Upper bound of the time-space non-commutative parameter from gravitational quantum well experiment

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Abstract. Starting from a field theoretic description of the gravitational well problem in a canonical non-commutative spacetime, we have studied the effect of time-space non-commutativity in the gravitational well scenario. The corresponding first quantized theory reveals a spectrum with leading order perturbation term of non-commutative origin. GRANIT experimental data are used to estimate an upper bound on the time-space non-commutative parameter.

1. Introduction

Non-commutative (NC) spacetime [1], where the coordinates x^μ satisfy the non-commutative algebra,

$$[x^\mu, x^\nu] = i\Theta^{\mu\nu}, \tag{1}$$

has regained prominence in the recent past, and field theories defined over the NC spacetime are currently a subject of very intense research [2, 3]. Various gauge theories including gravity are being studied in a NC perspective formally [4, 5, 6, 7, 8, 9, 10] as well as phenomenologically [11, 12, 13, 14]. A part of the endeavour is to find the order of the NC parameter and exploring its connection with observations [15, 16, 17, 18, 19, 20, 21].

In particular, GRANIT experimental data [22], which shows the quantum states of the neutrons trapped in earth's gravitational field, have been used to set an upper bound on the momentum space NC parameters [23, 24] by analyzing the gravitational well problem using NC quantum mechanics, where non-commutativity is introduced among the phase-space variables at the Hamiltonian level. Therefore, non-commutativity in the time-space sector, i.e., $\theta^{0i} \neq 0$ is not accounted for. Time-space non-commutativity poses certain difficulties regarding unitarity and causality [25, 26, 27], which could be avoided by a perturbative approach [28, 29, 30, 31]. Therefore, a search for any possible upper bound on the time-space NC parameter is very much desirable. In this paper, we have studied the effect of time-space NC (if any) on the spectrum of a cold neutron trapped in a gravitational quantum well, starting from a NC Schrödinger field theory.

2. The NC Schrödinger action

To model a non-relativistic particle with a constant background gravitational field in NC spacetime, we start with the NC Schrödinger action in the deformed phase-space, where the ordinary product is replaced by the star product [31, 32, 33]. Here, the fields are defined as functions of the phase-space variables and the redefined product of two fields $\hat{\phi}(x)$ and $\hat{\psi}(x)$ is given by:

$$\hat{\phi}(x) \star \hat{\psi}(x) = \left(\hat{\phi} \star \hat{\psi}\right)(x) = e^{\frac{i}{2}\theta^{\alpha\beta}\partial_\alpha \partial'_\beta} \hat{\phi}(x)\hat{\psi}(x')|_{x'=x} \,. \tag{2}$$

The action for the system in vertical $x - y$ $(i = 1, 2)$ plane and gravitational background in the x-direction reads:

$$\hat{S} = \int dx\, dy\, dt\, \hat{\psi}^\dagger \star \left(i\hbar\partial_0 + \frac{\hbar^2}{2m}\partial_i\partial_i - mg\hat{x}\right) \star \hat{\psi}. \tag{3}$$

Under $\star$ composition, the Moyal bracket between the coordinates is $[\hat{x}^\mu, \hat{x}^\nu]_\star = i\Theta^{\mu\nu}$, where the non-trivial components are: $\Theta^{12} = -\Theta^{21} = \theta$, $\Theta^{10} = -\Theta^{01} = \eta$ and $\Theta^{20} = -\Theta^{02} = \eta'$. Since the effect of non-commutativity is expected to be small, we have expanded the star product and considered only the first order correction terms. A physically irrelevant re-scaling[1] to the field variable, re-definition of the observable mass, and the partial derivative (∂_y) are given by:

$$\psi \mapsto \tilde{\psi} = \sqrt{\left(1 - \frac{\eta}{2\hbar}mg\right)}\ \psi, \qquad \tilde{m} = \left(1 - \frac{\eta}{2\hbar}mg\right)m, \quad \textit{and} \qquad \partial_y = \left(\partial_y - \frac{i\theta}{2\hbar^2}\tilde{m}^2 g\right), \tag{4}$$

give the final effective NC Schrödinger action as:

$$\hat{S} = \int dx\, dy\, dt\, \tilde{\psi}^\dagger \left[i\hbar\partial_t + \frac{\hbar^2}{2\tilde{m}}\left({\partial_x}^2 + {\partial_y}^2\right) - \tilde{m}gx - \eta\left(\frac{\tilde{m}^2 g^2}{\hbar}\right)x\right]\tilde{\psi}, \tag{5}$$

which gives the equation of motion for the field $\tilde{\psi}(x)$.

3. Reduction to first quantized theory

In a field theoretic setting, we have imposed non-commutativity and found the only non-trivial change in the Schrödinger equation is, indeed, originating from time-space non-commutativity. Specifically, it shows up only in the direction of the external gravitational field $\mathbf{g} = -g\mathbf{e_x}$. Since the first and second quantized formalisms are equivalent as far as Galilean systems are concerned, we can, hereafter, reinterprete $\tilde{\psi}$, the basic field, as a wave function and carry out an equivalent NC quantum mechanical analysis.

From Eq. (5), we easily read off the Hamiltonian as:

$$H = H_0 + H_1 = \frac{({p_x}^2 + {p_y}^2)}{2\tilde{m}} + \tilde{m}gx + \eta\frac{\tilde{m}^2 g^2}{\hbar}x. \tag{6}$$

The last term in Eq. (6) represents a perturbation H_1 in the gravitational quantum well problem described by H_0, which we now briefly review.

3.1. Ordinary gravitational quantum well

The first two terms in Eq. (6) describe the quantum states of a particle with mass $\tilde{m}$ trapped in a gravitational well. Since the particle is free to move in y-direction, its energy spectrum is continuous along y and the corresponding wave function will be a collection of plane waves $\tilde{\psi}(y) = \int_{-\infty}^{+\infty} g(k)e^{iky}dk$, where $g(k)$ determines the group's shape in phase-space. The analytical

[1] Such re-scalings are only viable in a region of spacetime, where variation of the external field is negligible. Since the results we have derived are to be compared with the outcome of a laboratory-based experiment, we can safely assume a constant external gravitational field throughout.

solution of the Schrödinger equation in x-direction $H_0\tilde{\psi}_n = E_n\tilde{\psi}_n$, is well known [34]. The eigenfunctions can be expressed in terms of the Airy function $\phi(z)$ as $\psi_n(x) = A_n\phi(z)$, with eigenvalues given by the roots of the Airy function α_n, with $n = 1, 2\dots$ as:

$$E_n = -\left(\tilde{m}g^2\hbar^2/2\right)^{1/3}\alpha_n\ . \tag{7}$$

The dimensionless variable z is related to the height x by $z = \left(2m^2g/\hbar^2\right)^{1/3}(x - E_n/\tilde{m}g)$. The normalization factor for the n-th eigenstate is given by:

$$A_n = \left[\left(\hbar^2/2m^2g\right)^{\frac{1}{3}}\int_{\alpha_n}^{+\infty} dz\phi^2(z)\right]^{-\frac{1}{2}}. \tag{8}$$

The wave function for a particle with energy E_n oscillates below the classically allowed height $x_n = \frac{E_n}{\tilde{m}g}$, and above x_n it decays exponentially. This was realized experimentally by Nesvizhevsky *et al* [22] by letting cold neutrons flow with horizontal velocity 6.5 ms^{-1} through a horizontal slit formed between a mirror below and an absorber above. The number of transmitted neutrons as a function of absorber height is recorded, and the classical dependence is observed to change into a stepwise quantum-mechanical dependence at a small absorber height. The experimentally found value of the classical height for the first quantum state is $x_1^{exp} = 12.2 \pm 1.8$ (syst.) $\pm\ 0.7$ (stat.) μm, and the corresponding theoretical value can be determined from Eq. (7) for $\alpha_1 = -2.338$, yielding $x_1 = 13.7$ μm. This value is contained in the error bars and allow for maximum absolute shift of the first energy level with respect to the predicted values:

$$\Delta E_1^{exp} = 6.55 \times 10^{-32}\ \mathrm{J} = 0.41\ \mathrm{peV}. \tag{9}$$

The values of the constants taken in this calculations are $\hbar = 10.59 \times 10^{-35}$ Js, $g = 9.81\ \mathrm{ms}^{-2}$, *and* $\tilde{m} = 167.32 \times 10^{-29}$ kg.

3.2. Analysis of the perturbed energy spectrum

Going back to the effective NCQM theory, we have now analyzed the perturbed system in Eq. (6). The perturbative potential given by $H_1 = \eta\left(\tilde{m}^2g^2/\hbar\right)x$ is a direct manifestation of time–space non-commutativity. So, we have worked out an upper bound for the time-space NC parameter following [23] by demanding that the correction in the energy spectrum should be smaller or equal to the maximum energy shift allowed by the experiment [22]. We have worked out the theoretical value of the leading order energy shift of the first quantum state numerically. It is just the expectation value of the perturbation potential, given by:

$$\Delta E_1 = \eta\frac{\tilde{m}^2g^2}{\hbar}\int_0^{+\infty} dx\ \tilde{\psi}_1^*(x)\ x\ \tilde{\psi}_1(x) = \eta\frac{\tilde{m}^2g^2}{\hbar}\left[\left(\frac{2\tilde{m}^2g}{\hbar^2}\right)^{-\frac{2}{3}}A_1^2I_1 + \frac{E_1}{\tilde{m}g}\right], \tag{10}$$

where $I_1 \equiv \int_{\alpha_1}^{+\infty} dz\phi(z)z\phi(z)$. The values of the first unperturbed energy level E_1 is determined from Eq. (7) with $\alpha_1 = -2.338$ as:

$$E_1 = 2.259 \times 10^{-31}\ (\mathrm{J}) = 1.407\ (\mathrm{peV}). \tag{11}$$

The normalization factor A_1 is calculated from Eq. (8). The integrals in A_n and I_1 are numerically determined for the first energy level $A_1 = 588.109$, *and* $I_1 = -0.383213$. The first order correction in the energy level is given by $\Delta E_1 = 2.316 \times 10^{-23}\eta$ J. Comparing with the experimentally determined value of the energy level from Eq. (9), we have found the bound on the time-space NC parameter is:

$$|\eta| \sim 2.83 \times 10^{-9}\ \mathrm{m}^2. \tag{12}$$

Interestingly, the value of the upper bound derived, here, can be shown [18] to be consistent with the results of [23, 24].

4. Conclusions

In this paper, we have obtained an effective NC quantum mechanics (NCQM) for the gravitational well problem starting from a NC Schrödinger action coupled to external gravitational field. We have re-interpreted this one particle field theory as a first quantized theory and obtained an effective NCQM. The outcome of our calculation shows that the time-space sector of the NC algebra introduces non-trivial NC effects in the energy spectrum of the system. We have demanded that the calculated perturbation in the energy level should be less than or equal to the maximum energy shift allowed by the GRANIT experiment [22]. This comparison leads to an upper bound on the time-space NC parameter. However, one should keep in mind that this value is only in the sense of an upper bound, and not the value of the parameter itself.

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