Journal of Physics: Conference Series **198** (2009) 012002 doi:10.1088/1742-6596/198/1/012002

# The Furry Picure

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**Abstract.** A short summary is given on the key points of the Furry picture. It describes quantum mechanical systems in external fields. It differs therefore from the quantum mechanical Schrödinger, Heisenberg and interaction representations.

## 1. Overview about the different quantum mechanical representations

### *1.1. Quantum mechanical description*

Any quantum mechanical state is given by its state vector $|\Psi>$. Usually these vectors are normalized $<\Psi|\Psi>=1$. The state vector is composed by its different eigenstates:

$$|\Psi>=\sum_i c_i|e_i> \tag{1}$$

Experimental observables are presented by Hermitian operators $O$

$$O|e_i>=\omega_i|e_i> \tag{2}$$

with the eigenvalues $\omega_i$. The set of all eigenstates build a complete orthonormal basis:

$$<e_i|e_i'>=\delta_{ii'}. \tag{3}$$

The expectation values of the operators correspond to experimental mean values of the observables and are given by:

$$\begin{aligned}<O>=<\Psi|O|\Psi> &= \sum_i\sum_j <\Psi|e_i><e_i|A|e_j><e_j|\Psi> && (4)\\ &= \sum_i\sum_j \omega_i <\Psi|e_i><e_i|e_j><e_j|\Psi> && (5)\\ &= \sum_i \omega_i|<e_i|\Psi>|^2 && (6)\end{aligned}$$

Two observables are simultaneously measurable if they commute:

$$[A,B]|\Psi>=0|\Psi>=A(B|\Psi>)-B(A|\Psi>). \tag{7}$$

The correspondence principle guarantees that the classical physics is obtained in the limit of large statistics. The quantum system is described via the same observables as in classical mechanics. However, these observables are functions of the space and momentum operators $X_i$, $P_j$ which correspond to the generalized conjugated coordinates $x_i$, $p_j$.

The momentum and space operators fulfill the Heisenberg commutation relations:

$$[x_i, x_j] = 0 \tag{8}$$
$$[p_i, p_j] = 0 \tag{9}$$
$$[x_i, p_j] = i\hbar\delta_{ij} \tag{10}$$
$$\tag{11}$$

The commutation relations eqn.(8)–(10) lead directly to the quantum mechanical uncertainty relation:

$$\Delta X \cdot \Delta P \leq \frac{\hbar}{2} \tag{12}$$

Space and momentum of a system are not simultaneously measurable.

There exist two presentations, the space– and the momentum representation. Measuring, for instance, the momentum spectrum of a quantum system and taking into account the uncertainty relation it is plausible that a continuous spectrum rather than a discrete spectrum can be expected. The Kronecker $\delta$ in the condition for orthonormal eigenstates, eq.3, can therefore be replaced by the $\delta$-function:

$$< \vec{p}|\vec{p'} >= \delta(\vec{p} - \vec{p'}). \tag{13}$$

The $\delta$-distribution is defined as

$$\int_{-\infty}^{+\infty} dx\delta(x - x_0)f(x) = f(x_0) \tag{14}$$

Applying a Fourier transformation, the $\delta$-function can be expressed:

$$\delta(x) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dp \exp^{-ipx/\hbar} \tag{15}$$

The summation over all possible eigenstates has to be replace by an integration over the full phase space:

$$\int d^3p|\vec{p} >< \vec{p}| = 1 \tag{16}$$

Analogously the spectrum of space operator can be regarded as being continuous.

The momentum and position operator act on the wave function in the momentum space:

$$X\Psi(\vec{p}) = i\hbar\nabla_{\vec{p}}\Psi(\vec{p}), \tag{17}$$
$$P\Psi(\vec{p}) = \vec{p}\Psi(\vec{p}). \tag{18}$$

The wave function in momentum space is the Fourier transform of the wave function in position space and vice versa:

$$\Psi(\vec{p}) = < \vec{p}|\Psi >= \frac{1}{\sqrt{2\pi\hbar}}\int d^3x \exp^{-i\vec{p}\vec{x}/\hbar}\Psi(\vec{x}) \tag{19}$$
$$\Psi(\vec{x}) = < \vec{x}|\Psi >= \frac{1}{\sqrt{2\pi\hbar}}\int d^3p \exp^{i\vec{p}\vec{x}/\hbar}\Psi(\vec{p}) \tag{20}$$

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### *1.2. Time-dependence in Quantum Mechanics*

There exist several possibilities to describe the time development of a quantum mechanical system. Analogous to the classical equations of motion (Newton, Lagrange or Hamilton), one applies a quantum mechanical equation of motions where momentum and position are replaced by their operators. But that requires that also the state vector and the properties of the quantum system on which the operators acts has to be taken into account when discussing the time development. Besides, time is not an observable of the system but just a parameter.

*1.2.1. Schrödinger picture* In the Schrödinger picture the complete time dependence is presented by the time development of the quantum state vector. The time development is given by an unitary time development operator,

$$|\Psi_S(t)>= U(t,t_0)|\Psi_S(t_0)>, \tag{21}$$

given by the time-dependent Schördinger equation,

$$\frac{d}{dt}|\Psi_S>= \frac{-i}{\hbar}\mathcal{H}|\Psi_S> . \tag{22}$$

The index 'S' denotes the Schrödinger picture. In case it is a closed system, i.e. $\mathcal{H} \neq \mathcal{H}(t)$, the unitary operator is given by

$$U(t,t_0) = \exp^{\frac{i}{\hbar}(t-t_0)\mathcal{H}} . \tag{23}$$

The observables and their eigensystem is constant in time. The mean value is constant if $|\Psi>$ is in an energy eigenstate, i.e. stationary state.

*1.2.2. Heisenberg picture* In the Heisenberg picture the complete time dependence is governed by the observables, i.e. the operators and their eigensystem:

$$|e_i,t> = U^\dagger(t)|e_i>, \tag{24}$$

$$O_H(t) = U^\dagger(t)O_S U(t), \tag{25}$$

index 'H' denoting the Heisenberg picture and corresponding to the equation of motion:

$$\frac{dO}{dt} = \frac{[O,\mathcal{H}]}{i\hbar} + \frac{\delta O}{\delta t}. \tag{26}$$

The state vector $|\Psi_H>$ is constant. The mean value is constant if observable $O$ is conserved quantity.

*1.2.3. Dirac picture ('Interaction picture')* The time dependence is split between the state vectors and the observables. The full Hamiltonian is given by

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{V}, \tag{27}$$

where $\mathcal{H}_0$ denotes the free Hamiltonian and $\mathcal{V}$ the contribution due to the interaction. The free Hamiltonian $\mathcal{H}_0$ is time-independent, therefore the time development operator is given by

$$U_0(t,t_0) = \exp^{-\frac{i}{\hbar}(t-t_0)\mathcal{H}_0} . \tag{28}$$

The eigenvalues and eigenstates of $\mathcal{H}_0$ are assumed to be known and given by the free system. Therefore the time dependence of the operators are fixed by $\mathcal{H}_0$,

$$i\hbar\frac{d}{dt}O_D = [O_D,\mathcal{H}_0] + \frac{\delta O}{\delta t}, \tag{29}$$

but time dependence of the states are influenced by the interaction Hamiltonian $\mathcal{V}$. In this picture it is more complicated to derive the time development operator for the states than in the Schrödinger picture. The measurements, i.e. the expectation values, however, have to be independent on the chosen presentation:

$$\begin{aligned} < O > (t) &= < \Psi_S(t)|O_S|\Psi_S(t) > = < \Psi_S(t_0)|U^\dagger O_S U|\Psi_S(t_0) > && (30)\\ &= < \Psi_S(t_0)|U^\dagger U_0 U_0^\dagger O_S U_0 U_0^\dagger U|\Psi_S(t_0) > && (31)\\ &= < \Psi_S(t_0)|U^\dagger U_0 O_D U_0^\dagger U|\Psi_S(t_0) > && (32)\\ &= < \Psi_D(t)|O_D|\Psi_D(t) >, && (33)\end{aligned}$$

with

$$|\Psi_D(t) > = U_0^\dagger(t)|\Psi_S(t) > = U_0^\dagger(t)U(t)|\Psi_S(t=0) > = U_D(t)|\Psi_D(t_0) > \quad (34)$$

and

$$O_D(t) = U_0^\dagger O_S U_0. \quad (35)$$

Calculating the explicit time dependence, one derives

$$\begin{aligned} \frac{d}{dt}|\Psi_D(t) > &= \frac{d}{dt}\left[U_0^\dagger(t)U(t)|\Psi_S(t=0) >\right] && (36)\\ &= \frac{i}{\hbar}\left[\mathcal{H}_0 - U_0^\dagger\mathcal{H}\right]U(t)|\Psi_S(t=0) > && (37)\\ &= \frac{i}{\hbar}\left[\mathcal{H}_0 - U_0^\dagger(t)\mathcal{H}U_0(t)\right]|\Psi_D(t) > && (38)\\ &= -\frac{i}{\hbar}\mathcal{V}_D|\Psi_D(t) >, && (39)\end{aligned}$$

i.e. $\mathcal{V_D}$ is given by

$$\mathcal{V}_D = U_0^\dagger(t)\mathcal{V}U_0(t) = \exp^{\frac{i}{\hbar}(t-t_0)\mathcal{H}_0}\mathcal{V}\exp^{-\frac{i}{\hbar}(t-t_0)\mathcal{H}_0}. \quad (40)$$

With eqn.(34) and (39), one derives the equation of motion for the development operator:

$$\frac{d}{dt}U_D = -\frac{i}{\hbar}\mathcal{V}_D(t)U_d(t) \quad (41)$$

and the chronology of the operators gets crucial. The solution has to fulfill

$$U(t,t_0) = U_0(t,t_0) + \int_{t_0}^{t} dt' \frac{U_0(t,t')VU(t',t_0)}{i\hbar} \quad (42)$$

and can be solved via stepwise integration. Assuming $U_0 \sim U$ leads to a perturbation series, where the first integration step leads to the Born part.

Note that in the interaction picture the eigenstates of the quantum system are assumed to be states of the free system.

## 2. Introduction in the Furry representation

Including an external field into the calculation requires an extension of the above mentioned description of a quantum system. The Furry representation [1] is a kind of interaction representation, however the main difference is that the eigenstates are assumed to be states of a bound state, caused by the external field. The Hamiltonian is similarly split as in eq.(27)

with the difference that the external field is added to free Hamiltonian and causes therefore 'new' eigenstates:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{ext}} + \mathcal{V} = \mathcal{H}_B + \mathcal{V}, \tag{43}$$

where $\mathcal{H}_{\text{ext}}$ denotes the Hamiltonian of a classical, external field that is independent of time, $\mathcal{H}_B$ the Hamiltonian causing the bound states and $\mathcal{V}$ the contribution of the perturbative interaction term.

The wave functions in the interaction representation and in the Furry picture differ, but are related via a canonical transformation:

$$\Psi_F(x) = M^{-1}\Psi_D(x)M, \quad \text{and} \quad \Psi_F^\dagger(x) = M^-1\Psi_D^\dagger(x)M. \tag{44}$$

However, one has to be aware that $\Psi_F$ and $\Psi_D$ are spanned by a different basis system. Consequently, the commutation relations differ in both representations,

$$\{\Psi_F, \Psi_F^\dagger\} \neq \{\Psi_D, \Psi_D^\dagger\}, \tag{45}$$

since $\Psi_F$, $\Psi_F^\dagger$ contain the effects of the external field. However, in the asymptotic limit $A^\mu_{ext} \to 0$, the usual commutation relations are recovered:

$$\{\Psi_F^\dagger(x), \Psi_F(x')\} = \sum_i \Psi_{F,i}^\dagger(x)\Psi_{F,i}(x') \stackrel{A^\mu_{ext}\to 0}{\longrightarrow} -i\delta(x - x') \tag{46}$$

### 2.1. Gauge transformation

Quantum field theory is a invariant under gauge transformation:

$$A^\mu(x) \to A^\mu(x) - \frac{\delta\Lambda(x)}{\delta x^\mu}, \tag{47}$$

$$\Psi(x) \to \exp^{-ie\Lambda(x)}\Psi(x), \quad \text{and} \quad \Psi^\dagger(x) \to \exp^{ie\Lambda(x)}\Psi^\dagger(x), \tag{48}$$

where the scalar function $\Lambda(x)$ has to fulfill $\Box^2\Lambda(x) = 0$.

In a bound-state representation, however, a gauge transformation may affect either the external potential $A^\mu_{ext}$, the interaction potential $A^\mu$ or even both. Therefore, *in addition* to eqn.(47), (48), the following transformation have to be performed:

$$A^\mu_{ext}(x) \to A^\mu_{ext} - \frac{\delta\Lambda_{ext}(x)}{\delta x^\mu}, \tag{49}$$

$$\Psi(x) \to \exp^{-ie\Lambda_{ext}}\Psi(x), \quad \text{and} \quad \Psi^\dagger(x) \to \exp^{ie\Lambda_{ext}(x)}\Psi^\dagger(x). \tag{50}$$

### 2.2. Charge conjugation

Another feature in the Furry picture is the absence of absolute symmetry between the wave functions and their charged conjugated wave functions which leads to physical consequences as, for instance, vacuum polarization, see next section.

The charged conjugated wave functions are defined by the usual relations:

$$\Psi'(x) = C\Psi^\dagger(x), \quad \text{and} \quad \Psi'^\dagger(x) = C^{-1}\Psi(x), \tag{51}$$

where

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T. \tag{52}$$

Since the matrix $C$ commutes with the canonical transformation $M$, the relation to the operators in the Dirac representation is:

$$\Psi'_F(x) = M^{-1}\Psi'_D(x)M, \quad \text{and} \quad \Psi'^\dagger_F(x) = M^-1\Psi'^\dagger_D(x)M. \tag{53}$$

Consequently both the resulting equation of motion as well as the commutation relations differ from those of the original operators in a change of a sign of the factor $e$.

## 3. Consequences of the Furry picture
In the following we restrict ourselves to a class of fields where the definition of a stable vacuum is possible [4].

### *3.1. Vacuum polarization*
Usually Feynman diagrams with self-closed electron lines, the tadpole diagrams, vanish as a consequence of Lorentz invariance. However, in the Furry picture the external field influences the vacuum: the vacuum expectation value of the current $j^\mu = \Psi^\dagger\gamma^\mu\Psi$ does not necessarily vanish any more. Therefore these kind of diagrams have to be included.

To make this plausible we split up the wave functions in states with positive and negative energy states:

$$\Psi_F^{(+)} = \sum_s c_s \Psi_{F,s}, \quad \text{and} \quad \Psi_F^{(-)} = \sum_\sigma c_\sigma \Psi_{F,\sigma}, \tag{54}$$

$$\Psi_F^{(\dagger,+)} = \sum_\sigma c_\sigma^* \Psi_{F,\sigma}^\dagger, \quad \text{and} \quad \Psi_F^{\dagger(-)} = \sum_s c_s^* \Psi_{F,s}^\dagger. \tag{55}$$

The vacuum expectation current can be expressed:

$$< j^\mu(x) > \sim \{\sum_s \Psi_s^\dagger \gamma^\mu \Psi_s - \sum_\sigma \Psi_\sigma^\dagger \gamma^\mu \Psi_\sigma\} \tag{56}$$

Eq.(56) is divergent. Using Pauli-Villar regularization [5], the equation can be transformed so that only in order $e^2$ logarithmically divergent but gauge invariant terms remain which can be removed via charge renormalization. Eq.(56) the whole vacuum polarization is contained up to the order $e^2$. In eq.(56)

### *3.2. Electron propagator*
Since the external field influences the vacuum, space and time is no longer homogeneous. This inhomogenity causes the electron operator dependent on $x$ and $x'$ instead of only $(x - x')$.

The electron propagator can be defined:

$$G(x, x') = -i < 0|T\Psi_F(x)\Psi_F^\dagger(x')|0 > \tag{57}$$

## 4. Conclusions
The Furry representation is an intermediate picture between the Heisenberg and the Dirac representation. The crucial difference is that the eigenstates in the Furry picture are influenced by the external field whereas in the other representations one assumes a free-particle system as basis. This significant change has several consequences:

- the commutation relations for the wave functions in the Furry picture differ from those in the Dirac picture;
- the gauge transformation have to be accomplished by transformations including the external field;
- the charge conjugation is different for the wave functions in the Furry representation from those in the Dirac representation. As a consequence one gets different commutation relations for the wave functions and their charged-conjugated functions;
- as consequence of this lack of symmetry one gets contributions to self-closed diagrams in the vacuum polarization;
- another consequence of the lack of symmetry is that the Green functions of the electron operator is described by $G(x, x')$ instead of $G(x - x')$.

**Acknowledgement**
GMP would like to thank the Cockcroft Institute for the kind hospitality and the IPPP of Durham for the generous financial support. The author is also very thankful to the administrative secretaries Linda Wilkinson from the IPPP and Sue Waller from the CI who managed to cope with all eventualities.

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