0}\right)\right]/\chi(\tau),
\end{aligned}\tag{4.12}$$

$$\begin{aligned}
&\mathrm{tr}\left(J^{a_1}(\rho_1)J^{a_2}(\rho_2)J^{a_3}(\rho_3)J^{a_4}(\rho_4)w^{L_0}\right)_{CS}\prod_{j=1}^4\rho_j\\
&= \mathrm{tr}\left(J_0^{a_1}J_0^{a_2}J_0^{a_3}J_0^{a_4}w^{L_0}\right)_S - 3\left[\mathrm{tr}\left(J_0^{a_1}J_0^{a_2}w^{L_0}\right)\mathrm{tr}\left(J_0^{a_3}J_0^{a_4}w^{L_0}\right)\right]_S/\chi(\tau).
\end{aligned}\tag{4.13}$$

From (3.78), this shows that ω_4 is given as a "connected part" of a trace of zero modes. In the next section we define such connected parts and show that ω_n is given in terms of them for all $n \geq 2$.

4.2 Connected parts of zero mode amplitudes

Because of the locality of the currents, $\mathscr{A}^A$, defined as in (3.32), and so $\mathscr{A}^A_C$, defined inductively by (3.36), is symmetric under simultaneous permutations of the indices a_j and the variables ν_j. We define the symmetrization $\mathscr{A}^A_S$ of $\mathscr{A}^A$ by symmetrizing on the a_j alone:

$$\begin{aligned}&\mathscr{A}_S^{a_{i_1}a_{i_2}\ldots a_{i_n}}(\nu_{i_1},\nu_{i_2},\ldots,\nu_{i_n},\tau)\\&\quad=\frac{1}{n!}\sum_{\rho\in\mathfrak{S}_n}\mathscr{A}^{a_{i_\rho(1)}a_{i_\rho(2)}\ldots a_{i_\rho(n)}}(\nu_{i_1},\nu_{i_2},\ldots,\nu_{i_n},\tau);\end{aligned}\tag{4.14}$$

equivalently we could symmetrize on the variables ν_j alone. We define $\mathscr{A}^A_{CS}$, the symmetrization of $\mathscr{A}^A_C$, similarly.

We consider the trace of zero modes of the currents,

$$\mathscr{Z}^A\equiv\mathscr{Z}^{a_{i_1}a_{i_2}\ldots a_{i_n}}(\tau)=\operatorname{tr}\left(J_0^{a_{i_1}}J_0^{a_{i_2}}\ldots J_0^{a_{i_n}}w^{L_0}\right)(2\pi i)^n,\tag{4.15}$$

and, more particularly, its symmetrization, $\mathscr{Z}^A_S$, defined as in (4.14). We can define a "connected part", $\mathscr{Z}_{CS}$, inductively for $\mathscr{Z}^A_S$, following (3.36),

$$\mathscr{Z}^A_{CS}=\mathscr{Z}^A_S-\sum_{P\in\mathfrak{P}'_A}\chi(\tau)^{1-|P|}\prod_{A_j\in P}\mathscr{Z}^{A_j}_{CS},\tag{4.16}$$

(where again $\mathfrak{P}'_A$ denotes the same collection of divisions of A into disjoint subsets but omitting the division of A into the single set consisting of itself) together with the vanishing of the one point function $\mathscr{Z}^{\{i\}}_{CS}=0$, and with the two-point function given by

$$\mathscr{Z}^{\{i,j\}}_{CS}=\mathscr{Z}^{\{i,j\}}_S=\operatorname{tr}\left(J_0^{a_i}J_0^{a_j}w^{L_0}\right)(2\pi i)^2.\tag{4.17}$$

For $A=\{i_1,i_2,\ldots,i_{2m}\}$, define

$$\mathscr{P}^A=\frac{k^m(2\pi i)^2}{2^m m!}\sum_{\rho\in\mathfrak{S}_{2m}}\prod_{j=1}^{m}\delta^{a_{i_\rho(2j-1)}a_{i_\rho(2j)}}\Delta_2(\nu_{i_\rho(2j-1)}-\nu_{i_\rho(2j)}),\tag{4.18}$$

and define $\mathscr{P}^A=0$ if A has an odd number of elements. Then

$$\mathscr{A}^{\{i,j\}}_{CS}=\mathscr{Z}^{\{i,j\}}_{CS}+\mathscr{P}^{\{i,j\}}\chi,\tag{4.19}$$

and the recurrence relation (4.6) leads to

$$\mathscr{A}^A_S=\mathscr{Z}^A_S+\sum_{B\in\mathfrak{R}_A}[\mathscr{P}^B\mathscr{Z}^{A\sim B}_S]_S,\tag{4.20}$$

where $\mathfrak{R}_A$ denotes the subsets of A, excluding the empty set but including A itself.

Now, symmetrizing (3.36),

$$\mathscr{A}^A_S=\mathscr{A}^A_{CS}+\sum_{P\in\mathfrak{P}'_A}\chi^{1-|P|}\left[\prod_{A_j\in P}\mathscr{A}^{A_j}_{CS}\right]_S.\tag{4.21}$$

If we assume, as the inductive hypothesis, that $\mathscr{A}^B_{CS} = \mathscr{Z}^B_{CS}$, for $2 < |B| < |A|$, and that (4.19) holds when $|B| = 2$, we have, on substituting for $\mathscr{A}^{A_j}_{CS}$ and symmetrizing, that

$$\begin{aligned}\mathscr{A}^A_S &= \mathscr{A}^A_{CS} + \sum_{P\in\mathfrak{P}'_A} \chi^{1-|P|}\left[\prod_{A_j\in P}\mathscr{Z}^{A_j}_{CS}\right]_S + \sum_{B\in\mathfrak{R}_A}\left[\mathscr{P}^B \sum_{R\in\mathfrak{P}_{A\sim B}} \chi^{1-|R|}\prod_{D_j\in R}\mathscr{Z}^{D_j}_{CS}\right]_S \\ &= \mathscr{A}^A_{CS} + \sum_{P\in\mathfrak{P}'_A} \chi^{1-|P|}\left[\prod_{A_j\in P}\mathscr{Z}^{A_j}_{CS}\right]_S + \sum_{B\in\mathfrak{R}_A}\left[\mathscr{P}^B\,\mathscr{Z}^{A\sim B}\right]_S\end{aligned} \tag{4.22}$$

by (4.16). Then using (4.20),

$$\mathscr{Z}^A_S = \mathscr{A}^A_{CS} + \sum_{P\in\mathfrak{P}'_A} \chi^{1-|P|}\left[\prod_{A_j\in P}\mathscr{Z}^{A_j}_{CS}\right]_S, \tag{4.23}$$

so that $\mathscr{A}^A_{CS}$ satisfies the recurrence relation (4.16) for $\mathscr{Z}^A_{CS}$ and we can conclude inductively that $\mathscr{A}^A_{CS} = \mathscr{Z}^A_{CS}$, for $|A| > 2$.

It follows from (3.78),

$$\omega_n(\tau) = \kappa^{a_1\ldots a_n}_{n,0}(\tau) = -\frac{(2\pi i)^n}{(n-1)!\chi(\tau)}\mathrm{tr}\left(J^{a_1}_0 J^{a_2}_0\ldots J^{a_n}_0 w^{L_0}\right)_{C,S}, \quad n\geq 3, \tag{4.24}$$

with ω_2 given by (4.8).

4.3 Traces of zero modes and characters

In this section we will relate the symmetrized traces of the zero modes

$$\mathscr{Z}^{a_{i_1}a_{i_2}\ldots a_{i_n}}_S(\tau) = \mathrm{tr}\left(J^{a_{i_1}}_0 J^{a_{i_2}}_0\ldots J^{a_{i_n}}_0 w^{L_0}\right)_S (2\pi i)^n, \tag{4.25}$$

to the character of the representation of the affine algebra, $\hat{\mathfrak{g}}$, defined by (1.1), in the space of states,

$$\chi(\theta,\tau) = \mathrm{tr}\left(e^{iH\cdot\theta}w^{L_0}\right). \tag{4.26}$$

Here H denotes the generators of a Cartan subalgebra, $\mathfrak{h}$, of the finite-dimensional algebra $\mathfrak{g}$ formed by the zero modes,

$$[J^a_0, J^a_0] = f^{ab}{}_c J^c_0. \tag{4.27}$$

For convenience of exposition, we shall take $\mathfrak{g}$ to be simple in what follows.

For fixed τ, $\mathrm{tr}\left(J^{a_{i_1}}_0 J^{a_{i_2}}_0\ldots J^{a_{i_n}}_0 w^{L_0}\right)_S$ is an invariant symmetric tensor for $\mathfrak{g}$, The space of symmetric tensors, $\mathscr{S}(\mathfrak{g})$ is isomorphic (as a vector space) to $\mathscr{U}(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$,

$$\omega^{a_1a_2\ldots a_n} \to \omega^{a_1a_2\ldots a_n} J^{a_{i_1}}_0 J^{a_{i_2}}_0\ldots J^{a_{i_n}}_0. \tag{4.28}$$

The invariant tensors $\mathscr{S}(\mathfrak{g})^{\mathfrak{g}} \subset \mathscr{S}(\mathfrak{g})$ correspond to the center $Z(\mathscr{U}(\mathfrak{g}))$ of $\mathscr{U}(\mathfrak{g})$, *i.e.* the elements of $\mathscr{U}(\mathfrak{g})$ that commute with $\mathfrak{g}$. This is a ring generated by rank $\mathfrak{g}$ elements (*e.g.* (28), p. 337), the basic Casimir operators, or primitive invariant tensors. These can be taken to be orthogonal,

$$\omega^{a_1 a_2 \ldots a_n} \omega'_{a_1 a_2 \ldots a_m} = 0, \tag{4.29}$$

where ω, ω' are primitive invariant symmetric tensors of orders n, m, $m < n$.

We can use a Cartan-Weyl basis for $\mathfrak{g}$, using Φ to denote the set of roots of $\mathfrak{g}$,

$$\begin{aligned}
[H^i, H^j] &= 0, && 1 \le i, j \le \text{rank}\mathfrak{g}; \\
[H^i, E^\alpha] &= \alpha^i E^\alpha, && \alpha \in \Phi, \quad 1 \le i \le \text{rank } \mathfrak{g}; \\
[E^\alpha, E^\beta] &= \varepsilon(\alpha,\beta) E^{\alpha+\beta}, && \alpha, \beta, \alpha+\beta \in \Phi; \\
&= \frac{2}{\alpha^2} \alpha \cdot H, && \beta = -\alpha \in \Phi \\
&= 0, && \text{otherwise.}
\end{aligned} \tag{4.30}$$

(We omit the suffix 0 on H, E^α.) With this choice of basis, the quadratic Casimir operator

$$\begin{aligned}
J^a J^a &= H^2 + \sum_{\alpha>0} \frac{\alpha^2}{2} (E_{-\alpha} E_\alpha + E_\alpha E_{-\alpha}) \\
&= H^2 + 2\delta \cdot H + \sum_{\alpha>0} \alpha^2 E_{-\alpha} E_\alpha, \qquad \delta = \frac{1}{2} \sum_{\alpha>0} \alpha, \\
&= (H+\delta)^2 - \delta^2 + \sum_{\alpha>0} \alpha^2 E_{-\alpha} E_\alpha.
\end{aligned} \tag{4.31}$$

The value of J^2 in a representation with highest weight λ can be obtained by evaluating this on the highest weight state $|\lambda\rangle$, which has $E^\alpha |\lambda\rangle = 0$ for $\alpha > 0$. Thus the value of J^2 in this representation is

$$\lambda \cdot (\lambda + 2\delta) = (\lambda + \delta)^2 - \delta^2. \tag{4.32}$$

If $\xi^{a_1 a_2 \ldots a_n}$ is any invariant tensor for $\mathfrak{g}$,

$$C_\xi = \xi^{a_1 a_2 \ldots a_n} J_0^{a_1} J_0^{a_2} \ldots J_0^{a_n} \in Z(\mathscr{U}(g)), \tag{4.33}$$

and we can evaluate its value in the representation with highest weight $|\lambda\rangle$ by expressing it in the Cartan-Weyl basis and moving the $E_\alpha, \alpha > 0$, to the right. Because $[H^i, C_\beta] = 0$, we can write

$$C_\xi = \phi_\xi(H) + \sum_{\alpha>0} F_{\xi,\alpha} E_\alpha, \qquad \text{for suitable } F_{\xi,\alpha} \in \mathscr{U}(\mathfrak{g}), \tag{4.34}$$

where $\phi_\xi(H)$ is a polynomial of degree n in the H^i. Then $C_\xi |\lambda\rangle = \phi_\xi(\lambda) |\lambda\rangle$, so that C_ξ takes the value $\phi_\xi(\lambda)$ in the representation with highest weight λ. Given two such invariant tensors ξ_1, ξ_2,

$$C_{\xi_1}C_{\xi_2} = \left(\phi_{\xi_1}(H) + \sum_{\alpha>0} F_{\xi_1,\alpha}E_\alpha\right)\left(\phi_{\xi_2}(H) + \sum_{\beta>0} F_{\xi_2,\beta}E_\beta\right)$$
$$= \phi_{\xi_1}(H)\phi_{\xi_2}(H) + \sum_{\alpha>0} F_{\xi_1,\alpha}\phi_{\xi_2}(H-\alpha)E_\alpha + \sum_{\beta>0}\phi_{\xi_1}(H)F_{\xi_2,\beta}E_\beta$$
$$+ \sum_{\alpha>0} F_{\xi_1,\alpha}\phi_{\xi_2}(H)E_\alpha \sum_{\beta>0}\phi_{\xi_1}(H)F_{\xi_2,\beta}E_\beta, \tag{4.35}$$

so that

$$\phi_{\xi_1\xi_2}(H) = \phi_{\xi_1}(H)\phi_{\xi_2}(H). \tag{4.36}$$

If $\phi_{\xi_1} = \phi_{\xi_2}$, then $C_{\xi_1} = C_{\xi_2}$ acting in each highest weight representation of $\mathfrak{g}$. It follows from this that $C_{\xi_1} = C_{\xi_2}$ as elements of $\mathscr{U}(\mathfrak{g})$ (see, e.g., (29), p. 251). Thus $C_\xi \mapsto \phi_\xi$ defines a map $Z(\mathscr{U}(\mathfrak{g})) \to \mathscr{S}(\mathfrak{h})$, which is an algebra homomorphism and is one-to-one.

The elements $\phi_\xi \in \mathscr{S}(\mathfrak{h})$ obtained in this way have an invariance under the Weyl group, W, of $\mathfrak{g}$ as we shall now show (see (30), p. 130, or (29), p. 246). Consider the action of ϕ_ξ in the infinite-dimensional representation, $\tilde{V}_\lambda$, with highest weight λ, where $\lambda \in \Lambda_\mathfrak{g}$, the weight lattice of $\mathfrak{g}$, with $\alpha\cdot\lambda \geq 0$ for $\alpha > 0$, whose states are generated by the action of $E_\alpha, \alpha > 0$, on a state $|\lambda\rangle$. The finite-dimensional representation, V_λ, is the quotient of $\tilde{V}_\lambda$ by its largest invariant subspace. Taking a basis of simple roots, $\alpha_1, \alpha_2, \ldots, \alpha_r$, $r = \text{rank}\,\mathfrak{g}$, $m_i = 2\alpha_i\cdot\lambda/\alpha_i^2 \in \mathbb{Z}$ and $\alpha_i\cdot\lambda \geq 0$, consider $E_{-\alpha_i}^{m_i+1}|\lambda\rangle$. Now

$$\alpha_i\cdot H E_{-\alpha_i}^s|\lambda\rangle = \tfrac{1}{2}\alpha_i^2(m_i - 2s)E_{-\alpha_i}^s|\lambda\rangle$$

so

$$E_{\alpha_i}E_{-\alpha_i}^{m_i+1}|\lambda\rangle = \sum_{s=0}^{m_i}(m_i-2s)E_{-\alpha_i}^{m_i}|\lambda\rangle = 0$$

and $E_{\alpha_j}E_{-\alpha_i}^{m_i+1}|\lambda\rangle = 0$ for $i \neq j$ because $[E_{\alpha_i}, E_{\alpha_j}] = 0$, $i \neq j$. It follows that $E_\alpha E_{-\alpha_i}^{m_i+1}|\lambda\rangle = 0$ for $\alpha > 0$ and so $E_{-\alpha_i}^{m_i+1}|\lambda\rangle = 0$ generate an invariant subspace of $\tilde{V}_\lambda$ (which is divided out in the construction of V_λ). Then

$$C_\xi E_{-\alpha_i}^{m_i+1}|\lambda\rangle = \phi_\xi(H)E_{-\alpha_i}^{m_i+1}|\lambda\rangle = \phi_\xi(\lambda - m_i\alpha_i - \alpha_i)E_{-\alpha_i}^{m_i+1}|\lambda\rangle. \tag{4.37}$$

But, on the other hand

$$C_\xi E_{-\alpha_i}^{m_i+1}|\lambda\rangle = E_{-\alpha_i}^{m_i+1}|C_\xi\lambda\rangle = \phi_\xi(\lambda)E_{-\alpha_i}^{m_i+1}|\lambda\rangle. \tag{4.38}$$

Thus, for each simple root, α_i,

$$\phi_\xi(\lambda) = \phi_\xi(\lambda - m_i\alpha_i - \alpha_i). \tag{4.39}$$

If σ_i denotes the element of the $W_\mathfrak{g}$ corresponding to reflection in the hyperplane orthogonal to α_i,

$$\sigma_i(\lambda) = \lambda - m_i\alpha_i, \quad \text{and} \quad \sigma_i(\delta) = \delta - \alpha_i,$$

because $2\delta\cdot\alpha_i/\alpha_i^2 = 1$ for each simple root α_i. Thus (4.39) can be rewritten

$$\phi_\xi(\lambda) = \phi_\xi(\sigma_i(\lambda+\delta)-\delta), \tag{4.40}$$

and, if we define

$$\tilde{\phi}_\xi(\lambda) = \phi_\xi(\lambda-\delta) = \phi_\xi(\sigma_i(\lambda)-\delta) = \tilde{\phi}(\sigma_i(\lambda)). \tag{4.41}$$

Because the reflections in the simple roots, σ_i, generate the Weyl group $W_\mathfrak{g}$, $\tilde{\phi}_\xi(\lambda) = \phi_\xi(\lambda-\delta)$ defines a function invariant under the whole Weyl group. Thus $C_\xi \mapsto \tilde{\phi}_\xi$ defines a homomorphism of $Z(\mathscr{U}(\mathfrak{g})) \to \mathscr{S}(\mathfrak{h})^W$, the polynomials in H invariant under the Weyl group. In fact, this map is an isomorphism, called the Harish-Chandra isomorphism. That $C_\xi \mapsto \tilde{\phi}_\xi$ is onto follows from the fact that $\mathscr{S}(\mathfrak{h})^W$ is spanned by ϕ_ξ for $\xi^{a_1a_2\cdots a_n} = \mathrm{tr}(t^{a_1}t^{a_2}\ldots t^{a_n})$, where the t^a are the representations of J_0^a in the finite-dimensional representation $V_\lambda, \lambda\in\Lambda_\mathfrak{g}$ (see, e.g., (29), p 253).

Now, writing

$$\mathrm{tr}\left(e^{iH\cdot\theta}w^{L_0}\right) = \chi(\theta,\tau) = \sum_{\lambda\in\Lambda_\mathfrak{g}^+} b_\lambda(w)\chi^\lambda(\theta), \tag{4.42}$$

where $\Lambda_\mathfrak{g}^+ = \{\lambda\in\Lambda_\mathfrak{g} : \alpha\cdot\lambda \geq 0 \text{ for } \alpha>0\}$,

$$\mathrm{tr}\left(C_\xi w^{L_0}\right) = \sum_{\lambda\in\Lambda_\mathfrak{g}^+} b_\lambda(w)\tilde{\phi}(\lambda+\delta)\dim V_\lambda, \tag{4.43}$$

and the character for the finite-dimensional representation V_λ of $\mathfrak{g}$, $\chi^\lambda(\theta)$ is given by the Weyl character formula,

$$\chi^\lambda(\theta) = \frac{1}{\Delta_\mathfrak{g}(\theta)}\sum_{\sigma\in W_\mathfrak{g}} \varepsilon(\sigma)e^{i\sigma(\delta+\lambda)\cdot\theta}, \tag{4.44}$$

with $\varepsilon(\sigma) = \pm 1$ being the determinant of σ, and the Weyl denominator being given by

$$\Delta_\mathfrak{g}(\theta) = \prod_{\alpha>0}\left(e^{\frac{i}{2}\alpha\cdot\theta} - e^{-\frac{i}{2}\alpha\cdot\theta}\right), \tag{4.45}$$

where the product is over the positive roots of $\mathfrak{g}$. (See (30), p. 139.) The dimension $\dim V_\lambda = \chi^\lambda(0)$, but to evaluate this from (4.44), we need to take a limit on the right-hand side. In fact $\Delta_\mathfrak{g}(\theta) = \mathscr{O}(\theta^{n^+})$, as $\theta\to 0$, where n^+ is the number of positive roots of $\mathfrak{g}$. Now

$$\begin{aligned}\prod_{\alpha>0}\alpha\cdot\partial_\theta \sum_{\sigma\in W_\mathfrak{g}}\varepsilon(\sigma)e^{i\sigma(\delta+\lambda)\cdot\theta} &= \prod_{\alpha>0}\sum_{\sigma\in W_\mathfrak{g}} i\varepsilon(\sigma)e^{i\sigma(\delta+\lambda)\cdot\theta}\sigma(\delta+\lambda)\cdot\alpha \\ &= \prod_{\alpha>0}\sum_{\sigma\in W_\mathfrak{g}} i\varepsilon(\sigma)(\delta+\lambda)\cdot\sigma^{-1}(\alpha), \\ &\quad \text{when } \theta = 0.\end{aligned} \tag{4.46}$$

As α runs over the positive roots of $\mathfrak{g}$, $\sigma(\alpha)$ will range over a set obtained from the positive roots by reversing some of their signs. The product of these sign changes equals $\varepsilon(\sigma) = \varepsilon(\sigma^{-1})$. Hence the sign changes cancel the effect of $\varepsilon(\sigma)$ in (4.46) and we have

$$\prod_{\alpha>0} \alpha\cdot\partial_\theta \sum_{\sigma\in W_\mathfrak{g}} \varepsilon(\sigma) e^{i\sigma(\delta+\lambda)\cdot\theta}\Big|_{\theta=0} = i^{n^+}|W_\mathfrak{g}| \prod_{\alpha>0}(\delta+\lambda)\cdot\alpha. \tag{4.47}$$

Since $\chi^0(\theta) = 1$,

$$\Delta_\mathfrak{g}(\theta) = \sum_{\sigma\in W_\mathfrak{g}} \varepsilon(\sigma) e^{i\sigma(\delta)\cdot\theta}, \tag{4.48}$$

and, hence,

$$\prod_{\alpha>0} \alpha\cdot\partial_\theta \Delta_\mathfrak{g}(\theta)\Big|_{\theta=0} = i^{n^+}|W_\mathfrak{g}| \prod_{\alpha>0}\delta\cdot\alpha, \tag{4.49}$$

and

$$\dim V_\lambda = \chi^\lambda(0) = \prod_{\alpha>0} \frac{(\delta+\lambda)\cdot\alpha}{\delta\cdot\alpha}. \tag{4.50}$$

Applying

$$(\beta\cdot\partial_\theta)^n \prod_{\alpha>0} \alpha\cdot\partial_\theta \tag{4.51}$$

to the equation

$$\chi^\lambda(\theta)\Delta_\mathfrak{g}(\theta) = \sum_{\sigma\in W_\mathfrak{g}} \varepsilon(\sigma) e^{i\sigma(\delta+\lambda)\cdot\theta}, \tag{4.52}$$

we obtain

$$\begin{aligned}(\beta\cdot\partial_\theta)^n \left(\prod_{\alpha>0}\alpha\cdot\partial_\theta\right)\chi^\lambda(\theta)\Delta_\mathfrak{g}(\theta)\Big|_{\theta=0} &= i^{n^+ +n}\prod_{\alpha>0}(\delta+\lambda)\cdot\alpha \sum_{\sigma\in W_\mathfrak{g}} (\sigma(\delta+\lambda)\cdot\beta)^n \\ &= i^{n^+ +n}\prod_{\alpha>0}\delta\cdot\alpha \sum_{\sigma\in W_\mathfrak{g}} (\sigma(\delta+\lambda)\cdot\beta)^n \dim V_\lambda,\end{aligned} \tag{4.53}$$

and so

$$\begin{aligned}&(\beta\cdot p_\theta)^n \left(\prod_{\alpha>0}\frac{\alpha\cdot p_\theta}{\alpha\cdot\delta}\right)\chi(\theta,\tau)\Delta_\mathfrak{g}(\theta)\Big|_{\theta=0} \\ &\quad = \sum_\lambda b_\lambda(w) \sum_{\sigma\in W_\mathfrak{g}} (\sigma(\delta+\lambda)\cdot\beta)^n \dim V_\lambda,\end{aligned} \tag{4.54}$$

where $p_\theta = -i\partial_\theta$.

If

$$\tilde{\phi}(\lambda) = \sum_{\sigma' \in W_{\mathfrak{g}}} \left(\beta \cdot \sigma'(\lambda)\right)^n = \sum_{\sigma' \in W_{\mathfrak{g}}} \left(\sigma'(\beta) \cdot \lambda\right)^n, \tag{4.55}$$

$$\begin{aligned}\tilde{\phi}(p_\theta) \left(\prod_{\alpha>0} \frac{\alpha \cdot p_\theta}{\alpha \cdot \delta}\right) \chi(\theta,\tau) \Delta_{\mathfrak{g}}(\theta)\Big|_{\theta=0} &= \sum_{\lambda} b_\lambda(w) \sum_{\sigma,\sigma' \in W_{\mathfrak{g}}} \left(\sigma(\delta+\lambda) \cdot \sigma'(\beta)\right)^n \dim V_\lambda \\ &= |W_{\mathfrak{g}}| \sum_{\lambda} b_\lambda(w) \tilde{\phi}(\delta+\lambda) \dim V_\lambda .\end{aligned} \tag{4.56}$$

The functions (4.55) span the polynomial functions $\tilde{\phi}(\lambda)$ invariant under the Weyl group and so (4.56) holds for any such function. From (4.43), it follows that

$$\operatorname{tr}\left(C_\xi w^{L_0}\right) = \frac{1}{|W_{\mathfrak{g}}|} \tilde{\phi}_\xi(p_\theta) \left(\prod_{\alpha>0} \frac{\alpha \cdot p_\theta}{\alpha \cdot \delta}\right) \chi(\theta,\tau) \Delta_{\mathfrak{g}}(\theta)\Big|_{\theta=0} . \tag{4.57}$$

The symmetrized products of the primitive symmetric invariant tensors form a basis for all symmetric invariant tensors. Suppose $\omega_j^{a_1 a_2 \ldots a_n}, 1 \le j \le N$ forms an orthonormal basis for the symmetric invariant tensors of order n, so that

$$\omega_j^{a_1 a_2 \ldots a_n} \omega_k^{a_1 a_2 \ldots a_n} = \delta_{jk}. \tag{4.58}$$

Then we can write

$$\operatorname{tr}\left(J_0^{a_{i_1}} J_0^{a_{i_2}} \ldots J_0^{a_{i_n}} w^{L_0}\right)_S = \sum_{j=1}^{N} f_j(w) \omega_j^{a_1 a_2 \ldots a_n}, \tag{4.59}$$

where

$$f_j(w) = \operatorname{tr}\left(C_{\omega_j} w^{L_0}\right) \tag{4.60}$$

and

$$\begin{aligned}&\operatorname{tr}\left(J_0^{a_{i_1}} J_0^{a_{i_2}} \ldots J_0^{a_{i_n}} w^{L_0}\right)_S \\ &= \frac{1}{|W_{\mathfrak{g}}|} \sum_{j=1}^{N} \omega_j^{a_1 a_2 \ldots a_n} \tilde{\phi}_{\omega_j}(p_\theta) \left(\prod_{\alpha>0} \frac{\alpha \cdot p_\theta}{\alpha \cdot \delta}\right) \chi(\theta,\tau) \Delta_{\mathfrak{g}}(\theta)\Big|_{\theta=0} \\ &= \frac{1}{|W_{\mathfrak{g}}|} \sum_{j=1}^{N} \omega_j^{a_1 a_2 \ldots a_n} \sum_{\sigma \in W_{\mathfrak{g}}} \varepsilon(\sigma)\, \tilde{\phi}_{\omega_j}(p_\theta + \sigma(\delta)) \\ &\quad \times \left(\prod_{\alpha>0} \frac{\alpha \cdot (p_\theta + \sigma(\delta))}{\alpha \cdot \delta}\right) \chi(\theta,\tau)\Big|_{\theta=0} \\ &= \frac{1}{|W_{\mathfrak{g}}|} \sum_{j=1}^{N} \omega_j^{a_1 a_2 \ldots a_n} \sum_{\sigma \in W_{\mathfrak{g}}} \phi_{\omega_j}(\sigma(p_\theta)) \left(\prod_{\alpha>0} \frac{\alpha \cdot (\delta + \sigma(p_\theta))}{\alpha \cdot \delta}\right) \chi(\theta,\tau)\Big|_{\theta=0} .\end{aligned} \tag{4.61}$$

5 Summary and Conclusions

In this paper, we have constructed a general formula for the loop amplitude

$$\mathrm{tr}\left(J^{a_1}(\rho_1)J^{a_2}(\rho_2)\dots J^{a_n}(\rho_n)w^{L_0}\right), \tag{5.1}$$

where the currents $J^a(\rho)$ satisfy the operator product expansion

$$J^a(z_1)J^b(z_2) \sim \frac{\kappa^{ab}}{(z_1-z_2)^2} + \frac{f^{ab}{}_c J^c(z_2)}{z_1-z_2}, \tag{5.2}$$

which is equivalent to the affine algebra $\hat{\mathfrak{g}}$, defined by (1.1). This formula extends the Frenkel-Zhu construction for tree amplitudes (1), described in Sect. 2.2, and generalizes the results obtained when the currents are given as bilinear expressions in fermionic fields, which are reviewed in 3.1.

The general formula is described graphically by summing over all graphs with n vertices where the vertices carry the labels $a_1, a_2, \dots, a_n$ and each vertex is connected by directed lines to other vertices, one of the lines at each vertex pointing towards it and one away from it. Each graph consists of a number, r, of directed "loops" or cycles, $\xi = (i_1, i_2 \dots i_\ell)$ with which we associate an expression f_ξ. The expression associated with the whole graph consists of a factor of $1/2\pi i\rho_j$ for each current $J^{a_j}(\rho_j)$ and $-f_{\xi_i}, 1 \le i \le r$, for each cycle,

$$\left[\prod_{j=1}^{n} \frac{1}{2\pi i\rho_j}\right] \sum_{\text{diagrams}} (-1)^r \prod_{i=1}^{r} f_{\xi_i}. \tag{5.3}$$

For $\xi = (i_1, i_2 \dots i_\ell)$,

$$f_\xi = \sum_{m=0}^{\ell} \kappa_{\ell,m}^{a_{i_1} a_{i_2} \dots a_{i_\ell}} H_{\ell,m}(\nu_{i_1 i_2}, \dots, \nu_{i_{\ell-1} i_\ell}, \nu_{i_\ell i_1}, \tau). \tag{5.4}$$

The functions $H_{\ell,m}$ are defined in terms of the Weierstrass σ function by

$$\frac{\nu^\ell}{\sigma(\nu,\tau)^\ell} \prod_{j=1}^{\ell} \frac{\sigma(\mu_j+\nu,\tau)}{\sigma(\mu_j,\tau)} = \sum_{m=0}^{\infty} H_{\ell,m}(\mu_1, \mu_2, \dots, \mu_\ell, \tau)\nu^m, \tag{5.5}$$

and the invariant tensors $\kappa_{\ell,m}$ are defined inductively by the equations

$$\kappa_{\ell,m}^{a_1 a_2 \dots a_\ell}(\tau) - \kappa_{\ell,m}^{a_2 a_1 \dots a_\ell}(\tau) = f^{a_1 a_2}{}_b \kappa_{\ell-1,m-1}^{b a_3 \dots a_\ell}(\tau), \tag{5.6}$$

$$\kappa_{\ell,m}^{a_1 a_2 \dots a_\ell}(\tau) = \kappa_{\ell,m}^{a_2 \dots a_\ell a_1}(\tau), \tag{5.7}$$

together with the requirement that $\kappa_{\ell,m}$ be orthogonal to all symmetric tensors for $m > 0$ and $\ell > 2$, and the initial condition that $\kappa_{2,2} = \kappa, \kappa_{2,1} = 0$ and $\kappa_{\ell,0}(\tau) = \omega_\ell(\tau)$, where the symmetric invariant tensor,

$$\omega_\ell^{a_1 a_2 \dots a_\ell}(\tau) = -\frac{(2\pi i)^\ell}{(\ell-1)!\chi(\tau)} \mathrm{tr}\left(J_0^{a_1} J_0^{a_2} \dots J_0^{a_\ell} w^{L_0}\right)_{C,S}, \quad \ell \ge 3; \tag{5.8}$$

$$\omega_2^{ab} = \frac{4\pi^2}{\chi(\tau)} \mathrm{tr}\left(J_0^a J_0^b w^{L_0}\right) - 2\delta^{ab} k\eta(\tau). \tag{5.9}$$

A proof that $\kappa_{\ell,m}$ exists and is defined uniquely by (5.6) and (5.7) is given in Sect. 2.3 and an algorithmic method for constructing them inductively using Young tableaux is given in Appendix A.

The results described so far in this section apply to the affine algebra, $\hat{\mathfrak{g}}$, associated with any finite-dimensional Lie algebra, $\mathfrak{g}$. In Sect. 4.3 we gave a method for calculating the traces of zero modes, necessary to determine the symmetric tensors ω, in terms of the character

$$\chi(\theta,\tau) = \mathrm{tr}\left(e^{iH\cdot\theta} w^{L_0}\right) \tag{5.10}$$

of the representation provided by the space of states of the theory. The method would apply to any compact $\mathfrak{g}$ but we took it to be simple for ease of exposition.

The "connected" symmetrized trace (5.8) is defined in Sect. 4.2 in terms of the "full" symmetrized traces, which themselves can be expanded in terms of an orthonormal basis of symmetric invariant tensors of order ℓ,

$$\mathrm{tr}\left(J_0^{a_{i_1}} J_0^{a_{i_2}} \ldots J_0^{a_{i_\ell}} w^{L_0}\right)_S = \sum_{j=1}^{N} f_j(w)\omega_j^{a_1 a_2 \ldots a_\ell}, \tag{5.11}$$

where N is the number of independent symmetric invariant tensors of order ℓ, $f_j(w) = \mathrm{tr}\left(C_{\omega_j} w^{L_0}\right)$ and the Casimir operator $C_{\omega_j} = \omega_j^{a_1 a_2 \ldots a_\ell} J_0^{a_1} J_0^{a_2} \ldots J_0^{a_\ell}$. In Sect. 4.3, we reviewed how a normal ordering of the J_0^a in C_{ω_j}, by writing $C_{\omega_j} = \phi_{\omega_j}(H) + N_j$, where H denotes the elements of a Cartan subalgebra and N_j annihilates highest weight states, so that $C_{\omega_j} \mapsto \phi_{\omega_j}(H)$ defines the Harish-Chandra isomorphism of the center of the enveloping algebra of $\mathfrak{g}$ (that is the ring of Casimir operators) onto the polynomials in H invariant under the action of the Weyl group, $W_{\mathfrak{g}}$ of $\mathfrak{g}$. This leads to the expression

$$\frac{1}{|W_{\mathfrak{g}}|} \sum_{j=1}^{N} \omega_j^{a_1 a_2 \ldots a_n} \sum_{\sigma \in W_{\mathfrak{g}}} \phi_{\omega_j}(\sigma(p_\theta)) \left(\prod_{\alpha>0} \frac{\alpha\cdot(\delta+\sigma(p_\theta))}{\alpha\cdot\delta}\right) \chi(\theta,\tau)\Big|_{\theta=0} \tag{5.12}$$

for the symmetrized trace (5.11), where $p_\theta = -i\partial_\theta$, α denotes a root of $\mathfrak{g}$ and δ denotes half the sum of positive roots. With this we have assembled all the elements of an explicit expression for the loop amplitude (5.1). In Appendix C, this is compared with expressions given previously (3) for $n = 2,3,4$.

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A Explicit Construction of the Tensor κ_n in Terms of κ_{n-1}

As a preparation for giving an explicit construction of the tensor κ_n in terms of κ_{n-1}, we review some salient features of the representation theory of $\mathfrak{S}_n$ (see *e.g.* (31), p. 44).

The number of inequivalent irreducible representations of $\mathfrak{S}_n$, the group of permutations of n objects, is $p(n)$, the number of partitions of n. Each partition, $p=(p^1,\ldots,p^m)$, with $p^i \geq p^j$, if $i \leq j$ and $\sum_{i=1}^m p^i = n$, determines a Young diagram, consisting of n boxes arranged into m rows and p^1 columns, with p^i boxes in the i^{th} row and the number of boxes in the j^{th} column equal to the number of $p^k \geq j$. We can identify the partition p with the corresponding Young diagram. The Young diagrams label the inequivalent irreducible representations.

Given a Young diagram p, a Young tableau, λ, is defined by an assignment of the integers $1,\ldots,n$ to the n boxes of p. This gives $n!$ Young tableaux associated with a given Young diagram. A standard Young tableau is one for which the numbers assigned to the boxes decrease along each row (from left to right) and down each column. The number of standard Young tableau associated with the Young diagram p,

$$d_p = \frac{n!}{\ell_1!\ldots\ell_m!}\prod_{i<j}(\ell_i-\ell_j), \qquad \text{where } \ell_j = p^j+m-j, \tag{A.1}$$

and this is also the dimension of the irreducible representation associated with p. The regular representation of $\mathfrak{S}_n$, V, which consists of linear combinations $\sum_{g\in\mathfrak{S}_n} x_g g$, of elements of $\mathfrak{S}_n$, contains d_p representations of the type labeled by p, so that $|\mathfrak{S}_n| = \sum_p d_p^2$, which we can regard as being labeled by the standard Young tableaux associated with p. We label these $\lambda_i^p, 1 \leq i \leq d_p$.

Given a Young tableau, λ, we define $\mathfrak{A}_\lambda$ of $\mathfrak{S}_n$ to be the subgroup of $\mathfrak{S}_n$ consisting those permutations which map each row of λ into itself and define $\mathfrak{B}_\lambda$ of $\mathfrak{S}_n$ to be the the subgroup of $\mathfrak{S}_n$ consisting those permutations which map each column of λ into itself. Let

$$a_\lambda = \sum_{\rho\in\mathfrak{A}_\lambda}\rho, \qquad b_\lambda = \sum_{\rho\in\mathfrak{B}_\lambda}\varepsilon(\rho)\rho, \tag{A.2}$$

where $\varepsilon(\rho)$ denotes the sign of the permutation ρ. Then

$$\rho a_\lambda = a_\lambda\rho = a_\lambda, \quad \rho\in\mathfrak{A}_\lambda; \qquad \rho b_\lambda = b_\lambda\rho = \varepsilon(\rho)b_\lambda, \quad \rho\in\mathfrak{B}_\lambda.$$

Define the Young symmetrizer

$$c_\lambda = a_\lambda b_\lambda. \tag{A.3}$$

Then

$$c_\lambda^2 = N_p c_\lambda, \quad \text{where } N_p = \frac{n!}{d_\lambda}, \tag{A.4}$$

and

$$c_\lambda c_\mu = 0, \tag{A.5}$$

if λ,μ have different shapes, *i.e.* are associated with different Young diagrams (partitions).

If the distinct Young tableaux λ,μ are associated with the same Young diagram, p, we can find a permutation $\sigma_{\lambda\mu}\in\mathfrak{S}_n$, which takes μ into λ; then

$$a_\lambda = \sigma_{\lambda\mu}a_\mu\sigma_{\mu\lambda}, \qquad b_\lambda = \sigma_{\lambda\mu}b_\mu\sigma_{\mu\lambda}, \qquad c_\lambda = \sigma_{\lambda\mu}c_\mu\sigma_{\mu\lambda}, \qquad \sigma_{\mu\lambda} = \sigma_{\lambda\mu}^{-1}. \tag{A.6}$$

Further (see (32), p. 393, or (33), p. 75), *either* there exists a pair (j,k) contained in a single column of λ and a single column of μ, in which case, if $t\in\mathfrak{S}_n$ is the transposition interchanging j and k, $t\in\mathfrak{B}_\lambda\cap\mathfrak{A}_\mu$, $t^2=1$, so that

$$b_\lambda a_\mu = b_\lambda t^2 a_\mu = -b_\lambda a_\mu, \quad \text{implying} \quad c_\lambda c_\mu = 0, \tag{A.7}$$

or the elements of each given column of λ are in different rows in μ, in which case

$$\sigma_{\lambda\mu} = \beta_{\lambda\mu}\alpha_{\lambda\mu}, \quad \text{for some } \alpha_{\lambda\mu}\in\mathfrak{A}_\mu, \beta_{\lambda\mu}\in\mathfrak{B}_\lambda, \tag{A.8}$$

so that

$$b_\lambda a_\mu = \varepsilon_{\lambda\mu} b_\lambda \beta_{\lambda\mu} \alpha_{\lambda\mu} a_\mu = \varepsilon_{\lambda\mu} b_\lambda \sigma_{\lambda\mu} a_\mu, \quad \text{where } \varepsilon_{\lambda\mu} = \varepsilon(\beta_{\lambda\mu}), \tag{A.9}$$

implying

$$c_\lambda c_\mu = N_p \varepsilon_{\lambda\mu} \sigma_{\lambda\mu} c_\mu = N_p \varepsilon_{\lambda\mu} c_\lambda \sigma_{\lambda\mu}.$$

If the normalized Young tableau $\hat{c}_\lambda = c_\lambda / N_p$, and $\hat{a}_\lambda = a_\lambda / \sqrt{N_p}, \hat{b}_\lambda = b_\lambda / \sqrt{N_p}$, and λ, μ have the same shape,

$$\hat{b}_\lambda \hat{a}_\mu = \varepsilon_{\lambda\mu} \hat{b}_\lambda \sigma_{\lambda\mu} \hat{a}_\mu, \qquad \hat{c}_\lambda \hat{c}_\mu = \varepsilon_{\lambda\mu} \hat{c}_\lambda \sigma_{\lambda\mu} = \varepsilon_{\lambda\mu} \sigma_{\lambda\mu} \hat{c}_\mu, \tag{A.10}$$

where $\varepsilon_{\lambda\lambda} = 1$, $\varepsilon_{\lambda\mu} = \varepsilon(\beta_{\lambda\mu})$ if the elements of each given column of λ are in different rows in μ, and $\varepsilon_{\lambda\mu} = 0$ otherwise.

Writing, $\lambda_i \equiv \lambda_i^p, 1 \le i \le d_p$ for the d_p standard Young tableaux of type p, in lexicographical order, that is if $i < j$ and we compare the entries of integers in the boxes of λ_i and λ_j reading along each row from left to right starting with the first row and proceeding to the second, and so on, then for the first discrepancy the integer in the relevant box in λ_j is greater than the one in the corresponding box in λ_i; in this case we write $\lambda_i < \lambda_j$ if $i < j$. Then, writing $a_i = a_{\lambda_i}, b_i = b_{\lambda_i}, \sigma_{ij} = \sigma_{\lambda_i \lambda_j}$, $b_i a_j = 0$ if $i > j$,

$$\hat{b}_i \hat{a}_j = \varepsilon_{ij} \hat{b}_i \sigma_{ij} \hat{a}_j, \tag{A.11}$$

where ε_{ij} is defined as in (A.10), for $i \le j$.

For each Young tableau λ of type p,

$$V_\lambda = V c_\lambda, \tag{A.12}$$

defines an irreducible representation subspace of the regular representation V of type p, dimension d_p. The spaces V_{λ_i}, $1 \le i \le d_p$, provide d_p irreducible representations of type p in V. In fact,

$$V \cong \bigoplus_p \bigoplus_{i=1}^{d_p} V_{\lambda_i^p}. \tag{A.13}$$

Corresponding to this decomposition into irreducible components, a basis for V is provided by

$$\{\sigma_{\lambda_i^p \lambda_j^p} \hat{c}_{\lambda_j^p} = \hat{a}_{\lambda_i^p} \sigma_{\lambda_i^p \lambda_j^p} \hat{b}_{\lambda_j^p} : 1 \le i, j \le d_p; p \in P(n)\}, \tag{A.14}$$

where $P(n)$ denotes the set of partitions of n. To establish that this is a basis, it is enough to show that the states

$$\hat{a}_i \sigma_{ij} \hat{b}_j = \sigma_{ij} \hat{c}_j, \quad 1 \le i, j \le d_p,$$

using the notation of (A.11), are linearly independent. If

$$\sum_{1 \le i,j \le d_p} x_{ij} \sigma_{ij} \hat{c}_j = 0, \quad \text{then} \quad \sum_{1 \le i,j \le d_p} x_{ij} \hat{c}_\ell \sigma_{ij} \hat{c}_j \hat{c}_k = 0,$$

implying

$$\sum_{\ell \le i,j \le k} x_{ij} \hat{c}_\ell \sigma_{ij} \hat{c}_j \hat{c}_k = 0. \tag{A.15}$$

Suppose some $x_{ij} \neq 0$; choose ℓ so that is the largest value of i for which this is true and then k so that it is the smallest value of j for which $x_{\ell j} \neq 0$. Then all the terms on the left hand side of (A.15) are zero except for one leaving

$$x_{\ell k} \hat{c}_\ell \sigma_{\ell k} \hat{c}_k \hat{c}_k = x_{\ell k} \hat{c}_\ell \sigma_{\ell k} = 0,$$

which implies $x_{\ell,k} = 0$, a contradiction. Thus, we conclude that $x_{ij} = 0$ for all i, j and so that the states (A.14) form a basis.

Now we seek to determine $x_{ij}, 1 \le i < j < d_p$, so that

$$P_p = \sum_{1\le i\le d_p} \hat{c}_i + \sum_{1\le i<j\le d_p} x_{ij}\sigma_{ij}\hat{c}_j \tag{A.16}$$

is the projection onto the spaces corresponding to the standard Young tableaux of shape p,

$$V_p = \bigoplus_{i=1}^{d_p} V_{\lambda_i^p}. \tag{A.17}$$

(See (33), p. 76.) A necessary and sufficient condition for this is $P_p\sigma_{ij}\hat{c}_j = \sigma_{ij}\hat{c}_j$ for $1 \le i, j \le d_p$. If this holds we will have $\sum_p P_p = 1$, because $P_{p'}\sigma_{ij}\hat{c}_j = 0$ if p' is another shape of Young tableaux:

$$\begin{aligned}
P_p\sigma_{k\ell}\hat{c}_\ell &= \sum_{1\le i\le d_p} \hat{c}_i\hat{c}_k\sigma_{k\ell} + \sum_{1\le i<j\le d_p} x_{ij}\sigma_{ij}\hat{c}_j\hat{c}_k\sigma_{k\ell} \\
&= \hat{c}_k^2\sigma_{k\ell} + \sum_{1\le i<k} \hat{c}_i\hat{c}_k\sigma_{k\ell} + \sum_{1\le i<k} x_{ik}\sigma_{ik}\hat{c}_k^2\sigma_{k\ell} + \sum_{1\le i<j<k} x_{ij}\sigma_{ij}\hat{c}_j\hat{c}_k\sigma_{k\ell} \\
&= \sigma_{k\ell}\hat{c}_\ell + \sum_{1\le i<k} \varepsilon_{ik}\sigma_{i\ell}\hat{c}_\ell^2 + \sum_{1\le i<k} x_{ik}\sigma_{i\ell}\hat{c}_\ell + \sum_{1\le i<j<k} x_{ij}\varepsilon_{jk}\sigma_{i\ell}\hat{c}_\ell^2 \\
&= \sigma_{k\ell}\hat{c}_\ell + \sum_{1\le i<k} \left(\varepsilon_{ik} + x_{ik} + \sum_{i<j<k} x_{ij}\varepsilon_{jk}\right)\sigma_{i\ell}\hat{c}_\ell.
\end{aligned} \tag{A.18}$$

Because the $\sigma_{i\ell}\hat{c}_\ell$ are linearly independent, the condition that $P_p\sigma_{k\ell}\hat{c}_\ell = \sigma_{k\ell}\hat{c}_\ell$ is

$$x_{ik} = -\varepsilon_{ik} + \sum_{i<j<k} x_{ij}\varepsilon_{jk}, \quad \text{for } 1 \le i < k. \tag{A.19}$$

We can solve this equation iteratively for increasing $k - i$, starting with $k - i = 1$:

$$\begin{aligned}
&x_{k-1,k} = -\varepsilon_{k-1,k}; \\
&x_{k-2,k} = -\varepsilon_{k-2,k} + \varepsilon_{k-2,k-1}\varepsilon_{k-1,k}; \\
&\dots\ \dots \\
&x_{ik} = -\varepsilon_{ik} + \sum_{i<j<k} \varepsilon_{ij}\varepsilon_{jk} - \sum_{i<j<\ell<k} \varepsilon_{ij}\varepsilon_{j\ell}\varepsilon_{\ell k} + \dots + (-1)^{k-i}\varepsilon_{i,i+1}\varepsilon_{i+1,i+2}\dots\varepsilon_{k-1,k},
\end{aligned} \tag{A.20}$$

for $i < k$. Then

$$P_p = \frac{d_p}{n!} \sum_{1\le j\le d_p} \xi_j c_j, \quad \text{where } \xi_j = 1 + \sum_{1\le i<j} x_{ij}\sigma_{ij} \tag{A.21}$$

and

$$1 = \frac{1}{n!}\sum_p d_p \sum_{1\le i\le d_p} \xi_{\lambda_i^p} c_{\lambda_i^p}. \tag{A.22}$$

In fact $\xi_j = 1$ for all j when $n \le 4$.

Every $\mathfrak{B}_\lambda \ne 1$ unless λ corresponds to the Young diagram p_1 with only one row; this corresponds to the identity representation and has $\mathfrak{A}_\lambda = \mathfrak{S}_n$. So

$$1 = \frac{1}{n!}\sum_{\rho\in\mathfrak{S}_n} \rho + \frac{1}{n!}\sum_{p\ne p_1} d_p \sum_{1\le i\le d_p} \xi_{\lambda_i^p} c_{\lambda_i^p}. \tag{A.23}$$

So, for given $\kappa_{n-1} \in \mathscr{K}_{n-1}$, if κ_n is the solution to (2.25) and (2.26) orthogonal to all symmetric invariant tensors,

$$\kappa_n = \frac{1}{n!} \sum_{p \neq p_1} d_p \sum_{1 \leq i \leq d_p} \xi_{\lambda_i^p} c_{\lambda_i^p} \kappa_n. \tag{A.24}$$

For each Young tableau λ not corresponding to the identity representation, choose a transposition $t_\lambda \in \mathfrak{B}_\lambda$. Then $c_\lambda t_\lambda = -c_\lambda$ so that

$$c_\lambda \kappa_n = \tfrac{1}{2} c_\lambda (1 - t_\lambda) \kappa_n = \tfrac{1}{2} c_\lambda \phi(t_\lambda, \kappa_{n-1}) \tag{A.25}$$

and

$$\kappa_n = \frac{1}{2n!} \sum_{p \neq p_1} d_p \sum_{1 \leq i \leq d_p} \xi_{\lambda_i^p} c_{\lambda_i^p} \phi(t_{\lambda_i^p}, \kappa_{n-1}). \tag{A.26}$$

We can express each of the transpositions t_λ as a product of the generating transpositions σ_i, defined as in (2.33); if t_λ is the transposition interchanging j and k, with $j < k$,

$$t_\lambda = \sigma_{k-1} \ldots \sigma_{j+1} \sigma_j \sigma_{j+1} \ldots \sigma_{k-1}.$$

We can then evaluate $\phi(t_\lambda, \kappa_{n-1})$ for $\lambda = \lambda_i^p$, using (2.43), to give an explicit expression for κ_n in terms of κ_{n-1}.

B Properties of H_n

In this Appendix we establish some properties of the generating function H_n, defined by (3.71). We noted that

$$\frac{1}{\nu^n} H_n(\nu_{12}, \ldots, \nu_{n1}, \tau; \nu) = \frac{1}{\sigma(\nu, \tau)^n} \prod_{j=1}^{n} \frac{\sigma(\mu_j + \nu, \tau)}{\sigma(\mu_j, \tau)} \tag{B.1}$$

is an elliptic function of ν and ν_j, $1 \leq j \leq n$. Viewed as function of ν, it has a pole of order n at the origin but it is otherwise regular,

$$\begin{aligned} \frac{1}{\nu^n} H_n = \; & \frac{1}{\nu^n} + \frac{1}{\nu^{n-1}} H_{n,1} + \cdots + \frac{1}{\nu^2} H_{n,n-2} + \frac{1}{\nu} H_{n,n-1} + H_{n,n} \\ & + \mathscr{O}(\nu) \quad \text{as } \nu \to 0. \end{aligned} \tag{B.2}$$

Writing the Weierstrass elliptic function

$$\mathscr{P}(\nu, \tau) = \nu^{-2} + \sum_{m=1}^{\infty} c_{2m}(\tau) \nu^{2m}, \tag{B.3}$$

we note that its derivatives

$$\mathscr{P}^{(n)}(\nu, \tau) = \frac{(-1)^n (n+1)!}{\nu^{n+2}} + n! \, c_n(\tau) + \mathscr{O}(\nu), \tag{B.4}$$

as $\nu \to 0$, where $c_\ell = 0$ if ℓ is odd. So

$$\frac{1}{\nu^n} H_n - \frac{(-1)^n}{(n-1)!} \mathscr{P}^{(n-2)}(\nu) + \frac{(-1)^n}{(n-2)!} \mathscr{P}^{(n-3)}(\nu) H_{n,1} + \cdots - \mathscr{P}(\nu) H_{n,n-2} \tag{B.5}$$

is an elliptic function of ν whose only potential singularity is a simple pole at the origin. This implies that it is constant as a function of ν (see (26), Prop. 4.11, p. 48) and the residue of the pole must vanish:

$$H_{n,n-1}(\nu_{12}, \ldots, \nu_{n1}, \tau) = 0, \quad \text{for } n \geq 2. \tag{B.6}$$

For $n=3$, this gives

$$\left[\sum_{r=1}^{3}\zeta(\mu_r)\right]^2=\sum_{r=1}^{3}\mathscr{P}(\mu_r),\quad \text{if } \sum_{r=1}^{3}\mu_r=0; \tag{B.7}$$

and, for $n=4$,

$$\left[\sum_{r=1}^{4}\zeta(\mu_r)\right]^3=3\left[\sum_{r=1}^{4}\zeta(\mu_r)\right]\left[\sum_{r=1}^{4}\mathscr{P}(\mu_r)\right]+\left[\sum_{r=1}^{4}\mathscr{P}'(\mu_r)\right],$$
$$\text{if } \sum_{r=1}^{4}\mu_r=0; \tag{B.8}$$

(see (27), p. 446 and 459, respectively).

We can equate (B.5) to its value at $\nu=0$, giving

$$\frac{1}{\nu^n}H_n=\frac{(-1)^n}{(n-1)!}\mathscr{P}^{(n-2)}(\nu)-\frac{(-1)^n}{(n-2)!}\mathscr{P}^{(n-3)}(\nu)H_{n,1}$$
$$+\cdots+\mathscr{P}(\nu)H_{n,n-2}+H_{n,n}-\sum_{\ell=1}^{[\frac{1}{2}n]-1}2\ell c_{2\ell}H_{n,n-2\ell-2}. \tag{B.9}$$

By direct calculation for $n=2$, we have that

$$\nu^{-2}H_2(\nu_{12},\nu_{21},\tau;\nu)=\frac{\sigma(\nu+\nu_{12})\sigma(\nu+\nu_{21})}{\sigma(\nu)^2\sigma(\nu_{12})\sigma(\nu_{21})}=\mathscr{P}(\nu)-\mathscr{P}(\nu_{12}), \tag{B.10}$$

so that

$$H_{2,0}(\nu_{12},\tau)=1;\quad H_{2,2}(\nu_{12},\tau)=-\mathscr{P}(\nu_{12},\tau); \tag{B.11}$$

$$H_{2,2m}(\nu_{12},\tau)=c_{2m-2}(\tau),\ m\geq 2. \tag{B.12}$$

In particular, from the expression given for $H_{2,4}$ given by (3.52), we can deduce the differential equation for $\mathscr{P}$,

$$\mathscr{P}''-6\mathscr{P}^2+\tfrac{1}{2}g_2(\tau)=0,$$

where

$$g_2(\tau)=\tfrac{2}{3}\pi^4\left(\theta_2(0,\tau)^8+\theta_3(0,\tau)^8+\theta_4(0,\tau)^8\right)=20c_2(\tau).$$

Consider the symmetrizations of $H_{n,m}$ and H_n,

$$H^S_{n,m}(\nu_{12},\ldots,\nu_{n1},\tau)=\frac{1}{n!}\sum_{\rho\in\mathfrak{S}_n}H_{n,m}(\nu_{\rho(1)\rho(2)},\ldots,\nu_{\rho(n)\rho(1)},\tau), \tag{B.13}$$

$$H^S_{n}(\nu_{12},\ldots,\nu_{n1},\tau;\nu)=\frac{1}{n!}\sum_{\rho\in\mathfrak{S}_n}H_{n}(\nu_{\rho(1)\rho(2)},\ldots,\nu_{\rho(n)\rho(1)},\tau;\nu). \tag{B.14}$$

Since $H_n(\nu_{12},\ldots,\nu_{n1},\tau;\nu)=H_n(-\nu_{1n},\ldots,-\nu_{21},\tau;\nu)$, H^S_n is an even function of ν and so

$$H^S_{n,m}(\nu_{12},\ldots,\nu_{n1},\tau)=0,\ m \text{ odd}. \tag{B.15}$$

For $n>2$,

$$\operatorname*{Res}_{\nu_{12}=0}H^S_{n,m}(\nu_{12},\ldots,\nu_{n1},\tau)=0;\quad \operatorname*{Res}_{\nu_{12}=0}H^S_{n}(\nu_{12},\ldots,\nu_{n1},\tau;\nu)=0. \tag{B.16}$$

Because $\nu^{-n}H^S_{n,m}(\nu_{12},\ldots,\nu_{n1},\tau)$ and $\nu^{-n}H^S_n(\nu_{12},\ldots,\nu_{n1},\tau;\nu)$ are elliptic as functions of ν_{ij}, but have no poles in these variables, it follows that they are independent of them, *i.e.*

$$H^S_{n,m}(\nu_{12},\ldots,\nu_{n1},\tau)=H^S_{n,m}(\tau),\quad \text{independent of } \nu_{ij}; \tag{B.17}$$

$$H^S_n(\nu_{12},\ldots,\nu_{n1},\tau;\nu)=H^S_n(\tau;\nu)=\sum_{m=0}^{\infty}H^S_{n,m}(\tau)\nu^m,\quad n>2. \tag{B.18}$$

For $n=2$, we have that

$$H^S_2(\nu_{12},\nu_{21},\tau;\nu)=H_2(\nu_{12},\nu_{21},\tau;\nu)=\nu^2[\mathscr{P}(\nu)-\mathscr{P}(\nu_{12})]. \tag{B.19}$$

So,

$$H^S_{2,0}(\nu_{12},\tau)=1;\quad H^S_{2,2}(\nu_{12},\tau)=-\mathscr{P}(\nu_{12},\tau); \tag{B.20}$$

$$H^S_{2,2m}(\nu_{12},\tau)=c_{2m-2}(\tau),\ m\geq 2. \tag{B.21}$$

By direct calculation for $n=3$, we have that

$$\begin{aligned}&H^S_3(\nu_{12},\nu_{23},\nu_{31},\tau;\nu)\\&\quad=\frac{\nu^3\sigma(\nu+\nu_{12})\sigma(\nu+\nu_{23})\sigma(\nu+\nu_{31})}{2\sigma(\nu)^3\sigma(\nu_{12})\sigma(\nu_{23})\sigma(\nu_{31})}+\frac{\nu^3\sigma(\nu+\nu_{21})\sigma(\nu+\nu_{13})\sigma(\nu+\nu_{32})}{2\sigma(\nu)^3\sigma(\nu_{21})\sigma(\nu_{13})\sigma(\nu_{32})}\\&\quad=-\tfrac{1}{2}\nu^3\mathscr{P}'(\nu)=\tfrac{1}{2}\nu^3\zeta''(\nu).\end{aligned} \tag{B.22}$$

We also have, by direct calculation, that

$$\begin{aligned}H^S_n(\tau;\nu)=\frac{(-1)^n}{(n-1)!}\nu^n\mathscr{P}^{(n-2)}(\nu)=\frac{(-1)^{n+1}}{(n-1)!}\nu^n\zeta^{(n-3)}(\nu)\\ \text{holds for } 3\leq n\leq 6.\end{aligned} \tag{B.23}$$

and conjecture that it holds for all $n\geq 3$. When (B.23) holds, we have that

$$H^S_{n,0}(\tau)=1;\quad H^S_{n,2m}(\tau)=0,\ 1\leq m<\tfrac{1}{2}n; \tag{B.24}$$

$$H^S_{n,2m}(\tau)=\frac{(-1)^n(2m-2)!}{(n-1)!(2m-n)!}c_{2m-2}(\tau),\ m\geq\tfrac{1}{2}n. \tag{B.25}$$

C Explicit Formulae for Two-, Three- and Four-Point Loops

In this Appendix we show how the formulae we have given previously (3) for two-, three- and four-point current algebra loops relate to the general result (5.3). For two- and three-point loops there is no distinction between the connected part and the whole loop amplitude.

C.1 Two-point current algebra loop

$$\begin{aligned}\mathrm{tr}\left(J^a(\rho_1)J^b(\rho_2)w^{L_0}\right) &= -\frac{\chi(\tau)}{4\pi^2\rho_1\rho_2}\delta^{ab}k\chi(\tau)\left[\left(\chi_{NS}^{12}\right)^2 - 4\pi^2 f(\tau)\right] \\ &= \frac{\chi(\tau)}{4\pi^2\rho_1\rho_2}\left(\kappa_2^{ab}H_{2,2} + \kappa_{2,0}^{ab}\right), \end{aligned} \tag{C.1}$$

where

$$H_{2,2} = -\mathscr{P}_{12} = -\left(\chi_{NS}^{12}\right)^2 - \frac{\pi^2}{3}\left[\theta_2^4(0,\tau) - \theta_4^4(0,\tau)\right], \tag{C.2}$$

$$\kappa_2^{ab} = k\delta^{ab}, \quad \text{for } \kappa^{ab} = k\delta^{ab}, \tag{C.3}$$

$$\begin{aligned}\kappa_{2,0}^{ab} &= \frac{4\pi^2}{\chi(\tau)}\,\mathrm{tr}(J_0^a J_0^b w^{L_0}) + k\delta^{ab}\frac{1}{3}\frac{\theta_1'''(0,\tau)}{\theta_1'(0,\tau)} \\ &= 4\pi^2 k\delta^{ab}\left[f(\tau) + \tfrac{1}{12}(\theta_2^4(0,\tau) - \theta_4^4(0,\tau)\right] \end{aligned} \tag{C.4}$$

and

$$f(\tau) = \frac{\chi^{(2)}(\tau)}{k\chi(\tau)} + \frac{1}{4\pi^2}\frac{\theta_3''(0,\tau)}{\theta_3(0,\tau)}, \quad \mathrm{tr}(J_0^a J_0^b w^{L_0}) = \delta^{ab}\chi^{(2)}(\tau), \tag{C.5}$$

$$\chi_{NS}^{ij} = \chi_{NS}(\nu_i - \nu_j, \tau), \quad \chi(\tau) = \mathrm{tr}\left(w^{L_0}\right). \tag{C.6}$$

C.2 Three-point current algebra loop

$$\begin{aligned}&\mathrm{tr}\left(J^a(\rho_1)J^b(\rho_2)J^c(\rho_3)w^{L_0}\right) \\ &= -\frac{i\chi(\tau)}{8\pi^3\rho_1\rho_2\rho_3}\left[\kappa_3^{abc}H_{3,3} + \kappa_{3,1}^{abc}H_{3,1} + \kappa_{3,0}^{abc}\right]_{\mathbb{S}} \\ &= \frac{ikf^{abc}\chi(\tau)}{8\pi^3\rho_1\rho_2\rho_3}\left[\chi_{NS}^{21}\chi_{NS}^{32}\chi_{NS}^{13} - 4\pi^2(\zeta^{21}+\zeta^{32}+\zeta^{13})f(\tau)\right] \\ &\quad + \frac{d^{abc}\chi^{(3)}(\tau)}{2\rho_1\rho_2\rho_3}, \end{aligned} \tag{C.7}$$

where $\zeta^{ij} = \zeta(\nu_j - \nu_i, \tau)$ and $\Phi_{\mathbb{S}}$ denotes the symmetrization

$$\begin{aligned}\Phi_{\mathbb{S}}^{a_{i_1}a_{i_2}\ldots a_{i_n}}(\nu_{i_1}, \nu_{i_2}, \ldots, \nu_{i_n}, \tau) &= \frac{1}{n}\sum_{\rho\in\mathfrak{S}_n}\mathscr{A}^{a_{i_{\rho(1)}}a_{i_{\rho(2)}}\ldots a_{i_{\rho(n)}}} \\ &\quad \times(\nu_{i_{\rho(1)}}, \nu_{i_{\rho(2)}}, \ldots, \nu_{i_{\rho(n)}}, \tau); \end{aligned} \tag{C.8}$$

and

$$
\begin{aligned}
H_{3,3} &= \frac{1}{6}\left[(\zeta^{21}+\zeta^{32}+\zeta^{13})^2 - 3(\zeta^{21}+\zeta^{32}+\zeta^{13})(\mathscr{P}^{21}+\mathscr{P}^{32}+\mathscr{P}^{13})\right.\\
&\quad \left.-(\mathscr{P}'^{21}+\mathscr{P}'^{32}+\mathscr{P}'^{13})\right],\\
&= \frac{1}{6}\left[-2(\zeta^{21}+\zeta^{32}+\zeta^{13})(\mathscr{P}^{21}+\mathscr{P}^{32}+\mathscr{P}^{13})\right.\\
&\quad \left.-(\mathscr{P}'^{21}+\mathscr{P}'^{32}+\mathscr{P}'^{13})\right], \qquad \text{since} H_{3,2}=0\\
&= -\chi^{21}_{NS}\chi^{32}_{NS}\chi^{13}_{NS} - \frac{\pi^2}{3}(\theta_2^4(0,\tau)-\theta_4^4(0,\tau))\,(\zeta^{21}+\zeta^{32}+\zeta^{13}),
\end{aligned}
\tag{C.9}
$$

$$
H_{3,1} = \zeta^{21}+\zeta^{32}+\zeta^{13}, \tag{C.10}
$$

$$
\kappa_3^{abc} = \tfrac{1}{2}kf^{abc}, \tag{C.11}
$$

$$
\kappa_{3,1}^{abc} = 2\pi^2 k f^{abc}\left[f(\tau)+\tfrac{1}{12}\left(\theta_2^4(0,\tau)-\theta_4^4(0,\tau)\right)\right], \tag{C.12}
$$

$$
\kappa_{3,0}^{abc} = 2\pi^3 i d^{abc}\frac{\chi^{(3)}(\tau)}{\chi(\tau)} = \frac{4\pi^3 i}{\chi(\tau)}\mathrm{tr}\left(J_0^a J_0^b J_0^c w^{L_0}\right)_{\mathbb{S}}, \tag{C.13}
$$

$$
\mathrm{tr}(J_0^a J_0^b J_0^c w^{L_0}) = \tfrac{1}{2}f^{abc}\chi^{(2)}(\tau)+\tfrac{1}{2}d^{abc}\chi^{(3)}(\tau), \tag{C.14}
$$

and we also use $\kappa^{abc}_{3,m}-\kappa^{bac}_{3,m} = f^{ab}{}_e\kappa^{ec}_{2,m-1}$, $\kappa^{abc}_{3,m}=\kappa^{cab}_{3,m}$, where $\kappa_{n,n}=\kappa_n$.

C.3 Four-point current algebra loop

In (3), we gave the general form of the four-point loop in the symmetric form

$$
\begin{aligned}
&\mathrm{tr}(J^a(\rho_1)J^b(\rho_2)J^c(\rho_3)J^d(\rho_4)w^{L_0})\rho_1\rho_2\rho_3\rho_4\\
&= \delta^{ab}\delta^{cd}\left(k^2\chi(\tau)\left[(\chi^{12}_{NS})^2/4\pi^2 - f(\tau)\right]\left[(\chi^{34}_{NS})^2/4\pi^2-f(\tau)\right]-\chi^{(2)}(\tau)^2/\chi(\tau)\right)\\
&\quad+\delta^{ac}\delta^{bd}\left(k^2\chi(\tau)\left[(\chi^{13}_{NS})^2/4\pi^2 - f(\tau)\right]\left[(\chi^{24}_{NS})^2/4\pi^2-f(\tau)\right]-\chi^{(2)}(\tau)^2/\chi(\tau)\right)\\
&\quad+\delta^{ad}\delta^{bc}\left(k^2\chi(\tau)\left[(\chi^{14}_{NS})^2/4\pi^2 - f(\tau)\right]\left[(\chi^{23}_{NS})^2/4\pi^2-f(\tau)\right]-\chi^{(2)}(\tau)^2/\chi(\tau)\right)\\
&\quad+\mathrm{tr}(J_0^aJ_0^bJ_0^cJ_0^dw^{L_0})_S\\
&\quad-\frac{1}{96}\left(\sigma^{abcd}+\sigma^{adcb}+\sigma^{acdb}+\sigma^{abdc}+\sigma^{adbc}+\sigma^{acbd}\right)\chi(\tau)\theta_2^4(0,\tau)\theta_4^4(0,\tau)\\
&\quad-\left(\sigma^{abcd}+\sigma^{adcb}\right)\frac{1}{32\pi^4}\chi(\tau)\left\{\chi^{12}_{NS}\chi^{23}_{NS}\chi^{34}_{NS}\chi^{41}_{NS}\right.\\
&\quad\left.-\pi^2 f(\tau)\left(\frac{\mathscr{P}'_{24}-\mathscr{P}'_{32}}{\mathscr{P}_{24}-\mathscr{P}_{32}}\right)\left(\frac{\mathscr{P}'_{24}-\mathscr{P}'_{41}}{\mathscr{P}_{24}-\mathscr{P}_{41}}\right)+4\pi^2 f(\tau)\mathscr{P}_{24}\right\}\\
&\quad-\left(\sigma^{acdb}+\sigma^{abdc}\right)\frac{1}{32\pi^4}\chi(\tau)\left\{\chi^{13}_{NS}\chi^{34}_{NS}\chi^{42}_{NS}\chi^{21}_{NS}\right.\\
&\quad\left.-\pi^2 f(\tau)\left(\frac{\mathscr{P}'_{24}-\mathscr{P}_{32}}{\mathscr{P}_{24}-\mathscr{P}_{32}}\right)\left(\frac{\mathscr{P}'_{21}-\mathscr{P}'_{32}}{\mathscr{P}_{21}-\mathscr{P}_{32}}\right)+4\pi^2 f(\tau)\mathscr{P}_{32}\right\}\\
&\quad-\left(\sigma^{adbc}+\sigma^{acbd}\right)\frac{1}{32\pi^4}\chi(\tau)\left\{\chi^{14}_{NS}\chi^{42}_{NS}\chi^{23}_{NS}\chi^{31}_{NS}\right.\\
&\quad\left.-\pi^2 f(\tau)\left(\frac{\mathscr{P}'_{24}-\mathscr{P}'_{32}}{\mathscr{P}_{24}-\mathscr{P}_{32}}\right)\left(\frac{\mathscr{P}'_{14}-\mathscr{P}'_{31}}{\mathscr{P}_{14}-\mathscr{P}_{31}}\right)+4\pi^2 f(\tau)\mathscr{P}_{34}\right\}\\
&\quad-\frac{i}{4\pi}\chi^{(3)}(\tau)\left[\sigma^{abcd}\left(\zeta^{21}+\zeta^{32}+\zeta^{43}+\zeta^{14}\right)+\sigma^{adcb}\left(\zeta^{41}+\zeta^{34}+\zeta^{23}+\zeta^{12}\right)\right.\\
&\quad\left.+\sigma^{acdb}\left(\zeta^{31}+\zeta^{43}+\zeta^{24}+\zeta^{12}\right)+\sigma^{abdc}\left(\zeta^{21}+\zeta^{42}+\zeta^{34}+\zeta^{13}\right)\right.
\end{aligned}
$$

$$+\sigma^{adbc}\left(\zeta^{41}+\zeta^{24}+\zeta^{32}+\zeta^{13}\right)+\sigma^{acbd}\left(\zeta^{31}+\zeta^{23}+\zeta^{42}+\zeta^{14}\right)\Big], \tag{C.15}$$

where $\mathscr{P}_{ij}=\mathscr{P}(\nu_j-\nu_i,\tau)$, $\mathscr{P}'_{ij}=\mathscr{P}'(\nu_j-\nu_i,\tau)$, and $\sigma^{abcd}=\mathrm{tr}(t^at^bt^ct^d)$, where t^a provides a representation of $\mathfrak{g}$ with $\sigma^{ab}=\mathrm{tr}(t^at^b)=2\kappa^{ab}=2k\delta^{ab}$ and $\sigma^{abc}=\mathrm{tr}(t^at^bt^c)=f^{ab}{}_e\kappa^{ec}+d^{abc}$; then

$$\begin{aligned}\sigma^{abcd} &= \frac{k}{3}\left(f^{ab}{}_ef^{cde}+f^{da}{}_ef^{bce}\right)\\ &\quad+\frac{1}{4}\left(f^{ab}{}_ed^{ecd}+f^{ac}{}_ed^{edb}+f^{bc}{}_ed^{eab}\right)+\bar{\omega}^{abcd},\end{aligned} \tag{C.16}$$

where $\bar{\omega}^{abcd}$ is a totally symmetric tensor, independent of τ. Now, using

$$\begin{aligned}&\frac{1}{2}\left(\frac{\mathscr{P}'_{24}-\mathscr{P}_{32}}{\mathscr{P}_{24}-\mathscr{P}_{32}}\right)\left(\frac{\mathscr{P}'_{24}-\mathscr{P}'_{41}}{\mathscr{P}_{24}-\mathscr{P}_{41}}\right)-2\mathscr{P}_{24}\\ &\quad=\left(\zeta^{13}+\zeta^{32}+\zeta^{21}\right)\left(\zeta^{13}+\zeta^{34}+\zeta^{41}\right)+\left(\zeta^{24}+\zeta^{41}+\zeta^{12}\right)\\ &\qquad\times\left(\zeta^{24}+\zeta^{43}+\zeta^{32}\right)-\mathscr{P}_{13}-\mathscr{P}_{24}\\ &\quad=\mathscr{P}_{12}+\mathscr{P}_{23}+\mathscr{P}_{34}+\mathscr{P}_{41}-(\zeta^{12}+\zeta^{23}+\zeta^{34}+\zeta^{41})^2,\end{aligned} \tag{C.17}$$

we have that

$$\begin{aligned}&\mathrm{tr}(J^a(\rho_1)J^b(\rho_2)J^c(\rho_3)J^d(\rho_4)w^{L_0})_C\\ &\quad=\mathrm{tr}(J^a(\rho_1)J^b(\rho_2)J^c(\rho_3)J^d(\rho_4)w^{L_0})-\mathrm{tr}(J^a(\rho_1)J^b(\rho_2)w^{L_0})\mathrm{tr}(J^c(\rho_3)J^d(\rho_4)w^{L_0})\\ &\qquad-\mathrm{tr}(J^a(\rho_1)J^c(\rho_3)w^{L_0})\,\mathrm{tr}(J^b(\rho_2)J^d(\rho_4)w^{L_0})\\ &\qquad-\mathrm{tr}(J^a(\rho_1)J^d(\rho_4)w^{L_0})\,\mathrm{tr}(J^c(\rho_3)J^b(\rho_2)w^{L_0})\\ &\quad=-\frac{\chi(\tau)}{16\pi^4\rho_1\rho_2\rho_3\rho_4}\left[\kappa_4^{abcd}H_{4,4}+\kappa_{4,2}^{abcd}H_{4,2}+\kappa_{4,1}^{abcd}H_{4,1}+\kappa_{4,0}^{abcd}\right]_{\mathbb{S}}\end{aligned} \tag{C.18}$$

with $\mathbb{S}$ denoting symmetrization as defined in (C.8), and

$$
\begin{aligned}
24H_{4,4} &= (\zeta^{21}+\zeta^{32}+\zeta^{43}+\zeta^{14})^4 - 6(\zeta^{21}+\zeta^{32}+\zeta^{43}+\zeta^{14})^2(\mathscr{P}_{21}+\mathscr{P}_{32}+\mathscr{P}_{43}+\mathscr{P}_{14}) \\
&\quad -4(\zeta^{21}+\zeta^{32}+\zeta^{43}+\zeta^{14})(\mathscr{P}'_{21}+\mathscr{P}'_{32}+\mathscr{P}'_{43}+\mathscr{P}'_{14}) \\
&\quad +3(\mathscr{P}_{21}+\mathscr{P}_{32}+\mathscr{P}_{43}+\mathscr{P}_{14})^2 \\
&\quad -(\mathscr{P}''_{21}+\mathscr{P}''_{32}+\mathscr{P}''_{43}+\mathscr{P}''_{14}) \\
&= -3(\zeta^{21}+\zeta^{32}+\zeta^{43}+\zeta^{14})^2(\mathscr{P}_{21}+\mathscr{P}_{32}+\mathscr{P}_{43}+\mathscr{P}_{14}) \\
&\quad -3(\zeta^{21}+\zeta^{32}+\zeta^{43}+\zeta^{14})(\mathscr{P}'_{21}+\mathscr{P}'_{32}+\mathscr{P}'_{43}+\mathscr{P}'_{14}) \\
&\quad +3(\mathscr{P}_{21}+\mathscr{P}_{32}+\mathscr{P}_{43}+\mathscr{P}_{14})^2-(\mathscr{P}''_{21}+\mathscr{P}''_{32}+\mathscr{P}''_{43}+\mathscr{P}''_{14}), \\
&\quad \text{since} H_{4,3}=0, \\
&= -\frac{\pi^2}{3}(\theta_2^4(0,\tau)-\theta_4^4(0,\tau))H_{4,2}+\chi_{NS}^{21}\chi_{NS}^{32}\chi_{NS}^{43}\chi_{NS}^{14}+\frac{\pi^4}{3}\theta_2^4(0,\tau)\theta_4^4(0,\tau); \\
2H_{4,2} &= (\zeta^{21}+\zeta^{32}+\zeta^{43}+\zeta^{14})^2-(\mathscr{P}_{21}+\mathscr{P}_{32}+\mathscr{P}_{43}+\mathscr{P}_{14}) \\
H_{4,1} &= \zeta^{21}+\zeta^{32}+\zeta^{43}+\zeta^{14} \\
\kappa_4^{abcd} &= \tfrac{1}{2}\sigma^{abcd} = \tfrac{1}{6}k\left(f^{ab}{}_e f^{cde}+f^{da}{}_e f^{bce}\right)+\mathring{\kappa}_{4,4}^{abcd} \\
\kappa_{4,2}^{abcd} &= 2\pi^2\left[f(\tau)+\frac{1}{12}(\theta_2^4(0,\tau)-\theta_4^4(0,\tau))\right]\sigma^{abcd} \\
&= \tfrac{2}{3}\pi^2 k\left(f^{ab}{}_e f^{cde}+f^{da}{}_e f^{bce}\right)\left[f(\tau)+\tfrac{1}{12}(\theta_2^4(0,\tau)-\theta_4^4(0,\tau))\right]+\mathring{\kappa}_{4,2}^{abcd} \\
\kappa_{4,1}^{abcd} &= i\pi^3\frac{\chi^{(3)}(\tau)}{\chi(\tau)}\left(f^{ab}{}_e d^{ecd}+f^{ac}{}_e d^{edb}+f^{bc}{}_e d^{eab}\right)+\mathring{\kappa}_{4,1}^{abcd} \\
\kappa_{4,0}^{abcd} &= -\frac{8\pi^4}{3}\left[\frac{1}{\chi(\tau)}\operatorname{tr}(J_0^aJ_0^bJ_0^cJ_0^dw^{L_0})-\frac{1}{\chi(\tau)^2}\operatorname{tr}(J_0^aJ_0^bw^{L_0})\operatorname{tr}(J_0^cJ_0^dw^{L_0})\right]_S \\
&= -\frac{8\pi^4}{3}\left[\frac{1}{\chi(\tau)}\operatorname{tr}(J_0^aJ_0^bJ_0^cJ_0^dw^{L_0})_S-\frac{\chi^{(2)}(\tau)^2}{\chi(\tau)^2}(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc})\right] \\
&= -\frac{8\pi^4}{3\chi(\tau)}\operatorname{tr}(J_0^aJ_0^bJ_0^cJ_0^dw^{L_0})_{CS},
\end{aligned}
\tag{C.19}
$$

where $\left[\mathring{\kappa}_{4,m}^{abcd}H_{4,m}\right]_{\mathbb{S}}=0$, and we have also used $H_{4,2}^S=H_{4,1}^S=0$ (which follows from (B.15) and (B.24)), and

$$
\begin{aligned}
&\chi_{NS}^{12}\chi_{NS}^{23}\chi_{NS}^{34}\chi_{NS}^{41}+\chi_{NS}^{13}\chi_{NS}^{34}\chi_{NS}^{42}\chi_{NS}^{21}+\chi_{NS}^{14}\chi_{NS}^{42}\chi_{NS}^{23}\chi_{NS}^{31} \\
&\quad =-\pi^4\theta_2(0,\tau)^4\theta_4(0,\tau)^4.
\end{aligned}
\tag{C.20}
$$

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