

Cosmic time evolution and propagator from a Yang–Mills matrix model

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Abstract

We consider a solution of a IKKT-type matrix model which can be considered as a 1+1-dimensional space-time with Minkowski signature and a Big Bounce (BB)-like singularity. A suitable $i\epsilon$ regularization of the Lorentzian matrix integral is proposed, which leads to the standard $i\epsilon$ -prescription for the effective field theory. In particular, the Feynman propagator is recovered locally for late times. This demonstrates that a causal structure and time evolution can emerge in the matrix model, even on non-trivial geometries. We also consider the propagation of modes through the BB, and observe an interesting correlation between the post-BB and pre-BB sheets, which reflects the structure of the brane in target space.

Keywords: matrix models, braneworlds, noncommutative field theory, quantum space-time

(Some figures may appear in colour only in the online journal)

1. Introduction

Matrix theory can be viewed as an alternative approach to string theory. There are two prominent matrix models which can be taken as starting point: the BFSS model [1] is a model of matrix quantum mechanics with a classical time variable, while the IKKT model [2] is a pure

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matrix model without any *a priori* notion of time. Both models admit solutions which can be interpreted in terms of noncommutative D branes with a B field, and fluctuations around such backgrounds lead to noncommutative gauge theory.

The absence of a classical time variable in the IKKT model leads to an intriguing question: how can time, and an effectively unitary time evolution, emerge from such a pure matrix model? Indeed a naive interpretation of time in the noncommutative field theory leads to some issues, which have been raised e.g. in [3]. However, to properly address this issue it is crucial to first identify the effective metric, which is dynamical in matrix models and depends on the background under consideration. This can be clarified by studying the propagation of modes on such backgrounds [4], which allows to identify a unique effective metric closely related to the open string metric on the D-brane. Only then a notion of time and time evolution can be identified. Moreover, a proper treatment of the quantum theory can only be attempted in the maximally supersymmetric IKKT model. From this perspective, the objections raised in [3] no longer apply.

In the present paper, we wish to elaborate some of these issues in more detail, and demonstrate that a low-energy field theory can indeed emerge from IKKT-type matrix models which displays the appropriate structures of causality and time evolution required in quantum field theory. We will restrict ourselves to a free noncommutative scalar field theory defined by a simplified model, i.e. ignoring loop corrections; the latter should be addressed only in the full-fledged IKKT model. More specifically, we will study a particular 1+1-dimensional solution of a reduced model, which can be viewed as a toy model for the 3+1-dimensional covariant space-time solution given in [5]. The present solution is obtained as a projection of 2-dimensional fuzzy hyperboloid, with structure reminiscent of a 1+1-dimensional Friedmann–Lemaître–Robertson–Walker (FLRW) cosmology with a Big Bounce (BB). It comprises a pre-BB and a post-BB sector, which are glued together at the BB through a well-defined matrix configuration³.

The main claim of the present paper is that once a suitable definition of the matrix path integral in Minkowski signature is implemented, the two-point correlation functions have indeed the correct structure of a Feynman propagator in quantum field theory. The Feynman $i\epsilon$ structure is obtained from a suitable regularization of the oscillatory matrix integral, which thus becomes absolutely convergent and well-defined, at least for finite-dimensional matrices. This prescription is slightly different from a similar regularization used in recent computer simulations of the Lorentzian IKKT model [7, 8], but is expected to be equivalent.

More explicitly, we obtain the full set of (on- and off-shell) fluctuation modes on the FLRW-type background under consideration. These modes stretch across the BB, and allow an explicit computation of the Bogoliubov coefficients which relate the asymptotic pre- and post-BB regime. Given these modes, we compute the propagator by performing the matrix ‘path’ integral, which displays the standard structure of a Feynman propagator at times far from the BB. This implies that the resulting effective field theory behaves as it should—at least at low energies—including the appropriate causality structure and time evolution. In particular, the continuation of the modes across the BB suggests a continuous time evolution across the mild singularity at the BB, with opposite ‘arrow of time’ on the two sheets. We also observe indications of some rather unexpected and intriguing correlations between the pre-BB and post-BB sheets.

³ In is interesting to note that a 1+1-dimensional space-time structure linked by an Euclidean phase was recently reported in numerical studies of the bosonic IKKT model with mass term [6].

The paper is organized as follows. In section 2, we define the matrix model and the $i\varepsilon$ prescription. In section 3 we review the definition of a fuzzy 2-hyperboloid, explicitly construct harmonics on the classical 2-hyperboloid and then use those to construct a harmonic basis for functions on the fuzzy hyperboloid. In section 4 we obtain our solution of interest, a fuzzy two dimensional space with a Minkowski signature, $\mathcal{M}^{1,1}$. In section 5 we describe dynamics of a single transverse fluctuation, solve the classical wave equation on $\mathcal{M}^{1,1}$ and study the Bogoliubov coefficients. Finally, in section 6 we put it all together, using the harmonic basis on the fuzzy hyperboloid to compute a matrix model two point function in the background of an emergent cosmological spacetime $\mathcal{M}^{1,1}$. Some further discussion is offered in section 7.

2. Definition of the model and quantization

We will consider the following three-dimensional IKKT-type matrix model

$$S[Y] = \frac{1}{g^2} \text{Tr} \left(- [Y^a, Y^b] [Y^{a'}, Y^{b'}] \eta_{aa'} \eta_{bb'} - 2m^2 Y^a Y^b \eta_{ab} \right). \quad (1)$$

Here $\eta_{ab} = \text{diag}(-1, 1, 1)$, and the $Y^a \in \text{End}(\mathcal{H})$ are Hermitian matrices acting on some (finite- or infinite-dimensional) Hilbert space \mathcal{H} . Throughout this paper, indices will be raised and lowered with η_{ab} . The action (1) is a toy model for the IKKT model [2], supplemented by a mass term m^2 which introduces a scale into the model and without fermions for simplicity. This model has the gauge invariance

$$Y^a \rightarrow U^{-1} Y^a U, \quad U \in U(\mathcal{H}), \quad (2)$$

which, as in Yang–Mills gauge theory, is essential to remove ghost contributions from the time-like direction, as well as a global $SO(2, 1)$ symmetry. The classical equations of motion are

$$\square_Y Y^a = m^2 Y^a, \quad (3)$$

where the matrix d’Alembertian is defined as

$$\square_Y = \eta_{ab} [Y^a, [Y^b, \cdot]]. \quad (4)$$

Equation (3) governs the propagation of scalar modes $\phi \in \text{End}(\mathcal{H})$ on the background defined by Y^a . Such scalar modes arise in the matrix model from transverse fluctuations of the background solution, while the tangential fluctuations give rise to gauge fields. However, such gauge fields are not dynamical in two dimensions, and we will focus on the scalar modes in the present paper.

Quantization of the model is defined via a matrix path integral,

$$Z = \int dY e^{iS[Y]}. \quad (5)$$

As is the case with the oscillatory path integral in Lorentzian QFT, this is not well defined as it stands. It was shown in [9] that, for pure bosonic Euclidean Yang–Mills matrix model, the matrix integral makes sense in $d \geq 3$ dimensions. In the case of Minkowski signature, one possibility to define the path integral is to put an IR cutoff in both space-like and time-like directions as was done in [10]. Here we propose a similar but more elegant regularization, giving the mass term $\text{Tr}(m^2 Y^a Y^b \eta_{ab})$ a suitable imaginary part as follows:

$$\text{Tr}(m^2 Y^a Y^b \eta_{ab}) \rightarrow \text{Tr}(-(m^2 + i\varepsilon) Y^0 Y^0 + (m^2 - i\varepsilon) Y^i Y^i). \quad (6)$$

We thus define

$$S_\varepsilon[Y] = \frac{1}{g^2} \text{Tr} \left([Y^0, Y^j]^2 - [Y^i, Y^j]^2 + 2(m^2 + i\varepsilon)(Y^0)^2 - 2(m^2 - i\varepsilon)(Y^j)^2 \right). \quad (7)$$

which reduces to (1) in the limit $\varepsilon \searrow 0$. Then, the integral

$$Z_\varepsilon = \int dY e^{iS_\varepsilon[Y]} \quad (8)$$

is absolutely convergent for any $\varepsilon > 0$. To prove this, it suffices to observe that

$$\int dY \left| e^{iS_\varepsilon[Y]} \right| = \int dY \left| e^{\frac{2i}{g^2} \text{Tr} \left((m^2 + i\varepsilon)(Y^0)^2 - (m^2 - i\varepsilon)(Y^j)^2 \right)} \right| < \infty \quad (9)$$

since the rhs is a Gaussian integral with good decay properties. Note that the integration is always over the space of Hermitian matrices $(Y^a)^\dagger = Y^a$, even for the time-like matrices. In view of (7), this regularization amounts to Feynman's $i\varepsilon$ -prescription in quantum field theory, and therefore automatically imposes the appropriate causality structure in the propagators. This will be verified explicitly in section 6, by computing the propagator in terms of the matrix path integral for a free scalar field.

3. The fuzzy 2-hyperboloid H_n^2

In analogy to the well-known case of the fuzzy sphere S_N^2 , the fuzzy 2-hyperboloid H_n^2 [11–13] is defined in terms of generators of $SO(1, 2)$ acting on a unitary irreducible representation. Let \mathcal{M}_{ab} be the generators of $\mathfrak{so}(1, 2)$, which satisfy

$$[\mathcal{M}_{ab}, \mathcal{M}_{cd}] = i(\eta_{ac}\mathcal{M}_{bd} - \eta_{ad}\mathcal{M}_{bc} - \eta_{bc}\mathcal{M}_{ad} + \eta_{bd}\mathcal{M}_{ac}) . \quad (10)$$

Fuzzy H_n^2 is then defined in terms of vector operators $K_a := \frac{1}{2}\epsilon_{abc}\mathcal{M}^{bc}$, which satisfy

$$[K_a, K_b] = i\epsilon_{abc}K^c$$

using the convention $\epsilon_{012} = 1$. Explicitly, $\mathcal{M}_{12} = K_0$, $\mathcal{M}_{20} = -K_1$ and $\mathcal{M}_{01} = -K_2$ satisfy

$$[K_1, K_2] = -iK_0, \quad [K_2, K_0] = iK_1, \quad [K_0, K_1] = iK_2 . \quad (11)$$

Here K_0 generates the compact $SO(2) \subset SO(1, 2)$ subgroup, while K_1 and K_2 generate non-compact $SO(1, 1) \subset SO(1, 2)$ subgroups. As usual, it is convenient to introduce the ladder operators

$$K_\pm = K_1 \pm iK_2 , \quad (12)$$

which satisfy

$$[K_0, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -2K_0. \quad (13)$$

The Casimir operator of $\mathfrak{so}(1, 2)$ is defined as

$$C^{(2)} = -\eta^{ab}K_aK_b = -K_1^2 - K_2^2 + K_0^2. \quad (14)$$

3.1. Fuzzy $H_n^2 \subset \mathbb{R}^{1,2}$ as brane in target space

For any unitary irrep \mathcal{H} of $SO(1, 2)$, define the Hermitian generators

$$X^a := rK^a, \quad a = 0, 1, 2 \quad (15)$$

where r is a parameter of dimension length. They satisfy

$$\begin{aligned} \eta_{ab}X^aX^b &= -X^0X^0 + X^1X^1 + X^2X^2 = -r^2C^{(2)}\mathbb{1}, \\ [X^a, X^b] &= ir\epsilon^{abc}X_c. \end{aligned} \tag{16}$$

Moreover, it follows easily from these Lie algebra relations that

$$\square_X X^a = [X_b, [X^b, X^a]] = -2r^2X^a, \quad a = 0, 1, 2. \tag{17}$$

Therefore these X^a provide a solution of the matrix model (1) for⁴

$$m^2 = -2r^2. \tag{18}$$

Finally we have to choose an appropriate representation. To obtain a one-sided hyperboloid, we should choose a discrete series positive-energy unitary irrep $\mathcal{H}_n := D_n^+$ of $SO(2, 1)$, as reviewed in appendix section ‘Unitary representations of $\mathfrak{so}(2, 1)$ ’. Then

$$-R^2 := \eta_{ab}X^aX^b = X^iX^i - X^0X^0 = -r^2n(n-1) < 0 \tag{19}$$

and $X^0 = rK^0 > 0$ has positive spectrum, given by

$$\text{spec}(X^0) = r\{n, n+1, \dots\}. \tag{20}$$

This structure will be denoted as H_n^2 .

3.1.1. Semi-classical limit. The semi-classical limit of H_n^2 is obtained by replacing the generator X^a with functions x^a satisfying the constraint

$$\eta_{ab}x^ax^b = -(x^0)^2 + (x^1)^2 + (x^2)^2 = -R^2 < 0, \tag{21}$$

and a $SO(2, 1)$ -invariant Poisson structure⁵

$$\{x^a, x^b\} = r\epsilon^{abc}x_c \tag{22}$$

corresponding to (16). Accordingly, we can interpret the X^a as quantized embedding functions of a one-sided Euclidean hyperboloid into $\mathfrak{so}(2, 1) \cong \mathbb{R}^{1,2}$,

$$X^a \sim x^a : H^2 \hookrightarrow \mathbb{R}^{1,2} \tag{23}$$

This is the quantization of the coadjoint orbit H^2 of $SO(2, 1)$, with the $SO(1, 2)$ -invariant Poisson bracket (or symplectic structure) (22). The operator algebra $\text{End}(\mathcal{H}_n)$ can thus be interpreted as quantized algebra of functions on $\mathcal{C}^\infty(H^2)$. Clearly H_n^2 has a finite density of microstates, according to the Bohr–Sommerfeld rule.

3.2. Functions and harmonics on classical H^2

The action of $SO(2, 1)$ on functions $\phi \in \mathcal{C}^\infty(H^2)$ is realized via the Hamiltonian vector fields

$$K^a \triangleright \phi = \frac{i}{r}\{x^a, \phi\} = -\frac{1}{2}\epsilon^{abc}\mathcal{M}_{bc} \triangleright \phi. \tag{24}$$

⁴ The negative mass does not imply an instability in the Minkowski case, it will merely lead to a cosmological solution with $k = -1$. The transverse fluctuations will be stabilized by a positive mass in section 5.

⁵ Note that $r \sim R/n$ can be viewed as deformation parameter, which for large n separates the semi-classical regime from the noncommutative regime.

In particular, the space of square-integrable functions $\phi(x)$ on H^2 forms a unitary representation, which decomposes into unitary irreps of $SO(2, 1)$. It follows that the Casimir

$$C^{(2)}\phi = -K^a K_a \phi = (K_0^2 - K_1^2 - K_2^2)\phi =: -\frac{1}{r^2}\square_H\phi$$

$$\square_H\phi := -\{x_a, \{x^a, \phi\}\} \tag{25}$$

coincides with the metric Laplacian Δ_H on H^2 up to a factor,

$$\Delta_H\phi = \frac{1}{R^2}C^{(2)}\phi = -\frac{1}{R^2r^2}\square_H\phi = \frac{1}{\sqrt{|g_{\mu\nu}|}}\partial_\mu\left(\sqrt{|g_{\mu\nu}|}g^{\mu\nu}\partial_\nu\phi\right), \tag{26}$$

where g is the induced metric on H^2 . This gives $C^{(2)}P_l(x) = l(l+1)R^2P_l(x)$ for irreducible polynomials of degree l in x^a ; for example,

$$C^{(2)}x^b = \frac{1}{r^2}\{x_a, \{x^a, x^b\}\} = 2x^b. \tag{27}$$

For square-integrable functions, the Casimir must be negative definite, which is indeed the case for functions in the principal series irreps.

3.2.1. Hyperbolic coordinates and eigenfunctions. To find the general eigenfunctions of Δ_H , consider the following coordinates⁶ on H^2 :

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = R \begin{pmatrix} \cosh(\eta) \begin{pmatrix} \cosh(\chi) \\ \sinh(\chi) \end{pmatrix} \\ \sinh(\eta) \end{pmatrix}, \tag{28}$$

for $\eta, \chi \in \mathbb{R}$. Then, the induced metric on H^2 is

$$ds^2|_H = g_{\mu\nu}dx^\mu dx^\nu = R^2(d\eta^2 + \cosh^2(\eta)d\chi^2) \tag{29}$$

with $\sqrt{g} = R^2 \cosh(\eta)$. Hence the metric Laplacian on H^2 is given by

$$\Delta_H\phi = \frac{1}{\sqrt{|g_{\mu\nu}|}}\partial_\mu\left(\sqrt{|g_{\mu\nu}|}g^{\mu\nu}\partial_\nu\phi\right) = \frac{1}{R^2}\left(\partial_\eta^2\phi + \tanh(\eta)\partial_\eta\phi + \frac{1}{\cosh^2(\eta)}\partial_\chi^2\phi\right). \tag{30}$$

Now consider eigenfunctions of Δ_H :

$$\Delta_H\phi = \lambda\phi. \tag{31}$$

The separation ansatz

$$\phi(\eta, \chi) = f(\eta)e^{ik\chi} \tag{32}$$

leads to

$$\left(\partial_\eta^2 + \tanh(\eta)\partial_\eta - \frac{k^2}{\cosh^2(\eta)} - R^2\lambda\right)f = 0. \tag{33}$$

To bring this to standard form, we can substitute $u = \tanh\eta \in (-1, 1)$ and define $f(u) = (1 - u^2)^{1/4}h(u)$, to obtain

$$(1 - u^2)h'' - 2uh' + \left(-\left(k^2 + \frac{1}{4}\right) - \frac{\frac{1}{4} + \lambda R^2}{1 - u^2}\right)h = 0. \tag{34}$$

⁶ These coordinates are compatible with the projection to $\mathcal{M}^{1,1}$ considered below.

The solutions are associated Legendre functions of the first and second kind, P_ν^μ and Q_ν^μ , with

$$\nu(\nu + 1) = -k^2 - \frac{1}{4} \quad \text{and} \quad \mu^2 = \frac{1}{4} + \lambda R^2. \tag{35}$$

We use the definitions and conventions given in [14], and all properties of these functions we require can be found therein. The first relation amounts to⁷

$$\nu(k) = -\frac{1}{2} + i|k|. \tag{36}$$

For $\lambda < -\frac{1}{4R^2}$, the solutions realize the principal series irreps P_s with

$$s = |\mu| = \sqrt{-\lambda R^2 - \frac{1}{4}} \quad \text{is positive.} \tag{37}$$

Indeed, the Casimir is

$$C^{(2)} = R^2 \lambda = -\left(s^2 + \frac{1}{4}\right) < -\frac{1}{4} \tag{38}$$

using (26), which corresponds precisely to the principal series (141).

For $\lambda \in (-\frac{1}{4R^2}, 0)$, the solutions correspond to the complementary series irreps P_j^c with $j = \frac{1}{2} + \sqrt{(\frac{1}{4} + \lambda R^2)}$ from equation (142), since $C^{(2)} = R\lambda^2 = j(j - 1) \in (-\frac{1}{4}, 0)$.

3.2.2. Principal series solutions and asymptotics. For $\mu^2 < 0$, the differential equation (34) has two linearly independent solutions, corresponding to the principal series. It will be convenient to use $\mu = \pm is$, so that these solutions are⁸ $P_{\nu(k)}^{is}(-u)$ and $P_{\nu(k)}^{-is}(-u)$ for every positive s . For later use, we consider their asymptotic behavior. As $x \rightarrow 1^-$, we have

$$P_{\nu(k)}^\mu(x) \underset{x \rightarrow 1^-}{\sim} \frac{1}{\Gamma(1 - \mu)} \left(\frac{2}{1 - x}\right)^{\mu/2}. \tag{39}$$

Therefore

$$P_{\nu(k)}^{\pm is}(-u) \underset{u \rightarrow 1^+}{\sim} \frac{1}{\Gamma(1 \mp is)} \left(\frac{2}{1 + u}\right)^{\pm is/2} \tag{40}$$

for $u \rightarrow -1^+$, or equivalently

$$P_{\nu(k)}^{\pm is}(-\tanh \eta) \underset{\eta \rightarrow -\infty}{\sim} \frac{e^{\mp is \eta}}{\Gamma(1 \mp is)}. \tag{41}$$

Hence these solutions behave like plane waves for $\eta \rightarrow -\infty$.

To obtain the behavior of the solutions for $\eta \rightarrow \infty$, we use the following identity:

$$\frac{\sin((\nu - \mu)\pi)}{\Gamma(\nu + \mu + 1)} P_\nu^\mu(x) = \frac{\sin(\nu\pi)}{\Gamma(\nu - \mu + 1)} P_\nu^{-\mu}(x) - \frac{\sin(\mu\pi)}{\Gamma(\nu - \mu + 1)} P_\nu^{-\mu}(-x). \tag{42}$$

We can thus write

$$P_\nu^{-\mu}(-x) = \frac{\sin(\nu\pi)}{\sin(\mu\pi)} P_\nu^{-\mu}(x) - \frac{\sin((\nu - \mu)\pi)}{\sin(\mu\pi)} \frac{\Gamma(\nu - \mu + 1)}{\Gamma(\nu + \mu + 1)} P_\nu^\mu(x) \tag{43}$$

⁷ Strictly speaking, it should be $\nu = -\frac{1}{2} \pm ik$, but as we will use associated Legendre functions of the first kind as our basis, this is irrelevant since $P_\nu^\mu = P_{-1-\nu}^\mu$.

⁸ $Q_{\nu(k)}^{\pm is}$ can be written as linear combinations of $P_{\nu(k)}^{\pm is}$ and is therefore not an independent solution. We can use either $P_\nu^{is}(-u)$ or $P_\nu^{is}(u)$, since the equation is invariant under $u \rightarrow -u$.

and asymptotically

$$\begin{aligned} \mathbf{P}_{\nu}^{-\mu}(-x) \underset{x \rightarrow 1^-}{\sim} & \frac{\sin(\nu\pi)}{\sin(\mu\pi)} \frac{1}{\Gamma(1+\mu)} \left(\frac{2}{1-x}\right)^{-\mu/2} \\ & - \frac{\sin((\nu-\mu)\pi)}{\sin(\mu\pi)} \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)} \frac{1}{\Gamma(1-\mu)} \left(\frac{2}{1-x}\right)^{\mu/2}. \end{aligned} \quad (44)$$

Therefore, for $\eta \rightarrow \infty$, we have

$$\begin{aligned} \mathbf{P}_{\nu(k)}^{-\mu}(-\tanh\eta) \underset{\eta \rightarrow \infty}{\sim} & \frac{\sin(\nu\pi)}{\sin(\mu\pi)} \frac{1}{\Gamma(1+\mu)} e^{-\mu\eta} \\ & - \frac{\sin((\nu-\mu)\pi)}{\sin(\mu\pi)} \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)} \frac{1}{\Gamma(1-\mu)} e^{\mu\eta} \\ = & \frac{\sin(\nu\pi)\Gamma(-\mu)}{\pi} e^{-\mu\eta} + \frac{\Gamma(\mu)}{\Gamma(\mu-\nu)\Gamma(\nu+\mu+1)} e^{\mu\eta} \end{aligned} \quad (45)$$

using $\Gamma(z)\Gamma(1-z)\sin(\pi z) = \pi$.

To summarize, a complete set of solutions of (31) is given by

$$\boxed{\Upsilon_k^{s\pm}(\eta, \chi) := \frac{1}{\sqrt{\cosh\eta}} e^{ik\chi} \mathbf{P}_{-\frac{1}{2}+i|k|}^{\pm is}(-\tanh(\eta)) \quad \text{for } s > 0, k \in \mathbb{R}}. \quad (46)$$

These $\Upsilon_k^{s\pm}$ realize the principal series irrep P_s (141). They are the analogs of the spherical harmonics, and the space of all square-integrable functions on H^2 is spanned by the Υ_k^s . We will find analogous solutions in the Minkowski case (see section 4) corresponding to propagating waves, where $\mathbf{P}_{\nu(k)}^{\pm is}(-u)$ will be interpreted as positive (\mathbf{P}^{is}) and negative (\mathbf{P}^{-is}) frequency modes in the far past.

3.2.3. Comment on the complementary series. We have seen that $s^2 > 0$ (or equivalently $\mu^2 < 0$) is the case where the functions oscillate for $\eta \rightarrow \pm\infty$. In contrast, the solutions with $s^2 < 0$ corresponding to the complementary series do not describe waves propagating in the far past or future. For this reason, we will not consider the complementary series solutions any further.

3.2.4. Symplectic form, integration and inner product. The $SO(2, 1)$ -invariant volume form (i.e. the symplectic form) is given by

$$\omega = \frac{R}{r} \cosh(\eta) d\eta d\chi, \quad (47)$$

corresponding to the Poisson bracket

$$\{\eta, \chi\} = -\frac{r}{R \cosh(\eta)}. \quad (48)$$

This is consistent with $\sqrt{|g|} = R \cosh(\eta)$ in the $\eta\chi$ coordinates, (29). The trace corresponds to the integral over the symplectic volume form on H^2 ,

$$2\pi \text{Tr}(\hat{\phi}) \sim \int_{H^2} \omega \phi(x). \quad (49)$$

In particular, we can define an $SO(2, 1)$ -invariant inner product via

$$\langle \phi, \psi \rangle := \int_{H^2} \omega \phi^* \psi, \tag{50}$$

which defines the space $L^2(H^2)$ of square-integrable functions. Then the eigenmodes (46) of \square satisfy orthogonality relations

$$\begin{aligned} \langle \Upsilon_{k'}^{s'\pm}, \Upsilon_k^{s\pm} \rangle &:= \int_{H^2} \omega (\Upsilon_{k'}^{s'\pm}(\eta, \chi))^* \Upsilon_k^{s\pm}(\eta, \chi) \\ &= \frac{R}{r} \int d\eta d\chi e^{-ik'\chi} \left(\mathbf{P}_{-\frac{1}{2}+i|k'|}^{\pm is'}(-\tanh(\eta)) \right)^* e^{ik\chi} \mathbf{P}_{-\frac{1}{2}+i|k|}^{\pm is}(-\tanh(\eta)) \\ &= \frac{R}{r} \int d\chi e^{i(k-k')\chi} \int du \frac{1}{1-u^2} \left(\mathbf{P}_{-\frac{1}{2}+i|k'|}^{\pm is'}(-u) \right)^* \mathbf{P}_{-\frac{1}{2}+i|k|}^{\pm is}(-u) \end{aligned} \tag{51}$$

using $d\eta = \frac{du}{1-u^2}$. The last integral can be evaluated explicitly using the orthogonality relations (143) if desired.

3.3. Functions on fuzzy H_n^2 and coherent states

3.3.1. Tensor product decomposition. The fuzzy analog of the algebra of functions $\mathcal{C}^\infty(H^2)$ is given by $\text{End}(\mathcal{H}_n)$. To understand the fluctuation spectrum, we should decompose this into irreps of $SO(2, 1)$. This is somewhat non-trivial since these are infinite-dimensional representations, as in the commutative case. However, we can use the fact that $SO(2, 1)$ acts on noncommutative functions $\hat{\phi}$ via the adjoint

$$K^a \triangleright \hat{\phi} = [K^a, \hat{\phi}] = \frac{1}{r} [X^a, \hat{\phi}], \quad \hat{\phi} \in \text{End}(\mathcal{H}_n). \tag{52}$$

Square-integrable functions $\phi \in L^2(H^2)$ correspond to Hilbert–Schmidt operators $\hat{\phi} \in \text{End}(\mathcal{H}_n)$, which form a Hilbert space, and accordingly decompose into unitary irreps of $SO(2, 1)$, defining fuzzy scalar harmonics $\hat{\Upsilon}_k^{s\pm}$.

The decomposition of Hilbert–Schmidt operators in $\text{End}(\mathcal{H}_n)$ is obtained from the unitary tensor product decomposition [15]:

$$\text{End}(\mathcal{H}_n) \cong D_n^+ \otimes D_n^- \cong \int_0^\infty ds P_s. \tag{53}$$

The P_s are principal series irreps which asymptotically correspond to plane waves, and the direct integral on the rhs means that square-integrable functions are obtained as usual by forming wave-packets of these.

3.3.2. Coherent states and an isometric quantization map. Due to the above unique decomposition, the quantization map between $\mathcal{C}^\infty(H^2)$ to $\text{End}(\mathcal{H}_n)$ is fixed by symmetry up to a set of normalization constants. To make this more explicit, we can use coherent states. These are defined in a natural way using the fact that \mathcal{H}_n is a lowest weight representations. Let

$$|x_0\rangle := |n, n\rangle \in \mathcal{H}_n \tag{54}$$

be the (unit length) lowest weight state. This is an optimally localized state at the ‘south pole’ $x_0 = (R, 0, 0) \in \mathbb{R}^{1,2}$ of H^2 . Then the coherent state

$$|x\rangle := U_g |x_0\rangle \in \mathcal{H}_n \tag{55}$$

is defined by acting with a $SO(2, 1)$ rotation U_g which rotates x_0 into $x \in H^2$. The ambiguity in the choice of the group element $g \in SO(2, 1)$ leads to a $U(1)$ phase ambiguity, so that the coherent states form a $U(1)$ bundle over H^2 .

With this, we can define any $SO(2, 1)$ -equivariant quantization map \mathcal{Q} through its action on the harmonics

$$\begin{aligned} \mathcal{Q} : L^2(H^2) &\rightarrow \text{End}(\mathcal{H}_n) \\ \Upsilon_k^{s\pm} &\mapsto \hat{\Upsilon}_k^{s\pm} := c_s \int_{H^2} \omega \Upsilon_k^{s\pm}(x) |x\rangle\langle x|, \end{aligned} \tag{56}$$

where c_s are (so far) undefined constants.

This map is one-to-one as a map from square-integrable functions to Hilbert-Schmidt operators, and its inverse is given by the symbol

$$\hat{\Upsilon}_k^{s\pm} \mapsto \Upsilon_k^{s\pm}(x) = d_s \langle x | \hat{\Upsilon}_k^{s\pm} | x \rangle. \tag{57}$$

where the coefficients d_s satisfy

$$c_s d_s \int_{H^2} \omega \Upsilon_k^{s\pm}(x) |\langle x | x' \rangle|^2 = \Upsilon_k^{s\pm}(x'). \tag{58}$$

Since \mathcal{Q} respects $SO(2, 1)$, it is an intertwiner of its generators

$$[X^a, \mathcal{Q}(\phi)] = \mathcal{Q}(i\{x^a, \phi(x)\}), \tag{59}$$

so that the Laplacian is respected as well:

$$\square_X \mathcal{Q}(\phi) = [X^a, [X_a, \phi]] = \mathcal{Q}(\square_H \phi). \tag{60}$$

Here \square_H (26) is the usual Laplacian on H^2 , which is essentially the quadratic Casimir.

When the coefficients c_s are all equal, this construction is the well known quantization map used, for example, on symmetric spaces,

$$\begin{aligned} \mathcal{C}(H^2) &\rightarrow \text{End}(\mathcal{H}_n) \\ \phi(x) &\mapsto \hat{\phi} = \int_{H^2} \omega \phi(x) |x\rangle\langle x|. \end{aligned} \tag{61}$$

Here, however, we are interested in a quantization which is an isometry with respect to the inner products defined by the trace and (50), respectively. This can be accomplished choosing suitable normalization constant c_s for each $\hat{\Upsilon}_k^{s\pm}$, such that

$$\langle \Upsilon_k^{s\pm}, \Upsilon_{k'}^{s\pm} \rangle = \int_{H^2} \omega (\Upsilon_k^{s\pm})^* \Upsilon_{k'}^{s\pm} = 2\pi \text{Tr} \left((\hat{\Upsilon}_k^{s\pm})^\dagger \hat{\Upsilon}_{k'}^{s\pm} \right). \tag{62}$$

When \mathcal{Q} is an isometric map, we must have $d_s = 2\pi c_s$. Coefficients c_s can therefore be computed from equation (58).

4. 1+1-dimensional (squashed) space-time $\mathcal{M}^{1,1}$

Following [16], we can obtain a space with Minkowski signature by projecting of H^2 onto the 0, 1 plane as follows

$$\begin{aligned} \Pi : H^2 &\rightarrow \mathbb{R}^{1,1} \subset \mathbb{R}^{1,2}, \\ (x^0, x^1, x^2) &\mapsto (x^0, x^1, 0). \end{aligned} \tag{63}$$

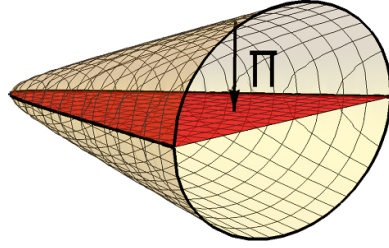


Figure 1. Sketch of the projection Π_x from H^2 to $\mathcal{M}^{1,1}$ with Minkowski signature.

The projected space $\mathcal{M}^{1,1} = \mathcal{M}^+ \cup \mathcal{M}^-$ consists of two sheets which are connected at the boundary, cf figure 1. This respects the $SO(1,1)$ generated by K_2 . In the fuzzy case, this projection is realized simply by dropping X^2 from the matrix background, and considering a new background through X^0 and X^1 only. Thus define⁹

$$Y^\mu = X^\mu, \quad \text{for } \mu = 0, 1. \quad (64)$$

The $\mathfrak{so}(2,1)$ algebra gives

$$\begin{aligned} [Y^0, [Y^0, Y^1]] &= ir^3 [K^0, K^2] = r^2 Y^1 \\ [Y^1, [Y^1, Y^0]] &= -ir^3 [K^1, K^2] = -r^2 Y^0 \end{aligned} \quad (65)$$

so that¹⁰

$$\square_Y Y^\mu = -[Y^0, [Y^0, Y^\mu]] + [Y^1, [Y^1, Y^\mu]] = -r^2 Y^\mu, \quad \mu = 0, 1. \quad (66)$$

This means that the Y^μ for $\mu = 0, 1$ provide a solution of the Lorentzian matrix model (3) with positive mass

$$m^2 = r^2. \quad (67)$$

This is the solution of interest here, which can be realized either in a 1+1-dimensional matrix model, or in the 3 (or higher)- dimensional model (1) by setting the remaining Y^a to zero. If we keep such extra matrices in the model, their fluctuations will play the role of scalar fields on the background, viewed as transverse fluctuations of the brane. This will be discussed in section 5. Note that $m^2 > 0$ suggests stability of this background, which should be studied in more detail elsewhere.

Y^μ transform as vectors of $SO(1,1)$, which can be realized by the adjoint i.e. through gauge transformations. Hence the solution admits a global $SO(1,1)$ symmetry. In the semi-classical limit, this defines a foliation of \mathcal{M} into one-dimensional space-like hyperboloids H_t^1 , more precisely

$$\mathcal{M}^{1,1} = \mathcal{M}^+ \cup \mathcal{M}^- = H_{t_0}^1 \cup 2 \bigcup_{t > t_0} H_t^1$$

one for each sheet except for $t = t_0 = R$. The two sheets $\mathcal{M}^+ \cup \mathcal{M}^-$ are connected at $t = R$, cf figure 1. We will see that the x^0 direction is time-like, and that $\mathcal{M}^{1,1}$ resembles a double-covered 1+1-dimensional FLRW space-time with hyperbolic ($k = -1$) spatial geometry, similar to that in [16]. Note that these time-slices are infinite in the space direction,

⁹ Note that X^0 and X^1 generate the full algebra of functions $\text{End}(\mathcal{H})$ on H_n^2 ; only the effective geometry defined by the matrix background is changed.

¹⁰ Note that Y^μ must be eigenvectors of \square_Y due to $SO(1,1)$ invariance.

even at the BB $t = t_0$. Therefore it is not unreasonable to expect a unitary time-evolution for all t .

4.1. Semi-classical geometry

4.1.1. *Induced metric.* Consider the semi-classical limit $Y^\mu \sim y^\mu$. On this projected space, the induced metric on $\mathcal{M}^{1,1} \subset \mathbb{R}^{1,1}$ is clearly Lorentzian,

$$g_{\mu\nu} = (-1, 1) = \eta_{\mu\nu}, \quad \mu, \nu = 0, 1 \tag{68}$$

in Cartesian coordinates y^μ . This is recognized as a $SO(1,1)$ -invariant FLRW metric with $k = -1$, by decomposing \mathcal{M} into 1-hyperboloids H_t ,

$$\begin{pmatrix} Y^0 \\ y^1 \end{pmatrix} = t \begin{pmatrix} \cosh(\chi) \\ \sinh(\chi) \end{pmatrix} \tag{69}$$

for $t = R \cosh(\eta) \in [R, \infty)$. In particular,

$$t^2 = -y^\mu y^\nu \eta_{\mu\nu} \geq R^2, \quad R^2 = t^2 - x_2^2 > 0 \tag{70}$$

where

$$x_2 = R \sinh(\eta) = \pm \sqrt{t^2 - R^2} \tag{71}$$

is a function on H^2 which allows to distinguish the two sheets of $\mathcal{M}^{1,1}$ for $\eta \in \mathbb{R}$. This gives the 2D flat Milne metric:

$$ds_g^2 = -dt^2 + t^2 d\chi^2 = -dy_0^2 + dy_1^2. \tag{72}$$

Here $\chi \in (-\infty, \infty)$ parametrizes the $SO(1,1)$ -invariant space-like H^1 with $k = -1$. The $(\eta, \chi) \in \mathbb{R}^2$ variables are very useful because they parametrize both sheets of the projected hyperboloid H^2 .

The induced metric g can be viewed as closed-string metric in target space. However as familiar from matrix models [4] and string theory [17], the fluctuation on the brane are governed by a different metric or kinetic term:

4.1.2. *Effective generalized d'Alembertian.* We will see in the next section that the kinetic term for a (transverse) scalar field on this background in the matrix model is governed by

$$\square_y = -\{y^\mu, \{y_\mu, \cdot\}\} = \square_H + \{y_2, \{y_2, \cdot\}\} \tag{73}$$

where \square_H is the Laplacian (25) on H^2 . The extra term is evaluated easily as

$$\{y_2, \{y_2, \phi\}\} = r^2 \partial_\chi^2 \phi \tag{74}$$

using (48). Together with (30) we obtain

$$\begin{aligned} \square_y \phi &= -r^2 \left(\partial_\eta^2 + \tanh(\eta) \partial_\eta - \tanh^2(\eta) \partial_\chi^2 \right) \phi \\ &= -\left(\gamma^{\mu\nu} \partial_\mu \partial_\nu + O(\partial) \right) \phi. \end{aligned} \tag{75}$$

This is a second-order hyperbolic differential operator with leading symbol $\gamma^{\mu\nu} p_\mu p_\nu$ where

$$\gamma^{\mu\nu} = r^2 \begin{pmatrix} 1 & 0 \\ 0 & -\tanh^2(\eta) \end{pmatrix} \tag{76}$$

in $(\eta\chi)$ coordinates. This governs the propagation of scalar fields on $\mathcal{M}^{1,1}$, and respects the $SO(2, 1)$ symmetry of a $k = -1$ FLRW space-time with time η . We also note the identity

$$\omega = \rho \sqrt{|\gamma_{\mu\nu}|} d\xi^0 d\xi^1 \quad \text{with} \quad \rho = rR|\sinh(\eta)| \tag{77}$$

in any local coordinates. In dimensions larger than 2, such a ‘matrix Laplacian’ can always be written in terms of a metric Laplacian (or d’Alembertian) for a unique effective metric [4]. This is not possible in two dimensions due to Weyl invariance¹¹. We will therefore study the operator \square_y directly, which will be referred to as generalized d’Alembertian. The metric $-\gamma^{\mu\nu}$ is that of a FLRW space-time and clearly governs the local propagation and causality structure, which is the main focus of the present paper. However it should not be considered as effective metric. The origin of $\gamma^{\mu\nu}$ will become clear in the next section.

5. Scalar harmonics on fuzzy $\mathcal{M}^{1,1}$

5.1. Transverse fluctuations in the matrix model

Scalar fields on $\mathcal{M}^{1,1}$ are realized by the transverse (space-like) matrix Y^a , $a = 2$ in the model (1) or (7) (possibly extended by further matrices Y^a):

$$S[Y] = S[Y^\mu] + \frac{2}{g^2} \text{Tr} \left(-[Y^\mu, Y^a][Y_\mu, Y^a] - (m_\phi^2 - i\varepsilon)Y^a Y_a \right) \tag{78}$$

Here we include an arbitrary scalar mass parameter m_ϕ^2 , independent of m^2 in (1). We focus on one such transverse matrix $Y^a =: \hat{\phi}$, viewed as scalar field on $\mathcal{M}^{1,1}$. Its effective action is accordingly

$$S[\hat{\phi}] = \frac{2}{g^2} \text{Tr} \left(\hat{\phi} \square_Y \hat{\phi} - (m_\phi^2 - i\varepsilon) \hat{\phi}^2 \right) = \frac{2}{g^2} \int_{\mathcal{M}} \omega \left(\phi \square_y \phi - (m_\phi^2 - i\varepsilon) \phi^2 \right) =: S[\phi] \tag{79}$$

with \square_Y for matrices given in (66), and \square_y for functions in (4). Here the matrix model $S[\hat{\phi}]$ is identified with the semi-classical action $S[\phi]$, which needs some explanation. The matrix $\hat{\phi} \in \text{End}(\mathcal{H}_n)$ is identified via the quantization map \mathcal{Q} (56) with a function $\phi \in L^2(H^2)$, which in turn is identified with $L^2(\mathcal{M}^{1,1})$ via Π (63). The symplectic form ω is also the same as on H^2 , so that the integral over $\mathcal{M}^{1,1}$ can be viewed as an integral over H^2 . As discussed in section 3.3, \mathcal{Q} is (by definition) an isometry between $L^2(H^2) = L^2(\mathcal{M}^{1,1})$ and (Hilbert–Schmidt operators in) $\text{End}(\mathcal{H}_n)$. Moreover, the $[Y^\mu, \cdot]$ are $SO(2, 1)$ generators which commute with the quantization map \mathcal{Q} . Therefore the free matrix model $S[\hat{\phi}]$ is identically mapped by \mathcal{Q} to the classical action $S[\phi]$.

In the same vein, the matrix equation of motion for the scalar field

$$\square_Y \hat{\phi} = m_\phi^2 \hat{\phi} \tag{80}$$

is equivalent to the semi-classical (Poisson) wave equation

$$\square_y \phi = m_\phi^2 \phi . \tag{81}$$

We will determine the classical eigenmodes of \square_y explicitly below.

¹¹ The conformal factor could be determined for gauge fields, which arise from tangential fluctuations of $\mathcal{M}^{1,1}$ in the matrix model. However we refrain from pursuing this direction in the present paper.

To understand the role of $\gamma^{\mu\nu}$ in (75), it is instructive to rewrite the above kinetic term as follows

$$S[\phi] = \frac{2}{g^2} \int_{\mathcal{M}} \omega \left(\gamma^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - (m_\phi^2 - i\varepsilon) \phi^2 \right) \tag{82}$$

in terms of a frame [18]

$$E^{a\mu} = \{Y^a, \xi^\mu\}, \quad \gamma^{\mu\nu} = \eta_{ab} E^{a\mu} E^{b\nu} \tag{83}$$

in any local coordinates ξ^μ . In view of (77), this can be interpreted as action for a scalar field non-minimally coupled to a dilaton [19], and it explains the origin and the significance of the metric $\gamma^{\mu\nu}$. In the case of 3 + 1 dimensions, this metric turns out to be conformally equivalent to the effective metric [20].

5.2. Eigenfunctions of \square_y

We want to solve the eigenvalue equation

$$\square_y \phi = \lambda \phi, \tag{84}$$

which should provide a complete set of eigenfunctions on our space-time. We will essentially recover the modes Υ_k^s (46) in the principal series of $SO(2, 1)$. In the adapted (t, χ) coordinates and using (75), this takes the form

$$\square_y \phi = r^2 \left(-\partial_\eta^2 - \tanh(\eta) \partial_\eta + \tanh^2(\eta) \partial_\chi^2 \right) \phi = \lambda \phi. \tag{85}$$

To solve this equation, we again make a separation ansatz

$$\phi = e^{ik\chi} \varphi_k(\eta) \tag{86}$$

which gives

$$-r^2 \left(\partial_\eta^2 + \tanh(\eta) \partial_\eta + k^2 \tanh^2(\eta) \right) \varphi_k = \lambda \varphi_k. \tag{87}$$

Clearly for $\eta \rightarrow \pm\infty$ this reduces to the ordinary wave equation

$$-r^2 \left(\partial_\eta^2 + \text{sign}(\eta) \partial_\eta + k^2 \right) \varphi_k \approx \lambda \varphi_k, \quad \eta \rightarrow \pm\infty \tag{88}$$

whose solutions for large k are exponentially damped plane waves,

$$\varphi_k^\pm(\eta) \rightarrow e^{\pm ik\eta - \frac{1}{2}|\eta|}, \quad \eta \rightarrow \pm\infty. \tag{89}$$

We can bring the exact equation (87) into a more familiar form by again substituting $u = \tanh(\eta) \in (-1, 1)$ and $f(u) = (1 - u^2)^{1/4} h(u)$ to obtain

$$(1 - u^2) h'' - 2uh' + \left(-\left(k^2 + \frac{1}{4}\right) + \frac{k^2 + r^{-2}\lambda - \frac{1}{4}}{1 - u^2} \right) h = 0. \tag{90}$$

This has the same structure as (34), replacing $-\lambda R^2 \rightarrow k^2 + r^{-2}\lambda$. It is hence solved again by associated Legendre functions of the first and second kind P_ν^μ and Q_ν^μ , as in section 3.2, for

$$\nu(\nu + 1) = -k^2 - \frac{1}{4} \quad \text{and} \quad \mu = \pm is, \quad s^2 = k^2 + r^{-2}\lambda - \frac{1}{4}. \tag{91}$$

Asymptotically oscillating solutions are obtained for $k^2 + \lambda/r^2 > \frac{1}{4}$ so that $\mu = \pm is$ is purely imaginary,

$$s = \sqrt{k^2 + \frac{\lambda}{r^2} - \frac{1}{4}} > 0. \tag{92}$$

A basis of solutions, as before, is given by

$$\mathbf{P}_{\nu(k)}^{is}(u) \tag{93}$$

which form the unitary reps of $SO(2, 1)$ of the principal series P_s . The degree of the Legendre function can be taken to be

$$\nu(k) = -\frac{1}{2} + i|k| \tag{94}$$

which should be compared with (36). As expected, we obtain the same basis of modes as we did for H^2 in (46),

$$\Upsilon_k^{s\pm}(\eta, \chi) := \frac{1}{\sqrt{\cosh \eta}} e^{ik\chi} \mathbf{P}_{-\frac{1}{2}+i|k|}^{\pm is}(\tanh(\eta)) \quad \text{for } s > 0, k \in \mathbb{R}. \tag{95}$$

To recap, above modes satisfy

$$\begin{aligned} \square_y \Upsilon_k^{s\pm} &= r^2 \left(s^2 - k^2 + \frac{1}{4} \right) \Upsilon_k^{s\pm}, \\ (\Upsilon_k^s)^* &= \Upsilon_{-k}^{-s}. \end{aligned} \tag{96}$$

These modes will be used to compute the path integral in section 6.

5.2.1. On-shell modes. Now we identify the on-shell modes among the above harmonics, which are the eigenmodes for $\lambda = m_\phi^2$. Then the eom (81) has the following solutions

$$\mathcal{Y}_k^{\pm s} = \frac{1}{\sqrt{\cosh \eta}} e^{ik\chi} \mathbf{P}_{\nu(k)}^{\pm is}(\tanh(\eta)) \tag{97}$$

where

$$\nu(k) = -\frac{1}{2} + i|k|, \quad s^2 = k^2 + \frac{m_\phi^2}{r^2} - \frac{1}{4}. \tag{98}$$

These are the positive and negative energy eigenmodes, which form principal series irreps.

5.2.2. Asymptotics and Bogoliubov coefficients. Since s depends now on k , the early and late time frequencies depend on k . On-shell, we have

$$s = \omega_k := \sqrt{k^2 + \frac{m_\phi^2}{r^2} - \frac{1}{4}}. \tag{99}$$

The asymptotic expansion (41) and (45) become

$$\mathbf{P}_{\nu(k)}^{\pm i\omega_k}(\tanh \eta) \underset{\eta \rightarrow \infty}{\sim} \frac{e^{\mp i\omega_k \eta}}{\Gamma(1 \mp i\omega_k)} \tag{100}$$

and

$$\mathbf{P}_{\nu(k)}^{i\omega_k}(\tanh \eta) \underset{\eta \rightarrow -\infty}{\sim} -\frac{\sin(\nu\pi)\Gamma(i\omega_k)}{\pi} e^{i\omega_k \eta} + \frac{\Gamma(-i\omega_k)}{\Gamma(-i\omega_k - \nu)\Gamma(-i\omega_k + 1 + \nu)} e^{-i\omega_k \eta}. \tag{101}$$

Therefore the modes $\mathcal{Y}_k^{+s} \sim e^{i(k\chi - \omega_k \eta)}$ are negative energy modes in the far future $\eta \rightarrow \infty$ (long after the BB), if we consider η as globally oriented time coordinate, while

$\mathcal{Y}_k^{-s} \sim e^{i(k\chi + \omega_k \eta)}$ are the positive energy modes. In the far past $\eta \rightarrow -\infty$, $\mathcal{Y}_k^{+s} \sim \alpha_k e^{i(k\chi - \omega_k \eta)} + \beta_k e^{i(k\chi + \omega_k \eta)}$ is then a superposition of positive- and negative-energy modes.

The transformation $\begin{pmatrix} \alpha_k & \beta_k \\ \beta_k^* & \alpha_k^* \end{pmatrix}$ is canonical i.e. it preserves the Poisson bracket. Comparing the coefficients in equations (101) and (100), we obtain the Bogoliubov coefficients:

$$\alpha_k = \frac{\Gamma(1 - i\omega_k)\Gamma(-i\omega_k)}{\Gamma(-i\omega_k - \nu)\Gamma(-i\omega_k + 1 + \nu)} = \frac{\Gamma(-i\omega_k)}{\Gamma(i\omega_k)\sin(i\pi\omega_k)} \frac{\pi}{\Gamma(-i\omega_k - \nu)\Gamma(-i\omega_k + 1 + \nu)}$$

$$\beta_k = -\frac{\sin(\nu\pi)\Gamma(1 - i\omega_k)\Gamma(i\omega_k)}{\pi} = -\frac{\sin \pi \nu}{\sin(i\pi\omega_k)}. \quad (102)$$

As a check, we can confirm that they satisfy $|\alpha_k|^2 - |\beta_k|^2 = 1$. To do so, we notice that, as long as $\mu = \pm i\omega_k$ is purely imaginary and $\text{Re}(\nu) = -\frac{1}{2}$, $\frac{1}{2} + \nu \pm \mu$ is purely imaginary, and

$$\frac{1}{|\Gamma(\mu - \nu)\Gamma(\nu + \mu + 1)|^2} = \frac{1}{\pi^2} (\sin^2(\pi\nu) - \sin^2(\pi\mu)). \quad (103)$$

We also have $|\sin(\mu\pi)|^2 = -\sin^2(\mu\pi)$ because μ is purely imaginary, and $|\sin(\nu\pi)|^2 = \sin^2(\nu\pi)$ because the real part of ν is $\frac{1}{2}$. Then,

$$|\alpha_k|^2 - |\beta_k|^2 = \frac{\sin^2(\pi\nu) - \sin^2(\pi\mu)}{|\sin \pi \mu|^2} - \frac{|\sin \pi \nu|^2}{|\sin \pi \mu|^2} = 1. \quad (104)$$

More explicitly, we have

$$|\beta_k|^2 = \frac{|\sin \pi \nu|^2}{|\sin \pi \mu|^2} = \frac{\cosh^2\left(\pi\sqrt{k^2 - \frac{1}{4}}\right)}{\sinh^2(\pi\omega_k)}. \quad (105)$$

Using the on-shell relation (99) we have $\sqrt{k^2 - \frac{1}{4}} \approx \omega_k$ in the relativistic regime, so that

$$|\beta_k|^2 \approx \left(\frac{e^{2\pi\omega_k} + 1}{e^{2\pi\omega_k} - 1}\right)^2 \approx 1, \quad |\alpha_k|^2 = 1 + \beta_k^2 \approx 2. \quad (106)$$

This means that the Bogoliubov transformation is ‘large’, and strongly mixes the positive and negative energy modes.

5.2.3. Fuzzy wavefunctions. As discussed before, we define the fuzzy harmonics through the map in equation (56) with coefficients c_s chosen so that (62) is satisfied,

$$\boxed{\{\hat{\Upsilon}_k^{\pm s} = \mathcal{Q}(\Upsilon_k^{\pm s})\}}. \quad (107)$$

These are the principal series modes in the unitary decomposition of $\text{End}(\mathcal{H}_n)$, cf (53), and satisfy (60)

$$\square_Y \hat{\Upsilon}_k^{\pm s} = r^2 \left(s^2 - k^2 + \frac{1}{4} \right) \hat{\Upsilon}_k^{\pm s},$$

$$(\hat{\Upsilon}_k^s)^\dagger = \hat{\Upsilon}_{-k}^{-s}. \quad (108)$$

The equivalence via \mathcal{Q} implies that the matrix configurations have the same properties as the classical ones, and satisfy a unique time-evolution once the appropriate semi-classical

boundary conditions are imposed via \mathcal{Q} . The local causality structure will be verified in the next section. In particular, the appearance of infinite time derivatives in a star product formulation is completely misleading in this respect, and the model with space-time noncommutativity has perfectly nice and reasonable properties¹².

6. Fluctuations and path integral quantization

The quantization of a matrix model is naturally defined via a path integral, which amounts to integrating over all matrices in $\text{End}(\mathcal{H}_n)$. On the above background $\mathcal{M}^{1,1}$, we can expand $\text{End}(\mathcal{H}_n)$ in the basis $\hat{\mathcal{Y}}_k^{\pm s}$ of $SO(2, 1)$ principal series modes (107),

$$\hat{\phi} = \int dsdk (\phi_{s,k}^+ \hat{\Upsilon}_k^{s+} + \phi_{s,k}^- \hat{\Upsilon}_k^{s-}) \in \text{End}(\mathcal{H}_n) \tag{109}$$

integrating over $s > 0$ and $k \in \mathbb{R}$. In the semi-classical limit, this reduces to

$$\phi(x) = \int dsdk (\phi_{s,k}^+ \Upsilon_k^{s+}(x) + \phi_{s,k}^- \Upsilon_k^{s-}(x)) . \tag{110}$$

We can now define correlation functions in the angular momentum basis as

$$\langle \hat{\phi}_{sk}^\sigma \hat{\phi}_{s'k'}^{\sigma'} \rangle := \frac{1}{Z} \int D\phi \phi_{sk}^\sigma \phi_{s'k'}^{\sigma'} e^{iS_\varepsilon[\phi]} \tag{111}$$

where $\sigma, \sigma' = \pm$ and $D\phi = \prod d\phi_{sk}$ is the integral over all modes, and the $i\varepsilon$ prescription (6) is understood. Using the correspondence between classical and fuzzy functions, we can associate to this a 2-point function in position space as follows

$$\langle \hat{\phi}(x) \hat{\phi}(y) \rangle := \sum_{s,s'; \sigma, \sigma' = \pm} \Upsilon_k^{s\sigma}(x) \Upsilon_{k'}^{s'\sigma'}(y) \langle \hat{\phi}_{sk}^\sigma \hat{\phi}_{s'k'}^{\sigma'} \rangle . \tag{112}$$

Since we only consider the free theory, the fuzzy case is equivalent to the semi-classical version on classical space-time. The only new ingredient inherited from the matrix model is a specific action and the $i\varepsilon$ prescription¹³ (6).

Now consider the action in terms of the eigenmodes, which in the semi-classical case has the form

$$\begin{aligned} S_\varepsilon[\phi] &= \int_{H_2} \omega \phi^* (\square_y - m_\phi^2 + i\varepsilon) \phi \\ &= Rr \int_{H_2} \cosh(\eta) (\phi_{s',k'}^+ \Upsilon_{k'}^{s'+} + \phi_{s',k'}^- \Upsilon_{k'}^{s'-})^* \left(s^2 - k^2 + \frac{1}{4} - \tilde{m}^2 + i\varepsilon \right) (\phi_{s,k}^+ \Upsilon_k^{s+} + \phi_{s,k}^- \Upsilon_k^{s-}) \end{aligned} \tag{113}$$

where the $\Upsilon_k^{s\pm} = (\Upsilon_{-k}^{s\mp})^*$ are given in (95), the eigenvalue of \square_y is $r^2(s^2 - k^2 + \frac{1}{4})$ (108) and

$$\tilde{m}^2 = \frac{m^2}{r^2} . \tag{114}$$

¹² Of course non-commutativity does have significant implications. Even though the correspondence defined via \mathcal{Q} is appropriate at low energies, it is quite misleading at high energies, where the fields acquire a string-like behavior [21]. This also implies that quantum effects in interacting theories typically exhibit a strong non-locality known as UV/IR mixing.

¹³ Since ϕ can be considered as a transverse (space-like) matrix of the underlying Yang–Mills matrix model (1), this prescription boils down to replacing the mass term as $m^2 \rightarrow m^2 - i\varepsilon$.

To evaluate the action, we need

$$\begin{aligned}
 \int d\chi d\eta \cosh(\eta) (\Upsilon_{k'}^{s'})^* \Upsilon_k^s &= \int_{-\infty}^{\infty} d\chi e^{i(k'-k)\chi} \int_{-1}^1 \frac{du}{1-u^2} \mathbf{P}_{\nu(k')}^{-is'}(u) \mathbf{P}_{\nu(k)}^{is}(u) \\
 &= (2\pi) \delta(k-k') \int_{-1}^1 \frac{du}{1-u^2} \mathbf{P}_{\nu(k)}^{-is'}(u) \mathbf{P}_{\nu(k)}^{is}(u) \\
 &= (2\pi) \delta(k-k') \left(a(k,s) \delta(s+s') + b(k,s) \delta(s-s') \right)
 \end{aligned} \tag{115}$$

see (51) using the orthogonality relations (143), where

$$\begin{aligned}
 a(k,s) &= \frac{2\pi}{\Gamma(\frac{1}{2} + i|k| - is) \Gamma(\frac{1}{2} - i|k| - is)} \frac{\cosh(\pi k)}{s \sinh(\pi s)} = a(k, -s)^* \\
 b(k,s) &= 2 \frac{\sinh(\pi s)}{s} \left(1 + \frac{\cosh^2(\pi k)}{\sinh^2(\pi s)} \right) = b(k, -s).
 \end{aligned} \tag{116}$$

Note that half of the terms in (113) will drop out since $s, s' > 0$. We thus obtain

$$\begin{aligned}
 S_\varepsilon[\phi] &= 2\pi Rr \int ds dk ds' dk' \delta(k-k') \delta(s-s') \left(s^2 - k^2 + \frac{1}{4} - \tilde{m}^2 + i\varepsilon \right) \\
 &\quad \times \left((\phi_{s',k'}^+)^*, (\phi_{s',k'}^-)^* \right) \begin{pmatrix} b(k,s) & a(k,-s) \\ a(k,s) & b(k,-s) \end{pmatrix} \begin{pmatrix} \phi_{s,k}^+ \\ \phi_{s,k}^- \end{pmatrix}.
 \end{aligned} \tag{117}$$

Inverting the 2×2 matrix, the propagator in ‘momentum space’ is

$$\begin{aligned}
 \left\langle \begin{pmatrix} \phi_{s,k}^+ \\ \phi_{s,k}^- \end{pmatrix} \left((\phi_{s',k'}^+)^*, (\phi_{s',k'}^-)^* \right) \right\rangle &= \frac{1}{\pi Rr} \delta(k-k') \delta(s-s') \frac{1}{s^2 - k^2 + \frac{1}{4} - \tilde{m}^2 + i\varepsilon} \\
 &\quad \times \frac{s^2}{\cosh(2\pi s) + \cosh(2\pi k)} \begin{pmatrix} b(k,-s) & -a(k,-s) \\ -a(k,s) & b(k,s) \end{pmatrix}
 \end{aligned} \tag{118}$$

using

$$\det \begin{pmatrix} b(k,s) & a(k,-s) \\ a(k,s) & b(k,-s) \end{pmatrix} = \frac{2}{s^2} (\cosh(2\pi k) + \cosh(2\pi s)). \tag{119}$$

6.1. Propagator in position space

In the $\eta\chi$ space-time coordinates of $\mathcal{M}^{1,1}$, the propagator takes the form

$$\langle \hat{\phi}(\eta, \chi) \hat{\phi}(\eta', \chi')^* \rangle = \int ds dk ds' dk' \sum_{\sigma, \sigma' = \pm} \Upsilon_k^{s\sigma}(\eta, \chi) \langle \hat{\phi}_{sk}^\sigma (\hat{\phi}_{s'k'}^{\sigma'})^* \rangle \Upsilon_{k'}^{s'\sigma'}(\eta', \chi')^*. \tag{120}$$

We can evaluate this explicitly in the late-time regime $\eta \rightarrow \infty$ using the asymptotic form (100), which gives

$$\Upsilon_k^{s\pm} = \frac{e^{ik\chi}}{\sqrt{\cosh\eta}} \mathbf{P}_{\nu(k)}^{\mp is}(\tanh\eta) \underset{\eta \rightarrow \infty}{\sim} \frac{e^{i(k\chi \pm s\eta)}}{\Gamma(1 \mp is)\sqrt{\cosh\eta}}. \quad (121)$$

Thus

$$\begin{aligned} \langle \hat{\phi}(\eta, \chi) \hat{\phi}(\eta', \chi')^* \rangle &\sim \frac{2}{2\pi Rr\sqrt{\cosh\eta\cosh\eta'}} \int ds dk \frac{e^{ik(\chi-\chi')}}{s^2 - k^2 + \frac{1}{4} - \tilde{m}^2 + i\varepsilon} \frac{s^2}{\cosh(2\pi s) + \cosh(2\pi k)} \\ &\times \begin{pmatrix} \frac{e^{-is\eta}}{\Gamma(1-is)}, \frac{e^{is\eta}}{\Gamma(1+is)} \\ -a(k,s), b(k,s) \end{pmatrix} \begin{pmatrix} b(k,-s) & -a(k,-s) \\ -a(k,s) & b(k,s) \end{pmatrix} \begin{pmatrix} \frac{e^{is\eta'}}{\Gamma(1+is)} \\ \frac{e^{-is\eta'}}{\Gamma(1-is)} \end{pmatrix} \\ &= \frac{1}{2\pi Rr\sqrt{\cosh(\eta)\cosh(\eta')}} \int ds dk \frac{e^{ik(\chi-\chi')}}{(s^2 - k^2 + \frac{1}{4} - \tilde{m}^2 + i\varepsilon)} \\ &\times \left[\frac{\cos(s(\eta-\eta'))}{\pi} + \frac{1}{2\pi^3} \cosh(\pi k) s \sinh(\pi s) \right. \\ &\times \left(e^{-is(\eta+\eta')} \Gamma(is)^2 \Gamma\left(ik-is+\frac{1}{2}\right) \Gamma\left(-ik-is+\frac{1}{2}\right) \right. \\ &\left. \left. + e^{is(\eta+\eta')} \Gamma(-is)^2 \Gamma\left(ik+is+\frac{1}{2}\right) \Gamma\left(-ik+is+\frac{1}{2}\right) \right) \right] \\ &=: \langle \hat{\phi}(\eta, \chi) \hat{\phi}(\eta', \chi')^* \rangle_0 + \langle \hat{\phi}(\eta, \chi) \hat{\phi}(\eta', \chi')^* \rangle_{op}. \quad (122) \end{aligned}$$

At late times $\eta, \eta' \rightarrow \infty$, the second term is rapidly oscillating and hence suppressed. Therefore the first term is the leading contribution in the late time regime.

6.1.1. Late time propagator for $\eta', \eta \rightarrow \infty$. Consider first the late time propagator

$$\langle \hat{\phi}(\eta, \chi) \hat{\phi}(\eta', \chi')^* \rangle_0 = c_\eta \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dk \frac{e^{i(k(\chi-\chi') + s(\eta-\eta'))}}{s^2 - k^2 + \frac{1}{4} - \tilde{m}^2 + i\varepsilon}. \quad (123)$$

The pre-factor

$$c_\eta := \frac{1}{4\pi^4 Rr\sqrt{\cosh\eta\cosh\eta'}} \quad (124)$$

reflects the non-canonical normalization, which can be traced to the exponential damping behavior in (89). Apart from this normalization, we recover precisely the Feynman propagator on flat 1+1-dimensional space-time at zero temperature, including the appropriate $i\varepsilon$ prescription which ensures local causality.

Notice that the formula applies equally in the opposite limit $\eta, \eta' \rightarrow -\infty$. Since the eigenmodes stretch continuously across the singularity at $\eta = 0$, the parameter η is expected to indicate the physical time evolution on both sides of the BB, so that the arrow of time points inwards (towards the BB) for $\eta < 0$. This strongly suggests to interpret the singularity as ‘Big Bounce’. A more profound justification e.g. via entropic considerations is beyond the scope of this paper.

6.1.2. *Non-local contribution for large $\eta \approx -\eta' \rightarrow \infty$.* To evaluate (120) in a limit where $\eta \rightarrow \infty$ but $\eta' \rightarrow -\infty$, we make use of the asymptotic form (101) and the Bogoliubov coefficients:

$$\begin{aligned} \langle \hat{\phi}(\eta, \chi) \hat{\phi}(\eta', \chi')^* \rangle &\sim \frac{2}{2\pi Rr\sqrt{\cosh\eta\cosh\eta'}} \int ds dk \frac{e^{ik(\chi-\chi')}}{s^2 - k^2 + \frac{1}{4} - \bar{m}^2 + i\varepsilon} \frac{s^2}{\cosh(2\pi s) + \cosh(2\pi k)} \\ &\times \left(\frac{e^{-is\eta}}{\Gamma(1-is)}, \frac{e^{is\eta}}{\Gamma(1+is)} \right) \begin{pmatrix} b(k, -s) & -a(k, -s) \\ -a(k, s) & b(k, s) \end{pmatrix} \begin{pmatrix} \alpha^*(k, s) & \beta^*(k, s) \\ \beta(k, s) & \alpha(k, s) \end{pmatrix} \\ &\times \begin{pmatrix} \frac{e^{is\eta'}}{\Gamma(1+is)} \\ \frac{e^{-is\eta'}}{\Gamma(1-is)} \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} \alpha(k, s) &:= \frac{\Gamma(1-is)\Gamma(-is)}{\Gamma(-is-\nu)\Gamma(-is+1+\nu)} = -i \frac{\Gamma(-is)}{\Gamma(is)\sinh(\pi s)} \frac{\pi}{\Gamma(-is-\nu)\Gamma(-is+1+\nu)} \\ \beta(k, s) &:= -\frac{\sin(\nu\pi)\Gamma(1-is)\Gamma(is)}{\pi} = i \frac{\cosh\pi k}{\sinh(\pi s)} = -\beta^*(k, s). \end{aligned} \quad (125)$$

Note that

$$a(k, s) = a(k, -s)^* = \frac{2}{-is} \frac{\Gamma(is)}{\Gamma(-is)} \cosh(\pi k) \alpha(k, s), \quad (126)$$

and

$$b(k, s) = b(k, -s) = \frac{2}{-is} \frac{\sinh^2(\pi s) + \cosh^2(\pi k)}{\cosh(\pi k)} \beta(k, s), \quad (127)$$

and, as before,

$$\det \begin{pmatrix} b(k, -s) & -a(k, -s) \\ -a(k, s) & b(k, s) \end{pmatrix} = \frac{4}{s^2} (\sinh^2(\pi s) + \cosh^2(\pi k)) = \frac{2}{s^2} (\sinh(2\pi s) + \cosh(2\pi k)) =: D,$$

where we defined a useful quantity D :

$$\begin{aligned} |\alpha(s, k)|^2 &= 1 + |\beta(s, k)|^2 = \frac{Ds^2}{4\sinh^2(\pi s)} \\ a(k, s) &= a(k, -s)^* = -\frac{2\cosh(\pi k)}{is} \frac{\Gamma(is)}{\Gamma(-is)} \alpha(k, s) \\ b(k, s) &= b(k, -s) = D \frac{is}{2\cosh(\pi k)} \beta(k, s) = \frac{is}{2\sinh(\pi s)} D, \end{aligned} \quad (128)$$

This allows us to evaluate:

$$-a(k, s)\alpha^*(k, s) + b(k, s)\beta(k, s) = -\frac{iDs}{2} \frac{\cosh(\pi k)}{\sinh^2(\pi s)} \left(\frac{\Gamma(is)}{\Gamma(-is)} + 1 \right), \quad (129)$$

which allow us to identify in the combination

$$\left(\frac{e^{-is\eta}}{\Gamma(1-is)}, \frac{e^{is\eta}}{\Gamma(1+is)} \right) \begin{pmatrix} b(k, -s) & -a(k, -s) \\ -a(k, s) & b(k, s) \end{pmatrix} \begin{pmatrix} \alpha^*(k, s) & \beta^*(k, s) \\ \beta(k, s) & \alpha(k, s) \end{pmatrix} \begin{pmatrix} \frac{e^{is\eta'}}{\Gamma(1+is)} \\ \frac{e^{-is\eta'}}{\Gamma(1-is)} \end{pmatrix},$$

terms that do not oscillate rapidly in the limit considered. One of these is

$$\begin{aligned}
 & -e^{is(\eta+\eta')} [-a(k,s)\alpha^*(k,s) + b(k,s)\beta(k,s)] \frac{1}{\Gamma(1+is)} \frac{1}{\Gamma(1+is)} \\
 & = -e^{is(\eta+\eta')} \frac{D \cosh(\pi k)}{2 \pi^2} [\Gamma(is) + \Gamma(-is)] \Gamma(1-is)
 \end{aligned} \tag{130}$$

and the other is its complex conjugate. The leading part of the propagator is therefore

$$\begin{aligned}
 \langle \hat{\phi}(\eta, \chi) \hat{\phi}(\eta', \chi')^* \rangle & \sim -\frac{1}{\pi^3 R r \sqrt{\cosh \eta \cosh \eta'}} \int_{s>0} ds dk \frac{e^{ik(\chi-\chi')} \cosh(\pi k)}{s^2 - k^2 + \frac{1}{4} - \tilde{m}^2 + i\epsilon} \\
 & \times \left[e^{is(\eta+\eta')} \Gamma(1-is) (\Gamma(is) + \Gamma(-is)) + \text{c.c.} \right] \\
 & = -\frac{1}{\pi^3 R r \sqrt{\cosh \eta \cosh \eta'}} \int ds dk \frac{e^{ik(\chi-\chi')} e^{is(\eta+\eta')}}{s^2 - k^2 + \frac{1}{4} - \tilde{m}^2 + i\epsilon} \cosh(\pi k) \Phi(s)
 \end{aligned} \tag{131}$$

(the integral is over $s \in \mathbb{R}$ in the last expression) for $\eta \rightarrow \infty$ but $\eta' \rightarrow -\infty$. Here

$$\Phi(s) := \Gamma(1-is) (\Gamma(is) + \Gamma(-is)) = \frac{1}{\sinh \pi s} \left(1 + \frac{\Gamma(-is)}{\Gamma(is)} \right) \tag{132}$$

is a regular function in $s \in \mathbb{R}$ which decays exponentially for large s :

$$|\Phi(s)| \sim e^{-\pi |s|}, \quad s \rightarrow \pm\infty. \tag{133}$$

However, the expression in equation (131) is pathological due to the $\cosh(\pi k)$ factor, which leads to a UV divergence of the space-like momentum k . This divergence can be cured by smearing the correlation functions by a space-like Gaussian $\psi_{\chi_0}(\chi) = \frac{1}{\sqrt{\sigma\pi}} e^{-(\chi-\chi_0)^2/2\sigma^2}$ with width σ :

$$\langle \hat{\phi}(\eta, \chi_0) \hat{\phi}(\eta', \chi'_0)^* \rangle_\sigma := \int d\chi d\chi' \psi_{\chi_0}(\chi) \langle \hat{\phi}(\eta, \chi) \hat{\phi}(\eta', \chi')^* \rangle \psi_{\chi'_0}(\chi'). \tag{134}$$

Noting that $\int d\chi e^{-(\chi-\chi_0)^2/2\sigma^2} e^{ik\chi} = e^{-\frac{\sigma k^2}{4}} e^{ik\chi_0}$ this space-like UV divergence then disappears:

$$\langle \hat{\phi}(\eta, \chi) \hat{\phi}(\eta', \chi')^* \rangle_\sigma = -\frac{1}{\pi^3 R r \sqrt{\cosh \eta \cosh \eta'}} \int ds dk \frac{e^{is(\eta+\eta')} e^{ik(\chi_0-\chi'_0)} \cosh(\pi k) e^{-\frac{\sigma k^2}{2}}}{s^2 - k^2 + \frac{1}{4} - \tilde{m}^2 + i\epsilon} \Phi(s). \tag{135}$$

Now the integrals are well-defined. Due to their oscillatory behavior, the correlators are peaked at $\eta \approx -\eta'$ and $\chi_0 \approx \chi'_0$ and strongly suppressed otherwise. We therefore obtain a non-trivial correlation between the fields before and after the BB, for points on the in- and out sheets which coincide in target space. This result will find a natural interpretation in terms of string states, as discussed below.

It is remarkable that the correlations between smeared wave-packets between the in- and out-sheets are perfectly well defined, while the point-like propagators are not¹⁴. This indicates that the Bogoliubov transformation relating the in- and out vacua on the two sheets strongly modifies the UV structure of the modes, which is also manifest in (125). The physical significance of this observation is not clear, and deserves further investigations.

6.2. Further remarks

In the noncommutative or matrix setting, the above calculation goes through for the free theory, because the spectrum of \square coincides with the commutative case, and the eigenmodes are in one-to-one correspondence via \mathcal{Q} . In the presence of interactions, only the IR modes behave as in the commutative theory, while the UV sector is better described by non-local string modes $|x\rangle\langle y|$ [21, 22]; these also provide a geometrical understanding of the spectrum of \square . In noncommutative field theory, such non-local string modes span the extreme UV sector of the theory with eigenvalues $\square \sim |x - y|^2 + \Lambda_{\text{NC}}^2$ far above the scale of noncommutativity Λ_{NC} , and they are responsible for UV/IR mixing.

Due to the 2-sheeted structure of the present $\mathcal{M}^{1,1}$ brane, there are in particular string modes of the structure

$$|x\rangle_+ \langle y|_- \in \text{End}(\mathcal{H}_n) \tag{136}$$

which connect the pre-BB and post-BB sheets; here $|x\rangle_+$ is a coherent state on the upper (post-BB) sheet and $|y\rangle_-$ is a coherent state on the lower (pre-BB) sheet. From the point of view of either sheet, they behave like point-like objects which are charged under $U(1)$. In particular, the antipodal points on the opposite sheets of $\mathcal{M}^{1,1}$ coincide in target space, so that the corresponding string modes have only ‘intermediate’ energy of the order Λ_{NC} . These modes appear to be responsible for the observed correlation for $\eta + \eta' \approx 0$, which are non-local from the intrinsic brane point of view, but local in target space. A similar phenomenon can be seen for the squashed fuzzy sphere, see [23].

Although the string states are typically UV states, they are important in the loops, and mediate long-distance interactions [21]. In particular, the inter-brane string states connecting the two branes will lead to gravity-like interactions between the pre-BB and post-BB branes at one loop. This effect is on top of the correlations observed in the previous section, which arise in the free theory. The same effects will apply in the more realistic 3+1-dimensional cosmological solution [5]. It is therefore conceivable that physically significant correlations and interactions exist between the pre-BB and post-BB branes. Such effects would be very intriguing, but they arise only for the specific embedding structure of the coincident branes in target space under consideration.

Finally, there is a subtlety in the signature of the effective metric, which is somewhat hidden in our analysis. The effective metric on noncommutative branes in Yang–Mills matrix models has the structure $G^{\mu\nu} = \theta^{\mu\mu'} \theta^{\nu\nu'} \eta_{\mu'\nu'}$ [4], which is closely related to the open string metric [17]. In the presence of time-like noncommutativity, the anti-symmetric structure of the Poisson tensor $\theta^{\mu\mu'}$ implies a flip of the causality structure, which in 1+1 dimensions amounts

¹⁴ Note that the same result holds also in the matrix case, since the free theory in the commutative and matrix framework are identical, related by a Weyl-type quantization map. However due to UV/IR mixing or the uncertainty relation on NC spaces, $k \rightarrow \infty$ inevitably entails $s \rightarrow 0$. Therefore the UV divergence in k would disappear e.g. on compact space-times, such as a cyclic cosmologies. The ramifications in the presence of interactions are unclear.

to a flip of the space- and time-like directions. In the scalar field theory under consideration, this can be accommodated simply by an appropriate choice of overall sign. This phenomenon disappears on the covariant quantum space-times discussed in [5, 24], which have a very similar 3+1-dimensional structure as the present background. Since the $i\varepsilon$ regularization of the matrix model is independent of the background, the conclusions of the present paper can be extended straightforwardly to these 3+1-dimensional backgrounds [20].

7. Conclusion

In this paper, we have demonstrated some new and remarkable features of field theory on Lorentzian noncommutative space-time in matrix models. In particular, we have shown that a suitable regularization of the Lorentzian (oscillatory) matrix path integral leads to the usual $i\varepsilon$ prescription for the emergent local quantum field theory, even on a curved background. We obtained the propagator on a non-trivial 1+1-dimensional FLRW-type background by computing the ‘matrix’ path integral (8), which is seen to reduce locally to the standard Feynman propagator.

This result demonstrates that the framework of Yang–Mills matrix models, including notably the IKKT model, can indeed give rise to a physically meaningful time evolution, even though there is no *a priori* time in the matrix model. This should be contrasted to models of matrix quantum mechanics such as the BFSS model [1, 25], which are defined in terms of an *a priori* notion of time. Even though we consider only a simple, free toy model in 1+1 dimensions, the result clearly extends to the interacting case. However then UV/IR mixing arises due to non-local string states, so that a sufficiently local theory should be expected only for the maximally supersymmetric IKKT model.

From a physics perspective, perhaps the most interesting conclusion is that the modes and the propagator naturally extend across the BB. It is therefore possible to study questions such as the propagation of physical modes across the BB, in a well-defined framework of quantum geometry provided by the matrix model. For the particular space-time solution under consideration, we also observe an intriguing correlation between the pre-BB and post-BB physics, which is attributed to the coincidence of the pre- and post-BB sheets in target space. All these results generalize to an analogous 3+1-dimensional solution [20]. However, we leave a more detailed investigation of these and other physical aspects to future work.

Data availability statement

No new data were created or analysed in this study.

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Appendix

Unitary representations of $\mathfrak{so}(2, 1)$

Unitary representations are characterized by Hermitian generators

$$K_a^\dagger = K_a, \quad a = 0, 1, 2. \tag{137}$$

acting on the weight basis as follows

$$\begin{aligned} K_3 |j, m\rangle &= m |j, m\rangle, \\ K_+ |j, m\rangle &= a_{m+1} |j, m+1\rangle, \\ K_- |j, m\rangle &= a_m |j, m-1\rangle, \end{aligned} \tag{138}$$

where

$$a_m = \sqrt{m(m-1) - j(j-1)}. \tag{139}$$

There are different classes of unitary irreps of $\mathfrak{so}(2, 1)$ (see e.g. [26]):

- The discrete series

$$\begin{aligned} D_j^+, \quad j \in \mathbb{N}_{>0}: \quad \mathcal{H}_j &= \{|j, m\rangle; m = j, j+1, \dots; m \in \mathbb{N}\}, \\ D_j^-, \quad j \in \mathbb{N}_{>0}: \quad \mathcal{H}_j &= \{|j, m\rangle; m = -j, -j-1, \dots; -m \in \mathbb{N}\}, \end{aligned} \tag{140}$$

characterized by a Casimir $C^{(2)} = j(j-1) \geq 0$. These are either lowest or highest weight irreps, which correspond to square-integrable positive or negative energy wavefunctions on AdS^2 . The states which span the positive energy (lowest weight) irreps are obtained by acting with K^+ on the lowest-weight state, and conversely for the negative energy (highest weight) irreps.

- The principal continuous series

$$P_s, \quad s \in \mathbb{R}, \quad 0 < s < \infty, \quad j = \frac{1}{2} + is: \quad \mathcal{H}_j = \{|j, m\rangle; m = 0, \pm 1, \dots; m \in \mathbb{Z}\}, \tag{141}$$

is labeled by a real number s and has $C^{(2)} = -(s^2 + \frac{1}{4}) < -1/4$. These correspond to wavefunctions on the hyperboloid H^2 , which is the space of interest this paper. Note that the eigenvalues of K^3 are real, but j is complex.

- The complementary series

$$P_j^c, \quad 1/2 < j < 1, \quad j \in \mathbb{R}: \quad \mathcal{H}_j = \{|j, m\rangle; m = 0, \pm 1, \dots; m \in \mathbb{Z}\} \tag{142}$$

with $-1/4 < C^{(2)} < 0$.

Normalization of the Legendre functions

In [27], the following formula was given

$$\begin{aligned}
 I(s, s') &= \int_{-1}^1 \mathbf{P}_\nu^{is}(t) \mathbf{P}_\nu^{is'}(t) \frac{1}{1-t^2} dt \\
 &= -\frac{2\Gamma(is)\Gamma(-is)}{\Gamma(1+\nu-is)\Gamma(-\nu-is)} \sin(\pi\nu) \delta(s-s') \\
 &\quad + \left(\frac{\pi}{\Gamma(1-is)\Gamma(1+is)} + \frac{\sin^2(\pi\nu)\Gamma(is)\Gamma(-is)}{\pi} \right. \\
 &\quad \left. + \frac{\pi\Gamma(is)\Gamma(-is)}{\Gamma(1+\nu-is)\Gamma(-\nu-is)\Gamma(1+\nu+is)\Gamma(-\nu+is)} \right) \delta(s+s') \\
 &= -\frac{2\pi}{\Gamma(1+\nu-is)\Gamma(-\nu-is)} \frac{\sin(\pi\nu)}{s \sinh(\pi s)} \delta(s-s') \\
 &\quad + 2 \left(\frac{\sinh(\pi s)}{s} + \frac{\sin^2(\pi\nu)}{s \sinh(\pi s)} \right) \delta(s+s') \tag{143}
 \end{aligned}$$

using the standard identities

$$\begin{aligned}
 \Gamma(is)\Gamma(-is) &= \frac{\pi}{s \sinh(\pi s)} \\
 \Gamma(1+is)\Gamma(1-is) &= \frac{\pi s}{\sinh(\pi s)} \\
 \Gamma(z)\Gamma(1-z) &= \frac{\pi}{\sin(\pi z)} \tag{144}
 \end{aligned}$$

and

$$\sin(\pi(-\nu-is)) \sin(\pi(-\nu+is)) = \sin^2(\pi\nu) + \sinh^2(\pi s) . \tag{145}$$

For $\nu = -\frac{1}{2} + i|k|$, this is

$$\begin{aligned}
 I(s, s') &= \frac{2\pi}{\Gamma(\frac{1}{2} + i(|k| - s))\Gamma(\frac{1}{2} - i(|k| + s))} \frac{\cosh(\pi k)}{s \sinh(\pi s)} \delta(s-s') \\
 &\quad + 2 \left(\frac{\sinh(\pi s)}{s} + \frac{\cosh^2(\pi k)}{s \sinh(\pi s)} \right) \delta(s+s') \tag{146}
 \end{aligned}$$

noting that

$$\sin^2 \left(\pi \left(-\frac{1}{2} + i|k| \right) \right) = \cosh^2(\pi k) . \tag{147}$$

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