

Modified Vlasov Equation and Collective Oscillations of Relativistic Radiating Electron Plasma

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Abstract. A Covariant Statistical Mechanics is developed for a system of electrons within the limits of plasma physics. By using the Landau-Lifshitz equation of motion for point charged particles, the phase space reduces to be a seven dimensional space instead of the ten dimensional space proposed by Hakim and Mangeney [J. Math. Phys. 1968, 9, 1, 116]. A modified Vlasov equation is obtained which permits one to deduce a general expression for the dispersion relations. Unlike other works, neither the Special Relativity nor any of the properties of the gauge are used to discard some of the wave modes. Instead, the continuity equation is used as a constraint embedded in the definition of the 4–vector current. This permits to make a mathematical analysis for calculating and discarding the transverse and longitudinal modes for any temperature. An expression for the longitudinal wave modes is deduced for any temperature and wave number. Usually, certain approximations are made to obtain such an expression that matches our result. But in general, without making such approximations, the result differs by a factor. The Landau damping and the damping due to the radiation reaction force are obtained and contrasted at low and high temperature cases. The results are compared with the ones obtained within Magnetohydrodynamics. Moreover, a numerical method is applied in order to compare the dispersion relations obtained with the approximations made. Surprisingly, it is sampled numerically that the dispersion relations are bounded for each temperature and that there is a cut-off for each wave number.

1. Introduction

During the sixteens of the last century, Hakim developed a manifestly Covariant Statistical Mechanics [1], [2]. The theory studied the electromagnetic interactions and described a rigorous hierarchy of equations for suitable reduced densities. Later, Hakim and Mangeney [3], including the radiation reaction force calculated by Dirac [5], obtained a relativistic Vlasov equation which permits one to obtain wave dispersion relations and the conductivity tensor. They also analyzed the irreversibility due to the interaction between the charges and the radiation reaction force and many other properties of a relativistic plasma. However, the use of the Lorentz-Dirac equation obligates to consider the hyper-acceleration as an independent variable since it plays the same role than the position and the velocity. The phase space results to be a $10N$ –dimensional space ($12N - 2N$, due to the kinematic relations). In an approximation process, by using the Lorentz-Dirac equation at first order in the characteristic time of the charges τ_o and by considering absence of correlation between particles, they obtained a modified Vlasov equation at first order in $ne^2\tau_o$ where n , e and $\tau_o = 2e^2/3mc^3$ represent the density of particles, the charge of the electron and the characteristic time of the electron, respectively. Among others properties that



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can be derived, as we mentioned above, it has to be highlighted that this modified Vlasov equation can be linearized in order to obtain dispersion relations for electromechanical waves. They obtained general expressions for the transverse and longitudinal collective modes. They gave a solution for the cases $T = 0$ and $T \rightarrow \infty$, discriminating certain results due to very vague physical reasoning. Indeed, for $T = 0$, they eliminated solutions for the longitudinal mode ($w^2 = w_p^2 + k^2$) arguing that this solution comes from Special Relativity. Moreover, for high temperature without giving any proof for the longitudinal mode, they mentioned "it seems that the extreme relativistic plasma behaves like the vacuum for both longitudinal and transverse electromagnetic waves". In a later article [4], they accepted the lack of severity in the choice of the solutions and they claimed that the dispersion relations do not take into account the fact that the 4-current vector must satisfy the continuity equation. Although for low temperatures ($T \simeq 0$) the usual expected results are obtained just for the neighborhood of $k \simeq w_p$, for the high temperature case ($T \rightarrow \infty$), in the first paper about collective oscillations [3] they just assured that $k^\mu k_\mu = 0$ without any proof and they argued that the damping effects predominate and a wave cannot propagate in a high temperature plasma. Nevertheless, these types of contradictions were corrected in a second article [4] where they made some approximations for high temperature showing that the plasma frequency is shifted. Indeed, this does not present a surprise since radiation emission becomes more effective when the temperature increases. Although, for Hakim and Mangeney [3] in extreme relativistic plasmas, their own expressions are not longer valid since the first order approximation in $ne^2\tau_o$ becomes meaningless if bigger order terms are considered and, consequently, the decay of the wave must be accentuated. However, the order of approximation is related to the value of $ne^2\tau_o$ and high temperature systems can be considered within their model as long as the parameter $ne^2\tau_o$ stays small (see figure 2). Although the Landau-Lifshitz equation [6] can be considered or not as an approximation to order τ_o of the Lorentz-Dirac equation [5], the reality is that the differences with the Lorentz-Dirac equation or with the Eliezer-Ford-O'Connell equation [12], [10], [11] are negligible except in the cases of self-accelerations and pre-accelerations [14], [15], [13], [16]. Therefore, the method may be used to deduce the behavior of the longitudinal modes for high temperatures. One of the purposes of this paper consists in demonstrating that the longitudinal modes disappear when the temperatures increase but accompanied by the trend in which the longitudinal waves are canceled; that is: a dispersion relation is proposed for high temperature such that for $T \rightarrow \infty$ the frequency w becomes imaginary. Then, a cut-off exists above a temperature and all the perturbations are damped. Particularly, a cut-off appears for high frequencies leaving just the existence of low frequencies that does not possess any physical important effect.

On the other hand, as we mentioned before, although the Landau-Lifshitz equation is considered as a first order approximation in τ_o of the Lorentz-Dirac equation [5], many authors refer to it as the exact equation of motion for a charge particle. Indeed, for Spohn [7] and Rohrlich [8], [9] the Lorentz-Dirac equation must be restricted to its critical surface yielding the Landau-Lifshitz equation and consequently this last one represents the correct equation of motion of a spinless classical point charge. Moreover, by considering a generalized quantum Langevin equation and by giving a structure to the electron with a factor form and a finite cut-off parameter, Ford and O'Connell [10], [11] derived a non relativistic equation of motion for charged particles. Generalizing this equation to Special Relativity [11] they obtained an equation that Eliezer [12] derived fifty years before using distinct arguments. Although, the Landau-Lifshitz equation [6] and the Eliezer-Ford equation are mathematically different, their implications are physically equivalent [14], [15], [13], [16] within the Shen's zone [17] where quantum effects are negligible. Recently, Hammond [18], [19], [20], [21] has proposed another approach for deducing the reaction force which eliminates the apparent paradox of the vanishing reaction force when a constant electric field is applied to a charge. Although for many authors [23], [24], [25], [26], [27], [28], this apparent paradox is explained by noticing that the radiation

exists at the infinity; that is, the energy radiated to infinity is taken from the attached fields (the Scott term or the acceleration energy) and consequently even if the total radiation term in the equation of motion vanishes, the radiation to the infinity (the irreversible emission of radiation) exists. Moreover, the regular Larmor formula does not describes the radiated energy and it is substituted by a new expression that coincides with it for low accelerations [24]. Although, Hammond theory is consistent, it possesses the inconvenience that the expression for the reaction term must be deduced for each kind of interaction making it unpractical to deal with it when a system of particles is considered. Apart of this paradox, both equations, Landau-Lifshitz and Eliezer-Ford-O'Connell, are second order differential equations which do not present unphysical solutions, as pre-accelerations and runaway solutions. Whether or not the Landau-Lifshitz equation represents an exact or approximate description of the motion of charged particle, it must be subjected to experimental tests. However, such experiments present a very large degree of difficulty for various reasons [13], [14], [15]. However, dispersion relations can be easier measured and comparing them with the ones deduced by using the modified Vlasov equation, obtained starting from the Landau-Lifshitz equation, will give validity or not to the last equation. Otherwise, theory such as Hammond's may have more physical acceptance because at high speeds (high temperatures) and high accelerations great differences are presented between the different equations [19].

Once the Landau-Lifshitz equation is chosen to describe the motion of the charges, it is possible to eliminate the degree of liberty represented by considering the hyper-acceleration as an independent variable in the deduction of the density function. Indeed, the hyper-acceleration does not appear anymore. Therefore, we can follow Hakim [3] in the deduction of the density just by considering the positions and the velocities of the particles as independent variables. The phase space results to be a $7N$ -dimensional space ($8N - N$, due to the kinematic relations) The final result will coincide with Hakim's generalization of the Vlasov equation in Special Relativity [3]. Moreover, a more general equation will be deduced by including a constant external electric field. The wave dispersion relations will be deduced and the effects of the coupling between the constant external electric field and the interaction of the particle will not play an important role and the dispersion relations will not be affected. Moreover, Landau damping can be deduced and compared with the damping due to the radiation reaction force. When a numerical method is used in order to solve the general expression for the longitudinal modes, it turns out that although the effect is much more pronounced in the case of very high temperatures, the dispersion relations have no real solution above a wave number for each temperature.

The article is organized as follows: in section 2, an equivalent expression of the Landau-Lifshitz equation of motion will be described in order to be used in the deduction of the modified relativistic Vlasov equation in section 3. In section 3, a modified relativistic Vlasov equation is deduced without considering the hyper-acceleration as independent variable and including an external field. In section 4, a perturbation is made in order to obtain a perturbed current and a constant electric field is considered in order to prove that no changes appear in the wave dispersion relations. In section 5, the dispersion relations will be deduced for different temperatures. The low temperature case will be analyzed for transverse and longitudinal modes and we give the reason why the wave solution ($w^2 = w_p^2 + k^2$) should be discarded for longitudinal modes without resorting to arguments based on not considering solutions that come from relativity or the absurd theory that they depend on the chosen gauge. Nor do we resort to the continuity equation as an extra constraint since it must already be contained in the Vlasov equation due to the fact that the last is deduced by using the definition of point particle current or the Maxwell's equations. Of course, Landau damping [4] is deduced for some values of k and temperatures. The damping due to the radiation reaction force is deduced for low temperatures and compared with the Landau damping. In this section, the high temperature case is analyzed obtaining the dispersion relations for longitudinal modes and explaining how

the wave disappears as long as the temperature increases. The existence of a cut-off for the frequencies is shown for high temperatures. Some concluding remarks are done in section 7.

2. Landau-Lifshitz Equation

For a system of N charged particles with the same mass m and equal charge e , for each particle, the Lorentz-Dirac equation [5] is

$$ma_i^\mu = \frac{e}{c} F_{(i)}^{\mu\nu} v_{\nu i} + m\tau_o \left[\dot{a}_i^\mu + \frac{a_i^2}{c^2} v_i^\mu \right], \quad i = 1, \dots, N \quad (1)$$

where c , $F_{(i)}^{\mu\nu}$ and $\tau_o = 2e^2/3mc^3$ are the speed of light, the field strength tensor due to the action of the other particles and the external field on the i -particle and the characteristic time of the charge e , respectively. The subscript “ i ” represents the i -particle and v_i^μ , a_i^μ and $\dot{a}_i^\mu = da_i^\mu/d\tau_i$ are the 4-velocity, the 4-acceleration, the 4-hyper-acceleration of the i -particle being τ_i the proper time of the i -particle, respectively. The equation can be expressed in a different way as:

$$ma_i^\mu = \frac{e}{c} F_{(i)}^{\mu\nu} v_{\nu i} + m\tau_o \Delta^{\mu\nu}(v_{\rho i}) \dot{a}_{\nu i}, \quad i = 1, \dots, N, \quad (2)$$

where

$$\Delta^{\mu\nu}(v_{i\rho}) = n^{\mu\nu} - \frac{v_i^\mu v_i^\nu}{c^2}. \quad (3)$$

In order to incorporate Landau-Lifshitz proposal on equation (2), the hyper-acceleration must be substituted by

$$\dot{a}_{\nu i} \rightarrow \frac{e}{mc} \frac{d}{d\tau_i} F_{\nu\alpha}^{(i)} v_i^\alpha = \frac{e}{mc} \left[F_{\nu\alpha}^{(i)} a_i^\alpha + v_i^\rho v_i^\alpha \frac{\partial F_{\nu\alpha}^{(i)}}{\partial x_i^\rho} \right], \quad (4)$$

and in a second step, in the first term of the right side of equation (4), the acceleration a_i^α must be substituted by the Lorentz force; that is:

$$\dot{a}_{\nu i} \rightarrow \frac{e}{mc} \left[\frac{e}{mc} F_{\nu\alpha}^{(i)} F_{(i)}^{\alpha\beta} v_{\beta i} + v_i^\rho v_i^\alpha \frac{\partial F_{\nu\alpha}^{(i)}}{\partial x_i^\rho} \right]. \quad (5)$$

Therefore, the Landau-Lifshitz equation of motion for each particle is

$$\begin{aligned} ma_i^\mu &= \frac{e}{c} F_{(i)}^{\mu\nu} v_{\nu i} + m\tau_o \Delta^{\mu\nu}(v_{\rho i}) \frac{e}{mc} \left[\frac{e}{mc} F_{\nu\alpha}^{(i)} F_{(i)}^{\alpha\beta} v_{\beta i} + v_i^\rho v_i^\alpha \frac{\partial F_{\nu\alpha}^{(i)}}{\partial x_i^\rho} \right], \\ i &= 1, \dots, N. \end{aligned} \quad (6)$$

On the other hand, the force applied on each particle must be decomposed in two parts: the external field and the field produced by the other particles on the i -particle; that is:

$$F_{(i)}^{\mu\nu} = F_{ext}^{\mu\nu} + \sum_{j \neq i} F_{ji}^{\mu\nu}(x_\rho), \quad (7)$$

where

$$\begin{aligned} F_{ji}^{\mu\nu}(x_\rho) &= \frac{4\pi e}{c} \int \int \int dx'^4 dv'^4 d\tau' [v'^\nu \partial^\mu - v'^\mu \partial^\nu] D_{ret}(x_\rho - x'_\rho) \\ &\quad \times \delta(x'_\rho - x_{\rho j}(\tau')) \delta(v'_\rho - v_{\rho j}(\tau')), \end{aligned} \quad (8)$$

where D_{ret} represents the retarded Green function [29].

Finally, it is important to notice that the equation of motion described by equation (6) does not depend on the hyper-acceleration and consequently the phase space is described by $8N$ -dimensional space with the regular N constraint, $v^\mu v_\mu = c^2$. This point represents one of the different approaches to obtain the relativistic Vlasov equation compared with the work done by Hakim and Mangeney [3].

3. Relativistic Vlasov Equation with External Field Including Radiation Reaction Force

Hakim [1], [2] developed a manifestly covariant Statistical Mechanics generating an equation of BBGKY hierarchy in order to obtain a kinetic equation [3]. The work was so important that many others relied on it to develop various ideas. For example, Horwitz [30] proposed a quantum mechanical derivation of a manifestly covariant relativistic Boltzmann equation. We will follow Hakim and Mangeney method [3] but using the Landau-Lifshitz equation.

The phase space in Special Relativity must be constructed by following the dynamics of the particles in the system. By looking at the equation of motion, equation (6), It is clear that is composed as:

$$\Gamma = \{x_\mu\}^N \times \{v_\mu\}^N, \quad (9)$$

accompanied by the constraint

$$v_\mu v^\mu = c^2, \quad (10)$$

which can be incorporated directly in the densities by using a delta function. Notice that unlike Hakim and Mangeney [3] the hyper-acceleration is not taken as a part of the phase space since the equation of motion to be considered is the Landau-Lifshitz equation which is a second order differential equation. Using the regular assumptions of the Gibbs ensemble [1], [2], it is possible to define a random one-particle density as

$$R_1(x_\mu, v_\mu; \tau) = \sum_{i=1}^N \delta(x_\mu - x_{\mu i}(\tau)) \delta(v_\mu - v_{\mu i}(\tau)), \quad (11)$$

where τ represents a parameter which will describe the trajectory of each particle by means of $v_{\mu i}(\tau) = dx_{\mu i}(\tau)/d\tau$. Also, following Hakim and Mangeney [3], we define the proper time-dependent density,

$$D_1(x_\mu, v_\mu; \tau) = N \langle R_1(x_\mu, v_\mu; \tau) \rangle, \quad (12)$$

where the brackets $\langle R_1 \rangle$ represents the regular average pondered by a random initials conditions [1], [2]. Also, we will need

$$\begin{aligned} R_2(x_\mu, v_\mu, \tau; x'_\mu, v'_\mu, \tau') = & \sum_{i \neq j} \delta(x_\mu - x_{\mu i}(\tau)) \\ & \times \delta(v_\mu - v_{\mu i}(\tau)) \delta(x'_\mu - x_{\mu i}(\tau')) \delta(v'_\mu - v_{\mu i}(\tau')), \end{aligned} \quad (13)$$

and

$$D_2 = (x_\mu, v_\mu, \tau; x'_\mu, v'_\mu, \tau') = N(N-1) \langle R_2(x_\mu, v_\mu, \tau; x'_\mu, v'_\mu, \tau') \rangle. \quad (14)$$

Finally, the relativistic one-particle density:

$$\mathcal{N}_1(x_\mu, v_\mu) = \int_{-\infty}^{\infty} D_1(x_\mu, v_\mu; \tau) d\tau. \quad (15)$$

Let us follow the Klimontovich method [31]: first, by using the continuity equation, we have

$$\frac{\partial R_1(x_\mu, v_\mu; \tau)}{\partial \tau} + v^\mu \frac{\partial R_1(x_\mu, v_\mu; \tau)}{\partial x^\mu} + a^\mu \frac{\partial R_1(x_\mu, v_\mu; \tau)}{\partial v^\mu} = 0, \quad (16)$$

which can be written as

$$\frac{\partial R_1(x_\mu, v_\mu; \tau)}{\partial \tau} + v^\mu \frac{\partial R_1(x_\mu, v_\mu; \tau)}{\partial x^\mu} + \frac{\partial a^\mu R_1(x_\mu, v_\mu; \tau)}{\partial v^\mu} = 0. \quad (17)$$

Notice that a difference appears with Hakim and Mangeney dependence of R_1 since they obtained $R_1(x_i^\mu, v_i^\mu, a_{\nu i})$ and our result is that $R_1 = R_1(x^\mu, v^\mu)$. This is obviously due to the fact that the equation of motion is taken from the beginning as the equation (6) which does not depend on $a_{\nu i}$ but just in the fields and velocities as we mention above. Therefore, by using equations (6), (7) and (8), we arrive at:

$$\begin{aligned} & \frac{\partial R_1(x_\mu, v_\mu; \tau)}{\partial \tau} + v^\mu \frac{\partial R_1(x_\mu, v_\mu; \tau)}{\partial x^\mu} \\ & + \frac{e}{mc} \frac{\partial}{\partial v^\mu} \left[+\tau_o \Delta^{\mu\nu}(v_\rho) \left[\frac{e}{mc} F_{\nu\alpha} F^{\alpha\beta} v_\beta + v^\rho v^\alpha \frac{\partial F_{\nu\alpha}}{\partial x^\rho} \right] R_1 \right] = 0. \end{aligned} \quad (18)$$

Let us analyze the second line of equation (18). First, the strength tensor $F^{\mu\nu}$ must be decomposed in two terms corresponding to the external force and the internal force, see equations (7) and (8); second, the non radiation terms are represented in the second line of equation (19). The radiation terms are represented in the third line of equation (19).

$$\begin{aligned} & \frac{\partial R_1(x_\mu, v_\mu; \tau)}{\partial \tau} + v^\mu \frac{\partial R_1(x_\mu, v_\mu; \tau)}{\partial x^\mu} \\ & + \frac{e}{mc} \frac{\partial}{\partial v^\mu} [F_{ext}^{\mu\nu} v_\nu R_1] + \frac{e}{mc} \frac{\partial}{\partial v^\mu} [F_{int}^{\mu\nu} v_\nu R_1] \\ & + \frac{e}{mc} \tau_o \frac{\partial}{\partial v^\mu} \left[\Delta^{\mu\nu}(v_\rho) \left[\begin{aligned} & \frac{e}{mc} [F_{\nu\alpha}^{ext} + F_{\nu\alpha}^{int}] \\ & \times [F_{ext}^{\alpha\beta} + F_{int}^{\alpha\beta}] v_\beta \\ & + v^\rho v^\alpha \frac{\partial [F_{\nu\alpha}^{ext} + F_{\nu\alpha}^{int}]}{\partial x^\rho} \end{aligned} \right] R_1 \right] = 0. \end{aligned} \quad (19)$$

By using equation (8), the first two lines of equation (19) are expressed in the first two lines of equation (20). The radiation term must be analyze as follows. Terms depending on bigger order of e^2 will be neglected. In this order of ideas, as we can notice in equation (8), the internal $F_{int}^{\alpha\beta}$ depends on e and consequently the product of the external $F_{ext}^{\alpha\beta}$ with the internal $F_{int}^{\alpha\beta}$ as well as the product of the internal $F_{ext}^{\alpha\beta}$ with the internal $F_{ext}^{\alpha\beta}$ will be neglected since there is a factor $e\tau_o/mc$ in the radiation reaction term. Finally, the conserved terms are described in the third and fourth lines of equation (20).

$$\begin{aligned} & \frac{\partial R_1(x_\mu, v_\mu; \tau)}{\partial \tau} + v^\mu \frac{\partial R_1(x_\mu, v_\mu; \tau)}{\partial x^\mu} + \frac{e}{mc} \frac{\partial}{\partial v^\mu} [F_{ext}^{\mu\nu} v_\nu R_1] \\ & + \frac{4\pi e^2}{mc^2} v_\nu \frac{\partial}{\partial v_\mu} \int d\tau' dx^{4'} dv^{4'} \left[\{v^{\nu'} \partial^\mu - v^{\mu'} \partial^{\nu'}\} D_{ret}(x_\rho - x'_\rho) \right] \\ & \times R_2(x_\mu, v_\mu, \tau; x'_\mu, v'_\mu, \tau') \\ & + \frac{e}{mc} \tau_o \frac{\partial}{\partial v^\mu} \left[\Delta^{\mu\nu}(v_\rho) \left[\frac{e}{mc} [F_{\nu\alpha}^{ext}] [F_{ext}^{\alpha\beta}] v_\beta + v^\rho v^\alpha \frac{\partial F_{\nu\alpha}^{ext}}{\partial x^\rho} \right] R_1 \right] \\ & + \tau_o \frac{\partial}{\partial v^\mu} \left\{ \Delta^{\mu\nu}(v_\rho) \frac{4\pi e^2}{mc^2} v^\rho v^\alpha \int d\tau' dx^{4'} dv^{4'} \right. \\ & \left. \left[\{v'_\alpha \partial_\nu - v'_\nu \partial_\alpha\} \frac{\partial}{\partial x^\rho} D_{ret}(x_\rho - x'_\rho) \right] R_2(x_\mu, v_\mu, \tau; x'_\mu, v'_\mu, \tau') \right\} \\ & = 0((ne^2\tau_o)^2). \end{aligned} \quad (20)$$

In the following, we will consider the case of an external constant electric field in the x -axis. Therefore, due to the antisymmetry of $F_{ext}^{\mu\nu}$ and remembering that the 4-current which produces the external field are outside of the volume, we have

$$\frac{e}{mc} \frac{\partial}{\partial v^\mu} [F_{ext}^{\mu\nu} v_\nu R_1] = \frac{e}{mc} F_{ext}^{\mu\nu} v_\nu \frac{\partial}{\partial v^\mu} R_1. \quad (21)$$

On the other hand, for a constant electric field it is a well known fact that the radiation term vanishes as it has been proved by many authors [23], [24], [25], [26], [27]. Indeed, consider the part of the external radiation reaction term, since the electric field is constant, we have

$$\Delta^{\mu\nu}(v_\rho) \left[\frac{e}{mc} [F_{\nu\alpha}^{ext}] [F_{ext}^{\alpha\beta}] v_\beta + v^\rho v^\alpha \frac{\partial F_{\nu\alpha}^{ext}}{\partial x^\alpha} \right] = \Delta^{\mu\nu}(v_\rho) \left[\frac{e}{mc} [F_{\nu\alpha}^{ext}] [F_{ext}^{\alpha\beta}] v_\beta \right], \quad (22)$$

then,

$$\begin{aligned} \Delta^{\mu\nu}(v_\rho) \left[\frac{e}{mc} [F_{\nu\alpha}^{ext}] [F_{ext}^{\alpha\beta}] v_\beta + v^\rho v^\alpha \frac{\partial F_{\nu\alpha}^{ext}}{\partial x^\alpha} \right] &= (n^{\mu\nu} - \frac{v^\mu v^\nu}{c^2}) \\ &\quad \times \left[\frac{e}{mc} [F_{\nu\alpha}^{ext}] [F_{ext}^{\alpha\beta}] v_\beta \right] \\ &= E^2 v^\mu \left(1 - \frac{c^2}{c^2} \right) = 0, \end{aligned} \quad (23)$$

where E represents the constant electric field. Notice that within Hammond theory [18], the radiation reaction force does not vanish for a constant electric field and this will predict different dispersion relations. We arrive at

$$\begin{aligned} &\frac{\partial R_1(x_\mu, v_\mu; \tau)}{\partial \tau} + v^\mu \frac{\partial R_1(x_\mu, v_\mu; \tau)}{\partial x^\mu} + \frac{e}{mc} F_{ext}^{\mu\nu} v_\nu \frac{\partial}{\partial v^\mu} R_1 \\ &+ \frac{4\pi e^2}{mc^2} v_\nu \frac{\partial}{\partial v_\mu} \int d\tau' dx^{4'} dv^{4'} [\{v^{\nu'} \partial^\mu - v^{\mu'} \partial^{\nu'}\} D_{ret}(x_\rho - x'_\rho)] \\ &\times R_2(x_\mu, v_\mu, \tau; x'_\mu, v'_\mu, \tau') \\ &+ \tau_o \frac{\partial}{\partial v^\mu} \left\{ \times [\{v'_\alpha \partial_\nu - v'_\nu \partial_\alpha\} \frac{\partial}{\partial x^\rho} D_{ret}(x_\rho - x'_\rho)] R_2(x_\mu, v_\mu, \tau; x'_\mu, v'_\mu, \tau') \right\} \\ &= 0((ne^2 \tau_o)^2). \end{aligned} \quad (24)$$

Considering that the correlation function vanishes or is of order $(ne^2 \tau_o)^2$; that is:

$$D_2(x_\mu, v_\mu, \tau; x'_\mu, v'_\mu, \tau') = D_1(x_\mu, v_\mu; \tau) D_1(x'_\mu, v'_\mu; \tau'). \quad (25)$$

By calculating the average of equation (12) and taking $N - 1 \simeq N$, we arrive at

$$\begin{aligned} &\frac{\partial D_1(x_\mu, v_\mu; \tau)}{\partial \tau} + v^\mu \frac{\partial D_1(x_\mu, v_\mu; \tau)}{\partial x^\mu} + \frac{e}{mc} F_{ext}^{\mu\nu} v_\nu \frac{\partial}{\partial v^\mu} D_1 \\ &+ \frac{4\pi e^2}{mc^2} N v_\nu \frac{\partial}{\partial v_\mu} \int d\tau' dx^{4'} dv^{4'} [\{v^{\nu'} \partial^\mu - v^{\mu'} \partial^{\nu'}\} D_{ret}(x_\rho - x'_\rho)] \\ &\times D_1(x_\mu, v_\mu; \tau) D_1(x'_\mu, v'_\mu; \tau') \\ &+ \tau_o \frac{4\pi e^2}{mc^2} N \frac{\partial}{\partial v^\mu} \left\{ \begin{aligned} &\Delta^{\mu\nu}(v_\rho) v^\rho v^\alpha \int d\tau' dx^{4'} dv^{4'} \\ &\times [\{v'_\alpha \partial_\nu - v'_\nu \partial_\alpha\} \frac{\partial}{\partial x^\rho} D_{ret}(x_\rho - x'_\rho)] \\ &\times D_1(x_\mu, v_\mu; \tau) D_1(x'_\mu, v'_\mu; \tau') \end{aligned} \right\} \\ &= 0 \end{aligned} \quad (26)$$

Considering that

$$\lim_{\tau \rightarrow \pm\infty} D_1(x_\mu, v_\mu; \tau) = 0, \quad (27)$$

and integrating over the proper time τ , we obtain

$$\begin{aligned} & v^\mu \frac{\partial \mathcal{N}_1(x_\mu, v_\mu)}{\partial x^\mu} + \frac{e}{mc} F_{ext}^{\mu\nu} v_\nu \frac{\partial}{\partial v^\mu} \mathcal{N}_1(x_\mu, v_\mu) \\ & + \frac{4\pi e^2}{mc^2} N v_\nu \frac{\partial}{\partial v_\mu} \mathcal{N}_1(x_\mu, v_\mu) \\ & \times \int dx^{4'} dv^{4'} [\{v^{\nu'} \partial^\mu - v^{\mu'} \partial^{\nu'}\} D_{ret}(x_\rho - x'_\rho)] \mathcal{N}_1(x'_\mu, v'_\mu) \\ & + \tau_o \frac{4\pi e^2}{mc^2} N \frac{\partial}{\partial v^\mu} \left\{ \begin{array}{l} \Delta^{\mu\nu}(v_\rho) v^\rho v^\alpha \int dx^{4'} dv^{4'} \\ [\{v'_\alpha \partial_\nu - v'_\nu \partial_\alpha\} \frac{\partial}{\partial x^\rho} D_{ret}(x_\rho - x'_\rho)] \\ \times \mathcal{N}_1(x_\mu, v_\mu) \mathcal{N}_1(x'_\mu, v'_\mu) \end{array} \right\} \\ & = 0, \end{aligned} \quad (28)$$

which represents a modified Vlasov equation with an external constant electric field up to the order $ne^2\tau_o$. It has to be highlighted that no collision term is obtained because all the interaction is electromagnetic and there are no collisions or friction forces. Nevertheless, the radiation reaction term can be taken as a collision term because without it we recover the usual Vlasov equation with no collisions.

4. Perturbation

4.1. Distribution function and the shielded potential

We assume now that the plasma is in an equilibrium state described by the Jüttner distribution function [32], [33], [34], [35] without an electric field. However, when a constant electric field is present the distribution function maybe pondered by a an exponential factor which takes into account the potential due to the electric constant field in the x -axis; that is:

$$\phi = \frac{\exp \frac{x}{\lambda} - \exp -\frac{x}{\lambda}}{\exp \frac{R}{\lambda}} \Rightarrow \mathcal{N}_{eq} = \mathcal{N}_J \exp -\frac{\phi}{\kappa T} \simeq \mathcal{N}_J, \quad (29)$$

where λ , R , κ , \mathcal{N}_J and T correspond to the Debye length of the plasma in equilibrium, the length between the plates of the capacitor which generates the electric constant field, the Boltzmann constant, the Jüttner distribution function and the temperature, respectively. Therefore, in an equilibrium plasma the electric field is shielded [36]. Therefore, the Jüttner distribution function maybe considered good enough to describe an equilibrium plasma and the electric field represented by $F_{ext}^{\mu\nu} v_\nu$ is much smaller than it would exist if there were no plasma due to the shielding.

4.2. First order perturbation

Accordingly, a small perturbation can be described

$$\mathcal{N} = \mathcal{N}_{eq} + \mathcal{N}_p \simeq \mathcal{N}_J + \mathcal{N}_p. \quad (30)$$

Therefore, applying this to equation (28), considering that \mathcal{N}_{eq} is solution of the unperturbed case which does not depend on x_μ , that is $\mathcal{N}_{eq} \neq \mathcal{N}_{eq}(x_\mu)$, and taking into account that both the derivatives of the perturbation and the product of perturbation can be neglected, we arrive

at

$$\begin{aligned}
 & v^\mu \frac{\partial \mathcal{N}_p(x_\mu, v_\mu)}{\partial x^\mu} \\
 & + \frac{4\pi e^2}{mc^2} N v_\nu \frac{\partial}{\partial v_\mu} \mathcal{N}_{eq}(v_\mu) \int dx^{4'} dv^{4'} \\
 & \times [\{v^{\nu'} \partial^\mu - v^{\mu'} \partial^\nu\} D_{ret}(x_\rho - x'_\rho)] \mathcal{N}_p(x'_\lambda, v'_\lambda) \\
 & = -\tau_o \frac{4\pi e^2}{mc^2} \frac{\partial}{\partial v^\mu} \left\{ \begin{aligned} & [\Delta^{\mu\nu}(v_\rho) v^\rho v^\alpha \int dx^{4'} dv^{4'} \\ & \times \{v'_\alpha \partial_\nu - v'_\nu \partial_\alpha\} \frac{\partial}{\partial x^\rho} D_{ret}(x_\rho - x'_\rho)] \\ & \times \mathcal{N}_{eq}(v_\lambda) \mathcal{N}_p(x'_\sigma, v'_\sigma) \end{aligned} \right\}. \quad (31)
 \end{aligned}$$

Notice that the term

$$\frac{e}{mc} F_{ext}^{\mu\nu} v_\nu \frac{\partial}{\partial v^\mu} \mathcal{N}_p(x_\mu, v_\mu), \quad (32)$$

has been withdrawn because for high volume as it is the analyzed case the electric field is shielded and it represents the product of two small quantities. This means that the perturbed plasma will not be affected by an external constant electric field. Let us use the Fourier transformation in order to obtain the Fourier 4-current. We have

$$\tilde{\mathcal{N}}_p(k_\mu, v_\mu) = \frac{1}{(2\pi)^2} \int dx^4 \exp[-ik^\mu x_\mu] \mathcal{N}_p(k_\rho, v'_\mu). \quad (33)$$

By using equations (31) and (33), we obtain

$$\begin{aligned}
 & ik^\mu v_\mu \tilde{\mathcal{N}}_p(k_\mu, v_\mu) \\
 & + i \left(\frac{4\pi e^2}{mc^2} \right) v_\nu \frac{\partial}{\partial v_\mu} \mathcal{N}_{eq}(v_\mu) \int dv^{4'} \frac{k^\mu v^{\nu'} - k^\nu v^{\mu'}}{-k^\lambda k_\lambda} \mathcal{N}_p(k_\rho, v'_\rho) \\
 & = \left(\frac{4\pi e^2}{mc^2} \tau_o \right) \frac{\partial}{\partial v^\mu} \{ \Delta^{\mu\nu}(v_\rho) v^\rho v^\alpha \mathcal{N}_{eq}(v_\lambda) \} \\
 & \times \int dv^{4'} \frac{v'_\alpha k_\nu - v'_\nu k_\alpha}{-k^\lambda k_\lambda} k_\rho \mathcal{N}_p(k_\sigma, v'_\sigma). \quad (34)
 \end{aligned}$$

Finally, we arrive at

$$\begin{aligned}
 \tilde{\mathcal{N}}_p(k_\mu, v_\mu) & = \frac{\omega_p^2}{n_o} v_\nu \frac{\partial}{\partial v_\mu} \mathcal{N}_{eq}(v_\mu) \\
 & \times \frac{1}{k^\mu v_\mu} \int dv^{4'} \frac{k^\mu v^{\nu'} - k^\nu v^{\mu'}}{k^\lambda k_\lambda} \mathcal{N}_p(x_\rho, v'_\rho) \\
 & + i \left(\frac{\omega_p^2}{n_o} \tau_o \right) \frac{\partial}{\partial v^\mu} \{ \Delta^{\mu\nu}(v_\rho) v^\rho v^\alpha \mathcal{N}_{eq}(v_\lambda) \} \\
 & \times \frac{1}{k^\mu v_\mu} \int dv^{4'} \frac{v'_\alpha k_\nu - v'_\nu k_\alpha}{k^\lambda k_\lambda} k_\rho \mathcal{N}_p(k_\sigma, v'_\sigma), \quad (35)
 \end{aligned}$$

where ω_p represents the regular plasma frequency. If we want to calculate the Fourier component of the 4-current vector $J^\beta(k_v)$, we have

$$J^\beta(k_v) = \int dv^4 v^\beta \mathcal{N}_p(k_\nu, v_\sigma). \quad (36)$$

Before continuing with the development, it is necessary to notice that some properties of the 4–vector current. Actually, we define the 4–vector point particles current for a system of point particles,

$$J^\alpha(x_\rho) = \int d\tau \left\{ \sum_{i=1}^n e_i \delta^4(x_\rho - x_{i\rho}(\tau)) \frac{dx_i^\alpha(\tau)}{d\tau} \right\}, \quad (37)$$

which represents a well defined 4–vector. It can be decomposed and written as:

$$\begin{aligned} \vec{J}(x, t) &= \sum_{i=1}^n e_i \delta^3(\vec{x} - \vec{x}_i(t)) \frac{d\vec{x}_i(t)}{dt} \\ \text{and} \\ \rho(x, t) &= \sum_{i=1}^n e_i \delta^3(\vec{x} - \vec{x}_i(t)). \end{aligned} \quad (38)$$

It is easy to show that

$$\frac{\partial \rho(x, t)}{\partial t} + \nabla \cdot \vec{J}(x, t) = 0. \quad (39)$$

That is, the continuity equation is satisfied. If we take the Fourier transform of the 4–vector point particles current, the continuity equation must be satisfied. If we want to pass to a continuous system of particles with the same charge e , we put

$$\frac{\partial e\mathcal{N}}{\partial t} + \nabla \cdot \vec{J}(x, t) = 0. \quad (40)$$

where \mathcal{N} represents a continuous particle density as the one introduced in equation (28). Of course, the continuity equation must be accomplished if we use the Maxwell equations. However, when a density of point particles is transformed to a continuous density, some properties can be lost and we must impose the continuity equation. Then, in equation (36) the continuity equation must be considered as a constraint embedded in the theory; that is:

$$k^\mu J_\mu = w J^0(k_v) - \vec{k} \cdot \vec{J}(k_v) = 0. \quad (41)$$

This is equivalent of the procedure used in equations (9) and (10). By using equation (35), we arrive at

$$J^\beta(k_v) = \frac{w_p^2}{k^\alpha k_\alpha} \left\{ \begin{array}{l} -I_\lambda k^\beta - k^\alpha k_\alpha I_\lambda^\beta + k^\alpha I_\alpha g_\lambda^\beta \\ + i\tau_o \left(k^\alpha k_\alpha J_\lambda^\beta + \xi^\alpha k_\alpha \xi^{-1} g_\lambda^\beta - k^\beta \xi_\lambda \xi^{-1} \right) \end{array} \right\} J^\lambda(k_v) \quad (42)$$

where

$$I_\nu(k_\rho, \xi_\rho) = -\frac{1}{n_o} \int dv^4 \frac{v_\nu}{k^\lambda u_\lambda} \mathcal{N}_{eq}(v_\mu), \quad (43)$$

and

$$I_\mu^\nu(k_\rho, \xi_\rho) = -\frac{1}{n_o} \int dv^4 \frac{v_\mu v^\nu}{(k^\lambda u_\lambda)^2} \mathcal{N}_{eq}(v_\rho), \quad (44)$$

and

$$J_\mu^\nu(k_\rho, \xi_\rho) = -\frac{1}{n_o} \int dv^4 \frac{v_\mu v^\nu}{k^\lambda u_\lambda} \mathcal{N}_{eq}(v_\rho), \quad (45)$$

4.3. The Jüttner distribution function

The Jüttner distribution function is expressed by Hakim and Mangeney [3] as

$$N_J = \frac{n_o m \xi}{4\pi K_2(m\xi)} [\exp -m\xi^\mu v_\mu] 2\Theta(v^0) \delta(v^\mu v_\mu - 1), \quad (46)$$

with

$$\xi = \frac{1}{\kappa T_o} \quad \text{and} \quad \xi^\mu = \xi \bar{v}^\mu, \quad (47)$$

where κ is the Boltzmann constant, K_2 represents the Kelvin function and $\gamma(\vec{u}) = (1 - u^2)^{-1/2}$ (Normally, the Hankel function of second order is used in order to describes the Jüttner function [33], [35] but some factors are changed) and Θ represents the Heaviside function. Noticed that T_o is defined in the comoving frame of the space where the particles are moving with a velocity \vec{u} where $\vec{u} = d\vec{r}/dt$ and $\vec{v} = \gamma(\vec{u})\vec{u}$ with $\gamma(\vec{u}) = (1 - u^2)^{-1/2}$; that is: the rest frame K_o where the system is at rest and the temperature is T_o . The existence of such rest frame is assured because in the free particle system there is no interaction and a rest frame can be defined. However, when there is interaction between the particles it is necessary to define a volume in a particular frame and in such a frame the instantaneity is considered [37], [38], [39], [34], [35]. This means that Jüttner is defining the temperature in the rest frame K_o and a volume V_o where $\vec{u} = \vec{0}$. Therefore, Jüttner takes into account the effects of Relativistic Thermodynamics.

5. Dispersion Relations

Once we obtain the expression for the 4–vector current, equation (42), we need just to calculate the determinant and equals it to zero; that is:

$$\det G_\lambda^\beta = \det \left\{ \begin{array}{l} g_\lambda^\beta \left(1 - w_p^2 \frac{k^\alpha I_\alpha + i\tau_o \xi^\alpha k_\alpha \xi^{-1}}{k^\alpha k_\alpha} \right) + w_p^2 \left(I_\lambda^\beta - i\tau_o J_\lambda^\beta \right) \\ + w_p^2 k_\lambda^\beta \frac{I_\lambda + i\tau_o \xi_\lambda \xi^{-1}}{k^\alpha k_\alpha} \end{array} \right\} = 0 \quad (48)$$

In a reference frame (the lab frame) where $\xi^\mu = (\xi, \vec{0})$ and choosing the z –axis as the direction of propagation of the perturbation mode, we have

$$k^\mu = (k^0, 0, 0, k^3) = (w, 0, 0, k). \quad (49)$$

For the longitudinal mode, obtains

$$\begin{aligned} & \left\{ \left[1 - \frac{w_p^2}{k^\alpha k_\alpha} G \right] + w_p^2 \{ I_0^0 - i\tau_o J_0^0 \} + \frac{w_p^2 k^0}{k^\alpha k_\alpha} (I_0 + i\tau_o) \right\} \\ & \times \left\{ \left[1 - \frac{w_p^2}{k^\alpha k_\alpha} G \right] + w_p^2 \{ I_3^3 - i\tau_o J_3^3 \} + \frac{w_p^2 k^3}{k^\alpha k_\alpha} (I_3) \right\} \\ & = w_p^4 \left\{ \frac{k^3}{k^\alpha k_\alpha} (I_0 + i\tau_o) + I_0^3 - i\tau_o J_0^3 \right\} \times \left\{ I_3^0 - i\tau_o J_3^0 + \frac{k^0 I_3}{k^\alpha k_\alpha} \right\}. \end{aligned} \quad (50)$$

where

$$G = k^\nu I_\nu + i\tau_o k^\nu \xi_\nu \xi^{-1}. \quad (51)$$

For the transverse mode, obtains

$$\left\{ 1 - \frac{w_p^2}{k^\alpha k_\alpha} G \right\} + w_p^2 \{ I_1^1 + i\tau_o J_1^1 \} = 0, \quad (52)$$

5.1. Dispersion relations at $T = 0$

5.1.1. *Real part of the dispersion relation for the longitudinal mode at $T = 0$* Let us consider equation (50) without taking into account the imaginary part which are multiplied by τ_o . As a first approximation since τ_o is too small, we will obtain a real dispersion relation; that is:

$$\begin{aligned} & \left\{ \left[1 - \frac{w_p^2}{k^\alpha k_\alpha} G \right] + w_p^2 I_0^0 + \frac{w_p^2 k^0 I_0}{k^\alpha k_\alpha} \right\} \\ & \times \left\{ \left[1 - \frac{w_p^2}{k^\alpha k_\alpha} G \right] + w_p^2 I_3^3 + \frac{w_p^2 k^3 I_3}{k^\alpha k_\alpha} \right\} \\ & = w_p^4 \left\{ \frac{k^3 I_0}{k^\alpha k_\alpha} + I_0^3 \right\} \left\{ I_3^0 + \frac{k^0 I_3}{k^\alpha k_\alpha} \right\}. \end{aligned} \quad (53)$$

For a temperature very close to zero, we can suppose that $N_{eq}(v_\rho)$,

$$N_{eq}(v_\rho) = n_o \delta(v_\rho - \bar{v}_\rho) 2\Theta(\bar{v}^0) \delta(\bar{v}^\rho \bar{v}_\rho - 1). \quad (54)$$

It has to be noted that in our case since the lab frame is at rest with the plasma,

$$\bar{v}^\rho = (\bar{v}^0, \vec{0}) \quad \text{with} \quad \bar{v}^0 > 1 \quad \text{and} \quad \bar{v}^\rho \bar{v}_\rho = 1. \quad (55)$$

Therefore, by using equation (54) in equations (43), (44) and (45), we arrive at

$$I_\nu(k_\rho, \xi_\rho) = n_o^{-1} \int d^4v \frac{v_\nu n_o \delta(\bar{v}_\rho - v_\rho)}{k^\alpha v_\alpha} = \int dv_0 d^3v \frac{v_\nu \delta(\bar{v}_0 - v_0) \delta(\vec{v} - \vec{v}')}{{wv_o - \vec{k} \cdot \vec{v}}}, \quad (56)$$

and

$$\begin{aligned} I_\mu(k_\rho, \xi_\rho) &= -n_o^{-1} \int d^4v \frac{v_\mu v^\nu n_o \delta(\bar{v}_\rho - v_\rho)}{(k^\alpha v_\alpha)^2} \\ &= - \int dv_0 d^3v \frac{v_\mu v^\nu \delta(\bar{v}_0 - v_0) \delta(\vec{v} - \vec{v}')}{{(wv_o - \vec{k} \cdot \vec{v})}^2}. \end{aligned} \quad (57)$$

By substituting equation (55) in equations (56) and (57), we obtain

$$I_0 = \frac{1}{w} \quad \text{and} \quad I_i = 0 \quad \text{for} \quad i = 1, 2, 3, \quad (58)$$

and

$$I_0^0 = -\frac{1}{w^2} \quad \text{and} \quad I_\mu^\nu = 0 \quad \text{otherwise} \quad (59)$$

By using equations (58) and (59) in equation (50), we obtain

$$\left\{ \left[1 - \frac{w_p^2}{k^\alpha k_\alpha} G \right] - \frac{w_p^2}{w^2} + \frac{w_p^2 k^0}{k^\alpha k_\alpha w} \right\} \left\{ \left[1 - \frac{w_p^2}{k^\alpha k_\alpha} G \right] \right\} = 0. \quad (60)$$

Since we are not considering terms proportional to τ_o , from equation (51) G can be considered as

$$G = k^\nu I_\nu = w \frac{1}{w} = 1. \quad (61)$$

Therefore, equation (60) can be expressed as

$$(1 - \frac{w_p^2}{w^2})(1 - \frac{w_p^2}{k^\alpha k_\alpha}) = 0. \quad (62)$$

Hence, we have two solutions, the first one is

$$1 - \frac{w_p^2}{k^\alpha k_\alpha} = 0 \Rightarrow w^2 = w_p^2 + k^2, \quad (63)$$

and the second is

$$(1 - \frac{w_p^2}{w^2}) = 0 \Rightarrow w^2 = w_p^2. \quad (64)$$

We can conclude that for longitudinal mode, the dispersion relations are

$$w_{l1}^2 = w_p^2 + k^2 \quad \text{and} \quad w_{l2}^2 = w_p^2. \quad (65)$$

The second corresponds to the regular non-propagating longitudinal mode which is also found in the non-relativistic case. However, the first dispersion relation corresponds to a mode which does not satisfy the continuity equation described in equation (41).

5.1.2. Real part of the dispersion relation for the transverse mode at $T = 0$ Let us consider equation (52) which corresponds to the transverse modes in general. If we apply the same treatment than above using equations (59), (61) and not considering the imaginary part which multiply the characteristic time τ_o , we arrive at

$$1 - \frac{w_p^2}{k^\alpha k_\alpha} = 0, \quad (66)$$

which corresponds to

$$w_T^2 = w_p^2 + k^2. \quad (67)$$

5.1.3. Eigenvectors and the continuity equation In order to understand the reason why we have to discard one of the longitudinal mode, it is necessary to remember that equation (41) must be included in the deduction of the modes. The eigenvectors of equation (42) are

$$A_\alpha^\beta J^\alpha(k_v) = \lambda J^\beta(k_v) \quad \text{with} \quad \lambda = 1, \quad (68)$$

and for $T = 0$,

$$A_\alpha^\beta = \begin{pmatrix} \frac{w_p^2}{w^2} & 0 & 0 & 0 \\ 0 & \frac{w_p^2}{k^\mu k_\mu} & 0 & 0 \\ 0 & 0 & \frac{w_p^2}{k^\mu k_\mu} & 0 \\ -\frac{w_p^2 k}{w k^\mu k_\mu} & 0 & 0 & \frac{w_p^2}{k^\mu k_\mu} \end{pmatrix}. \quad (69)$$

The eigenvalues and eigenvectors correspond to:

Transverse modes

$$\lambda = \frac{w_p^2}{k^\mu k_\mu} = 1 \Rightarrow \begin{pmatrix} 0 \\ J_1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ J_2 \\ 0 \end{pmatrix} \quad (70)$$

The eigenvectors correspond to the dispersion relations of equation (63).

Longitudinal modes

$$\lambda = \frac{w_p^2}{w^2} = 1 \Rightarrow \begin{pmatrix} J^0 \\ 0 \\ 0 \\ J^3 \end{pmatrix} = \begin{pmatrix} J^0 \\ 0 \\ 0 \\ \frac{w}{k} J^0 \end{pmatrix} \quad \text{and} \quad \lambda = \frac{w_p^2}{k^\mu k_\mu} = 1 \Rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ J^3 \end{pmatrix} \quad (71)$$

The first eigenvalue of equation (71) corresponds to the second dispersion relation of equation (65) and the second eigenvalue corresponds to the first dispersion relation of equation (65). However, in our case with $k^\mu = (w, 0, 0, k)$, the continuity equation turns to be

$$wJ^0 - kJ^3 = 0. \quad (72)$$

It is clear that the two transverse modes and the first transverse mode satisfy such equation. Nevertheless, the second longitudinal mode does not satisfy the continuity equation and must be discarded. This is why this mode does not exist and the reasons claimed by Hakim and Mangeney [3] of arguing that it was due to the choice of the gauge or because it was a solution coming from Special Relativity, are wrong. It is true that few years later, Hakim and Mangeney [4] used the continuity equation to discard this mode. However, the equation was introduced as an extra constraint of the theory and not as embedded in the theory itself as we explain in subsection 5.2. For non vanishing temperatures we will directly incorporate the continuity equation in the deduction of the normal modes.

5.1.4. Damping due to the radiation reaction force By neglecting the radiation term in equation (50), we obtain the dispersion relations from equation (60) at zero temperature ($T = 0$) but if we want to know the damping effect due to the radiation effect it is necessary to consider the term containing $i\tau_o$ in equation (50). By using equations (58) and (59) in subsection 5.2 and some identities of the Appendix of Hakim and Mangeney [3], for the values

$$J_0^0 = 0, J_3^3 = 0, J_3^0 = 0, J_0^3 = 0 \text{ and } J_1^1 = 0. \quad (73)$$

We have

For the transverse mode

$$\{w^2 - k^2 - w_p^2\} + i\tau_o \{w^2 - k^2 - w_p^2\} w_p^2 J_1^1 = 0. \quad (74)$$

Since $J_1^1 = 0$ (see equation (73)), we arrive at

$$\{w^2 - k^2 - w_p^2\} = 0, \quad (75)$$

which corresponds to the normal transverse mode which does not suffer any damping effect due to the radiation reaction force.

For the longitudinal mode

$$1 - \frac{w_p^2}{k^\alpha k_\alpha} (1 + \tau_o w) + w_p^2 \left(-\frac{1}{w^2} - i\frac{\tau_o}{w} \right) + \frac{w_p^2 w}{k^\alpha k_\alpha} \left(\frac{1}{w} + \tau_o \right) = 0. \quad (76)$$

Then,

$$w^2 k^\alpha k_\alpha - w_p^2 k^\alpha k_\alpha (1 + i\tau_o w) = 0. \quad (77)$$

Finally, we arrive at

$$w^2 = w_p^2(1 + i\tau_o w). \quad (78)$$

In order to find the damping term, we need to put

$$w_s = w_o + i\gamma, \quad (79)$$

where w_o represents the dispersion relation at zero order that in this case is $w_o^2 = w_p^2$. Therefore,

$$(w_o + i\gamma)^2 = w_p^2(1 + i\tau_o(w_o + i\gamma)). \quad (80)$$

We obtain

$$w_o^2 + 2iw_o\gamma - \gamma^2 = w_p^2(1 - \tau_o\gamma) + i\tau_o w_p^2 w_o \quad (81)$$

Since $\gamma \ll w_p \simeq w_o$, we can conclude that

$$\gamma = \tau_o \frac{w_p^2}{2} \Rightarrow w_s = w_o + i\tau_o \frac{w_p^2}{2}. \quad (82)$$

We conclude that the damping effect corresponds to a constant damping close to zero temperature. This result will be verified when the damping effect is calculated for small temperatures in subsection 6.2.

5.2. Dispersion relation for the longitudinal mode at small temperature with $T \neq 0$ with damping effect (Landau and radiation reaction dampings)

5.2.1. The continuity equation and the imaginary integral Let us put on a side the transverse mode and put our attention just in the longitudinal modes. We have assumed that I_0 and I_0^0 are real and consequently we obtained real dispersion relations. However, if we look at equations (56) and (57), imaginary parts can be deduced. Indeed, returning to equation (53) and putting

$$I_o = \text{Re}I_0 + i\text{Im}I_0 \quad \text{and} \quad I_0^0 = \text{Re}I_0^0 + i\text{Im}I_0^0 \quad (83)$$

and note that the frequency must have a damping term; that is:

$$w = w' + i\gamma, \quad (84)$$

where

$$\gamma \ll w'. \quad (85)$$

We will now incorporate the continuity equation in the calculation of the dispersion relations. Indeed, if we consider equations (42), (48), (38) and (72), we have a system of equations

$$\left\{ \begin{array}{l} G_\lambda^\beta J_\lambda = 0 \\ \det G_\lambda^\beta = 0 \\ k^\mu J_\mu = w J_0^0 - k J^3 = 0 \end{array} \right\}, \quad (86)$$

for which (see equation (48))

$$G_\lambda^\beta = \begin{pmatrix} G_0^0 & G_3^0 \\ G_0^3 & G_3^3 \end{pmatrix}. \quad (87)$$

We obtain

$$w G_3^0 + k G_0^0 = 0 \quad \text{and} \quad w G_3^0 + k G_0^0 = 0. \quad (88)$$

However, these two equations are equivalent as we can see easily. Therefore, we will just use the first of the two equations. In this case, and after a long way and using the relations obtained in the Appendix by Hakim and Mangeney [4], we arrive at

$$\begin{aligned} \frac{k^2 + w_p^2 k^\alpha I_\alpha}{k^\alpha k_\alpha} &= \frac{2w w_p^2 I_o}{k^\alpha k_\alpha} + w_p^2 \frac{\partial I_o}{\partial w} \\ &+ i\tau_o \frac{w_p^2}{k^\alpha k_\alpha} \left(w + k^\alpha k_\alpha \frac{\partial I_o}{\partial w} - k^\alpha k_\alpha \left(k^\alpha I_\alpha + \frac{3}{\alpha} \right) I_o \right), \end{aligned} \quad (89)$$

where

$$\alpha = m\xi = \frac{m}{\kappa T_o}.$$

This result differs from Hakim and Mangeney [2] by an extra factor w/k in the imaginary part. However, they always calculated the dispersion relations for $w \simeq k$ and the results are equivalent.

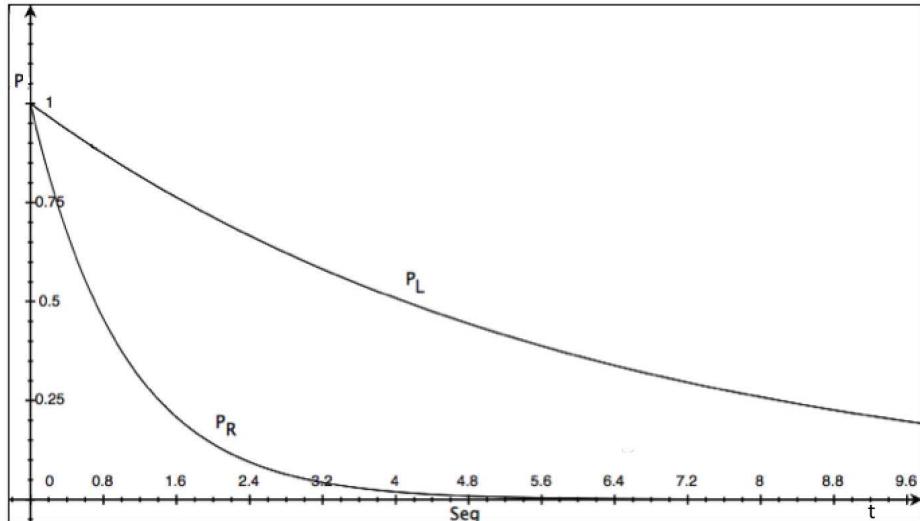


Figure 1. In a typical Tokamak, P_L is greater than P_R which means that the losses due to the radiation reaction damping are bigger than the one due to the Landau damping.

5.2.2. The suprathermic case the suprathermic case normally is described by using the relation

$$v_T < v_\phi < 1, \quad (90)$$

where v_T and v_ϕ represent the thermic velocity $\sqrt{m\xi} = \sqrt{m/\kappa T_o} = \sqrt{\alpha}$ (for non relativistic case) and the phase velocity w/k , respectively. However, to obtain analytical expressions, it is necessary to add other constraints as

$$\begin{aligned} 1/m^2 \xi^2 (1 - v_\phi^2) &\ll 1 \\ (1 - v_\phi^2) &\ll m^2 \xi^2, \end{aligned} \quad (91)$$

which means that

$$w \simeq k \quad \text{and} \quad w \simeq w_p \quad (92)$$

Plasma	Density $N_{0e}(\text{m}^{-3})$	Electron's temperature $T(\text{Kelvin})$	α	$N_{0e}e^2\tau_0$
Solar core	10^{32}	10^7	593.56	$1.6 \cdot 10^{-29}$
Tokamak	10^{20}	10^8	59.35	$1.6 \cdot 10^{-41}$
Glow discharge	10^{16}	10^4	593565.05	$1.6 \cdot 10^{-45}$
Solar wind	10^6	10^5	593565.0	$1.6 \cdot 10^{-55}$
Ionosphere	10^{12}	10^3	5935650.50	$1.6 \cdot 10^{-58}$
Sun's crown	$5.5 \cdot 10^{14}$	10^7	593.56	$8.8 \cdot 10^{-47}$
Chromosphere	10^{15}	10^4	593565.05	$1.6 \cdot 10^{-46}$

Figure 2. Some characteristic plasmas

and accompanied by

$$m\xi > 1. \quad (93)$$

That is: the temperature is not small. Accordingly, we can now develop around a new parameter

$$\eta^2 = \alpha^{-2} (1 - v_\phi^2). \quad (94)$$

Without considering the imaginary quantities (Landau and radiation reaction damping terms are neglected in first instance), equation (84) can be written as

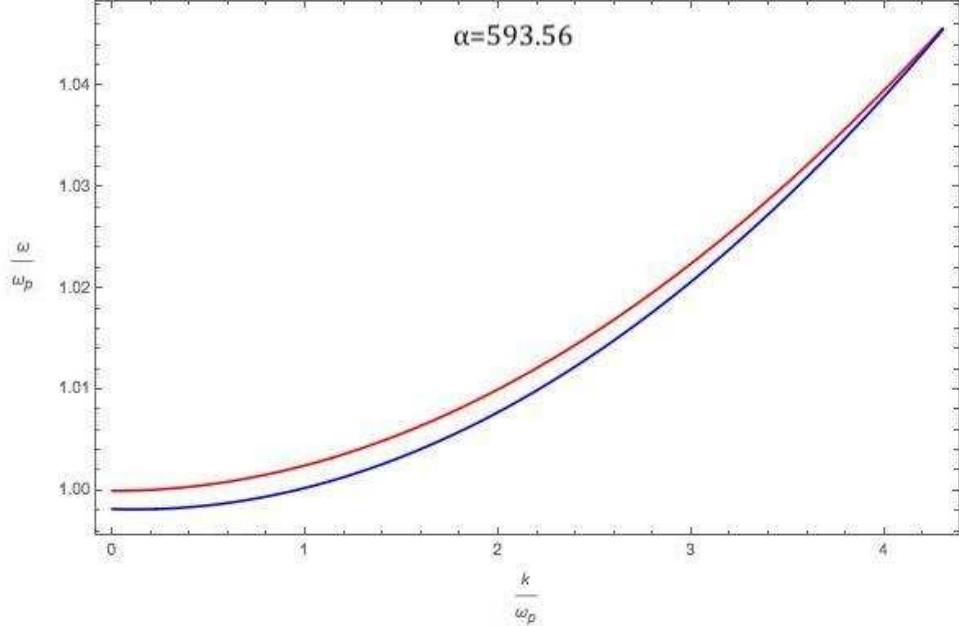


Figure 3. For α bigger than 500, the approximation (red line) coincides with the numerical method (blue line) as it is expected.

$$k^2 + w_p^2 k^\alpha I_\alpha = 2w_p^2 w I_o + w_p^2 (w^2 - k^2) \frac{\partial I_o}{\partial w}. \quad (95)$$

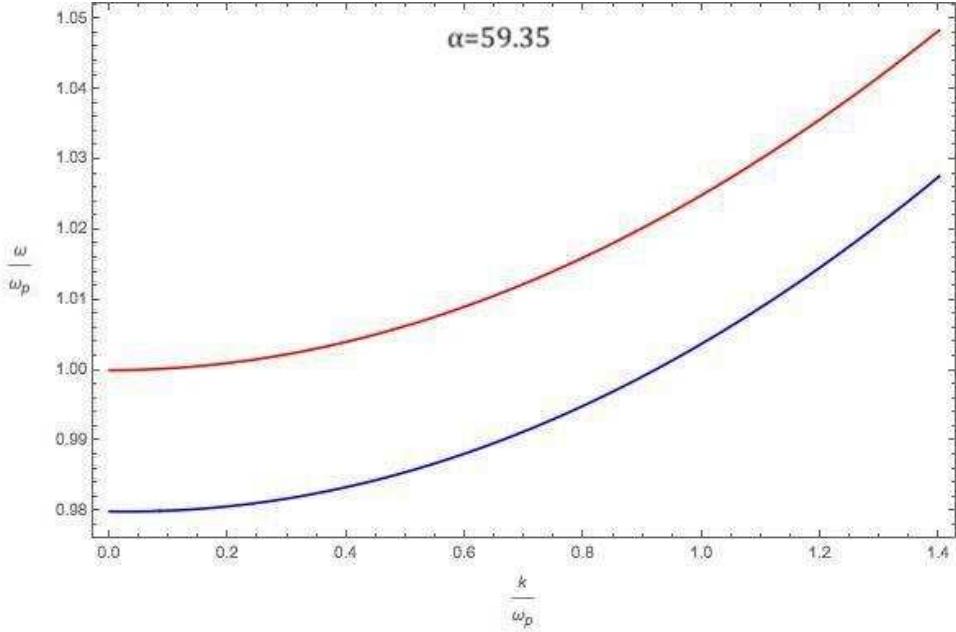


Figure 4. For a typical Tokamak, the approximation is still valid. But it has to be noticed that the numerical method (blue line) predicts that the value when $k \rightarrow 0$, $w = 0.98w_p$

In order to continuous with real part of equation (96), just $\text{Re}I_o$ is considered, we follow the same calculation done by Hakim and Mangeney [2] which make an expansion in η^2 , we arrive at

$$w^2 = w_p^2 + \frac{3w_p^4}{k^2} \alpha^{-1} = w_p^2 + 3(kv_T)^2, \quad (96)$$

which represents the classical result. Now if we consider, the complex terms in equation (97), (remember that the radiation reaction term are not considered) we obtain

$$k^\alpha k_\alpha - w_p^2 + k^\alpha k_\alpha w_p^2 (\text{Re}I_0^0 + i\text{Im}I_0^0) + w_p^2 k^0 (\text{Re}I_0 + i\text{Im}I_0) = 0. \quad (97)$$

Equation (97) can be written as

$$k^\alpha k_\alpha - w_p^2 + k^\alpha k_\alpha w_p^2 \text{Re}I_0^0 + w_p^2 k^0 \text{Re}I_0 + i(k^\alpha k_\alpha w_p^2 \text{Im}I_0^0 + w_p^2 k^0 \text{Im}I_0) = 0. \quad (98)$$

By substituting the values of the real part, see equation (64), we have

$$(w^2 - w_p^2) k^\alpha k_\alpha + i w^2 w_p^2 (k^\alpha k_\alpha \text{Im}I_0^0 + k^0 \text{Im}I_0) = 0. \quad (99)$$

Putting

$$\varepsilon = k^\alpha k_\alpha \text{Im}I_0^0 + k^0 \text{Im}I_0, \quad (100)$$

we obtain

$$(w^2 - w_p^2) k^\alpha k_\alpha + i w^2 w_p^2 \varepsilon = 0. \quad (101)$$

This is equivalent to

$$(w^2 - w_p^2) (w^2 - k^2) + i w^2 w_p^2 \varepsilon = 0. \quad (102)$$

By using equation (84), we arrive at

$$[(w' + i\gamma)^2 - w_p^2] [(w' + i\gamma)^2 - k^2] + i (w' + i\gamma)^2 w_p^2 \varepsilon = 0. \quad (103)$$

Finally,

$$(w'^2 + 2i\gamma w' - \gamma^2 - w_p^2) (w'^2 + 2i\gamma w' - \gamma^2 - k^2) + i(w' + i\gamma)^2 w_p^2 \varepsilon = 0. \quad (104)$$

By neglecting the term γ^2 , $w' = w_p$ and for the damping part, we obtain

$$\begin{aligned} 2(\gamma w_p)(w_p^2 - k^2) + w_p^4 \varepsilon &= 0 \\ \gamma &= -\frac{w_p^3 \varepsilon}{2(w_p^2 - k^2)}. \end{aligned} \quad (105)$$

Obtaining the value of ε in this approximation, we arrive to

$$\gamma = \frac{1}{2} \sqrt{\frac{\pi}{2}} \frac{w_p^4}{k^3} (m\xi)^{3/2} \exp{-(m\xi v_\phi^2)/2}, \quad (106)$$

which represents the Landau damping term. It has to be noticed that within this approximation, the radiation reaction damping is equal to the one calculated for zero temperature. We can affirm that the intensity of the perturbation P is damped as

$$P = \exp{-\gamma t} \exp{-\tau_o \frac{w_p^2}{2} t}. \quad (107)$$

Therefore, If we consider a typical Tokamak case with

$$w_p = 5.6 \times 10^{11} \text{ Hz} \quad \text{and} \quad \beta = \frac{v_T}{c} = \alpha^{-1/2} = 0.12, \quad (108)$$

we can compare graphically both dampings (see figure 1)

$$P_L = \exp{-\gamma t} = \exp{-0.16t} \quad \text{and} \quad P_{\tau_o} = \exp{-\tau_o \frac{w_p^2}{2} t} = \exp{-0.98t}, \quad (109)$$

which shows that in this case the Bremsstrahlung generates a more important damping.

5.3. The superluminic and relativistic cases (longitudinal modes)

Let us now analyze the results which can be obtained when

$$v_\phi \gg 1 \quad (110)$$

In this case, let us rewrite equation (89) in another form without any approximation

$$\begin{aligned} \Omega_p^2 + k^2 &= 2w_p^2 w I_o + w_p^2 (w^2 - k^2) \frac{\partial I_o}{\partial w} \\ &\quad + i\tau_o w_p^2 \left[w + (w^2 - k^2) \frac{\partial I_o}{\partial \alpha} + (w^2 - k^2) \left(G + \frac{3}{\alpha} \right) I_o \right], \end{aligned} \quad (111)$$

where

$$\Omega_p^2 = w_p^2 [K_1(\alpha)/K_2(\alpha)] = w_p^2 k^\nu I_\nu = w_p^2 \text{Re}G = w_p^2 \Im. \quad (112)$$

We can make an expansion in $(k/w)^2 = v_\phi^{-2} \ll 1$; we arrive to

$$\begin{aligned} \Omega_p^2 + k^2 &= 2w_p^2 \left[\sum_{l=0}^{\infty} \left(\frac{k}{w} \right)^{2l} \phi_l(\alpha) \right] \\ &\quad - w_p^2 \left(1 - \frac{k^2}{w^2} \right) \left[\sum_{l=0}^{\infty} \left(\frac{k}{w} \right)^{2l} (2l+1) \phi_l(\alpha) \right] \\ &\quad + i\tau_o w_p^2 \frac{w^2}{k} \left\{ \begin{array}{l} 1 + \left(1 - \frac{k^2}{w^2} \right) \left[\sum_{l=0}^{\infty} \left(\frac{k}{w} \right)^{2l} \frac{d\phi_l(\alpha)}{d\alpha} \right] \\ + \left(1 - \frac{k^2}{w^2} \right) (\Im + \frac{3}{\alpha}) \left[\sum_{l=0}^{\infty} \left(\frac{k}{w} \right)^{2l} \phi_l(\alpha) \right] \end{array} \right\}, \end{aligned} \quad (113)$$

where $\phi_l(\alpha)$ are defined in the Appendix by Hakim and Mangeney [3]. It has to be remembered that

$$\tau_o \frac{w^2}{k} \ll 1 \quad \text{and} \quad k \ll w. \quad (114)$$

If we do not consider the imaginary part, we arrive at

$$w_o^2 = (\Im - \phi_1) w_p^2 + 3k^2 [(\phi_1 - \phi_2) / (\Im - \phi_1)]. \quad (115)$$

5.3.1. Low temperature For low temperatures, we obtain

$$w_o^2 = w_p^2 + 3(\kappa T/m) k^2. \quad (116)$$

which coincides with the suprathermic case and suspects that this result is valid for any value of k .

5.3.2. High temperature For high temperatures, and just for the first order relativistic expansion, we need to continuous the expansion in the ϕ_l 's and it can be deduced that (we recover the value of the speed of light c)

$$w_o^2 = w_p^2 + 3k^2 (\kappa T/m) - \frac{5}{2} w_p^2 (\kappa T/m c^2) - \frac{33}{2} \left(\frac{k^2}{c^2} \right) (\kappa T/m)^2, \quad (117)$$

which means that when $k \rightarrow 0$, the w_o does not go to w_p but to

$$w_o^2 = w_p^2 \left(1 - \frac{5}{2} (\kappa T/m c^2) \right). \quad (118)$$

This relation shows that there exists a shift of the plasma frequency by a factor $(1 - (5/2)(\kappa T/m c^2))$. This approximation is valid for not such high temperature since when $T \rightarrow \infty$, $w_o^2 \rightarrow -\frac{5}{2}(\kappa T/m c^2)$ and w_o will be imaginary which corresponds to a damping. Nevertheless, high expansion in the ϕ_l 's will turn on a similar effect as we will see in the ultra-relativistic case. Considering the imaginary part (the radiation reaction term), we have

$$w = w_o + i\gamma. \quad (119)$$

We arrive at

$$\gamma = \tau_o w_p^2 (w_o^3/k^3) [\Im (\Im + 3/\alpha)]. \quad (120)$$

which represents a tremendous damping effect. The Landau damping vanishes for high temperature because the imaginary part of the integrals can be neglected.

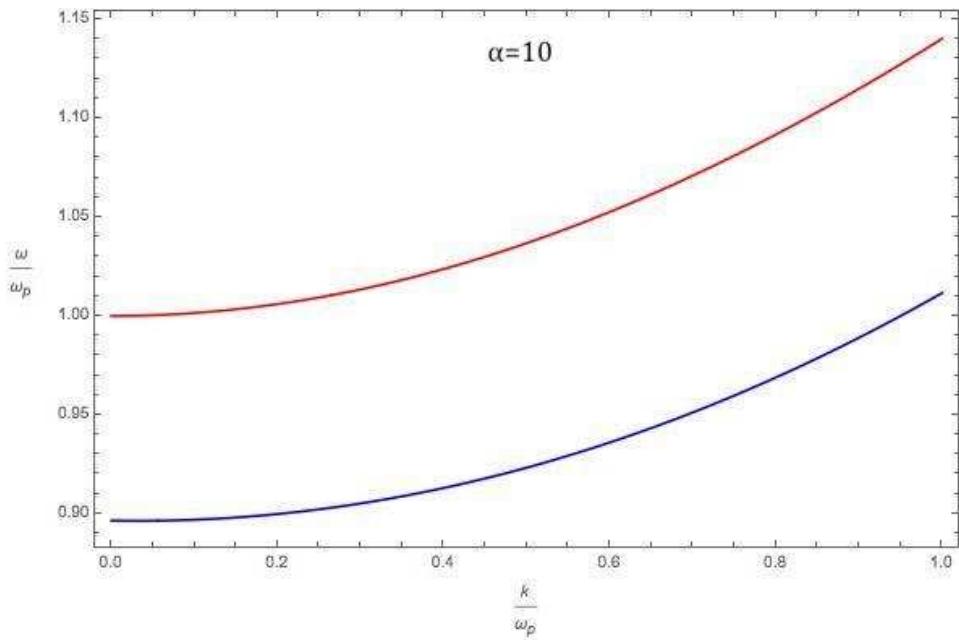


Figure 5. The approximation is good but the numerical method (blue line) shows that as α decays the approximation is less accurate (red line).

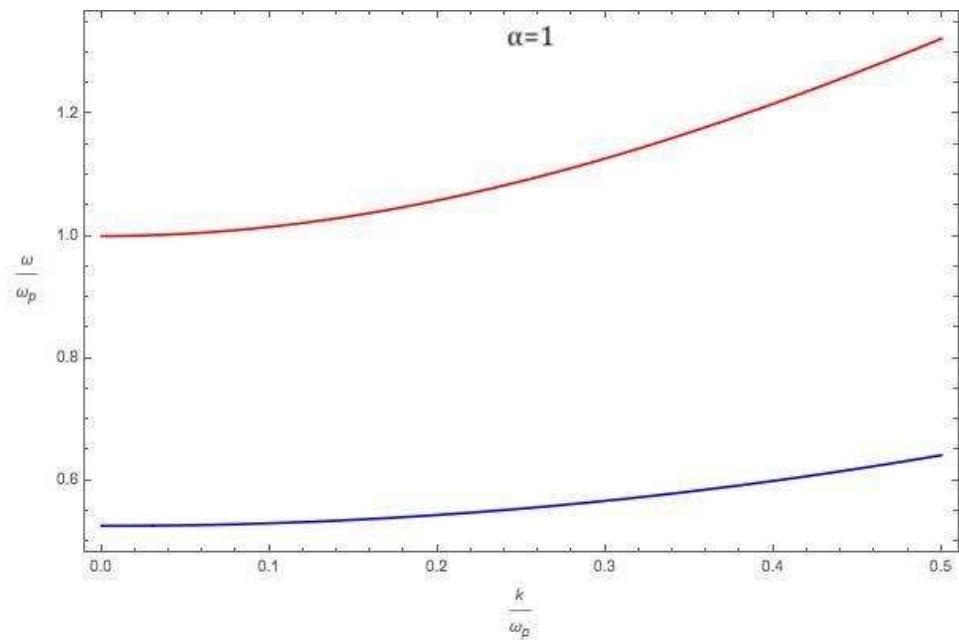


Figure 6. For $\alpha \simeq 1$, equation (116) is not anymore valid (red line) and it has to be substituted by an equation close to equation (117) or by the numerical solution (blue line)

5.3.3. The ultra relativistic case

The relation between α and v_T We have to notice that when the temperature is ultra-relativistic, the normal relation between the $\alpha = m\xi = \frac{m}{\kappa T_o}$ and v_T must be changed. For non relativistic temperature, the relation is (the 3 degree of liberty are taken into account)

$$v_T^2 = 3/\alpha \quad \text{with} \quad \alpha^{-1} = \frac{\kappa T_o}{m}, \quad (121)$$

with the particularity that

$$\mathfrak{S} = 1 - \frac{3}{2\alpha} + 0(\alpha^{-2}) \quad \text{for} \quad \alpha \gg 1. \quad (122)$$

However, when the temperature is ultra-relativistic, it is clear that $v_T = 1$ and α must constraint to a minimum value. Nevertheless, the thermal velocity has to be calculated in different form. Indeed, following the Special Relativity, the average energy,

$$\langle E \rangle = \frac{m}{\sqrt{1 - v_T^2}}, \quad (123)$$

represents an appropriate relativistic definition of the thermal velocity. On the other hand, Synge [40] has proved that

$$\langle E \rangle = m(\mathfrak{S} + 3/\alpha). \quad (124)$$

Noticing that

$$\mathfrak{S} = \frac{1}{2}\alpha + 0(\alpha^2) \quad \text{for} \quad \alpha \ll 1 \quad (125)$$

which gives

$$v_T^2 \simeq 1 - \frac{1}{9}\alpha^2. \quad (126)$$

Therefore, $\alpha \rightarrow 0$, $v_T^2 \rightarrow 1$ as it was expected.

Dispersion relation for the ultra-relativistic case Now, we can continuous analyzing the ultra-relativistic case. Taking the real part of equation (89), we arrive to ($\alpha \rightarrow 0$)

$$k^2 + w_p^2 k^\alpha I_\alpha = 2w w_p^2 \left[\frac{\alpha}{4k} \log \left(\frac{w - k}{w + k} \right) \right] + w_p^2 k^\alpha k_\alpha \frac{\alpha}{2k^\alpha k_\alpha}. \quad (127)$$

In this case,

$$k^\alpha I_\alpha \simeq \frac{\alpha}{2}. \quad (128)$$

We obtain

$$k^2 = w w_p^2 \left[\frac{\alpha}{2k} \ln \left(\frac{w - k}{w + k} \right) \right]. \quad (129)$$

In the superlumic case, $k \rightarrow 0$, we have

$$k^2 = -w w_p^2 \alpha \left[\frac{\ln(w + k) - \ln(w - k)}{2k} \right], \quad (130)$$

that is:

$$k^2 = -w w_p^2 \alpha \frac{d \ln w}{dw} = -w w_p^2 \frac{1}{w} = -w_p^2. \quad (131)$$

which means that for $k \rightarrow 0$, there is no perturbation as we will show in section 6 (Numerical Results of the Dispersion Relations).

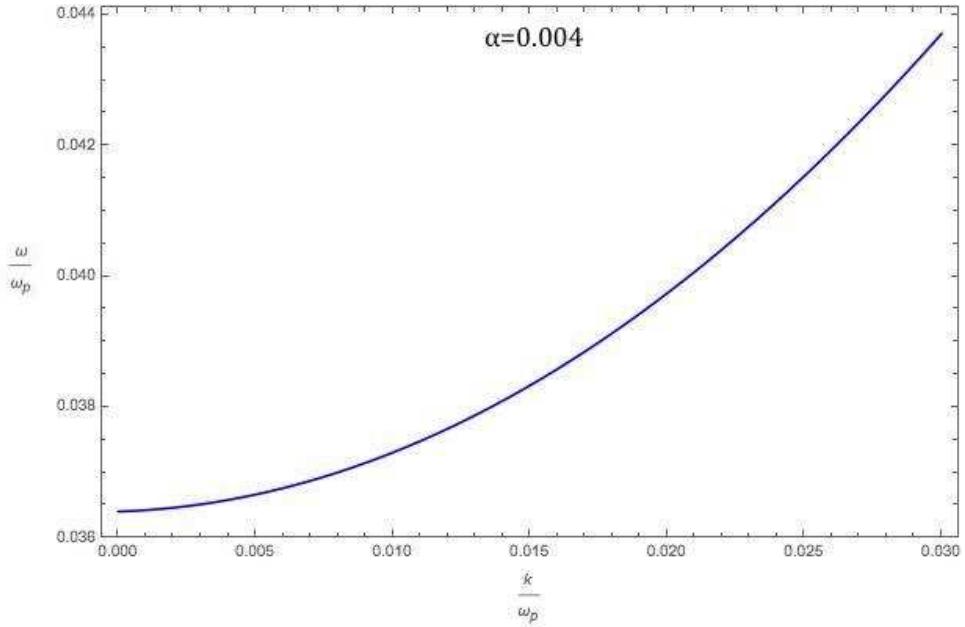


Figure 7. For very low α , there is not exist a good approximation and just the numerical method is valid. It has to be highlighted that when $k \rightarrow 0$, $w = 0.37w_p$

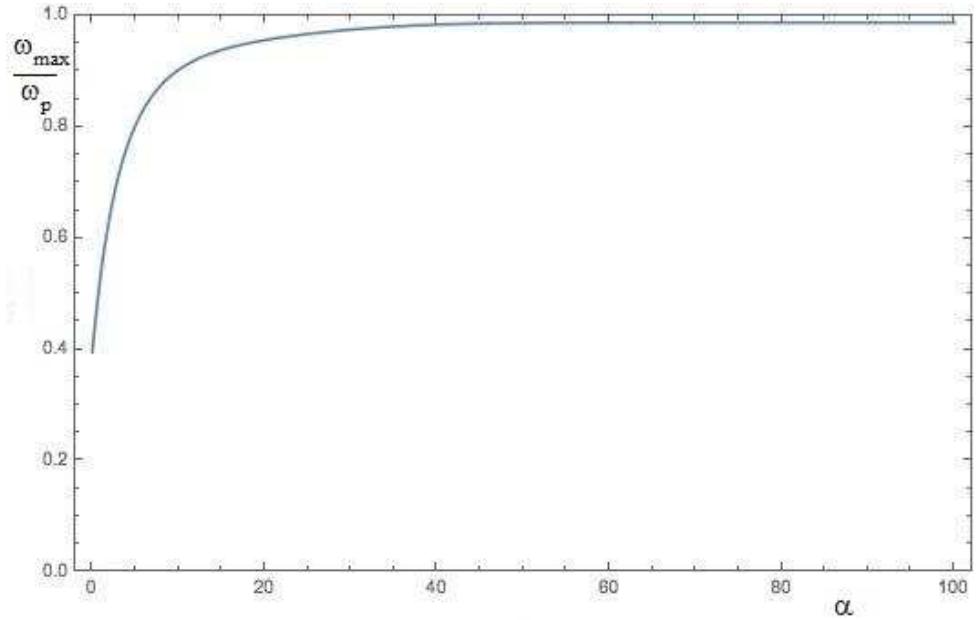


Figure 8. The value of w when $k \rightarrow 0$ as a function of α . It has to be highlighted that the value of $w \rightarrow 3$ as $\alpha \rightarrow 0$.

6. Numerical Results of the Dispersion Relations

Until now we have given dispersion relations in many ranges of validity but no for any α neither k in general. Due to difficulty to solve in general equation (89), it is necessary to use numerical methods to describe the dispersion relations. Let us present some important results. In figure 2,

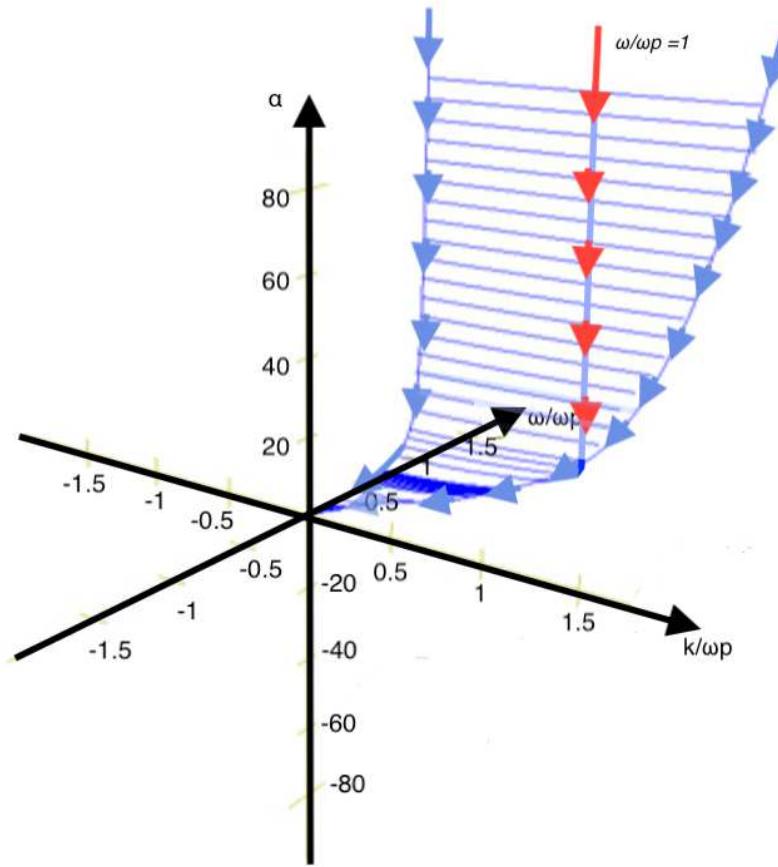


Figure 9. Dispersion relations surface in the 3-dimension space k, w, α . The red arrow line which crosses the surface, represent $(w/w_p) = 1$

some characteristic plasmas are described showing that all of them satisfy the required condition $ne^2\tau_o \ll 1$.

Some dispersion relations for different α 's are shown from figure 3 to figure 7. A comparison is made between the dispersion relations obtained by making some approximations, equation (116) and the numerical method for different values of α . Notice that the relation described in equation (118) is much accurate for decreasing α . The red lines in the figures just corresponds to equation (116) which does not describe the shift in the plasma frequency.

In figure (8), it is numerically showed that there exists a limit for the value of w when $k \rightarrow 0$ for ultra-relativistic temperatures. However, the most interesting fact consists on noticing that when α is decreasing the dispersion relation has a cut-off for the wave number k (see figure 9); that is: for each α there is a maximum wave number k_α such that above it the wave no longer exists. Moreover, below a certain value of α , it can be said that there is not anymore a dispersion relation but unique values of w and k ; that is

$$\alpha < \alpha_p \rightarrow w = w(\alpha) \quad \text{and} \quad k = k(\alpha). \quad (132)$$

7. Conclusion

We follow Hakim [3] in the deduction of the density of particles just by considering the positions and the velocities of the particles as independent variables without taking into account the hyper-acceleration as an independent variable. This is due to the fact that Landau-Lifshitz equation of motion for charged particles is used to describe the motion of the charges. The final result coincides with Hakim generalization of the Vlasov equation in Special Relativity [3]. Moreover, a more general equation is deduced by including a constant external electric field. The wave dispersion relations are obtained and the effect of the coupling between the constant external electric field and the interaction of the particle does not play an important role since the dispersion relations are not affected.

We recover well-known dispersion relations for different range of values and approximations comparing their with numerical solutions. The original results are threefold:

A- There exist a cut-off wave number k_α for each temperature for which above this wave number there is no wave.

B- For temperatures above $T_{\alpha=1}$ (for $\alpha < 1$), the value of w at $k \rightarrow 0$ decreases below $0.5w_p$

C- For temperatures above T_p ($\alpha < \alpha_p$), the dispersion relation does not exist and we have a unique pair, $w = w(\alpha)$ and $k = k(\alpha)$.

It remains to do a similar study using the Hammond model. Indeed, although Hammond model does not give an expression for the radiation reaction force, we will make an attempt in order to deduce an effective Hammond radiation reaction force for the case of an external constant electric field in a plasma. The radiation reaction force must not vanish as it happens for the other equations and consequently different dispersion relations will be obtained to test the viability of Hammond model. It has to be noted that Hammond proposed an experiment to check the difference between the different equations by measuring the gain of kinetic energy under a high laser intensities ($10^{22} W cm^{-2}$) [19], [22]. We think that dispersion relations are easier to be measured.

Acknowledgement : This work was partially supported by COFAA, EDI SIP - IPN and SNI CONACYT.

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