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Letter

Integral Involving Bessel Functions Arising in Propagation Phenomena

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Abstract: An new integral identity involving the product of two modified Bessel functions K_0 and K_1, previously unreported, is presented and proved.

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1. Introduction

Bessel functions are ubiquitous in mathematical physics and engineering. Yet they continue to surprise us with amazing new properties that, to the best of our knowledge, seem not to have been recorded anywhere, at least not in classic literature on the subject [1–3]. In this paper we state and prove one such property. We also speculate on a possible application of this new property to an eventual theory of quantum gravity.

Concerning our use of Bessel functions, we follow the notations and conventions of ref. [2].

2. Statement and Proof of Theorem

Theorem 1. *The following identity holds:*

$$K_0\left(\sqrt{x^2+y^2}\right) = \frac{|y|}{\pi}\int_{-\infty}^{\infty} K_0\left(|x-\xi|\right)\frac{K_1\left(\sqrt{y^2+\xi^2}\right)}{\sqrt{y^2+\xi^2}}\mathrm{d}\xi, \tag{1}$$

where $x, y \in \mathbb{R}$ *with* $y \neq 0$.

Proof. Let us consider the contour integral in the complex ξ-plane

$$f_C(x,y) := \frac{|y|}{\pi}\int_C K_0\left(|x-\xi|\right)\frac{K_1\left(\sqrt{y^2+\xi^2}\right)}{\sqrt{y^2+\xi^2}}\mathrm{d}\xi \tag{2}$$

along a contour C to be specified presently. For a fixed x, the change of variables

$$\xi := z + x \tag{3}$$

transforms (2) into

$$f_C(x,y) = \frac{|y|}{\pi}\int_C K_0\left(|z|\right)\frac{K_1\left(\sqrt{y^2+(z+x)^2}\right)}{\sqrt{y^2+(z+x)^2}}\mathrm{d}z. \tag{4}$$

To begin with, we assume $y > 0$. Let the contour C extend along the straight segment from $-x + \mathrm{i}y$ to $x + \mathrm{i}y$, while closing on the upper half-plane along a semicircle centred at the point $(0, y)$ with radius R. By Cauchy's theorem

$$f_C(x,y) = 2\pi \mathrm{i} \sum \mathrm{Res} \left\{ K_0\left(|z|\right) \frac{K_1\left(\sqrt{y^2 + (z+x)^2}\right)}{\sqrt{y^2 + (z+x)^2}} \right\} \tag{5}$$

where the sum extends over all the residues enclosed by C.

We will need the asymptotics of $K_0(z)$ and $K_1(z)$ on the complex plane. As $z \to 0$ we have

$$K_0(z) = -\ln z + \ldots, \qquad K_1(z) = \frac{1}{z} + \ldots \tag{6}$$

while for $z \to \infty$

$$K_0(z) = \mathrm{e}^{-z}\sqrt{\frac{\pi}{2z}} + \ldots, \quad K_1(z) = \mathrm{e}^{-z}\sqrt{\frac{\pi}{2z}} + \ldots, \quad |\arg(z)| < \pi. \tag{7}$$

Away from the negative real axis in the complex plane, the functions K_0 and K_1 are analytic. In Equation (7), $K_0(|z|)$ and $K_1(\sqrt{y^2 + (z+x)^2})$ both vanish as $R \to \infty$. Then the sum over residues in Equation (5) reduces to the single residue located at

$$z_0 = -x + \mathrm{i}y. \tag{8}$$

In Equation (6), in a neighbourhood of z_0 the function on the right-hand side of (5) behaves as

$$K_0(|z|) \frac{K_1\left(\sqrt{y^2 + (z+x)^2}\right)}{\sqrt{y^2 + (z+x)^2}} \simeq \frac{K_0(|z|)}{y^2 + (z+x)^2} = \frac{K_0(|z|)}{(z + x + \mathrm{i}y)\,(z + x - \mathrm{i}y)}. \tag{9}$$

Thus

$$\sum \mathrm{Res} \left\{ K_0\left(|z|\right) \frac{K_1\left(\sqrt{y^2 + (z+x)^2}\right)}{\sqrt{y^2 + (z+x)^2}} \right\} = \frac{K_0(|z_0|)}{2\mathrm{i}y}. \tag{10}$$

Altogether

$$\lim_{R\to\infty} f_C(x,y) = \frac{y}{\pi} 2\pi \mathrm{i} \frac{K_0(|z_0|)}{2\mathrm{i}y} = K_0(|z_0|), \tag{11}$$

which proves the statement made in Equation (1) when $y > 0$. An analogous reasoning holds when $y < 0$, the contour C now being closed by a semicircle in the lower half plane and the residue being located at $z_0 = -x - \mathrm{i}y$, which introduces an extra minus sign. This completes the proof. □

3. Quantum Gravity?

The authors' own interest in the subject arose out of contact [4–7] with problems in the theory of heat conduction [8] and quantum field theory [9]. Without aspirations to completeness (the subject is too vast to summarise) let us briefly mention one such occurrence of Bessel functions.

In the quantum field theory of a scalar field with mass m in Euclidean spacetime $\mathbb{R}^d$ [9] one is interested in the Feynman propagator G_d. The latter is a Green function for the Klein-Gordon operator on $\mathbb{R}^d$ satisfying a specific set of boundary conditions. With $\mathbf{x}$ denoting a vector in $\mathbb{R}^d$ and $||\mathbf{x}|| = \sqrt{\mathbf{x}^2}$ its Euclidean norm, it turns out that

$$G_d(\mathbf{x}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\exp(\mathrm{i}\mathbf{p}\mathbf{x})}{\mathbf{p}^2 + m^2}\, \mathrm{d}^d\mathbf{p} = \frac{m^{d-2}}{(2\pi)^{d/2}} \left(m||\mathbf{x}||\right)^{1-d/2} K_{1-d/2}\left(m||\mathbf{x}||\right). \tag{12}$$

We see that the propagator G_d depends on the vector argument $\mathbf{x} \in \mathbb{R}^d$ only through the norm $||\mathbf{x}||$. The latter is a real number that we may denote by x. Further setting $m = 1$ for convenience, we simplify Equation (12) to a function of a single real variable $x := ||\mathbf{x}||$:

$$G_d(x) = \frac{1}{(2\pi)^{d/2}} x^{1-d/2} K_{1-d/2}(x). \tag{13}$$

The Bessel functions $K_{1-d/2}$ of lowest integer order arise when $d = 2$ and $d = 4$, respectively. When $d = 2$ we have that the Feynman propagator $G_2(x)$ is proportional to the function $K_0(x)$:

$$G_2(x) = \frac{1}{2\pi} K_0(x). \tag{14}$$

When $d = 4$, the identity $K_{-1}(x) = K_1(x)$ yields

$$G_4(x) = \frac{1}{(2\pi)^2} \frac{K_1(x)}{x}. \tag{15}$$

The new identity (1) is thus reminiscent of the self-reproducing property of integral kernels (such as the propagators G_2 and G_4), yet it is definitely different, because it involves propagation in *different* dimensions. In fact we speculate that it might be related to propagation phenomena in an (eventual) theory of quantum gravity. In order to see how this might come about, let us recall the following facts [10,11].

The propagators (14) and (15) have been obtained under the assumption that spacetime remains a continuum all the way down to the shortest classical length, namely zero length. Now there are good theoretical reasons to believe that the continuous nature of spacetime breaks down when one reaches lengths of the order of the Planck scale $L_P = \sqrt{\hbar G/c^3}$. Here $\hbar$ is Planck's quantum of action, G is Newton's gravitational constant, and c is the speed of light. The numerical value of L_P is roughly 10^{-33} cm, well beyond what current technology can probe. All this notwithstanding, it has been convincingly argued [10,11] that an effective, or low-energy, manifestation of quantum gravity effects on the propagator functions $G_d(x)$ can be captured by the simple replacement

$$x = \sqrt{x^2} \longrightarrow \sqrt{x^2 + L_P^2} \tag{16}$$

within the argument of G_d. In other words, a true propagator that encodes overall quantum-gravity effects will no longer be given by $G_d(x) = G_d(\sqrt{x^2})$, but instead by $G_d(\sqrt{x^2 + L_P^2})$. Thus even after setting $x = 0$, spacetime points will continue to be separated by a nonzero length L_P. Among other desirable properties, this has the welcome effect that the singularities present in the propagators (14) and (15) as $x \to 0$ are smoothed out. For the sake of simplicity let us assume choosing units such that $L_P = 1$. Then the *quantum-gravity corrected propagator functions* (14) and (15) become

$$G_2^{(\mathrm{QG})}(x) = \frac{1}{2\pi} K_0\left(\sqrt{x^2+1}\right), \qquad G_4^{(\mathrm{QG})}(x) = \frac{1}{(2\pi)^2} \frac{K_1\left(\sqrt{x^2+1}\right)}{\sqrt{x^2+1}}. \tag{17}$$

The above corrected propagators duly reduce to their counterparts (14) and (15) for large x, while no longer exhibiting a singular behaviour at the origin. This correction is denoted by the superindex QG, standing for *quantum gravity*. As a rule one generally expects quantum effects to mollify (at least some of) the singular behaviour that classical theories may exhibit.

Altogether, substituting Equations (14) and (17) into Equation (1) and setting $y = 1 = L_P$ in the latter, the identity we have just proved reads

$$G_2^{(\mathrm{QG})}(x) = 4\pi \int_{-\infty}^{\infty} G_2\left(|\xi - x|\right) G_4^{(\mathrm{QG})}(\xi) \mathrm{d}\xi \tag{18}$$

in terms of scalar propagation functions in Euclidean space. Although reminiscent of the self-reproducing property of Feynman propagators, it is intriguing for several reasons. First, it involves different dimensions (2 and 4). Second, it links propagators in the *presence* of quantum-gravity effects to propagators in the *absence* of quantum-gravity effects. Last but not least, it requires that y be nonvanishing, so it suggests setting $y = L_P$—a key feature of quantum-gravity theories is the existence of a quantum of length.

One possible reading of Equation (18) is the following: starting from the 2-dimensional, quantum-gravity free propagator G_2, it suffices to compute its convolution with the 4-dimensional, quantum-gravity corrected propagator $G_4^{(QG)}$, in order to obtain the quantum-gravity corrected propagator $G_2^{(QG)}$ in two dimensions. We hope to report on this issue in the future [12].

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