The Stochastic Basis of The Lund Model

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Abstract

I exhibit The Lund Model Fragmentation process as a process, similar to Brownian motion along the typical space-time and energy momentum space breakup hyperbola and derive a general process for transverse momentum generation (transverse wrt to the force-field-string direction) for the final state hadrons. I show that the equation for the x-curve, which generalises the typical hyperbola to the case of multi-parton fragmentation situations are the same as the equations which governs the transverse momentum generating process (the Ornstein-Uhlenbeck process for the velocity ditribution of a particle undergoing Brownian motion) besides the fact that those wee and collinear gluons which do not affect the longitudinal process instead occur as a random gaussian noice in the transverse momentum process.

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1 Introduction

The Lund Model contains at the basis a stochastical process for multi-particle production of a Markovian character. Given a set of partonic excitations, ie color-connected charges $q = 3$, $\bar{q} = \bar{3}$ and $g = 8$, stemming from a fundamental interaction and the ensuing bremsstrahlung, there is a unique state of the massless relativistic string that can be constructed.

The process corresponds to the breakup of this string state and works on the corresponding space-time (or equivalently the energy-momentum space) history, [1], [2], [3]. It is derived, [2], by semi-classical probability concepts but the final result can be given several analagous interpretations inside quantum field theory and statistical mechanics. I will in this talk also describe some recent developments and in particular derive a corresponding process for transverse (wrt the above-mentioned "longitudinal" direction along the string) momentum generation. This introduces a correlation length in the model which can be related to the properties of the longitudinal process.

The transverse momentum process is actually equivalent to a well-known stochastical process, the Ornstein-Uhlenbeck process for the velocity distributions in time of a particle undergoing Brownian motion. I will show that the "time"-variable in the process can be identified with the generalised rapidity variable, λ, which is introduced in Ref [4] and further considered in detail in Refs [5], [6], [7]. The size of the correlation length means that it is the "small" pieces of the breakup string that will be affected, ie in particular the final state pions with their small mass. This has a direct application to the phenomenological understanding of the present experimental results.

Eddi De Wolf, [8], has at this meeting brought up the problem with the small mass part of the two-particle correlation function. The present Lund Model exaggerates the mass-bins around the ρ-peak but the predictions go below the experimental data for smaller two-particle masses. We feel that this is firstly due to the model production of too many direct ρ's. But secondly, if we suppress the ρ's, then to conserve the total probability we will enhance not only direct pions but also the η- and η'-signals. And then we get back the problem at the ρ-peak because the η' has a large branching ratio to ρ!

The way to solve all the problems is to suppress both the direct ρ and η' (cf also Ref [9]), thereby enhancing the direct pions. At the same time it is necessary to endow these direct pions with a generally smaller transverse momentum. In that way they will populate the mass region in the two-particle correlation between $\sim 300 - 700\ MeV$, bringing the predictions close to the data. (Although I am not a friend of "intermittency" in multi-particle production I should mention that The Lund Model in this way will be even closer to these data). This is achieved by means of the above-described correlated transverse momentum generation as we will show in a forthcoming publication, [10], where we also provide some possible mechanisms for the necessary (ρ, η')-suppression.

I start with an introduction to The Lund Model longitudinal process (hoping that you have some previous knowledge because it must due to space limitations be very brief). After that I consider the mean behaviour of a general string state breakup, introducing both the generalised rapidity variable λ mentioned above and the x-curve, which can be directly constructed from a knowledge of the underlying partonic states, [4],[5]. (At this point we assume that you have by means of your favorite partonic cascade mechanism, eg the Webber-Marchesini HERWIG, [11], Sjöstrand's JETSET, [12], or Lönnblad's ARIADNE, [13], Monte Carlo simulations, produced an ensemble of such states).

The x-curve has an everywhere time-like tangent. If we partition the curve into pieces, each with the length corresponding to the final state hadron masses, then we obtain the average distribution in the energy momenta of the final state particles for the particular state. When we average over the ensemble of states this procedure provides all the single particle inclusive features of The Lund Model

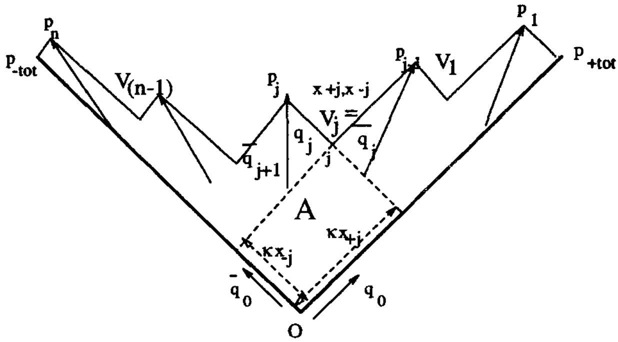

Figure 1: The breakup of a Lund String into an n-particle state with energy momenta p_j. The vertex V_j divides the event into "rightmovers" along the positive lightcone with total energy momentum $(p_{+tot} - \kappa x_{+j}, \kappa x_{-j})$ and "leftmovers" with $(\kappa x_{+j}, p_{-tot} - \kappa x_{-j})$.

fragmentation process. This is The Lund Model correspondence to the Local Parton-Hadron Duality concept, introduced by Dokshitzer and Troyan, [14]. They use it to calculate in analytical form the inclusive hadronic distributions from the partonic ones [15].

After that we go over to a more detailed description of the transverse momentum generation process in order to show (at some point) the general considerations behind The Lund Model formulas. All the details can be found in the papers cited before and in a forthcoming book (this is a shameless commercial!), [16].

2 The Longitudinal Breakup Process in The Lund Model

2.1 Preliminaries

The general formula for the probability to obtain the set $\{p_j\}$ of energy-momentum vectors for particles with masses $\{m_j\}$, $j = 1, \ldots, n$ from a $(q\bar{q})$-string with total energy momentum p_{tot} ($p_{tot}^2 = s$) is

$$dP = dPh_n|\mathcal{M}|^2, |\mathcal{M}|^2 = \exp(-bA) \tag{1}$$

$$dPh_n = \prod_{j=1}^{n} N dp_j \delta(p_j^2 - m_j^2)\delta(\sum_{j=1}^{n} p_j - p_{tot})$$

Here dPh_n is the n-particle phase space and A is the area spanned by the string before the decay (cf Fig 1) and b and N are parameters. The formula is introduced in a somewhat fancy way in order to make you remember Fermi's Golden Rule to calculate cross sections by multiplying the square of the transition matrix element with the available number of final states.

Before we consider this analogy we note that, [1], [2]

I The string breakup corresponds to the production at every "vertex" (called V_j in Fig 1) of a pair of $(q_j\bar{q}_j)$. After production the pair is dragged apart by the string force field tension ($\kappa \simeq 1$

GeV/fm) in opposite directions. The color force fields, modelled by the string, end on these charges (*confinement*, note that in QED a produced (e^+e^-)-pair continue to interact even if they are dragged apart by external forces).

II The final state mesons stem from a q_j and a $\bar{q}_{j+1}$ from adjacent vertices. In order that the hadron should have a positive mass the space-time difference between the vertices must be space-like. Therefore time-ordering in the process has no meaning, all vertices occur in a relativistically invariant and causal theory "at the same time". There is no "preferred, first" vertex. Each vertex divides the process into the hadrons produced "to the right" and "to the left" of it.

III A convenient way to order the process is given by "rank-ordering", defined so that the first rank particle (along the positive light cone) contains the original q_0 and the $\bar{q}_1$ from the "first" vertex along the lightcone, the second rank $(q_1, \bar{q}_2)$ etc. Assuming that the q and $\bar{q}$ from the same vertex together have the quantum numbers of the vacuum, ie are particle and anti-particle, rank evidently corresponds to flavor connection.

We may just as well order the process along the negative lightcone, $(\bar{q}_0, q_{n-1})$, $(\bar{q}_{n-1}, q_{n-2})$ etc. *The Lund Model result in Eq (1) stems from the requirement that the stochastic process should give the same result in both cases.* Actually the result is even more general, you may start on any vertex and "peel off" your final state hadrons one by one in any order to the left and to the right and still obtain the same formula. This is tantamount to *local gauge-invariance*, cf Ref [16].

IV The factors of the formula in Eq (1) act in opposite ways. To obtain a large phase space one should produce many particles but in order to obtain a small area suppression from the "squared matrix element" one should produce few. This means that Nature on the average picks a compromise and the vertices tend to be produced along a typical hyperbola in space-time or energy momentum space, cf below.

The $(q\bar{q})$-pairs exhibited in Fig 1 are massless, which means that the process can be envisioned as (semi-)classical, ie conserving energy momentum at every point. A massive pair (mass μ) needs a region $\delta l \simeq 2\mu/\kappa$ to be produced. If the pair is endowed with transverse momentum $\pm\vec{p}_\perp$ then it needs a string region $2\mu_\perp/\kappa$ with $\mu_\perp = \sqrt{\mu^2 + p_\perp^2}$. In that case one can take recourse to tunneling in a constant force-field. This was calculated by Heisenberg and Euler, [17] in the 30's, by Schwinger, [18], in the 50's and by a host of authors in the 70's and 80's.

It is a nice example of The Law of The Conservation of The Useful Dynamics which says that every new generation of theorists reinvent, reuse and usually rename all the useful dynamics of the earlier generation. There are few situations of dynamical interest, which allow a result in terms of closed expressions in elementary functions. The primary tunneling probability is given by

$$\simeq d^2 p_\perp \exp - \left(\frac{\pi \mu_\perp^2}{\kappa} \right) \tag{2}$$

It is useful to note that Heisenberg's undeterminancy relation would give the probability for a quantum fluctuation containing a pair of mass $\mu_\perp$ emerging at the space-like distance $2\mu_\perp/\kappa$

$$|\Delta_F(2\mu_\perp, \mu_\perp)|^2 \sim \exp - \left(\frac{4\mu_\perp^2}{\kappa} \right) \tag{3}$$

with Δ_F the Feynman propagator. We have used a simple approximation for the ensuing Bessel-function in the second line. Thus the "push apart" of the pair by the force-field changes $4 \rightarrow \pi$, making it a little bit "easier", ie less suppressed.

2.2 The Dynamical Analogues of The Fragmentation Formulas

It is possible to make a case for the matrix element interpretation of Eq (1) in several ways.

M1 In terms of Schwinger's formula, [18], cf also [20] and [21], for the break-down of the vacuum in a constant (external) electric field. One obtains for a field with longitudinal energy density κ spanned over the space-time region V during the time T

$$\propto \exp(-VT\kappa^2\Xi) \tag{4}$$

Writing $V = LA_\perp$ we may identify the area A along the string space-time history with $LT \equiv A$. Then we obtain a formula for the parameter b in the Eq (1) (note that usually the Lund formulas are written with $A \to \kappa^2 A$ implying $b \to b/\kappa^2 \equiv b'$)

$$b = \kappa^2 \Xi A_\perp \tag{5}$$

Here Ξ a number calculable from a knowledge of the quanta coupled to the field and $A_\perp$ is the (space) transverse size of the string force field.

M2 In terms, [19], of Wilson's loop operators necessary to ensure gauge invariance in an expression when the final state hadrons stem from a q and a $\bar{q}$ from different vertices. There must be a gauge-connector $\exp(ig \int A_\mu dx^\mu)$ with the integral along the field between the vertices in terms of the coupling constant g and the gauge field A_μ. For the whole state that means that there must be a total exponential loop integral $\exp(ig \oint A_\mu dx^\mu)$ around the field region.

Wilson gave as a criterium for confinement that the loop operators should behave as exponentials of the area over which the field exists. This has been confirmed by extensive simulations of confined field situations in the lattice approximation. One obtains the same expression for the parameter b from the Schwinger method and from Wilson's loop integrals

$$b = \kappa \frac{n_f \alpha_s}{12} \tag{6}$$

in terms of the number of massless flavors, n_f, and the QCD coupling constant α_s. It would take us too far to provide the proof (cf [16]). But it is interesting that by the Wilson method we also obtain a phase for the matrix element which has a direct relevance to the Hanbury-Brown-Twiss effect (or it is also called the Bose-Einstein effect), [19], in multi-particle production.

Before we consider another interpretation we note that one can also give an interpretation to the decay formulas inside statistical physics, [16]. In that way The Lund Model final state hadrons behave like an almost ideal gas in rapidity space (another "old" phenomenological friend, the Feynman-Wilson gas).

We end this section with a final analogy, this time between The Lund Model production formula and the perturbation theoretical multiperipheral graphs, which are at the basis of Gribov's Reggeon theory. This was firstly suggested by Artru, [22].

Consider the vertex $V_j = (x_{+j}, x_{-j})$ marked out in Fig (1). The particles produced "to the right" of that vertex $p_1, \ldots, p_{j-1}$ will evidently have a total energy momentum $\sum_1^{j-1} p_k = (p_{+tot} - \kappa x_{+j}, \kappa x_{-j})$ while those "to the left" $p_j, \ldots, p_n$ similarly have $(\kappa x_{+j}, p_{-tot} - \kappa x_{-j})$. The situation may then be described as in Fig 2a that the q_0 comes in with p_{+tot} and is transformed to the right-movers while the $\bar{q}_0$ similarly is transformed into the left-movers. The momentum transfer across is then given by $q_j = \kappa(x_{+j}, -x_{-j})$.

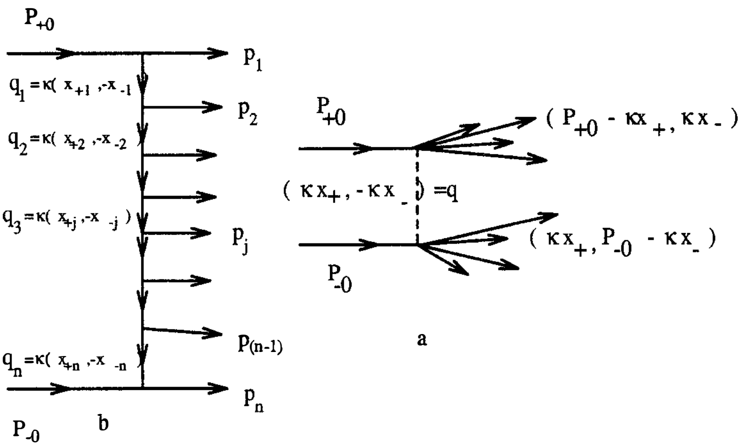

Figure 2: The dual description in terms of momentum transfer "links", corresponding to the vertex positions in space-time

In the same way the whole production can be subdivided down to the individual particles like in Fig 2b. We note the resemblance to the multiparticle production diagrams, which in Ref [23] are used to obtain Regge-behaviour through unitarity.

Thus in the production process *there is a dual relationship between the vertices V_j in space time and the "linking" momentum transfers q_j*. The fact that the vertices occurs along a hyperbola (the locus of constant proper time squares $x_{+j}x_{-j} \equiv t_j^2 - x_{lj}^2$ with the index l for longitudinal along the string) is mirrored by the property that the major contributions of the ladder graphs stem from limited values of the squared momentum transfers $-q_j^2 = \kappa^2 x_{+j}x_{-j}$. Similarly the short range correlations among the vertices are mirrored by the short range correlations among the momentum transfers along the chain. Finally a large rapidity gap means a large fluctuation down to small proper times in the production process, ie a behaviour again far away from the main contributions.

In a production model with these three properties it is straightforward to prove, [23], Regge behaviour for the elastic scattering amplitude $T_{AB} \equiv T(p_A, p_B; p_A + q, p_B - q)$ with q the momentum transfer. Thus the unitarity equations will be

$$Im(T_{AB}) = 1/2 \sum_N \int T_N(p_A, p_B; \{p_j\}) T_N^*(p_A + q, p_B - q; \{p_j\}) d\tau_N \tag{7}$$

The right hand side corresponds to a sum and phase space-($d\tau_N$)integral over all states, which can be reached from the initial and final (AB). This sum behaves as $s^{\alpha(q^2)}$ with $\alpha(q^2)$ a Regge trajery.

The correspondence to the right hand side of Eq (7) in The Lund Model is a sum and integral over the (un-normalised) probability in Eq (1). It should come as no surprise that one obtains the result

$$\sum \int dPh |\mathcal{M}|^2 \propto s^a \tag{8}$$

The parameter a is according to experiments very stable around $a \simeq 0.5$, also when one includes gluon radiation. The ρ-trajectory should in accordance with the flavor composition of the ρ be related to the the most common flavors. It should be more than a coincidence that $\alpha_\rho(0) \simeq 0.5 \simeq a$.

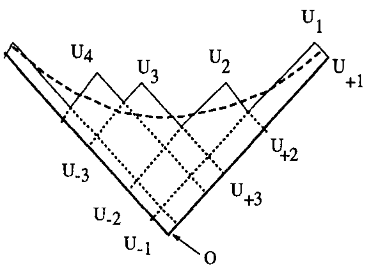

Figure 3: The Lund string breakup process with the typical hyperbola exhibited and with notations described in the text.

2.3 Extension to Multi-gluon States, the λ-measure and the x-curve

From the result in Eq (8) we may deduce the distribution in $-q^2 = \kappa^2 x_+ x_- \equiv \Gamma$ along the chain. With respect to the vertex V_j in Fig (1) the total result can be subdidvided into an integral over all the right-movers $p_1, \ldots, p_{j-1}$, another over all the left-movers with $p_j, \ldots, p_n$ and finally the remaining area-exponential of the region "below" the hatched lines. Thus fixing the value of the V_j-coordinates we obtain for the required distribution H

$$s^a H \propto s_r^a s_l^a \exp(-b'\Gamma)$$
$$s_r = (p_{+tot} - \kappa x_+)(\kappa x_-) \quad \text{and} \quad s_l = \kappa x_+ (p_{-tot} - \kappa x_-) \tag{9}$$

(we have for simplicity written $x_{\pm j} \equiv x_\pm$). For large values of (s_r, s_l) (which for practical phenomenology means a few GeV^2) we obtain that $\Gamma \simeq s_r s_l / s$. We then obtain the distribution H as

$$H(\Gamma) = C\Gamma^a \exp(-b'\Gamma) \tag{10}$$

which is The Lund Model result, [2].

We next reconsider the longitudinal process and *describe it as a process along the typical hyperbola* according to the distribution H. Consider Fig (3) where we again exhibit the decay of the string state in Fig (1) but this time with the typical hyperbola (parametrised as $U_+ U_- = B$) drawn out. We will now calculate the difference between the area A and the area corresponding to the typical hyperbola, to be called AB in Fig (3). It is possible to subdivide this region into contributions from the areas spanned between $U_j \equiv (U_{+j}, U_{-j})$ and O and into regions "between" the points U_j. We note the occurrence of regions both "above" and "below" the curve and also "overlapping regions".

We start to consider that part of AB, which is spanned by the region $(U_{+1}, U_{-1}) \to O$ and note that it has the size

$$U_{+1}U_{-1} - \int_{B/U_{-1}}^{U_{+1}} \frac{Bdx}{x} - U_{+1}\frac{B}{U_{+1}} = B(\rho_{11} - \log(\rho_{11}) - 1) \quad \text{with} \quad \rho_{11} = \frac{U_{+1}U_{-1}}{B} \tag{11}$$

The next part of AB is below the hyperbola and its size is

$$-(\frac{B}{U_{+2}} - U_{-1})U_{+2} + \int_{U_{+2}}^{B/U_{-1}} \frac{Bdx}{x} = B(\rho_{21} - \log(\rho_{21}) - 1) \equiv B(\mathcal{D}_{21}) \tag{12}$$

It is easy to continue along the same lines. For AB it is evident that the areas below the hyperbola should be subtracted just as those which are overlapping. These parts are always of the "mixed" kind $\mathcal{D}_{j+1j}$, while all the"straight" areas $\mathcal{D}_{jj}$ should be added. Defining a function $G(y)$ by

$$G(y) = \exp -bB(\exp(y) - y - 1) \quad \text{with} \quad y = \log(\rho) \tag{13}$$

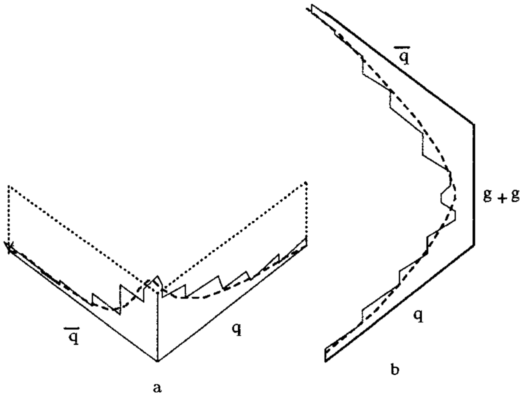

Figure 4: The space-time decay (a) of a $(qg\bar{q})$-state with the hadrons marked out in the lightcone segements between the qg and $g\bar{q}$ lightcone directions. The typical hyperbola is shown hatched. The same production process exhibited (b) with hyperbolas spanned along the directrix. Note that as the string force works as 2κ on the g, the space-time "lengths" of the g is half the directrix length (alternatively think about the gluon as "two-sided", with one side towards q and $qbar$,respectively.

the whole suppression of the area AB can be written as

$$\exp(-b(AB)) = \frac{G_{11}G_{22}G_{33}\cdots}{G_{21}G_{31}\cdots} \tag{14}$$

The variable y defined in Eq (13) is evidently a coordinate along the typical hyperbola and we have in this way obtained a representation of the Lund string fragmentation as a process on the hyperbola. Note that the function G is close to a Gaussian and therefore the process is also close to a Brownian motion in the coordinate. In case we identify $bB = a$ then the function $G(y) = H(\log(\Gamma))$.

We will next consider the generalisation of this result to the decay of an arbitrary string, obtained from a multi-parton state. We start with the next simplest case when there is a $qg\bar{q}$-state. According to The Lund Model prescription a g, ie a gluon color-connected to the q and $\bar{q}$ is an internal excitation in the string spanned between them.

Then there is one string segment spanned between (qg) and another spanned between the $(g\bar{q})$. Due to the motion of the partonic excitations (they are all mass-less and move out along different lightcones) the segments also move out in different directions during the string decay, [24]. The corresponding asymmetry in the final state, [25] is known as "the string effect". It was first observed by the JADE collaboration at PETRA and is nowadays, since the start of LEP, very well established.

We describe in Fig (4a) the decay in space-time of such a state. It is implemented according to Sjöstrand's model, [3], in the JETSET Monte Carlo as an extension to the string space-time history of the above-described longitudinal fragmentation model process. It is in that version not a completely unique process. But we have found no observable differences, as of now, of changing "the small details" in Sjöstrand's model.

In the region between the q-lightcone and the g-lightcone the decay looks just as "before" (in case one boosts to the rest-frame of this string segment it is easy to convince oneself that in this frame the q moves in one direction and the g in the oppsite along the string, [16]). The same goes for the region between the g and the $\bar{q}$. Thus in each such part, as shown in Fig (4a), the produced hadrons come out close to the typical hyperbolas (exhibited), spanned between the lightcones as before.

We note, however, that the length of the hyperbola for a state with total cms energy s is $\delta y \propto \log(s)$. (Therefore the mean multiplicity, correponding to a brekup along the hyperbola, is also $\propto \log(s)$ in The Lund Model). If the mass square in the segment (qg) is s_{12} and in $(g\bar{q})$ is s_{23} then the combined hyperbolas will have the length

$$\lambda = \log(s_{12}) + \log(s_{23}) = \log(s) + \log(k_\perp^2) \quad \text{with} \quad k_\perp^2 = \frac{s_{12}s_{23}}{s} \tag{15}$$

The variable $k_\perp^2$ corresponds to an invariant definition of the transverse momentum of the gluon in the QCD bremsstrahlung emission models. It is evidently a reasonable size of the "peak" sticking out along the gluon lightcone direction in the particle fragmentation process. The variable λ can be directly generalised to multi-gluon emission situations, [5]. It is also possible to introduce analytical calculation schemes for the distributions in λ, [5], [6] and [7], which are equivalent to calculating the multiplicities in the Leading Log and Modified Leading Log approximations of perturbative QCD, [15].

The variable λ as defined above is, however, not an infrared stable variable in its present shape. In order to mend this we will provide a different description, [4] of the final state particle distributions. To that end consider Fig (4b) where the distributions in Fig (4a) is shown in a different way. It is of course the same final state particles. But instead of showing them along the lightcone segments (qg) and $(g\bar{q})$ they are now shown as *a band stretched out along the directrix curve.*

This curve (which is unique for and compleley describes, [16], a particular string state) is built by placing the parton energy momentum vectors one after the other in color order. For an n-parton state (index 1 for the q and index n for the $\bar{q}$) it is defined as

$$\begin{aligned}
A(\xi) &= \xi\frac{k_1}{e_1} \quad &\text{if} \quad 0 \le \xi \le e_1 \\
A(\xi) &= \sum^{j-1} k_l + \frac{(\xi - \sum^{j-1} e_l)k_j}{e_j} \quad &\text{if} \quad \sum^{j-1} e_l \le \xi \le \sum^{j} e_l \\
A(\xi) &= -A(-\xi) \\
A(2E+\xi) &= A(\xi) + 2\sum^{n} k_l \quad &\text{with} \quad E = \sum^{n} e_l
\end{aligned} \tag{16}$$

with a parametrisation using the energies $\{e_j\}$ of the partons. The directrix (as well as the string state) is periodic with a period translation $2p_{tot}$ over the period $2E$. It has in every point always a light-like tangent.

The curve exhibited in Fig (4b), which corresponds to two hyperbolas spanned along the directrix parts in Eq (16) $0 \le \xi \le e_1 + e_2/2$ and $e_1 + e_2/2 \le \xi \le e_1 + e_2 + e_3$ respectively, can be generalised to the x-curve. The general equations are given by $(d\lambda = q dA/m_0^2)$

$$\frac{dx}{d\lambda} = q \quad \text{and} \quad \frac{dq}{d\lambda} = -q + \frac{dA}{d\lambda} \tag{17}$$

It is easy to show that the x-curve defined in this way has an everywhere time-like tangent q and (cf [4], [5] and [16]) the variable λ corresponds to the length along the x-curve from its starting point $\xi = 0$. The variables (x, q) looks just as the position and the momentum vector for a particle which on the one hand is dragged along by the directrix pull, on the other also feels a friction force proportional to its momentum.

The general idea is that, although the curve feels the directrix direction at every point, it is not sensitive to small and collinear changes (ie wee and collinear gluon emissions) due to the friction term

$-q$. There is a "resolution scale" m_0, which corresponds to the hyperbola parameter, ie it is a measure of how closely the x-curve approximates the directrix.

3 The Transverse Momentum Generating Process

3.1 Preliminaries

We will next derive a general transverse momentum generating process using methods of the same kind as we used in Ref [2] to exhibit the longitudinal breakup. We assume that there is a set of hadrons produced along the positive lightcone direction, each of them provided with the transverse momentum $\vec{k}_j$. After n steps the total transverse momentum is $\vec{p}$:

$$\sum_1^n \vec{k}_j = \vec{p} \tag{18}$$

We now take one more "step" and produce a hadron with $\vec{k}$ thereby reaching the next vertex with

$$\vec{p}' = \vec{k} + \vec{p} \tag{19}$$

We further assume

- After many steps the distribution in $\vec{p}$ "saturates" and becomes independent of the earlier steps, $f(\vec{p})d^2p$ and there is no preferred transverse direction.
- The transverse momentum production of the next hadron only depends upon the the value reached, $\vec{p}$, and the step, $\vec{k}$. We will in particular assume *that there is an anticorrelation* so that it depends upon the distribution $g(\vec{k}+\gamma\vec{p})d^2k$, with γ a positive number determined by the hadron produced.

We may evidently also consider the vector $\vec{p}'$ as the result of having produced a set of hadrons along the negative lightcone with transverse momenta $\vec{l}_j$ thereby reaching the point

$$\vec{p}' = -\sum \vec{l}_j \tag{20}$$

(the minus sign is necessary to conserve the total transverse momentum, cf Eq (19)). The probability to do that is then again given by the saturating distribution $f(\vec{p}')d^2p'$. The next step from $\vec{p}'$ to $\vec{p}$ with $\vec{p} = \vec{p}' - \vec{k}$, thereby producing the hadron with $\vec{k}$, is given by $g(-\vec{k} + \gamma\vec{p}')d^2k$.

3.2 The Resulting Distributions

If we equate the two probabilities to produce the hadron with the transverse momentum $\vec{k}$ we obtain (exchanging $\vec{k}$ for $\vec{p}' - \vec{p}$):

$$f(\vec{p})g(\vec{p}' + (\gamma - 1)\vec{p})d^2pd^2p' = f(\vec{p}')g(\vec{p} + (\gamma - 1)\vec{p}')d^2pd^2p' \tag{21}$$

After removing the differentials we may take logarithms of the functions ($\log(f) = F, \log(g) = G$) and then partial differentials of the result with respect to the same components of $(\vec{p}, \vec{p}')$. We note the similarity to the methods used in Ref [2] to derive the longitudinal fragmentation function! Again one set of functions vanishes, this time the f's.

We are then as in that case left with a (2nd order) differential equation (dots equal to derivatives):

$$(\gamma - 1)\ddot{G}(\vec{p}' + (\gamma - 1)\vec{p}) = (\gamma - 1)\ddot{G}(\vec{p} + (\gamma - 1)\vec{p}') \quad (22)$$

Therefore if $\gamma \neq 0, 1$ we conclude, due to the independence of $\vec{p}$ and $\vec{p}'$, that the two sides must be equal to the same constant, to be called $-\beta/\gamma(2-\gamma)$. We obtain directly the result

$$G(\vec{P}) = \log(N) - \frac{\beta \vec{P}^2}{\gamma(2-\gamma)} \rightarrow g(\vec{P}) = N \exp - \frac{\beta \vec{P}^2}{\gamma(2-\gamma)} \quad (23)$$

We have now invoked (euclidean) invariance, ie the assumption that there is no preferred direction. Therefore there is no linear term in the vectors.

The result for F (f) is obtained by noting that in case we introduce the result for G (g) in Eq (21) we may gather the contributions depending upon $\vec{p}$ and $\vec{p}'$ on each side. The two sides must therefore again be equal to the same constant, $\log(N')$:

$$F(\vec{P}) = \log(N') - \beta \vec{P}^2 \rightarrow f(\vec{P}) = N' \exp(-\beta \vec{P}^2) \quad (24)$$

The constants (N, N') are evidently normalisation constants, while (β, γ) are dynamical quantities.

The result can be written in different ways. One interesting way is to write it as a squared matrix element (defining $\gamma = 1 - \exp(-\tau)$)

$$f(\vec{p})g(\vec{p}' - \exp(-\tau)\vec{p}) = |\mathcal{M}|^2 \quad \text{with} \quad \mathcal{M} = <0|\vec{p}'> p_\tau(\vec{p}', \vec{p}) < \vec{p}|0> \quad (25)$$

We have then introduced the harmonic oscillator groundstates in the momentum representation:

$$<\vec{p}|0> = C \exp(-\frac{\beta \vec{p}^2}{4}) = <0|\vec{p}> \quad (26)$$

The symmetrical function $p_\tau(\vec{p}', \vec{p})$ is equal to the transition matrix from the state $\vec{p}$ to the state $\vec{p}'$, cf eg Ref [26] and Eq (36) below:

$$p_\tau(\vec{p}', \vec{p}) = <\vec{p}'| \exp(-H\tau)|\vec{p}> \quad (27)$$

with H the harmonic oscillator hamiltonian in terms of the canonically conjugate variables $(\vec{p}, \vec{b})$:

$$H = \frac{\sqrt{\beta}\vec{p}^2}{4} + \frac{\vec{b}^2}{\sqrt{\beta}} \quad (28)$$

This brings out the symmetry of the results between transverse momentum and impact parameter, $\vec{b}$. At the same time we note the similarity to Feynman's path-integral formulation of quantum mechanics, although this is in an "imaginary" time τ. The imaginary time formalism fits into a statistical physics scenarium and we will now show the close relationship between the results above and the velocity distributions of a particle undergoing Brownian motion.

It is nevertheless true that motion in spacelike directions (ie this time in impact parameter space) is formally equivalent to an imaginary time formalism because in this case the proper time $\tau \rightarrow i\sqrt{\vec{b}^2 - t^2}$.

3.3 The Relation to Brownian Motion, the Ornstein-Uhlenbeck Process

Another way to understand the results is to consider a seemingly different problem, [27] the motion of a Brownian particle, mass m, under the influence of friction (the friction force proportional to the velocity, $-m\sigma v$) and a *gaussian random force*, mR. This is governed by the Langevin equation:

$$\frac{dv}{dt} = -\sigma v + R \tag{29}$$

In this way v is obtained as *a stochastical variable* defined by R. We assume that there is an ensemble of states which are used to perform measurements on. The ensemble average of a measurement on the combination a will be denoted by brackets, $< a >$.

We may evidently write as a general solution for the Eq (29):

$$v(t) = v(t_0)\exp(-\sigma(t-t_0)) + \int_{t_0}^{t} dt' R(t')\exp(-\sigma(t-t')) \tag{30}$$

The gaussian randomness assumption on R means that it contains only "white" noice. It has always vanishing mean value and it contains no correlations in time with I_R a constant:

$$\begin{aligned} < R(t) > &= 0 \\ < R(t)R(t') > &= 2\pi I_R \delta(t-t') \end{aligned} \tag{31}$$

From this we conclude that the mean value and *the time correlation function* of v are given by

$$\begin{aligned} &< v(t) >=< v(t_0) > \exp(-\sigma(t-t_0) \\ &< v(t)v(t') >= \frac{\pi I_R}{\sigma}\exp(-\sigma|t-t'|) \end{aligned} \tag{32}$$

We have then assumed that the distribution is"thermalised", at the starting time t_0 so that according to the Maxwell velocity distribution

$$< v^2(t_0) >= \frac{\pi I_R}{\sigma} = \frac{kT}{m} \equiv \frac{1}{2\beta} \tag{33}$$

Thus there is an exponentially falling correlation between the value of v obtained at one time and at another. The correlation function only depends upon the time-difference and therefore it is referred to as a *stationary stochastic process.* The variance of $v(t)$ is therefore

$$< (v(t)- < v(t) >)^2 >= \frac{(1-\exp(-2\sigma(t-t_0))}{2\beta} \tag{34}$$

In case we define $u(t) = v(t)- < v(t) >$ then $u(t)$ considered as a stochastical variable will have the particular *gaussian property that all higher order correlation functions are determined by the two-point correlations:*

$$< \prod_{j=1}^{2n} u(t_n) >= \sum_{perm} \prod_{j\neq k} < u(t_j)u(t_k) > \tag{35}$$

ie only contain all possible two-point correlations in time (the notation "*perm*" means all possible permutations of $1,\ldots,2n$). The expectation values of an odd number of u's vanishes.

The transition probability $P(v_0,t_0|v,t)$ to go from the value v_0 at t_0 to v at t is then given by the gaussian distribution

$$\begin{aligned} P(v_0,t_0|v,t) &= \left(\frac{\beta}{\pi(1-\exp(-2\sigma\delta t)}\right)^{d/2}\exp(-\frac{\beta(v-v_0\exp(-\sigma(\delta t))^2}{(1-\exp(-2\sigma(\delta t)})) \\ \int d^d v_1 P(v_0,t_0|v_1,t_1)P(v_1,t_1|v_1,t_1) &= P(v_0,t_0|v,t) \qquad \text{if} t_0 \leq t_1 \leq t \end{aligned} \tag{36}$$

Here we have introduced the number of dimensions, d, in which the process goes on and written for the time difference $\delta t = (t - t_0)$.

It is no difficulty to recognize the distribution g derived before (with $\gamma = 1 - \exp(-\sigma\delta t) \equiv 1 - \exp(-\tau)$) in the transition probability for the change in velocity of a Brownian particle, in thermal equilibrium, under the influence of a gaussian white noice.

4 Conclusions

I have shown that

- the general behaviour of The Lund Model fragmentation process can be described as a process on the typical hyperbola (with properties similar to a gaussian process)
- the typical hyperbola can for a multi-parton fragmentation situation be generalised to the x-curve with the equations

$$\frac{dx}{d\lambda} = q \quad \text{and} \quad \frac{dq}{d\lambda} = -q + \frac{dA}{d\lambda} \tag{37}$$

  with λ the measure along the x-curve.
- the most general transverse momentum process, we can imagine, corresponds to the Ornstein-Uhlenbeck process which is governed by the Langevin equation

$$\frac{dx}{dt} = v \quad \text{and} \quad \frac{dv}{dt} = -\sigma v + R \tag{38}$$

  In case we identify the variable $v \to \vec{p}_\perp$ and the time variable $t \to \lambda$ the two sets of equations completely match and we may identify the gaussian noice as those wee and collinear gluons which should be there but is not noticable in the construction of the x-curve.

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