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Bounds on the effective theory of gravity in models of particle physics and cosmology

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Submitted for the degree of Doctor of Philosophy

University of Sussex

May 2013

Declaration

I hereby declare that this thesis has not been and will not be submitted in whole or in part to another University for the award of any other degree.

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Summary

The effective theory of gravity coupled to matter represents a fully consistent low energy theory of quantum gravity coupled to the known particles and forces of the standard model. In recent years this framework has been extensively used to make physical predictions of phenomena in high energy physics and cosmology. In this thesis we use theoretical tools and experimental data to place constraints on various popular models which utilise this framework. We specifically derive unitarity bounds in grand unified theories, models of low scale quantum gravity, models with extra dimensions and models of Higgs inflation. We also derive a bound on the size of the Higgs boson's non-minimal coupling to gravity. This represents an important area of research because it helps us to better understand the theories and models that many physicists are currently working on and crucially it can inform us where we can reliably use the effective theory approach and where it breaks down.

Acknowledgements

First and foremost I would like to thank my supervisor Xavier Calmet, for all his help, support and encouragement throughout my PhD. I would also like to thank all the other members of faculty with whom I have shared valuable discussions, in particular David Bailin, Stephan Huber, Sebastian Jaeger, Mark Hindmarsh and Daniel Litim. I am very grateful to have shared my time at Sussex with some wonderful fellow students with whom I have enjoyed learning, researching and socialising with - I wish them all the best of luck in the future.

I would also like to thank all my friends and family who have supported me throughout my PhD and who have provided me with welcome breaks and distractions from my work. Particular thanks goes to all my house mates over the years who have put up with my moods, listened to my rants and provided me with delicious meals. Finally, I want to say a huge thank you to my partner Isy, who has supported and encouraged me throughout and has inspired me to complete this project.

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Introduction

The two greatest triumphs of $20^{\rm th}$ century physics are undoubtedly Einstein's theory of general relativity and the quantum field theory description of the standard model of particle physics. Together they provide a predictive framework to understand virtually all phenomena in the observable universe, from the interactions of high energy subatomic particles to the behaviour of superclusters of galaxies at the furthest reaches of the cosmos. The great challenge of $21^{\rm st}$ century physics is to unify these two theories into a fully consistent theory of quantum gravity. Many promising steps have been made in this direction but there may still be a long way to go to realising most physicist's dream of discovering a true theory of everything.

While the quest for a full theory of quantum gravity still remains somewhat in the realm of conjecture with many competing ideas - string theory, loop quantum gravity, asymptotic safety etc.[1] - and few if any measurable predictions, the effective field theory approach offers a fully consistent quantum field theory treatment of the low energy physics of quantum gravity. Despite our ignorance of the true theory of quantum gravity, effective field theory techniques allow us to reliably predict many of the low energy phenomena of quantum gravity and its interactions with the standard model. Recently, there has been an explosion in research in observable consequences of the effective theory of gravity in the fields of both particle physics and cosmology, with a healthy exchange of ideas and techniques between the two. This new activity has come about for two main reasons: Firstly, in particle physics there has been a huge amount of work dedicated to trying account for the seemingly unnatural hierarchy between the scale of quantum gravity and the electroweak scale. Some of the more exciting models which address this problem

[1]For a general review of different approaches to quantum gravity see Ref. [1] and for a clear assessment of how far we still have to go towards a full theory see Ref. [2].

propose that the scale of quantum gravity could be much lower than typically expected and may even be observable in high energy collisions at the Large Hadron Collider (LHC). Secondly, with the recent precise measurements of the cosmic microwave background radiation (CMB), cosmologists are beginning to be able to test theories of the early universe which involve energies approaching the scale of quantum gravity. In order to make reliable predictions in both of these scenarios requires the careful use of the effective theory of gravity consistently coupled to models of particle physics. Comparing these predictions to experimental data, physicists have already been able to rule out and strongly constrain various models and ideas. However, most exciting is the prospect that predictions based on the low energy effective theory of gravity may begin to point us in the direction of the full theory of quantum gravity and its effects may even be observed in the near future.

In this thesis we will study the effective field theory of gravity and its interplay with particle physics in a variety of situations, including grand unified theories, extra dimensions and cosmological inflation. The main focus will be on using both theoretical tools and experimental data to place bounds and constraints on various parameters in these models. This represents an important area of research because it helps us to better understand the theories and models that many physicists are currently working on and crucially it can inform us where and when we can reliably use the effective theory approach and where it breaks down.

In the rest of this introductory chapter, we review the framework of the effective theory of gravity coupled to matter in preparation for the following chapters. We also introduce and review one of the main tools used throughout the thesis: perturbative unitarity bounds. The rest of the thesis is laid out as follows:

- In Chapter 2 we derive unitarity bounds on models of particle physics coupled to gravity in four dimensions such as grand unified theories (GUTs). We also employ an original renormalisation group approach here. This is based mostly on work published by the author and Xavier Calmet in Ref. [3].

- In Chapter 3 we derive unitarity bounds on models with extra dimensions with low scales of quantum gravity. We specifically look at the ADD model, the Randall-Sundrum model and the linear dilaton model. This work is based in large part on work published by the author and Xavier Calmet in Ref. [4] and also work published by the author, Xavier Calmet and Ignatios Antoniadis in Ref. [5]. The unitarity bounds derived in the linear dilaton model represent original work not published elsewhere.

- In Chapter 4 we investigate two models of Higgs inflation and again derive unitarity bounds in these models. We show that the unitarity bounds pose serious problems for the predictivity of these models and discuss asymptotic safety as a framework in which these problems could be addressed. This chapter is based on work published by the author and Xavier Calmet in Refs. [3, 4, 6].

- In Chapter 5 we use data from the LHC to derive the first ever bound on the size of the non-minimal coupling between the Higgs boson and gravity. This is based mainly on work published by the author and Xavier Calmet in Ref. [7].

The rest of this chapter will be dedicated to reviewing important background material and setting the notation in preparation for the main body of work. We begin with a review of the effective field theory of quantum gravity.

1.1 Effective theory of gravity

In this section we review the treatment of quantum general relativity coupled to matter as an effective field theory. There already exist a number of good reviews of this topic in the literature, see for example Refs. [8, 9, 10, 11], many of which we have used as a guide in writing this section.

Before we delve into the specifics of gravity, let us first review the basic concepts underlying effective field theories in general. Again, many good reviews exist on this subject, see for example Refs. [12, 13, 14, 15, 16]. The treatment given here is specific to situations where we do not know the full high energy theory, as is the case for gravity. Despite this lack of knowledge, the effective field theory can still be predictive. We list here a procedure for constructing and using an effective theory:

- Identify the low energy degrees of freedom and symmetries.

- Using only these fields construct the most general Lagrangian including all possible operators consistent with the symmetries. The operators can be ordered in an energy expansion in terms of increasing dimension.

- Quantise the fields and identify the propagators.

- We may now proceed to compute observables in the usual way treating all the additional operators as interactions. Loops can be calculated and renormalisation can be carried out by absorbing divergences into the renormalisation of terms in the

Lagrangian. Note that an effective theory requires an infinite amount of terms in order to absorb all divergences.

- Because we do not know the full theory we have to determine the coefficients of the operators by matching to experiment. This only needs to be done once per term and once fixed each coefficient can be reliably used to compute further observables.

- Use the resulting effective field theory up to the required order in the energy expansion to make reliable predictions to within a specified accuracy.

When one considers the above, there seems little difference between a conventional renormalisable field theory and an effective field theory. In fact it is now commonly believed that renormalisable theories simply represent the leading order terms in an effective theory, where the operators of dimension greater than four are heavily suppressed by some large mass scale which can be arbitrarily high. The main difference is that in a renormalisable theory, the leading order terms can be renormalised without having to introduce higher dimensional operators, while in a so called "non-renormalisable" effective field theory, we technically require an infinite number of counterterms in order to absorb all the divergences during renormalisation. Despite this, only a finite number of terms are required to make predictions to within a required accuracy and if the terms are ordered in an efficient energy expansion this can easily be determined.

The effective theory will only be valid up to the scale at which the full high energy theory manifests itself. We therefore expect that there is a cutoff to the effective theory and it is this scale that the higher order operators in the effective field theory are suppressed by. We will see later that we may be able to use the concept of unitarity to provide an estimate of this cutoff.

Turning now to gravity, we assume that the full unknown theory of quantum gravity must have general relativity as the low energy limit. So the degrees of freedom in the effective field theory are massless gravitons which are the quantum fluctuations of the metric, and the symmetry is general coordinate invariance. The action is built from the curvature tensors which in turn are derived from the connection which is given in terms of the metric as

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right) \tag{1.1.1}$$

and the Ricci tensor is given by

$$R_{\mu\nu} = \partial_\mu \Gamma^{\sigma}_{\nu\sigma} - \partial_\sigma \Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\mu\rho} \Gamma^{\rho}_{\nu\sigma} - \Gamma^{\sigma}_{\mu\nu} \Gamma^{\rho}_{\rho\sigma}. \tag{1.1.2}$$

Considering that the curvature tensor $R_{\mu\nu}$ contains two derivatives of the metric, the effective theory can be ordered in an energy expansion in the following way:

$$S = -\int d^4x\sqrt{g}\left(-\Lambda + \frac{M_P^2}{2}R + c_1R^2 + c_2R_{\mu\nu}R^{\mu\nu} + \ldots + \mathcal{L}_{\text{matter}}\right), \tag{1.1.3}$$

where $R = g^{\mu\nu}R_{\mu\nu}$. Λ is the cosmological constant which has been found experimentally to have the tiny positive value 10^{-47} GeV^4 [17] and M_P is the Planck mass, defined in terms of Newton's constant G_N as

$$M_P = \frac{1}{\sqrt{8\pi G_N}} \simeq 2.435 \times 10^{18} \text{ GeV}. \tag{1.1.4}$$

Note that this definition of the Planck mass is often referred to as the reduced Planck mass and will be used throughout this thesis. The action (1.1.3) may also contain gauge fixing and ghost terms, however they have been suppressed here as we will not be considering graviton loops and will therefore not require such terms. The coefficients c_1 and c_2 are dimensionless parameters. It has been shown by Stelle in 1977 [18] that the terms c_1R^2 and $c_2R^{\mu\nu}R_{\mu\nu}$ lead to Yukawa-like corrections to the Newtonian potential of a point mass m:

$$\Phi(r) = -\frac{G_Nm}{r}\left(1 + \frac{1}{3}e^{-m_0r} - \frac{4}{3}e^{-m_2r}\right), \tag{1.1.5}$$

where

$$m_0^{-1} = \sqrt{32\pi G\left(3c_1 - c_2\right)}, \quad m_2^{-1} = \sqrt{16\pi Gc_2}. \tag{1.1.6}$$

Using recent experimental advances [19], one finds that the coefficients c_1 and c_2 are constrained to be less than 10^{61} [20] in the absence of accidental fine cancellations between both Yukawa terms. Attempts to bound these terms using astrophysical measurements have been reviewed in [21]. The fact that this constraint is so weak demonstrates what a small effect these terms have in low energy physics and how effective the energy expansion is. This is also seen by the fact that the energy expansion is in powers of E/M_P, and the large scale of M_P acts to heavily suppress the contribution of higher order terms relative to the leading order terms.

In order to quantize the theory, the metric is expanded around a background $\bar{g}_{\mu\nu}$ in the following way

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \frac{\sqrt{2}}{M_P}h_{\mu\nu}(x), \tag{1.1.7}$$

where $h_{\mu\nu}$ is the graviton. For our purposes in this thesis, we will only need to expand around flat Minkowski space[2] and so we take $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ from now on. We use the metric

[2] Despite the use of Minkowski space here, recent astrophysical data coming from type Ia supernovae [22,

signature $(+,-,-,-)$ throughout. The second term in the action (1.1.3) can be expanded in terms of the graviton to give

$$\frac{M_P^2}{2}\sqrt{g}R = -\frac{1}{4}h^{\mu\nu}\Box h_{\mu\nu} + \frac{1}{4}h^{\mu}_{\mu}\Box h^{\nu}_{\nu} - \frac{1}{2}h^{\mu\nu}\partial_\mu\partial_\nu h^{\rho}_{\rho} + \frac{1}{2}h^{\mu\nu}\partial_\mu\partial_\alpha h^{\alpha}_{\nu} + \mathcal{O}(M_P^{-2}). \quad (1.1.8)$$

From this the graviton propagator can be determined. In harmonic gauge ($\partial^\lambda h_{\mu\lambda} = \frac{1}{2}\partial_\mu h$) the propagator is given by

$$i\Delta_{\mu\nu\rho\sigma} = \frac{\frac{i}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma})}{q^2 + i\epsilon}. \quad (1.1.9)$$

From here, one may proceed to expand to higher orders in the graviton and use the resulting interaction terms to calculate quantum effects in pure gravity. Examples of such calculations are $2 \to 2$ graviton scattering at one-loop [25], quantum corrections to Newton's potential [26, 27] and quantum corrections to Reissner-Nordström and Kerr-Newman metrics [28].

For the purposes of this thesis we require to focus on the coupling of the graviton to matter. The leading order terms in the matter part of the action for scalar fields ϕ, fermions ψ and vector bosons A_μ are given by

$$\mathcal{L}_{\text{matter}} = \sqrt{g}\left(\frac{1}{2}g^{\mu\nu}D_\mu\phi D_\nu\phi + \frac{1}{2}\xi\phi^2 R - \frac{1}{2}m_\phi^2\phi^2 + ie\bar{\psi}\gamma^\mu\mathcal{D}_\mu\psi - m_\psi\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right), \quad (1.1.10)$$

where e is the vierbein defined by $e^a_\mu e^b_\nu\eta_{ab} = g_{\mu\nu}$ and $e = \det(e^a_\mu), \gamma^\mu = e^\mu_a\gamma^a$ and $\mathcal{D}_\mu = D_\mu + \frac{1}{2}\omega^{ab}_\mu\sigma_{ab}$ with ω the spin connection. We have suppressed additional interaction terms that may be present. Most of the matter Lagrangian is standard but there is one unique term which will play an important role later. Because of the low mass dimension of the scalar field, it is possible to include the second term in the Lagrangian which is a dimension four operator and couples the scalar field to the Ricci curvature. The parameter ξ is an unknown dimensionless parameter and when $\xi \neq 0$ this term is referred to as a non-minimal coupling. This coupling will be of importance at a number of points throughout this thesis, and the non-minimally coupled scalar field will often be the Higgs boson.

Given the matter Lagrangian (1.1.10), one can derive the Feynman rules for the interactions of gravitons with matter. These have been worked out in Refs. [29, 30] and

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23] indicate that the expansion of the universe is accelerating and point towards a small but nonvanishing positive cosmological constant. This would mean that our universe might currently be in a de Sitter phase. Also during the inflationary era, one assumes that the universe was also described by a de Sitter phase. For these reasons there is much current research into describing quantum gravity in de-Sitter space, see for example Ref. [24].

are reproduced in Appendix E. We will later use the graviton propagator and Feynman rules to calculate scattering amplitudes for matter fields via graviton exchange. But first we turn our attention to an important tool for investigating the regime of validity of an effective theory which is perturbative unitarity.

1.2 Unitarity

One of the fundamental requirements of a quantum theory is unitary time evolution. This simple fact ensures consistency of the theory through the enforcement of the conservation of probability. In quantum field theory it is embodied in the unitarity of the S-matrix, $S^\dagger S = SS^\dagger = 1$. From this simple assumption, we are able to derive bounds on the partial wave amplitudes and cross sections for different scattering processes. These bounds can then be applied at any order in perturbation theory as a consistency check on the validity of perturbative calculations. We first derive the expressions for the unitarity bounds and will then discuss the interpretation of the breakdown of unitarity in perturbation theory. We will also review a classic example of the use of such a unitarity bound as applied to longitudinal W boson scattering.

There are many ways to go about the derivation, here we follow closely the derivation given in Ref. [31]. Defining the matrix T via $S \equiv 1 + iT$, the unitarity of the S-matrix can be recast as $T^\dagger T = 2\mathrm{Im}T$. Taking the matrix element of both sides of this expression between identical 2-body initial and final states and inserting a complete set of states into the left hand side we have,

$$\int \sum_n d\Pi_n \langle 2|T|n\rangle\langle n|T^\dagger|2\rangle = 2\mathrm{Im}\langle 2|T|2\rangle. \tag{1.2.1}$$

where $d\Pi_n$ denotes the n-body phase space integration measure. Separating out the elastic (internal quantum numbers not changed during scattering) channel from all inelastic channels and denoting $\langle a|T|b\rangle \equiv (2\pi)^4\delta(p_{in} - p_{out})\mathcal{M}(a \to b)$ gives

$$\int d\Pi_{2'}|\mathcal{M}_{\mathrm{el}}(2 \to 2')|^2 + \sum_n \int d\Pi_n |\mathcal{M}_{\mathrm{inel}}(2 \to n)|^2 = 2\mathrm{Im}\left[\mathcal{M}_{\mathrm{el}}(2 \to 2)\right] \tag{1.2.2}$$

where now the sum in the second term is over all possible *inelastic* channels. Note that in the term on right hand side the scattering is between identical initial and final states and so is in the forward direction (the term on the left is distinguished in this respect by the prime). To separate the angular dependence of the amplitudes we decompose them into partial waves a_j^{el} via

$$\mathcal{M}_{\mathrm{el}}(2 \to 2') = 16\pi e^{i(\lambda'-\lambda)\varphi} \sum_j (2j+1) d^j_{\lambda'\lambda}(\cos\theta)\, a_j^{\mathrm{el}} \tag{1.2.3}$$

where θ and φ are the standard scattering angles, $d^j_{\lambda'\lambda}(\cos\theta)$ are the Wigner d-functions (see Appendix B) and $(\lambda,\lambda') = (\lambda_1-\lambda_2,\ \lambda_3-\lambda_4)$ where λ_1,λ_2 and λ_3,λ_4 are the helicities of the incoming and outgoing particles respectively. From here on we specialise to scattering in the $\phi = 0$ plane where in the massless limit, the 2-body phase space integral is given by $\int d\Pi_2 = \frac{1}{16\pi}\int_{-1}^{1} d\cos\theta$. The d-functions obey an orthogonality relation

$$\int_{-1}^{1} dx\, d^j_{\lambda'\lambda}(x) d^{j'}_{\lambda'\lambda}(x) = \frac{2\delta_{jj'}}{2j+1} \tag{1.2.4}$$

which we can use to invert Eq. 1.2.3 to obtain

$$a^{\rm el}_j = \frac{1}{32\pi}\int_{-1}^{1} d\cos\theta\, d^j_{\lambda'\lambda}(\cos\theta)\,\mathcal{M}_{\rm el}(2\to 2') \tag{1.2.5}$$

It then follows that

$$\int d\Pi_{2'}|\mathcal{M}_{\rm el}(2\to 2')|^2 = \frac{32\pi}{\rho}\sum_j (2j+1)\, d^j_{\lambda'\lambda}\, |a^{\rm el}_j|^2, \tag{1.2.6}$$

where ρ is a symmetry factor and $\rho = 1!\,(2!)$ if the final state contains non-identical (identical) particles. Using Eqs. (1.2.5) and (1.2.6) we find the unitarity condition Eq. (1.2.2) is

$$\sum_j (2j+1)\frac{1}{\rho}\left[\frac{\rho^2}{4} - \left(\mathrm{Re}\, a^{\rm el}_j\right)^2 - \left(\mathrm{Im}\, a^{\rm el}_j - \frac{\rho}{2}\right)^2\right] = \frac{1}{32\pi}\sum_n \int d\Pi_n |\mathcal{M}_{\rm inel}(2\to n)|^2. \tag{1.2.7}$$

Now the right hand side is non-negative, and so we find for each j

$$\left(\mathrm{Re}\, a^{\rm el}_j\right)^2 - \left(\mathrm{Im}\, a^{\rm el}_j + \frac{\rho}{2}\right)^2 \leq \frac{\rho^2}{4}. \tag{1.2.8}$$

Which can be reinterpreted as

$$\left|\mathrm{Re}\, a^{\rm el}_j\right| \leq \frac{\rho}{2}, \qquad \left|\mathrm{Im}\, a^{\rm el}_j\right| \leq \rho. \tag{1.2.9}$$

These are the unitarity bounds on the partial wave amplitudes we require.

Unitarity of a superposition of states

In order to derive the lowest possible unitarity bound it is often useful to consider the scattering of a superposition of states. Consider the set of normalised states $\{A_1, A_2, \ldots, A_n\}$ and the matrix of partial wave scattering amplitudes $(a_j)_{kl} = a_j(A_k \to A_l)$. For any vector $v \in \mathbb{C}^n$ such that $v^\dagger v = 1$, we find

$$\left|\mathrm{Re}\left[v^\dagger a_j v\right]\right| \leq \frac{1}{2}, \qquad \left|\mathrm{Im}\left[v^\dagger a_j v\right]\right| \leq 1. \tag{1.2.10}$$

If v is an eigenvector of a_j then the above conditions apply to the eigenvalues λ_i of a_j in the following way

$$|\text{Re}(\lambda_i)| \leq \frac{1}{2}, \qquad |\text{Im}(\lambda_i)| \leq 1. \tag{1.2.11}$$

This derivation assumes that we have non-identical particles in the initial/final state. If there exist identical particles then an additional factor of $1/\sqrt{2}$ can be included in the normalisation for these states to compensate for the factor of 2 contained in ρ in the unitarity bound.

As a simple example of this technique, consider the scattering of n identical particles ϕ_i. If the amplitude for $\phi_i\phi_i \to \phi_i\phi_i$ is given by $a_{\phi,j}$ then we may consider the scattering of the normalised state (including additional factors of $1/\sqrt{2}$ to compensate ρ)

$$a_j\left(\sum_{l=1}^{n} \frac{1}{\sqrt{2n}}\phi_l\phi_l \to \sum_{k=1}^{n} \frac{1}{\sqrt{2n}}\phi_k\phi_k\right) = \frac{n}{2}a_{\phi,j}, \tag{1.2.12}$$

and the unitarity bounds become

$$|\text{Re}(a_{\phi,j})| \leq \frac{1}{n} \qquad |\text{Im}(a_{\phi,j})| \leq \frac{2}{n}. \tag{1.2.13}$$

For a large number of fields the unitarity bound can be significantly reduced using this technique.

Perturbative unitarity bounds

We may apply the unitarity bounds Eq. (1.2.9) to amplitudes at any order in perturbation theory. However, we need to be clear in our interpretation of what it means if we find that unitarity breaks down at a finite order in the perturbation expansion. It is clear that if the perturbative unitarity bound is exceeded then something extra must act to cure the unitarity problem or the theory will be inconsistent. There are two options and it is not normally possible to determine a priori which of the two paths might be taken. The first is that some new degrees of freedom may enter at or before the scale at which unitarity is violated and act to restore unitarity, at least until some higher scale. The second option is simply that the theory becomes strongly coupled at the scale at which perturbative unitarity breaks down and all orders in perturbation theory become equally relevant and higher order effects act to restore unitarity. In fact there are active areas of research such as the asymptotic safety program [32, 33] and the idea of classicalisation [34], in which a strongly coupled effective field theory is proposed to offer the full UV completion to the theory.

Regardless of what ultimately fixes the perturbative unitarity problem, the true utility of unitarity bounds is that they can inform us of the energy regime of validity of an effective theory. The lowest possible unitarity bound provides a cutoff to the effective theory, above which the energy expansion breaks down. This scale is then likely the scale by which the higher order terms in the effective Lagrangian are suppressed. We will use the tool of perturbative unitarity throughout this thesis in exactly this way - to determine the regime of validity of effective field theories that are studied in the literature.

We will often refer to the energy scale at which unitarity breaks down in a specific model using the notation $E_\star$ and if we interpret this as a cutoff to the effective theory we may refer to the cutoff with the symbol Λ.

Before we move on to presenting our original work, we review a classic example of the use of perturbative unitarity for both pedagogical purposes and also because we will later require the process in Chapter 3.

1.2.1 Example - unitarity of WW scattering

The classic example of the utility of perturbative unitarity bounds is the Lee-Quigg-Thacker (LQT) bound which was used to place a bound on the mass of the Higgs boson in the standard model [35, 36] (see also Ref. [37]). They considered the tree level scattering of longitudinal W bosons, $W_L W_L \to W_L W_L$. Without the Higgs boson, this process would take place only via the s and t-channel exchange of the photon and Z boson and the four point contact interaction (see the first three diagrams of Fig. 1.1). The leading order $j = 0$ partial wave amplitude for this process is

$$a_0^{(\text{gauge})} = -\frac{g^2\, s}{128\pi m_W^2} + \mathcal{O}\left(\left(\frac{m_W}{s}\right)^0\right), \tag{1.2.14}$$

where g is the weak coupling constant and m_W is the mass of the W boson. Applying the unitarity bound $|\text{Re}(a_0)| \leq 1/2$, it is found that without the Higgs boson, tree level unitarity would break down at a scale $E_\star = 1.7$ TeV. This alone provided a strong argument for the expectation that we should see something at the TeV scale at the LHC connected to the symmetry breaking sector of the standard model. Either some new degrees of freedom would have had to appear below this scale or we would have expected to see the effects of strongly coupled W bosons.

However, in the standard model there is a physical Higgs boson which can also be exchanged in s and t-channel processes in WW scattering (see the last two diagrams of

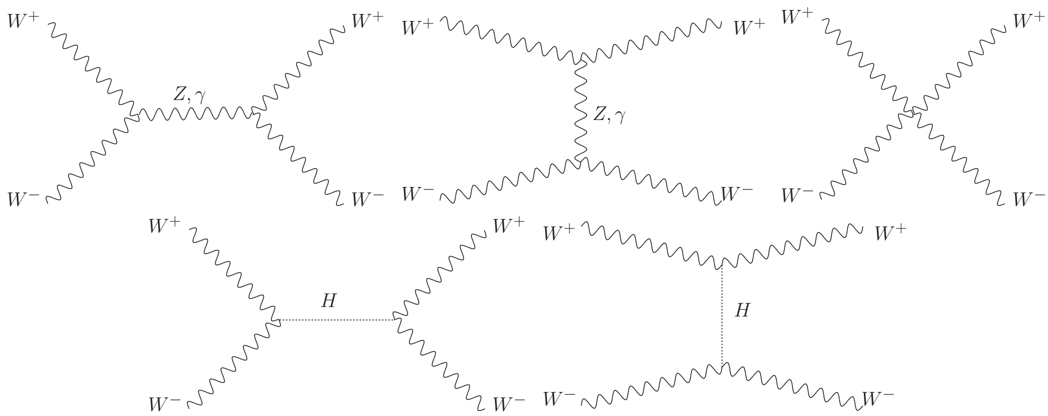


Figure 1.1: Processes contributing to the WW scattering amplitude.

Fig. 1.1). The leading order $j = 0$ partial wave amplitude from Higgs exchange is

$$a_0^{(\text{higgs})} = \frac{g^2\, s}{128\pi m_W^2} + \mathcal{O}\left(\frac{m_W^0}{s^0}\right), \tag{1.2.15}$$

which exactly cancels the gauge boson piece.

After this cancellation, if one assumes the Higgs boson mass is large ($m_H^2 \gg m_W^2$), the leading order contribution to the total amplitude is given by

$$a_0^{(\text{total})} = -\frac{g^2\, m_H^2}{32\pi\, m_W^2}, \tag{1.2.16}$$

and applying the unitarity bound to this amplitude, it was found that

$$m_H^2 \leq \frac{32\pi\, m_W^2}{g^2}, \tag{1.2.17}$$

which led to a bound on the Higgs boson mass of $m_H \lesssim 1.2$ TeV. This bound should be interpreted as largest the Higgs boson mass could have been for the standard model to remain weakly coupled.

In Ref. [35] an even lower bound was found by considering the scattering of a superposition of states. The individual scattering amplitudes for each process are given in Table 1.1, note that a factor of $-\frac{g^2 m_H^2}{128\pi m_W^2}$ has been extracted from each of the amplitudes. The largest eigenvalue of this matrix is

$$\lambda = -\frac{3g^2 m_H^2}{64\pi m_W^2}, \tag{1.2.18}$$

and applying the unitarity bound (1.2.13) it was found that

$$m_H \leq \sqrt{\frac{32\pi m_W^2}{3g^2}} \simeq 700 \text{ GeV}. \tag{1.2.19}$$

In 2012, the Higgs boson was indeed discovered with a mass of around 125 GeV [38, 39] and so W boson scattering is expected to remain weakly coupled up to high energies.

$\to$	W^+W^-	$\frac{1}{\sqrt{2}}ZZ$	$\frac{1}{\sqrt{2}}hh$	Zh
W^+W^-	4	$\sqrt{2}$	$\sqrt{2}$	0
$\frac{1}{\sqrt{2}}ZZ$	$\sqrt{2}$	3	1	0
$\frac{1}{\sqrt{2}}hh$	$\sqrt{2}$	1	3	0
Zh	0	0	0	2

Table 1.1: $j = 0$ partial wave scattering amplitudes for W, Z and Higgs bosons. A factor of $-\frac{g^2 m_H^2}{128\pi m_W^2}$ has been extracted from each of the amplitudes.

Goldstone boson equivalence principle

We describe here one more interesting result of Ref. [35]: the Goldstone boson equivalence principle (see also Refs. [40, 41] and for a comprehensive review see Ref. [42]). First, consider the WW scattering amplitude described above. If one applies a naive power counting analysis, we might expect that the amplitude in fact grows as E^4/m_W^4. The polarisation vector of a longitudinally polarised W boson in the high energy limit behaves as (see Appendix A)

$$\epsilon^\mu(p) = \frac{p^\mu}{m_W} + \mathcal{O}\left(\frac{m_W}{E}\right), \tag{1.2.20}$$

and each vertex for Z, γ exchange is proportional to E. Combining these, we would expect the scattering amplitude to grow as E^4/m_W^4. However, as we have seen, the $\mathcal{O}(E^4)$ terms cancel due to the gauge structure and if a physical Higgs boson is present, the $\mathcal{O}(E^2)$ terms also cancel. Why does this happen? One way to understand this is to remember that in the high energy limit, the longitudinal components of the massive gauge bosons are effectively just the scalar degrees of freedom of the Goldstone bosons coming from the electroweak symmetry breaking sector. The Goldstone boson equivalence theorem says that an amplitude involving external massive gauge bosons V_i can be written in terms of the respective Goldstone bosons φ_i as [42]

$$\mathcal{M}(V_{i,1}, V_{i,2}, \ldots, V_{i,n}, A \to V_{f,1}, V_{f,2}, \ldots, V_{f,n_f}, B) \tag{1.2.21}$$

$$= \mathcal{M}(\varphi_{i,1}, \varphi_{i,2}, \ldots, \varphi_{i,n_i}, A \to \varphi_{f,1}, \varphi_{f,2}, \ldots, \varphi_{f,n_f}, B) \times i^{n_i - n_f} C \left(1 + \mathcal{O}\left(\frac{m_V}{E}\right)\right) \tag{1.2.22}$$

where C is a constant that does not depend on energy and appears from renormalisation effects, ($C = 1$ at tree level), A and B represent any other fields that may be present and m_V is the mass of the heaviest gauge boson V_i. The Goldstone bosons do not have polarisation vectors so the power counting is made more simple and reliable, it can then be shown that at tree level, the degree of divergence is at most $\mathcal{O}(E^2)$ [31].

In the case of WW scattering above, without the physical Higgs boson, the gauge symmetry is realised non-linearly on the Goldstone bosons and the Lagrangian for the Goldstone bosons contains terms with derivative couplings which lead to the amplitude being of $\mathcal{O}(E^2)$. With the inclusion of the physical Higgs boson, the gauge symmetry is realised linearly on the Higgs sector and the interactions do not contain any derivatives, leading to the amplitude in Eq. (1.2.16).

Chapter 2

Unitarity of gravity coupled to models of particle physics

As discussed in the opening chapter, one of the best ways to understand the realm of validity for an effective theory is to calculate the energy scale where perturbative unitarity breaks down. In the first section of this chapter we do exactly this for the effective theory of gravity coupled to matter as given by the action (1.1.10). In the second section we apply the bound to various grand unified theories. In the third section we incorporate renormalisation group (RG) effects into the bounds and are then able to compare the scale at which unitarity breaks down with the scale of strong coupling. We discuss the consequences of the RG improved bounds for various models of particle physics and introduce two models which can lower the scale of quantum gravity in four dimensions. The unitarity bound derived here will also provide an important basis for later chapters.

2.1 Unitarity of linearised general relativity

In this section we calculate the unitarity bound for the effective theory of gravity coupled to matter as given by the action (1.1.10). The calculation was first performed by Han and Willenbrock [43]. We have verified their calculation using `FeynCalc` and `Mathematica`.

The first step is to calculate $2 \to 2$ graviton exchange amplitudes for tree level scattering of complex scalars s, Weyl fermions ψ and vector bosons V in the high energy (massless) limit. We restrict ourselves to the case where initial and final states consist of different particles. This simplifies the calculations tremendously since only s-channel processes need to be considered. The amplitudes for all possible such processes are given in Table (2.1) and agree with those obtained in Ref. [43]. Note that a factor of $-\frac{1}{4}sM_P^{-2}$

$\to$	$s'\bar{s}'$	$\psi'_+\bar{\psi}'_-$	$\psi'_-\bar{\psi}'_+$	$V'_+V'_-$	$V'_-V'_+$
$s\bar{s}$	$2/3\ d^2_{0,0} - 2/3(1+6\xi)^2\ d^0_{0,0}$	$\sqrt{2/3}\ d^2_{0,1}$	$\sqrt{2/3}\ d^2_{0,-1}$	$2\sqrt{2/3}\ d^2_{0,2}$	$2\sqrt{2/3}\ d^2_{0,-2}$
$\psi_+\bar{\psi}_-$	$\sqrt{2/3}\ d^2_{1,0}$	$d^2_{1,1}$	$d^2_{1,-1}$	$2\ d^2_{1,2}$	$2\ d^2_{1,-2}$
$\psi_-\bar{\psi}_+$	$\sqrt{2/3}\ d^2_{-1,0}$	$d^2_{-1,1}$	$d^2_{-1,-1}$	$2\ d^2_{-1,2}$	$2\ d^2_{-1,-2}$
V_+V_-	$2\sqrt{2/3}\ d^2_{2,0}$	$2\ d^2_{2,1}$	$2\ d^2_{2,-1}$	$4\ d^2_{2,2}$	$4\ d^2_{2,-2}$
V_-V_+	$2\sqrt{2/3}\ d^2_{-2,0}$	$2\ d^2_{-2,1}$	$2\ d^2_{-2,-1}$	$4\ d^2_{-2,2}$	$4\ d^2_{-2,-2}$

Table 2.1: Scattering amplitudes for scalars, fermions and vector bosons via s-channel graviton exchange in terms of the Wigner d-functions in the massless limit. A factor of $-\frac{1}{4}sM_P^{-2}$ has been extracted from each of the amplitudes.

has been extracted from each of the amplitudes. We have used the helicity basis[1] for the 'in' and 'out' states and the subscripts + and − refer to helicity. The spinors and polarisation vectors in this basis are given in Appendix A. We also use the Feynman rules of Ref. [29] which are reproduced in Appendix E.

2.1.1 j=2 partial wave amplitude

The partial wave amplitudes are found using Eq. (1.2.6) and so are simply proportional to the entries in Table (2.1) with the Wigner d-functions removed. To obtain the lowest unitarity bound we wish to find the eigenvalues of the matrix of partial wave amplitudes for N_s complex scalars, N_ψ fermions and N_V vector bosons. Since all entries contain a $j = 2$ partial wave, this is what will be focussed on here. The $j = 0$ partial wave will be considered separately later. Because the partial waves for opposite helicity processes are identical, the matrix can be simplified by only considering the $+,-$ helicity combinations and not $-,+$. With N_φ degrees of freedom we may consider the normalised state obtained by including all N_φ particles in the initial and final states: $(1/N_\varphi)\sum \varphi_+\varphi_-$. The matrix of partial waves thus obtained is given in Table 2.2.

Due to the symmetric nature of the matrix it only has a single eigenvalue, given by the trace

$$a_2 = -\frac{1}{320\pi}\,\frac{s}{M_P^2}N, \tag{2.1.1}$$

[1]Note that as in the case for WW scattering in the standard model (see Section 1.2.1) we might think that including longitudinally polarised vector bosons in the external states may lead to the largest high energy behaviour of the scattering amplitudes. However, as can be seen by considering the Goldstone boson equivalence theorem, there should be cancellations that happen in the calculation of such amplitudes so that the high energy behaviour is no stronger than for transversely polarised vector bosons. Indeed this is the case and we have verified it for the scattering amplitudes presented here.

$\rightarrow$	$\frac{1}{\sqrt{N_s}}\Sigma s'\bar{s}'$	$\frac{1}{\sqrt{N_\psi}}\Sigma\psi'_+\bar{\psi}'_-$	$\frac{1}{\sqrt{N_V}}\Sigma V'_+V'_-$
$\frac{1}{\sqrt{N_s}}\Sigma s\bar{s}$	$2/3N_s$	$\sqrt{2/3}\sqrt{N_S N_\psi}$	$2\sqrt{2/3}\sqrt{N_S N_V}$
$\frac{1}{\sqrt{N_\psi}}\Sigma\psi_+\bar{\psi}_-$	$\sqrt{2/3}\sqrt{N_S N_\psi}$	N_ψ	$2\sqrt{N_\psi N_V}$
$\frac{1}{\sqrt{N_V}}\Sigma V_+V_-$	$2\sqrt{2/3}\sqrt{N_s N_V}$	$2\sqrt{N_\psi N_V}$	$4N_V$

Table 2.2: $j = 2$ partial wave amplitudes for N_s scalars, N_ψ fermions and N_V vector bosons via s-channel graviton exchange. A factor of $-\frac{1}{320\pi}sM_P^{-2}$ has been extracted from each of the amplitudes.

where [43]

$$N = \frac{2}{3}N_s + N_\psi + 4N_V. \tag{2.1.2}$$

This amplitude is the main result of this chapter. Using this amplitude it is possible to test where tree level unitarity breaks down in models of particle physics coupled to linearised general relativity. Requiring that $|\text{Re}(a_2)| \leq \frac{1}{2}$ leads to the unitarity bound

$$\sqrt{s} \leq M_P\sqrt{\frac{160\pi}{N}}. \tag{2.1.3}$$

2.1.2 j=0 partial wave amplitude

For models with large numbers of scalar fields or large non-minimal couplings, it may also be of interest to consider the unitarity bound obtained from from the $j = 0$ partial wave amplitude. First, consider the scattering of N_s scalar fields, all with identical non-minimal coupling ξ. The partial wave amplitude for this process can be read off from Table (2.1) giving

$$a_0 = \frac{(1+6\xi)^2}{96\pi}\frac{s}{M_P^2}N_s. \tag{2.1.4}$$

Applying the unitarity bound to this amplitude, $|\text{Re}(a_0)| \leq \frac{1}{2}$ for complex scalars or $|\text{Re}(a_0)| \leq 1$ for real scalars gives

$$\sqrt{s} \leq \frac{M_P}{1+6\xi}\sqrt{\frac{96\pi}{N_s}} \tag{2.1.5}$$

where N_s is the number of complex scalar fields (or twice the number of real scalar fields).

2.2 Unitarity of models of particle physics

Given the unitarity bounds (2.1.3) and (2.1.5) it is possible to find where tree level unitarity breaks down for any model by considering its matter content. For example, in the standard model, $N_s = 2$, $N_\psi = 45$ (we only include left handed neutrinos), $N_V = 12$ and so we find $N = 283/3$ and we find unitarity breaks down at a scale $E_\star = 2.3 M_P$ from the $j = 2$ partial wave amplitude.

In Ref. [3] we calculated the scale at which unitarity breaks down in a variety of models. The results are presented in Table 2.3. The last two columns show the scale at which unitarity is violated for both the $j = 0$ and $j = 2$ partial wave amplitudes as a fraction of the Planck mass M_P. It is assumed that $\xi = 0$ in all models. Note that in some models unitarity breaks down below M_P. Naively, one may expect gravitational effects to become strongly coupled at M_P, so it may be a surprise to see unitarity problems appearing below this scale. However, to properly interpret these results we need to analyse more carefully the scale at which we expect gravity to become strongly coupled. This subject will be taken up in the next section using the techniques of the renormalisation group.

2.3 Running of the Planck mass and renormalisation group improved unitarity bound

Within the effective field theory framework of gravity, it is possible to define the Planck mass as a coupling that runs under renormalisation group (RG) effects, analogously to the well established RG running of the gauge couplings in the standard model. For example, based on calculations of the renormalisation of Newton's constant by Larsen and Wilczek [44] (see also Ref. [45]), Calmet, Hsu and Reeb defined a running Planck mass which depends on the RG scale μ in the following way [20]:

$$M_P(\mu)^2 = M_P(0)^2 - \frac{1}{96\pi^2}\mu^2 N_l, \tag{2.3.1}$$

where

$$N_l = N_s + N_\psi - 4N_V. \tag{2.3.2}$$

Note that N_l is not the same as N and noticeably the sign for the contribution of vector bosons is opposite in the two cases.

This result has been rigorously derived using heat kernel techniques (see Refs. [44, 20] for more details). Here, we give a brief illustration of how this effect can be seen to arise. Consider the one loop self energy correction to the graviton propagator, Fig. 2.1, where matter particles may run in the loop. Neglecting the index structure, this correction to the propagator is

$$\Delta(q^2) \sim \frac{i}{M_P^2 q^2} + \frac{i}{M_P^2 q^2}\Sigma\frac{i}{M_P^2 q^2} + \cdots, \tag{2.3.3}$$

particle physics model	N	N_S	j=2 bound	j=0 bound
standard model	283/3	4	2.3	8.7
MSSM	425/3	98	1.9	1.8
SU(5) w/ **5**, **24**	457/3	34	1.8	3.0
SU(5) w/ **5**, **200**	211	210	1.5	1.2
SU(5) w/ **5**, **24**, **75**	532/3	109	1.7	1.7
SU(5) w/ **5**, **24**, **75**, **200**	244	309	1.4	0.99*
SO(10) w/ **10**, **16**, **45**	781/3	97	1.4	1.8
SO(10) w/ **10**, **16**, **210**	946/3	262	1.3	1.1
SO(10) w/ **10**, **16**, **770**	502	822	1.0	0.61*
SUSY-SU(5) w/ **5**, $\overline{\mathbf{5}}$, **24**	755/3	158	1.4	1.4
SUSY-SU(5) w/ **5**, $\overline{\mathbf{5}}$, **24**, **75**	1130/3	308	1.2	0.99*
SUSY-SU(5) w/ **5**, $\overline{\mathbf{5}}$, **200**	545	510	0.96*	0.77*
SUSY-SO(10) w/ **10**, **16**, $\overline{\mathbf{16}}$, **45**, **54**	540	378	0.96*	0.89*
SUSY-SO(10) w/ **10**, **16**, $\overline{\mathbf{16}}$, **210**	725	600	0.83*	0.71*
SUSY-SO(10) w/ **10**, **16**, $\overline{\mathbf{16}}$, **770**	4975/3	1720	0.55*	0.42*

Table 2.3: Different grand unified models which have been considered in the literature. The last two columns show the scale at which tree level unitarity breaks down as a fraction of M_P in each model due to the bound from the $j = 2$ partial wave bound (2.1.3) or the $j = 0$ partial wave bound (2.1.5). It is assumed that $\xi = 0$. Entries marked with $*$ highlight where tree level unitarity breaks down below M_P. This can be compared with the approximate scale at which one expects strong coupling, see for example Eq. (2.3.9).

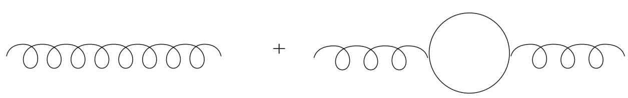


Figure 2.1: The one loop contribution to the running Planck mass. The curly line represents the graviton and the straight lines represents the matter particles running in the loop.

where Σ is the self energy insertion. Σ can be estimated from the Feynman diagram:

$$\Sigma \sim -iq^2 \int^{\Lambda} d^4p \Delta_m (p^2)^2 p^2 + \cdots , \qquad (2.3.4)$$

where Δ_m is the propagator for the matter particle running in the loop and Λ is the ultraviolet cutoff of the loop. For scalar fields, the loop integral is quadratically divergent, and by absorbing the divergence in the redefinition of M_P we obtain

$$M^2_{P(ren)} = M^2_{P(bare)} + c\Lambda^2. \qquad (2.3.5)$$

Taking $\Lambda = \mu$ we recover the form of the rigorously derived running Planck mass Eq. (2.3.1)[2].

It is simple to incorporate a running Planck mass into the tree level unitarity bounds Eqs. (2.1.1) and (2.1.4) in order to give an 'RG improved' unitarity bound:[3]

$$\sqrt{s} \leq M_P(\sqrt{s})\sqrt{160\pi/N}, \qquad (2.3.6)$$

$$\sqrt{s} \leq M_P(\sqrt{s})\sqrt{96\pi/N_s}. \qquad (2.3.7)$$

In Ref. [3], we argued that since the running Planck mass incorporates quantum effects into the definition of the coupling constant, a running Planck mass gives a good indication of when quantum gravitational effects become strong. The scale at which quantum gravitational effects become strong is therefore defined as $\mu_\star$, where

$$\mu_\star^2 \simeq M_P(\mu_\star)^2. \qquad (2.3.8)$$

This criteria ensures that the scale $\mu_\star$ is the scale at which the expansion parameter for the effective theory, $E/M_P(E)$, is equal to one, i.e. the theory becomes strongly coupled. Since loop effects are normally accompanied by a factor of $1/16\pi^2$ (coming from the integral over unconstrained loop momenta), it could even be argued that the criteria (2.3.8) is rather conservative and in fact the scale at which gravitational effects are expected to become strong could easily be an order of magnitude higher than $\mu_\star$. For example in Ref. [48]

[2] We remark here that despite the rigorous heat kernel derivation of Eq. (2.3.1), a recent publication [46] has criticised attempts to define a running Planck mass. The main argument is that a precise definition of the running is not independent of the process from which it was derived. This therefore leads to difficulty in defining a universally applicable running. If true, these criticisms could cast doubt on the validity of our arguments here. However, we only consider a single process, s-channel scattering via graviton exchange, and so we need not worry about universality of the definition of the running. The running we employ is defined from exactly the process we wish to consider and should therefore be applicable everywhere we have used it, even if it were not applicable for other processes.

[3] A similar procedure of defining an RG improved unitarity bound was given in Ref. [47] for the bound on the Higgs boson mass from WW scattering as outlined in Section 1.2.1.

(see also Ref. [49]) a careful power counting analysis is carried out and it is shown that a generic sufficient condition for successive loops of interactions to be smaller than preceding ones is

$$\frac{E}{4\pi M_P} < 1. \tag{2.3.9}$$

Despite this, we will retain the criteria (2.3.8) as the scale at which we expect gravitational effects to become strongly coupled, safe in the knowledge that this is a conservative estimate.

Accepting that $\mu_\star$ is the scale at which gravitational effects are expected to become strong, we argued in Ref. [3] that if either of the RG improved unitarity bounds, Eq. (2.3.6) or (2.3.7), showed unitarity problems below $\mu_\star$, then the unitarity problem could not be fixed by strong coupling effects. We are still in the weakly coupled regime and so higher order effects can not be sizeable enough to counteract the rapid growth with energy of the amplitudes. Additionally, higher order effects coming from the graviton self energy have already been incorporated into the bound via the RG. The interpretation is then that the unitarity problem is a clear sign that either new physics that fixes the unitarity problem must enter at or below the unitarity violation scale, or the model would be inconsistent (suffer from an incurable unitarity problem). The requirement to distinguish such cases is therefore whether or not the theory remains unitary up to the point when $\sqrt{s} = M_P(\sqrt{s})$. Clearly this will occur if

$$N \leq 160\pi \tag{2.3.10}$$

and

$$N_s \leq 96\pi. \tag{2.3.11}$$

Note that these criteria are completely independent of the specific running of the Planck mass[4].

In Ref. [3] we distinguish the models analysed in Table 2.3 according to the above criteria. Since the criteria are independent on the details of the running Planck mass, the models in Table 2.3 can be distinguished by whether or not either of the two unitarity bounds are below M_P. Entries in Table 2.3 where unitarity breaks down below M_P have been marked with $*$. All the models for which the unitarity bound is below M_P are therefore classified as being inconsistent without the addition of new physics below the

[4] We remark again that the concerns raised in Ref. [47] about defining a universal running Planck mass are not relevant here since the argument given in this section turns out to be independent of the specific running employed.

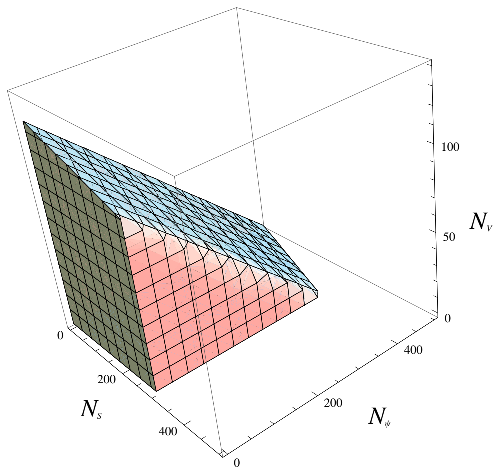


Figure 2.2: The parameter space for models in which unitarity is maintained up to the scale $\mu_\star$.

scale at which gravity becomes strong. The parameter space for all models for which unitarity is maintained up to the scale $\mu_\star$ is plotted in Fig 2.2.

Model with large number of fields

The main motivation for investigating the running of the Planck mass in Ref. [20] was to utilise the running to propose a model which could offer a solution to the seemingly unnatural hierarchy between the electroweak and quantum gravity scales. By introducing an extremely large number of scalar or fermion fields the scale at which gravity becomes strong can be significantly lowered (note that due to the sign of the vector boson contribution to N_l, a large number of spin one particles will act to increase the scale of strong coupling). If the scale $\mu_\star$ is identified as the scale of quantum gravity, then the hierarchy problem will not exist if $\mu_\star$ can be lowered to the electroweak scale.

If $M_P(\mu_\star) \ll M_P(0)$ then we find the value of N_l required to have $M_P(\mu_\star) = \mu_\star$ is given by

$$N_l = 96\pi^2 \frac{M_P(0)^2}{\mu_\star^2}. \tag{2.3.12}$$

In order to have $M_P(\mu_\star) = \mu_\star = 1$ TeV we require $N_l \simeq 5 \times 10^{33}$. Assuming that the entire contribution to N_l is made up of scalars, i.e. $N_l = N_s$, we find unitarity is violated (using the $j = 0$ bound) at a scale

$$E_\star = \frac{\mu_\star}{\sqrt{1+\pi}} \simeq 0.5\mu_\star. \qquad (2.3.13)$$

This is below the scale at which gravity is expected to become strong and so new physics will need to enter at this scale in order to fix the unitarity problem.

Model with a large non-minimal coupling

In Ref. [4], we noted that one could also define a running Planck mass based on the results of Ref. [44] for models with non-minimally coupled scalar fields. The running Planck mass defined in this way for N_s scalar fields with non-minimal coupling ξ is given by

$$M_P(\mu)^2 = M_P(0)^2 - \frac{(1+6\xi)}{96\pi^2}\mu^2 N_s. \qquad (2.3.14)$$

As a result of this running, it was observed that not only could one lower the Planck mass by introducing a large number of fields, one could also achieve this by introducing one or more scalar fields with very large non-minimal couplings. This opened the door to yet another model offering a solution to the hierarchy problem. If we require $M_P(\mu_\star) = \mu_\star$ and assuming that $M_P(0) \gg M_P(\mu_\star)$ and $\xi \gg 1$ we find

$$\xi N_s = 16\pi^2 \frac{M_P(0)^2}{\mu_\star^2}. \qquad (2.3.15)$$

In order to have $M_P(\mu_\star) = \mu_\star = 1$ TeV, we would require $\xi N_s \simeq 9 \times 10^{32}$. Assuming $N_s > 2$, so that the s-channel unitarity bound is valid, we then find unitarity is violated (using the $j = 0$ bound) at a scale

$$E_\star = \frac{\mu_\star}{\sqrt{1+6\pi\xi}} \ll \mu_\star. \qquad (2.3.16)$$

Again this is below the scale at which gravity is expected to become strong and so new physics will need to enter at this scale in order to fix the unitarity problem.

Chapter 3

Unitarity of models with extra dimensions

In this chapter we investigate a number of models that utilise extra dimensions in order to address the seemingly unnatural hierarchy between the electroweak and Planck scales. We specifically calculate the scale at which perturbative unitarity breaks down in order to understand the energy regime for which the effective theory used to study these models is valid. These models have been extensively researched and there are many experimental searches for signatures of these models at the LHC. The search strategies rely on comparing experimental data to predictions calculated using the effective theory and for this reason alone it is crucial to have a firm understanding of when the effective theory is valid.

In the first section we introduce the general idea of extra dimensions and the concept of Kaluza-Klein modes. We discuss how these models can be viewed as an effective theory with a cutoff and how we may attempt to use perturbative unitarity in such models to provide an upper limit to the size of the cutoff. We then derive a unitarity bound in a general model independent way.

Following this we introduce and calculate unitarity bounds in three different popular models of extra dimensions: the ADD model, the Randall-Sundrum model and the linear dilaton model.

3.1 Extra Dimensions and Kaluza-Klein modes

The idea that there may be extra space dimensions that we have not yet observed was first proposed by Kaluza in 1921 [50] and then expanded on by Klein in 1926 [51]. The theory was introduced as a novel geometrical unification of electromagnetism and general relativity. Extra dimensions and so called Kaluza-Klein theories received renewed interest in the 1980's with the growth of research in supergravity and string theory and the understanding that string theories require extra dimensions in order to be internally consistent. However, the extra dimensional models I will be discussing here are not motivated by the idea of unification but instead were introduced to resolve the seemingly unnatural hierarchy between the electroweak and Planck scales. As such they were received with great excitement as they offered the possibility to observe strong and quantum gravitational phenomena in particle collisions at accelerators such as the Tevatron and the LHC. The first model of this type, introduced by Antoniadis, Arkani-Hamed, Dimopoulos and Dvali in 1998 became known as the ADD model [52, 53]. The extra dimensions are flat and the hierarchy problem is addressed by the observation that, in the presence of the extra dimensions, the fundamental Planck mass can be lowered to the TeV scale, thus eliminating the hierarchy between the electroweak and Planck scales. Following this, Randall and Sundrum developed a model with a single extra dimension with a warped geometry [54]. Here the hierarchy problem is resolved by having all the fundamental scales, including the Higgs VEV, at the Planck scale. The warped geometry then acts to exponentially suppress the Higgs VEV to the electroweak scale where it couples to standard model particles. Both of these models generated huge excitement and spawned large industries of research. The original papers [52, 54] have now received well over 4,000 citations each.

In addition to the intense research into the ADD and Randall-Sundrum (RS) models, a number of other extra dimensional models have since been introduced to offer solutions to the hierarchy problem. In this thesis we will consider the ADD, RS and linear dilaton models [55].

We will expand on the specific details of each of these models in the introductions to their respective sections below. First we will discuss a few general features of extra dimensional models, why they need to be viewed as effective field theories, introduce the concept of Kaluza-Klein modes and investigate what can be said about unitarity of extra dimensional models in the most general model independent case.

3.1.1 Extra dimensional models as effective theories with a low cutoff

The common feature of the extra dimensional models discussed here is that they offer a solution to the hierarchy problem. They do this in different ways but essentially they remove the hierarchy by matching the extra dimensional Planck scale to the extra dimensional electroweak scale[1]. For this reason, a common feature of the models is that strong gravitational effects will appear around the electroweak scale. We will see how this comes about for each individual model separately. However, because we expect strong gravitational effects at this very low scale, the cutoff for the effective theory of gravity coupled to matter will be reduced from the Planck scale (see Section 1.1) to the TeV scale. We would like to reliably calculate observable quantities in these models and so it is essential to have a good idea of where we expect this cutoff to appear. As discussed in Section 1.2, the scale at which unitarity in scattering amplitudes breaks down should provide a good estimate for this cutoff. The main aim of this chapter is to determine the strongest unitarity bounds available in these models in order to understand the regime of validity of the effective field theory approach.

One of the exciting prospects of the extra dimensional models presented here is that with strong gravitational effects appearing at the TeV scale, they offer the prospect of observing quantum gravity and other phenomena such as black hole formation at particle accelerators such as the LHC.

3.1.2 Kaluza-Klein modes

An important feature of compact extra dimensions is the concept of Kaluza-Klein (KK) modes. For this reason we will now give a brief review of KK modes. For illustrative purposes we first use the example of a scalar field living in a single extra dimension. We will then briefly discuss the extension to the graviton field. The specific couplings of the graviton field to matter are model dependent and will be introduced at the beginning of each respective section.

For simplicity, let us consider a single flat extra dimension compactified on a circle of radius r. We can denote the usual four dimensional Minkowski spacetime coordinates by x^μ and the coordinate in the extra dimension by y. The action for a complex scalar field $\Phi(x,y)$ living in the full five dimensional bulk is then given by

$$S = \int d^4x \int_0^{2\pi r} dy \left(\frac{1}{2}\partial_M \Phi^* \partial^M \Phi - \frac{1}{2} m_0^2 \Phi^* \Phi \right). \tag{3.1.1}$$

Capital letter indices such as M run over the full five dimensional coordinates, greek letters will be retained for the standard four dimensional Minkowski space and lower case latin indices (i,j etc.) will represent the extra dimensional coordinates. Compactification on

[1]The removal of the hierarchy can often come at the price of introducing a fine tuning in the geometry of the extra dimension. The amount of fine tuning required is model dependent as we will see later in this chapter.

a circle imposes periodic boundary conditions. For example Φ must be periodic under $y \to y + 2\pi r$. This means that we can decompose Φ with the following mode expansion

$$\Phi(x, y) = \frac{1}{\sqrt{2\pi r}} \sum_{n=-\infty}^{\infty} \phi_n(x) e^{iny/r}. \tag{3.1.2}$$

Substituting this into Eq. (3.1.1), we find

$$S = \frac{1}{2\pi r} \int d^4x \; dy \sum_{m,n} \left(\frac{1}{2} \partial_\mu \phi_m^* \partial^\mu \phi_n - \frac{1}{2} \frac{mn}{r} \phi_m^* \phi_n - m_0^2 \phi_m^* \phi_n \right) e^{i(n-m)y/r}. \tag{3.1.3}$$

Now using the orthogonality of the exponential:

$$\int_0^{2\pi r} dy \; e^{i(n-m)y/r} = 2\pi r \delta_{mn} \tag{3.1.4}$$

we find

$$S = \int d^4x \sum_n \left(\frac{1}{2} \partial_\mu \phi_n^* \partial^\mu \phi_n - \frac{1}{2} \left(m_0^2 + \frac{n^2}{r^2} \right) \phi_n^* \phi_n \right). \tag{3.1.5}$$

This is a 4D action for an infinite tower of 4D fields with masses

$$m_n^2 = m_0^2 + \frac{n^2}{r^2} \qquad , \qquad n \in \mathbb{Z}. \tag{3.1.6}$$

The fields ϕ_n are known as Kaluza-Klein (KK) modes. The same procedure can be carried out for fields of any spin with no real extra complication.

The extension to more than one extra dimension is straightforward. If we extend the idea to δ extra dimensions, all of which have common radius r, we would obtain the same action but with a mass spectrum

$$m_{\vec{n}}^2 = m_0^2 + \frac{\vec{n}^2}{r^2} \qquad , \qquad \vec{n} = (n_1, n_2, \ldots, n_\delta), \tag{3.1.7}$$

$$m_{\vec{n}}^2 = m_0^2 + \frac{\vec{n}^2}{r^2} \qquad , \qquad \vec{n} = (n_1, n_2, \ldots, n_\delta), \tag{3.1.8}$$

where we see that the index $\vec{n}$ is now a vector in a discretised δ-dimensional lattice.

Kaluza-Klein gravitons

The formalism for dealing with the graviton degrees of freedom in extra dimensions has been well developed in in Refs. [29, 30]. It is also covered in many good reviews such as Ref. [56] which we will follow closely here. We will give a brief overview of the main important features, some of the model dependent features will be separately developed at the beginning of the respective sections.

In δ extra dimensions, the graviton is a D by D symmetric tensor, where $D = 4 + \delta$ is the total number of dimensions. After consideration of the gauge symmetries of general coordinate invariance, the graviton is found to have $D(D-3)/2$ independent degrees of freedom. So from a 4D perspective there will be more particles than just the 4D graviton.

The full D dimensional metric g_{MN} can be expanded in fluctuations (the graviton modes) around flat space

$$g_{MN} = \eta_{MN} + \frac{1}{2M_*^{1+\delta/2}} h_{MN}, \tag{3.1.9}$$

where M_* represents the D-dimensional Planck mass. We can then expand the fluctuations into KK modes

$$h_{MN}(x,y) = \sum_{\vec{n}} \frac{1}{N_{\vec{n}}} h^{\vec{n}}_{MN}(x) \chi_{\vec{n}}(y), \tag{3.1.10}$$

where $N_{\vec{n}}$ is a normalisation factor which may be dependent on the mode number $\vec{n}$, and $\chi_{\vec{n}}(y)$ is the wavefunction of the KK mode in the extra dimension.

The metric g_{MN} can be generically decomposed into tensor, vector and scalar modes in the following way

$$g_{MN} = \left(\begin{array}{c|c} g^{\vec{n}}_{\mu\nu} & V^{\vec{n}}_{\mu j} \\ \hline V^{\vec{n}}_{i\nu} & S^{\vec{n}}_{ij} \end{array} \right). \tag{3.1.11}$$

It can be shown [29, 30] that for compactification on a torus, some of the vector and scalar modes are "eaten" by the graviton modes in a Higgs like mechanism to provide the extra degrees of freedom required by the massive KK gravitons. In this compactification it will also be seen that the remaining vector modes completely decouple from the theory and so can be disregarded. The same is true for many of the scalar modes. The only scalar field that remains and couples to matter is the $S_i^{(\vec{n})i}$ field which is related to the size of the extra dimensional volume. The fluctuation of this field $h_i^{(\vec{n})i}$ is known as the radion. In more complex geometries, the behaviour of vector and scalar modes may be more complicated.

In the models that will be considered here, the matter particles will be confined to a brane with three space dimensions. They will experience an induced metric on the brane $g_{\mu\nu}(x)$. If the action for the matter on the brane is denoted S_m then the definition of the energy momentum tensor is as usual given by

$$\sqrt{g} T^{\mu\nu} = \frac{\delta S_m}{\delta g_{\mu\nu}}, \tag{3.1.12}$$

and the coupling between the graviton and matter is given by

$$\mathcal{L}_{int} = T^{\mu\nu}\frac{h_{\mu\nu}}{M_*^{1+\delta/2}}. \tag{3.1.13}$$

Substituting in the KK mode expansion we find

$$\mathcal{L}_{int} = \sum_{\vec{n}} T^{\mu\nu}\frac{1}{M_*^{1+\delta/2}}\frac{h^{\vec{n}}_{\mu\nu}}{N_{\vec{n}}} = \sum_{\vec{n}} T^{\mu\nu}\frac{h^{\vec{n}}_{\mu\nu}}{\Lambda_{\vec{n}}}. \tag{3.1.14}$$

Where $\Lambda_{\vec{n}}^{-1}$ represents the coupling of the associated KK graviton. We will see that it is always the case that $\Lambda_0 = M_P$ so that the massless zero mode graviton couples to matter in the same way as the normal 4D graviton as required to reproduce general relativity on large scales. Using this, the Feynman rules for KK gravitons coupled to matter are derived in Refs. [29, 30] and reproduced in Appendix E.

The propagator for a massive KK graviton $h^{\vec{n}}_{\mu\nu}$ in harmonic gauge, $\partial^\mu(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h) = 0$, is given by [29]

$$\Delta_{\mu\nu,\rho\sigma}(k) = \frac{B_{\mu\nu,\rho\sigma}(k)}{k^2 - m^2_{\vec{n}} + i\epsilon} \tag{3.1.15}$$

where

$$\begin{aligned} B_{\mu\nu,\rho\sigma}(k) = &\left(\eta_{\mu\rho} - \frac{k_\mu k_\rho}{m^2_{\vec{n}}}\right)\left(\eta_{\nu\sigma} - \frac{k_\nu k_\sigma}{m^2_{\vec{n}}}\right) + \left(\eta_{\mu\sigma} - \frac{k_\mu k_\sigma}{m^2_{\vec{n}}}\right)\left(\eta_{\nu\rho} - \frac{k_\nu k_\rho}{m^2_{\vec{n}}}\right) \\ &- \frac{2}{3}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{m^2_{\vec{n}}}\right)\left(\eta_{\rho\sigma} - \frac{k_\rho k_\sigma}{m^2_{\vec{n}}}\right). \end{aligned} \tag{3.1.16}$$

It will be useful later to make the separation

$$\Delta_{\mu\nu,\rho\sigma}(k) = B_{\mu\nu,\rho\sigma}(k)\Delta_{\vec{n}}(k^2) \tag{3.1.17}$$

where

$$\Delta_{\vec{n}}(k^2) = \frac{1}{k^2 - m^2_{\vec{n}} + i\epsilon}. \tag{3.1.18}$$

In the high energy limit we find

$$B_{\mu\nu,\rho\sigma}(k) = \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \frac{2}{3}\eta_{\mu\nu}\eta_{\rho\sigma}. \tag{3.1.19}$$

Note the factor of 2/3 that differs from Eq. (1.1.9) differentiates the massless propagator to the massive propagator in the massless limit (this is known as the Van Dam-Veltman-Zakharov (VDVZ) discontinuity [57, 58]). The Feynman rules for KK gravitons coupling to matter are thus identical to those for massless gravitons with the exception of the coupling strength and the expression for the propagator.

$\to$	$s'\bar{s}'$	$\psi'_+\bar{\psi}'_-$	$\psi'_-\bar{\psi}'_+$	$V'_+V'_-$	$V'_-V'_+$
$s\bar{s}$	$2/3d^2_{0,0}$	$\sqrt{2/3}\ d^2_{0,1}$	$\sqrt{2/3}\ d^2_{0,-1}$	$2\sqrt{2/3}\ d^2_{0,2}$	$2\sqrt{2/3}\ d^2_{0,-2}$
$\psi_+\bar{\psi}_-$	$\sqrt{2/3}\ d^2_{1,0}$	$d^2_{1,1}$	$d^2_{1,-1}$	$2d^2_{1,2}$	$2d^2_{1,-2}$
$\psi_-\bar{\psi}_+$	$\sqrt{2/3}\ d^2_{-1,0}$	$d^2_{-1,1}$	$d^2_{-1,-1}$	$2d^2_{-1,2}$	$2d^2_{-1,-2}$
V_+V_-	$2\sqrt{2/3}\ d^2_{2,0}$	$2d^2_{2,1}$	$2d^2_{2,-1}$	$4d^2_{2,2}$	$4d^2_{2,-2}$
V_-V_+	$2\sqrt{2/3}\ d^2_{-2,0}$	$2d^2_{-2,1}$	$2d^2_{-2,-1}$	$4d^2_{-2,2}$	$4d^2_{-2,-2}$

Table 3.1: Scattering amplitudes for scalars, fermions, and vector bosons via s-channel KK graviton exchange in terms of the Wigner d functions in the massless limit. A factor of $-\frac{1}{4}s^2\Lambda_n^{-2}\Delta_n(s)$ has been extracted from each of the amplitudes.

3.1.3 Partial wave amplitude for KK graviton exchange

Using the Feynman rules of Refs. [29, 30] the complete set of tree level s-channel scattering amplitudes between all different types of matter particles via KK graviton exchange can be calculated and are given in Table (3.1). We have used the helicity basis in Appendix A and the Feynman rules of Appendix E. We also work in the high energy limit where external particles but not KK gravitons are taken to be massless. Note that a factor of $-\frac{1}{4}s^2\Lambda_n^{-2}\Delta_n(s)$ has been extracted from each of the amplitudes.

The main difference between the entries in Table (3.1) and the amplitudes for the massless graviton in Table (2.1) occurs in the $ss \to s's'$ entry. The exchange of a massive graviton occurs only in the $j = 2$ channel (there is no $j = 0$ partial wave). This difference is related to the VDVZ discontinuity as explained in Ref. [59]. Diagonalising the matrix of partial wave amplitudes we find the $j = 2$ partial wave amplitude

$$a_2^{(n)} = -\frac{1}{320\pi}\frac{s^2}{\Lambda_n^2}\Delta_n(s)N. \tag{3.1.20}$$

The amplitudes in Table 3.1 can occur via exchange of any of the large number of KK gravitons and the total amplitude is the sum over all the KK modes. Incorporating this, the total diagonalised partial wave amplitude is

$$a_2 = -\frac{Ns^2}{320\pi}\mathcal{S}'(s), \tag{3.1.21}$$

where

$$\mathcal{S}'(s) = \sum_n \frac{1}{\Lambda_n^2}\Delta_n(s) = \sum_n \frac{1}{\Lambda_n^2(s - m_n^2 + i\epsilon)}. \tag{3.1.22}$$

The sum over KK modes will be model dependent and may not even converge. The consequences for the sum of the different spectrums of KK gravitons will be discussed separately in the three different extra dimensional models below. In many situations, the coupling Λ_n will be independent of n. We can therefore extract the coupling from the sum and we only need consider a simplified propagator sum which we will call $\mathcal{S}$ defined by

$$\mathcal{S}(s) = \sum_n \Delta_n(s) = \sum_n \frac{1}{s - m_n^2 + i\epsilon}. \tag{3.1.23}$$

3.1.4 Width of KK gravitons

Using the Feynman rules one can also determine an expression for the decay rate of a KK graviton. Expressions for the decay rate to different final states are given in Ref. [29], combining these, the total decay rate of a KK graviton is

$$\Gamma(m_n) = \frac{N m_n^3}{320\pi\Lambda_n^2}, \tag{3.1.24}$$

where $N = \frac{1}{3}N_s + N_\psi + 4N_V$ where N_s, N_ψ and N_V are respectively the number of real scalar fields, Weyl fermions and vector bosons which the KK graviton can decay into. Note that this is the same factor N as that appearing in (2.1.2). For the standard model with no right handed neutrinos and treating decay products as massless we have $N = 283/3$ and the total KK graviton width is

$$\Gamma(m_n) = \frac{283 m_n^3}{960\pi\Lambda_n^2}. \tag{3.1.25}$$

3.2 Unitarity of KK graviton resonances

Using only what has been developed so far, it is possible to show that if more than one KK graviton mode exists in a model, then the model will suffer from unitarity problems at the energy scale of the first KK mode. In this section, this simple observation will be derived. We will briefly discuss that this problem appears to stem from simply adding Breit-Wigner resonances. We then show that there are hints that if the resonances are added by fully taking into account interference effects between different KK modes, that it appears that this unitarity violation does not occur.

3.2.1 Sum of Breit-Wigner resonances

As mentioned, the sum over KK modes (3.1.22) will be model dependent. However, we can already make an important observation in any general model with KK gravitons. The tree level amplitude (3.1.20) for exchange of a single KK graviton diverges when the graviton is on shell, $k^2 = m_n^2$. To regulate this type of divergence (resonance) the standard technique is to introduce a Breit-Wigner width into the propagator [60],

$$\Delta_n(p^2) = \frac{1}{p^2 - m_n^2 + im_n\Gamma(m_n)}, \tag{3.2.1}$$

and the sum over modes Eq. 3.1.22) will become

$$\mathcal{S}'(s) = \sum_n \frac{1}{\Lambda_n^2(s - m_n^2 + im_n\Gamma(m_n))}. \tag{3.2.2}$$

Inserting (3.2.1) into the amplitude for the exchange of a *single* KK graviton Eq. (3.1.20) the imaginary part of the amplitude is given by

$$\mathrm{Im}\big[a_2^{(n)}(s)\big] = \frac{Ns^2}{320\pi\Lambda_n^2}\frac{m_n\Gamma(m_n)}{(s-m_n^2)^2 + m_n^2\Gamma(m_n)^2} \geq 0. \tag{3.2.3}$$

The imaginary part is positive for $s \neq 0$. Now, if we look at the amplitude when the exchanged graviton is on shell by setting $s = m_n^2$ and using Eq. (3.1.24) we have

$$\mathrm{Im}\big[a_2^{(n)}(m_n^2)\big] = 1. \tag{3.2.4}$$

The amplitude saturates unitarity exactly for exchange of a single on shell mode.

Now, the important point is that the addition of any further KK modes, will push this amplitude to exceed the unitarity bound. The total amplitude is given by the sum (3.2.2) and all modes contribute with a positive sign, Eq. (3.2.3). As a consequence, the contribution of the exchange of any extra modes will positively increase the imaginary part, pushing it to violate unitarity $|\mathrm{Im}[a_2(m_n)]| > 1$. We immediately see that the presence of more than one KK graviton leads to problems with unitarity at the first KK mode, $s = m_1^2$!

Comment on the massless limit

We will apply the above unitarity bound to three different models below. We note here that in the above we have used the massless limit. A similar derivation could be made including the masses of the external particles but this would complicate things greatly. If the first KK mode of the model under consideration lies well above the mass of the top quark $m_t \simeq 173$ GeV (the heaviest standard model particle) then the bound applies without restriction. However, if the first KK mode is lower lying, then strictly speaking we should consider a more careful analysis. However, even when including the masses of the external particles, to a first approximation the analysis should be the same as the massless limit and if we are to find unitarity problems below 173 GeV, this would be a big problem for the model since the perturbative standard model has now been well tested to much higher energies at the Tevatron and the LHC. If we find a breakdown of unitarity using the massless limit at energies below 173 GeV, we need not worry about determining this scale with high accuracy, this approximation is simply enough to know that the model suffers from serious inconsistencies with experimental observations.

3.2.2 Beyond the Breit-Wigner approximation

The unitarity problem described in the previous section comes about from summing amplitudes with Breit-Wigner widths. If resonances are far apart from each other (relative to the size of the widths) this procedure is normally a very good approximation. However, if the resonances significantly overlap, a more sophisticated procedure can be employed which takes into account interference between the resonances. The technology for dealing with nearby resonances has been developed in Ref. [61]. A full treatment of this topic goes beyond the scope of this thesis, however I will show that in the simple case of two nearby resonances, using the propagator of Ref. [61] we find that unitarity is maintained where it would be violated if we were to naively sum the separate Breit-Wigner amplitudes as we have done in Eq. (3.2.2) above.

First let us simply consider the behaviour of two degenerate modes with masses $m_1^2 = m_2^2 = s$. If we simply add the amplitudes as in Eq (3.2.2) we will find $\text{Im}[a_2(m_n)] = 2$, clearly exceeding the unitarity bound. However, since these two modes are degenerate in mass, they fully overlap and we should consider the interference between the two modes.

Following Ref. [61], we should replace the Δ_n part of the propagator for the exchange of two nearby resonances with common width Γ and KK mode numbers i and j with

$$\Delta_{ij} = \left(K^{-1}\right)_{ij} \tag{3.2.5}$$

where K is the matrix given by

$$K_{ij} = \left(p^2 - m_i^2\right)\delta_{ij} + i\Gamma. \tag{3.2.6}$$

This corrected propagator represents the possibility of interference between the resonances and the subscripts i, j represent the fact that the KK gravitons coupling to the external particles can be of different KK mode number as shown in Fig. 3.1. For example, the 'in' states could be coupled to KK mode number i and the 'out' states could be coupled to KK mode number j. When we consider the self energy correction to the graviton propagator (the shaded bubble in Fig. 3.1) which is ultimately the source of the Breit-Wigner width, we see that we must now also consider the case where $i \neq j$. The sum over KK modes is now a sum over the subscripts i and j.

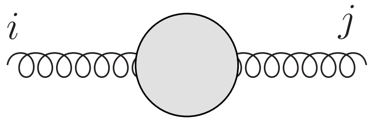


Figure 3.1: The corrected propagator for KK graviton exchange.

Using this corrected propagator to calculate the amplitudes for two nearby resonances we have plotted the size of the amplitude for different values of $\sqrt{s}$ in Figure 3.2. The dashed line comes from naively adding the separate amplitudes and clearly violates unitarity. The solid line uses the corrected propagator (3.2.5) which includes the effect of the interference between the modes and clearly remains unitary. Note we have chosen an arbitrary constant value of the coupling Λ_n in this example, the qualitative statements made here are not affected by the size of the coupling.

On further investigation it appears that this effect continues as we add in more and more modes, and the amplitude continues to remain unitary in the limit that the number of modes goes to infinity. Clearly this would imply that the full sum (3.1.21) is in fact unitary and finite at all energies. This is an extremely strong statement and relies on non-perturbative effects in order to be reliably verified. Also, for resonances far from the scattering energy we should incorporate the full expression for the graviton self energy (not just the width). For these reasons, further study of this effect goes beyond the scope of this thesis. It is however certainly a claim that warrants further research, and if true would overcome the unitarity problem described above which as we will see can have severe consequences for certain extra dimensional models.

Until these interesting observations have been more fully investigated, we will work throughout the rest of this thesis applying the unitarity bound from the previous section where we found that for models with more than one KK graviton unitarity breaks down at the first KK mode resonance. However, we keep in mind the caveat that this unitarity problem may be cured by the complicated non-perturbative interference effects outlined in this section, which could therefore provide an interesting avenue for future research.

3.3 Unitarity in the ADD model

In this section we introduce the ADD model and use a variety of approaches to attempt to calculate the scale of unitarity violation in the model.

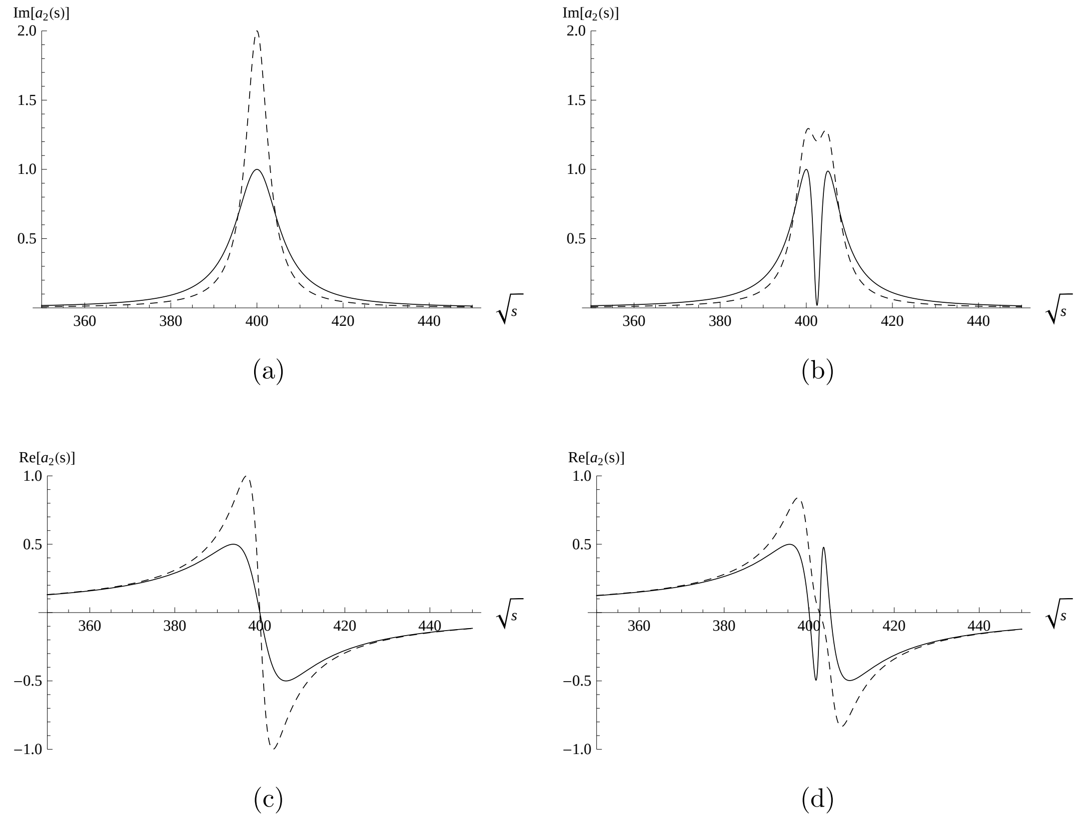


Figure 3.2: Partial wave amplitudes for the exchange of two nearby resonances: (a) $\mathrm{Im}[a_2(s)]$ both with masses $m_1 = m_2 = 400$ GeV (b) $\mathrm{Im}[a_2(s)]$, $m_1 = 400$ GeV and $m_2 = 405$ GeV, (c) $\mathrm{Re}[a_2(s)]$, $m_1 = m_2 = 400$ GeV and (d) $\mathrm{Re}[a_2(s)]$, $m_1 = 400$ GeV and $m_2 = 405$ GeV. The dashed line comes from naively adding the separate amplitudes and clearly violates unitarity. The solid line includes the effect of the interference between the modes and remains unitary.

3.3.1 Introduction to the ADD model

The ADD model was proposed in 1998 as a novel approach to addressing the hierarchy problem [52, 53]. The setup consists of δ flat compact extra dimensions with the standard model fields confined to a 3-brane. The hierarchy problem is addressed by allowing the fundamental Planck scale in the extra dimensions to be at the TeV scale. At high energies one would probe distances smaller than the size of the extra dimensions and experience strong gravitational effects coming from the low Planck scale. At low energies (large length scales), the gravitational force is diluted in the extra dimensions reproducing the usual weak force of gravity we experience.

To see how this works in more detail (again following Ref. [56]), we start with the Einstein-Hilbert action in the full extra dimensional spacetime:

$$S_{4+\delta} = -\frac{M_*^{\delta+2}}{2} \int d^{4+\delta}x \sqrt{g^{(4+\delta)}} R^{(4+\delta)}. \tag{3.3.1}$$

We would like to see how this is related to the usual 4D Einstein-Hilbert action

$$S_4 = -\frac{M_P^2}{2} \int d^4x \sqrt{g^{(4)}} R^{(4)}. \tag{3.3.2}$$

Since the space is flat, the full metric can be written as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu - r^2 d\Omega_{(\delta)}^2. \tag{3.3.3}$$

From this we can see how the extra dimensional quantities in (3.3.1) relate to the 4D quantities in (3.3.2). We find

$$\sqrt{g^{(4+\delta)}} = r^n \sqrt{g^{(4)}} \quad , \quad R^{(4+\delta)} = R^{(4)}. \tag{3.3.4}$$

We then have

$$S_{4+\delta} = -\frac{M_*^{\delta+2}}{2} \int d^{4+\delta}x \sqrt{g^{(4+\delta)}} R^{(4+\delta)} = -\frac{M_*^{\delta+2}}{2} \int d\Omega_{(\delta)} r^n \int d^4x \sqrt{g^{(4)}} R^{(4)}. \tag{3.3.5}$$

The factor $\int d\Omega_{(\delta)} r^n$ is just the extra dimensional volume, $V_{(\delta)}$. The simplest compactification geometry is a δ-torus with common radius r. Using this we have $V_{(\delta)} = (2\pi r)^\delta$. Integrating over the extra dimensions, we find the relationship

$$M_P^2 = M_*^{\delta+2} V_{(\delta)} \equiv M_D^{\delta+2} r^\delta. \tag{3.3.6}$$

This definition of M_D was first given in Ref. [30] and is now standard in the literature. From here on, in the context of the ADD model, we will refer to M_D as the fundamental Planck scale, rather than M_*.

We can now see that if the extra dimensional volume is large, the fundamental Planck scale can be much lower than the 4D Planck scale. However, on large length scales (when we can effectively integrate out the extra dimensions), we still reproduce weak Planck scale gravity. In order to offer a solution to the hierarchy problem we wish to lower the fundamental Planck scale M_D to the TeV scale. To do this we need to fix the size of the extra dimensions. Imposing $M_D \sim 1$ TeV, we find the size of the extra dimensions to be

$$r \sim 10^{-19} 10^{32/\delta} \text{ m.} \tag{3.3.7}$$

For $\delta = 1$ extra dimension, this gives $r = 10^{13}$ m. This is roughly the size of the solar system and is clearly ruled out. For $\delta = 2$ extra dimensions, we find $r \sim 1$ mm. The best direct experimental constraints coming from searches for deviations from Newton's inverse square law place the bound $r < 37\mu$m [62]. Interestingly, allowing $r = 37\mu$m would correspond to $M_D > 3.6$ TeV, which would only produce a very small hierarchy between the electroweak and fundamental Planck scales. $\delta > 2$ extra dimensions are not ruled out by direct experiments. There are however further bounds coming from astrophysical and cosmological constraints. The most stringent arises from the requirement that neutron stars are not excessively heated by KK decays into photons and leads to $M_D > 1700$ TeV for $\delta = 2$ and $M_D > 76$ TeV for $\delta = 3$ [63]. Also, the LHC has been able to place strong bounds on the model in searches for jets plus missing energy which would be associated with graviton production The CMS experiment places a lower bound on M_D of 4.54 TeV for $\delta = 2$, 2.98 TeV for $\delta = 4$ and 2.51 TeV for $\delta = 6$ [64]. The ATLAS experiment places a lower bound on M_D of 4.37 TeV for $\delta = 2$, 2.97 TeV for $\delta = 4$ and 2.53 TeV for $\delta = 6$ [65].

The graviton KK spectrum and couplings

We now turn to deriving the graviton KK spectrum and the couplings of KK gravitons to matter. In δ flat extra dimensions, the normalisation for the KK modes (3.1.10) is given by

$$h_{MN}(x, y) = \sum_{\vec{n}} \frac{1}{\sqrt{V_{(\delta)}}} h^{\vec{n}}_{MN}(x) e^{\vec{n}\cdot\vec{y}/r}. \tag{3.3.8}$$

Inserting this into the equations of motion, we find the KK spectrum

$$m_n^2 = \left(\frac{\vec{n}}{r}\right)^2 \quad , \quad \vec{n} = (n_1, n_2, \ldots, n_\delta). \tag{3.3.9}$$

Note that due to the large size of the extra dimensions the spacing between the modes is very fine and we have approximately 10^{32} KK modes below M_D.

Substituting the mode expansion into the expression for the graviton coupling to matter (3.1.14) we find

$$\mathcal{L}_{int} = \sum_{\vec{n}} T^{\mu\nu} \frac{1}{M_*^{1+\delta/2}} \frac{h^{\vec{n}}_{\mu\nu}}{\sqrt{V_{(\delta)}}} = \sum_{\vec{n}} \frac{1}{M_P} T^{\mu\nu} h^{\vec{n}}_{\mu\nu}. \tag{3.3.10}$$

All of the KK modes couple to matter with strength M_P^{-1}. At low energies and large length scales, standard model matter still feels a very weak gravitational force. At high energies the huge number of KK modes conspire to produce strong gravitational effects.

Now that we know the mass spectrum and the couplings of the KK gravitons in the ADD model, we may proceed to calculate the partial wave amplitudes and unitarity bounds in this model.

3.3.2 Unitarity in the ADD model

We saw in Section 3.2.1 that if an extra dimensional model contains more than one KK graviton, unitarity will break down at the first KK mode. This is certainly the case for the ADD model. Because the spacing between the modes in the ADD model is very fine, the lowest lying KK mode is at an extremely low energy. For example, in $\delta = 4$ extra dimensions, the lowest lying KK mode has a mass of approximately 20 keV. If unitarity completely breaks down at this scale it would spell disaster for the ADD model as no reliable perturbative calculations could be performed above the scale of the first KK mode. We note however, that the derivation of Section 3.2.1 finds unitarity problems when an exchanged KK graviton is on shell. In the ADD model, the width of the resonances is extremely small, $\Gamma \sim m_n^3/M_P^2$. We also see that the width is much smaller than the spacing between the modes $\Gamma \ll \delta m \simeq 1/r$. So we see that despite the problems with unitarity at the resonance peaks as explained in Section 3.2.1, there is a large range of energies between each resonance where the scattering amplitude will be much smaller and may not suffer from problems with unitarity until a much higher scale. In fact, the spacing between the modes, let alone the narrow width of the resonances, is much smaller than the energy resolution of detectors at the LHC. For this reason, and for calculational ease, it is extremely common in the ADD model to approximate the sum over modes by an integral (see Ref. [66] for discussions on the validity of these approximations in light of the finite detector resolution).

So in addition to our understanding that unitarity breaks down at the first resonance peak, in this section we attempt to find out if we can determine a unitarity bound coming from the parts of the amplitude which are not near resonances. We will use the standard technique of approximating the sum over modes by an integral which we also review here. Unfortunately we will see that despite our best efforts, it is extremely hard to separate non-resonant behaviour from the unitarity problems associated with resonances.

The following sections are based closely on our work in Refs. [4] and [5]. We first present an analysis in the zero width approximation where we will see that we have to deal with problems due to the divergent nature of the sum over KK modes as well as divergences from on shell KK graviton exchange. In the following section we will attempt to separate these problems by the introduction of a Breit-Wigner width. However, we will ultimately see that the unitarity problems still stem primarily from the resonances and that it is extremely difficult to separate the behaviour of the resonances from the behaviour away from resonances without introducing strong dependence on an arbitrary cutoff. We will summarise our findings in full in Section 3.3.5.

3.3.3 KK sum and unitarity in the zero width approximation

In order to see when unitarity breaks down in the ADD model we analyse the $j = 2$ partial wave amplitude

$$a_2 = -\frac{Ns^2}{320\pi M_P^2}\mathcal{S}(s), \tag{3.3.11}$$

where $\mathcal{S}(s)$ represents the sum over the KK graviton propagators. We will first consider this sum in the zero width approximation:

$$\mathcal{S}(s) = \sum_{\vec{n}} \frac{1}{s - m_{\vec{n}}^2 + i\epsilon}. \tag{3.3.12}$$

Due to the high density of massive modes in the ADD model, it is well known that this sum does not converge for $\delta > 1$ extra dimensions. It is therefore assumed that some sort of cutoff (Λ) to the theory exists, above which new degrees of freedom appear and the sum over modes should be curtailed here. In the ADD model the KK modes have a very fine spacing, $\delta m \sim 1/r$, and it is common practice to approximate the sum (3.1.22) by an integral [29, 30] (see also Refs. [66] for detailed analysis of the validity of this approximation). The number of modes with masses between m and $m + dm$ is given by

$$dN = S_{\delta-1} m^{\delta-1} r^{\delta} dm \tag{3.3.13}$$

where $S_{\delta-1} = 2\pi^{\delta/2}/\Gamma(\delta/2)$ is the surface of a unit-radius sphere in δ dimensions. Summing all the modes with masses $m_n \leq \Lambda$, we find

$$\mathcal{S}(s) = \sum_{\vec{n}} \frac{1}{s - m_{\vec{n}}^2 + i\epsilon} \simeq \int_0^{\Lambda} \frac{m^{\delta-1}}{s - m^2 + i\epsilon} S_{\delta-1} r^{\delta} dm. \tag{3.3.14}$$

Approximating the sum by an integral effectively smooths out the mass distribution of the KK modes and hence also smooths out the resonance peaks. We therefore hope that this method will be able to provide a unitarity bound for energies away from the resonances. Unfortunately as we will soon see this is not possible without introducing a strong dependence on the arbitrary cutoff.

The integral clearly diverges in the limit that the cutoff is taken to infinity for $\delta > 1$. However, with a finite cutoff, it can be evaluated exactly [29, 66][2] . By identifying

$$\frac{1}{s - m^2 + i\epsilon} = P\left(\frac{1}{s - m^2}\right) - i\pi\delta(s - m^2) \tag{3.3.15}$$

where P signifies the Cauchy principal value, we have

$$\mathcal{S}(s) = \frac{\pi^{\delta/2} r^{\delta} s^{\delta/2-1}}{\Gamma(\delta/2)} \left(-i\pi + 2I(\Lambda/\sqrt{s})\right). \tag{3.3.16}$$

where

$$I(\Lambda/\sqrt{s}) = P\int_0^{\Lambda/\sqrt{s}} \frac{y^{\delta-1}}{1 - y^2} dy. \tag{3.3.17}$$

This integral can be evaluated (see Appendix C) to give

$$I(\Lambda/\sqrt{s}) = -\sum_{k=1}^{\delta/2-1} \frac{(\Lambda/\sqrt{s})^{2k}}{2k} - \frac{1}{2}\log\left(\frac{\Lambda}{\sqrt{s}} - 1\right) \qquad \delta = \text{even}, \tag{3.3.18}$$

$$= -\sum_{k=1}^{(\delta-1)/2} \frac{(\Lambda/\sqrt{s})^{2k-1}}{2k-1} - \frac{1}{2}\log\left(\frac{\Lambda + \sqrt{s}}{\Lambda - \sqrt{s}}\right) \qquad \delta = \text{odd}. \tag{3.3.19}$$

The UV divergence of the sum (3.3.14) can now clearly be seen for $\delta > 1$, note that for $\delta = 2$ it is only logarithmically divergent. The UV divergence comes from the high density of states of KK gravitons and is related to the unconstrained momenta allowed to propagate into the extra dimensions from the brane. The imaginary part of (3.3.16) arises from the exchange of on shell KK gravitons. It is tempting to derive a unitarity bound from this part of the amplitude as is done in Ref. [30], however, since the KK gravitons are unstable particles, a proper expression for the width should be used and the zero width approximation will simply give misleading results. The inclusion of a Breit-Wigner width will be taken up in the following section, but first we look at the possibility of deriving a unitarity bound from the real part of the zero width amplitude.

In Ref. [4] we attempted to bound the real part of the amplitude via $|\text{Re}(a_2)| \leq 1/2$, i.e. find a maximum energy at which unitarity is violated. However, because the amplitude is

[2] Some authors choose to evaluate the divergent integral using dimensional regularisation (see e.g.[30]). However, since the the physical picture is that we are actually cutting off all KK modes with masses $m_n > \Lambda$, we consider it more physically meaningful to calculate the integral with a hard cutoff.

strongly dependent on both the centre of mass energy and the cutoff, it is not immediately clear how to go about this. The argument given in Ref. [4] is that in a standard effective field theory approach, the cutoff can be estimated as the lowest energy at which tree level unitarity is violated. The procedure is then to set $\Lambda = \sqrt{s}$ and vary $\sqrt{s}$ to see at what energy unitarity is violated. It can clearly be seen from Eqs. (3.3.18) and (3.3.19) that setting $\Lambda = \sqrt{s}$ results in divergences in the logarithms. (In fact under the principal value prescription the point $s = m^2$ is removed from the integral, and one should only consider the limit $\Lambda \to \sqrt{s}$). In Ref. [4] it was therefore suggested to take Λ to range from $0.9\sqrt{s}$ to $0.999\sqrt{s}$ and show that with such a choice unitarity was violated at an energy $\sqrt{s} < M_D$. However, the choice of how close to take $\sqrt{s}$ to Λ is completely arbitrary and renders the unitarity bound derived in such a manner meaningless since one can make the amplitude as large as one wishes by choosing Λ arbitrarily close to $\sqrt{s}$.

It should also be noted that by setting $\Lambda = \sqrt{s}$, the divergence does not come from a part of the amplitude due bad high energy behaviour of off shell amplitudes, instead the logarithmic divergences are infrared. The divergences come from attempting to integrate up to and not beyond the singular point $s = m^2$ in (3.3.14). Any integration in this region is strongly dependent on the form of the amplitude where an on-shell KK graviton is exchanged. This is the resonance region and so we in fact end up probing the unitarity problems associated with resonances discussed in Section 3.2. Despite using the integral approximation which smooths out the mass distribution, by choosing $\sqrt{s} \sim \Lambda$ the integral is still strongly dependent on a resonant peak. Because of this fact, it was decided in Ref. [4] and later in Ref. [5] that to take the analysis further, the resonance region must be dealt with properly and so a Breit-Wigner width needs to be introduced. This will be taken up in the following section.

Of course, one could attempt to avoid the contribution from the resonances by taking $\Lambda \gg \sqrt{s}$, however doing so will mean any unitarity bound (in $\delta > 1$ extra dimensions) will be strongly dependent on the choice of Λ and therefore completely arbitrary.

3.3.4 KK sum and unitarity including Breit-Wigner width

As discussed in Section 3.1.4, KK gravitons are unstable particles and as such, the propagator near the on-shell region can be approximated by the inclusion of a Breit-Wigner width:

$$\Delta_{\vec{n}}(s) = \frac{1}{s - m_{\vec{n}}^2 + i m_{\vec{n}} \Gamma_G(m_{\vec{n}})}, \tag{3.3.20}$$

where $\Gamma_G(m_{\vec{n}})$ in the ADD model is given by

$$\Gamma_G(m_{\vec{n}}) = \frac{N m_{\vec{n}}^3}{320\pi M_P^2}. \tag{3.3.21}$$

Including the width, the sum over propagators can again be converted to an integral

$$\mathcal{S}(s) = \sum_n \frac{1}{s - m_n^2 + i m_n \Gamma_G(m_n)} \simeq \int_0^\Lambda \frac{m^{\delta-1}}{s - m^2 + i m \Gamma_G(m)} S_{\delta-1} r^\delta dm. \tag{3.3.22}$$

It is now possible to set $\Lambda = \sqrt{s}$ without encountering divergences. We also now have an accurate expression for the imaginary part of the amplitude. We will derive unitarity bounds from the imaginary part later in this section, but first we revisit attempts to place unitarity bounds on the real part of the amplitude.

Real part with Breit-Wigner width

The integral (3.3.22) has to be evaluated numerically. Following the procedure outlined for the zero width case, we wish to find the highest energy at which unitarity is violated. This will define the cutoff, so we set $\Lambda = \sqrt{s}$ and vary $\sqrt{s}$ to see at what scale unitarity is violated. Note that the Breit-Wigner width will now regulate the divergence at $s = m^2$ and so we can follow this procedure without encountering divergences.

Evaluating the $j = 2$ partial wave amplitude for $M_D = 1$ TeV, setting $\Lambda = \sqrt{s}$, and imposing $|\text{Re}(a_2)| \leq 1/2$, we find unitarity breaks down at the following energies for δ extra dimensions:

δ	1	2	3	4	5	6	7
$E_\star$ (TeV)	0.41	0.39	0.41	0.44	0.48	0.51	0.55

Assuming that the leading order behaviour is the same as for the zero width case, it is possible to see how the unitarity violation scale varies with different values of M_D. First, from Eqs. (3.3.11) and (3.3.16) we can see that $a_2 \sim s^{\delta/2+1}$. Next, from (3.3.22), we also see that $a_2 \sim r^\delta \sim 1/M_D^{\delta+2}$. Combining these we find

$$a_2(s) \sim \left(\frac{\sqrt{s}}{M_D}\right)^{\delta+2}. \tag{3.3.23}$$

From this we can see that the scale of unitarity violation is proportional to M_D. This scaling behaviour has been numerically verified to hold with better than 10% accuracy for 1 TeV $< M_D < 10^4$ TeV. For example, for $\delta = 4$ extra dimensions we find that unitarity breaks down at around $0.4 M_D$ for 1 TeV $< M_D < 10^4$ TeV

In Ref. [4] we interpret this result as a clear breakdown of unitarity at the given scale ($E_\star \sim M_D/2$). In Refs. [29, 67] it is shown that by naive dimensional analysis, one expects gravity to become strong at around

$$\Lambda_{\text{strong}} = \left[\Gamma\left(2+\delta/2\right)\right]^{1/(2+\delta)} \left(4\pi\right)^{\frac{4+\delta}{4+2\delta}} M_D. \tag{3.3.24}$$

Note that in four dimensions this agrees with our requirement given in Eq. (2.3.9). For any number of extra dimensions, it is found that $\Lambda_{\text{strong}} > 7.2 M_D$. For this reason we stated in Ref. [4] that it is very unlikely that higher orders in perturbation theory will be able to fix this breakdown of unitarity and new physics would be required to enter at around $M_D/2$ in order to fix the unitarity problem. Two comments on this are however in order. Firstly, the estimate of Λ_{strong} given above does not take into account factors of N_s, N_ψ and N_V that will be present when a large number of particles can circulate in the loops. Secondly, as we properly identified in Ref. [5], the unitarity bound found using the method described above is again appearing as a result of the large resonances. Setting $\Lambda = \sqrt{s}$ would appear to be a sensible procedure to deal with the cutoff dependence of the sum, however it means that we are only summing up to and not beyond the point where $s = m^2$. The contribution from modes above $\sqrt{s}$ come with opposite sign and so will act to reduce the unitarity bound. For this reason, the amplitude is extremely sensitive to the rapidly changing behaviour near the resonance peak. Increasing the cutoff by only a tiny amount will change the amplitude significantly. Despite our best efforts, we find that by using the real part of the amplitude, we can neither remove the extreme sensitivity to the cutoff nor separate resonance from the non-resonance regions.

For these reasons, we decided in Ref. [5] that it is wiser to concentrate on the imaginary part of the amplitude which appears entirely from the resonances but is not sensitive to the cutoff. We will now focus on the imaginary part of the amplitude and derive a more robust unitarity bound from it.

Imaginary part with Breit-Wigner width

With the principal value prescription for dealing with the sum in the zero width approximation, an imaginary part is generated. As mentioned above, a unitarity bound can be derived from this (see Ref. [30]), however since this part of the amplitude is appearing from the exchange of an on shell graviton, we choose here to derive the unitarity bound using the full expression for the width of KK gravitons for much greater accuracy.

Including the Breit-Wigner width, we again have to evaluate the partial wave amplitude numerically in order to determine the imaginary part. Due to the extremely small width of TeV scale KK gravitons ($\Gamma_G \sim m^3/M_P^2$), by far the dominant contribution to the imaginary part of the amplitude comes from the resonant region. For this reason one can set the cutoff to $\Lambda = 2\sqrt{s}$ and capture all the important behaviour of the imaginary part of the amplitude. Increasing the cutoff will not effect any of the bounds derived here.

Evaluating the $j = 2$ partial wave amplitude for $M_D = 1$ TeV, setting $\Lambda = 2\sqrt{s}$, and imposing $|\text{Im}(a_2)| \leq 1$, we find unitarity breaks down at the following energies for δ extra dimensions:

δ	1	2	3	4	5	6	7
$E_\star$ (TeV)	1.5	1.02	0.89	0.84	0.83	0.83	0.84

Again, and for the same reasons as for the real part, these bounds scale with M_D, and so we find for $4 \leq \delta \leq 8$ that unitarity breaks down at around $0.8M_D$. This bound is not at all sensitive to the cutoff and is therefore a much more robust bound. However it is generated entirely from the exchange of on-shell KK gravitons and therefore fails to capture the behaviour of the amplitude away from resonances. Comparing the difference between this unitarity bound and Λ_{strong} we again concluded in Ref. [5] that new physics would have to enter at this scale in order to fix the unitarity problem.

3.3.5 Summary of the unitarity bounds

For the sake of clarity we now review the different unitarity bounds and comment on them in turn. We first address the bounds generated by approximating the sum $\mathcal{S}$ as an integral:

- In the zero width approximation, we cannot meaningfully determine a unitarity bound on the real part of the amplitude. If we try to set $\Lambda = \sqrt{s}$ we encounter an IR divergence coming from the pole in the propagator. The unitarity bound would be strongly dependent on any other choice for Λ and so would be completely arbitrary.

- In the zero width approximation we can bound the imaginary part but since this part of the amplitude is being generated by exchange of an on shell graviton, the correct way to determine this is with the inclusion of the KK graviton widths.

- Including a Breit-Wigner width we can now set $\Lambda = \sqrt{s}$ in order to bound the real part of the amplitude. Doing this we find in general that unitarity breaks down at about $M_D/2$. However, the bound is still extremely sensitive to any variation in the choice of cutoff and to the details of the resonance region. It is therefore more robust to consider a bound coming from the imaginary part.

- Including a Breit-Wigner width, we can bound the imaginary part of the amplitude. This bound is insensitive to the choice of cutoff and is therefore considered to be the only robust unitarity bound obtained in the integral approximation. For $4 \leq \delta \leq 8$ we find unitarity breaks down at around $0.8 M_D$. This bound is coming entirely from the resonant exchange of on shell gravitons.

All the above bulleted items are calculated by approximating the sum $\mathcal{S}$ by an integral. This is an extremely common technique used in the literature to calculate cross sections from KK graviton exchange. However, it should be noted that none of the bounds agree with the discussion given in Section 3.2.1 where it was shown on very general grounds that the presence of more than one KK graviton will mean a breakdown of unitarity at the first KK mode. Why is this? The answer is that by converting the sum to an integral, we effectively smooth out the spectrum of KK masses. For this reason, the contribution of multiple KK modes near a resonance is smoothed out and only appears to cause problems with unitarity at a much higher scale. Approximating the sum by an integral may be a perfectly valid procedure for calculating phenomenological observables, particularly when the spacing between the modes is much smaller than the detector resolution (see for example the discussion in Ref. [66]). However, it seems that this approximation is not suitable when one is trying to place theoretical bounds on the model, particularly when the bounds cannot be separated from the contributions coming from resonances.

Ultimately the set up of the ADD model is clear. The extra dimensions are compact and so the sum over KK modes is discrete. Performing this discrete sum we find that unitarity should break down at the first KK mode as explained in Section 3.2.1. There may be situations where approximating the sum by an integral is a satisfactory approximation, unfortunately this does not seem to be the case for deriving unitarity bounds. The unitarity bounds derived from the integral approximation are far from the scale obtained by performing the discrete sum. It has to be concluded that despite using a very common technique, the unitarity bounds derived in this way grossly overestimate the scale at which unitarity breaks down in the ADD model.

If the bounds from approximating the sum by an integral are inaccurate, and in fact when the KK modes are simply summed unitarity breaks down at the first KK mode, where does this leave the ADD model? Due to the large size of the extra dimensions in the ADD model, the first KK mode has an extremely small mass. In $\delta = 4$ extra dimensions for example, the lowest lying KK mode has a mass of approximately 20 keV. If perturbation theory really broke down at such a low energy, it would spell disaster for the model. The only way out we can foresee is to return to the idea outlined in Section 3.2.2 and resum the propagator properly taking into account the interference between different

KK modes. As mentioned previously, when this is done we begin to see a fascinating hint that the amplitude remains unitary and the sum over modes is finite in any number of extra dimensions! This would not only resolve all the unitarity problems in the ADD model, it would also be of great interest to phenomenologists and experimentalists who use the process of graviton exchange to place bounds on the fundamental scale M_D. At present, all such bounds are given in terms of an unknown cutoff. If the claim that the sum $\mathcal{S}$ is finite is indeed true, it would remove all cutoff dependence and a much cleaner extrapolation from experimental data would be possible. As mentioned before, pursuing this idea in detail goes beyond the scope of this thesis.

3.4 Unitarity in the Randall-Sundrum Model

Following soon after the publication of the ADD model, Randall and Sundrum proposed a new extra dimensional approach to solving the hierarchy problem [54]. The RS model has only a single extra dimension with a large curvature, in contrast to the multiple flat extra dimensions of the ADD model. The large curvature, known as warping, allows for a large hierarchy of scales in the model with only a very small tuning of the model parameters. In this section we will introduce the basics of the RS model, including the KK graviton spectrum and couplings. We discuss the consequences for the RS model of the unitarity bound derived in Section 3.2.1, we will see that they are far less severe than for the ADD model. Finally we review the stabilisation mechanism which gives a mass to the radion and review unitarity bounds derived from radion exchange.

This section is intentionally kept short since our original contribution is simply to apply the generalised unitarity bound derived in Section 3.2.1 to the RS model. The rest of this section is included for pedagogical reasons and for comparison to the ADD model and later to the linear dilaton model.

3.4.1 Introduction to the Randall-Sundrum model

In this section we again follow closely the presentation in Ref. [56]. The RS model consists of a single extra dimension, compactified on an $S^1/\mathbb{Z}_2$ orbifold, that is, the geometry of the circle whose upper and lower halves are identified. This provides two fixed points at $y = 0$ and $y = \pi r \equiv b$, with a 3-brane at each fixed point. The warping is provided by a cosmological constant Λ and so the Einstein Hilbert action looks like

$$S = -\int d^4x \int_{-b}^{+b} dy \sqrt{g} \left(M_*^3 R - \Lambda \right). \tag{3.4.1}$$

In order that the observed 4D universe is ordinary flat Minkowski space, the components of the 5D metric can only depend on the fifth coordinate y. The general ansatz for such a metric is

$$ds^2 = e^{-A(y)}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2. \tag{3.4.2}$$

Solving Einstein's equations for this setup, one finds

$$A(y) = k|y|, \tag{3.4.3}$$

where k determines the curvature and is given by

$$k^2 \equiv -\frac{\Lambda}{12M_*^3}. \tag{3.4.4}$$

However, in order to match the Einstein tensor at the end points, there also need to be localised energy densities V_1 and V_2 on the branes. The full action then looks like

$$S = -\int d^5x\sqrt{g}\left(M_*^3 R - \Lambda + V_1\delta(y) + V_2\delta(y-b)\right) \tag{3.4.5}$$

where we find

$$V_1 = -V_2 = 12kM_*^3. \tag{3.4.6}$$

We see that the branes have to have equal and opposite tensions in order for the Einstein equations to be consistently solved. This then involves two fine tunings. One can be associated with the requirement of a vanishing 4D cosmological constant. As we will see later, the other is associated with stabilising the size of the extra dimension which will be addressed in more detail in Section 3.4.3.

We are now able to see how the RS model addresses the hierarchy problem. Consider the Higgs field H confined to the negative tension brane at $y = b$, the action will be

$$S_{\text{Higgs}} = \int d^4x\sqrt{\tilde{g}}\left[\tilde{g}_{\mu\nu}D^\mu H^\dagger D^\nu H - \lambda\left(H^\dagger H - v^2\right)^2\right] \tag{3.4.7}$$

$$= \int d^4x\, e^{-4kb}\left[e^{2kb}\eta_{\mu\nu}D^\mu H^\dagger D^\nu H - \lambda\left(H^\dagger H - v^2\right)^2\right], \tag{3.4.8}$$

where $\tilde{g}_{\mu\nu}$ is the induced metric on the negative tension brane. In order for the Higgs field to be canonically normalised, we perform a field redefinition $H = e^{kb}\tilde{H}$ and the action for the properly normalised Higgs then looks like

$$S_{\text{Higgs}} = \int d^4x\left[D_\mu\tilde{H}^\dagger D^\mu\tilde{H} - \lambda\left(\tilde{H}^\dagger\tilde{H} - e^{-kb}v^2\right)^2\right]. \tag{3.4.9}$$

We can now see that the effective Higgs VEV is exponentially suppressed

$$v_{\text{eff}} = e^{-kb}v. \tag{3.4.10}$$

As a consequence, all mass scales on the negative tension brane get warped down by the exponential suppression factor e^{-kb}. If the bare Higgs VEV v is of order the Planck scale, the effective Higgs VEV could be warped down to the weak scale $v \simeq 10^{-16} M_P$ by choosing $kb \simeq 35$.

It can easily be shown that the relationship between the 4D and 5D Planck scales in the RS model is given by

$$M_P^2 = \frac{M_*^3}{k}\left(1 - e^{-2kb}\right). \tag{3.4.11}$$

For moderately large sizes of kb this expression is almost independent of the size of the extra dimension. This means that we can have all the bare parameters M_*, Λ, k, V_1 and crucially v at a scale of order the Planck scale, but the physical Higgs VEV can easily be exponentially suppressed down to the weak scale with only a moderately large value of b. Thus the hierarchy problem is addressed in the RS model by having no large hierarchy of scales between the bare parameters. As mentioned above, we will see later in Section 3.4.3 how the size of the extra dimension can be stabilised at the required value without introducing significantly extra fine tuning.

Because of the different values of the mass scales on the two branes, the positive tension brane at $y = 0$ is often referred to as the ultra-violet (UV) brane whilst the negative tension brane at $y = b$ is known as the infra-red (IR) brane. In the original version of the RS model, all the SM fields are confined to the IR brane along with the Higgs. For the purposes of this thesis, we will only be considering this scenario. We note here however that later extensions of the model allow the SM fields (excluding the Higgs) to live in the bulk which provides further interesting phenomenology and also provides an explanation for the fermion mass hierarchy [68, 69, 70].

The graviton KK spectrum and couplings

Computing the KK graviton wave functions and mass spectrum is somewhat involved and was originally derived in Ref. [71, 72]. We present the main results only here.

The graviton KK spectrum consists of a massless zero mode, which is to be identified with the usual 4D graviton, plus a tower of massive modes with masses given by

$$m_n = kx_n e^{-kb}, \qquad n = 1, 2, 3, \ldots \tag{3.4.12}$$

where the x_n are roots of the Bessel function: $J_1(x_n) = 0$. Note that with k of order the Planck scale and $kb \simeq 35$ the spacing between the KK modes is of order TeV. In particular the first KK mode is of order TeV. This is in strong distinction to the ADD model where the modes are very finely spaced. Also, note that the KK mode numbers run only over the positive integers and so each KK mass is non-degenerate, again in contrast to the ADD model.

The couplings between KK modes and matter are found to be

$$\mathcal{L}_{int} = \frac{1}{M_P} T^{\mu\nu} h^{(0)}_{\mu\nu} + \frac{1}{M_P e^{-kb}} \sum_{n=1}^{\infty} T^{\mu\nu} h^{(n)}_{\mu\nu}. \tag{3.4.13}$$

We see that the massless zero mode couples with correct strength for the normal 4D graviton, however the coupling of the KK modes is much stronger and of order TeV^{-1}. So like the ADD model we will see strong gravitational effects around the TeV scale. However, unlike the ADD model, where the huge number of weakly coupled modes provided for strong gravitational effects, in the RS model individual widely spaced modes couple strongly to matter and produce the strong effects.

The widely spaced strongly coupled modes should appear as clear resonances in processes that involve s-channel exchange of a KK graviton, such as $pp \to \text{graviton} \to e^+e^-$. The current lowest limits on the mass of the first KK graviton coming from the LHC are 2.23 (1.89) TeV from the ATLAS detector [73] and 2.390 (2.030) TeV from the CMS detector [74], both these results assume values of $k/M_P = 0.1$ (0.05).

3.4.2 Unitarity from graviton exchange

Because the RS model has only one extra dimension, the sum over KK modes of s-channel amplitudes in the RS model is finite. We can see this by considering the sum (3.1.22) in this model:

$$\mathcal{S}'(s) = \sum_n \frac{1}{\Lambda_n} \Delta_n(s) = \frac{1}{M_P^2 e^{-2kb}} \sum_n \frac{1}{s - m_n^2 + i\epsilon}. \tag{3.4.14}$$

In the limit $m_n^2 \gg s$, the sum over KK modes becomes $(kM_P)^{-2} e^{4kb} \sum_n x_n^{-2}$ which rapidly converges [72].

As already discussed, s-channel scattering of KK gravitons will produce clearly defined individual resonances. The general procedure outlined in Section 3.2.1 where we found that unitarity breaks down at the first KK resonance in models with more than one KK graviton applies to the RS model. The lowest lying KK mode has a mass of $m_1 = x_1 k e^{-kb} \simeq 3.8 k e^{-kb}$. This is therefore expected to be of order TeV and we have seen that current experimental searches place a lower bound of around 2 TeV on this mass.

We want to know whether the consequence of unitarity breaking down at the first KK mode poses a problem for the RS model as it does for the ADD model? In the ADD model, the first KK mode is extremely light and so unitarity breaks down at a very low scale invalidating the effective field theory well below the scale at which we would expect strong gravitational effects. However, in the RS model, the fist KK mode is at a much higher scale and coincides with the scale at which we expect strong gravitational effects. For this reason it is not unexpected for the effective theory to break down around this scale and there still remains a large low energy regime where the effective theory is valid and calculations are reliable.

3.4.3 The radion and unitarity

So far, the size of the extra dimension has been fixed by hand to be $b \simeq 35/k$, we have not yet introduced any mechanism to stabilise the size. Additionally, as we saw in Section 3.1.2, there is an extra scalar degree of freedom in the 5D metric associated with fluctuations in the size of the extra dimension called the radion. Currently there is no potential for this degree of freedom and so it is massless and would contribute to violations of Newton's law which have not been observed. Both of these problems will be solved if the radion can obtain a potential with a minimum. The radion will want to sit at the minimum thus stabilising the size of the extra dimension and it will then have a mass and so there will be no additional long range forces.

In this section we will discuss briefly the most common mechanism of stabilising the extra dimension and show how this also compensates for the additional fine tuning which we were presented with in matching the potentials on the branes, Eq. (3.4.6). We will then review unitarity bounds derived for the RS model by considering processes which involve radion exchange.

The simplest and most commonly used solution to stabilising the size of the extra dimension is known as the Goldberger-Wise mechanism [75]. Qualitatively the mechanism works by introducing an additional bulk scalar field with a bulk mass term and a non-trivial VEV which changes with the extra dimensional coordinate. This is achieved by potentials for the scalar field on both of the branes with different minima. The non-trivial bulk profile then acts to stabilise the size of the extra dimension by balancing the forces from the mass term tending to minimise the size of the extra dimension, with forces from the kinetic term trying to flatten the potential in order to minimize the kinetic energy.

The formalism for dealing with additional bulk scalar fields is non-trivial since the scalar field will mix with the radion and the trace of the graviton. This mixing ultimately gives mass to the radion. If the values of the scalar field on the two branes are Φ_1 and Φ_2

then it can be shown that

$$b = \frac{1}{u} \ln \frac{\Phi_1}{\Phi_2}, \tag{3.4.15}$$

where u is a parameter which provides the mass term to the scalar field. To generate the hierarchy between the Planck and weak scale, we require $kb \simeq 35$ and so we need

$$\frac{k}{u} \ln \frac{\Phi_1}{\Phi_2} \simeq 35 \tag{3.4.16}$$

which can easily be obtained with only a modest tuning of the ratio u/k.

Generally the radion is much lighter than the first graviton mode. The coupling of the radion to matter is given by

$$\mathcal{L} = \frac{1}{\sqrt{6} M_P e^{-kb}} r(x) T^{\mu}_{\mu}, \tag{3.4.17}$$

where $r(x)$ represents the canonically normalised 4D radion field and T^{μ}_{μ} is the trace of the energy momentum tensor. We see that similarly to the Higgs field, the radion couples to the mass terms of the SM fields. One also has to consider the trace anomaly term which contributes to T^{μ}_{μ} for gauge fields. This means that unlike the Higgs, the radion additionally couples to massless gauge fields such as the gluon and the photon [76].

An additional complication can arise if the Higgs boson has a non-minimal coupling to gravity

$$\mathcal{L} = \sqrt{\tilde{g}} \xi H^{\dagger} H R^{(4)}. \tag{3.4.18}$$

The Higgs now mixes with the radion and the system needs to be diagonalised to find the physical degrees of freedom. After the system has been properly diagonalised, the Higgs boson will no longer have standard model like couplings and this can alter the phenomenology of Higgs physics.

With the realisation that the the radion couples similarly to the Higgs boson, it is natural to ask if it will have any significant effects on perturbative unitarity bounds in the SM such as those presented in Section 1.2.1. A few papers have investigated this issue [77, 76, 78] but by far the most comprehensive analysis was performed in Ref. [79] and we briefly present the main results here.

In Ref. [79] the contribution of the radion to longitudinal WW scattering is considered. The radion can be exchanged in the s and t-channels, just like the Higgs and so adds an extra component to the amplitude presented in Section 1.2.1. The cutoff for the effective theory is defined to be $\Lambda = e^{-kb} M_P$ and they test whether the contribution of the radion to WW scattering causes a violation of unitarity at energies lower than Λ.

For the case $\xi = 0$ they find “no significant constraint on the radion mass or coupling”. However, with the introduction of a non-minimal coupling between the Higgs and the curvature, they find "with a mixing coefficient $|\xi| \gtrsim 2.7$, the partial wave amplitude for W scattering does exceed the unitarity bound for scattering energies lower than the cutoff scale".

Note that comparing the definition of the cutoff used in Ref. [79], to the mass of the first KK graviton we find $m_1/\Lambda \simeq 3.8k/M_P$. We therefore find that unitarity breaks down below Λ from the process involving graviton exchange for values of $k \lesssim 0.26\, M_P$.

3.5 Unitarity in the linear dilaton model

3.5.1 Introduction to the linear dilaton model

More recently a new extra dimensional solution to the hierarchy problem has been introduced and will be referred to here as the linear dilaton model [80]. This model was constructed as a holographic dual to TeV little string theory, a string theory where the string scale and all the compact dimensions can be at the electroweak scale [55]. From the string theoretic relation:

$$M_P^2 = \frac{1}{g_s^2} M_s^8 V_6 \tag{3.5.1}$$

where g_s is the string coupling, M_s is the fundamental string scale and V_6 is the extra six-dimensional volume - it can be seen that the fundamental scale can be at the TeV scale if it is compensated by a tiny string coupling (note that this offers an alternative to the large volume compensation provided by the ADD model).

The linear dilaton model contains a single extra dimension compactified on a $S_1/\mathbb{Z}_2$ orbifold bounded by two 3-branes much like the RS model. The presence of a dilaton field with a linear profile provides a unique KK spectrum with a mass gap followed by a near continuum of modes. A stabilisation mechanism is also required to solve the hierarchy problem. However, here the dilaton can play the role of a stabilising field without having to introduce any additional new fields. Once this has been achieved the couplings of the radion are known and its phenomenology can be studied [81].

In this section I review the linear dilaton model and and its graviton and radion phenomenology. Following this I calculate the perturbative unitarity bounds that arise from both graviton exchange and radion exchange.

The linear dilaton model contains a single extra dimension compactified on a $S_1/\mathbb{Z}_2$ orbifold with 3-branes positioned at $y = 0$ and $y = b$. The fundamental 5D gravity scale M_* is of order a TeV. The SM fields are confined to the visible brane at $y = 0$ and the model contains a single extra bulk scalar field φ called the dilaton. Following the notation of reference [81], the action for the linear dilaton model is given by

$$S_{bulk} = -\int d^5x\sqrt{-g}\, e^{-\varphi}\left(M_*^3 R + (\nabla\varphi)^2 - \Lambda\right) \tag{3.5.2}$$

$$S_{brane} = \int d^4x\sqrt{-g_4}\, e^{-\varphi}\left(\mathcal{L}_{SM} - V_{vis}\right) - \int d^4x\sqrt{-g_4}\, e^{-\varphi} V_{hid} \tag{3.5.3}$$

where $V_{vis(hid)}$ are the potentials on the visible ($y = 0$) and hidden ($y = b$) branes. The easiest way to analyse this model is to transform to the Einstein frame, $\tilde{g}_{\mu\nu} = e^{-\frac{2}{3}\varphi} g_{\mu\nu}$ (see Appendix D). The action then reads

$$S_{bulk} = -\int d^5x\sqrt{-\tilde{g}}\left[M_*^3\left(\tilde{R} - \frac{1}{3}(\nabla\varphi)^2\right) - e^{\frac{2}{3}\varphi}\Lambda\right] \tag{3.5.4}$$

$$S_{brane} = \int d^4x\sqrt{-\tilde{g}_4}\, e^{\frac{1}{3}\varphi}\left(\mathcal{L}_{SM} - V_{vis}\right) - \int d^4x\sqrt{-\tilde{g}_4}\, e^{\frac{1}{3}\varphi} V_{hid}\,. \tag{3.5.5}$$

The dilaton background ϕ is given a linear profile $\phi = \alpha|y|$, which is a solution to the equations of motion in conjunction with the following metric which solves the gravitational equations of motion

$$ds^2 = e^{-\frac{2}{3}\alpha|y|}(\eta_{\mu\nu}dx^\mu dx^\nu + dy^2) \tag{3.5.6}$$

with the following constraints

$$\Lambda = -M_*^3\alpha^2\,, \quad V_{vis} = -V_{hid} = 4\alpha M^3. \tag{3.5.7}$$

If we require that the 5D curvature is smaller than the fundamental scale, we also have the constraint [81]

$$|\alpha| < \frac{3M_*}{2\sqrt{7}}. \tag{3.5.8}$$

The 4D Planck mass is found by integrating over the extra dimension

$$M_P^2 = 2\int_0^b dz\, e^{-\alpha|y|} M_*^3 = -2\frac{M_*^3}{\alpha}\left(e^{-\alpha b} - 1\right)\,. \tag{3.5.9}$$

First it is seen that $\alpha < 0$. Secondly, with $M_* \sim \alpha \sim \mathcal{O}(\text{TeV})$, we must have $|\alpha b| \sim 70$ to produce the required value of M_P. So we see that similarly to the ADD model, the fundamental Planck scale is of order TeV and there is no hierarchy between the electroweak and quantum gravity scales.

The graviton KK spectrum and couplings

In order to find the graviton KK spectrum the graviton fluctuations are parametrised as $h_{\mu\nu}$ with

$$ds^2 = e^{-\frac{2}{3}\alpha|y|}\left((\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu + dy^2\right)\,. \tag{3.5.10}$$

Expanding the fluctuations as $h_{\mu\nu}(x,y) = \sum h^{(n)}_{\mu\nu}(x) f^{(n)}_h(y)$ and requiring the 4D modes to be mass eigenstates, $\Box h^{(n)}_{\mu\nu} = m_n^2 h^{(n)}_{\mu\nu}$, the equations of motion for the KK modes are

$$\partial_y^2 f_h^{(n)} - \alpha \partial_y f_h^{(n)} = -m_n^2 f_h^{(n)} \,. \tag{3.5.11}$$

The orbifold symmetry imposes Neumann boundary conditions which result in a massless zero mode which is flat in the extra dimension and the rest of the KK spectrum has solutions

$$f_h^{(n)}(y) = N_n e^{\frac{\alpha}{2}|y|} \left(\sin \frac{n\pi|y|}{b} - \frac{2n\pi}{\alpha r} \cos \frac{n\pi|y|}{b} \right) \tag{3.5.12}$$

where N_n is a normalisation constant. The mass spectrum is then found to be

$$m_n^2 = \frac{\alpha^2}{4} + \left(\frac{n\pi}{b}\right)^2, \qquad n = 1, 2, 3, \ldots \tag{3.5.13}$$

The novel graviton mass spectrum has a mass gap above the zero mode of $|\alpha|/2$, and for $|\alpha|/2 \sim M_* \sim 1$ TeV, we find $b \sim (30 \text{ GeV})^{-1}$ and the KK modes are closely spaced above the mass gap. We therefore see that the linear dilaton model shares aspects of both the ADD model with a near continuum of modes and the RS model with a distinct gap between the zero mode and first KK mode.

The KK modes couple to the standard model fields via the stress energy tensor

$$\mathcal{L} = \frac{1}{M_P} h^{(0)}_{\mu\nu} T^{\mu\nu} + \sum_{n\geq 1} \frac{1}{\Lambda_n} h^{(n)}_{\mu\nu} T^{\mu\nu} \tag{3.5.14}$$

where Λ_n^{-1} is the coupling of the n^{th} KK mode and is given by

$$\frac{1}{\Lambda_n} = \frac{|\alpha|^{1/2}}{M_*^{3/2}} \frac{1}{|\alpha b|^{1/2}} \left(\frac{4n^2\pi^2}{4n^2\pi^2 + (\alpha b)^2} \right)^{\frac{1}{2}} . \tag{3.5.15}$$

3.5.2 Unitarity from graviton exchange

For large n the coupling and the spacing between the graviton KK modes tends to a constant and so the sum of s-channel amplitudes (Eq. 3.1.22)

$$\mathcal{S}'(s) = \sum_n \frac{1}{\Lambda_n} \frac{1}{s - m_n^2 + i\epsilon}. \tag{3.5.16}$$

is finite. This can also be seen by approximating the sum by an integral in the following way [82]. The spacing between the modes is approximately $\delta m \simeq \pi/b$. so the sum over n is replaced by

$$\sum_n f(m_n) \simeq \int dx \frac{b}{\pi} f\left(\sqrt{\alpha^2/4 + x^2}\right) = \int dm \frac{m}{(m^2 - \alpha^2/4)^{1/2}} \frac{b}{\pi} f(m). \tag{3.5.17}$$

Substituting m_n from (3.5.13) into (3.5.15) we find the compact expression

$$\mathcal{S}'(s) \simeq \frac{1}{\pi M_*^3} \int \frac{dm}{m} \frac{\left(m^2 - \alpha^2/4\right)^{1/2}}{s - m^2 + i\epsilon} \tag{3.5.18}$$

which is clearly finite.

As with the RS and ADD models, the general unitarity bound derived in Section 3.2.1 applies to the linear dilaton model and we can again say that unitarity breaks down at the mass of the first KK mode. In the linear dilaton model this is at $m_1 \simeq |\alpha|/2$. In general, α is a free parameter which can take any value. However, for very small values of $|\alpha| \ll M_*$ the mass gap is reduced and the linear dilaton model can become constrained from astrophysical considerations. In Ref. [82] it is shown that $|\alpha|$ needs to be at least of order GeV to evade constraints from big bang nucleosynthesis and Supernova 1987A.

With M_* being the extra dimensional Planck mass, we can expect that the effective theory should break down at about this scale due to entering the strongly coupled/quantum gravity regime. We would therefore like to test whether unitarity holds up to this scale. For large values of $|\alpha| \sim M_*$, the first KK mode is near to M_* and it is therefore expected for the effective theory to break down here. Also there is a large mass gap and there is therefore still a large regime of validity for the low energy effective theory, much like the case for the RS model. However, for small values $|\alpha| \ll M_*$, the first KK mode is extremely light and the effective theory breaks down at a very low scale well before strong gravitational effects appear. Similarly to the ADD model this could cause problems to reliably perform calculations in the linear dilaton model with a small mass gap.

3.5.3 The radion and dilaton modes and the associated unitarity bounds

Similarly to the RS model, the size of the extra dimension in the linear dilaton model needs to be stabilised. Completely analogously, this will provide a mass to the radion. However, unlike in the Goldberger-Wise mechanism, we do not need to add an additional bulk scalar, instead the dilaton field can itself act as the stabilising field. In this section we review how this mechanism can be achieved via the dilaton, and then we derive unitarity bounds on WW scattering coming from exchange of the radion in the LDM.

The formalism for stabilising the extra dimension in the linear dilaton model was developed in Ref. [81] and we follow closely this reference here. Potentials for the dilaton field are added to the branes, and if ϕ_v and ϕ_h are the background values of the dilaton field on the visible and hidden branes respectively, it can be shown that

$$\alpha b = \phi_h - \phi_v, \tag{3.5.19}$$

i.e. the interbrane distance is stabilised. Given that to produce the Planck-weak scale hierarchy, we have already argued that $|\alpha b| \sim 70$, this does not represent any strong fine tuning.

Similarly to the RS model, the dilaton mixes with the radion and the trace of the 5D graviton. The Einstein equations can be solved and provide constraint equations which effectively reduce the radion/dilaton degrees of freedom to a single degree of freedom. The original radion part we will call Φ, and the fluctuations of the dilaton field $\delta\phi$, where $\varphi = \phi + \delta\phi$ are related to Φ through the constraint $\delta\phi = \frac{9}{2\alpha}\partial_y \Phi - 3\Phi$. Performing a KK decomposition and labelling the y dependent extra dimensional profiles $\Phi_n(y)$, we find the equation of motion for the Φ_n is given by

$$\left[\frac{d^2}{dy^2} + m_n^2 - \frac{\alpha^2}{4}\right]\left(e^{-\frac{1}{2}\alpha y}\Phi_n\right) = 0. \tag{3.5.20}$$

The boundary conditions are non-trivial and contain the free parameters $\mu_{vis(hid)}$ which appear in the brane potentials for the dilaton field. Solving the equation of motion subject to the boundary conditions, we obtain the following solution for Φ_n,

$$\Phi_n(z) = N_n e^{\frac{1}{2}\alpha y}\left[\sin(\beta_n y) - \frac{6\beta_n \mu_{vis}}{4\beta_n^2 + \alpha(\alpha - \mu_{vis})}\cos(\beta_n y)\right], \tag{3.5.21}$$

where $\beta_n^2 \equiv m_n^2 - \frac{\alpha^2}{4}$ and N_n is normalisation factor. The full expression for N_n is rather involved and is given in the appendix of Ref. [81].[3] We will call the zero mode the "radion" from now on and the KK modes ($n = 1, 2, 3, \ldots$) will be referred to as the dilaton KK modes.[4] An analytic expression can be obtained for the radion mass in the limit $|\alpha b| \gg 1$ and we find

$$m_r^2 = \frac{\alpha^2}{4} - \frac{\alpha^2}{16\epsilon_v^2}\left(3 - \sqrt{9 + 4\epsilon_v + 4\epsilon_v^2}\right)^2, \qquad 0 < \epsilon_v < \infty, \tag{3.5.22}$$

where $\epsilon_v \equiv |\alpha|/\mu_{vis}$ and $\mu_{vis(hid)} > 0$ is required to ensure no tachyonic modes. The radion mass has a maximum for $\epsilon_v \to 0$ which is equal to $\max(m_r^2) = 2\alpha^2/9$. If we incorporate the requirement of Eq. (3.5.8) then we find that $\max(m_r^2) < M_*^2/126$ and so it will always be safe to assume $m_r \ll M_*$.

The couplings for the radion and the dilaton KK modes to the SM matter on the visible brane can also be worked out. They contain a part which couples to the trace of

[3] We note here a typo in the appendix of Ref. [81], the factor 3 at the beginning of the last line of their Eq. (A.5) should be removed.

[4] This notation is somewhat obscure since there is essentially only one 5D degree of freedom coming from the mixed dilaton/radion modes. However, in the limit $\mu_{vis(hid)} = 0$, the model corresponds to the unstabilised case where we would expect a single massless radion field and the dilaton field with a KK tower of modes. In this limit the terminology will coincide.

the energy momentum tensor as in the RS models but in the linear dilaton model there is also a part which comes from the dilaton coupling to the full SM lagrangian. Following Ref. [81] we will refer to these separate couplings as $\kappa_{\Phi,n}$ and $\kappa_{\phi,n}$ respectively. They are given by the values of $\Phi_n(y)$ and the fluctuations of the dilaton field $\delta\phi_n(y)$ by their values on the visible brane in the following way:

$$\frac{\kappa_{\Phi,n}}{M_*} \equiv \frac{\Phi_n(0)}{2}, \qquad \frac{\kappa_{\phi,n}}{M_*} \equiv \frac{\delta\phi_n(0)}{3}. \tag{3.5.23}$$

The full expressions for $\kappa_{\Phi,n}$ and $\kappa_{\phi,n}$ are rather involved due to the complicated form of the wavefunctions and in particular the normalisation factors. Approximate expressions for the radion couplings are given in Ref. [81] for when ϵ_v is small. For our purposes here, we also require the limit of large ϵ_v where we find the radion couplings asymptote to a maximum value given by

$$\lim_{\epsilon_v\to\infty} \kappa_\Phi = \lim_{\epsilon_v\to\infty} \kappa_\phi = \frac{1}{6}\sqrt{\frac{-\alpha}{M}}. \tag{3.5.24}$$

The Feynman rules for the radion coupled to gauge bosons are presented in Appendix E.

We now turn to deriving unitarity bounds on SM processes in the presence of the radion, first in the case where the Higgs boson is minimally coupled to the curvature and secondly with a non-minimal coupling

Minimal coupling

We first consider the case where the Higgs boson is minimally coupled. The radion coupling to massive vector bosons is proportional to p^2/M_* (see Appendix E). As a result, one might naively expect, by a power counting analysis, that the amplitude for $W_LW_L \to W_LW_L$ scattering via radion exchange would be proportional to $s^3/M_*^2 m_W^4$. For $s, M_* \gg m_W^2$ this could easily exceed the unitarity bound well before the scattering energy reaches M_*. For this reason we carefully derive the full amplitude for WW scattering via radion exchange. We will see that in fact the amplitude is only proportional to s/M^2 (as would be expected via the Goldstone boson equivalence principle) and there are no unitarity problems for energies below M_*.

With zero non-minimal coupling for the Higgs boson, the SM contribution to WW scattering is proportional to $g^2m_H^2/m_W^2$ and can be ignored here. We now consider the contribution of radion exchange to the process $W_LW_L \to W_LW_L$. There are two Feynman diagrams which contribute to this process at tree level with s- and t-channel exchanges of the radion. The invariant amplitudes for these diagrams are

$$\mathcal{M}_s = \frac{\left(Am_W^2 + B(s-2m_W^2)\right)^2}{M_*^2(s-m_r^2)}, \tag{3.5.25}$$

$$\mathcal{M}_t = \frac{\left(A(t^2 - sm_W^2) + Bs(t + 2m_W^2)\right)^2}{s^2 M_*^2 (t - m_r^2)}, \tag{3.5.26}$$

where

$$A = \kappa_\phi \qquad \text{and} \qquad B = \frac{\kappa_\phi}{2} - \kappa_\Phi. \tag{3.5.27}$$

The corresponding $j = 0$ partial wave amplitudes, neglecting terms of $\mathcal{O}(m_W^2/s)$, are

$$\begin{aligned}
a_0^{(s)} &= \frac{m_W^2}{16\pi M_*^2}\left[\frac{B^2 s}{m_W^2} + \frac{B^2 m_r^2}{m_W^2} + 2B(A-2B)\left(1 + \frac{m_r^2}{s}\right) + B^2\frac{m_r^2}{m_W^2}\left(\frac{m_r^2}{s - m_r^2}\right)\right.\\
&\quad \left. + 2B(A-2B)\frac{m_r^2}{s}\left(\frac{m_r^2}{s-m_r^2}\right)\right],\\
a_0^{(t)} &= \frac{m_W^2}{16\pi M_*^2}\left[-\frac{s}{12 m_W^2}(3A^2 - 8AB + 6B^2) + \frac{m_r^2}{3m_W^2}(A^2 - 3AB + 3B^2) + (A-2B)^2\right.\\
&\quad + \frac{m_r^6}{s^2 m_W^2}A^2 - \frac{1}{s^3 m_W^2}\left(A(m_r^4 - sm_W^2) + Bs(m_r^2 + 2m_W^2)\right)^2 \log\left(1 + \frac{s}{m_r^2}\right)\\
&\quad \left. - \frac{m_r^4}{2s^2}A(A-4B)\right].
\end{aligned} \tag{3.5.28}$$

We will see that for $m_r^2, m_W^2 \ll M_*^2$, the unitarity bound is at such a high energy we will only require the leading order terms in this expansion, namely

$$a_0^{(s+t)} \simeq \frac{6B^2 - 3A^2 + 8AB}{192\pi}\frac{s}{M_*^2}. \tag{3.5.29}$$

In order to find the lowest possible unitarity bound, we are interested in the largest possible amplitude. We find that the maximum value of $|6B^2 - 3A^2 + 8AB|$ occurs in the $\epsilon_v \to \infty$ limit. The couplings in this limit are given by Eq. (3.5.24), and inserting these into Eq. (3.5.29) we find

$$\max(|a_0|) = \frac{11|\alpha| s}{13,824\pi M_*^3} \tag{3.5.30}$$

which means that the lowest possible scale at which unitarity breaks down is

$$E_\star \simeq 44 M_* \sqrt{\frac{M_*}{|\alpha|}}. \tag{3.5.31}$$

If we further require the condition (3.5.8), then we find

$$E_\star \gtrsim 59 M_*. \tag{3.5.32}$$

This bound is far above the scale M_* and so we find there are no unitarity problems from radion exchange in WW scattering until well above M_*.

3.5.4 Higgs-radion mixing and the associated unitarity bounds

If the Higgs boson has a non-minimal coupling to gravity

$$\mathcal{L} = \sqrt{\tilde{g}}\, e^{\frac{\delta\phi}{3}} \xi H^\dagger H R^{(4)} \tag{3.5.33}$$

then the Higgs boson will mix with the radion and the system will need diagonalising in order to find the physical 4D degrees of freedom. The Higgs boson no longer has standard model couplings and therefore no longer exactly cancels the $\mathcal{O}(s/m_W^2)$ contribution to the WW scattering amplitude as described in Section 1.2.1. This now makes it possible that in the presence of $\xi \neq 0$ the amplitude for $W_L W_L \to W_L W_L$ could exceed the unitarity bound at energies below M_*. We will see that this is in fact the case for certain large values of ξ.

Ref. [81] also developed the formalism for dealing with the radion in the presence of a non-minimally coupled Higgs. We briefly present here the main points and then we use the couplings to derive unitarity bounds. The kinetic terms are diagonalised by a field redefinition and then the mass matrix is diagonalised by a rotation by an angle θ. In unitary gauge, we denote the gauge eigenstates of the Higgs boson and the radion that appear in the original lagrangian by h and r respectively. After the field redefinitions, the physical mass eigenstates $(h_m,\, r_m)$ can be expressed in terms of the gauge eigenstates in the following way:

$$\begin{aligned} h &= \left(\cos\theta - \frac{6\xi\kappa_\Phi v}{\Omega M_*}\sin\theta\right) h_m + \left(\sin\theta + \frac{6\xi\kappa_\Phi v}{\Omega M_*}\cos\theta\right) r_m, \\ r &= -\frac{\sin\theta}{\Omega} h_m + \frac{\cos\theta}{\Omega} r_m, \end{aligned} \tag{3.5.34}$$

where v is the Higgs VEV and Ω is given by

$$\Omega^2 = 1 + \frac{6\xi\kappa_\Phi v^2}{M_*^2}\big((1-6\xi)\kappa_\Phi - \kappa_\phi\big). \tag{3.5.35}$$

This expression for Ω already provides a constraint on ξ. Ω^2 must be positive in order that the radion mass term remains positive. This constraint implies

$$\frac{1}{12\kappa_\Phi}\left(\rho - \sqrt{\rho^2 + \frac{4M_*^2}{v^2}}\right) \leq \xi \leq \frac{1}{12\kappa_\Phi}\left(\rho + \sqrt{\rho^2 + \frac{4M_*^2}{v^2}}\right), \tag{3.5.36}$$

where $\rho \equiv \kappa_\Phi - \kappa_\phi$. In the limit of large ϵ_v, we find from Eq.(3.5.24) that $\rho = 0$ and then $-\sqrt{3}M_*/v \leq \xi \leq \sqrt{3}M_*/v$.

For notational convenience, we will also write the relationship between h and r and h_m and r_m in the following way

$$\begin{aligned} h &= a_0 h_m + a_1 r_m, \\ r &= b_0 h_m + b_1 r_m. \end{aligned} \tag{3.5.37}$$

We now proceed to calculate the amplitude for WW scattering. The $j = 0$ partial wave amplitude for this process without the Higgs or radion is (see Eq. (1.2.14) in Section 1.2.1)

$$a_{0,\text{gauge}} = -\frac{g^2}{128\pi}\frac{s}{m_W^2}. \tag{3.5.38}$$

The radion contribution to the amplitude is still given by Eq. (3.5.28) but now A and B will include contributions from the Higgs in the following way

$$A = b_1\kappa_\phi \qquad \text{and} \qquad B = b_1\left(\frac{\kappa_\phi}{2} - \kappa_\Phi\right) + a_1\frac{M_*}{v}. \tag{3.5.39}$$

With $\xi \neq 0$, the Higgs boson contribution to the amplitude is equivalent to the radion contribution (Eq. 3.5.28) but with the replacements $m_r \to m_h$, $A \to C$ and $B \to D$, where

$$C = b_0\kappa_\phi \qquad \text{and} \qquad D = b_0\left(\frac{\kappa_\phi}{2} - \kappa_\Phi\right) + a_0\frac{M_*}{v}. \tag{3.5.40}$$

So at leading order we have

$$a_{0,\text{higgs}} = \frac{6D^2 - 3C^2 + 8CD}{192\pi}\frac{s}{M_*^2}. \tag{3.5.41}$$

Combining the gauge, Higgs and radion amplitudes and taking the maximum values for κ_Φ and κ_ϕ the total leading order contribution to $W_LW_L \to W_LW_L$ scattering is

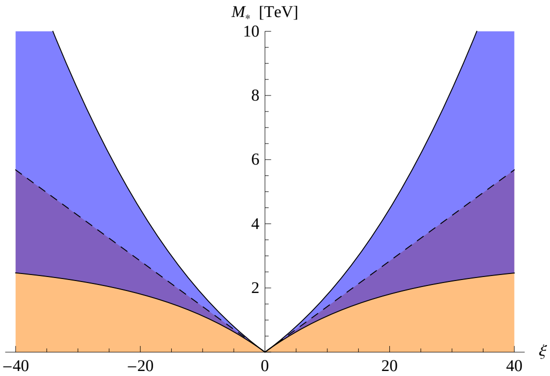
$$\begin{aligned} a_0 &= \frac{1}{13824\pi M_*^3}\left[108g^2(a_0+a_1)^2M_*^3 + 12g\sqrt{-\frac{\alpha}{M_*}}(a_0b_0+a_1b_1)M_*^2m_W \right.\\ &\qquad \left. + 11(b_0+b_1^2)\alpha m_W^2\right]\frac{s}{m_W^2} - \frac{g^2}{128\pi}\frac{s}{m_W^2} \\ &= -\frac{\left(432\xi^2+24\xi-11\right)\alpha\, s}{13824\pi\left(M_*^3+\xi^2v^2\alpha\right)}. \end{aligned} \tag{3.5.42}$$

We see that for $\xi = 0$ the contribution from the Higgs cancels the gauge part of the amplitude and we are left with the radion contribution as given in Eq. (3.5.30).

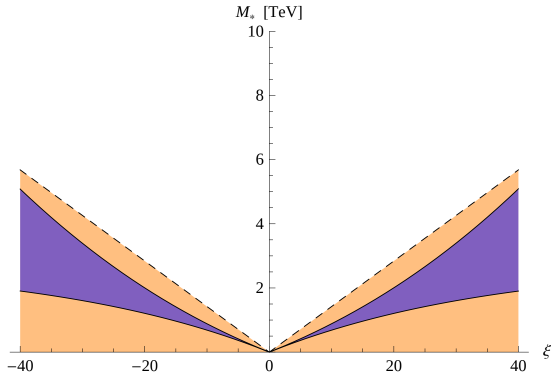
We would like to test whether unitarity holds up to $\sqrt{s} = M_*$. Figure 3.3 shows an exclusion plot for values of ξ and M_*. The blue shaded region represents regions of the parameter space where unitarity breaks down in WW scattering before M_*. The orange shaded region lying below the dashed line is excluded by the constraint Eq. (3.5.36). The plots are given for two different values of α. We note that for large $|\alpha| = M_*/3$, a significant proportion of the parameter space is excluded by the unitarity bound, however there remains a sizeable parameter space allowed. In particular there is no problem with $\xi \sim \mathcal{O}(1)$ for $M_* > 1$ TeV. For $|\alpha| = M_*/10$ the region excluded by the unitarity bound lies entirely within the region excluded by Eq. (3.5.36) and so no new constraints are imposed within the region plotted. For smaller values of ϵ_v the total amplitude is larger and the unitarity bounds are even less restrictive.

Note regarding the dilaton KK modes

As mentioned previously, the mixed radion/dilaton field has a KK tower of modes. These can also be exchanged in processes such as WW scattering and will contribute to the



(a) $|\alpha| = M_*/3$



(b) $|\alpha| = M_*/10$

Figure 3.3: Bound on ξ as a function of M_* (TeV). The blue shaded region is excluded by the requirement that the unitarity bound is not exceeded before M_*, i.e. the blue region represents $E_\star < M_*$. The orange shaded region below the dashed line is excluded by the requirement that the radion mass remains positive, Eq. (3.5.36). We take $\epsilon_v \to \infty$ and (a) $|\alpha| = M_*/3$, (b) $|\alpha| = M_*/10$.

amplitude. We have carried out calculations of these effects and found that due to the extremely small size of the couplings between the KK modes and matter (much smaller than for the radion) they do not make any significant contribution to the bounds derived above. The same was noted for the KK modes of the stabilising scalar in the Goldberger-Wise mechanism in the RS model in Ref. [79].

Chapter 4

Higgs Inflation

In this chapter we review the exciting idea that the Higgs boson of the standard model could have caused a period of rapid inflation in the early universe. We derive unitarity bounds in two separate models of Higgs inflation and discuss the consequences of these bounds for producing reliable predictions in these models. We also discuss how the paradigm of asymptotic safety may offer an ideal framework for the original Higgs inflation model to accommodate the unitarity problems.

4.1 Inflation and the Higgs boson as the inflaton

Standard big bang cosmology fails to explain why today's universe appears flat, homogeneous, and isotropic. However, these problems are easily solved if the early universe went through a period of rapid accelerated expansion known as inflation [83]. During inflation, a small causally connected area of the early universe would expand to the size of the observable universe and all inhomogeneities would be smoothed out. Inflation has now been widely accepted as the paradigm for the early universe, especially following the understanding that quantum fluctuations during inflation grow to become the observed cosmological fluctuations required to seed the formation of large scale structure in the universe.

The mechanism that drove inflation is unknown and currently the subject of intense research in cosmology. Arguably the simplest and certainly the most prolific models of inflation fall into the category of "slow roll inflation" [84, 85]. Here, a scalar field (the inflaton) slowly rolls down a flat potential, driving inflation. A vast array of models implement a multitude of hypothetical scalar fields to play the role of the inflaton. However, with the Higgs boson being the only observed fundamental scalar particle, it is an important question whether or not the Higgs could have caused inflation in addition to its electroweak symmetry breaking role. We will see that with a large non-minimal coupling to gravity, the Higgs could indeed have a flat enough potential to produce viable inflation. However, as we will show, there are questions as to whether the classical approximation used to calculate the inflationary predictions are valid during the inflationary period. Before discussing Higgs inflation itself and its potential problems, we review the basic idea of slow roll inflation.

Assuming a Friedmann, Robertson Walker (FRW) metric for the universe,

$$ds^2 = dt^2 - a(t)^2 d\vec{x}^2 \tag{4.1.1}$$

where $a(t)$ is the scale factor, using Einstein's equations, we find

$$\frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3}(\rho + 3p) \tag{4.1.2}$$

where ρ and p are the density and pressure appearing in the stress energy tensor of the vacuum of the universe. Inflation can be described as the condition $\ddot{a} > 0$ and we see that this will occur if $p < -\rho/3$, i.e. a negative pressure vacuum energy. With a certain form of potential, $V(\phi)$, a scalar field ϕ can provide such a vacuum. Comparing the energy-momentum tensor of the scalar field with that of a perfect fluid, we find

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{4.1.3}$$

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{4.1.4}$$

and so we see

$$\dot{\phi}^2 < V(\phi) \iff \ddot{a} > 0. \tag{4.1.5}$$

With a flat enough potential, the above criteria will be met and inflation will occur as the scalar field "slowly rolls" down the slope. The potential also requires a minimum where inflation can eventually end. During the period of inflation the universe is supercooled. Following inflation, the inflaton will oscillate around its final minimum transferring its potential energy into the standard model particles that fill the universe including electromagnetic radiation which starts the radiation dominated phase of the universe. This period after inflation ends and before the inflaton comes to rest is known as reheating.

The standard model Higgs potential is of the form

$$V(H) = \lambda \left(H^\dagger H - \frac{v^2}{2} \right)^2, \tag{4.1.6}$$

where H is the standard model Higgs doublet, v = 246 GeV and the self coupling is assumed to be $\lambda \sim \mathcal{O}(10^{-1})$ (required to produce the observed Higgs boson mass of about

125 GeV). This potential is far from flat and is certainly not capable of producing enough inflation to solve the cosmological problems mentioned above. However, it was observed by Bezrukov and Shaposhnikov [86] that if the Higgs field has a large non-minimal coupling to gravity then the potential will be modified near the Planck scale allowing for slow roll inflation. The Higgs boson's non-minimal coupling to gravity takes the form

$$S_\xi = -\int d^4x\,\sqrt{-g}\left(\frac{1}{2}M^2 + \xi H^\dagger H\right)R \tag{4.1.7}$$

where ξ is an unknown constant and M is a mass scale (the Higgs boson's kinetic term, potential and other interaction terms have been suppressed). In the context of Higgs inflation we will assume that $1 < \xi \ll 10^{32}$ in which case $M = M_P$ to a very good approximation. We will now see how this coupling can effect the potential and cause inflation.

The action for the Higgs in unitary gauge, $H = \frac{1}{\sqrt{2}}(0,h)^\top$, with the non-minimal coupling is

$$S = -\int d^4x\,\sqrt{-g}\left[\frac{1}{2}\left(M_P^2 + \xi h^2\right)R - \frac{1}{2}\partial_\mu h \partial^\mu h + \frac{\lambda}{4}\left(h^2 - v^2\right)^2\right], \tag{4.1.8}$$

where we have ignored gauge and other interactions with standard model particles. The simplest way to analyse the potential in this model is to make a transformation to the Einstein frame (see Appendix D), $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, where $\Omega^2 = 1 + \xi h^2/M_P^2$. The action in the Einstein frame then reads

$$S = -\int d^4x\,\sqrt{-\tilde{g}}\left[\frac{1}{2}M_P^2\tilde{R} - \frac{3\xi^2}{M_P^2\Omega^4}h^2\partial_\mu h\partial^\mu h - \frac{1}{2\Omega^2}\partial_\mu h\partial^\mu h + \frac{1}{\Omega^4}V(h)\right]. \tag{4.1.9}$$

In order to have a canonically normalized kinetic term for the Higgs boson we need to transform to a new field χ where

$$\frac{d\chi}{dh} = \sqrt{\frac{1}{\Omega^2} + \frac{6\xi^2 h^2}{M_P^2\Omega^4}}. \tag{4.1.10}$$

The action then looks like

$$S = -\int d^4x\,\sqrt{-\tilde{g}}\left[\frac{1}{2}M_P^2\tilde{R} - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + U(\chi)\right], \tag{4.1.11}$$

where

$$U(\chi) = \frac{1}{\Omega(\chi)^4}\frac{\lambda}{4}\left(h(\chi)^2 - v^2\right)^2. \tag{4.1.12}$$

For small field values, the potential is the same as that for the normal Higgs potential (4.1.6). However, for $h \gg M_P/\sqrt{\xi}$ (corresponding to $\chi \gg \sqrt{6}M_P$) we have

$$h = \frac{M_P}{\sqrt{\xi}}\exp\left(\frac{\chi}{\sqrt{6}M_P}\right) \tag{4.1.13}$$

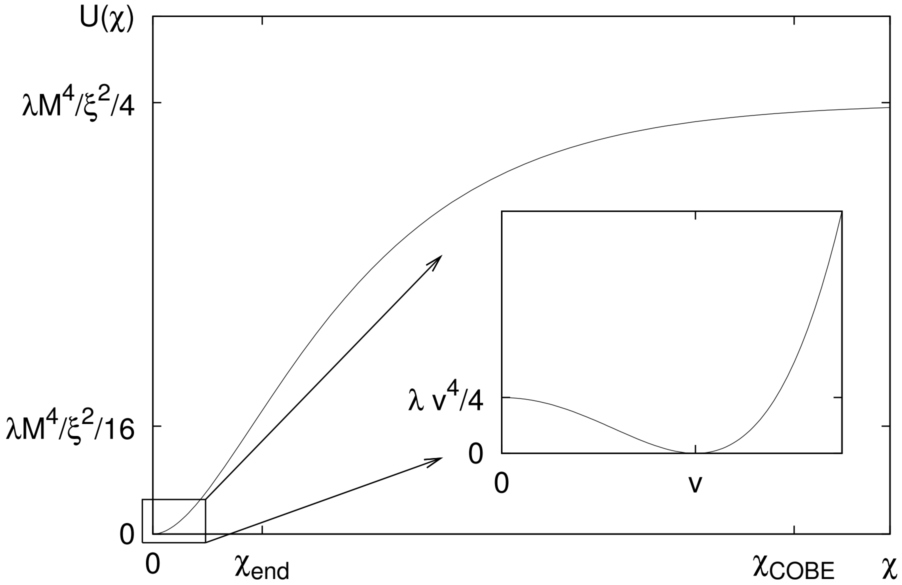


Figure 4.1: The Higgs potential in the Einstein frame with non-minimal coupling ξ. Diagram taken from Ref [86]. χ_{end} is the field value at which slow roll inflation is defined to end, χ_{COBE} is value at which the CMB radiation was produced.

and the potential takes the form

$$U(\chi) = \frac{\lambda M_P^4}{4\xi^2}\left(1 + \exp\left(-\frac{2\chi}{\sqrt{6}M_P}\right)\right)^{-2}. \qquad (4.1.14)$$

The form of the potential is plotted in Figure 4.1. So in the large field regime ($h \gg M_P/\sqrt{\xi}$) it can be seen that the potential becomes exponentially flat. This is perfect for slow roll inflation. It was shown in Ref. [86] that matching the predicted CMB density fluctuations to observation requires a value of $\xi \sim 10^4$ and the predicted spectral index and the tensor to scalar ratio are well within the observed limits.

It was later checked whether the above analysis stood up under quantum corrections. The effective potential for the Higgs under renormalisation group effects was considered at two-loops [87, 88] and it was concluded that the inflationary predictions are still well with in the experimental limits at the time and continues to be in good agreement with the latest data released from the Planck satellite [89].

With the observation of the Higgs boson at a mass of around 125 GeV [38, 39] it is possible to extrapolate the Higgs effective potential up to the Planck scale. Two three-loop analysis have been made [90, 91] which show that the current data lead to the conclusion that the Higgs potential is not stable up to the Planck scale, which would rule out the possibility of Higgs inflation without introducing other degrees of freedom to stabilise the potential. However, there is still a small window of hope for the Higgs inflation model. If the top quark mass lies right at the bottom of the 98% C.L. window and the strong coupling constant lies right at the top of its 98% C.L. window then the Higgs potential would remain stable all the way up to the Planck scale [90, 91] and the Higgs inflation model could again be a viability.

4.2 Unitarity of Higgs inflation

In addition to serious issues with the stability of the Higgs potential, the large size of the non-minimal coupling ($\xi \sim 10^4$) required for Higgs inflation raises the question whether the model remains perturbative at the high energies at which inflation takes place. It is critical that this is the case for the inflationary model to be predictive since all the above calculations (including the higher order corrections to the effective potential) assume perturbation theory is valid during the inflationary era.

A relatively simple and straightforward way to test whether perturbation theory is valid in this model is to see whether tree level processes are unitary. In the calculation of s-channel scattering via graviton exchange presented in Sec. (2.1) we saw that the $j = 0$ partial wave amplitude for $\phi\phi \to \phi'\phi'$ with a non-minimal coupling ξ is

$$a_0(s) = \frac{(1+6\xi)^2}{96\pi} \frac{s}{M_P^2}. \tag{4.2.1}$$

This amplitude is directly applicable to the Higgs inflation model where the Higgs field is a complex doublet, allowing us to consider different in and out states (ϕ and ϕ') and therefore restrict the amplitude to an s-channel process only. Note the calculation is carried out in the Jordan frame.

In order for the tree level amplitude to be unitary we require $|\mathrm{Re}(a_0)| \leq 1/2$ which means that unitarity is violated at an energy

$$E_\star = \frac{4\sqrt{3\pi}}{1+6\xi} M_P. \tag{4.2.2}$$

In the Higgs inflation model $\xi \sim \mathcal{O}(10^4)$ and we can approximate the bound to be

$$E_\star \simeq \frac{M_p}{\xi}. \tag{4.2.3}$$

We derived this result in Ref. [3] and it was separately found in Refs. [48, 92].

This result was interpreted as a major problem for the Higgs inflation model. In order to maintain unitarity, one of two things must happen at or below $E_\star$: either the effective theory enters a strongly coupled regime, or new degrees of freedom appear which couple to the Higgs. But inflation takes place when the Higgs field is in the regime $h \gg M_P/\sqrt{\xi} > E_\star$ so both of these scenarios jeopardise the existence of the flat potential required for slow roll inflation. If the theory becomes strongly coupled near $E_\star$, perturbation theory breaks down in the inflationary regime and the leading order and loop corrected calculations of the potential are no longer valid. If new degrees of freedom appear at or below $E_\star$ which couple to the Higgs they would be represented in the effective theory as higher order operators such as [6, 92]

$$\frac{c_{n,m}}{\Lambda^{2(n+m)-4}}(H^\dagger H)^n R^m, \qquad n \geq 3, \tag{4.2.4}$$

where $c_{n,m}$ are dimensionless coefficients expected to be of order one and and $\Lambda \simeq E_\star$. This infinite tower of operators will significantly alter the shape of the potential in a completely unpredictable way.

The lack of predictability of the Higgs potential in the inflationary regime certainly appears to be a major problem for the Higgs inflation model. However, there have been two subsequent claims that the unitarity bound derived above is either incorrect or irrelevant for the Higgs inflation model. The first claim is that the unitarity bound is frame dependent and the second is that the unitarity bound is background dependent. The first turns out to be incorrect but the second is true and leaves open the possibility of reliable predictions in the Higgs inflation model. I will discuss both of these claims in turn.

Frame dependence

In Ref. [93] Lerner and McDonald made the claim that the unitarity bound derived above does not appear in the Einstein frame and therefore the Jordan frame calculation is incorrect. Their argument is as follows: If we expand the potential in the Einstein frame Eq. (4.1.12) for small field values we find it has the form (setting $v = 0$)

$$U(\chi) = \frac{\lambda}{4}\chi^4 - \frac{3\lambda\xi^2}{M_P^2}\chi^6 + \dots \tag{4.2.5}$$

The scattering amplitude corresponding to Eq. (4.2.1) would come from $\chi\chi \to \chi\chi$ scattering which arises from the four point term in the potential and is simply proportional to λ. In the Einstein frame χ is minimally coupled to gravity so the only problems with unitarity coming from $\chi\chi \to \chi\chi$ scattering would seem to appear when gravity becomes strongly coupled at M_P. Because of this the authors of Ref. [93] concluded that there were no unitarity problems and no need for new physics below M_P.

Because the transformation between the Jordan frame and the Einstein frame is simply a change of variables, scattering amplitudes should not be frame dependent (at the classical level at least). So how can the seeming mismatch be reconciled and which, if either, point of view is correct? The answer, it was later pointed out [94, 95], is that both calculations are correct, but they are in fact calculating different things. In the Jordan frame, the amplitude is calculated for a complex Higgs doublet allowing for different 'in' and 'out' states and therefore only s-channel scattering needed to be considered. In contrast, if we only had a singlet scalar we would not be able to make this restriction and s, t and u-channel diagrams would also have to be included. When this is done, it turns out that there is a remarkable cancellation between the diagrams and the amplitude is only proportional to s/M_P^2 [96], meaning unitarity only breaks down at around M_P as it does in the Einstein frame calculation above.

The Einstein frame calculation above is certainly valid for a singlet scalar, so in this case the two calculations match. But what happens if we have multiple degrees of freedom in the Einstein frame. It turns out that in this case, although the transformation from the Jordan to Einstein frame can proceed as above, once in the Einstein frame there is no field transformation which can simultaneously make multiple scalars canonically normalised. Terms such as the second term in Eq. (4.1.9) remain, producing an amplitude proportional to $\xi^2 s/M_P^2$ and therefore unitarity breaks down at around M_P/ξ as it does in the Jordan frame with multiple non-minimally coupled scalar fields.

The upshot of all this is that calculations in either frame are consistent and the unitarity bound is not frame dependent. For a singlet non-minimally coupled real scalar, unitarity breaks down at $E_\star \simeq M_P$. For multiple non-minimally coupled scalar fields, unitarity breaks down at $E_\star \simeq M_P/\xi$. The Higgs of the standard model is a complex doublet and therefore of the latter type and so the unitarity bound $E_\star \simeq M_P/\xi$ remains a potential problem for the Higgs inflation model.

Background dependence

The amplitude in Eq. (4.2.1) is calculated by expanding around $h = 0$. It was pointed out in Ref. [97] that the cutoff calculated from this amplitude is not the correct bound to consider, since during inflation the Higgs field takes a large value ($h \gg M_P/\sqrt{\xi}$) and so the expansion should be done around the field values in the inflating background. This idea was fully developed in Ref. [98] where the cutoff was calculated in both the Jordan and Einstein frames by separately expanding h around three different backgrounds: $h \ll M_P/\xi$, relevant for today's universe, $M_P/\xi \ll h \ll M_P/\sqrt{\xi}$, relevant for reheating and $h \gg M_P/\sqrt{\xi}$, relevant for inflation. When the Higgs field, with non-minimal coupling, is properly expanded around a non-zero background in the Jordan frame, there is a mixing between the graviton and Higgs degrees of freedom. Once these fields are diagonalised and canonically normalised the coupling between two Higgs and a graviton is proportional to

$$\frac{\xi\sqrt{M_P^2+\xi {h_0}^2}}{M_P^2+\xi {h_0}^2+6\xi^2 {h_0}^2} \tag{4.2.6}$$

where h_0 is the background value of the Higgs field. Using this background dependent coupling at large field values, $h_0 \gg M_P/\sqrt{\xi}$, we find the scattering amplitude for $hh \to hh$ scattering is proportional to $s/\xi {h_0}^2$, leading to unitarity breaking down at $E_\star \simeq \sqrt{\xi} h_0$. With $\xi \sim \mathcal{O}(10^4)$ this is always well above the size of the Higgs field during inflation. The same cutoff was also found in the Einstein frame and the analysis in the reheating regime also finds a cutoff above the size of the Higgs field during reheating. The conclusion is that the effective theory remains perturbative during the full history of inflation, reheating and in today's universe and hence the Higgs potential can be reliably extrapolated to the inflationary regime and Higgs inflation remains a predictive model.

The above discussion of background dependence is indeed true and shows that during inflation the Higgs inflation model remains weakly coupled. However, we pointed out in Ref. [6] that the original unitarity bound, Eq. (4.2.3), could still spell problems for the Higgs inflation model. Regardless of the background dependent nature of the cutoff, we still find in today's universe, expanding around $h = 0$, that unitarity breaks down at $E_\star \simeq M_P/\xi$. The essential question is then whether or not the tree level unitarity problem is fixed by new degrees of freedom or whether the theory simply becomes strongly coupled at M_P/ξ and no new physics enters until around M_P. As discussed in the opening chapter of this thesis, there is no sure fire way to determine which of the two paths will be chosen by nature to cure perturbative unitarity problems. However, if it would turn out that new physics does indeed appear at or below M_P/ξ then we would be back in the situation where the unknown physics could be characterised by higher dimensional operators such as Eq. (4.2.4) making the potential unpredictive again. There is no reason to believe that these new degrees of freedom would not also be present at this scale during the inflationary era and the Higgs inflation model would again be in trouble.

The conclusion is that if we were able to determine without ambiguity that the effective theory simply heals its unitarity problem by becoming strongly coupled at M_P/ξ, then the model would be fully consistent and predictive. However, there remains the strong possibility that new physics appears instead to fix the unitarity problem and the model becomes unpredictive. There is no obvious way to determine which of the two scenarios would happen, and with this in mind, all one can say is that the Higgs inflation model remains consistent with the caveat that it heals its perturbative unitarity problem simply by entering a non-perturbative regime of the effective theory, and no new physics enters until at least M_P.

Unitarising Higgs inflation

Following the understanding that the original model of Higgs inflation [86] suffered from unitarity problems, a number of models were proposed with the aim of providing a Higgs inflation model which was free from unitarity problems. Two models did so via the introduction of higher dimensional operators. Ref. [99] introduced additional interactions which are proportional to products of the derivatives of the Higgs doublet. These interactions were specifically introduced to counteract the parts of the amplitude in the original model that caused problems with unitarity. Ref. [100] introduced a new coupling between the kinetic term of the Higgs and the Einstein tensor. We will see in Section 4.4 that despite the claims made in the paper the model in fact suffers from unitarity problems in both the inflating background and today's universe. Both of these models could be criticised for employing very specific choices of higher dimensional operators. If they allow the specific operators that are introduced to give the desired effects they offer no argument as to why all other operators of the same dimension are suppressed. As such they can both be considered finely tuned.

A third proposal was introduced in Ref. [101] were an additional heavy scalar coupled to the Higgs is introduced to unitarise Higgs inflation. When we integrate out the heavy scalar, the low energy theory looks like the model of original Higgs inflation with a large Higgs non-minimal coupling. However, when we look at the full action we see that in the high energy regime, the Higgs actually has a small non-minimal coupling and the new scalar has a large non-minimal coupling and is the field that plays the dominant role of the inflaton. As such we do not really consider this a true model of Higgs inflation, see also Ref. [102] for similar criticisms.

Finally, there has been one more recent model of Higgs inflation proposed and is known as generalised Higgs inflation [103]. In this model an extensive set of higher dimensional operators has been introduced and there exists a broad area of the parameter space for which the inflationary predictions agree with experiment. However, there has not yet been any study of the scale of unitarity violation in this model and it would be a useful avenue of research to determine which parts of the large parameter space are free from unitarity problems.

4.3 Asymptotic safety and Higgs inflation

Because it is essential for the consistency of the Higgs inflation model that no new physics spoils the Higgs potential, it is of interest that the asymptotic safety scenario offers a paradigm for quantum gravity which requires no new physics even above the Planck scale. As such it offers an ideal framework in which the Higgs inflation model can exist. The scenario of asymptotically safe gravity, first proposed by Weinberg [104], provides a fully renormalisable UV completion to gravity (for reviews see [32, 33]). In this scenario, the dimensionless gravitational coupling approaches a non trivial fixed point in the UV under renormalisation group effects. The Planck mass is expected to become larger in the UV and the growth of amplitudes with energy of type $\xi^2 s/M_P^2$ could be compensated by the running of the Planck mass (see e.g. Ref. [105]). When gravity is coupled to matter the existence of the fixed point is even more difficult to establish, however detailed investigations have recently been carried out into scalar fields coupled to gravity [106, 107]. These studies incorporate the non-minimal coupling used in the Higgs inflation model and indicate that in the presence of these couplings a Gaussian matter fixed point could exist. A further result of their work is that if a non-trivial fixed point for gravity does exist, when scalar fields are introduced all the non-minimal couplings will be zero at the fixed point. This implies that ξ gets smaller in the UV and would further counter the growth with energy of amplitudes.

Although asymptotically safe gravity can provide its own paradigm for inflation [108], it could also be the perfect framework in which the Higgs inflation model could be realised. At high energies, the theory does not get replaced with new physics (as happens in string theory for example), instead the theory becomes strongly coupled and all possible higher dimensional operators become important. The theory remains predictive because it is hypothesised that only finitely many couplings are relevant as the UV fixed point is approached. No new physics beyond the standard model plus gravity is required and so operators such as those in Eq. (4.2.4) need not be present to spoil the potential. When we consider Higgs inflation, the background dependent cutoff still holds, meaning that perturbation theory remains valid during inflation. Thus asymptotically safe gravity with the standard model Higgs boson could provide a fully consistent inflationary scenario without having to introduce any new degrees of freedom.

Interestingly it was also suggested in [109], that if gravity were asymptotically safe and assuming there are no intermediate energy scales between the electroweak and Planck scales, the Higgs boson's mass could be predicted. For a positive gravity induced anomalous dimension (as is suggested by calculations in the literature) the Higgs boson mass would have to sit right at the bottom of the window which allows for the effective potential to be stable up to the Planck scale. In [109] this mass was calculated to be 126 GeV. As mentioned above, more recent studies, taking into account the observed Higgs boson mass [90, 91], allow for this to happen if the top quark mass lies right at the bottom of the 98% C.L. window and the strong coupling constant lies right at the top of its 98% C.L. window. If this turns out to be the case and the effective potential of the Higgs boson remains just stable right up to the Planck mass, then the standard model with the Higgs boson non-minimally coupled and asymptotically safe gravity could provide a complete theory of all the known forces of nature and a fully predictive model for inflation.

4.4 Unitarity of new Higgs inflation

To overcome the unitarity problems associated with the original proposal for Higgs inflation, Germani and Kehagias proposed a new model where the Higgs boson has a derivative coupling to the Einstein tensor [100]. They claimed that this new model was free of unitarity problems and could produce successful inflation. In a later paper [110] they calculated the cosmological perturbations in the model and showed that they were consistent with the latest WMAP data. Since the prime motivation for the new model was to overcome the unitarity problems associated with the original model of Higgs inflation, it is important to carry out a thorough analysis of the scale of unitarity violation in this model. We do this here and find that contrary to the original claims, the new model of Higgs inflation also suffers from unitarity problems during the inflationary period.

In Ref. [100] it is shown that the unique non-minimal derivative coupling of the Higgs boson to gravity, propagating no more degrees of freedom than general relativity minimally coupled to a scalar field, is given by the action

$$S = -\int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R - \frac{1}{2}(g^{\mu\nu} - w^2 G^{\mu\nu})\partial_\mu h \partial_\nu h + \frac{\lambda}{4}h^4\right], \qquad (4.4.1)$$

where $G^{\mu\nu} = R^{\mu\nu} - \frac{R}{2}g^{\mu\nu}$ is the Einstein tensor, w is an inverse mass scale, and h represents one of the real degrees of freedom of the standard model Higgs doublet.

To calculate the scale at which unitarity is violated in such a theory we consider $hh \to hh$ scattering via graviton exchange. As in Refs. [3, 43], we simplify the calculation by only considering s-channel scattering. This is justified for the case of the standard model Higgs doublet, which in the high energy regime being considered, appears as four real scalars. Expanding around the inflating background $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}/M_P$ where

$\bar{g}_{\mu\nu} = \text{diag}(-1, a(t), a(t), a(t))$ is the Friedmann-Robertson-Walker (FRW) metric, to lowest order in $h_{\mu\nu}$ the Einstein tensor is $G_{\mu\nu} = -3\mathcal{H}^2\bar{g}_{\mu\nu}$ where $\mathcal{H} \equiv \dot{a}/a$ is the Hubble constant.

For $w\mathcal{H} \gg 1$, we can expand h around its background value during inflation h_0. We have $h = h_0 + \frac{1}{\sqrt{3wH}}\delta h$ where δh is canonically normalized. As in Ref. [100], we find an interaction term

$$I \simeq \frac{1}{2H^2 M_P}\partial^2 h^{\mu\nu}(\partial_\mu \delta h)(\partial_\nu \delta h). \tag{4.4.2}$$

A power counting analysis then gives the scale at which unitarity is violated to be

$$E_\star \simeq (2\mathcal{H}^2 M_P)^{1/3}. \tag{4.4.3}$$

In Ref. [110], by direct comparison with the WMAP data and considering the allowed range of the standard model Higgs boson self coupling, the size of the background fields during inflation are found to be

$$R \simeq 5.6 \times 10^{-8} M_P^2\ , \tag{4.4.4}$$

$$2.1 \times 10^{-2} M_P < h_0 < 2.7 \times 10^{-2} M_P\ . \tag{4.4.5}$$

In order for higher dimensional operators such as Eq. (4.2.4) to be suppressed, we must ensure that during inflation, $R < E_\star^2$ and $h_0 < E_\star$. We can determine $\mathcal{H} \simeq \sqrt{R/12}$ from Eq. (4.4.4) and we find

$$E_\star \simeq 2 \times 10^{-3} M_P. \tag{4.4.6}$$

In Ref. [100] only the condition $R < E_\star^2$ was considered and the model was said to be free of unitarity problems. However, considering the bound on the Higgs field, Eq. (4.4.5), we see that $h_0 > E_\star$ during inflation and the model in fact suffers from unitarity problems.

It is also of interest to calculate $E_\star$ around today's background since this gives us the lowest energy at which new physics must appear in order to unitarise the theory. Expanding around a flat background $g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{2}h_{\mu\nu}/M_P + \mathcal{O}(M_P^{-2})$ and the standard model Higgs boson vacuum expectation value (which we take to be zero in the high energy limit being considered), the cut off is found to be

$$E_\star \simeq \left(\frac{5M_P}{w^2}\right)^{1/3}. \tag{4.4.7}$$

In Ref. [110], by comparison with the WMAP data the value of the dimensionful parameter w is found to lie in the range

$$7 \times 10^{-8}\, M_P < w^{-1} < 8.8 \times 10^{-8}\, M_P. \tag{4.4.8}$$

Taking the upper bound for w^{-1} we find that unitarity is violated at

$$E_\star = 3.4 \times 10^{-5} M_P \tag{4.4.9}$$

which is smaller than both $\sqrt{R}$ and h_0 during inflation.

We conclude that, during the inflationary period, new physics must be present to cure the unitarity problem and would likely spoil the inflationary potential.

Chapter 5

Bound on the Non-minimal Coupling of the Higgs Boson to Gravity

Three examples of the non-minimal coupling of the Higgs boson to gravity producing interesting physics have already been presented: model of low scale quantum gravity (Section 2.3), Higgs-radion mixing (Sections 3.4.3 and 3.5.4) and Higgs inflation (Chapter 4). There has been much additional interest in this coupling over the years. It could play an important role in cosmological models [111], inflationary scenarios [112] and models of induced gravity [113, 114]. Also, as mentioned in the introduction, this coupling should be generically present in the effective theory expansion unless it is forbidden by a symmetry.

The recent discovery of the Higgs boson at the LHC [38, 39] motivates the question of how to measure the size of the Higgs boson's non-minimal coupling ξ. In this chapter we derive the first known bound on the size of the non-minimal coupling. The approach utilises a decoupling effect between the physical Higgs boson and the standard model particles that accompanies a large non-minimal coupling and the effect this would have on the production and decay of the Higgs boson at the LHC. We also estimate the expected reach of future high energy, high luminosity runs at the LHC and proposed International Linear Collider (ILC) to improve the bounds on ξ. Finally we add some comments on Higgs boson decays to gravitons, the effect of a large non-minimal coupling on the Higgs boson's mass and the consequences of these results for various models found in the literature.

5.1 The decoupling effect

The action for the standard model Lagrangian ($\mathcal{L}_{SM}$) coupled to gravity, including the Higgs non-minimal coupling is

$$S = -\int d^4x\,\sqrt{-g}\left[\left(\frac{1}{2}M^2 + \xi H^\dagger H\right) R - (D^\mu H)^\dagger (D_\mu H) + \mathcal{L}_{SM} + \mathcal{O}(M_P^{-2})\right] \quad (5.1.1)$$

where the cosmological constant term has been suppressed. In the first term the Planck scale has been replaced by a generic mass scale to be fixed below. The kinetic term for the Higgs field, which is normally contained in $\mathcal{L}_{SM}$ has been explicitly written. After electroweak symmetry breaking, the Higgs boson gains a non-zero vacuum expectation value, $v = 246$ GeV, M and ξ are then fixed by the relation

$$(M^2 + \xi v^2) = M_P^2. \quad (5.1.2)$$

From this it is clear that $\xi \leq M_P^2/v^2 \simeq 10^{32}$. Note that ξ can be of arbitrary size if negative. One might naively expect that if $|\xi|$ is much below 10^{32} then its effects would not be observable in low energy experiments. This however turns out to be false as will be shown below.

The easiest way to see the decoupling effect of the Higgs boson[1] is to make a transformation to the Einstein frame (see Appendix D), $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, where $\Omega^2 = (M^2 + 2\xi H^\dagger H)/M_P^2$. The action in the Einstein frame then reads

$$S = -\int d^4x\,\sqrt{-\tilde{g}}\left[\frac{1}{2}M_P^2\tilde{R} - \frac{3\xi^2}{M_P^2\Omega^4}\partial^\mu(H^\dagger H)\partial_\mu(H^\dagger H) - \frac{1}{\Omega^2}(D^\mu H)^\dagger(D_\mu H) + \frac{\mathcal{L}_{SM}}{\Omega^4}\right] \quad (5.1.3)$$

Expanding around the Higgs vacuum expectation value and specializing to unitary gauge, $H = \frac{1}{\sqrt{2}}(0, h+v)^\top$, in order to have a canonically normalized kinetic term for the physical Higgs boson we need to transform to a new field χ where

$$\frac{d\chi}{dh} = \sqrt{\frac{1}{\Omega^2} + \frac{6\xi^2 v^2}{M_P^2\Omega^4}}\,. \quad (5.1.4)$$

Expanding Ω^{-1}, at leading order the field redefinition simply has the effect of a wave function renormalisation of

$$h = \frac{1}{\sqrt{1+\beta}}\,\chi, \quad (5.1.5)$$

[1] This effect was first realized for the Higgs boson in a paper by Van der Bij [114] where it was assumed that $M = 0$ and the Planck scale is generated entirely by the Higgs boson's vacuum expectation value with $\xi \simeq 10^{32}$. An earlier reference to the same effect in grand unified theories was made by Zee [115] where he assumed the Higgs boson's vacuum expectation value that breaks the Grand Unified Theory gauge symmetry could dynamically generate the Planck scale. See also [116] and references in [117], where the Planck scale is generated via a symmetry breaking mechanism.

where

$$\beta = 6\xi^2 v^2/M_P^2. \tag{5.1.6}$$

As a result, the Higgs boson's couplings to all the standard model particles get suppressed. For example, a Yukawa coupling to one of the standard model fermions ψ will become

$$yh\bar{\psi}\psi \to \frac{y}{\sqrt{1+\beta}}\,\chi\bar{\psi}\psi. \tag{5.1.7}$$

For $\xi^2 \gg M_P^2/v^2 \simeq 10^{32}$ the Higgs boson effectively decouples from the rest of the standard model.

This effect can also be understood in the original Jordan frame action (5.1.1) as arising from a mixing between the kinetic terms of the Higgs and gravity sectors. After fully expanding the Higgs boson (in unitary gauge) around its vacuum expectation value and also the metric around a fixed background, $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, the quadratic part of the Lagrangian becomes

$$\begin{aligned}\mathcal{L}^{(2)} = -\frac{M^2+\xi v^2}{8}\left(h^{\mu\nu}\Box h_{\mu\nu} + 2\partial_\nu h^{\mu\nu}\partial^\rho h_{\mu\rho} - 2\partial_\nu h^{\mu\nu}\partial_\mu h^\rho_\rho - h^\mu_\mu \Box h^\nu_\nu\right)\\ + \frac{1}{2}(\partial_\mu h)^2 + \xi v(\Box h^\mu_\mu - \partial_\mu\partial_\nu h^{\mu\nu})h.\end{aligned} \tag{5.1.8}$$

The final term represents a kinetic mixing between the higgs and graviton. To canonically normalise the fields requires the following change of variables

$$h = \frac{1}{\sqrt{1+\beta}}\,\chi, \tag{5.1.9}$$

$$h_{\mu\nu} = \frac{1}{M_P}\tilde{h}_{\mu\nu} - \frac{2\xi v}{M_P^2\sqrt{1+\beta}}\bar{g}_{\mu\nu}\chi. \tag{5.1.10}$$

We again find the physical Higgs boson gets renormalised by a factor $1/\sqrt{1+\beta}$.

5.2 Higgs Boson Production and Decay

At the LHC, the Higgs boson production and decay will be affected by the above suppression. At each vertex involving the Higgs boson coupled to standard model particles, a factor of $1/\sqrt{1+\beta}$ will be introduced. Clearly if $\beta \gg 1$ the Higgs boson would simply not be produced in a large enough abundance to be observed.

In the following we will make the assumption that there are no other degrees of freedom beyond those present in the standard model and Einstein gravity. We will refer to the usual standard model total cross section for Higgs boson production and decay with $\beta = 0$ as σ_{SM}. If the cross section including a non-zero β is given by σ, we are interested in the ratio $\sigma/\sigma_{\text{SM}}$. The LHC experiments produce fits to the data assuming that all

Higgs boson couplings are modified by a single parameter κ [118] which in our model corresponds to $\kappa = 1/\sqrt{1+\beta}$. Using the narrow width approximation, the cross section for Higgs production and decay from any initial i to final state f is given by

$$\sigma(ii \to h \to ff) = \sigma(ii \to h) \cdot \mathrm{BR}(h \to ff) = \kappa^2\, \sigma_{\mathrm{SM}}(ii \to h) \cdot \mathrm{BR}_{\mathrm{SM}}(h \to ff). \quad (5.2.1)$$

One might naively expect the cross section to be proportional to κ^4, but in the narrow width approximation this is not the case. The presence of the branching fraction, which is independent of a universal suppression of the couplings, leads to the cross section being proportional to κ^2. For a 125 GeV Higgs the narrow width limit is an excellent approximation and is used in the determination of the signal strength at the LHC.

The ATLAS detector has currently measured the global signal strength $\mu = \sigma/\sigma_{\mathrm{SM}} = 1.4 \pm 0.3$ [39] and CMS has measured this as $\mu = 0.87 \pm 0.23$ [38]. Combining these results gives $\mu = 1.07 \pm 0.18$. This excludes $|\xi| > 2.6 \times 10^{15}$ at the 95% C.L.

Reference [119] estimates the expected reach in the accuracy of the measurement of the Higgs boson couplings in a large number of processes in future runs at the LHC and the proposed ILC. Combining these results gives an estimated uncertainty in the global signal strength μ. Assuming a central value of $\mu = 1$, at a 14 TeV LHC with an integrated luminosity of 300 fb^{-1}, the uncertainty in the measurement of μ is expected to be 0.07 which would lead to a bound on $|\xi| < 1.6 \times 10^{15}$. At the ILC with a center of mass of 500 GeV and an integrated luminosity of 500 fb^{-1}, the expected uncertainty on μ is 0.005, which gives a bound of $|\xi| < 4 \times 10^{14}$. Despite expected measurements of the total cross section to an accuracy better than 1% at future high energy runs at the ILC, one cannot expect to push the constraints on $|\xi|$ below about 10^{14}.

5.3 Effects of a large non-minimal coupling on missing energy and the Higgs mass

Given a large non-minimal coupling to gravity, one might also expect to have decreased observable rates (missing energy) for Higgs decays at the LHC arising from unobserved decays to gravitons. The effect is in fact very small as we will now discuss. The lowest order vertex in ξ is a three point vertex connecting a single graviton line to two Higgs boson lines. This could introduce the possibility of a Higgs boson radiating a single graviton before decaying to standard model particles. While this process is kinematically allowed for an off shell Higgs boson, it turns out that due to the nature of the derivative coupling, the amplitude for this process is always proportional the four-momentum squared of the emitted graviton and is therefore zero. There is no vertex allowing for a Higgs boson decaying to two gravitons after the kinetic terms have been properly normalized. All other higher order processes will involve multiple Higgs bosons and as such will be extremely rare at any future collider. This leads to the conclusion that the decoupling effect is the primary method available at particle colliders to put constraints on ξ. It would be of great interest if any cosmological or astrophysical effects were found that could compete with the bound derived here.

We would like to make a short comment on the effect of the wave function renormalisation on the Higgs boson self coupling. Clearly the wave function renormalisation will also act to reduce the mass of the Higgs boson. This effect would have to be compensated by an increase in the Higgs boson self coupling. The increased self coupling would unfortunately not show up in direct searches attempting to measure the four point Higgs boson vertex since this will be further suppressed by a factor of $1/(1+\beta)$ coming from the additional two Higgs boson lines.

As mentioned above, there has been considerable interest in the Higgs boson non-minimal coupling to gravity in the literature. This coupling is particularly important in models of "induced gravity" where the Planck scale is generated spontaneously by setting $M = 0$ and requiring that $\xi \simeq 10^{32}$ [111, 114, 113]. Such a setup was also shown to be able to produce good inflation with the standard model Higgs boson acting as the inflaton [112]. Clearly the discovery of the Higgs boson rules out such models on the grounds that with such a large ξ the Higgs boson would be almost completely decoupled from the rest of the standard model and would never be produced at a collider. In fact the decoupling effect for the Higgs boson used here was first observed in Ref. [114]. As we saw in chapter 4, later models of Higgs inflation used a much smaller value of the non-minimal coupling of the order of 10^4. Unfortunately the results here imply that colliders will not be able to probe the size of the non-minimal coupling down to these scales in the foreseeable future.

Comment on a recent publication

We would like to make a brief comment here on a very recent publication related to the work in this chapter. In Ref. [120] Xianyu, Ren and He go through the same process as we have done above to establish the decoupling effect and their results agree with ours. They then specifically study the effect of a large non-minimal coupling on the unitarity of gauge boson scattering. They do so by using both the Goldstone boson equivalence principle and directly confirm the results for gauge boson scattering. They find that amplitudes such as

$WW \to ZZ$ are proportional to $\xi^2 E^2/M_P^2$ and are then able to bound ξ by requesting that unitarity holds up to a chosen energy scale. At low energies their bound is not competitive with ours, however by requiring that unitarity holds up to high energy scales they are able to place more stringent bounds on the size of ξ. For example, requiring unitarity to hold all the way up to the Planck scale produces a bound of $\xi \lesssim \mathcal{O}(10)$. We also note here that the bound they have derived for processes such as $WW \to ZZ$ using the Goldstone boson equivalence principle is essentially the same as the bound we have derived for the Higgs inflation model above.

Chapter 6

Conclusions

The effective theory of gravity coupled to matter represents a fully consistent quantum mechanical framework in which to study the low energy gravitational interactions of the standard model. This framework has been extensively used in recent years in both particle physics and cosmology. For example, in particle physics, there has been huge interest in the last fifteen years in the idea that extra space dimensions might bring the scale of quantum gravity within reach of experiments at the LHC. There have been thousands of papers published discussing the consequences of these models and experimental searches for signatures of extra dimensions are ongoing. In cosmology, one of the central areas of investigation is into the hypothesis that the early universe went through a period of exponential expansion. Again, hundreds of papers have been published proposing models which could have caused this inflation and using the effective theory of gravity coupled to matter to predict observable consequences of the models. With the recent data from the Planck satellite, comparison of these predictions with experimental data has reached new levels of accuracy.

With such a large amount of research time invested into projects which utilise the framework of the effective field theory of gravity coupled to matter it is of utmost importance that we understand the theory as well as we can. In particular, since every effective theory has a cutoff above which the theory breaks down, knowledge of the cutoff is vital in order to be able to trust perturbative calculations. In this thesis, we have looked at a wide variety of models which rely on this framework to make predictive calculations and have used a variety of tools in order to place bounds and constraints on parameters in the models. In particular we have made extensive use of perturbative unitarity bounds to find lower bounds on the cutoff in many different models, providing new information about the regime of validity of the effective theory in each example.

The tool of perturbative unitarity is a powerful tool to place simple and clear bounds on models. In the introductory chapter, we reviewed the use of this tool by Lee, Quigg and Thacker to calculate a bound on the Higgs boson mass. Many such bounds have been calculated since and we have extended this work by applying unitarity bounds to models coupled to the effective theory of gravity.

In Chapter 2, we presented our general framework for many of the following chapters by deriving the lowest unitarity bounds for all types of matter fields coupled to gravity by considering s-channel scattering via graviton exchange. We then applied these bounds to various grand unified models found in the literature. We improved on this bound by incorporating renormalisation group effects into a running Planck mass. By incorporating quantum effects, this not only gives us a much more accurate determination of the scale of unitarity violation, it also allows us to define a notion of the scale at which we expect gravity to become strongly coupled. We are then able to compare this to the scale of unitarity violation in order to classify models by whether or not unitarity breaks down before the scale at which gravity becomes strongly coupled which would then require new physics to appear at this scale in order for the model to remain consistent. We found that grand unified theories with particularly large field contents can fall into the category of requiring new physics before the scale of strong gravity in order to remain unitary. We also looked briefly at two models which utilise the running Planck mass in order to lower the scale of quantum gravity to near the electroweak scale. This is possible by introducing either a very large number of fields or a huge non-minimal coupling. We found that in both these models, unitarity breaks down below the low scale of quantum gravity.

In Chapter 3, we looked at the exciting idea that there may be extra dimensions of space that could be observable at experiments at the LHC. These models have been proposed because they resolve the seemingly unnatural hierarchy between the Planck and the electroweak scale. They have been studied extensively in the literature and there are many ongoing searches for their experimental signatures. For this reason it is crucial to understand when and where we can reliably use the effective theory and trust our calculations. We made the model independent observation that if we have more than one KK graviton present in a model, then we find that the partial wave amplitude for graviton exchange exceeds the unitarity bound at the first KK mode resonance. This general bound relies on the addition of Breit-Wigner resonances and we also showed hints that if we include the effects of interference between the resonances this unitarity problem may be cured. This however is a complicated non-perturbative effect and the full consequences go beyond the scope of this thesis. It remains an interesting avenue for future research. We then looked more closely at three different extra dimensional models.

We first looked at the ADD model which is characterised by an extremely fine spacing of KK modes with the lowest lying mode having an extremely small mass. For this reason, the general unitarity bound has severe consequences for this model. If perturbative unitarity does break down completely at the first KK mode it means there is only a tiny energy regime in which one can reliably perform calculations. We point out however that the unitarity problem only exists near the top of the resonance peaks and this leaves a broad range of energies between the peaks where unitarity will likely still hold up to much higher energies. We have made extensive attempts to ascertain this 'off resonance' unitarity bound, using the common technique of approximating the sum over modes by an integral. Unfortunately we are unable to separate the effects of the resonances from the unitarity bound without introducing strong dependence on an arbitrary cutoff. The most robust bound that can be calculated using this method, comes from the imaginary part of the amplitude and in general we find from this bound that unitarity breaks down at about $0.8M_D$. However, this bound comes exclusively from the resonant exchange of on-shell KK gravitons. As such it is in competition with the bound calculated in the model independent way without using any approximation. Since these bounds show a big disagreement we have to conclude that approximating the sum over modes by an integral offers a very poor approximation for the purposes of calculating unitarity bounds.

We are left to conclude in the ADD model that unitarity breaks down at the first KK mode but there is likely a large energy range between each resonance where unitarity is maintained to much higher energies. Unfortunately, as with most observables in the ADD model, the specifics of this bound are strongly dependent on an arbitrary cutoff. Because the problems with unitarity happen only very near the resonances, and the energy resolution of current detectors is bigger than the spacing between the modes, it is likely that our bound does not in fact pose any serious problems for phenomenology of the ADD model and in particular the technique of approximating the sum over modes by an integral may remain valid for calculating experimental observables. However, it is still of theoretical interest how the ADD model deals with this unitarity problem and if there is indeed any mechanism by which it remains unitary at the resonances. We believe that the inclusion of interference effects between the resonances may hold the key to restoring theoretical consistency to the ADD model and as such deserves further research.

We also looked briefly at the RS model. Applying the general model independent bound in this scenario does not have such severe consequences as it does for the ADD model. The first KK mode resonance coincides with the scale at which we expect gravity to become strong and so it is not surprising that unitarity breaks down here. Also there is a large mass gap below the lowest lying KK mode and so there remains a large energy range in which the effective theory remains valid. We also reviewed attempts to derive unitarity bounds arising from exchange of the radion following a suitable stabilisation mechanism. With a minimally coupled Higgs boson, no significant bounds can be derived, however if the Higgs boson is non-minimally coupled, the size of the non-minimal coupling is constrained to be $|\xi| \lesssim 2.7$ by requiring that the unitarity bound is not exceeded below a cutoff defined as $\Lambda = e^{-kb} M_P$.

Finally we looked at the linear dilaton model. This model is distinguished by its unique graviton KK spectrum which has a mass gap of $|\alpha|/2$ followed by a near continuum of modes. For large $|\alpha|$ the application of the general unitarity bound does not pose much of a problem for this model for the same reasons as in the RS case. We expect gravity to become strongly coupled around the first KK mode and there remains a sizeable region in which the effective theory remains valid. However, for small values of $|\alpha|$, the prospect of unitarity breaking down at the first KK mode could pose problems for the model. The first KK mode will be light and so unitarity breaks down at a very low scale and calculations above this scale may not be reliable. We also derived unitarity bounds arising from the exchange of the radion in a stabilised linear dilaton model. Similarly to the RS model, we find that if the Higgs boson is minimally coupled then no significant constraints can be placed on the model from unitarity. However, with a non-minimally coupled Higgs boson, we find that for large values of $|\alpha|$, unitarity constrains the size of the non-minimal coupling. However, there remains a large part of the parameter space which does not suffer from problems with unitarity.

In Chapter 4 we looked at models which propose the idea that the standard model Higgs boson could play the role of the inflaton and have caused a period of exponential inflation in the early universe. The original model of Higgs inflation requires a rather large value of the non-minimal coupling $\xi \sim 10^4$. We showed that this large non-minimal coupling means that when the Higgs field is expanded around a small VEV, unitarity breaks down at about M_P/ξ. Inflation takes place for values of the Higgs field $h > M_P/\sqrt{\xi}$. If new physics appears at or below the unitarity violation scale in order to fix the unitarity problem it would have unknown effects on the Higgs potential above this scale and destroy the predictability of the model.

We reviewed the background dependent bound which shows that despite unitarity breaking down below the inflationary scale in today's universe, during the inflationary epoch the Higgs has a large background value and in this regime there are no unitarity problems. Despite this claim, we maintain that if in today's universe new physics appears at the scale M_P/ξ, there is no reason why it would not still be present at that scale during the inflationary period and would interfere with the potential. With this in mind, there are still major concerns for the consistency of the Higgs inflation model even though it does remain perturbative during the inflationary period.

We briefly introduced the idea that asymptotic safety could provide a perfect framework in which the Higgs inflation model could exist without having to introduce new physics. Following this we also looked at a new model of Higgs inflation which relies on a coupling between the kinetic term of the Higgs and the Einstein tensor. This model was specifically introduced to overcome the unitarity problems of the original model of Higgs inflation. However, after a thorough analysis, we find that in fact this model exceeds the unitarity bound below the inflationary scale both in the inflating background and in today's universe.

Finally, in Chapter 5, we derived the first ever bound on the size of the Higgs boson's non-minimal coupling to gravity. We observe a decoupling effect between the standard model particles and the Higgs boson in the presence of a large non-minimal coupling. Using this effect and the latest data from both the ATLAS and CMS experiments at the LHC we are able to place the bound $|\xi| < 2.6 \times 10^{15}$. We also predict the reach of future experiments to improve on this bound.

We have seen throughout this thesis, many places where bounds derived on the effective theory of gravity coupled to matter have provided a better understanding of a large variety of models. Amongst other things, this knowledge can provide us with confidence about when we are able to reliably use the effective theory and more importantly warn us when it is no longer valid. This remains an open area of research and this thesis provides motivation for a number of interesting new avenues. We list four possible future research directions here:

- The disparity between the scale at which unitarity breaks down and the scale at which the running Planck mass becomes strongly coupled is not fully understood. It would be of interest to properly understand why these scales are sometimes separate and whether this signifies a true inconsistency in the model
- The breakdown of unitarity at the first KK mode in models with extra dimensions

can be a serious problem for models with a small mass gap such as ADD. We have seen hints that if we take into account the interference effects between resonances it may cure this problem. This obviously provides motivation to develop this idea further.

- Because the Higgs boson is the only observed fundamental scalar field, it is of real interest whether it could also play the role of the inflaton in the early universe. Unfortunately Higgs inflation models tend to suffer from unitarity problems. A new model called generalised Higgs inflation [103] has recently been developed with a large parameter space of couplings. It would be useful to carry out a thorough investigation of this model to find out what areas of the parameter space are free from unitarity problems.

- We have shown that future particle accelerators will not be able to place bounds on the size of the Higgs boson's non-minimal coupling below about $|\xi| \simeq 10^{14}$. It is possible that cosmological or astrophysical observations could improve on these bounds and as such provide a further important research direction.

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Appendix A

Polarisations of External Particles

For the calculation of scattering amplitudes throughout this thesis, we have taken the following values for the initial and final state four momenta:

$$k_1^\mu = (E,\, 0,\, 0,\, -p)\,, \qquad k_3^\mu = (E,\, -p\sin\theta,\, 0,\, -p\cos\theta)\,,$$
$$k_2^\mu = (E,\, 0,\, 0,\, p)\,, \qquad k_4^\mu = (E,\, p\sin\theta,\, 0,\, p\cos\theta)\,.$$

The corresponding polarisation vectors for spin one fields are (0 represents longitudinal polarisation and $\pm$ transverse polarisation):

$$\epsilon_1^\mu(0) = (-p,\, 0,\, 0,\, E)\,/m, \qquad \epsilon_3^{\mu *}(0) = (p,\, -E\sin\theta,\, 0,\, -E\cos\theta)\,/m,$$
$$\epsilon_1^\mu(\pm) = (0,\, -1,\, \pm i,\, 0)\,/\sqrt{2}\,, \qquad \epsilon_3^{\mu *}(\pm) = (0,\, -\cos\theta,\, \mp i,\, \sin\theta)\,/\sqrt{2}\,,$$

$$\epsilon_2^\mu(0) = (-p,\, 0,\, 0,\, -E)\,/m, \qquad \epsilon_4^{\mu *}(0) = (p,\, E\sin\theta,\, 0,\, E\cos\theta)\,/m,$$
$$\epsilon_2^\mu(\pm) = (0,\, 1,\, \pm i,\, 0)\,/\sqrt{2}\,, \qquad \epsilon_4^{\mu *}(\pm) = (0,\, \cos\theta,\, \mp i,\, -\sin\theta)\,/\sqrt{2}.$$

In the helicity basis, the Dirac fermion spinors are of the following form:

$$u_+(p) = \begin{pmatrix} \sqrt{E-\mathrm{p}} & \xi_+ \\ \sqrt{E+\mathrm{p}} & \xi_+ \end{pmatrix} \qquad u_-(p) = \begin{pmatrix} \sqrt{E+\mathrm{p}} & \xi_- \\ \sqrt{E-\mathrm{p}} & \xi_- \end{pmatrix}$$

$$v_+(p) = \begin{pmatrix} \sqrt{E+\mathrm{p}} & \eta_+ \\ -\sqrt{E-\mathrm{p}} & \eta_+ \end{pmatrix} \qquad v_-(p) = \begin{pmatrix} \sqrt{E-\mathrm{p}} & \eta_- \\ -\sqrt{E+\mathrm{p}} & \eta_- \end{pmatrix}$$

and the Weyl spinors ξ and η are

$$\xi_+ = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}, \qquad \xi_- = \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}, \qquad \eta_\pm = \pm\xi_\mp\,.$$

Appendix B

Wigner d-functions

$$
\begin{aligned}
d^0_{0,0} &= 1 \\
d^2_{0,0} &= \frac{1}{2}(3\cos^2\theta \\
d^2_{1,0} &= -\sqrt{\frac{3}{8}}\sin 2\theta \\
d^2_{1,1} &= \frac{1}{2}(2\cos^2\theta + \cos\theta - 1) \\
d^2_{1,-1} &= \frac{1}{2}(-2\cos^2\theta + \cos\theta + 1) \\
d^2_{2,0} &= \sqrt{\frac{3}{8}}\sin^2\theta \\
d^2_{2,1} &= -\frac{1}{2}\sin\theta(1 + \cos\theta) \\
d^2_{2,-1} &= -\frac{1}{2}\sin\theta(1 - \cos\theta) \\
d^2_{2,2} &= \frac{1}{4}(1 + \cos\theta)^2 \\
d^2_{2,-2} &= \frac{1}{4}(1 - \cos\theta)^2
\end{aligned}
$$

$$
d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}
$$

Appendix C

Integrals

To solve the integral

$$\int \frac{y^{d-1}}{1-y^2} dy$$

we use the general formula

$$\int x^m \, (a + b\, x^n)^p \, dx = \frac{x^{m-n+1} \, (a + b\, x^n)^{p+1}}{b(m + n\, p + 1)} - \frac{a(m-n+1)}{b(m + n\, p + 1)} \int x^{m-n} \, (a + b\, x^n)^p \, dx$$

which for the simplified case considered here becomes

$$\int x^m (1 - x^2)^{-1} dx = -\frac{x^{m-1}}{m-1} + \int x^{m-2} (1 - x^2)^{-1} dx.$$

Iterating this for even m will reduce it to

$$-\sum_{k=1}^{m/2} \frac{x^{2k-1}}{2k-1} + \int \frac{1}{1-x^2} dx = -\sum_{k=1}^{m/2} \frac{x^{2k-1}}{2k-1} + \frac{1}{2} \log \left| \frac{1+x}{1-x} \right|$$

and for odd m it will reduce to

$$-\sum_{k=1}^{m/2} \frac{x^{2k}}{2k} + \int \frac{x}{1-x^2} dx = -\sum_{k=1}^{m/2} \frac{x^{2k}}{2k} - \frac{1}{2} \log \left| 1 - x^2 \right| .$$

Appendix D

Transforming between Einstein and Jordan frames

In this appendix general expressions are given in n dimensional spacetime. An action defined in the Jordan frame where a scalar field is coupled to the Ricci scalar can be transformed to a minimally coupled Einstein frame via a conformal transformation of the metric

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$$

$$\tilde{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu}\,, \quad \sqrt{-\tilde{g}} = \Omega^d \sqrt{-g}\,.$$

Under such a transformation, the Ricci scalar transforms as

$$R = \Omega^2 \left[\tilde{R} - 2(n-1)\tilde{\Box}\omega - (n-1)(n-2)\tilde{g}^{\mu\nu}\partial_\mu\omega\partial_\nu\omega\right]$$

where

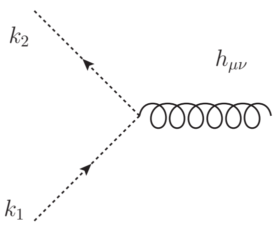
$$\omega \equiv \ln\Omega\,, \quad \tilde{\Box}\omega = \frac{1}{\sqrt{-\tilde{g}}}\partial_\mu(\sqrt{-\tilde{g}}\,\tilde{g}^{\mu\nu}\partial_\nu\omega)\,.$$

Note that the second term will often appear as a total derivative and can then be discarded.

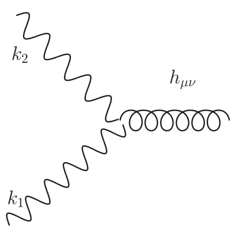
Appendix E

Feynman Rules

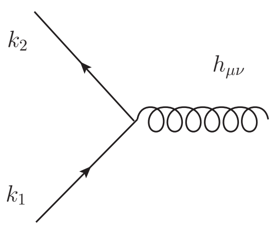
The first three Feynman rules are for gravitons coupled to scalars, vector bosons and fermions respectively. They are reproduced from Ref. [29].



$$\frac{-i}{2\sqrt{2}\Lambda_n}(m_\phi^2\eta_{\mu\nu}C_{\mu\nu,\rho\sigma}k_1^\rho k_2^\sigma)$$



$$\frac{-i}{2\sqrt{2}\Lambda_n}((m_A^2 + k_1 \cdot k_2)C_{\mu\nu,\rho\sigma} + D_{\mu\nu,\rho\sigma}(k_1, k_2) + \xi^{-1}E_{\mu\nu,\rho\sigma}(k_1, k_2))$$



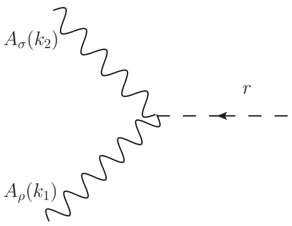
$$\frac{-i}{8\sqrt{2}\Lambda_n}(\gamma_\mu(k_{1\nu} + k_{2\nu}) + \gamma_\nu(k_{1\mu} + k_{2\mu}) - 2\eta_{\mu\nu}(\not{k}_1 + \not{k}_2 - 2m_\psi))$$

Where, Λ_n is the coupling and $\Lambda_n = M_P$ for the ususal massless 4D graviton. ξ is the gauge fixing parameter and

$$\begin{aligned} C_{\mu\nu,\rho\sigma} &= \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma} \ , \\ D_{\mu\nu,\rho\sigma}(k_1,k_2) &= \eta_{\mu\nu}k_{1\sigma}k_{2\rho} - \Big[\eta_{\mu\sigma}k_{1\nu}k_{2\rho} + \eta_{\mu\rho}k_{1\sigma}k_{2\nu} - \eta_{\rho\sigma}k_{1\mu}k_{2\nu} + (\mu \leftrightarrow \nu)\Big] \ , \end{aligned}$$

$$\begin{aligned} E_{\mu\nu,\rho\sigma}(k_1,k_2) &= \eta_{\mu\nu}(k_{1\rho}k_{1\sigma} + k_{2\rho}k_{2\sigma} + k_{1\rho}k_{2\sigma}) \\ &\quad - \Big[\eta_{\nu\sigma}k_{1\mu}k_{1\rho} + \eta_{\nu\rho}k_{2\mu}k_{2\sigma} + (\mu \leftrightarrow \nu)\Big] \ . \end{aligned}$$

The following Feynman rule is for the radion in the linear dilaton model coupled to gauge bosons and is reproduced from Ref. [81].



$$\frac{ib_1\kappa_\phi}{M_*}\left(k_2{\cdot}k_3\eta^{\sigma\rho} - k_2^\rho k_3^\sigma\right) + 2im_V^2\left(\frac{b_1}{M_*}\left(\frac{\kappa_\phi}{2} - \kappa_\Phi\right) + \frac{a_1}{v}\right)\eta^{\sigma\rho}$$