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Asymptotic Dynamics of Two-Dimensional Dilaton Gravity

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Asymptotic Dynamics of Two-Dimensional Dilaton Gravity

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Kurzfassung

Mehr als einhundert Jahre sind seit der Entstehung von Quantenmechanik und allgemeiner Relativitätstheorie vergangen und noch immer ist keine endgültige Theorie der *Quantengravitation* in Aussicht, welche die beiden grundlegenden Pfeiler der modernen Physik miteinander in Einklang bringen könnte. In den letzten Jahren hat sich jedoch das *holographische Prinzip* als wichtiges Werkzeug für die Suche nach dieser Theorie herausgestellt. Das holographische Prinzip besagt, dass jede Quantentheorie der Gravitation in D Raumzeitdimensionen äquivalent zu einer Quantentheorie ohne Gravitation in $(D-1)$ Raumzeitdimensionen ist. Damit ist es möglich, Eigenschaften der Quantengravitation mithilfe einer äquivalenten, im Allgemeinen besser verstandenen Theorie zu erforschen. Obwohl es noch nicht geklärt ist, ob das holographische Prinzip tatsächlich eine fundamentale Eigenschaft der Natur darstellt, gibt es einige theoretische Modelle, in denen es realisiert ist. Zu diesen zählt die *AdS/CFT Korrespondenz*, welche besagt, dass Quantengravitation in einer anti-de Sitter (AdS) Raumzeit äquivalent zu einer konformen Quantenfeldtheorie (CFT) am Rand der Raumzeit ist.

Diese Dissertation untersucht die AdS/CFT Korrespondenz für eine große Klasse zweidimensionaler Gravitationstheorien, genannt *Dilatongravitation*, für welche die Korrespondenz noch unzureichend verstanden ist. Dies rührt von den ungewöhnlichen Eigenschaften her, welche eine passende Randtheorie aufweisen müsste; als möglicher Kandidat wurde erst vor kurzem das *Sachdev–Ye–Kitaev* (SYK) Modell vorgeschlagen. Ein besseres Verständnis der Korrespondenz in zwei Dimensionen ist entscheidend für die Untersuchung der Mikrozustände vierdimensionaler, extremaler schwarzer Löcher im Zuge der *near horizon holography*.

Nach einer Einführung in das Thema zweidimensionaler Gravitationstheorien und deren Formulierung als nichtlineare Eichtheorien in der Form von *Poisson Sigma Modellen* wird in dieser Arbeit gezeigt, dass der *konstante Dilatonsektor* jeder Quantendilatongravitationstheorie notwendigerweise trivial ist. Daraus folgt, dass eine holographische Korrespondenz im Sinne der AdS/CFT nur im *linearen Dilatonsektor* möglich ist. Als Beispiel hierfür wird das *Jackiw–Teitelboim Modell* untersucht. Es wird gezeigt, dass die asymptotische Dynamik dieses Modells durch die *Schwarzsche Wirkung* bestimmt wird, die ein Kennzeichen des SYK Modells ist. Schließlich werden *verallgemeinerte Schwarzsche Wirkungen* konstruiert, welche die asymptotische Dynamik verallgemeinerter Jackiw–Teitelboim Modelle in der Anwesenheit weiterer Felder wie *Yang–Mills* oder *höherer Spins* beschreiben.

Abstract

Almost one hundred years after the formulation of quantum theory and general relativity, a definitive framework for the unification of these theories is still not at hand. One of the most fruitful approaches to the problem of quantum gravity that has emerged in the past years is the *holographic principle.* This principle conjectures that a quantum theory of gravity in D spacetime dimensions is equivalent to a theory without gravity in $(D-1)$ spacetime dimensions. This opens up the possibility of understanding properties of quantum gravity without detailed knowledge of the underlying fundamental theory. The *AdS/CFT correspondence*, that relates quantum gravity in anti-de Sitter (AdS) space to a conformal quantum field theory (CFT) on the boundary of AdS space, currently provides the best developed example of the holographic principle.

This thesis studies the AdS/CFT correspondence for a class of two-dimensional theories of gravity called *two-dimensional dilaton gravity.* The motivation for this is two-fold: First, the AdS/CFT correspondence in two dimensions appears to be more subtle than its higher-dimensional relatives. This is partly due to the elusive nature of its one-dimensional boundary theory that appears to be related to the *Sachdev–Ye–Kitaev (SYK) model*, as was proposed only recently. The second motivation for studying the AdS/CFT correspondence in two dimensions derives from its close relation to *near horizon holography* of four-dimensional *extremal black holes* that could provide a step towards a deeper understanding of their *black hole microstates.*

Following a thorough introduction to two-dimensional dilaton gravity and its reformulation as a non-linear gauge theory in the form of a *Poisson sigma model*, it is shown that quantum dilaton gravity is trivial in its *constant dilaton sector.* A non-trivial holographic correspondence is therefore only possible in the *linear dilaton sector.* As a particular example the *Jackiw–Teitelboim model* is studied in close detail. It is shown that its asymptotic dynamics are governed by the *Schwarzian action* that is a hallmark of the SYK model, thus providing evidence for the above mentioned correspondence. Finally, certain generalizations of the Jackiw–Teitelboim model are studied, by coupling it to further fields such as *Yang–Mills* or *higher spin fields.* For these models, boundary actions governing the asymptotic dynamics are derived in the form of *generalized Schwarzian actions.*

Acknowledgments

> Yes, to dance beneath the
> diamond sky
> with one hand waving free...
>
> "Mr. Tambourine Man",
> Bob Dylan

This thesis would not have been possible without a number of people to whom I am deeply indebted.

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Contents

Notes to the Reader

This thesis is based on the following original research articles.

[1] H. A. González, D. Grumiller, and J. Salzer, "Towards a bulk description of higher spin SYK," *JHEP* **(to appear)** (2018) arXiv:1802.01562 [hep-th].

[2] D. Grumiller, R. McNees, J. Salzer, C. Valcárcel, and D. Vassilevich, "Menagerie of AdS_2 boundary conditions," *JHEP* **10** (2017) 203, arXiv:1708.08471 [hep-th].

[3] D. Grumiller, J. Salzer, and D. Vassilevich, "Aspects of AdS_2 holography with non-constant dilaton," in *International Workshop on Strong Field Problems in Quantum Theory Tomsk, Russia, June 6-11, 2016*. 2016. arXiv:1607.06974 [hep-th].

[4] D. Grumiller, J. Salzer, and D. Vassilevich, "AdS_2 holography is (non-)trivial for (non-)constant dilaton," *JHEP* **12** (2015) 015, arXiv:1509.08486 [hep-th].

In addition to the above I also contributed to the following publications during my studies.

[5] S. Prohazka, J. Salzer, and F. Schöller, "Linking Past and Future Null Infinity in Three Dimensions," *Phys. Rev.* **D95** no. 8, (2017) 086011, arXiv:1701.06573 [hep-th].

[6] A. Bagchi, D. Grumiller, J. Salzer, S. Sarkar, and F. Schöller, "Flat space cosmologies in two dimensions - Phase transitions and asymptotic mass-domination," *Phys.Rev.* **D90** no. 8, (2014) 084041, arXiv:1408.5337 [hep-th].

[7] D. Grumiller, R. McNees, and J. Salzer, "Cosmological constant as confining U(1) charge in two-dimensional dilaton gravity," *Phys.Rev.* **D90** no. 4, (2014) 044032, arXiv:1406.7007 [hep-th].

[8] D. Grumiller, R. McNees, and J. Salzer, "Black holes and thermodynamics - The first half century," *Fundam. Theor. Phys.* **178** (2015) 27–70, arXiv:1402.5127 [gr-qc].

Although the articles [5–8] are also concerned with aspects of lower-dimensional gravity the choice of taking [1–4] as basis of this thesis was guided by the aim of having a comprehensive story emerging naturally. Part II of this thesis largely reproduces the content of these papers, partly verbatim, but their structure was edited in order to suit the above mentioned aim. Furthermore, I have attempted to highlight connections to the introductory material in part I or additional insights that were not included in the original publications.

I have tried to harmonize the different conventions used in the articles with each other, and I apologize for instances that have slipped my attention.

Chapter One

Introduction

> Como todos los hombres de la Biblioteca, he viajado en mi juventud; he peregrinado en busca de un libro, acaso del catálogo de catálogos… [1]
>
> "La Biblioteca de Babel"
> Jorge Luis Borges

The laws of physics do not pertain to the world as we observe it; it is a place by far too messy to be understood in all its details and interconnections. For instance, it is evident to every child rolling marbles on the floor, as it was evident to Aristotle, that a moving body returns to a state of rest, in complete contrast to the laws of Newton. Yet physics has surpassed in precision any other method employed by man to gather knowledge about their world. The insight that allowed physics to become so successful was to turn from the perceived phenomena of the world to the underlying regularities.[2] The way physics achieves this is by abstraction, separation in *dynamical system* and *background* that can influence the system but is not influenced itself, and *symmetries*.[3]

Symmetries and background structure (i). From the time of Newton until the beginning of the twentieth century it was assumed that space and time are a universal background structure for every physical system. It was furthermore observed, already by Galilei, that every physical system is invariant under certain *symmetry transformations*: for instance, the system behaves in the same way when moved to a different place or studied at a different time. These symmetry transformations can be identified with the *symmetry group* of the underlying background structure of space and time that is taken to be a geometric object. A

[1] Like all men of the Library, I have traveled in my youth; I have wandered in search of a book, perhaps the catalogue of catalogues …

[2] In his Nobel prize lecture [9], Wigner called this "specification of the explainable [...] the greatest discovery of physics so far", that seems to have occurred sometime between Kepler and Newton.

[3] "It is only slightly overstating the case to say that physics is the study of symmetry." Philip W. Anderson [10]

physical theory that describes a dynamical system placed on this background structure must be compatible with these underlying symmetries.

Our conceptions of space and time have changed throughout history, and so have the associated symmetries that we deem to be fundamental; starting from Aristotle's geocentric model over Newton's absolute space and time whose symmetries are called the *Galilei group*, under which Newtonian physics is invariant, to the unification of space and time in a four-dimensional entity, *Minkowski space*, that is invariant under the *Poincaré group*.

Most triumphs of twentieth century physics came from the offspring of the marriage of Poincaré invariance to quantum mechanics, in particular quantum field theory and the resulting standard model of particle physics (that again is almost completely described by the symmetries of its background structure, i.e., a fiber bundle over Minkowski space).

Yet in spite of these successes of quantum mechanics and Poincaré invariance just mentioned, we know that this is not the full story. It was once again Einstein who overthrew the underlying concept of space and time. In his theory of general relativity spacetime ceases to be a mere background but becomes a dynamical system itself that interacts with every other system in the universe. This interaction is gravity. When studying gravity the distinction between background and dynamical system is thus more subtle, as everything couples to gravity.

Almost immediately after the publication of his general theory of relativity Einstein pointed out that it would require modifications due to quantum effects [11], and ever since physicists have been trying to find this theory of *quantum gravity*. The reason for the difficulty of this problem partly stems from the lack of a fixed background that is needed in most approaches to quantum mechanics.

Symmetries and background structures (ii). Before continuing let us phrase the discussion of the previous paragraphs about the relation between symmetries and background structure, in a somewhat more mathematical language. Consider an arbitrary Lagrangian n-form $\mathbf{L}(\phi, \phi_0)$ on a spacetime manifold M that depends on a number of dynamical fields ϕ and further non-dynamical background structure ϕ_0. This background structure might come in the form of a fixed spacetime metric, as is the case for non-gravitational theories.

Almost every physical theory can be written in a coordinate-independent, i.e., geometrical way. More precisely, given a diffeomorphism $f : M \to M$ the Lagrangian transforms as an n-form under the diffeomorphism if one transforms both dynamical and non-dynamical fields

$$f^*\mathbf{L}(\phi, \phi_0) = \mathbf{L}(f^*\phi, f^*\phi_0) \tag{1.1}$$

up to a possible boundary term. This is nothing but the statement that one can choose arbitrary coordinates on M to describe a physical theory; it might only be that ϕ_0 suggests a particular set of natural coordinates. Of course, we have learned nothing new in this process.

More interesting is the case when one transforms the dynamical fields only but leaves the background structure unchanged. A diffeomorphism s is called a *symmetry* if

$$s^*\mathbf{L}(\phi, \phi_0) = \mathbf{L}(s^*\phi, \phi_0)\,. \tag{1.2}$$

If the Lagrangian $\mathbf{L}$ is diffeomorphism covariant in the sense of (1.1) for arbitrary f, choosing $f = s^{-1}$ and acting on (1.2) one finds

$$\mathbf{L}(\phi, \phi_0) = \mathbf{L}(\phi, (s^{-1})^*\phi_0)\,, \tag{1.3}$$

again up to a possible boundary term. Thus, the spacetime symmetries of the dynamical fields ϕ are necessarily given by the diffeomorphisms that leave the background structure invariant. For instance, in the case of field theory in Minkowski spacetime one finds that the theory is invariant under the Poincaré group as the transformations that leave the Minkowski metric, i.e., the background structure, invariant.

But what are the interesting symmetries in the case of gravity when the metric becomes a dynamical variable? It turns out that by restricting one's attention to spacetimes of a certain asymptotic form, the *boundary conditions* themselves can be regarded as providing some sort of background structure. The symmetries preserving this asymptotic form are called the *asymptotic symmetries* of the gravitational theory. In this way one finds the *Bondi–Metzner–Sachs group (BMS)* [12] as asymptotic symmetries of asymptotically flat spacetimes, i.e., of spacetimes of an isolated source. This result came as quite a surprise as it was expected that the asymptotic symmetries of asymptotically flat spacetimes would reduce to the Poincaré group. Instead, the BMS group encompasses the Poincaré group and contains an infinite number of additional symmetries, the *supertranslations*.

On the other hand, in the presence of a negative cosmological constant one finds that the asymptotic symmetries of these asymptotically anti-de Sitter (AdS) spacetimes form the *AdS group* [13, 14]. The fact that the AdS group in D dimensions is the same as the *conformal group* in $D-1$ dimensions has very interesting consequences as we will see below.[4]

Black holes and holography. *Black holes* have turned out to be our guiding stars in the long search for quantum gravity. An important clue for their relevance in this program comes from the proposal of Bekenstein [15, 16] that black holes have an entropy given by

$$S_{\mathrm{BH}} = \frac{k_B c^3 A}{4G\hbar}, \tag{1.4}$$

where A is the area of the black hole's horizon. According to Boltzmann's formula

$$S = k_B \log \Omega \tag{1.5}$$

the entropy of a macroscopic state is related to the number of microstates, Ω, that are compatible with it. The entropy formula (1.4) suggests therefore that a black hole, being a macroscopic state, is made up of a huge number of (quantum mechanical) microstates. In fact, this number is larger than for any other known object. Concurrently it was shown by Hawking that black holes have a temperature T_H and are therefore subject to the laws of thermodynamics like any other macroscopic system in the universe [17, 18]. For a historical overview of black hole thermodynamics, see for instance [8, 19].

This profound insight immediately led to another conundrum concerning the marriage of quantum mechanics and gravity. A physical system of finite temperature necessarily loses energy due to radiation: black holes evaporate. This has the puzzling consequence that black holes apparently destroy information. After the black hole has vanished, the whole information about the matter that originally collapsed into the black hole should be stored in the *Hawking radiation* emitted from the black hole. However, Hawking radiation is thermal and cannot accommodate enough information to achieve this [20].

[4] The author acknowledges thorough discussions with Friedrich Schöller on the point of view presented in this paragraph.

The Bekenstein–Hawking entropy (1.4) has another unusual feature: it is proportional to the area of the horizon not the black hole's volume, as one might have naïvely guessed. It appears that information is not stored inside the black hole but on its surface. In an attempt to solve the above mentioned information loss problem, this observation led 't Hooft to conjecture that every (quantum) gravitational system could have an effective description in terms of a lower-dimensional ordinary quantum field theory [21]. Susskind called this idea the *holographic principle* and suggested that it can be realized in *string theory* [22].

AdS/CFT correspondence. A concrete implementation of the holographic principle has emerged in the form of the *AdS/CFT correspondence* [23–25]. This is a conjectured duality between a gravitational theory on asymptotically anti-de Sitter space (AdS) and a *conformal quantum field theory* (CFT) on its boundary. As was already mentioned above, the fact that the symmetry groups of these two theories coincide can be regarded as a first hint for this conjecture. However, the conjecture goes much further than just equating the symmetry groups of the two theories. In its strongest version it states an exact duality between the two theories, i.e., every observable on the gravity side has a corresponding quantity on the CFT side and vice versa. While the *AdS/CFT dictionary* relating the respective observables already contains a large number of entries (in fact an infinite number), we have not yet arrived at a complete understanding of the correspondence.

The AdS/CFT correspondence has attracted a substantial amount of interest due to the prospect that it can help to explain quantum gravity using well known techniques of ordinary quantum field theories. In particular in regards to the above mentioned information loss paradox, if the duality is correct, it shows that information loss cannot occur since the boundary theory is known to be unitary.[5]

Holography in lower dimensions. The AdS/CFT correspondence has emerged as a valuable tool to study quantum gravity. But many of its features are still not well-understood. Since most conceptual issues arise independently of the number of dimensions it is a good strategy to consider simpler models that allow to study, and ideally resolve, these issues. For theories of gravity, this is usually achieved by studying theories of gravity in two or three dimensions.

The AdS/CFT correspondence in three dimensions is one of the prime examples of holography. Its roots can be traced back to the famous result by Brown and Henneaux [26] who showed that the asymptotic symmetries of three-dimensional AdS space are enhanced to an infinite-dimensional group, namely two copies of the Virasoro group. Again, these are precisely the symmetries of two-dimensional CFTs. These theories are rather well-understood due to the large number of available symmetries.

Einstein gravity in three dimensions has no propagating degrees of freedom, i.e., there exist no gravitational waves. However, it came as quite a surprise when it was shown that the theory has black hole solutions in AdS space, the so-called Bañados–Teitelboim–Zanelli (BTZ) black holes [27, 28]. The derivation of the Bekenstein–Hawking entropy (1.4) for these black holes from properties of the CFT on the boundary [29] by use of the Cardy formula [30, 31] is a great achievement of the AdS/CFT correspondence.

[5]Of course, it would still be necessary to show where precisely the usual argument in the gravitational theory breaks down.

But a number of puzzling features of the AdS_3/CFT_2 correspondence still remains. In particular, the nature of the boundary theory is still elusive. It is known due to the result of Brown and Henneaux, at least semi-classically, that the boundary theory is a two-dimensional CFT but which one precisely is not clear. It has been shown that the asymptotic dynamics of Einstein gravity in AdS_3 reduces to Liouville theory [32] that, however, does not correctly reproduce the Bekenstein–Hawking entropy [33, 34]. Furthermore, the partition function of Einstein gravity on AdS_3 does not produce sensible results unless further unknown contributions are taken into account [35]. This would mean that Einstein gravity cannot be consistently quantized but needs further matter fields, such as string theory would provide.

Given this status of the AdS/CFT correspondence in three dimensions one might hope that holography in two dimensions could help to answer some of the questions. Unfortunately, AdS_2 holography is somewhat less understood and more subtle than its higher-dimensional cousin. One immediate question is the nature of the boundary theory. Is it conformally invariant quantum mechanics or is it one chiral half of a CFT? Scaling arguments show that a truly conformal one-dimensional quantum theory necessarily has a vanishing Hamiltonian, thus having no dynamics [36]. On the other hand, having one half of a two-dimensional theory as a boundary theory is difficult to interpret. We will see below that recently another contender for a boundary theory has emerged in the form of the *Sachdev–Ye–Kitaev (SYK) model*.

Near horizon holography. There are two motivations why holography in two-dimensional AdS space is interesting to consider. The first one was mentioned above: it can be regarded as a toy model for the higher-dimensional case. This will be the point of view taken in this thesis. But there is a second one pertaining to its relation to higher-dimensional black holes that we want to mention.

In addition to its mass a black hole is also characterized by other conserved quantities such as angular momentum J, electrical charges Q etc. A black hole with the smallest possible mass that is compatible with a given set of charges $J, Q, \ldots$, is called *extremal black hole*.[6] For a large number of these black holes one finds that the region near the horizon is of the form $AdS_2 \times K$ where K is a compact space (for a comprehensive review of near horizon geometries see [39]).The appearance of an AdS_2 factor in the near horizon geometry of an extremal black hole led to the hope that the AdS/CFT correspondence could be applied to the (almost) realistic set-up of extremal Kerr black holes [40]. The symmetry algebra was shown to be one copy of the Virasoro group thus suggesting a chiral half of a two-dimensional CFT as boundary theory, and applying a chiral half of the Cardy formula indeed reproduces the entropy of the extremal Kerr black hole.[7] However, it was later shown that this dual theory presumably does not exist, as it is only capable of describing the ground state but not any non-trivial excitations [42–44]. This is well in line with an argument concerning the near horizon region of extremal Reissner–Nordström black holes [45]. Also in that case it was found that the set-up does not allow for any finite energy states. The reason for this can be shown by the following argument: Let M be the mass of the black hole, Q its charge, and $\ell_P = \sqrt{G}$ the Planck length. Then the energy above extremality is given by $E = M - \frac{Q}{\ell_P}$.

[6] Observations suggest that some black holes are very close to extremality [37], however the case is not settled yet; cf. [38].

[7] This result is also reproduced by the attractor mechanism of [41].

The near horizon or decoupling limit is controlled by the dimensionful parameter ℓ_P. Near extremality, one finds the following relation between E, ℓ_P, and Hawking temperature T_H

$$E = 2\pi^3 Q^3 \ell_P T_H^2 \,. \tag{1.6}$$

It is not possible to have $\ell_P \to 0$, i.e., to go to the decoupling limit while keeping T_H and E finite. For a thermodynamic description to be valid we would expect $E \gg T_H$ so that the radiation of a Hawking quantum of energy T_H does not drastically change the macrostate. The thermodynamic description therefore breaks down if E is of the order of $\ell_P^{-1} Q^{-3}$. This can be regarded as the *mass gap* of the black hole between the ground state and the first excited states. In the near horizon limit $\ell_P \to 0$ all excitations are sent to infinity and the only accessible state left is the ground state.

This suggests that in order to obtain any non-trivial dynamics one has to consider small deviations from the near horizon limit.

The SYK model. Interest in AdS_2 holography saw a recent rise due to the SYK model that was introduced by Kitaev and discussed in more detail by Maldacena and Stanford [46, 47]. The origins of this model go back to the original work [48] that was subsequently simplified by Kitaev. The SYK model is a quantum mechanical model of N Majorana fermions having a four-point interaction with random coupling constant. In the strong coupling, or similarly IR, limit at large N the theory exhibits local conformal symmetry in one dimension, i.e., time reparametrization invariance, that is spontaneously broken to $\mathrm{SL}(2,\mathbb{R})$ by the ground state. As always with spontaneous symmetry breaking, one expects Nambu–Goldstone bosons to arise that however have zero action in the present case. In this strict conformal limit the theory is not well-defined but shows a divergence in the four-point function. This divergence can be lifted by considering a small deviation from the IR limit that gives rise to a non-vanishing action for the Nambu–Goldstone bosons, the *Schwarzian action.* This pattern of symmetry breaking and the Schwarzian action can be recovered from the gravity side, as we will see in more detail in later chapters.[8]

The SYK model has attracted a lot of attention due to the fact that it is a strongly coupled theory that is solvable at large N. Various extensions and generalizations have been considered recently [49–70] that showed similar properties as the SYK model. One motivation for considering other models with similar features is the presence of random coupling constants in the original model. Since every observable requires an average over these coupling constants, the SYK model is not a quantum mechanical model in the strict sense as emphasized by [57]. This could pose some difficulties when studying subtle questions about black holes and holography.

While the SYK model provides a strong motivation for reconsidering AdS_2 holography this thesis is mostly concerned with the gravity side of this possible correspondence. Only the Schwarzian action, as somewhat of a promontory of the SYK model, will feature prominently. Appendix D contains a discussion of the SYK model explaining how the Schwarzian action arises from it.

[8] Note also that the requirement to treat the SYK model slightly away from the conformal limit is very similar to the above considerations concerning triviality of the dynamics in the near horizon limit of extremal black holes.

Generalizations of AdS/CFT. The AdS/CFT correspondence is currently the most developed realization of the holographic principle. However, if the holographic principle is correct, it should apply also to more realistic set-ups given that the cosmological constant in our universe is not negative. After all, in the end we want to understand *our* world, not only *some* world that would be possible by the laws of physics. There have been attempts to apply the holographic principle to de Sitter space [71] or asymptotically flat spacetimes [72–75]. In particular, in the latter case new interesting avenues have opened up in recent times sparked by the work [76]. Therein it was shown that, after linking the two separate BMS groups at past and future null infinity of four-dimensional asymptotically flat spacetimes, this "diagonal" subgroup is a symmetry of the S-matrix.[9] The associated Ward identity is the well-known soft graviton theorem of Weinberg [77, 78]. These new ideas regarding the asymptotic structure of asymptotically flat spacetimes, collectively dubbed the *infrared triangle*, promise to lead to new insights concerning flat space holography. We will not have much to say about these developments in this thesis but comment on some possible extensions in the conclusions.

The aim of this thesis. This thesis aims to present a comprehensive discussion of AdS_2 holography from a intrinsically two-dimensional perspective. We will see that the theory has two sectors with very different properties: the *constant dilaton sector* and the *linear dilaton sector*. We will show in full generality that the dynamics associated to the constant dilaton sector are trivial. This is in line with the above mentioned results concerning the near horizon region of extremal black holes but independent of a particular higher-dimensional set-up or two-dimensional gravity model. In order to find non-trivial holography we will then turn to the linear dilaton sector. We will first discuss the Jackiw–Teitelboim model [79, 80] from which we will derive the Schwarzian action that links this model to the SYK model, as mentioned above. In a second step we study suitable generalizations of the Jackiw–Teitelboim model by coupling the theory to Yang–Mills fields and fields of spin greater than two. We will present generalizations of the Schwarzian action for these cases. Our results can be regarded as mapping the space of theories in AdS_2 that can have a boundary dual with properties similar to the SYK model.

In all of this we find symmetry to be our guiding principle.

Structure of this thesis

This thesis is separated into two parts. In part I we collect necessary tools that we will need for our journey to the boundary of AdS_2 (or perhaps in search of the catalogue of catalogues). Some of the material presented therein will not be used explicitly but will provide valuable background information. Part II contains the original research of this thesis, heavily based on the works [1, 2, 4]. Part III contains the appendix.

Chapter 2 This chapter deals with symplectic structures and their generalization, Poisson structures. These will feature prominently throughout the thesis. The action of Lie groups on symplectic structures and Poisson structures is discussed in some detail.

[9] A similar linking in three dimensions was presented in [5].

Chapter 3 The concept of a two-dimensional theory of gravity requires some explanation. We motivate and define a certain two-dimensional theory of gravity, called dilaton gravity, and discuss its properties. A reformulation of dilaton gravity as a (non-linear) gauge theory is presented.

Chapter 4 Since this thesis is concerned with gravity on AdS_2 spacetimes we discuss these in close detail. It is shown that some solutions have properties that allow for a black hole interpretation. We show that hyperbolic geometry and knowledge of the coadjoint orbits of the Virasoro group are helpful for understanding these spacetimes.

Chapter 5 This chapter is based on the original work [4]. We show that the constant dilaton sector of any dilaton gravity theory in AdS_2 is necessarily trivial. This is in line with previous results but presented here without referring to any particular model. A one-loop calculation establishes that the result remains true when quantum mechanical effects are taken into account.

Chapter 6 This chapter is based on the original work [2] but is presented in a shortened and restructured form. We discuss the JT model that is the simplest dilaton gravity model with linear dilaton solutions in AdS_2 in the second order formulation. An action with a well-defined variational principle is presented. The symmetries of the JT model are discussed and it is shown that the action reduces to the Schwarzian action, thus establishing the relation to the SYK model as a possible boundary theory.

Chapter 7 This chapter is based on the original work [1]. The JT model is reformulated as a gauge theory of the gauge group SL(2) along the lines presented in chapter 3 and it is shown how to obtain the Schwarzian action in this formulation. The gauge theoretic formulation of the JT model lends itself to a straightforward generalization to gauge groups having an $\mathrm{SL}(2,\mathbb{R})$ subgroup such as $\mathrm{SL}(2,\mathbb{R}) \times \mathcal{K}$ with $\mathcal{K}$ an arbitrary (compact) Lie group or $\mathrm{SL}(N,\mathbb{R})$. We construct and discuss generalizations of the Schwarzian action for these actions.

Chapter 8 In this chapter we summarize our results and point out open lines of research.

Appendix A In this appendix a canonical analysis based on an ADM split for the second order formulation of dilaton gravity is presented. The resulting Hamiltonian is used to construct the canonical charges in section 3.5.

Appendix B One of the main protagonists of this thesis will be two-dimensional anti-de Sitter space, AdS_2. Here we collect various coordinate systems that will be used throughout the thesis.

Appendix C This appendix is based on unpublished results that were obtained during work leading up to [1]. We present a different way of constructing the generalized Schwarzian actions of chapter 7 using the Iwasawa decomposition of elements of $\mathrm{SL}(N,\mathbb{R})$ that shows interesting similarities to work in three dimensions.

Appendix D This appendix discusses aspects of the SYK model and its relation to the Schwarzian action.

Part I

Tools for the Journey

Chapter Two

Symplectic and Poisson Structures

We are going to start our collection of suitable tools for the second part of this thesis with an introduction to symplectic geometry which can be regarded as a geometric reformulation of the Hamiltonian picture of classical physics. This purely geometric point of view is the reason for the formalism's usefulness as it gets rid of particular parametrizations (or particular coordinate systems on the symplectic manifold) that might be useful for some physical systems but impeding for others.

In section 2.2 we will discuss Poisson geometry defined as manifolds that allow for the construction of a Poisson bracket. Apart from being a natural generalization of symplectic geometry, the main reason for discussing these manifolds is their appearance in a reformulation of two-dimensional gravity that we will heavily use in the remainder of this thesis.

All systems physicists are usually interested in come with some symmetry. Thus it makes sense to see how these symmetries interact with the geometric reformulation of classical physics provided by symplectic geometry. This will be the topic of section 2.3.

Due to the rather mathematical content of this section it lends itself to a presentation using definitions, theorems, and examples; proofs will be consistently omitted but attempts on elucidating the content of some theorems will be made. This section is based on textbooks [81–87] in which proofs and further material can be found.

2.1 Symplectic manifolds

Before defining the concept of symplectic manifolds let us make the following definition:

Definition 1 (Strong/weak nondegeneracy)**.** A two-form Ω defined on a manifold M is called *weakly non-degenerate* if the map

$$\Omega^\flat : T_pM \to T_p^*M \qquad \Omega^\flat(X)(Y) = \Omega(X,Y) \qquad \forall X,Y \in T_pM \tag{2.1}$$

is injective, i.e., $\Omega^\flat(X) = 0$ implies $X = 0$.

If the map $\Omega^\flat$ is injective and surjective thus defining an isomorphism between T_pM and T_p^*M, the form is called *strongly non-degenerate*.

Notice that for a finite dimensional manifold M weak and strong degeneracy are equivalent. However, in some applications the manifold M will be infinite dimensional. In most of these cases Ω is weakly non-degenerate only.

Using the above distinction between weak and strong non-degeneracy the definition of a symplectic manifold is stated promptly:

Definition 2 (Symplectic manifold)**.** A *symplectic manifold* is given by the pair (M, Ω) where M is a manifold and Ω is a non-degenerate two-form on M that is closed

$$d\Omega = 0. \tag{2.2}$$

If Ω is strongly non-degenerate, the pair is called *strong symplectic manifold*.

In a coordinate patch given by $z^I = (z^1, ..., z^D)$ on a finite-dimensional manifold the symplectic two-form Ω can be written as

$$\Omega = \Omega_{IJ}\, dz^I \wedge dz^J, \tag{2.3}$$

where Ω_{IJ} is a regular antisymmetric matrix that obeys the closedness condition (2.2)

$$\partial_K \Omega_{IJ} + \partial_I \Omega_{JK} + \partial_J \Omega_{KI} = 0. \tag{2.4}$$

Since an antisymmetric matrix in odd dimensions necessarily contains a zero eigenvalue we find that all (finite-dimensional) symplectic manifolds have to be even-dimensional, $D = 2n$.

Having defined a distinguished tensor on the manifold M some diffeomorphisms on M will be special in the sense that they preserve the symplectic structure.

Definition 3 (Canonical transformation)**.** If (M_1, Ω_1) and (M_2, Ω_2) are two symplectic manifolds, a diffeomorphism $\varphi : M_1 \to M_2$ is called *symplectic* if its pull back preserves the symplectic structure

$$\varphi_* \Omega_2 = \Omega_1. \tag{2.5}$$

If $(M_1, \Omega_2) = (M_2, \Omega_2)$ then the condition

$$\varphi_* \Omega = \Omega \tag{2.6}$$

defines a subgroup of all diffeomorphisms of M called *symplectomorphisms*.

It is worthwhile to pause for a moment and compare the above definition of symplectic manifolds with the familiar notion of a Riemannian manifold. On a superficial level both structures appear to be quite similar: In both cases one is given a manifold M with a distinguished non-degenerate element of the tangent space $T^*M \otimes T^*M$, in the former case an antisymmetric element that obeys (2.2) in the latter case a symmetric tensor. Interestingly, this is about as far as the similarities go: We already saw above that even-dimensionality is a necessary criterion for a manifold to be symplectic; on the other hand, any (paracompact) manifold can be given the structure of a Riemannian manifold. Another necessary condition on closed symplectic manifolds is non-vanishing of the second cohomology group $H^2(M) \neq 0$.[1]

[1] This is actually quite easy to see: Any symplectic form Ω defines a natural volume element ε by $\varepsilon = \Omega^n$. Now assume that $H^2(M) = 0$. Then there must exist some α such that $\Omega = d\alpha$, which implies $\varepsilon = d(\alpha \wedge \Omega^{n-1})$. The volume of any M would then be given by $\int_M \varepsilon = \int_M d(\alpha \wedge \Omega^{n-1}) = \int_{\partial M} \alpha \wedge \Omega^{n-1} = 0$ where the last equality follows from the fact that M is closed. This is a contradiction. Thus, $H^2(M) \neq 0$.

Thus, not even the $2n$-sphere (except $n = 1$) is a symplectic manifold. Loosely speaking, symplectic manifolds are much rarer than Riemannian manifolds.

Let us finally state another important difference. While one can define interesting local properties of Riemannian manifolds via curvature, all interesting properties of symplectic manifolds must be necessarily of global origin, as the following, well-known theorem shows.

Theorem 1 (Darboux theorem)**.** *Let (M, Ω) be a strong symplectic manifold. Then in a neighborhood of each point $p \in M$, there is a local coordinate chart in which Ω is constant.*

For a proof see any book on symplectic geometry, e.g., [84]. In contrast, in Riemannian geometry a metric with non-vanishing curvature can be transformed to the unit metric at a single point only.

From the above follows that, on a finite-dimensional manifold, there always exists a coordinate system $(q^1, .., q^n, p_1, ..., p_n)$ called *canonical coordinates* in which the symplectic form is given by

$$\Omega = dq^i \wedge dp_i. \tag{2.7}$$

In equation (2.7) by using the conventional symbols (q, p) we already suggested a relation of symplectic manifolds to physical systems. We will now turn to some examples that will clarify the above concepts.

Example 1 (Two-dimensional symplectic manifolds)**.** Any two-dimensional manifold is symplectic, given that one can define a non-degenerate two-form Ω on it, since $d\Omega = 0$ trivially. Thus, the torus with coordinates (θ, φ) and $\Omega = d\theta \wedge d\varphi$ and the two-sphere with coordinates (θ, φ) and the symplectic form given by the volume form $\Omega = -\, d(\cos\theta) \wedge d\varphi$ are symplectic.

Example 2 (Cotangent bundles)**.** We will now come to the example that will provide the relation of the above framework with the usual set-up in physics [85].

Consider a configuration manifold Q whose points describe all kinematically possible states of a physical system. Assuming that the differential equations governing the evolution of the system are of second-order, one can define a Lagrangian function, viewed as a function on the tangent bundle $L : TQ \to \mathbb{R}$ that governs the dynamics of the system. However, it turns out that it is the cotangent bundle T^*Q, i.e., the *phase space* of the system, that has the structure of a symplectic manifold. This is due to the existence of the *tautological* or *canonical one-form* θ on any cotangent bundle. From this form one can define the symplectic structure as

$$\Omega = -\, d\theta, \tag{2.8}$$

where the minus sign is conventional, which is obviously closed and can be shown to be non-degenerate.

More precisely, let π denote the canonical projection from the cotangent bundle to the base manifold

$$\pi : T^*Q \to Q \qquad (q, \alpha) \mapsto q\,, \tag{2.9}$$

where $(q, \alpha) \in T^*Q\,(q \in Q)$, and let $\pi_* : T(T^*Q) \to TQ$ be the push-forward (or differential) of π. Then the canonical one-form at a point $(q, \alpha) \in T^*Q$ is defined as

$$\langle \theta_{(q,\alpha)}, w \rangle = \langle \alpha, \pi_* w \rangle \qquad \forall w \in T(T^*Q), \tag{2.10}$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing between tangent and cotangent space on Q.

This definition is somewhat opaque. Thus, let us evaluate (2.10) for a finite-dimensional manifold Q. Choosing coordinates $(q^1,...,q^n)$ in a coordinate chart U on Q and coordinates $(p_1,...,p_n)$ for the cotangent space T_q^*Q at point $q\in U$, an element α in the fiber $\pi^{-1}(U)$ has coordinates $(q^1,...,q^n;p_1,..,p_n)$ and can be written as $\alpha=p_i(dq^i)_q$, where the subscript denotes that the (dq^i) belong to the tangent space at q. An arbitrary vector $w\in T(T_q^*Q)$ is therefore of the form

$$w=u^i\frac{\partial}{\partial q_i}+\beta_i\frac{\partial}{\partial p^i} \tag{2.11}$$

with the push-forward of the projection acting as

$$\pi_*:w\mapsto u^i\frac{\partial}{\partial q_i}. \tag{2.12}$$

Plugging this into equation (2.10) we are left with

$$\langle\theta_{(q,\alpha)},u^i\frac{\partial}{\partial q_i}+\beta_i\frac{\partial}{\partial p^i}\rangle=p_iu^j\langle dq^i,\frac{\partial}{\partial q^j}\rangle=p_iu^i, \tag{2.13}$$

which implies that $\theta_{(q,\alpha)}=p_i\,dq^i$. Consistently, we recover by (2.8) the symplectic two-form in canonical coordinates (2.7).

We have seen in the last example that the natural arena for classical mechanics in the phase space formulation is indeed provided by symplectic geometry. But up to this point we have studied kinematics only. In order to study the evolution of physical quantities on the symplectic manifold we introduce the concept of *Hamiltonian vector fields.*

Definition 4 (Hamiltonian vector field)**.** A vector field X_H on the symplectic manifold (M,Ω) is called *Hamiltonian* if there exists a function H such that

$$i_{X_H}\Omega\equiv\Omega(X_H,\cdot)=dH. \tag{2.14}$$

Equivalently, the symplectic structure defines a map $\Omega^\sharp:T^*M\to TM$ by

$$\Omega(\alpha,\beta)=\langle\alpha,\Omega^\sharp\beta\rangle,\qquad\forall\alpha,\beta\in T^*M \tag{2.15}$$

using which the Hamiltonian vector field can be written as

$$X_H=\Omega^\sharp\,dH\,. \tag{2.16}$$

The set of all Hamiltonian vector fields is denoted $\mathrm{Ham}(M)$.

A vector field X is called *locally Hamiltonian* if

$$d(i_{X_H}\Omega)=0, \tag{2.17}$$

since then, by the Poincaré lemma, there exists locally a function H such that $i_{X_H}\Omega=dH$.

Let us denote by φ_t the flow defined by a vector field X. Using Cartan's magic formula the Lie derivative $\pounds_X$ acting on the symplectic form can be written as

$$\pounds_X \Omega = i_X(d\Omega) + d(i_X\Omega)\,. \tag{2.18}$$

Using $d\Omega = 0$ and the above definition, this shows that the set of vector fields whose flows leave the symplectic structure invariant

$$\varphi_t^* \Omega = \Omega \tag{2.19}$$

is precisely given by locally Hamiltonian vector fields. Thus, all symplectomorphisms are generated by Hamiltonian vector fields.

With the notion of Hamiltonian vector fields at our disposal we can write down Hamilton's equations in the concise form

$$\dot{z} = X_H(z). \tag{2.20}$$

It is important to stress that the Hamiltonian H generating the Hamiltonian vector field X_H is not necessarily the generator of time translations but can be any function on phase space. Notice that while it is clear that there exists a Hamiltonian vector field X_H for any choice of H in finite dimensions, this is not guaranteed in the infinite-dimensional case. However, we will with the physicist's grace brush over these details.

As a final application of symplectic geometry let us define another important ingredient of classical mechanics.

Definition 5 (Poisson bracket)**.** Let f, g be two smooth functions on the symplectic manifold (M, Ω), $f, g : C^\infty(M) \to \mathbb{R}$. Then the *Poisson bracket* at a point $z \in M$ is defined as

$$\{f, g\}(z) = \Omega(X_f(z), X_g(z)) \tag{2.21}$$

where X_f and X_g are the Hamiltonian vector fields associated to f and g, respectively.

Note that the above definition is equivalent to

$$\{f, g\} = (i_{X_f}\Omega)(X_g) = df(X_g) = X_g[f] \tag{2.22}$$

where we used the defining equation for Hamiltonian vector fields (2.14) in the first step.

The Poisson bracket has a number of important properties. The most important is the fact that it gives the functions defined on the phase space the structure of a Lie algebra.

Theorem 2 (Poisson algebra)**.** *The pair* $\{C^\infty(M), \{\cdot,\cdot\}\}$ *defines an infinite-dimensional Lie algebra called* Poisson algebra.

The only non-trivial step in the proof of the above theorem is the verification of the Jacobi identity which can be established using the fact

$$X_{\{f,g\}} = -[X_f, X_g]\,. \tag{2.23}$$

This can be proved by applying Ω to both sides and then using the above definitions for Hamiltonian vector fields. In particular, closedness of Ω is crucially needed for the Jacobi identity of the Poisson bracket.

Equation (2.23) makes explicit the Lie algebra antihomomorphism $f \mapsto X_f$ between the Poisson algebra and the Lie algebra of Hamiltonian vector fields. The kernel of this antihomomorphism consists of the functions that generate trivial dynamics, i.e., the constant functions.

This ends our discussion of symplectic structures. We saw that the phase space of a physical system is naturally endowed with a symplectic structure that can be used to elegantly describe classical mechanics in geometrical terms. However, we saw in the beginning of the section that the requirements on symplectic manifolds are actually quite restrictive. In particular, if all one cares about is to obtain a Poisson bracket from the symplectic structure by equation (2.21), then we might consider loosening the conditions on the symplectic structure. We saw above that closedness of the symplectic structure translates to the Jacobi identity for the Poisson bracket which is a property that we would not like to discard. But the property of non-degeneracy can be loosened without sacrificing a well-defined Poisson bracket. Consider, for instance, the two-tensor $\omega = dx \wedge dy$ on $\mathbb{R}^3$. This does not define a symplectic structure as it is obviously degenerate. But let us make the following ad-hoc construction: The Poisson bracket for functions f, g depending on x, y is calculated according to the usual prescription (2.21). For functions of z only the associated Hamiltonian vector field will be defined to be the zero vector field such that these functions have vanishing Poisson bracket with every other function. This definition is extended to all functions by requiring the Poisson bracket to obey the Leibniz identity. As can be easily checked, this bracket satisfies all the requirements on a Poisson bracket, i.e., it is an antisymmetric, bilinear derivation that obeys the Jacobi identity.

Thus, without sacrificing the Poisson bracket we can pass to a structure, called Poisson structure, that still shows many of the interesting properties of a symplectic structure but is more general. In particular, as we will see in later sections, Poisson structures have an intimate relation to Lie algebras. We will explore Poisson structures in detail in the next section.

2.2 Poisson manifolds

In the last section we saw that the fundamental object on a symplectic manifold is a closed two-form. It turns out, similarly, that a Poisson manifold is defined by having a distinguished contravariant, antisymmetric two-tensor.[2]

Definition 6 (Poisson manifold)**.** A *Poisson structure* on a manifold M is a bilinear operation

$$\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M) \tag{2.24}$$

that obeys the following properties

(1) antisymmetry

$$\{f, g\} = -\{g, f\}, \tag{2.25}$$

(2) the Jacobi identity:

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0, \tag{2.26}$$

[2]This is sometimes called a bi-vector.

(3) the Leibniz identity (derivation property):

$$\{f, gh\} = \{f, g\}h + \{f, h\}g \tag{2.27}$$

for all functions $f, g, h \in C^\infty(M)$.

The pair $(M, \{\cdot, \cdot\})$ defines a *Poisson manifold*.

Equivalently, one can define a Poisson manifold by specifying an antisymmetric tensor field $P \in \wedge^2 TM$ on M such that

$$\{f, g\} = P(df, dg). \tag{2.28}$$

for all $f, g \in C^\infty(M)$. The tensor field P is called the *Poisson tensor* on M. The pair (M, P) defines a Poisson manifold.

In a coordinate system $(z^1, ..., z^n)$ definition (2.28) reads

$$\{f, g\} = P^{IJ} \frac{\partial f}{\partial z^I} \frac{\partial g}{\partial z^J}, \tag{2.29}$$

where

$$P^{IJ} = \{z^I, z^J\}, \tag{2.30}$$

is the expression for the Poisson tensor in local coordinates. Using definition (2.28), requirements (2.25) and (2.27) are automatically satisfied for an arbitrary antisymmetric tensor field P^{IJ}. The Jacobi identity (2.26) translates to the non-trivial equation

$$\circlearrowleft_{IJK} P^{LI} \partial_L P^{JK} \equiv P^{LI} \partial_L P^{JK} + P^{LJ} \partial_L P^{KI} + P^{LK} \partial_L P^{IJ} = 0. \tag{2.31}$$

In the symplectic case we saw that the Jacobi identity of the Poisson bracket is equivalent to vanishing of the three-form $d\Omega$. Similarly, one can formulate the requirement (2.31) on the Poisson tensor geometrically by demanding vanishing of an antisymmetric, contravariant three-tensor. An equivalent statement is that the Schouten bracket (a generalization of the Lie bracket to antisymmetric, contravariant tensors) of the Poisson tensor with itself vanishes. Since these concepts will not be used in the following we refer to, e.g., [83].

Note that the above definition says nothing about non-degeneracy of the Poisson tensor. In fact, often not even the rank of P will be constant on M. On an odd-dimensional manifold the Poisson tensor is necessarily degenerate since the rank of an antisymmetric matrix is always even.

Let us study some immediate consequences of the above definition. Since the Poisson bracket is bilinear and obeys the Jacobi identity it follows immediately that the pair $(C^\infty(M), \{\cdot, \cdot\})$ forms a Lie algebra. As a second point consider fixing a function $f \in C^\infty(M)$. Then $\{f, \cdot\}$ defines a linear map from $C^\infty(M) \to C^\infty(M)$ that obeys the Leibniz rule, i.e., it defines a vector field on M. We can therefore define:

Definition 7 (Hamiltonian vector fields, Casimir functions)**.** Consider a function $H \in C^\infty(M)$. The vector field X_H defined by

$$\pounds_{X_H} g = X_H(g) = \{g, H\} \qquad \forall g \in C^\infty(M) \tag{2.32}$$

is called the *Hamiltonian vector field* of the *Hamiltonian function* H. Equivalently, the Poisson tensor P defines a map $P^\sharp : T^*M \to TM$ by

$$P(\alpha, \beta) = \langle \alpha, P^\sharp \beta \rangle, \qquad \forall \alpha, \beta \in T^*M, \tag{2.33}$$

using which the Hamiltonian vector field can be written as

$$X_H = P^\sharp \, dH \,. \tag{2.34}$$

In a coordinate system we would write for the components of X_H

$$X_H^I = P^{IJ} \partial_J H. \tag{2.35}$$

A function C that has vanishing Poisson bracket with every other function

$$\{C, f\} = 0 \qquad \forall f \in C^\infty(M) \tag{2.36}$$

and thus generates a zero Hamiltonian vector field is called *Casimir function.* These functions generate the center of the Poisson algebra.

The set of Hamiltonian vector fields at a point p, $\mathrm{Ham}_p(M)$, equipped with the commutator defines a Lie subalgebra of the Lie algebra of tangent vectors. In fact, the map $f \mapsto X_f$ of $C^\infty(M) \to \mathrm{Ham}_p(M)$ is again a Lie algebra antihomomorphism

$$[X_f, X_g](h) = -X_{\{f,g\}}(h) \qquad \forall h \in C^\infty(M) \tag{2.37}$$

as can be checked straightforwardly using the Jacobi identity. The dimension of the vector space $\mathrm{Ham}_p(M)$ at a point $p \in M$ is equal to the rank of P at this point.

The existence of Casimir functions is related to the fact that the Poisson tensor P is not required to be non-degenerate. This is in contrast to the symplectic case where only constant functions have vanishing Hamiltonian vector field. Below we will see that set of Casimir functions can be used to define a foliation of the Poisson manifold into leaves on each of which one has a well-defined symplectic structure. Before looking at two simple examples of Poisson manifolds where we can study the above features, we want to make explicit the relationship between Poisson structure and symplectic structures.

Symplectic structure and non-degenerate Poisson structures should coincide in their definition of the Poisson bracket, i.e., equating (2.21) and (2.28) we find

$$P(df, dg) = \Omega(X_f, X_g) \tag{2.38}$$

which, using the maps defined in (2.15) and (2.33), is equivalent to

$$\langle df, P^\sharp \, dg \rangle = \langle df, \Omega^\sharp \, dg \rangle \tag{2.39}$$

which implies

$$\Omega^\sharp = P^\sharp \,. \tag{2.40}$$

In a local coordinate system this is equivalent to the statement

$$P^{IJ} = -\Omega^{IJ} \tag{2.41}$$

where Ω^{IJ} is defined as $\Omega^{IJ}\Omega_{JK} = \delta^I{}_K$. As expected, the inverse of a symplectic structure defines a regular Poisson structure on a manifold, and vice versa given that the Poisson structure is non-degenerate.

Analogous to symplectic manifolds we define:

Definition 8 (Poisson (canonical) maps)**.** Let (M_1, P_1) and (M_2, P_2) be Poisson manifolds and $\varphi : M_1 \to M_2$ a smooth map. Then φ is called *canonical* or *Poisson* if

$$\varphi^* P_1 = P_2 \,. \tag{2.42}$$

If $(M_1, P_1) = (M_2, P_2)$ this defines the *Poissonmorphisms* of the manifold (cf. the analogue statement (2.6) for symplectic manifolds). Analogous to the symplectic case one can prove that the flow generated by a Hamiltonian vector fields leaves the Poisson structure invariant. In particular, $L_{X_H} P = 0$ if X_H is a Hamiltonian vector field. The converse is, in general, not true. There are more vector fields leaving the Poisson structure invariant than Hamiltonian vector fields.[3]

Example 3 (Poisson structures on $\mathbb{R}^2$)**.** We choose as Poisson manifold $M = \mathbb{R}^2$ with global coordinates (x, y) and the antisymmetric tensor

$$P = \partial_x \wedge \partial_y \tag{2.43}$$

as our Poisson tensor. For the Poisson brackets of smooth functions $f, g \in C^\infty(\mathbb{R}^2)$ we find therefore

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}. \tag{2.44}$$

This is, of course, nothing but the standard Poisson bracket of a point particle on the configuration space $\mathbb{R}^2$ under relabeling $(x, y) \to (q, p)$. Consistently, we find that the symplectic structure given by (2.40) is the canonical structure $\Omega = dq \wedge dp$.

The Hamiltonian vector field associated to the Hamiltonian function $H \in C^\infty(\mathbb{R}^2)$ is given by

$$\xi_H = \{\cdot, H\} = -\frac{\partial H}{\partial x}\frac{\partial}{\partial y} + \frac{\partial H}{\partial y}\frac{\partial}{\partial x}. \tag{2.45}$$

These vector fields span the entire tangent space since any vector field on $\mathbb{R}^2$ can be written in this way. The Casimir functions that generate trivial dynamics are given by the constant functions $H = const$.

Example 4 (Poisson structure on $\mathbb{R}^3$)**.** Let us only marginally crank up the level of sophistication and consider the manifold $M = \mathbb{R}^3$ with global coordinates (x, y, z). This is the example alluded to at the end of section 2.1. Since an odd-dimensional antisymmetric matrix necessarily contains a zero eigenvalue the Poisson tensor will be singular. Choosing the same Poisson tensor as in (2.43) we find the same Poisson brackets and the same Hamiltonian vector fields (2.45) with H being replaced by an arbitrary function in $C^\infty(\mathbb{R}^3)$. However, the set of Casimir functions is now given by arbitrary functions in z. There exists no symplectic structure on this Poisson manifold since (2.40) yields a non-invertible matrix for Ω^{IJ}. However, planes of constant z, i.e., planes of constant Casimir functions, provide a foliation of the Poisson manifold into slices on each of which one can define a symplectic structure. This is called *symplectic foliation* and a generic feature of Poisson manifolds.

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[3] This can be formulated as a cohomological problem. The space of all vector fields leaving the Poisson structure invariant that do not derive from a Hamiltonian vector field is then called the *first (Lichnerowicz–) Poisson cohomology group*. The zeroth cohomology group corresponds to the Casimir functions while the second cohomology group encodes certain obstructions to quantization of the Poisson manifold. For further details consult [82, 83].

Before describing the global properties of Poisson manifold, in particular its symplectic foliation, we quote a theorem concerning the local structure of Poisson manifolds. For symplectic manifolds the Darboux theorem showed that locally every symplectic manifold looks the same, i.e., locally the symplectic form can be always brought into canonical form (2.7). An analogue theorem in the case of Poisson manifold was proven by Weinstein [88]. The theorem states that given a point $p \in M$ on which the Poisson tensor has rank $2r$ there exists a coordinate system centered on p in which the Poisson tensor splits into a regular part of rank $2r$ and a part that has rank zero at the origin. Consequently, as in the symplectic case locally every Poisson manifold looks the same apart from the rank of the Poisson tensor that can vary. This local splitting of Poisson manifolds works even in many infinite-dimensional applications, i.e., one finds that the Poisson manifolds splits in an infinite-dimensional symplectic space and a finite-dimensional Poisson space.

The fact that Poisson manifolds have an almost trivial local structure implies that one can define Darboux-like coordinates on P.

Theorem 3 (Darboux theorem for Poisson manifolds [83])**.** *Let (M, P) be a Poisson manifold of dimension d and suppose that p is a point where the rank of P is locally constant [i.e., constant in a neighborhood of p] and equal to $2r$. There exists a coordinate neighborhood U of p with coordinates $(q^1, .., q^r, p_1, ..., p_r, z_1, ..., z_s)$ with $d = 2r + s$ such that on U*

$$P = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} \,. \tag{2.46}$$

We will call coordinates in which the Poisson tensor takes this form *(Casimir–) Darboux coordinates.*

We now turn to the symplectic foliation mentioned in example 4. There we saw that a degenerate Poisson tensor becomes non-degenerate when restricted to certain subspaces of the Poisson manifold called *symplectic leaves.* This is true for any Poisson manifold as the following theorem shows.

Definition 9 (Symplectic leaves)**.** Let (M, P) be a Poisson manifold. Given a point $p \in M$, a point p' is said to be on the same *symplectic leaf* Σ_p if there exists an integral curve of a Hamiltonian vector field connecting p and p'.

Theorem 4 (Symplectic foliation [83])**.** *Every Poisson manifold (M, P) is the disjoint union of injectively immersed submanifolds*[4]*, whose tangent spaces are spanned by the Hamiltonian vector fields of (M, P). The Poisson structure, restricted to each of these submanifolds, yields a Poisson structure of maximal rank (symplectic structure). This decomposition is called the* symplectic foliation *of M and the immersed submanifolds are the symplectic leaves of M.*

Instead of giving a proof we want to clarify the content of the theorem. We stated above that the rank of the Poisson tensor at a point p is equal to the dimension of $\mathrm{Ham}_p(M)$. Since the commutator of two Hamiltonian vector fields yields another Hamiltonian vector field,

[4] A map $f : M \to N$ is called *immersion* if the tangent map $T_p f$ is injective at every point $p \in M$. It is an *injective immersion* if f is additional injective. Note that this is not the same as an *embedding.* As examples think of the Klein bottle immersed in $\mathbb{R}^3$ (that, however, can be *embedded* in $\mathbb{R}^4$), or the lemniscate injectively immersed in $\mathbb{R}^2$.

the Hamiltonian vector fields define an integrable submanifold (Frobenius' theorem).[5] If this procedure is performed at every point $p \in M$ we obtain a foliation of M.[6]

What role do the Casimir functions play in the above construction as they have no Hamiltonian vector field associated to them? It is easy to show that a Casimir function C is constant on a symplectic leaf.

Theorem 5 (Symplectic leaves and Casimirs (I))**.** *Let C be a Casimir function and Σ a symplectic leaf. Then C is constant on Σ.*

This suggests that we can identify symplectic leaves by looking for the level-sets of all Casimir functions of (M, P), i.e., constant Casimir functions imply a non-degenerate Poisson structure. As always, the situation is not that simple since there exist counter-examples to this naïve expectation. Yet in most well-behaved cases (and these are the cases of interest to us) the above expectation is true as stated in the following theorem that is quoted for completeness.

Theorem 6 (Symplectic leaves and Casimirs (II) [83])**.** *Let (M, P) be a Poisson manifold of dimension d, let U be a non-empty open subset of M and let $f_1, ..., f_s \in C^\infty(U)$, satisfying:*

(1) The rank of P is constant on U and is equal to $d - s$;

(2) The functions $f_1, ..., f_s$ are Casimirs of the restriction of P to U;

(3) For every point $p \in U$, the differentials $df_1, .., df_s$ are independent.

Then the symplectic foliation of the restriction of P to U coincides with the foliation defined on U by the map $(f_1, ..., f_s) : M \to \mathbb{R}^s$.

In other words, under the above natural assumptions we can conclude that being on surfaces of constant Casimir in M is equivalent to being on a symplectic leaf.

Before concluding this section on the general structure of Poisson manifolds we want to discuss an example.

Example 5. Consider the manifold $\mathbb{R}^3$ with global coordinates (x_1, x_2, x_3) and Poisson bracket given by

$$\{x_i, x_j\} = \varepsilon_{ij}{}^k x_k, \tag{2.47}$$

where $\varepsilon_{ij}{}^k$ is the Levi–Civitá tensor on $\mathbb{R}^3$. It is straightforward to check that this fulfills all the requirements on a Poisson bracket. The Poisson tensor is degenerate since $\mathbb{R}^3$ is odd-dimensional. The rank of P is two everywhere except at the origin where its rank is zero. At the origin the coordinates x^i themselves are Casimir functions since they trivially commute by vanishing of the Poisson tensor. Let us therefore focus on $\mathbb{R}^3 - \{0\}$. It should be clear, either by inspection or by solving the differential equations $\{x_i, C(x_k)\} = \{x_i, x_j\}\partial^j C = 0$,

[5] As a reminder: Frobenius' theorem is concerned with the higher dimensional analogue of finding the integral curve of a vector field. Given a subbundle W of the tangent bundle TM of a manifold M, under what condition can the tangent vectors be integrated to yield a submanifold such that its tangent space coincides with the given subbundle W? A necessary and sufficient condition is that the elements of the W involute, i.e., if $X, Y \in W$ then $[X, Y] \in W$ (cf. eg. [89]).

[6] To account for a varying rank of the Poisson tensor over M one has to employ a generalized (singular) version of Frobenius' theorem. A proof for the general case can be found in [83].

that any function $C(r)$ of $r^2 = \sum_{i=1}^{3} x_i x_i$ is a Casimir. Since all prerequisites of theorem 6 are (trivially) met spheres of constant radius $r = const$ provide the symplectic leaves for (M, P). We can verify this explicitly by going to Casimir–Darboux coordinates. If we introduce spherical coordinates (r, θ, φ),then the only non-vanishing Poisson bracket will be

$$\{r \cos\theta, \varphi\} = 1\,. \tag{2.48}$$

For constant r we reproduce the Darboux coordinates on a two-sphere (cf. example 1).

The attentive reader certainly has noticed that example 5 contains a lot more structure than we have actually used. Most importantly, the Poisson algebra (2.47) is related to the Lie algebra $\mathfrak{so}(3)$. This is not a coincidence! *Any* Lie algebra is naturally related to a Poisson algebra with its coadjoint orbits determining the symplectic leaves of the Poisson manifold. The beautiful theory of coadjoint orbits that we are going to develop in the next section finds ample application in physics, as we will see in some of the following chapters.

2.3 Poisson structures and Lie groups

Recall the following facts about Lie groups.[7] We are interested in a *representation* of a Lie group G on a vector space V, i.e., a map $\varphi : G \times V \to V, (g, v) \mapsto gv$ for $g \in G, v \in V$. The pair (V, φ) is called a (real) representation of G on the (real) vector space V. Very often the vector space V is endowed with more structure, e.g., an inner product in which case the representation is called *unitary* if the inner product is invariant under the action of G.

A natural choice for a vector space V to define the representation is the Lie algebra $\mathfrak{g}$ of G. We can define the representation in the following way: for every element $g \in G$ one can define an automorphism of the form $c_g = gxg^{-1}$, where $x \in G$. The differential of c_g at the unit element of G defines an automorphism of the Lie algebra $\mathfrak{g}$, $\mathrm{Ad}_g \in \mathrm{Aut}(\mathfrak{g})$. In other words, for an element $X \in \mathfrak{g}$ we will have

$$X \mapsto \mathrm{Ad}_g(X) = \frac{d}{dt}\big(g \exp(tX) g^{-1}\big)_{t=0}. \tag{2.49}$$

The *adjoint representation* of G is given by the homomorphism

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}), g \mapsto \mathrm{Ad}_g \tag{2.50}$$

The set of elements $\{\mathrm{Ad}_g(X) | g \in G\}$ is called the *adjoint orbit* of G through $X \in \mathfrak{g}$.

The differential of the map $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ at the identity element of G, defines a map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$

$$\mathrm{ad} : X \mapsto [\,\cdot\,, X], \tag{2.51}$$

where $[\cdot, \cdot]$ is the Lie bracket defined on $\mathfrak{g}$. This yields the *adjoint representation* of the Lie algebra $\mathfrak{g}$.

[7] For infinite-dimensional Lie groups, one usually assumes that the group manifold is a Fréchet manifold. This allows to define concepts such as vector fields, tangent spaces, differential forms etc. We will not go into the details of this construction.

Another natural representation of the Lie group G and its Lie algebra $\mathfrak{g}$ is associated with the dual vector space of the Lie algebra, $\mathfrak{g}^*$.[8] Elements of the dual Lie algebra are called (generalized) *moments* in this context, consistent with the use of the word in example 2. Denote the pairing between the Lie algebra and its dual by $\langle\cdot,\cdot\rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$. Then one can define the *coadjoint action* Ad^* of G on $\mathfrak{g}^*$ as

$$\langle \mathrm{Ad}^*_g(\mu), X\rangle := \langle \mu, \mathrm{Ad}_{g^{-1}}(X)\rangle \tag{2.52}$$

for all $\mu \in \mathfrak{g}^*, X \in \mathfrak{g}$. This defines the *coadjoint representation* of the Lie group G on $\mathfrak{g}^*$, the dual of its Lie algebra. Similarly, one can define the *coadjoint action* of the Lie algebra $\mathfrak{g}$, ad^*:$\mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$ for an element $Y \in \mathfrak{g}$

$$\langle \mathrm{ad}^*_Y(\mu), X\rangle := -\langle \mu, \mathrm{ad}_Y(X)\rangle \tag{2.53}$$

as the differential at the identity of (2.52). In general, the adjoint and coadjoint representation of a Lie group will differ. If and only if the Lie algebra admits an invariant, non-degenerate pairing, for instance in the form of a Cartan–Killing metric, $\mathfrak{g}$ and $\mathfrak{g}^*$ can be identified and the representations coincide.

Let $\mu \in \mathfrak{g}^*$ be a moment. Then the set

$$\mathcal{O}_\mu = \{\mathrm{Ad}^*_g(\mu) | \forall g \in G\} \tag{2.54}$$

is called the *coadjoint orbit* of G through μ. The set of elements of G leaving μ invariant, i.e.,

$$G_\mu = \{\mathrm{Ad}^*_g(\mu) = \mu | \forall g \in G\} \tag{2.55}$$

is called the *stabilizer* or *little group* of G at μ. A coadjoint orbit can also be defined intrinsically as $\mathcal{O}_\mu \simeq G/G_\mu$. The tangent space of the orbit $\mathcal{O}_\mu$ consists of elements in $\mathfrak{g}^*$. In particular, one has

$$T_\mu\mathcal{O}_\mu = \{\mathrm{ad}^*_{\mathfrak{g}}(\mu)\}. \tag{2.56}$$

Therefore, all elements $\delta_X\mu$ of the form $\delta_X\mu = \mathrm{ad}^*_X(\mu)$ for some $X \in \mathfrak{g}$ belong to the tangent space; thus, an element of $\mathfrak{g}$ fully specifies an element of $T_\mu\mathcal{O}_\mu$. Defining by

$$T_\mu(G_\mu) \simeq \mathfrak{g}_\mu = \{\mathrm{ad}^*_X(\mu) = 0 | X \in \mathfrak{g}\} \tag{2.57}$$

the *isotropy algebra*, an element $\delta\nu \in T_\mu\mathcal{O}_\mu$ specifies an element in $\mathfrak{g}$ up to an element of $\mathfrak{g}_\mu$. The tangent space T_μ is thus identified with the quotient $T_\mu(G/G_\mu) \simeq \mathfrak{g}/\mathfrak{g}_\mu$.

Let us study the coadjoint orbits of SO(3) as an instructive warm-up exercise.

Example 6 (Coadjoint orbits of SO(3))**.** The group SO(3) is the group of 3×3 orthogonal matrices, i.e., $A^\intercal = A^{-1}$, with determinant one. Its Lie algebra $\mathfrak{so}(3)$ is the three-dimensional vector space of antisymmetric 3×3 matrices which we can identify (using the Euler map) with the vector space $\mathbb{R}^3$ together with the bracket

$$[t_i, t_j] = \varepsilon_{ij}{}^k t_k\,. \tag{2.58}$$

[8]Again, care has to be taken in the infinite-dimensional case since the dual of a Fréchet space is, in general, not a Fréchet space. And again, being physicists we will brush over these details. For details see [81].

of the generators t_i. The Cartan-Killing form

$$\langle t_i, t_j \rangle = -2\delta_{ij} \tag{2.59}$$

provides a non-degenerate metric on $\mathfrak{so}(3)$ that can be used to identify the Lie algebra with its dual $\mathfrak{g}^*$. How does an element $g \in \mathrm{SO}(3)$ act on an element in $\mathfrak{so}(3)^* \simeq (\mathbb{R}^3)^* \simeq \mathbb{R}^3$? It does so by rotation. Given an element $(x^1, x^2, x^3) \in \mathbb{R}^3$ the orbit of the group is a sphere of radius $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$. The coadjoint orbits of SO(3) are therefore spheres of radius r in $\mathbb{R}^3$, apart from the origin that is an orbit by itself as it does not transform under rotations.

Example 7 (Coadjoint orbits of $\mathrm{SL}(2,\mathbb{R})$)**.** The group $\mathrm{SL}(2,\mathbb{R})$ is the group of 2×2 real matrices of unit determinant

$$\mathrm{SL}(2,\mathbb{R}) \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad ad - bc = 1, \qquad a, b, c, d \in \mathbb{R}. \tag{2.60}$$

The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of $\mathrm{SL}(2,\mathbb{R})$ is the set of 2×2 real matrices of zero trace

$$\mathfrak{sl}(2,\mathbb{R}) \ni X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \qquad a, b, c \in \mathbb{R}. \tag{2.61}$$

By equation (2.49) the Lie group $\mathrm{SL}(2,\mathbb{R})$ acts on its algebra by conjugation $X \mapsto \mathrm{Ad}_g(X) = gXg^{-1}$, thus defining the adjoint representation of the group on its algebra. Since $\mathfrak{sl}(2,\mathbb{R})$ is semi-simple there exists an invariant metric that provides an isomorphism between the Lie algebra and its dual. The adjoint and coadjoint representations therefore coincide. All matrices $X \in \mathfrak{sl}(2,\mathbb{R})$ related by $\mathrm{SL}(2,\mathbb{R})$-conjugation lie on the same orbit. Since the determinant of X is unchanged under conjugation the quantity

$$-C = (bc + a^2) \tag{2.62}$$

is an *orbit invariant*. One can distinguish between the following surfaces of constant C in $\mathfrak{sl}(2,\mathbb{R})$:

- $C < 0$: one-sheeted hyperboloids given by $a^2 + bc = |C|$;
- $C > 0$: the two connected components of the two-sheeted hyperboloid $a^2 + bc = -|C|$;
- $C = 0$: the two connected components of the cone $a^2 = -bc$ without the origin;
- the origin $a = b = c = 0$.

The discussion of the coadjoint orbits of SO(3) should be compared with example 5. We see that the coadjoint orbits of SO(3) coincide with the symplectic leaves of the Poisson structure defined in that example. Furthermore, on every coadjoint orbit one finds a natural symplectic structure. Of course, the Poisson structure in example 5 was chosen deliberately so that its symplectic leaves coincide with the coadjoint orbits of the underlying Lie group. It provides the first example of a so-called Lie–Poisson bracket which we are now going to discuss.

Definition 10 (Lie–Poisson or Kirillov–Kostant bracket)**.** Let $\mathfrak{g}^*$ be the dual of a Lie algebra $\mathfrak{g}$ and $f, g \in C^\infty(\mathfrak{g}^*)$. Then the *Lie–Poisson* or *Kirillov–Kostant bracket* on $\mathfrak{g}^*$

$$\{\cdot,\cdot\}_{LP} : C^\infty(\mathfrak{g}^*) \times C^\infty(\mathfrak{g}^*) \to C^\infty(\mathfrak{g}^*) \tag{2.63}$$

is defined as

$$\{f, g\}_{LP}(\mu) := \langle \mu, [d_\mu f, d_\mu g]\rangle \tag{2.64}$$

for any $\mu \in \mathfrak{g}^*$.

The defining equation for the Kirillov–Kostant bracket (2.64) might seem a bit opaque at first sight, thus let us explain it in some detail. The pairing $\langle\cdot,\cdot\rangle$ is the usual pairing between the Lie algebra and its dual. The differential of f at the point $\mu \in \mathfrak{g}^*$ $d_\mu f$, sometimes also written as $\frac{\delta f}{\delta \mu}$ in this context, is an element of the co-tangent space $T^*_\mu \mathfrak{g}^* \simeq (\mathfrak{g}^*)^*$. In finite dimensions this can be identified with $\mathfrak{g}$; otherwise we can always define the functional derivative as

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[F(\mu + \varepsilon\delta\mu) - F(\mu)] =: \langle \delta\mu, \frac{\delta F}{\delta \mu}\rangle\,, \tag{2.65}$$

i.e., $\frac{\delta F}{\delta \mu}$ is defined as an element of $\mathfrak{g}$; although some care has to be taken here in infinite dimensions.

With the bracket (2.64) the pair $(\mathfrak{g}^*, \{\cdot,\cdot\}_{LP})$ defines a Poisson manifold. Properties (1) and (3) of definition 6 are obvious from the antisymmetry of the Lie bracket and the derivation property of d_μ since $d_\mu(fg) = d_\mu f g + f\, d_\mu g$. The only non-trivial check concerns the Jacobi identity.[9]

In order to get a better understanding we are going to evaluate the bracket in the case of a finite-dimensional Lie algebra.

Example 8 (Lie–Poisson bracket for finite-dimensional Lie algebras [87])**.** Let $[t_a, t_b] = f_{ab}{}^c t_c$ be a finite-dimensional Lie algebra $\mathfrak{g}$ with $\{t_1, ..., t_n\}$ being a basis of the vector space. A general element in $\mathfrak{g}$ can then be written as $q = q^a t_a$. We can define a dual basis on $\mathfrak{g}^*$ by $\langle t^a, t_b\rangle = t^a(t_b) = \delta^a{}_b$. Any element of $\mathfrak{g}^*$ can then be expressed as $p = p_a t^a$ and the p_a's serve as coordinates on $\mathfrak{g}^*$. Choosing $\mu = p_a t^a$ we find

$$\{f, g\}(\mu) = \langle p_c t^c, [\frac{\partial f}{\partial p_a} t_a, \frac{\partial g}{\partial p_b} t_b]\rangle = p_c \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial p_b} f_{ab}{}^d \langle t^c, t_d\rangle = f_{ab}{}^c p_c \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial p_b} \tag{2.66}$$

Choosing the coordinate functions p_a and p_b for f and g, respectively, we find

$$\{p_a, p_b\} = f_{ab}{}^c p_c\,. \tag{2.67}$$

We recover the Poisson bracket (2.47) of example 5 if we choose for $f_{ab}{}^c$ the structure functions of $\mathfrak{so}(3)$.

[9] Instead of just defining the Poisson bracket on $\mathfrak{g}^*$ to be the Kirillov–Kostant bracket one can also derive it by reduction. In example 2 of section 2.1 it was shown that there is a natural symplectic structure, and therefore also a natural Poisson structure, on every cotangent bundle. In particular, there is a natural Poisson structure on the cotangent bundle of a Lie group T^*G. The quotient of the cotangent bundle by the Lie group can be identified with $\mathfrak{g}^*$, $T^*G/G \simeq (G \times \mathfrak{g}^*)/G \simeq \mathfrak{g}^*$. Thus, there is a natural projection $\pi : T^*G \to T^*G/G$ and a unique Poisson structure on $T^*G/G \simeq \mathfrak{g}^*$ such that π is canonical. This defines the Kirillov–Kostant bracket.

Notice that this implies that any (finite-dimensional) linear Poisson tensor can be interpreted as the Kirillov–Kostant bracket of some Lie algebra. Vice versa any (finite-dimensional) Lie algebra leads to a linear Kirillov–Kostant bracket via (2.67). Under suitable conditions, including a definition what is meant by a linear Poisson structure, this is true even in the infinite-dimensional case.

The Hamiltonian vector fields at a point $\mu \in (\mathfrak{g}^*, \{\cdot,\cdot\}_{LG})$ corresponding to a Hamiltonian function $H \in C^\infty(\mathfrak{g}^*)$ can be found using (2.32)

$$\pounds_{X_H} g(\mu) = \{g, H\}_{LP}(p) = \langle \mu, [d_\mu g, d_\mu H]\rangle = -\langle \mu, \mathrm{ad}_{d_\mu H}(d_\mu g)\rangle = \langle \mathrm{ad}^*_{d_\mu H}(\mu), d_\mu g\rangle \,. \quad (2.68)$$

Since we have $\pounds_{X_H} g(\mu) = \langle X_H, d_\mu g\rangle$ by definition of the Lie derivative we find

$$X_H = \mathrm{ad}^*_{d_\mu H}(\mu)\,. \quad (2.69)$$

As a last point we want to establish the equivalence between symplectic leaves of the Kirillov–Kostant bracket and the coadjoint orbits of the Lie group G, that we saw in the example of the Lie group SO(3).

Let $\mathcal{O}_\mu$ denote the coadjoint orbit of G through $\mu \in \mathfrak{g}^*$, explicitly given by $\mathcal{O}_\mu = \mathrm{Ad}^*_G(\mu)$. The tangent space of this orbit at μ is therefore $T_\mu \mathcal{O}_\mu = \mathrm{ad}^*_{\mathfrak{g}}(\mu)$. Notice that any element $v \in \mathfrak{g}$ can always be written as $v = d_\mu H$ by choosing an appropriate function H. Thus, all vectors in $\mathrm{ad}^*_{\mathfrak{g}}(\mu)$ can be obtained as Hamiltonian vectors. The tangent space $T_\mu \mathcal{O}_\mu$ is therefore spanned by all Hamiltonian vectors. But this is the defining property of a symplectic leaf! We have therefore established.

Theorem 7 (Symplectic leaves are coadjoint orbits)**.** *The symplectic leaves of the Kirillov–Kostant bracket $\{\cdot,\cdot\}$ on $\mathfrak{g}^*$ coincide with the coadjoint orbits of G. This implies that all finite-dimensional coadjoint orbits are of even dimension.*

This theorem is remarkable in two ways. On one hand, it provides a very efficient way of finding the symplectic leaves for a (possibly infinite-dimensional) Poisson manifold. This problem is solved by giving the symplectic leaves a nice geometric interpretation. On the other hand, it shows that coadjoint orbits, objects that we meet quite often in physics, come with a natural non-degenerate Poisson structure, and thus a natural symplectic structure.

Theorem 8 (Kirillov–Kostant–Souriau symplectic structure)**.** *Let G be a Lie group and let $\mathcal{O} \in \mathfrak{g}^*$ be a coadjoint orbit. Then for any point $\mu \in \mathfrak{g}^*$ and any two tangent vectors $\delta\nu_X, \delta\nu_Y \in \mathfrak{g}^*$*

$$\Omega(\mu)(\delta\nu_X, \delta\nu_Y) = \langle \mu, [X, Y]\rangle \quad (2.70)$$

defines a symplectic structure. The symplectic structure is invariant under G.

We will return to the subject of coadjoint orbits in section 4.4 where we will discuss them in the case of a particular infinite-dimensional group, and section 7.4 where we will discuss the boundary action for certain two-dimensional theories of gravity.

This concludes our short review on symplectic geometry. Due to lack of space and time we are unable to discuss further interesting issues such as moment maps or geometric quantization of symplectic manifolds; consult references [84, 85, 87, 90] for further details. We will now turn to the discussion of two-dimensional gravity theories where we will see some of the material of this chapter at work.

Chapter Three

Gravity in Two Dimensions

Constructing a theory of gravity in two dimensions is not as straightforward as it might seem. In three dimensions the Einstein–Hilbert action yields a well-defined theory of gravity that exhibits a surprisingly rich structure. In two dimensions this naïve approach does not lead to a well-defined theory, as we will show in the following.

Consider the Einstein–Hilbert action with Gibbons–Hawking–York boundary term on a pseudo-Riemannian manifold $(\mathcal{M}, g)$ in two dimensions

$$I_{Ein} = \frac{1}{16\pi G}\int_{\mathcal{M}} d^2x\,\sqrt{|g|}\,R + \frac{1}{8\pi G}\int_{\partial\mathcal{M}}\sqrt{\gamma}K. \tag{3.1}$$

Varying this action with respect to the metric ones finds

$$\delta I_{Ein} = \frac{1}{16\pi G}\int_{\mathcal{M}} d^2x\,\sqrt{|g|}\,(R^{ab} - \frac{1}{2}Rg^{ab})\delta g_{ab}, \tag{3.2}$$

due to the fact that $K_{ab} = \gamma_{ab}K$, where γ_{ab} is the induced metric on the one-dimensional boundary. However, in two dimensions the Riemann tensor R_{abcd} is uniquely determined by the Ricci scalar R

$$R_{abcd} = \frac{R}{2}(g_{ac}g_{bd} - g_{ad}g_{bc})\,, \tag{3.3}$$

and thus the Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab}$ vanishes identically. Therefore, *any* two-dimensional metric is a solution of the theory defined by the action principle (3.1).

Similarly, one can come to the same conclusion using the famous Gauss–Bonnet theorem. The original theorem is, strictly speaking, true only for Riemannian manifolds [91] but there exist various generalizations to pseudo-Riemannian manifolds [92, 93]. In the version of [93]: Consider a region D of a two-dimensional space-time with Lorentzian metric bounded by a piece-wise smooth boundary ∂D consisting of non-null smooth curves with exterior angles[1] θ_i, then

$$-\int_{\partial D} dsK - \frac{1}{2}\int_D R + \sum_i \theta_i = 2\pi i\,. \tag{3.4}$$

[1] Formulations of the theorem differ in the definition of the exterior angles, which influences the right-hand side of the theorem. In this form of the theorem, angles can become imaginary that cancel the imaginary contributions from the right-hand side.

Thus, the Einstein–Hilbert action with Gibbons–Hawking–York boundary term can be rewritten, in any region D, as a *number* that depends only on the boundary of D.

Although one might consider a theory of this type it is not expected to exhibit a rich physical structure. Yet, the very point of examining gravity in two dimensions is the study of phenomena such as singularities, black holes and their formation, Hawking radiation, and holography which the Einstein–Hilbert action (3.1) is unable to capture. Since the appearance of the above phenomena is largely independent of the precise form of the gravitational theory one should turn to theories with more structure than (3.1).

The outline of this chapter is as follows: In section 3.1 we are going to define two-dimensional dilaton gravity. We are going to show that various ways to obtain a two-dimensional theory of gravity all lead to some sort of dilaton gravity which warrants its study. In section 3.2 we are going to motivate a reformulation of dilaton gravity in first order form as a particular kind of non-linear gauge theory called Poisson sigma model (PSM). Section 3.3 studies properties of PSMs and shows the equivalence of dilaton gravity to a certain subclass of these. In section 3.5 we are going to study the canonical charges of dilaton gravity both in the PSM formulation and in the second order formulation.

Although we will be exclusively concerned with the Euclidean theory in the remaining parts of this thesis, in this chapter we will treat metrics of both Lorentzian ($\sigma = -1$) and Euclidean ($\sigma = 1$) signature.

3.1 Why dilaton gravity?

There are various ways to motivate more general gravitational theories in two dimensions. Interestingly, most of these turn out to be equivalent to so-called *generalized two-dimensional dilaton gravity* that includes a scalar field in addition to the metric. The action of this theory, that will accompany us for the rest of this thesis, is given by

$$I = -\frac{\sigma}{16\pi G}\int_{\mathcal{M}} d^2x\sqrt{\sigma g}\Big(XR - \sigma U(X)(\nabla X)^2 - 2V(X)\Big)\,. \tag{3.5}$$

The scalar field X is called *dilaton*. The functions $U(X), V(X)$ determine the specific model. The equations of motion following from (3.5) are

$$\sigma(U\nabla_\mu X\nabla_\nu X - \tfrac{1}{2}g_{\mu\nu}U(\nabla X)^2) - g_{\mu\nu}V + \nabla_\mu\nabla_\nu X - g_{\mu\nu}\nabla^2 X = 0 \tag{3.6a}$$

$$R + \sigma\left(\partial_X U(\nabla X)^2 + 2U\,\nabla^2 X\right) - 2\partial_X V = 0\,. \tag{3.6b}$$

At first sight the differential equations (3.6) look intimidating. However, we have a lot of symmetries available in two dimensions that can be used to solve the system analytically. Since this thesis is mostly concerned with a gauge-theoretic formulation of dilaton gravity, to be introduced in the next section, we will not discuss the second order action in more details.

There is one important conclusion that can be drawn without calculation. Regardless of the specific form, every dilaton gravity model has no local degrees of freedom. This follows from a simple counting argument. The freedom to choose two coordinates, guaranteed by the obvious diffeomorphism invariance of the action (3.5), allows to eliminate two of the three components of the symmetric tensor field $g_{\mu\nu}$. Together with the single degree of freedom of the scalar field X, we thus obtain two off-shell degrees of freedom. Turning now to the equations of motion, the analogue of the Bianchi identities, stemming from the diffeomorphism invariance of the theory, imposes two conditions on the system (3.6) that are associated to two constraints thus leaving us with *zero degrees of freedom* on the constraint surface.[2] An ADM split of dilaton gravity in the second order formulation is performed in appendix A. Despite having no propagating degrees of freedom the theory allows for interesting phenomena, at least interesting enough, if you believe the biased author, to warrant the read of the remaining parts of this thesis.

Dilaton gravity is an example of a *scalar-tensor theory*; the prime example for such theories is *Jordan–Brans–Dicke theory* in four dimensions [94].[3] While the geometric properties of a solution to a scalar-tensor theory are still governed by the metric [for instance, the (weak) equivalence principle is still valid since there always exists a local reference frame in which the metric is flat] the effective coupling strength of gravity is no longer given by Newton's constant G but by the combination X/G that can vary from place to place.[4]

Let us comment on a source of confusion regarding the form of the action (3.5). In a higher-dimensional context, an action in which the Ricci scalar couples to a scalar through $\sqrt{-g}\,X\,R$ as in (3.5) is said to be written in the *Jordan frame*. It can be transformed to the *Einstein frame* by an X-dependent conformal transformation so that the coupling of the Ricci scalar is given by $\sqrt{-\tilde{g}}\tilde{R}$ with

$$\widetilde{g}_{\mu\nu} = X^{-\frac{2}{D-2}} g_{\mu\nu}\,. \tag{3.7}$$

Under such a transformation the Ricci scalar transforms as

$$\widetilde{R} = X^{\frac{2}{D-2}} \left(R + 2\frac{D-1}{D-2} g^{\mu\nu}\nabla_\mu\nabla_\nu \ln X - \frac{D-1}{D-2}(\nabla \ln X)^2 \right), \tag{3.8}$$

which yields the wanted result in higher dimensions. This transformation is not possible for $D = 2$. Thus there exists no Einstein frame for dilaton gravity in two dimensions.

In the following we will list various ways to construct two-dimensional theories of gravity and we will find that all of them reduce to a theory of the form (3.5). We will be brief on the various points; more details can be found in, e.g., [99, 100].

Spherical reduction. Consider a D-dimensional manifold with metric $g^{(D)}$ that is spherically symmetric, i.e., its isometry group contains an $\mathrm{SO}(D-1)$ subgroup, the orbits of which are $(D-2)$-dimensional spheres. In a coordinate system adapted to this symmetry the metric can be written as

$$ds^2 = g^{(D)}_{\mu\nu}\, dx^\mu\, dx^\nu = g^{(2)}_{\alpha\beta}(x^\alpha)\, dx^\alpha\, dx^\beta + \Phi^2(x^\alpha)\, d\Omega^2_{(D-2)}\,, \tag{3.9}$$

where $g^{(2)}_{\alpha\beta}$ is a two-dimensional metric, and $d\Omega^2_{(D-2)}$ is the metric of the $D-2$ sphere. Due to spherical symmetry the metric component Φ depends on x^α only. A straightforward

[2] Using the slogan that "the gauge always strikes twice" we see that the four free functions contained in $g_{\mu\nu}, X$ are killed by the two diffeomorphisms.

[3] The two-dimensional counterpart of this theory corresponds to the choice $U(X) = \omega X^{-1}, V(X) = 0$ in (3.5).

[4] Scalar tensor theories are thus more in line with Mach's principle, which was the original motivation of Brans and Dicke. Having said that, these theories are now under great strain due to the joint detection of gravitational waves and light from the neutron star merger GW170817 [95–98].

calculation allows to express the Ricci tensor of the D-dimensional metric in terms of the Ricci tensor of the two-dimensional metric plus additional terms depending on the metric component Φ. Introducing

$$\Phi = \lambda^{-1} X^{\frac{1}{D-2}} \tag{3.10}$$

and regarding X as an independent scalar field of the theory, the D-dimensional Einstein–Hilbert action becomes (3.5) with

$$U(X) \propto \frac{D-3}{D-2}\,\frac{1}{X}\,, \qquad V(X) \propto (D-2)(D-3)\,X^{\frac{D-4}{D-2}} \tag{3.11}$$

after a trivial integration over the $D-2$ sphere. The constant λ is a parameter of mass dimension one that can be thought of as length scale of the compactification. Since the s-wave sector of Einstein gravity has no local degrees of freedom (there are no gravitational s-waves), spherical reduction necessarily leads to a theory without propagating degrees of freedom, which is consistent with the above counting argument.

$f(R)$ **theories.** We saw above that the Einstein–Hilbert action in two dimensions is too naïve a guess for a theory of gravity. A natural generalization of (3.1) is to consider the class of actions

$$I = \frac{1}{16\pi G} \int_{\mathcal{M}} d^2x\, \sqrt{|g|} f(R)\,, \tag{3.12}$$

where f is, in general, an arbitrary function of R. Define the quantity $X = f'(R)$ and assume that this relation is invertible such that $R = R(X)$. Introducing

$$V(X) = f(R(X)) - R(X)X\,, \tag{3.13}$$

the action (3.12) is equivalent to

$$I = \frac{1}{16\pi G} \int_{\mathcal{M}} d^2x\, \sqrt{|g|}(XR - V(X)) \tag{3.14}$$

upon integrating out the auxiliary field X, and thus to the generalized dilaton model (3.5). For further discussions of two-dimensional actions of the form (3.12) and the issue of invertibility of the relation $X = f'(R)$ see [101–103].

Gauge theory of Gravity. Despite numerous attempts Einstein gravity in four dimensions cannot be formulated as gauge theory of the Poincaré group ISO(3, 1); on the contrary, Einstein gravity in three dimensions allows for such a formulation. Similarly, gauging the two-dimensional Poincaré or AdS group, ISO(1, 1) or SO(2, 1), respectively, yields well-defined, non-trivial theories of two-dimensional gravity that have a second-order formulation falling into the class of models (3.5). These models were among the first instances of theories of that type [104–106]. Since large parts of this thesis deal with this formulation of two-dimensional dilaton gravity we defer details to later chapters.

String theory. The non-linear sigma model

$$I = \frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d^2\sigma \Big(\partial_\mu X^I \partial_\nu X^J (\sqrt{-h}\, g_{IJ} h^{\mu\nu} + \epsilon^{\mu\nu} B_{IJ}) + \alpha' \sqrt{-h} R^{(2)} \Phi\Big) \tag{3.15}$$

describes a string in a curved background geometry under the influence of a B-field. Here $h_{\mu\nu}$ is a metric on the world-sheet parametrized by the coordinates σ^μ, X^I are coordinates on the D-dimensional target space Σ that is a (pseudo-) Riemannian manifold with metric g_{IJ} and two-form gauge field B_{IJ}; the Ricci scalar of the world sheet $R^{(2)}$ couples to the dilaton Φ. The quantities g_{IJ}, B_{IJ}, and Φ are coupling constants from the point of view of the world-sheet theory. The action (3.15) is conformal invariant only if the beta functions associated with these coupling constants, that can be calculated perturbatively in α', vanish. As is well-known [107], it is possible to write down an effective action in the D-dimensional target space that encodes the vanishing of these beta functions as equations of motion. Setting the B-field to zero this target space action is given by

$$I = \int_\Sigma d^D x \sqrt{-g} e^{-2\Phi} \left(R + 4(\nabla\Phi)^2 + \frac{D-26}{3\alpha'}\right). \tag{3.16}$$

In the case of a two-dimensional target space $D = 2$ this becomes a model of the form (3.5) under the identification $X = e^{-2\Phi}$. This model describes the *Witten black hole* that produces to lowest order in α' the target space geometry of the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ coset model [108].[5] It is also the (conformally transformed) gravity part of the *Callan–Giddings–Harvey–Strominger (CGHS) model* that was seminal in the study of two-dimensional black hole evaporation [111].

This concludes our short tour of various approaches to two-dimensional theories of gravity. A bestiarium of dilaton gravity theories can be found in [112, 113]. We turn now to the gauge-theoretic formulation of dilaton gravity in which most of the original results of this thesis will be developed.

3.2 Dilaton gravity as a gauge theory

The purpose of this section is to motivate a gauge-theoretic formulation of dilaton gravity. In the same way as the Chern-Simons formulation of three-dimensional gravity [114, 115] enriches and, in many cases, simplifies the discussion of Einstein gravity in three dimensions, a reformulation of dilaton gravity as a topological quantum field theory is expected to do the same. It turns out that two-dimensional dilaton gravity models can be regarded as particular instances of so-called *Poisson sigma models* (PSMs) [116, 117]. While we are going to show their classical equivalence in the next section here we will try to motivate how these models arise. Readers not interested in heuristics are welcome to skip to section 3.3 where the classical equivalence of dilaton gravity and PSM models is shown explicitly.

[5] A solution for the target space geometry to all orders in α' was given in [109]. An action for this exact string black hole was provided in [110].

Gauge theories of gravity

Let us step back for a moment and remind ourselves what we mean by a gauge-theoretic formulation of gravity. In the past there have been many attempts to formulate Einstein gravity as a gauge theory. Naïvely one would expect that the theory can be written as the gauge theory of the D-dimensional Poincaré group $\mathrm{ISO}(D-1,1)$ starting from the first-order formulation of the theory. Indeed, taking the vielbein e^a and the spin-connection $\omega^a{}_b$ one can construct a gauge connection A as

$$A = e^a P_a + \frac{1}{2}\omega^{ab} J_{ab}\,. \tag{3.17}$$

Here P_a and J_{ab} are generators of the Poincaré algebra. Under a gauge transformation $\delta_\lambda A = d\lambda + [A, \lambda]$ with the parameter $\lambda = \lambda^a P_a + \frac{1}{2}\lambda^{ab} J_{ab}$ the components of the connection transform as

$$\delta_\lambda e^a = d\lambda^a + \lambda^a{}_b e^b \qquad \delta_\lambda \omega^{ab} = (d\lambda^{ab} + \omega^a{}_c \lambda^{cb} + \omega^b{}_c \lambda^{ac})\,. \tag{3.18}$$

On the other hand, under infinitesimal diffeomorphisms generated by the vector field ξ the fields e_a and $\omega^a{}_b$ transform as

$$\delta_\xi e^a = d(\xi \cdot e^a) + \xi \cdot de^a \qquad \delta_\xi \omega^{ab} = d(\xi \cdot \omega^{ab}) + \xi \cdot d\omega^{ab}\,. \tag{3.19}$$

At first glance this is very different from the transformation behavior given in (3.18). However, setting $\lambda^a = \xi \cdot e^a$ one sees that a gauge transformation of A becomes a combination of diffeomorphisms and local Lorentz transformations if the torsion $T^a \equiv de^a + \omega^a{}_b \wedge e^b$ vanishes. Thus, if it is possible to construct a variational principle for A from which vanishing torsion, $T^a = 0$, follows, one would have established an on-shell equivalence between the two symmetries (3.18) and (3.19). Yet, the crucial point is that it is impossible to construct a gauge-invariant variational principle for the connection A with gauge group $\mathrm{ISO}(D-1,1)$ unless $D \neq 3$. This is related to the fact that its Lie algebra $\mathfrak{iso}(D-1,1)$ is not semi-simple and thus does not come with a non-degenerate invariant metric.

The situation is very different in three dimensions where it is possible to write down the Chern–Simons action

$$I_{CS} = \frac{k_{\mathrm{CS}}}{2\pi} \int_{\mathcal{M}} \langle A \wedge (dA + \frac{2}{3}[A, A]) \rangle \tag{3.20}$$

for the gauge connection (3.17), where $\langle \cdot, \cdot \rangle$ denotes the trace using the non-degenerate invariant metric [114,115]. The decisive difference to the four-dimensional case is precisely the existence of this metric.[6] If in the Einstein–Hilbert action a positive or negative cosmological constant is included, the gauge group changes to $\mathrm{SO}(3,1)$ or $\mathrm{SO}(2,2)$, respectively. The coupling constant k is inversely proportional to Newton's constant. Similar constructions are also available in higher odd dimensions, see [121] for a pedagogic introduction into this topic.

After this longer digression let us return to two-dimensional dilaton gravity and try to see what kind of gauge theory we expect. We saw already in the last chapter that two-dimensional dilaton gravity in the second order formulation does not have any propagating

[6] Although $\mathfrak{iso}(2,1)$ is not semi-simple it is a *double extension* of the trivial Lie algebra by $\mathfrak{so}(2,1)$. It is possible to show that double extensions allow for a non-degenerate invariant metric [118] (under mild conditions presented in [119]). For a pedagogic exposition to double extensions, cf. e.g. [120].

degrees of freedom; this should be true as well for its gauge-theoretic description if the theories are supposed to be equivalent. This suggests that it will be described in terms of a diffeomorphism invariant *topological field theory*, cf. [122] for an extensive review. As in (3.17) we will assume that zweibein e^a and spin connection ω^{ab}, having one independent component only in two dimensions, are combined in a gauge connection A. In addition to the connection we expect to have a scalar field that is related to the dilaton. In fact, we will see that it is more convenient to consider a theory of more scalar fields, only one of which is identified with the dilaton.

Since we want the whole geometric structure on the spacetime manifold $\mathcal{M}$ to be described by the connection A, the action should have no explicit dependence on a metric on $\mathcal{M}$. According to the classification in [122] we are thus looking for a *topological theory of Schwarz type*, that is constructed from scalar fields and a connection.

The simplest action one can construct that obeys these requirements is given by

$$S_0 = \frac{k}{2\pi}\int_{\mathcal{M}} A_I \wedge dX^I = -\frac{k}{2\pi}\int_{\mathcal{M}} X^I F_I \qquad F_I = dA_I\,. \tag{3.21}$$

Here we dropped a boundary term and assumed that $I = 1, 2, \ldots, D$ although in the case relevant to gravity we will always have $D = 3$. In analogy to the Chern–Simons action (3.20) we have introduced a coupling constant k. This action is called Abelian BF action [123]. It is invariant under the gauge transformations

$$\delta_0 A_I = d\epsilon_I, \qquad \delta_0 X^I = 0\,. \tag{3.22}$$

On first sight neither the action nor the gauge symmetries have anything to do with two-dimensional dilaton gravity or diffeomorphism invariance. And indeed, they have not, apart from the field content and the correct number of gauge symmetries (two diffeomorphism plus one local Lorentz transformation for $D = 3$). In the three-dimensional case of Chern–Simons theory this would correspond to having only the first term in (3.20), i.e., a sum of Abelian Chern–Simons terms. We would like to add interactions to (3.21) that keep the field content and the number of gauge symmetries but change the form of the latter such that the resulting action is still gauge invariant. Luckily, this question can be systematically studied using the tools of the BV/BRST formalism [124, 125]. In the three dimensional case this process singles out non-Abelian Chern–Simons theory (3.20) as the unique consistent deformation of Abelian Chern–Simons theory [124]. Similarly, Yang–Mills theory in four dimensions is the unique consistent deformation of an Abelian gauge theory [126]. In the following we want to show starting from (3.21) that this process will lead us to Poisson sigma models [127] a subclass of which will describe two-dimensional dilaton gravity.

PSMs from non-linear deformation theory

We are now interested in the following problem. Can we add interaction terms S_1 to the action S_0 and simultaneously deform the gauge symmetry to $\delta = \delta_0 + \delta_1$ such that the resulting action $S = S_0 + S_1$ is gauge-invariant $\delta S = 0$ and the gauge symmetries close as an algebra? We require that the resulting action be diffeomorphism invariant and local in the fields. Obviously, we are not interested in trivial deformations that can be obtained by local redefinitions of the fields. The general principles of this construction will be explained using the example at hand; the interested reader is referred to [124, 125], conventions follow [128]. Detailed introductions to the BV formalism can be found in [129, 130].

Let us start by determining the field content, including ghosts and antifields, for the BF action (3.21). For every gauge parameter ϵ_I we introduce a Grassmann-odd ghost field of ghost number one. Together with the original fields A_I, X^I we thus have

$$\mathrm{gh}(A_I) = 0 \qquad \mathrm{gh}(X^I) = 0 \qquad \mathrm{gh}(C_I) = 1 \tag{3.23}$$

$$\deg(A_I) = 1 \qquad \deg(X^I) = 0 \qquad \deg(C_I) = 0\,, \tag{3.24}$$

with deg denoting form degree and gh denoting ghost number. The Grassmann parity of a field is even (odd) for ghost number even (odd). Antifields are now introduced according to

$$\mathrm{gh}({}^*\!A^I) = -1 \qquad \mathrm{gh}({}^*X_I) = -1 \qquad \mathrm{gh}({}^*C^I) = -2 \tag{3.25}$$

$$\deg({}^*\!A^I) = 1 \qquad \deg({}^*X_I) = 2 \qquad \deg({}^*C^I) = 2\,. \tag{3.26}$$

The antibracket between two functionals F, G of the fields is defined as

$$(F,G) = \frac{F\overleftarrow{\partial}}{\partial\Phi}\frac{\overrightarrow{\partial}G}{\partial\,{}^*\Phi} - (-1)^{\deg\Phi}\frac{F\overleftarrow{\partial}}{\partial\,{}^*\Phi}\frac{\overrightarrow{\partial}G}{\partial\Phi} \tag{3.27}$$

where Φ and ${}^*\Phi$ collectively denote fields and antifields, respectively. The classical master equation for the Abelian BF action

$$(S^{(0)}, S^{(0)}) = 0 \tag{3.28}$$

has the solution

$$S^{(0)} = S_0 + \frac{k}{2\pi}\int_{\mathcal{M}} {}^*\!A^I \wedge dC_I = \frac{k}{2\pi}\int_{\mathcal{M}} \big(A_I \wedge dX^I + {}^*\!A^I \wedge dC_I\big)\,. \tag{3.29}$$

This equation holds the full information of the gauge-invariance of the original action S_0. In particular, BV transformations are now defined by

$$s^{(0)}F = (F, S^{(0)})\,. \tag{3.30}$$

From this we find

$$s^{(0)}A_I = dC_I \qquad s^{(0)}X_I = 0 \qquad s^{(0)}C_I = 0 \tag{3.31}$$

$$s^{(0)}\,{}^*\!A^I = dX^I \qquad s^{(0)}\,{}^*C_I = -\,dA_I \qquad s^{(0)}\,{}^*C^I = d\,{}^*\!A^I\,, \tag{3.32}$$

which reproduces the gauge transformations (3.22) under the replacement $C_I \to \epsilon_I$. It is clear from these transformations that $(s^{(0)})^2 = 0$.

We are now looking for a deformation of this action, i.e., introducing a deformation parameter g we will study an action of the form

$$S = S^{(0)} + gS^{(1)} + g^2S^{(2)} + \dots\,, \tag{3.33}$$

that will generate a BV transformation

$$sF = (F,S) = (F,S^{(0)}) + g(F,S^{(1)}) + O(g^2) = s^{(0)}F + gs^{(1)}F + O(g^2)\,. \tag{3.34}$$

The deformed action itself has to satisfy

$$sS = (S, S) \tag{3.35}$$

in order to be invariant under the BV transformations.

We can solve this order by order in g

$$(S, S) = (S^{(0)}, S^{(0)}) + 2g(S^{(1)}, S^{(0)}) + g^2[(S^{(1)}, S^{(1)}) + 2(S^{(2)}, S^{(0)})] + O(g^3)\,. \tag{3.36}$$

The first term in the series is the classical master equation for the action $S^{(0)}$ which is solved by (3.29). By our general requirements $S^{(1)}$ has to be local and diffeomorphism invariant, thus we have $s^{(0)}S^{(1)} = s^{(0)}\int \mathcal{L}^{(1)} = 0$ from the second term. From this we find that $s^{(0)}\mathcal{L}^{(1)}$ has to vanish up to a possible boundary term

$$s^{(0)}\mathcal{L}^{(1)} + da^{(1)} = 0\,, \tag{3.37}$$

where $a^{(1)}$ is a one-form. Acting on (3.37) repeatedly with $s^{(0)}$ and using $(s^{(0)})^2 = s\,d + ds = 0$ one is led to the set of *descent equations*

$$s^{(0)}a^{(1)} + da^{(0)} = 0 \qquad s^{(0)}a^{(0)} = 0\,, \tag{3.38}$$

with $a^{(0)}$ a zero-form. We can solve this system from the bottom up. The term $a^{(0)}$ is a form of degree zero and ghost number two. The only diffeomorphism invariant term, up to $s^{(0)}$-exact terms that solves equation (3.38) must be built from the X^I, C_I as

$$a^{(0)} = -\frac{1}{2}P^{IJ}(X)\,C_I\,C_J \qquad P^{IJ} = -P^{JI}\,, \tag{3.39}$$

where $P^{IJ}(X)$ is an arbitrary function of X^I. The antisymmetry follows from the anti-commuting nature of the ghosts C_I. The next term $a^{(1)}$ in the series is then given by

$$a^{(1)} = \frac{1}{2}\partial_K P^{IJ}\,{}^*\!A^K C_I C_J + P^{IJ} A_I C_J\,, \tag{3.40}$$

from which we are finally able to determine $\mathcal{L}^{(1)}$ and thus, up to $s^{(0)}$-exact terms $S^{(1)}$. The resulting action $S = S^{(0)} + S^{(1)}$ is given by

$$\begin{aligned} S = \frac{k}{2\pi}\int_{\mathcal{M}} \Big[& A_I \wedge dX^I + {}^*\!A^I \wedge dC_I - \frac{1}{4}\partial_K\partial_M P^{IJ}\,{}^*\!A^K \wedge {}^*\!A^M\,C_I C_J \\ & - \partial_K P^{IJ}({}^*\!A^K \wedge A_I C_J + \frac{1}{2}\,{}^*C^K C_I C_J) + P^{IJ}({}^*X_I C_J + \frac{1}{2}A_I \wedge A_J)\Big]\,, \end{aligned} \tag{3.41}$$

where g was absorbed in P^{IJ}. Now consider the terms proportional to g^2 in (3.36). The second term leads to a set of equation identical to (3.38) and, consequently, to solutions of the same form. This means that $S^{(2)}$ can be reabsorbed by changing $S^{(1)} \to S^{(1)} + gS^{(2)}$. Similarly, one can redefine $S^{(1)}$ such that $S^{(i)} = 0, i \geq 2$. The only remaining condition is then

$$(S^{(1)}, S^{(1)}) = 0\,. \tag{3.42}$$

A straightforward, though tedious, calculation shows that this is equivalent to the condition

$$\circlearrowleft_{IJK} P^{IL}\partial_L P^{JK} = 0\,. \tag{3.43}$$

Notice that this is identical to (2.31) that is, together with antisymmetry, the defining condition of a Poisson tensor. In a moment we will see that, indeed, the quantity P^{IJ} allows for such an interpretation.

The action (3.41) together with condition (3.43) on P^{IJ} is thus the most general consistent deformation of the Abelian BF action (3.21). Setting the antifields to zero we arrive at the action

$$I_{\text{PSM}} = \frac{k}{2\pi}\int_{\mathcal{M}}\Big(A_I\wedge dX^I + \frac{1}{2}P^{IJ}A_I\wedge A_J\Big). \tag{3.44}$$

This action is called *Poisson sigma model* (PSM). Based on the arguments put forth in the previous section we expect a subclass of PSMs to be equivalent to dilaton gravity; we will show this explicitly in 3.4 in a straightforward calculation. The next section is concerned with a detailed investigation of (3.44) to get acquainted with our new-found fellow that will accompany us for the rest of this thesis.

3.3 Properties of PSMs

The end of last section saw our new companion (3.44) arising as a ghostly deformation of the Abelian BF action (3.21). Here, we want to provide a different interpretation for the action (3.44), in particular we want to emphasize the fact that it is a sigma model.

Let $\mathcal{M}$ be a two-dimensional manifold called *worldsheet* or *base space* with coordinates x^μ and Σ a D-dimensional manifold called *target space* with coordinates X^I. We will take the latter as dynamical fields of the theory, i.e., we have a map from the worldsheet to the target space: $x^\mu \mapsto X^I(x^\mu)$. If both $\mathcal{M}$ and Σ are (pseudo-)Riemannian manifolds with metric $\gamma_{\mu\nu}$ and g_{IJ}, respectively, one can define the well-known Polyakov action

$$I_P = \int_{\mathcal{M}} d^2x\sqrt{|\gamma|}\gamma^{\mu\nu}\partial_\mu X^I\partial_\nu X^J g_{IJ}\,, \tag{3.45}$$

which was encountered above as the starting point for string theory.

For our purposes we do not assume that $\mathcal{M}$ and Σ come equipped with a metric, rather Σ is taken to be a Poisson manifold (Σ, P) with Poisson tensor $P = \frac{1}{2}P^{IJ}\frac{\partial}{\partial X^I}\wedge\frac{\partial}{\partial X^J} \equiv \frac{1}{2}P^{IJ}\partial_I\wedge\partial_J$. Without further structure, i.e., without introducing a metric on the world-sheet, we are not able to construct a theory that is invariant under target space diffeomorphisms from P^{IJ} and X^I alone. The need for further structure can be bypassed by introducing an additional field $A = A_I\,dX^I$ that is both a one-form on the worldsheet and the pullback of a section of $T^*\Sigma$ by the map $X(x)$. In particular, we have $A = A_I\,dX^I = A_{I\mu}\,dx^\mu\wedge dX^I = A_{I\mu}\partial_\nu X^I\,dx^\mu\wedge dx^\nu$. We can use the field A to saturate the indices on P, so that we obtain the action (3.44).[7]

Since the time of their inception in [116, 117] PSM models have garnered great interest in the mathematical physics community. Even more so after their connection to Kontsevich's

[7]In the following we assume that $\Sigma \simeq \mathbb{R}^D$ which is the case relevant to our applications. For topologically non-trivial target spaces various issues concerning global well-definedness of gauge transformations and equations of motion arise; cf. [131, 132] for discussions on this point.

formality theorem [133] was shown in [134]. The work of Kontsevich is concerned with deformation quantization of arbitrary Poisson manifolds. Deformation quantization is a formalization of the concept of quantization in the following sense: given a Poisson manifold (M, P) one is looking for an associative product $*$ (the "star product") such that for $f, g \in C^\infty(M)$ one has

$$f * g = fg + \frac{i\hbar}{2}\{f, g\} + O(\hbar^2)\,. \tag{3.46}$$

The classical algebra of observables $C^\infty(M)$ is deformed to a non-commutative "quantum algebra" of observables; focusing on observables it is therefore related to the Heisenberg picture of quantum mechanics.[8] Kontsevich proved that such a star product can be constructed perturbatively in $\hbar$ for every Poisson manifold. In [134] it was shown that this perturbation series corresponds to the perturbative expansion of the path integral of the PSM model on a disk, having (M, P) as a target space. The formula for the star product (3.46) is therefore written as

$$(f * g)(x) = \int_{X(\infty)=x} \mathcal{D}X\,\mathcal{D}A\, f(X(0))\, g(X(1)) e^{\frac{i}{\hbar} I_{\mathrm{PSM}}[X,A]}, \tag{3.47}$$

where $0, 1, \infty$ are points on the boundary of the disk. Depending on one's definition of holography, this can be regarded as one of the few examples of a precise holographic correspondence in the sense that it relates a two-dimensional theory (PSM model) to a one-dimensional theory (quantum mechanics defined by the Poisson manifold) on its boundary. Unfortunately, the boundary conditions used in this derivation are not pertinent to the cases we are interested in since they would translate into a singular metric on the boundary. It would certainly be interesting to find a connection between the developments in part II of this thesis and the result (3.47).

The equations of motion that follow from the PSM action (3.44) are

$$dX^I + P^{IJ} A_J = 0 \tag{3.48a}$$

$$dA_I + \frac{1}{2}\partial_I P^{JK} A_J \wedge A_K = 0\,. \tag{3.48b}$$

Notice that the need for P^{IJ} to obey the defining property of a Poisson tensor (3.43) follows also as a consistency condition from the equations of motion (3.48). Applying d to (3.48a) and inserting both (3.48a) and (3.48b) yields

$$\frac{1}{2}\big(\circlearrowleft_{IJK} P^{LK}\partial_L P^{JI}\big) A_J \wedge A_K = 0, \tag{3.49}$$

which can be satisfied for generic A_J only if the quantity in parenthesis vanishes.

Symmetries of the PSM model

The action (3.44) is invariant under the non-linear gauge transformations

$$\delta_\lambda X^I = P^{IJ}\lambda_J\,, \tag{3.50a}$$

$$\delta_\lambda A_I = -\,d\lambda_I - \partial_I P^{JK} A_J \lambda_K \tag{3.50b}$$

[8] It is debatable whether deformation quantization indeed corresponds to our physicists' notion of quantization [135]. For instance, the perturbation series (3.46) is only formal.

up to the boundary term

$$\delta_\lambda I = -\frac{k}{2\pi}\int_{\mathcal{M}} d(\lambda_I\, dX^I)\,. \tag{3.51}$$

The transformations (3.50) form a *generating set* of all gauge symmetries of the action (3.44) in the sense of [136], i.e., every other gauge symmetry can be obtained from these by a (possibly field-dependent) choice of gauge parameter λ_I. Notice that the transformations (3.50a) do not form an algebra since

$$[\delta_{\lambda_1},\delta_{\lambda_2}]X^I = P^{IJ}([\lambda_1,\lambda_2])_J \tag{3.52a}$$

$$\begin{aligned}[\delta_{\lambda_1},\delta_{\lambda_2}]A_I = &-d([\lambda_1,\lambda_2])_I - \partial_I P^{ML}A_M([\lambda_1,\lambda_2])_L \\ &- \partial_I\partial_L P^{JK}(\lambda_1)_J(\lambda_2)_K\big(dX^L + P^{LM}A_M\big)\,,\end{aligned} \tag{3.52b}$$

where

$$([\lambda_1,\lambda_2])_I = \partial_I P^{JK}(\lambda_1)_J(\lambda_2)_K \tag{3.53}$$

due to the presence of the last term in (3.52b). The generating set is therefore said to define an *open algebra*.[9]

Let us now discuss the other symmetries of the action (3.44) to see how they are related to the gauge transformations generated by (3.50).

Target space diffeomorphisms. The PSM action is clearly invariant under target space diffeomorphisms $X^I \mapsto \tilde{X}^I(X^K)$ if the Poisson tensor P^{IJ} transforms as a contravariant two-tensor. However, the Poisson tensor is not a dynamical field of the theory but considered to be part of the *background structure*. Consequently, the target-space diffeomorphisms that leave this background structure invariant ought to be symmetries of the model (cf. the discussion in the introduction). Before discussing the PSM model (3.44) let us look at the slightly simpler sigma model (3.45) in the light of the above. This expression is clearly invariant under diffeomorphisms of the target space. However, the target space metric g_{IJ} is not a dynamical field; as was said above, it is supposed to be viewed as a coupling constant from the point of view of the two-dimensional theory. Thus, not all of the target space diffeomorphisms will correspond to symmetries but only those that leave the metric g_{IJ} invariant. We can show this explicitly by doing an infinitesimal field redefinition $X^I \to X^I + v^I$, that would correspond to a target-space diffeomorphism. Since g_{IJ} can in general depend on X^I one has

$$\delta_\xi I_P = \int_{\mathcal{M}} d^2x\sqrt{|\gamma|}\gamma^{\mu\nu}\big(g_{KJ}\partial_I v^K + g_{IK}\partial_J v^K + v^K\partial_K g_{IJ}\big)\partial_\mu X^I\partial_\nu X^J \tag{3.54}$$

$$= \int_{\mathcal{M}} d^2x\sqrt{|\gamma|}\gamma^{\mu\nu}\partial_\mu X^I\partial_\nu X^J(\nabla_I v_J + \nabla_J v_I)\,, \tag{3.55}$$

where one recognizes the Killing equation for the metric g_{IJ} written in terms of the compatible connection ∇_I. Thus, the target-space diffeomorphism v will generate a symmetry only if it leaves g_{IJ}, i.e., the background structure, invariant. Choosing $g_{IJ} = \eta_{IJ}$ one recovers the global Poincaré invariance of string theory.

[9] It is worth stressing that the full set of gauge transformations of any theory *always* forms a Lie algebra even if the generating set fails to do so, as is the case here.

With this in mind let us return to the PSM action (3.44). Again, not every target space diffeomorphism $X^I \to X^I + v^I$ is a symmetry of the theory; only those that leave the Poisson tensor invariant. This is what we are going to show in the following.

We are going to perform a target space diffeomorphism that acts only on the dynamical fields. In particular, we have

$$X^I \to X^I + v^I \qquad A_I \to A_I - v^K \partial_K A_I - A_K \partial_I v^K = A_I - A_K \partial_I v^K \,, \tag{3.56}$$

where $v^I(X^K)$ should be regarded as a vector on target space. The last equality follows from the fact that the A_I's do not have an explicit dependence on X^I. Under this target space diffeomorphism the action transforms as

$$\delta_v I = \frac{k}{4\pi}\int_{\mathcal{M}} \big(v^K \partial_K P^{IJ} A_I \wedge A_J - P^{IJ}\partial_I v^K\, A_K \wedge A_J - P^{IJ}\partial_J v^K\, A_I \wedge A_K\big) \tag{3.57}$$

$$= \frac{k}{4\pi}\int_{\mathcal{M}} \mathcal{L}_v P^{IJ}\, A_I \wedge A_J \,. \tag{3.58}$$

In order to be a symmetry the target space diffeomorphism has to obey

$$\pounds_v P^{IJ} = 0 \,, \tag{3.59}$$

that is, it has to generate a *canonical transformation* leaving the Poisson structure invariant. For symplectic structures a vector field generates a canonical transformation if and only if it is a Hamiltonian vector field [we showed this around equation (2.18)]. In the case of Poisson manifolds the set of vector fields obeying condition (3.59) is in general larger than the set of Hamiltonian vector fields as we argued below definition 8.

However, in the following it will be enough to consider vector fields of the form

$$v^I = P^{IJ}\lambda_J \,, \tag{3.60}$$

with λ_J a one-form on the target space that obeys $\partial_I\lambda_J - \partial_J\lambda_I = 0$. Since we assume that the target space is topologically trivial every λ_J can be written as $\lambda_J = \partial_J\mathcal{H}$ for some function $\mathcal{H}$ on target space. Equation (3.60) yields then precisely the Hamiltonian vector fields (2.34) on the target space. Using this fact, we can rewrite the transformation (3.56) of A_I and X_I under target space diffeomorphisms as

$$\delta_v X^I = P^{IJ}\lambda_J \tag{3.61a}$$

$$\begin{aligned}\delta_v A_I &= -A_J \partial_I P^{JK}\lambda_K - A_J P^{JK}\partial_I \lambda_K \\ &= -A_J\partial_I P^{JK} - dX^K \partial_K\lambda_I + \partial_K\lambda_I\big(dX^K + P^{JK}A_K\big) \\ &= -\,d\lambda_I - \partial_I P^{JK} A_J \lambda_K + \partial_K\lambda_I\big(dX^K + P^{KJ}A_J\big),\end{aligned} \tag{3.61b}$$

where $\partial_I\lambda_J = \partial_I\partial_J H = \partial_J\lambda_I$ was used in the third line. These transformations are of the same form as the gauge transformations (3.50) up to a term proportional to the equation of motion (3.48a). Forgetting about this term for a moment the difference between the transformations (3.61) and (3.50) lies in the parameter λ_I. While in the latter case λ_I is defined on the worldsheet $\mathcal{M}$, $\lambda_I = \lambda_I(x)$, it is defined as a one-form on the target space in the former case via $\lambda_I = \partial_I \mathcal{H}(X^K)$. Thus, it depends on the base space coordinates only implicitly via the map X^K: $\lambda_I = \lambda_I(X^K(x))$.

Now, what about the last term in (3.61b)? This term corresponds to a so-called *trivial gauge transformation* according to the terminology of [136]. Consider a gauge transformation of an arbitrary field Φ_i, of the form

$$\delta_\mu \Phi_i(x) = \int d^2x' \mu_{ij}(x, x') \frac{\delta I}{\delta \Phi_j(x')} \tag{3.62}$$

where $\delta I/\delta\Phi_i$ denotes the Euler-Lagrange derivative of S with respect to Φ_i and μ_{ij} is an arbitrary (possibly field-dependent) parameter that is (graded) antisymmetric. It is easy to see that this transformation is always a symmetry regardless of the specific form of μ_{ij}. These symmetries exist for any action and act trivial on the space of solutions and therefore carry no relevant physical information about the theory, hence the name. Furthermore, it is possible to show that any transformation of the fields that vanishes on-shell is a trivial gauge transformation in the sense of (3.62) (cf. Theorem 3.1 in [136]).

To summarize, we have shown that target space diffeomorphism generated by Hamiltonian vector fields are symmetries that can be treated for the non-linear gauge transformations (3.50) up to trivial gauge transformations.

Base manifold diffeomorphisms. Let us now turn to the other important symmetry of the Poisson sigma model which is diffeomorphism invariance on the base manifold $\mathcal{M}$. Under a diffeomorphism generated by ξ^μ the fields transform as

$$\delta_\xi X^I = \pounds_\xi X^I = \xi \cdot dX^I, \tag{3.63}$$

$$\delta_\xi A_I = \pounds_\xi A_I = d(\xi \cdot A_I) + \xi \cdot dA_I. \tag{3.64}$$

The first term in the second equations suggests that one should set $\lambda_I = -\xi \cdot A_I$. Indeed, using this substitution one finds

$$\delta_\xi X^I = P^{IJ}\lambda_J + \xi \cdot (dX^I + P^{IJ} A_J), \tag{3.65a}$$

$$\delta_\xi A_I = -d\lambda_I - \partial_I P^{JK} A_J \lambda_K + \xi \cdot \left(dA_I + \frac{1}{2} \partial_I P^{JK} A_J \wedge A_K \right). \tag{3.65b}$$

The terms in brackets are recognized as the equations of motion of the PSM model (3.48). These terms therefore generate only trivial gauge transformations in the sense of (3.62) due to the theorem mentioned under that equation.

Thus, diffeomorphisms on the world-sheet are equivalent to the non-linear gauge transformation (3.50) up to trivial gauge transformations. In the next section 3.4 we will provide the map between PSMs and two-dimensional theories of gravity. Relation (3.65) will allow us to treat diffeomorphism invariance of these theories for the easier to handle gauge invariance of PSM models.

Casimir functions

Before turning to this map let us add one more, general remark on PSM models. We saw in section 2.2 that generic Poisson tensors have a non-trivial kernel which implies the existence of conserved Casimir functions. Let us examine this statement in the context of PSM models. Suppose $C(X^K)$ is a Casimir function on the target space. By the defining equation (2.36) this function has to satisfy

$$0 = \{C, f\} = \partial_I C \partial_J f \{X^I, X^J\} = \partial_I C \partial_J f P^{IJ} \qquad \forall f(X^K) \in C^\infty(\Sigma)\,. \tag{3.66}$$

from which follows

$$P^{IJ}\partial_I C = 0\,, \tag{3.67}$$

i.e., $\partial_I C$ is in the kernel of P^{IJ}. The equations of motion of the PSM model (3.48) then imply that

$$dC = \partial_I C\; dX^I = -P^{IJ} A_J \partial_I C = 0\,. \tag{3.68}$$

Thus, Casimir functions are conserved quantities for the system. The gauge transformations (3.50) leave C invariant

$$\delta_\lambda C = \delta_\lambda X^I \partial_I C = P^{IJ}\lambda_J \partial_I C = 0\,. \tag{3.69}$$

The dynamics governed by a PSM model take place on surfaces of constant Casimir and therefore if the requirements of theorem 6 are met, on a symplectic leaf of the target space Σ.

The simple local structure of Poisson manifolds, explained in more detail in section 2.2, can be used to simplify the solution of PSM models. More precisely, theorem 3 asserts the existence of Casimir–Darboux coordinates. Suppose that the Poisson tensor is of rank $2r$, denote the coordinates by $X^I = (X^i, P^\alpha, Q^\alpha)$ where $\alpha = 1, \ldots, r$ and $i = r+1, \ldots D$ and the corresponding components of the fields $A_I = (A_i, A^P_\alpha, A^Q_\alpha)$. Since the PSM action is target space covariant we can write it in the form

$$I^{(\mathrm{CD})}_{\mathrm{PSM}} = \frac{k}{2\pi}\int_{\mathcal{M}} \left(A_I \wedge dX^I + A^Q_\alpha \wedge A^P_\alpha\right). \tag{3.70}$$

The equations of motion show that the X^i are conserved and are therefore identified with the Casimir functions. The gauge fields A_i are not determined by the equations of motion and, consequently, can be set to zero by the gauge transformations (3.50). While this coordinate system is useful in many contexts, in the applications we have in mind in which the target space is (the dual of) a Lie algebra, the coordinates associated to that structure will better suit our purpose.

3.4 Poisson sigma models and dilaton gravity

In this section we want to explicitly provide the map between PSM models and dilaton gravity in its first order formulation.

To this end, choose a three-dimensional target space $D = 3$ and denote the coordinates by $X^I = (X^a, X), a = 0, 1$. Furthermore, choose a Poisson tensor of the form

$$P^{ab} = \mathcal{V}\epsilon^{ab} \qquad P^{aX} = \sigma \epsilon^a{}_b X^b\,, \tag{3.71}$$

with $\mathcal{V}$ given by

$$\mathcal{V} = -\sigma\frac{U(X)}{2}X^c X_c - 2V(X)\,. \tag{3.72}$$

The map between the gravitational variables, zweibein e^a and spin connection ${\omega^a}_b \equiv \omega\, {\epsilon^a}_b$, is given by

$$A_I = (e_a, \omega) \tag{3.73}$$

and we assume the existence of a tangent space metric η^{ab} of Euclidean ($\sigma = 1$) or Lorentzian ($\sigma = -1$) signature. Under this map the PSM action (3.44) becomes

$$I = \frac{k}{2\pi}\int_{\mathcal{M}} \big(X^a(de_a + \sigma\epsilon_{ab}\omega\wedge e^b) + X\,d\omega + \epsilon\mathcal{V}(X^cX_c, X)\big) - \frac{k}{2\pi}\int_{\mathcal{M}} d(e_aX^a + \omega X), \tag{3.74}$$

where we introduced the volume form

$$\epsilon = \frac{1}{2}\epsilon_{ab}e^a\wedge e^b\,. \tag{3.75}$$

This is the action of 2d dilaton gravity in a first order formulation up to a boundary term that will be discarded in the following.

As discussed in section 3.2 the symmetries of this action are local Lorentz transformations (3.18) and diffeomorphisms (3.19). According to the discussion in the previous section these symmetries of the gravitational theory get mapped to the non-linear gauge symmetries of the PSM model (3.50). We have therefore succeeded in providing a reformulation of dilaton gravity as a (somewhat non-standard) gauge theory.

Since the target space is three-dimensional the antisymmetric Poisson tensor P^{IJ} cannot be of full rank. This implies the existence of at least one Casimir function, that we can determine explicitly. The equation of motion (3.48a) becomes

$$dX^a + \mathcal{V}\epsilon^{ab}e_b + \sigma{\epsilon^a}_bX^b\omega = 0 \tag{3.76}$$

$$dX + \sigma\epsilon_{ab}X^ae^b = 0 \tag{3.77}$$

for the Poisson tensor (3.71). Multiplying the first equation by e^QX_a, where Q is, at the moment, an arbitrary function and using the second equation one obtains

$$e^Q\,d\left(\frac{X^aX_a}{2}\right) + e^Q\left(\frac{X^aX_a}{2}U(X) + \sigma 2V(X)\right)dX = 0\,. \tag{3.78}$$

Defining the functions

$$Q := \int^X d\tilde{X}\,U(\tilde{X}) + Q_0 \qquad w(X) = -\sigma\,2\int^X d\tilde{X}\,e^{Q(\tilde{X})}\,V(\tilde{X}) + w_0 \tag{3.79}$$

equation (3.78) can be rewritten as

$$d\mathcal{C} \equiv d\big(e^QX^aX_a - w(X)\big) = 0\,. \tag{3.80}$$

It can be checked that $\partial_I\mathcal{C}$ thus defined is in the kernel of P^{IJ} and therefore a Casimir function. Notice that (3.80) in fact defines a two parameter family of Casimir functions since $\mathcal{C}$ is conserved for any choice of Q_0 or w_0.

Relation to the second order theory

Let us end this section by making the connection to the second order formulation explicit. To achieve this, one has to eliminate the auxiliary fields X^a and the spin connection ω in terms of their own equations of motion. They are

$$de^a + \sigma\epsilon^a{}_b\omega \wedge e^b + \epsilon\partial_a\mathcal{V} = 0\,, \tag{3.81a}$$
$$dX + \sigma X^a\epsilon_{ab}e^b = 0\,. \tag{3.81b}$$

One notices from the first equation and (3.72) that the connection ω has non-zero torsion if $U(X) \neq 0$. It is convenient to introduce a connection $\tilde{\omega}$

$$\tilde{\omega} \equiv \sigma\omega + \partial_c\mathcal{V}e^c \tag{3.82}$$

for which the first equation of (3.81) implies vanishing torsion. The action expressed in terms of this quantity reads

$$\begin{aligned} I = \frac{k}{2\pi}\int_{\mathcal{M}}\Big(X^a(de_a + \epsilon_{ab}\tilde{\omega}\wedge e^b) + \sigma X\, d\tilde{\omega} + \epsilon\mathcal{V} + \sigma\partial_c\mathcal{V}(dX + \sigma X^a\epsilon_{ab}e^b)\wedge e^c\Big) \\ - \frac{k}{2\pi}\int_{\mathcal{M}} d(-X\partial_c\mathcal{V}e^c)\,. \end{aligned} \tag{3.83}$$

Imposing the equations of motion (3.81) the first and fourth term in (3.83) vanishes and one can express

$$X^a = -\epsilon^{ab}\nabla_b X\,. \tag{3.84}$$

For the torsionless connection we find

$$d\tilde{\omega} = \sigma\frac{R}{2}\epsilon \tag{3.85}$$

such that the final form of the action is

$$I = \frac{k}{4\pi}\int_{\mathcal{M}} d^2x\sqrt{\sigma g}\Big(XR - \sigma U(X)(\nabla X)^2 - 2V(X)\Big), \tag{3.86}$$

which coincides with (3.5) when the overall sign for Euclidean and Lorentzian theories is introduced by hand and the coupling constant is related to Newton's constant as

$$\frac{k}{2\pi} \equiv \frac{1}{8\pi G}\,. \tag{3.87}$$

This establishes the equivalence of PSM models of the form (3.71) and two-dimensional dilaton gravity theories up to boundary terms.

Solution sectors of dilaton gravity

The equations of motion of dilaton gravity (3.48) with Poisson tensor (3.71) allow for two distinct sets of solution called *linear dilaton solutions* and *constant dilaton vacua*. The former are generic solutions that exist for any well-defined dilaton gravity model. The latter are defined by the condition that the dilaton field is on-shell constant $X = \bar{X}$. Since the zweibeine e^a are required to be linearly independent, as is apt for a gravity interpretation, the equations of motion imply that the auxiliary fields have to vanish

$$X = \bar{X} \qquad X^a = 0\,. \tag{3.88}$$

Due to the form of the Poisson tensor (3.71) for gravity models this is equivalent to the condition that the Poisson tensor vanishes on-shell. Constant dilaton vacua have necessarily constant curvature. This is also obvious from the second order equations (3.6). Constant dilaton solutions are not generic solutions, in the sense that not every dilaton model allows for these solution. More precisely, constant dilaton solutions are available for models where $V(X) = 0$ has a solution $X = \bar{X}$.

In section 5 we will discuss constant dilaton solutions and their asymptotic structure in some detail.

Coupling to (non)-Abelian Gauge fields. Supergravity. The PSM formulation of dilaton gravity makes the coupling to non-Abelian gauge fields of a gauge group $\mathcal{G}$ straightforward.[10] Instead of a three-dimensional target space one extends the target space by the dimension of $\mathcal{G}$ with additional coordinates Z^α and gauge fields A_α. The Poisson tensor obtains new entries of the form

$$P^{\alpha\beta} = f^{\alpha\beta}{}_\gamma Z^\gamma\,. \tag{3.89}$$

with $f^{\alpha\beta}_\gamma$ being the structure functions of the Lie algebra of $\mathcal{G}$. The potential $\mathcal{V}$ can be extended to depend on the Casimir elements of $\mathcal{G}$. We will consider more concrete examples of this kind in chapter 7.

As another possible generalization one can consider supersymmetric extensions of dilaton gravity in the PSM formulation. In this approach the target space obtains additional fermionic directions with the Poisson tensor subject to a graded version of the Jacobi identity. These models will not be discussed in the following. The interested reader is referred to the original works [137, 138] and the review article [99].

3.5 He do the charges in different voices

In this section we want to further explore the structure of dilaton gravity theories in the PSM formulation. The main result is a derivation of the canonical charges of the theory which will allow us to determine the asymptotic symmetry algebra in later sections.

There are many approaches to define charges for gauge theories including Hamiltonian methods [139], cohomological methods [140, 141], boundary counter-terms [142, 143], and covariant phase space methods [144–146]. We will first construct the charges for the PSM model using covariant phase space methods. These charges will be one of our main tools in the following chapters. However, for completion we will also add a calculation of the canonical charges in the second order formulation using Hamiltonian methods.

[10] Notice that this does not destroy the topological nature of the model, since these fields do not propagate in two dimensions.

Covariant Phase space method

Our starting point is an n-dimensional spacetime $\mathcal{M}$ with Lagrangian n-form $\mathbf{L}(\phi)$ depending on the fields ϕ and their derivatives up to a finite order. We denote by $\mathcal{F}$ the collection of all kinematically allowed field configurations on $\mathcal{M}$ that is assumed to have the structure of a Banach manifold. Note that a choice of field space $\mathcal{F}$ also includes the specification of boundary conditions for the fields ϕ. Now let $\phi(\lambda)$ be an arbitrarily parametrized curve of field configurations on $\mathcal{F}$, let $\phi(\lambda = 0) = \phi_0$ and define

$$\delta\phi_0 \equiv \frac{d\phi(\lambda)}{d\lambda}|_{\lambda=0} \,. \tag{3.90}$$

This equation defines a tangent vector $\delta\phi_0$ at the field configuration ϕ_0. We will sometimes write $\delta\phi^A$ with contravariant abstract index A in order to emphasize that a variation can be viewed as tangent vector in field space $\mathcal{F}$.

Varying the action, and hence the Lagrangian, with respect to the fields ϕ produces

$$\delta\boldsymbol{L} = \boldsymbol{E}(\phi)\,\delta\phi + d\boldsymbol{\theta}(\phi,\delta\phi)\,, \tag{3.91}$$

after integrating by parts. The first term defines the equations of motion $\boldsymbol{E} = 0$, whereas the latter term defines the $n-1$-form $\boldsymbol{\theta}(\phi,\delta\phi)$ called *presymplectic potential current*. The reason for the prefix “pre” will become clear in a moment. The solutions to the equations of motion specify a subspace $\bar{\mathcal{F}} \subset \mathcal{F}$.

Note that $\boldsymbol{\theta}$ is defined only up to an exact term $\boldsymbol{\theta} \to \boldsymbol{\theta} + d\boldsymbol{Y}$ and can be changed by adding a boundary term to the Lagrangian. Since none of these two have any consequences for the following we will not discuss them further.[11] Choose now an equal-time slice[12] Σ, that is assumed to be a Cauchy surface for $\mathcal{M}$, and define the *presymplectic potential* as

$$\theta(\phi,\delta\phi) = \int_\Sigma \boldsymbol{\theta}(\phi,\delta\phi). \tag{3.92}$$

Since the presymplectic potential θ depends linearly on the variation $\delta\phi$ and yields a number it can be regarded as a one-form θ_A in field space. This should be compared to the canonical one-form of section 2.1 that allowed us to define a symplectic structure in the finite-dimensional case via equation (2.8).

From this analogy it is clear how to proceed. Define an $n-1$ form $\boldsymbol{\omega}$ called *pre-symplectic current* as

$$\boldsymbol{\omega}(\phi,\delta_1\phi,\delta_2\phi) = \delta_1\boldsymbol{\theta}(\phi,\delta_2\phi) - \delta_2\boldsymbol{\theta}(\phi,\delta_1\phi) \tag{3.93}$$

and the *pre-symplectic structure* as

$$\Omega(\delta_1\phi,\delta_2\phi) = \int_\Sigma \boldsymbol{\omega}\,. \tag{3.94}$$

[11] Requiring that the variational principle for the Lagrangian $\mathbf{L}$ be well-defined fixes these ambiguities to some extent. More details on this point and the relation to (non)-conservation of charges can be found in [147].

[12] Although the formalism presented in this section applies equally well to Euclidean theories or radial evolution we will stick to this name in the following. The author apologizes for using Σ again for the time-slice, as is conventional. This should not be confused with the target space of the PSM model.

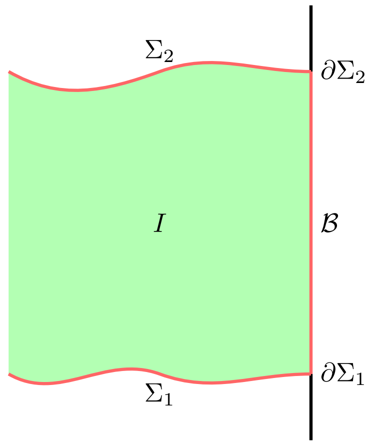

Figure 3.1: The integration regions of equation (3.96). In the two-dimensional case there could be a second boundary on the left hand side or the two time slices meet in the center of the disk, in the Euclidean finite temperature case.

Again, Ω can be regarded as a two-form on field space $\Omega_{AB} = -\Omega_{BA}$ since it depends linearly on the variations $\delta_1\phi^A$ and $\delta_2\phi^B$. This is in complete analogy to the finite-dimensional case. The pre-symplectic structure Ω is closed on $\mathcal{F}$ by construction, qualifying as symplectic structure. In general, however, it will fail to be non-degenerate which explains the label "pre-symplectic". The pair $(\mathcal{F}, \Omega)$ is therefore not a well-defined symplectic manifold although it is always possible to construct one by *symplectic reduction*.[13] In most cases it is more convenient to deal with the degenerate directions of Ω than to perform the reduction explicitly.

From equation (3.94) it is not immediately clear that the left hand side is independent of the time-slice on which $\boldsymbol{\omega}$ is evaluated. If this were not the case, this equation would provide a bad definition for a symplectic structure as it would change from one instant to another. To show the well-definedness of (3.94) consider the antisymmetrized, second variation of $\mathbf{L}$

$$0 = \delta_1 \boldsymbol{E} \delta_2 \phi - \delta_2 \boldsymbol{E} \delta_1 \phi + d\boldsymbol{\omega}(\delta_1\phi, \delta_2\phi)\,. \tag{3.95}$$

Thus, if the variations $\delta_1\phi, \delta_2\phi$ satisfy the linearized equations of motion, i.e., they are tangent to an element $\phi \in \bar{\mathcal{F}}$, then $d\boldsymbol{\omega} = 0$. Evaluate now (3.94) on two time-slices Σ_1, Σ_2 and consider their difference. Then we find using Stoke's theorem

$$\int_{\Sigma_2} \boldsymbol{\omega} - \int_{\Sigma_1} \boldsymbol{\omega} + \int_{\mathcal{B}} \boldsymbol{\omega} = \int_I d\boldsymbol{\omega}\,. \tag{3.96}$$

[13]In this process one identifies the degenerate directions of Ω and, integrating along these, one can define an equivalence relation $\phi_1 \simeq \phi_2$ between different field configurations. The quotient of $\mathcal{F}$ by this relation, $\Gamma = \mathcal{F}/\simeq$, induces a projection $\pi : \mathcal{F} \to \Gamma$. The pullback of the presymplectic form Ω by this projection yields a non-degenerate *symplectic form* $\pi^*\Omega$ such that the pair $(\Gamma, \pi^*\Omega)$ is a symplectic manifold that serves as phase space of the theory.

Here, $\mathcal{B}$ denotes the part of $\partial\mathcal{M}$ enclosed by the boundary of the two hypersurfaces and I is the subset of $\mathcal{M}$ in the interior of the two hypersurfaces, cf. figure 3.1. If the manifold has no boundary the third term on the left hand side of (3.96) is not present. Otherwise, one has to assume that either the symplectic structure is modified in such a way that this term is not present[14] or that the boundary conditions on ϕ are such that this term does not contribute.[15] Thus, we find for variations tangent to the solution manifold $\bar{\mathcal{F}}$

$$\int_{\Sigma_1} \boldsymbol{\omega} = \int_{\Sigma_2} \boldsymbol{\omega}\,, \tag{3.97}$$

due to equation (3.95).

Having a symplectic structure on phase space, we are now able to define Hamiltonian functions associated to symmetry transformations. Assume that $\hat{\delta}_\lambda \phi$ is a symmetry of the Lagrangian $\boldsymbol{L}$ depending on the (local) parameter λ, i.e., applying the symmetry leaves the Lagrangian invariant up to a boundary term

$$\hat{\delta}_\lambda \boldsymbol{L} = d\alpha(\phi, \hat{\delta}_\lambda \phi)\,. \tag{3.98}$$

Again, we can view $\hat{\delta}\phi$ as a vector field on $\mathcal{F}$. Since a symmetry of the Lagrangian is also a symmetry of the equations of motions [149], one has

$$\hat{\delta}_\lambda \boldsymbol{E} = 0\,, \tag{3.99}$$

i.e., the symmetry variation solves the linearized equations of motion [144]. In analogy to the definition of Hamilton's equations in the finite-dimensional case (2.14), we can now define based on the above

$$\delta H[\lambda] = \int_\Sigma \boldsymbol{\omega}(\delta\phi, \hat{\delta}_\lambda \phi)\,. \tag{3.100}$$

This equation defines the Hamiltonian, or the *charge*, as the generator of the (local) symmetry $\hat{\delta}_\lambda \phi$.

We will now apply the above to the PSM case. Starting from the Lagrangian (3.44) the presymplectic potential current is

$$\boldsymbol{\theta} = -\frac{k}{2\pi} A_I \delta X^I\,. \tag{3.101}$$

Choosing an equal time slice Σ we can derive the presymplectic potential

$$\theta = -\frac{k}{2\pi} \int_\Sigma A_I \delta X^I \tag{3.102}$$

and the presymplectic structure

$$\Omega = -\frac{k}{2\pi} \int_\Sigma \left(\delta_1 A_I \delta_2 X^I - \delta_2 A_I \delta_1 X^I\right)\,. \tag{3.103}$$

[14] Notice that the requirement that this term vanishes is very similar to the requirement for a well-defined variational principle for $\boldsymbol{L}$. Thus, for a the symplectic structure coming from a Lagrangian with well-defined variational principle this term will usually not be present, cf. e.g. [148].

[15] In some cases, however, there is a physical reason for this term being present and thus non-conservation of the symplectic current; for instance, in the case of asymptotically flat spacetimes for nonvanishing Bondi news [146].

Although it is certainly possible to proceed completely covariant, let us introduce coordinates (t, r) on $\mathcal{M}$ for convenience and assume that Σ is a constant r-slice. Then the symplectic structure is

$$\Omega = -\frac{k}{2\pi}\int_\Sigma dr\,\big(\delta_1(A_I)_r\delta_2 X^I - \delta_2(A_I)_r\delta_1 X^I\big)\,. \tag{3.104}$$

We can use this to understand the phase space Γ of the PSM model. We see immediately that Ω is degenerate in the direction of variations along $(A_I)_t$. Thus, this component of A_I can be regarded as pure gauge. Furthermore, variations of $(A_I)_r$ and X^I whose pullback to Σ vanish, i.e., which are localized away from the hypersurface, lead also to vanishing Ω. From this we can conclude that one should identify all $((A_I)_r, X)$ that differ only by their values away from Σ. The phase space Γ of the PSM model is given by all $((A_I)_r, X^I)$ having distinct values on Σ. However, the equation of motion (3.48a) constrains the possible elements of Γ to fulfill

$$\partial_r X^I + P^{IJ}(A_J)_r = 0\,. \tag{3.105}$$

The phase space Γ is therefore further reduced to the *constraint manifold* $\bar{\Gamma}$ that consists of all elements of Γ that obey equation (3.105).

We saw in our discussion in section 3.3 that all symmetries of PSM models are contained in the non-linear gauge transformations (3.50). Thus, we should be able to construct the charges for these symmetries by substituting them in Hamilton's equation (3.100). This yields immediately

$$\delta H[\lambda] = -\frac{k}{2\pi}\int_\Sigma \lambda_I\,\delta\big(dX^I + P^{IJ}A_J\big) + \frac{k}{2\pi}\lambda_I\delta X^I|_{\partial\Sigma}\,, \tag{3.106}$$

or in the above system of coordinates

$$\delta H[\lambda] = -\frac{k}{2\pi}\int_\Sigma dr\lambda_I\,\delta\big(\partial_r X^I + P^{IJ}(A_J)_r\big) + \frac{k}{2\pi}\lambda_I\delta X^I|_{\partial\Sigma}\,. \tag{3.107}$$

This equation shows that the constraint (3.105) is the generator of gauge transformations in the bulk. This is an instant of the general credo that *first class constraints generate gauge symmetries*.

If the variations are tangent to the constraint manifold $\bar{\Gamma}$ we find

$$\delta H[\lambda] = \frac{k}{2\pi}\lambda_I\,\delta X^I|_{\partial\Sigma}\,. \tag{3.108}$$

The charge is thus given by a pure boundary term. It is a general result that the charge associated to a gauge symmetry is the integral of an $n-2$ form where n is the spacetime dimension. The standard example for this behavior is Gauss law for the U(1) symmetry of electrodynamics. Since $n = 2$ in our case, the charge is given by a function evaluated at the boundary of the time slice. Notice that we cannot immediately functionally integrate the charges since λ can in principle contain state-dependent quantities, i.e., it might be a function on $\mathcal{F}$.

If one were to construct (3.108) using the Hamiltonian method pioneered in [139] one would arrive at an equation similar to (3.107). However, the interpretation would be that the bulk term is the generator of gauge symmetries that needs to be enhanced by the boundary term in order to be functional differentiable with respect to the canonical variables X^I and $(A_I)_r$.

Before we turn to this method to calculate the charge for dilaton gravity in the second order formalism, let us add one more observation regarding equation (3.108). Recall the target-space diffeomorphisms of the PSM model. We saw that they had to leave the Poisson structure invariant in order for them to be symmetries of the theory. Among them were, in particular, diffeomorphisms associated to Hamiltonian vector fields on the Poisson manifold (3.60) with $\lambda_J = \partial_J \mathcal{H}$ with $\mathcal{H}(X^K)$ being a Hamiltonian function on the target space. For symmetry parameters of this form we obtain from (3.108)

$$\delta H[\lambda] = \frac{k}{2\pi} \partial_I \mathcal{H} \, \delta X^I |_{\partial\Sigma}, \tag{3.109}$$

which can be trivially integrated to

$$H[\lambda] = \frac{k}{2\pi} \mathcal{H}(X^K(x))|_{\partial\Sigma} \,. \tag{3.110}$$

The Hamiltonian functions of the PSM model thus coincide with the Hamiltonian functions on the Poisson manifold depending on the X^K evaluated at the boundary of the base manifold.

Regge–Teitelboim method

The starting point for the Regge–Teitelboim method of constructing canonical charges [139] is the Hamiltonian of the theory, that is a sum of constraints for diffeomorphism invariant theories. Assuming that all second class constraints have been solved, the first-class constraints generate gauge transformations via the Poisson (or Dirac brackets) with the canonical variables. The crucial insight of [139] is that the first class constraints are in most cases not functionally differentiable on a manifold with boundary, for gauge parameters that do not vanish at infinity. The generators therefore have to be improved by a boundary term that is interpreted as the charge associated to the symmetry transformation.

It is a nice exercise to calculate the charges for dilaton gravity in the second order formulation. A straightforward ADM split (cf. appendix A) leads to the gravitational Hamiltonian

$$\begin{aligned} H = \alpha \int_\Sigma \sqrt{h} \Big[& \alpha^{-1} N^c (\pi_X D_c X - 2 D_a \pi^a{}_b) + \\ & N \big(\alpha^{-2} \sigma \pi^{ab} h_{ab} \pi_X + \alpha^{-2} U(X) \pi^{ab} \pi_{ab} + \sigma U(X) h^{ab} D_a X \, D_b X + 2V(X) - 2 D^2 X \big) \Big], \end{aligned} \tag{3.111}$$

where the prefactor of the action (3.5) was denoted by α. The quantities multiplied by N^c and N correspond to diffeomorphism and Hamiltonian constraint, respectively. Given a canonical variable Φ, defined on the hypersurface Σ the Hamiltonian acts as

$$\dot{\Phi} = \{\Phi, H\} \,. \tag{3.112}$$

However, the Hamiltonian is not functionally differentiable with respect to most of the canonical variables if the hypersurface Σ has an (asymptotic) boundary. Therefore, one has to add certain boundary terms to the constraints appearing in the Hamiltonian.

The variation of the Hamiltonian (3.111) produces the following terms

$$\delta H = \alpha \int_\Sigma \sqrt{h}(\dots) \\ + \Big[-2\delta\pi^a{}_b N^b r_a + \delta X r_c (N^c \pi_X + \sigma U(X) h^{ac} D_a X + 2D^c N) - 2N r^c D_c \delta X \Big]_{\partial\Sigma} \tag{3.113}$$

where the bulk terms were not written, and r_c denotes the (inward pointing) normal vector of the boundary of Σ. In order for for the Hamiltonian to be functionally differentiable one has to add a boundary term Q such that the variation of Q cancels the boundary term in (3.113). In the present case one finds

$$\delta Q = \Big[2\delta\pi^a{}_b N^b r_a - \delta X r_c (N^c \pi_X + \sigma U(X) h^{ac} D_a X + 2D^c N) + 2N r^c D_c \delta X \Big]_{\partial\Sigma} . \tag{3.114}$$

As above, we are in general not able to integrate expression (3.114) for the charge unless we specify certain boundary conditions on the fields. Lapse and shift should be chosen such that they preserve these asymptotic conditions. If it is possible to integrate δQ one defines the *improved Hamiltonian* as the sum of bulk term and boundary term Q.[16]

Since the bulk term (3.111) is a sum of constraints it vanishes for every solution, and the form of N and N^c in the bulk is completely arbitrary. The boundary term Q evaluated on a solution is then interpreted as charge of the solution associated to the asymptotic transformation generated by N, N^a.

In most cases we are interested in a covariant version of (3.114) in which a diffeomorphism is not split into temporal and spatial components. For this, we demand that the evolution of the dilaton X (or any other canonical variable) with Hamilton's equation (3.112) should correspond to the transformation coming from a diffeomorphism ξ^a acting via the Lie derivative. We find from Hamilton's equations

$$\dot{X} = N\sigma\alpha^{-1}\pi^{ab}h_{ab} + N^c D_c X = (Nn^a + N^a)\nabla_a X \tag{3.115}$$

Comparing this to $\pounds_\xi X = \xi^a \nabla_a X$ we can relate lapse and shift appearing in (3.114) to a diffeomorphism ξ^a.

Asymptotic Symmetries

Expressions (3.114) and (3.108), in particular the latter, will be our main tools in the study of asymptotic dynamics of dilaton gravity since one can derive the *asymptotic symmetries* of the theories from these. This can be seen as follows:

Since the configuration manifold $\mathcal{F}$ contains a choice of boundary conditions for the fields it is necessary to check that the symmetry transformations of the theory do not lead out of $\mathcal{F}$. This will put restrictions on the gauge parameters. The set of symmetries with these restricted parameters is called *allowed symmetries*. Among these one finds that certain

[16] Notice that by Hamiltonian we do not necessarily mean that it generates time translations. The interpretation depends on the asymptotic form of lapse and shift.

symmetries will lead to vanishing charges. In particular if the parameters vanish at the boundary of spacetime, the charges are zero. These symmetries are called *genuine or proper gauge symmetries*. On the other hand, if the charge associated to an allowed symmetry does not vanish on-shell, the symmetry is said to define an *improper* or *non-trivial gauge symmetry*. The associated charge can be used to distinguish physically different configurations; thus the symmetry cannot describe a redundancy of the theory.

The *asymptotic symmetries* of a theory are then defined as the quotient of the non-trivial symmetries by the trivial ones. The algebra of these symmetries is called *asymptotic symmetry algebra*.

Chapter Four

Two-dimensional Anti-De Sitter Space

The main part of this work is concerned with the asymptotic dynamics of dilaton gravity in AdS_2 spacetime. We want to use this chapter to obtain a deeper understanding of this spacetime. The outline is as follows. In section 4.1 we are going to discuss AdS_2 solutions in Lorentzian signature. We will see that, although all solutions have constant curvature, depending on the behavior of the dilaton field they will have different interpretations. In particular, we find that some of the solutions can be interpreted as black holes. In section 4.2 we will turn to the Euclidean sector. We will introduce and motivate the Fefferman–Graham gauge for the metric that will play a crucial role in the rest of this thesis. The discussion of two-dimensional geometries of constant negative curvature naturally leads into the realm of hyperbolic geometry, to be introduced in section 4.3, that will help us in obtaining a deeper understanding of the two-dimensional geometries we will be dealing with. In the last section 4.4 we will provide another viewpoint on Euclidean AdS_2 geometries in Fefferman–Graham gauge, deriving from their interpretation as elements of codadjoint orbits of the Virasoro group.

Throughout this section we will repeatedly refer to various coordinate systems on AdS_2 collected in appendix B. The curvature of AdS_2 is taken to be $R = -2$ in the following.

4.1 Lorentzian signature solutions

The most convenient starting point for the discussion of Lorentzian solutions in two-dimensional dilaton gravity is Eddington–Finkelstein gauge [150]

$$ds^2 = 2\,du\,dX - \xi(X)\,du^2\,, \tag{4.1}$$

where the dilaton X is used as a coordinate, and $\xi(X)$ is a function of the dilaton that determines the geometry. Note that every solution of the action (3.5) can be written in this form with the model defining functions $U(X), V(X)$ only entering in the precise form of $\xi(X)$. The derivation of this result will not be reviewed here as it can be found in, e.g., [99, 151].

Starting from (4.1) a short calculation yields $R = -\xi''(X)$. Thus, $\xi(X)$ is a quadratic function of the dilaton if the metric is to be AdS_2. Introducing the radial coordinate r we find for an AdS_2 solution

$$ds^2 = 2\,du\,dr - (r^2 - M_0)\,du^2, \qquad X = r\,. \tag{4.2}$$

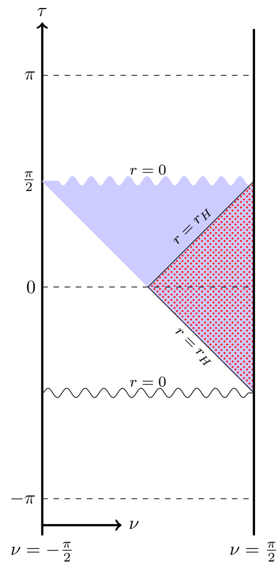

Figure 4.1: The AdS_2 black hole as a quotient of global AdS_2. The coordinate patch covered by (4.2) is painted in blue; the portion of spacetime in the exterior of the horizon (red hatched) is covered by the coordinate system (B.11). The squiggly lines denote the "singularities" where the dilaton vanishes.

where M_0 is an arbitrary integration constant.

We can draw a number of immediate conclusions from this solution. First, the vector ∂_u is a Killing vector for the combined metric-dilaton system. For $M_0 > 0$ the metric has a Killing horizon at spatial coordinate $r_H = \sqrt{M_0}$. Furthermore, the dilaton vanishes at $r = 0$. Since the dilaton can be regarded as an effective inverse Newton's constant, as was mentioned in section 3.1, one can think of this spacelike line as a strong coupling region. With this interpretation, the geometry has the flavor of a black hole geometry, at least for $M_0 > 0$ where a Killing horizon shields the strong coupling region. On the other hand, the dilaton field diverges for $r \to \infty$ which corresponds to a weakly coupled, i.e., asymptotic region. In this sense, the dilaton field provides boundary conditions for the interpretation of the metric (4.2) as a spacetime. However, the geometry (4.2) is locally AdS, thus there necessarily exists a coordinate transformation to global AdS_2. The explicit form of this transformation depends on the sign of M_0.

Starting with the case $M_0 = 0$, the simple replacement $r = z^{-1}, u = t - z$ brings the metric into Poincaré patch form (B.7) which covers the portion of AdS_2 shown in figure B.1. With the above interpretation, we should not extend the spacetime beyond that patch, despite being obviously possible, since the dilaton vanishes at these lines.

Consider now the case $M_0 = -|M_0| < 0$ and define $r = \sqrt{|M_0|}\tan\nu, u = \sqrt{|M_0|}^{-1}(\tau + \nu)$. The metric is then global AdS_2 (B.4). The dilaton diverges at $\nu = \pm\frac{\pi}{2}$ which corresponds to the boundaries but vanishes at $\nu = 0$. This time-like line is thus interpreted as a naked singularity.

Finally, the transformation

$$r = \sqrt{M_0}\,\frac{\cos\tau}{\cos\nu}, \qquad u = \sqrt{M_0}^{-1}\left(\tanh^{-1}\left(\frac{\sin\tau}{\sin\nu}\right) - \tanh^{-1}\left(\frac{\cos\tau}{\cos\nu}\right)\right) \tag{4.3}$$

brings the metric again into the form of global AdS_2 but now the dilaton vanishes at the time-like lines $\tau = n\pi$ hidden behind the horizon $r_H = \sqrt{M_0}$ (cf. figure 4.1). In this sense geometries (4.2) with $M_0 > 0$ describe black hole solutions. This black hole interpretation of a regular two-dimensional geometry was given first in [152, 153] wherein these solutions were obtained by dimensional reduction. Since the coordinate transformation (4.3) is periodic in τ, ν, global AdS_2 space accommodates an infinite number of the same black hole. This procedure of obtaining a black hole by identifying points of AdS is reminiscent of the three-dimensional case where it was shown that the Bañados–Teitelboim–Zanelli (BTZ) black hole can be obtained from AdS_3 by a quotient [28, 28]. We will see in the following that a similar statement holds in two dimensions, although the roles seem to be somewhat reversed. This is more straightforward to discuss in Euclidean signature to which we now turn.

4.2 Euclidean signature solutions

The aim of this section is to obtain a better understanding of Euclidean spacetimes of constant negative curvature, i.e., hyperbolic manifolds. This is deeply related to the study of Riemann surfaces since, by the uniformization theorem almost all Riemann surfaces are hyperbolic. Even the attempt to provide an overview over this topic would be widely beyond the scope of this work. Nevertheless, we will find that some very basic constructions are advantageous for the understanding of the following chapters.

Before turning to hyperbolic manifolds, we specify the asymptotic form of the metrics we are interested in. Since we are mainly considering finite temperature applications, we assume that Euclidean time is periodic with periodicity of inverse temperature β

$$\tau \sim \tau + \beta\,. \tag{4.4}$$

Thus, the two-dimensional hyperbolic space should admit (at least) one boundary with the topology of a circle. We will take $\beta = 2\pi$ in the following.

The spacetimes we are looking for have constant negative curvature $R = -2$. A well-known theorem by Fefferman and Graham (cf. [154] for a review) states that any asymptotically AdS spacetime admits an asymptotic coordinate system of the form

$$ds^2 = \frac{dr^2}{r^2} + \frac{1}{r^2}\left(g_0 + O(r)\right)d\tau^2. \tag{4.5}$$

This can be understood in the conformal framework of Penrose [155] in the sense that one has chosen a conformal factor $\Omega = r$ and then introduced Gaussian normal coordinates emanating from the boundary. The induced metric on the boundary is then given by g_0. Notice that the choice of conformal factor $\Omega = r$ is not unique as a change of conformal factor by $\Omega' = \omega\Omega$, where $\omega > 0$, yields another conformal factor. This induces a conformal transformation on the boundary metric $g_0 \to \omega^2 g_0$. Thus, as part of the boundary conditions one has to fix a conformal class of boundary metrics, e.g., the class of conformally flat metrics in the seminal work by Brown and Henneaux [26]. Since in the present case the topology of the boundary is a circle and all one-dimensional metrics belong to the same conformal class we can fix g_0 to any convenient value. Solving the curvature condition $R = -2$ in the gauge (4.5) order by order in r, one finds

$$ds^2 = d\rho^2 + \left(\frac{1}{2}e^{\rho} - \mathcal{L}(\tau)e^{-\rho}\right)^2 d\tau^2\,, \tag{4.6}$$

which, in contrast to the higher-dimensional case, is exact. Notice, however, that this is not the most general solution as will be discussed in chapter 6 where the boundary conditions of (asymptotically) AdS_2 will be discussed in more detail.

For the constant values $\mathcal{L} = +\frac{1}{2}, 0, -\frac{1}{2}$ the metric (4.6) can be recognized as the Euclidean versions of the two-dimensional black hole metric (B.11), the Poincaré patch (B.8), and global AdS_2 (B.5), respectively.

In three-dimensional AdS-space all solutions, in particular the BTZ solution, can be understood as quotients of global AdS by finite subgroups of the isometry group $SO(2,2)$. Figure 4.1 suggests, similarly, that the two-dimensional black hole, (4.2) with $M_0 > 0$, can be regarded as a particular quotient of global AdS_2 space. We now want to see explicitly how the different solutions in the Euclidean case, parametrized by $\mathcal{L}$ in Fefferman–Graham gauge, arise as identifications. We will start with *zero modes* $\mathcal{L} = \text{const}$ in the next section, and then turn to $\mathcal{L}$ with arbitrary τ dependence in section 4.4.

4.3 Some hyperbolic geometry

The hyperbolic plane $\mathbb{H}$ is defined as the upper half-plane $\mathbb{H} := \{z = x + iy \in \mathbb{C} : y > 0\}$ together with the hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}\,. \tag{4.7}$$

The boundary of $\mathbb{H}$ is taken to be the real line compactified by the point at infinity $\partial\mathbb{H} := \mathbb{R} \cup \{\infty\}$, i.e., the boundary has the topology of a circle. This becomes more apparent in the second model for hyperbolic space which is given by the Poincaré disc $\mathbb{D}$. The *Cayley transform*

$$z \mapsto \frac{z - i}{z + i} \tag{4.8}$$

maps $\mathbb{H}$ to the disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with metric

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - |z|^2)^2}\,. \tag{4.9}$$

This defines the Poincaré disc model of hyperbolic space. Using polar coordinates on the unit disc

$$x = \tanh(\tfrac{\rho}{2})\cos\tau \qquad y = \tanh(\tfrac{\rho}{2})\sin\tau \tag{4.10}$$

this becomes the metric of the Euclidean black hole $\mathcal{L} = \frac{1}{2}$ in (4.6). We can thus identify this geometry with the hyperbolic plane $\mathbb{H}$.

The isometries of (4.7) are given by the (real) *Möbius transformations* PSL$(2,\mathbb{R})$

$$z \mapsto \frac{az+b}{cz+d} \qquad ad-bc=1\,. \tag{4.11}$$

We will often be less precise, speaking of the isometries of AdS_2 as SL$(2,\mathbb{R})$ which is in fact the double cover of (4.11). Thus, there exists a two-to-one map from SL$(2,\mathbb{R})$ elements

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad ad-bc=1 \tag{4.12}$$

to Möbius transformations (4.11) [$\pm M \in \mathrm{SL}(2,\mathbb{R})$ lead to the same transformation in (4.11)].

Elements of (4.11) are classified according to the conjugacy class of the associated SL$(2,\mathbb{R})$ element M. Since the trace is invariant under conjugation M is called

- *elliptic* if $|\mathrm{tr}\,M| < 2$;
- *parabolic* if $|\mathrm{tr}\,M| = 2$;
- *hyperbolic* if $|\mathrm{tr}\,M| > 2$.

An elliptic matrix M is conjugate to a rotation matrix

$$\begin{pmatrix} \cos(2\pi\omega) & \sin(2\pi\omega) \\ -\sin(2\pi\omega) & \cos(2\pi\omega) \end{pmatrix} \tag{4.13}$$

with $\omega \in\ (0,1/2)\cup(1/2,1)$. A hyperbolic matrix is conjugate to

$$\pm\begin{pmatrix} e^{2\pi\omega} & 0 \\ 0 & e^{-2\pi\omega} \end{pmatrix} \tag{4.14}$$

with $\omega > 0$ and a parabolic matrix is conjugate to one of the following six matrices

$$\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \pm\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}. \tag{4.15}$$

Elements of SL$(2,\mathbb{R})$ are equivalently distinguished by the number and loci of their fixed points: a parabolic transformation has one fixed point in $\partial\mathbb{H}$, a hyperbolic transformation has two fixed points in $\partial\mathbb{H}$, an elliptic transformation has one fixed point in $\mathbb{H}$ and one in the complement of $\mathbb{R}^2/\mathbb{H}$. This simply follows from solving the equation $z = \frac{az+b}{cz+d}$. Let Γ be a subgroup of the isometries (4.11) of $\mathbb{H}$. For a sufficiently well-behaved subgroup Γ, the quotient $X = \mathbb{H}/\Gamma$ is another hyperbolic manifold. In order to study this in more detail we first quote the following theorem [156]:

Theorem 9. *Fuchsian groups. A subgroup of PSL*$(2,\mathbb{R})$ *acts properly discontinuously on* $\mathbb{H}$ *if and only if it is discrete. Such a group is called* Fuchsian group.[1]

[1] As a reminder: A subgroup H of a topological group G is called *discrete* if there exists an open cover of G such that every open set contains exactly one element of H, i.e., the induced topology on H is the discrete topology. The action of a group G on a manifold M is *properly discontinuous* if for all compact subsets $K \subset M$ the set $\gamma \in G$ such that $\gamma K \cap K \neq \{\}$ is finite.

Figure 4.2: The first three identifications for the parabolic transformation $z \mapsto z + 1$ (left) and the hyperbolic transformation $z \mapsto 4z$ (right), respectively, on the Poincaré disk. Regions bounded by two respective geodesics are identified with each other.

The importance of Fuchsian groups lies in a theorem by Hopf stating that all hyperbolic surfaces can be obtained as quotients $X = \mathbb{H}/\Gamma$ where Γ is a Fuchsian group containing no elliptic elements (this is a particular case of the classic theorem proved, e.g., in [91] that pertains to arbitrary manifolds of constant sectional curvature). Let us show explicitly how to obtain the various zero-mode geometries of (4.6).

The only non-trivial identification that can come from a parabolic transformation (4.15) is conjugate to $z \sim z + 2\pi$, where the constant was chosen conveniently. We can convince ourselves that the group generated by this parabolic transformation acts properly discontinuous and is therefore Fuchsian. This identification leaves fixed the point $z = \infty$ or, equivalently by the Cayley map (4.8), the point $(1, 0)$ on the Poincaré disc. The geometry obtained in this identification is called a *cusp* (cf. figure 4.2). A coordinate system adapted to this identification is

$$y = e^{-\rho} \qquad x = \tau \sim \tau + 2\pi\,. \tag{4.16}$$

In this coordinate system the hyperbolic metric (4.7) becomes

$$ds^2 = d\rho^2 + e^{2\rho}\, d\tau^2\,, \tag{4.17}$$

which is recognized as $\mathcal{L} = 0$ in (4.6).

Let us now turn to hyperbolic transformations (4.14). These are conjugate to identifications $z \sim e^{2\pi\omega} z$, where the prefactor was fixed conveniently. It is again straightforward to show that this group is Fuchsian. The two fixed points of this transformations are $z = 0, \infty$ lying on the boundary $\partial\mathbb{H}$. The corresponding geometry is known as *funnel* (cf. figure 4.2). An appropriate coordinate system realizing this identification is

$$x = e^{\omega\tau}\frac{e^{2\rho} - \omega^2}{e^{2\rho} + \omega^2} \qquad y = e^{\omega\tau}\frac{2e^{2\rho}\omega}{e^{2\rho} + \omega^2} \qquad \tau \sim \tau + 2\pi\,. \tag{4.18}$$

The hyperbolic metric then becomes (4.6) with $\mathcal{L} = -\frac{\omega^2}{2}$. We have thus found an interpretation for $\mathcal{L} = \frac{1}{2}$ and all non-positive zero-modes of (4.6) as the hyperbolic plane and quotients thereof by parabolic and hyperbolic subgroups.

But what about the other positive zero modes? Following the logic in the previous paragraphs, one would expect that these can be generated from quotients of $\mathbb{H}$ by elliptic subgroups. However, the classification mentioned below theorem 9 explicitly excluded those. Thus, we expect that the geometries described by $\mathcal{L} > 0$ are flawed. The reason behind this is the following theorem [157]:

Theorem 10 (Good quotient theorem)**.** *Let Γ be a group acting on a manifold X. The quotient space X/Γ is a Hausdorff manifold with $X \to X/\Gamma$ a covering map if and only if Γ acts freely and properly discontinuously.*

The manifolds produced from quotienting by an elliptic subgroup are not Hausdorff manifolds.

Let us look at this in a bit more detail. An elliptic geometry is conjugate to a rotation matrix (4.13). As we mentioned above, these groups have a fixed point in $\mathbb{H}$ and thus do not act freely. If the rotation parameter ω in (4.13) is a rational number p/q, the group is cyclic to order q and one can show that it is discrete and, by theorem 9, acts properly discontinuously. Since the requirements of theorem 10 are met everywhere except at one point, an identification leads to a manifold that is well-behaved everywhere except at one single point. This construction defines an *orbifold* with a conical deficit angle of $(1 - \frac{1}{q})2\pi$ at the fixed point of the elliptic transformation. On the other hand, a rotation by an irrational multiple of 2π does not act properly discontinuously.[2] Such an identification will lead to a non-Hausdorff space. We can therefore conclude that most of the positive zero modes of $\mathcal{L}(\tau)$ lead to sick geometries.

This concludes our inspection of the zero-mode solution of (4.6). In order to understand $\mathcal{L}$ with arbitrary τ-dependence we will turn to the discussions of coadjoint orbits of the Virasoro group.

4.4 Asymptotic symmetries and coadjoint orbits of the Virasoro group

In the previous subsection we understood the zero-modes of the asymptotically AdS_2 line element as quotients of the hyperbolic plane $\mathcal{L} = \frac{1}{2}$. But what about geometries with arbitrary $\mathcal{L}$, not necessarily constant? In order to understand these solutions let us first discuss the asymptotic symmetries of (4.6).

We said above that the boundary metric is only fixed up to a conformal factor. Using the terminology introduced at the end of the previous chapter, the allowed symmetries of the system are therefore given by the transformations leaving this background structure invariant. They are therefore expected to be the conformal transformations of the circle which is equivalent to the full diffeomorphism group of the circle $\mathrm{Diff}(S^1)$. Let us stress that without a concrete theory we do not know whether these symmetries are trivially canonically realized, in which case they would be proper gauge symmetries, or non-trivially, thus being improper gauge symmetries.

The allowed symmetries are determined in a straightforward calculation. Let ξ^μ be a vector in $\mathcal{M}$. The Lie derivative of the metric with respect to ξ^μ has to obey the boundary conditions set by (4.6)

$$\pounds_\xi g_{\mu\nu} = \delta g_{\mu\nu} \tag{4.19}$$

[2]Think of the action of such a rotation on a circle of fixed radius in the hyperbolic disc. Starting from any point on the circle, applying the rotation often enough we can come arbitrarily close to every other point on the circle. The orbit of a rotation by an irrational multiple of 2π is dense. All of these points are identified with each other.

where

$$\delta g_{\rho\rho} = 0 \qquad \delta g_{\rho\tau} = \delta g_{\tau\tau} = O(e^{-2\rho})\,. \tag{4.20}$$

This leads to the asymptotic Killing vector

$$\xi^\rho = -\sigma'(\tau) \qquad \xi^\tau = \sigma(\tau) - 2e^{-2\rho}\sigma''(\tau)\,. \tag{4.21}$$

At the boundary $\rho \to \infty$ this reduces to the generator of an arbitrary infinitesimal diffeomorphism $x \to x - \sigma(\tau)$. The function $\mathcal{L}$ changes according to

$$\delta_\sigma \mathcal{L} = 2\sigma'\mathcal{L} + \sigma\mathcal{L}' + \sigma'''\,. \tag{4.22}$$

This infinitesimal transformation behavior is the hallmark for the appearance of the *Virasoro group*. In particular, it shows that $\mathcal{L}$ belongs to a *coadjoint orbit* of the Virasoro group, as we will discuss below. We will not be able to cover every aspect of this. More details can be found in [158, 159] and in the detailed pedagogical accounts [86, 87]. We will mostly stick to the conventions of [87].

The Virasoro Group

An orientation-preserving diffeomorphism of the circle $f : S^1 \to S^1, \varphi \mapsto f(\varphi)$ is given by a function that obeys

$$f(\varphi + 2\pi) = f(\varphi), \qquad f'(\varphi) > 0\,, \tag{4.23}$$

where the circumference of the circle was fixed to 2π. These diffeomorphisms form the group $\mathrm{Diff}(S^1)$ under composition. The identity transformation is given by $f(\varphi) = \varphi$; the existence of an inverse element is guaranteed by the second condition in (4.23). This is an example of an infinite-dimensional Lie group. Let us determine the adjoint representation of this group, i.e., the action on its Lie algebra given by the tangent space at the identity. Choose the following curve in $\mathrm{Diff}(S^1)$

$$\gamma_t = \varphi + tX(\varphi) + O(t^2) \tag{4.24}$$

that reduces to the identity at $t = 0$. $X(\varphi)$ is a periodic function on the circle. The adjoint action is calculated as

$$\mathrm{Ad}_f(X)(\varphi) = \frac{d}{dt}\big(f\big(\gamma_t\big(f^{-1}(\varphi)\big)\big)\big)|_{t=0}, \tag{4.25}$$

according to (2.49). Taylor expanding around $t = 0$ yields

$$\mathrm{Ad}_f(X)(\varphi) = \frac{X(f^{-1}(\varphi))}{(f^{-1}(\varphi))'}, \tag{4.26}$$

or, equivalently,

$$\mathrm{Ad}_f(X)(f(\varphi)) = f'(\varphi)X(\varphi)\,. \tag{4.27}$$

We can recognize this as the transformation behavior of a vector fields $X(\varphi)\partial_\varphi$ under diffeomorphisms. The Lie algebra of the Lie group $\mathrm{Diff}(S^1)$ is therefore identified with the space of vector fields on the circle $\mathrm{Vect}(S^1)$. The Lie bracket of this algebra is given by the usual bracket of vector fields

$$[X(\varphi)\partial_\varphi, Y(\varphi)\partial_\varphi] = (Y'X - X'Y)\partial_\varphi, \tag{4.28}$$

which can established by calculating (4.25) for infinitesimal f [cf. equation (2.51)]. Defining the Fourier modes of the generators $L_m = e^{im\varphi}\partial_\varphi$ one recovers the Witt algebra from (4.28)

$$[L_m, L_n] = (m-n)L_{m+n}\,. \tag{4.29}$$

The central extensions of a Lie algebra are in one-to-one correspondence to the non-trivial elements of its second cohomology group [160]. In case of the Lie algebra $\mathrm{Vect}(S^1)$ the second-cohomology group is one-dimensional and spanned by the nontrivial cocycle

$$\omega(X,Y) = \frac{1}{24\pi}\int_0^{2\pi} d\varphi X\, Y''' \tag{4.30}$$

called *Gel'fand–Fuks cocyle*. This allows to define the unique central extension of $\mathrm{Vect}(S^1)$, called *Virasoro algebra*, as $\widehat{\mathrm{Vect}}(S^1) = \mathrm{Vect}(S^1) \oplus \mathbb{R}$ with Lie bracket

$$[(X,\lambda),(Y,\mu)] = ([X,Y],\omega(X,Y))\,. \tag{4.31}$$

In terms of the generators introduced above, adding the cocycle (4.30) to equation (4.29) reproduces the well-known Lie bracket of the Virasoro algebra.

Having defined a central extension of the Lie algebra of $\mathrm{Diff}(S^1)$ it is a fair question if there also exists a central extension of the Lie group the infinitesimal version of which reduces to $\widehat{\mathrm{Vect}}(S^1)$. The answer to this question is affirmative. Let $f, g \in \mathrm{Diff}(S^1)$ and define the *Bott–Thurston cocyle*

$$C(f,g) = \frac{1}{48\pi}\int_{S^1} \log(f'\circ g)\, d\log(g')\,. \tag{4.32}$$

It can be shown that (4.32) is the "integral" of (4.30) and is the only non-trivial cocyle of $\mathrm{Diff}(S^1)$. We can now define the *Virasoro group* $\widehat{\mathrm{Diff}}(S^1) = \mathrm{Diff}(S^1)\times\mathbb{R}$ with group operation

$$(f,\lambda)\cdot(g,\mu) = (f\circ g, \lambda+\mu+C(f,g)) \qquad f,g\in\mathrm{Diff}, \qquad \lambda,\mu\in\mathbb{R}\,. \tag{4.33}$$

The adjoint representation of this group on its Lie algebra $\widehat{\mathrm{Vect}}(S^1)$ is given by

$$\widehat{\mathrm{Ad}}_f(X,\lambda) = \left(\mathrm{Ad}_f X, \lambda + \frac{1}{24\pi}\int_{S^1} S[f]X\right) \tag{4.34}$$

where

$$S[f] = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 \tag{4.35}$$

is the *Schwarzian derivative* (also denoted by $\{f;\varphi\}$) that will play a leading role in what follows. It shows two important properties

- $S[f\circ g] = (g')^2 S[f]\circ g + S[g]$ (cocycle condition)
- $S[f](\tau) = 0$ if and only if f is of the form $f = \frac{a\tau+b}{c\tau+d}$.

The latter implies in particular that the Schwarzian derivative vanishes for Möbius transformations (4.11).

As mentioned in section 2.3 the coadjoint action of a Lie group is defined on the dual of its Lie algebra. The dual of the space $\mathrm{Vect}(S^1)$ is the space of quadratic differentials on the circle $p(\varphi)(d\varphi)^2$ with scalar product

$$\langle p, X\rangle = \frac{1}{2\pi}\int_{S^1} p\, X\, d\varphi\,. \tag{4.36}$$

Similarly, the dual space of $\widehat{\mathrm{Vect}}(S^1)$ is taken to be the pair $(p(\varphi)(d\varphi)^2, c)$ where c is a real number, and the scalar product is naturally generalized to

$$\langle (p,c),(X,\lambda)\rangle = \frac{1}{2\pi}\int_{S^1} p\, X\, d\varphi + c\lambda\,. \tag{4.37}$$

It is now a straightforward exercise using the definition (2.52) to calculate the coadjoint representation of the Virasoro group given by

$$f\cdot p \equiv \left(\widehat{\mathrm{Ad}}^*_f p\right)(f(\varphi)) = \frac{1}{(f'(\varphi))^2}\left[p(\varphi) - \frac{c}{12}S[f](\varphi)\right]. \tag{4.38}$$

The coadjoint representation of the Virasoro algebra, on the other hand, is given by

$$\widehat{\mathrm{ad}}^*_X p = Xp' + 2X'p + \frac{c}{12}X'''\,. \tag{4.39}$$

As promised above, this is precisely the transformation law of $\mathcal{L}$ for $c = 12$. We can therefore understand different metrics (4.6) as coadjoint vectors $(\mathcal{L}(\varphi)(d\varphi)^2, c = 12)$ at fixed central charge. While infinitesimal asymptotic symmetries act on this like (4.39), a finite asymptotic symmetry transforms this coadjoint vector according to (4.38). Notice that not every transformation $f\cdot p$ will lead to a different solution. The transformations f for which $f\cdot p = p$ determine the *stabilizer group* (cf. equation (2.55) in section 2.3). Similarly, the elements of the Lie algebra that obey $\widehat{\mathrm{ad}}^*_X p = 0$ define the Lie algebra of the stabilizer group. Since the expression for $\widehat{\mathrm{ad}}^*_X p$ contains third derivatives, the stabilizer equation has always three solutions locally. However, the requirement for periodicity 2π puts a global restriction on the solution. It is possible to show that this equation has either one or three globally well-defined solutions. The stabilizer group is invariant along the orbit and therefore yields a rough classification of orbits. In fact, the orbit can be written as a quotient of $\mathrm{Diff}(S^1)$ by the stabilizer group.

Coadjoint orbits of the Virasoro Group

A classification of coadjoint orbits of the Virasoro group will reveal which geometries are related by an asymptotic symmetry transformation. We will only give a rough classification and refer the reader to the above mentioned literature for more details.

Let us first introduce the second-order differential equation

$$\frac{c}{6}\psi''(\varphi) + p(\varphi)\psi(\varphi) = 0 \tag{4.40}$$

called *Hill's equation* associated to a coadjoint vector (p, c). Assume that $\psi(\varphi)$ is a differential form of weight $-1/2$, i.e., it transforms as

$$(f \cdot \psi)(f(\varphi)) = (f'(\varphi))^{1/2}\psi(\varphi) \tag{4.41}$$

under a diffeomorphism $f(\varphi)$. If ψ solves Hill's equation associated to the coadjoint vector (p, c), then it is straightforward to show that $(f \cdot \psi)$ solves Hill's equation associated to $(f \cdot p, c)$. Thus, Hill's equation is an orbit invariant.

Let ψ_1, ψ_2 be two independent solutions to Hill's equation. Then the Wronskian

$$W[\psi_1, \psi_2] = \det\begin{pmatrix} \psi_1' & \psi_2' \\ \psi_1 & \psi_2 \end{pmatrix} \tag{4.42}$$

is constant and can be set to $W[\psi_1, \psi_2] = 1$. Now let $\psi_i(\varphi), i = 1, 2$ be a solution vector to Hill's equation and consider $\psi_i(\varphi + 2\pi)$. Since $p(\varphi)$ is 2π periodic $\psi_i(\varphi + 2\pi)$ will be another solution. Therefore, there exists a linear transformation relating the solutions $\psi_i(\varphi)$ and $\psi_i(\varphi + 2\pi)$

$$\begin{pmatrix} \psi_1(\varphi) \\ \psi_2(\varphi) \end{pmatrix} = M \begin{pmatrix} \psi_1(\varphi + 2\pi) \\ \psi_2(\varphi + 2\pi) \end{pmatrix} \tag{4.43}$$

Furthermore, since both sides have Wronskian equal to one, the matrix M is an element of $\mathrm{SL}(2, \mathbb{R})$ called *monodromy matrix.* Since Hill's equation is linear all pairs of solutions are related by a linear matrix of determinant one. Choosing a different pair of solution therefore changes the the monodromy matrix (4.43) by conjugation. Thus, every Hill's equation with given (p, c) is associated to a monodromy matrix of fixed conjugacy class. Looking at (4.43) shows that the transformed solution $f \cdot \psi$ has a monodromy matrix in the same conjugacy class. Consequently, the conjugacy class of the monodromy matrix M is an orbit-invariant.

Using Hill's equation one can therefore give a rough classification of coadjoint orbits of the Virasoro group using conjugacy classes of $\mathrm{SL}(2, \mathbb{R})$. Orbits that arise by applying (4.38) to a constant p_0, are said to have a *constant representative.*[3] Given a generic point on an orbit with constant representative, it is always possible to find the transformation that brings it to its constant representative (cf. [87, 159] for a more detailed discussion). We can therefore discuss Hill's equation for constant representatives only, which is then given by

$$\frac{c}{6}\psi'' + p_0\psi = 0. \tag{4.44}$$

$p_0 < 0$: In this case a basis of solutions is given by

$$\psi_1 = (2\omega)^{-1/2} e^{\omega\tau}, \qquad \psi_2 = (2\omega)^{-1/2} e^{-\omega\tau} \qquad \omega^2 = \frac{6|p_0|}{c}\,. \tag{4.45}$$

The monodromy matrix is hyperbolic, given by (4.14) with ω as above. The stabilizer equation (4.38) has a single solution $X = \text{const}$ that is globally well-defined. The stabilizer group is therefore $\mathrm{U}(1)$.

[3] In addition to these orbits there are two classes of orbits without constant representatives that we will not discuss further.

$p_0 = 0$: A normalized basis of solutions is given by

$$\psi_1 = \tau, \qquad \psi_2 = 1 \tag{4.46}$$

with parabolic monodromy matrix given by the second matrix in (3.10). The stabilizer group is again U(1).

$p_0 > 0$: For these orbits a set of solutions is

$$\psi_i = \sqrt{\omega}^{-1/2}\sin(\omega\varphi) \qquad \psi_2 = \sqrt{\omega}^{-1/2}\cos(\omega\varphi) \qquad \omega = \sqrt{6p_0/c}\,. \tag{4.47}$$

The monodromy matrix is then given by (4.13) with ω as above. The orbit is elliptic. The only globally well-defined solution to the stabilizer equation is $X = \text{const}$. The stabilizer group is one-dimensional and given by U(1).

We find therefore that every p_0 is associated to a separate orbit. The classification into hyperbolic, elliptic and parabolic orbits is consistent with the one for zero modes given in the previous section.

Notice that for the values

$$p_0 = \frac{n^2 c}{24} \tag{4.48}$$

the monodromy matrix is not elliptic but the (negative) unit matrix. These *exceptional orbits* (called degenerate parabolic in the terminology of [87]) have a three-dimensional stabilizer group given by $\mathrm{PSL}^{(n)}(2,\mathbb{R})$, i.e., the n-fold cover of (4.11). Consistently, the geometry associated to the exceptional point $n = 1$ at $c = 12$ is the hyperbolic plane or the Euclidean black hole geometry (B.11).

As mentioned in the the section 2.3, a coadjoint orbit comes with a natural symplectic structure in the form of the Kostant–Kirillov–Souriau bracket (2.70). This can be used to better understand the boundary dynamics of three-dimensional Einstein gravity in AdS_3. In that case, the algebra of canonical charges is given by (two copies of) the Virasoro algebra. This is precisely the structure that Hamiltonian vector fields associated to canonical transformations on the coadjoint orbit are given from the symplectic structure. This allows a classification of boundary gravitons along the above lines [161]. Furthermore, the classification of coadjoint orbits yields a positive energy theorem for three-dimensional Einstein gravity in AdS_3 [162].

From what we have seen, it seems that we will recover a similar structure in two dimensions. Indeed, the asymptotic symmetries of the two-dimensional Euclidean metric (4.6) form the group of diffeomorphisms of the circle and the function $\mathcal{L}$ defining the geometry transforms according to this symmetry. From this, one could expect that the asymptotic phase space is again an orbit of the Virasoro group, just as in the three-dimensional case. However, this is too hasty a conclusion. Without specifying the symplectic structure of our gravitational theory it is not clear if the transformations are proper or improper gauge transformations, to use the terminology introduced at the end of the last chapter. Furthermore, we have not included the dilaton in our considerations. As we will see explicitly in the next chapters, the boundary conditions on the dilaton will play a crucial role.

We have now collected enough tools to turn to the second part of this thesis containing the original research results wherein some protagonists of this part will be met again.

Part II

Asymptotic Dynamics of Dilaton Gravity

Chapter Five

Constant Dilaton Holography

After collecting and examining our tools we are finally ready to start our discussion of the asymptotic dynamics of two-dimensional dilaton gravity, and thus possible starting points for AdS_2 holography.

In section 3.4 we discussed two solution sectors for dilaton gravity: linear dilaton and constant dilaton solutions. Naïvely, one would expect that the latter is the more basic set-up, thus it will be the starting point in our search for AdS_2 holography. However, there is an immediate caveat: For constant dilaton, the theory (3.5) essentially reduces to a theory of the metric alone, and there is a well-known argument, going back to [45] and rederived coming from various directions [42–44, 163], that pure AdS_2 does not allow for finite energy excitations. We will not repeat this argument here, but rederive it from another perspective. In summary, this avenue, i.e., having constant dilaton, appears not to be very promising as starting point for a holographic duality.

Nevertheless, there have been some attempts to circumvent this conclusion. In particular, a concrete proposal for a model of AdS_2 holography with constant dilaton was made in [164]. They introduced an additional Maxwell field and, using the similarity of the theory to a two-dimensional CFT, found an anomalous transformation behavior for the stress tensor twisted by the $U(1)$ current. Their results were confirmed through a holographic renormalization procedure [165]. This is in line with the discussion at the beginning of section 4.4 in the previous chapter.

However, when calculating the canonical boundary charges á la Brown and Henneaux [26] it turns out that they vanish for this particular model in the classical approximation, see for instance appendix A of [166], reminiscent of the situation in near horizon extremal Kerr [42–44] as was mentioned in the introduction. Thus, from an intrinsic two-dimensional perspective there are no physical states, which is consistent with the general argument of [45] mentioned above.

What was not known until the publication of [4], on which this chapter is based, is to what extent these conclusions are specific to the chosen model, the chosen boundary conditions and/or the classical approximation. It could be that *some* model exhibits non-trivial constant dilaton holography.

Thus, we are going to study an extension of the Euclidean dilaton gravity action (3.5) by a Maxwell field

$$I_{\text{bulk}} = -\frac{k}{4\pi} \int d^2x\sqrt{g}\,\big(XR - U(X)(\partial X)^2 - 2V(X) - \tfrac{1}{4}\,F(X)f_{\mu\nu}f^{\mu\nu}\big) \qquad (5.1)$$

where $k = 1/(4G)$ is inversely proportional to the Newton constant and where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ is the Abelian field strength. The main aim of the present chapter is to clarify the issue raised in the above paragraph by a comprehensive holographic analysis of all the models described by the bulk action (5.1) that allow for AdS_2 solutions with a constant dilaton field. We will find that the PSM formulation will be very convenient to study these constant dilaton vacua (CDVs).

The main conclusion we will find is that AdS_2 holography is trivial for constant dilaton boundary conditions for any choices of the functions U, V and F in the bulk action (5.1). We will show the robustness of our conclusion by considering looser boundary conditions, non-linear interactions of the Maxwell field with the dilaton, inclusion of higher spin fields or generic gauge fields. Therefore, if one would like to study AdS holography in a purely two-dimensional context one has to give up the condition of a constant dilaton. This will be subject of the following chapters.

The outline of this chapter is as follows. In section 5.1 we are going to define the model and determine conditions on our solutions for them to be CDVs in AdS_2. Using the boundary conditions on the metric (4.6) as inspiration we propose AdS_2 constant dilaton boundary conditions. In section 5.2 we discuss the canonical charges for our boundary conditions and show that all of them vanish. We try to circumvent this result by considering various generalizations but always end up with a similar conclusion. In order to determine if the theory can be non-trivial if one considers quantum effects we calculate the one-loop partition function in section 5.3, finding again that the result is trivial.

As everywhere else in this thesis, the viewpoint taken is an intrinsic two-dimensional one, without any relations to higher-dimensional theories, since we are interested in genuine AdS_2 holography. Naturally, our perspective and scope differ from papers that try to connect with AdS_3 holography, see, e.g., [166–168] and Refs. therein.

5.1 Poisson-sigma model formulation of CDVs

Introducing Cartan variables as in section 3.4 converts the second order action (5.1) into a first order action that depends on the zweibein e_a, the dualized spin-connection ω, the gauge connection a, the dilaton X, Lagrange multipliers for the torsion constraint X^a and an auxiliary field f, which on-shell becomes essentially the electric field E. The bulk action is given by

$$I_{\text{bulk}} = -\frac{k}{2\pi} \int \big(X^a\,(de_a + \epsilon_a{}^b\omega\wedge e_b) + X\,d\omega + f\,da + \frac{1}{2}\epsilon^{ab}e_a\wedge e_b\,\mathcal{V}(X^c,\,X,\,f)\big) \qquad (5.2)$$

with

$$\mathcal{V}(X^c,\,X,\,f) = -\tfrac{1}{2}\,X^aX^b\delta_{ab}\,U(X) - V(X) + f^2/F(X)\,. \qquad (5.3)$$

The only difference to the first order action presented in section 3.4 is the inclusion of additional fields related to the Maxwell term in (5.1) and a corresponding change of the potential function $\mathcal{V}$. We will be working in Euclidean signature from now on so that the tangent space metric is just the Kronecker delta. The metric follows in the usual way from the zweibein, $g_{\mu\nu} = e^a_\mu e^b_\nu \delta_{ab}$.

Using the map between dilaton gravity and PSM formulation given in section 3.4 in the other direction we can formulate (5.2) as a PSM model (3.44). The target space is now a four-dimensional space with coordinates $X^I = (X,\, X^a,\, f)$. The connection 1-form A_I has four components $A_X = \omega$, $A_a = e_a$, $A_f = a$ and the Poisson tensor acquires additional (vanishing) entries

$$P^{Xb} = X^a \epsilon_a{}^b \qquad P^{ab} = \mathcal{V}(X^c,\, X,\, f)\epsilon^{ab} \qquad P^{fX} = P^{fa} = 0 \qquad P^{IJ} = -P^{JI}\,. \tag{5.4}$$

For convenience, we write down again the equations of motion that follow from this action:

$$dX^I + P^{IJ} A_J = 0 \tag{5.5a}$$

$$dA_I + \frac{1}{2}\partial_I P^{JK} A_J \wedge A_K = 0\,. \tag{5.5b}$$

As in this chapter we are only interested in CDV solutions of the equations of motion we find

$$X^c = 0 \qquad X = \bar{X} \qquad f = \bar{f} \tag{5.6}$$

in accordance with the discussion in section 3.4. The constants $\bar{X}$ and $\bar{f}$ are related by the condition

$$\mathcal{V}(X^c = 0,\, X = \bar{X},\, f = \bar{f}) = 0 \tag{5.7}$$

which is dictated by the requirement that the Poisson tensor vanishes on-shell. The first set of equations (5.5a) then holds trivially. Equations (5.5b) for the connection 1-forms A_I imply vanishing torsion T_a, constant curvature R and constant electric field E

$$de_a + \epsilon_a{}^b\, \omega \wedge e_b = 0 = T_a, \tag{5.8a}$$

$$* \, d\omega = -\partial_X \mathcal{V} = \frac{1}{2}\, R, \tag{5.8b}$$

$$* \, da = -\partial_f \mathcal{V} = -E\,. \tag{5.8c}$$

To obtain an AdS_2 solution with unit AdS radius, $R = -2$, we additionally demand

$$\partial_X \mathcal{V}(X^c,\, X,\, f)\big|_{X^c=0,\, X=\bar{X},\, f=\bar{f}} = 1\,. \tag{5.9}$$

Since $\mathcal{V}$ has dimension of inverse length squared, condition (5.9) can be achieved always by a rescaling of units, provided the quantity $\partial_X \mathcal{V}$ is positive on a given CDV, which is necessarily the case for AdS vacua.

Boundary conditions in the PSM formulation

The line-element for AdS_2 in Fefferman–Graham gauge was already introduced in the previous section. Based on equation (4.6) we define CDV boundary conditions in the PSM formulation that lead asymptotically to the above solutions for the line-element and constant dilaton and electric field.

$$X^0 = 0 \qquad e_{\tau 0} = \tfrac{1}{2}\, e^{\rho} - \, e^{-\rho}\mathcal{L}(\tau) + \mathcal{O}(e^{-3\rho}) \qquad e_{\rho 0} = 0 \tag{5.10a}$$
$$X^1 = 0 \qquad e_{\tau 1} = 0 \qquad e_{\rho 1} = 1 \tag{5.10b}$$
$$X = \bar{X} \qquad \omega_\tau = -\tfrac{1}{2}\, e^{\rho} - \, e^{-\rho}\mathcal{L}(\tau) + \mathcal{O}(e^{-3\rho}) \qquad \omega_\rho = 0 \tag{5.10c}$$
$$f = \bar{f} \qquad a_\tau = E\,\omega_\tau + j(\tau) + \mathcal{O}(e^{-2\rho}) \qquad a_\rho = 0 \tag{5.10d}$$

The constants $\bar{X}$ and $\bar{f}$ are related through the conditions (5.7) and the electric field E is given by (5.8c). We have gauge-fixed as much as possible, using Fefferman–Graham gauge for the zweibein and axial gauge for spin- and gauge-connections. These are essentially the same boundary conditions as used in [165], reformulated in PSM language and generalized to arbitrary dilaton gravity models (5.1). There are two free functions of the Euclidean time τ appearing in our boundary conditions, $\mathcal{L}(\tau)$ and $j(\tau)$.

We consider now all transformations (3.50) that preserve the gauge and boundary conditions (5.10) and find

$$\lambda_0 = \tfrac{1}{2}\,\lambda(\tau)e^{\rho} + \big(- \lambda(\tau)\mathcal{L}(\tau) + \lambda''(\tau)\big)\, e^{-\rho} \tag{5.11a}$$
$$\lambda_1 = -\lambda'(\tau) \tag{5.11b}$$
$$\lambda_X = -\tfrac{1}{2}\,\lambda(\tau)e^{\rho} + \big(- \lambda(\tau)\mathcal{L}(\tau) + \lambda''(\tau)\big)\, e^{-\rho} \tag{5.11c}$$
$$\lambda_f = E\,\lambda_X + \mu(\tau)\,. \tag{5.11d}$$

Thus, we have two free functions, $\lambda(\tau)$ and $\mu(\tau)$, parametrizing all allowed boundary condition preserving transformations. In the nomenclature given at the end of section 3.5 these correspond to the allowed symmetry transformations that preserve the space of field configurations $\mathcal{F}$ defined by the boundary conditions (5.10).

Under the transformations (5.11) the free functions $\mathcal{L}$ and j change according to

$$\delta\mathcal{L} = \mathcal{L}'\lambda + 2\mathcal{L}\lambda' + \lambda''' \tag{5.12}$$
$$\delta j = -\mu'\,. \tag{5.13}$$

These results are compatible with the ones in [164–166], but now are valid for arbitrary dilaton gravity models with an AdS CDV. In particular, the presence of a Maxwell field is in no way essential for the appearance of the infinitesimal Schwarzian derivative in (5.12), that we already saw in the study of allowed symmetries in the second order formulation (4.22).

Introducing the normalization factor α for the Virasoro zero mode, $L_0 = \alpha\mathcal{L}_0$, we find that our result (5.12) is compatible with the assumption that the asymptotic symmetry algebra contains a Virasoro algebra $[L_n,\, L_m] = (n-m)\,L_{n+m} + c/12\,(n^3-n)\,\delta_{n+m,\,0}$ with central charge

$$c = 12\alpha \tag{5.14}$$

again in concordance with the results in [164]. The chiral Cardy formula would then yield an entropy

$$S_{\text{Cardy}} = \frac{\pi^2 c T}{3} = 2\pi\sqrt{\frac{c\mathcal{L}_0}{6}} = 2\pi\alpha\sqrt{\mathcal{L}_0}\,. \tag{5.15}$$

However, we have not checked yet whether there is a non-trivial asymptotic symmetry algebra in the first place; it could be that the transformations (5.12), (5.13) are pure gauge, in which case the theory would contain no physical states besides the vacuum. Therefore, we will now turn to the charges constructed in section 3.5.

5.2 Canonical charges of the model

Repeating expression (3.108), the canonical boundary currents of the theory are given by

$$\delta Q[\lambda_I] = \frac{k}{2\pi}\,\delta X^I\,\lambda_I\big|_{\rho\to\infty}\,. \tag{5.16}$$

The canonical currents (5.16) vanish identically for our boundary conditions (5.10) since $\bar{X}$ is assumed to be a fixed quantity for the whole phase space. *The canonical charges are state-independent and hence trivial.* This means that the asymptotic symmetry algebra is empty, and the boundary condition preserving transformations (5.12), (5.13) are pure gauge.

Another interesting property of the canonical currents (5.16) is their gauge invariance

$$\delta_{\lambda^2_J} Q[\lambda^1_I] = \frac{k}{2\pi}\,P^{IJ}\lambda^2_J\,\lambda^1_I\big|_{\rho\to\infty} = 0 \tag{5.17}$$

due to the CDV conditions (5.6). This shows that even non-infinitesimal transformations (connected with the identity) cannot make the canonical currents non-trivial.

Another indication that the theory is empty comes from the on-shell action. Notice that the PSM action (3.44) already has a well-defined variational principle for our boundary conditions, without the need to add a further boundary term

$$\delta\Gamma\big|_{\text{CDV}} = \frac{k}{2\pi}\int_{\partial\mathcal{M}} A_I\,\delta X^I = 0\,. \tag{5.18}$$

According to the saddle-point approximation to the Euclidean path integral an action with a well-defined variational principle, evaluated on a solution yields the free energy of that configuration. We find for all solutions encompassed by our boundary conditions (5.10)

$$\Gamma\big|_{\text{CDV}} = 0\,. \tag{5.19}$$

All of this can be regarded as a reformulation of the statement in [45] mentioned above that there exist no finite energy states above AdS_2. It is worth stressing again that this is completely independent of the chosen model and of the presence or absence of a Maxwell field.

Possible Generalizations

We will now try to circumvent these conclusions by considering looser boundary conditions and more complicated interactions.

Looser boundary conditions. Perhaps the boundary conditions (5.10) are simply too strict. Indeed, we have switched off all fluctuations of the target space coordinates, but we could have allowed instead some asymptotic fall-off. From the canonical currents (5.16) we see that a fall-off behavior of the dilaton field of the form $\delta X = \mathcal{O}(e^{-\rho})$ could produce finite canonical charges, since the gauge parameter λ_X in (5.11) diverges like e^{ρ}. Motivated by this observation we consider now looser boundary conditions that allow for such terms.

$$X^0 = X^0_{(1)}(\tau)e^{-\rho} + \mathcal{O}(e^{-2\rho}) \tag{5.20a}$$
$$X^1 = X^1_{(1)}(\tau)e^{-\rho} + \mathcal{O}(e^{-2\rho}) \tag{5.20b}$$
$$X = \bar{X} + X_{(1)}(\tau)e^{-\rho} + \mathcal{O}(e^{-2\rho}) \tag{5.20c}$$
$$f = \bar{f} + f_{(1)}(\tau)e^{-\rho} + \mathcal{O}(e^{-2\rho}) \tag{5.20d}$$

$$e_{\tau 0} = \tfrac{1}{2}\, e^{\rho} + e^{(0)}_{\tau 0}(\tau) + e^{(1)}_{\tau 0}(\tau)e^{-\rho} + \mathcal{O}(e^{-2\rho}) \tag{5.20e}$$
$$e_{\rho 0} = e^{(1)}_{\rho 0}(\tau)e^{-\rho} + \mathcal{O}(e^{-2\rho}) \tag{5.20f}$$
$$e_{\tau 1} = e^{(0)}_{\tau 1}(\tau) + e^{(1)}_{\tau 1}(\tau)e^{-\rho} + \mathcal{O}(e^{-2\rho}) \tag{5.20g}$$
$$e_{\rho 1} = 1 + e^{(1)}_{\rho 0}(\tau)e^{-\rho} + \mathcal{O}(e^{-2\rho}) \tag{5.20h}$$
$$\omega_{\tau} = -\tfrac{1}{2}\, e^{\rho} + \omega^{(0)}_{\tau}(\tau) + \omega^{(1)}_{\tau}(\tau)e^{-\rho} + \mathcal{O}(e^{-2\rho}) \tag{5.20i}$$
$$\omega_{\rho} = \omega^{(1)}_{\rho}(\tau)e^{-\rho} + \mathcal{O}(e^{-2\rho}) \tag{5.20j}$$
$$a_{\tau} = E\,\omega_{\tau} + a^{(0)}_{\tau}(\tau) + a^{(1)}_{\tau}(\tau)e^{-\rho} + \mathcal{O}(e^{-2\rho}) \tag{5.20k}$$
$$a_{\rho} = a^{(1)}_{\rho}(\tau)e^{-\rho} + \mathcal{O}(e^{-2\rho}) \tag{5.20l}$$

Again, the constants $\bar{X}$ and $\bar{f}$ are related through the conditions (5.7) and the electric field E is given by (5.8c). Note that we particularly allow for fluctuations

$$\delta X^I = \mathcal{O}(e^{-\rho})\,. \tag{5.21}$$

Since we are mostly interested in the evaluation of the canonical currents (5.16), we impose on-shell conditions on all the fluctuation terms that we have written explicitly. The EOM impose the conditions

$$X_{(1)} = X^0_{(1)} \tag{5.22}$$
$$f_{(1)} = 0 \tag{5.23}$$

on the subleading components of the target space coordinates. There are further restrictions on the functions appearing in the loose boundary conditions (5.20), but we do not need them for our conclusions. Note that the conditions above imply

$$\delta X^0 = \delta X + \mathcal{O}(e^{-2\rho}) \qquad \delta f = \mathcal{O}(e^{-2\rho})\,. \tag{5.24}$$

The gauge parameters that preserve the boundary conditions (5.20) can be similarly expanded

$$\lambda_0 = \tfrac{1}{2}\,\lambda(\tau)e^{\rho} + \lambda_0^{(0)} + \lambda_0^{(1)}e^{-\rho} + \mathcal{O}(e^{-2\rho}) \tag{5.25a}$$
$$\lambda_1 = \lambda_1^{(0)} + \lambda_1^{(1)}e^{-\rho} + \mathcal{O}(e^{-2\rho}) \tag{5.25b}$$
$$\lambda_X = -\tfrac{1}{2}\,\lambda(\tau)e^{\rho} + \lambda_X^{(0)} + \lambda_X^{(1)}e^{-\rho} + \mathcal{O}(e^{-2\rho}) \tag{5.25c}$$
$$\lambda_f = E\,\lambda_X + \mu(\tau) + \lambda_f^{(1)}e^{-\rho} + \mathcal{O}(e^{-2\rho})\,. \tag{5.25d}$$

Again, there will be restrictions on the functions appearing in the gauge parameters (5.25), and again we do not need them for our conclusions.

With the boundary conditions and gauge parameters above the canonical currents (5.16) expand to a sum of order unity terms [due to (5.21)] and subleading terms

$$\delta Q[\lambda_I] = \frac{k\lambda(\tau)}{4\pi}\,e^{\rho_c}\big(\delta X^0 - \delta X - E\,\delta f\big) + \mathcal{O}(e^{-\rho_c})\,. \tag{5.26}$$

Here $\rho_c \gg 1$ is the cut-off surface where the charges are evaluated. Taking the cut-off to infinity, $\rho_c \to \infty$, removes the subleading terms $\mathcal{O}(e^{-\rho_c})$. However, due to the relations (5.24) the order unity terms cancel precisely and the canonical currents vanish.

Therefore, even for the looser set of boundary conditions (5.20) the canonical charges are trivial.

Let us discuss one final generalization of our boundary conditions. We can allow X and f to fluctuate to $\mathcal{O}(1)$, as long as the condition (5.7) remains intact. This modifies the previous boundary conditions by making $\bar{X}$ and $\bar{f}$ state dependent, so that the following fluctuations are allowed additionally

$$\delta X = -E\,\delta f \qquad \delta f = \mathcal{O}(1)\,. \tag{5.27}$$

In this case there is a non-trivial, integrable and finite $U(1)$ charge.

$$Q[\mu] = \frac{k}{2\pi}\,f\mu \tag{5.28}$$

However, there are still no diffeomorphism charges, and the asymptotic symmetry algebra is trivial, since any gauge variation of the charge (5.28) vanishes due to $\delta_{\lambda_I} f = P^{fI}\lambda_I = 0$. We consider this case as somewhat artificial, as the boundary electric field (5.8c) is allowed to vary.[1]

Having failed in circumventing the triviality statement by considering looser boundary conditions one could conceive of adding interactions in the form of Yang–Mills or higher spin fields to find non-vanishing charges. However, as is argued in [4] also these fields do not change the conclusion.

In all the examples so far we have seen that the canonical diffeomorphism charges are trivial classically. In the next section we check indirectly if this statement also holds at the quantum level by calculating the full quantum gravity partition function.

[1] This choice of boundary conditions was also discussed in [168].

5.3 Quantum gravity partition function

The canonical analysis of the previous section was classical. It is conceivable that switching on quantum effects makes the theory non-trivial. After all, the asymptotic symmetry algebra and the canonical charges could receive quantum corrections, so even if the classical results show triviality the quantum mechanical results might be non-trivial. In this section we rule out this possibility by considering the full quantum gravity partition function and showing that it is unity.

Classical partition function

We use the Euclidean path integral formulation [169, 170]. Our aim is to determine the full quantum gravity partition function

$$Z = \int_{\text{bc}} (\mathcal{D}X^I)(\mathcal{D}A_I)(\text{measure}) \, \exp\big(-\Gamma[X^I,\, A_I]\big) \tag{5.29}$$

where 'bc' denotes that we evaluate the path integral for certain boundary and smoothness conditions, 'measure' refers to the ghost- and gauge-fixing part, and Γ is the full action, i.e., including boundary terms such that the variational principle is well-defined. Results for the exact path integral have shown quantum triviality, i.e., the quantum partition function equals the classical one [171]. However, the previous calculations did not take into account asymptotic boundary conditions, nor possible global effects, nor instanton contributions. This is why we re-evaluate the path integral. As we shall see, the local results of [171] are not modified globally for CDVs.

We make now an expansion of the path-integral into classical contribution (c), perturbative corrections (p) and non-perturbative corrections (n).

$$Z = Z_{\text{c}} \times Z_{\text{p}} \times Z_{\text{n}} \tag{5.30}$$

We start with the classical piece.

$$Z_{\text{c}} = \exp\big(-\Gamma|_{\text{CDV}}\big) \tag{5.31}$$

With the result (5.19) we then obtain

$$Z_{\text{c}} = 1\,. \tag{5.32}$$

Thus, the classical partition function is trivial, which concurs of course with the conclusions of section 5.2 that the canonical charges are trivial.

Perturbative corrections

Let us consider now the perturbative corrections Z_{p} to the classical partition function (5.32). Given that our theory is a topological field theory of Schwarz type, one can argue that the theory should be one-loop exact, along similar lines as [35] (who applied this to 3-dimensional gravity). We assume that there is no relevant subtlety with these arguments, so that the one-loop partition function captures the full information about all perturbative corrections.

In the one-loop calculation we use bars to denote classical values, while un-barred quantities will be quantum fluctuations. The action quadratic in quantum fluctuations reads

$$S_2 = -\frac{k}{2\pi}\int d^2x \Big[\tilde{\epsilon}^{\mu\nu} X^I(\partial_\mu A_{\nu I} + \Omega_{\mu I}{}^J A_{\nu J}) + (\bar{e})\frac{1}{2}\frac{\partial^2\mathcal{V}(\bar{X}^K)}{\partial X^I \partial X^K} X^I X^K\Big] \tag{5.33}$$

where $(\bar{e}) = \frac{1}{2}\tilde{\epsilon}^{\mu\nu}\epsilon^{ab}\bar{e}_{\mu a}\bar{e}_{\nu b}$ and

$$\Omega_{\mu a}{}^b = \bar{\omega}_\mu \epsilon_a{}^b \tag{5.34}$$
$$\Omega_{\mu a}{}^X = -\epsilon_a{}^b \bar{e}_{\mu b} \qquad \Omega_{\mu X}{}^a = -\epsilon^{ab}\bar{e}_{\mu b} \tag{5.35}$$
$$\Omega_{\mu f}{}^a = -\bar{E}\epsilon^{ab}\bar{e}_{\mu b}\,. \tag{5.36}$$

All other components of the connection Ω vanish. To derive (5.33) we used that the classical fields satisfy (5.8)-(5.9) for an AdS_2 CDV with unit AdS radius.

The same connection appears in the linearized gauge transformations

$$\delta_\lambda X^J = 0 \qquad \delta_\lambda A_{\mu I} = -\partial_\mu \lambda_I - \Omega_{\mu I}{}^J \lambda_J \equiv -D_\mu \lambda_I\,. \tag{5.37}$$

The invariance of the quadratic action (5.33) under gauge transformations (5.37) implies that the connection Ω is flat.

$$[D_\mu, D_\nu] = \partial_\mu \Omega_\nu - \partial_\nu \Omega_\nu + [\Omega_\mu, \Omega_\nu] = 0 \tag{5.38}$$

The flatness of the connection can also be verified by direct calculation.

Let us expand the fluctuations $A_{\mu I}$ into a sum of gauge (λ) and transverse (χ) parts.

$$A_{\mu I} = -D_\mu \lambda_I + \varepsilon_\mu{}^\nu D_\nu^\dagger \chi_I + A_{\mu I}^{(\mathrm{h})} \tag{5.39}$$

Here $\varepsilon_\mu{}^\nu$ is the Levi–Civitá tensor. $A_{\mu I}^{(\mathrm{h})}$ correspond to square integrable harmonic one-forms, that are both longitudinal and transverse and are given by gradients (with D_μ) of non-normalizable zero modes of the scalar operator $D_\mu^\dagger D_\mu$ (cf., e.g., [172]). As argued in [173], the harmonic one-forms correspond to boundary modes of the theory. Interestingly, these modes do not contribute to the quadratic action (5.33). These facts hint to the holographic triviality of CDVs. This is in contrast to the model considered in [174] where non-integrable scalar modes on the hyperbolic plane generate physical boundary states.

The presence of infinitely many harmonic one-forms complicates computations of the partition function on $\mathbb{H}^2$. To avoid this difficulty we will analytically continue the partition function to the sphere S^2. The unit S^2 is a CDV corresponding to the zeros of $\mathcal{V}(X, X^a, f)$ where $\partial_X \mathcal{V} = -1$ instead of $+1$ in Eq. (5.9). Non-vanishing components of the zweibein and spin-connection read: $\bar{e}_{\rho 1} = 1$, $\bar{e}_{\tau 0} = \sin(\rho)$, $\bar{\omega}_\tau = -\cos(\rho)$. The only modification of the connection Ω_μ is the sign flip of $\Omega_{\mu X}{}^a$, that becomes

$$\Omega_{\mu X}{}^a = -\epsilon^{ab}\bar{e}_{\mu b} \tag{5.40}$$

on S^2.

We define the path integral measure $\mathcal{D}A_{\mu J}$ by the identity

$$\int \mathcal{D}A_{\mu J}\, e^{-\langle A, A\rangle} = 1\,. \tag{5.41}$$

The path integral measure is, therefore, defined by the inner product $\langle\ ,\ \rangle$. We take an ultralocal product

$$\langle A, A' \rangle = \int d^2x(\bar{e})\delta^{IJ}\bar{g}^{\mu\nu}A_{\mu I}A'_{\nu J}\,. \tag{5.42}$$

The connection Ω is not hermitian with respect to the inner product (5.42). This implies in particular that $D^\dagger \neq -D$. However, it can be transformed to a hermitian one,

$$D_\mu = \Phi^{-1}\hat{D}_\mu\Phi \qquad \hat{D} = -\hat{D}^\dagger \tag{5.43}$$

with the field

$$\Phi = \ \mathrm{Id} + \phi \qquad {\phi_f}^X = -\bar{E} \tag{5.44}$$

where Id is the identity. Notice that for $E = 0$, we have $\Phi = \mathrm{Id}$ and the connection is Hermitian.

The change of variables $A_{\mu J} \to \lambda_J,\ \chi_J$ induces a Jacobian factor, $\mathcal{D}A_{\mu J} = \mathcal{J}\,\mathcal{D}\lambda_J\mathcal{D}\chi_J$, which can be easily found by substituting the decomposition (5.39) in the definition of the measure (5.41) and performing Gaussian integrals over λ and χ. This yields the Jacobian

$$\mathcal{J} = \det(D^\dagger_\mu D^\mu)^{\frac{1}{2}} \cdot \det(D_\mu D^{\mu\dagger})^{\frac{1}{2}}\,. \tag{5.45}$$

The one-loop partition function then decomposes into path integrals over X, λ and χ.

$$\begin{aligned} Z &= \int \mathcal{D}X\,\mathcal{D}A\,\exp(-S_2) \\ &= \int \mathcal{D}X\,\mathcal{D}\lambda\,\mathcal{D}\chi\,\mathcal{J}\exp\left[\frac{k}{2\pi}\int d^2x(\bar{e})\left(-X^I D^\mu D^\dagger_\mu \chi_I + \frac{1}{2}\frac{\partial^2\mathcal{V}(\bar{X}^K)}{\partial X^I \partial X^K}X^I X^K\right)\right] \end{aligned} \tag{5.46}$$

The integration over λ is performed trivially, yielding an infinite volume of the gauge group, which we discard. The integration over X^I and χ_I gives

$$Z = \mathcal{J}\cdot\det(D^\mu D^\dagger_\mu)^{-1} = \frac{\det(D^\dagger_\mu D^\mu)^{\frac{1}{2}}}{\det(D_\mu D^{\mu\dagger})^{\frac{1}{2}}}\,. \tag{5.47}$$

Interestingly, the terms in S_2 that are quadratic in fluctuations of the target space coordinates X^I have no influence on the partition function. This means that our results are universal for AdS_2 CDVs, regardless of the specific properties of the potentials in the action. Note that for $\bar{E} = 0$ we have $D = -D^\dagger$, and the partition function is trivial, $Z = 1$. This means we have proven that the one-loop partition function is trivial if the electric field vanishes.

The only reason why the partition function (5.47) has a chance to be non-trivial is that the transformation Φ is not unitary. In order to prove that the partition function vanishes for non-zero electric field one parametrizes the transformation Φ in terms of a parameter α and calculates $\delta_\alpha Z$. We will not go into the details of these rather involved calculations based on [175, 176] but refer to the original paper [4]. As final result we find

$$\delta_\alpha Z = 0\,. \tag{5.48}$$

Thus, the partition function is trivial, independently of the presence or absence of the electric field. A similar conclusion can be reached based on the Euclidean path integral in the second order approach [4].

Using our arguments above on one-loop exactness we have then the result

$$Z_{\mathrm{p}} = 1\,. \tag{5.49}$$

Thus, there are no perturbative corrections to the classical partition function.

Nonperturbative Corrections

Let us finally consider non-perturbative corrections. These come from all classical saddle points consistent with our boundary conditions (5.10), given a periodicity β of the boundary coordinate τ, and smoothness conditions, which we now specify. We allow all smooth Euclidean saddle points; in particular we prohibit conical singularities. Thus, only two saddle points are possible, namely global AdS_2 and the Poincaré disk corresponding to values $\mathcal{L} = -\frac{1}{2}$ and $\mathcal{L} = \frac{1}{2}$, respectively, in Fefferman–Graham gauge (4.6). Note that these two saddle points have different topologies: the former is topologically a cylinder, the latter topologically a plane.

Thus, for fixed topology[2] there is only one allowed saddle point and we find no instanton corrections.

$$Z_{\mathrm{n}} = 1 \tag{5.50}$$

In summary, the results (5.32), (5.49) and (5.50) together show that the full partition function (5.30) is trivial,

$$Z = Z_{\mathrm{c}} \times Z_{\mathrm{p}} \times Z_{\mathrm{n}} = 1\,. \tag{5.51}$$

We conclude that AdS_2 holography is trivial for CDVs not just classically, but also in the full quantum theory, which has only one physical state, the vacuum.

Since we originally set out to find AdS_2 holography we have to admit that our quest failed for CDVs. The only state to match to a possible boundary theory would be the vacuum. Thus, we showed in this chapter that the constant dilaton sector is just too trivial to allow for interesting dynamics to match to a possible boundary theory. Having no other options left, we return to the crossroads between constant and linear dilaton sectors, to take the path of linear dilaton solutions.

[2] It is conceivable to sum over both topologies. Then each saddle point contributes with a trivial partition function to the full partition function. However, it would still be a state-independent number and thus of no physical significance.

Chapter Six

Jackiw-Teitelboim Model

We saw in the previous chapter that CDVs in generic models of two-dimensional dilaton gravity are too simple to allow any physical states. It seems therefore that one has to turn to models with linear dilaton solutions in order to find more interesting physics. Unfortunately, one has to pay for this increase in complexity with the ability to treat all models in one go, as we did in the previous chapter. In this chapter we are going to start with the simplest model of dilaton gravity that allows for linear dilaton solutions in AdS_2: the *Jackiw–Teitelboim model* [79, 80].

The study of dilaton gravity in two dimensions began with this model in the 1980's and has been punctuated by periods of increased interest in the community. For instance, in the early 1990s, work on the string theory black hole [108, 109, 177, 178] and the CGHS model [111] triggered a new round of activity and led to the emergence of a host of new models [179, 180]. Neglecting global effects, the path integral for all of them was calculated in [171]. See the book by Brown [181] for an account of the first five years, the review [99] for a summary of the first eighteen years, and table 1 in [113] for a (non-exhaustive) list of models.

Naturally, only after the late 1990s dilaton gravity was revisited in the context of AdS/CFT [23–25] and holographic renormalization [142, 182–184]. Interest in AdS_2 holography has been re-invigorated by recent work [47] on the Sachdev–Ye–Kitaev (SYK) model [46, 48, 185]. The main reason for this is that the latter model shows an emergent reparametrization invariance in the infrared that is spontaneously broken to a $SL(2, \mathbb{R})$. As we will see, a similar pattern of symmetry breaking is found in the JT model. In the SYK model, the resulting Goldstone modes are governed by an effective action, the *Schwarzian action*, that emerges naturally as a boundary action in the case of dilaton gravity. Appendix D contains an introduction to the SYK model.

The outline of this chapter is as follows. Section 6.1 introduces the JT model and defines the boundary conditions we are going to use. They differ in some respects from those found in the previous literature. In particular, we assume that the dilaton is not fixed at the boundary but transforms under the symmetry transformations. This circumvents the usual argument that only constant dilaton vacua are consistent with all of the isometries of AdS_2. This possibility was discussed for the first time in [186], to the best of the author's knowledge. A short detour into the conformal boundary framework of Penrose sheds some light on the geometric interpretation of our boundary conditions. In section 6.2 we discuss the solutions of the JT model. Section 6.3 shows that the JT model has a well-defined variational principle when certain boundary terms are added to the action. We will furthermore discuss the asymptotic Killing vectors of the model. In section 6.4 we will come to one of the main points of this chapter, namely a derivation of the Schwarzian action from the JT model. To this end, some material of section 4.4 on the coadjoint orbit of the Virasoro group will come in handy. We will end this chapter with a discussion of the canonical charges of the JT model in section 6.5, where we will see another indication for the spontaneous symmetry breaking mentioned above.

Throughout this chapter we work with the second order action of dilaton gravity in Euclidean signature at finite temperature β. The JT model is reformulated as a PSM model in section 7. Based on this we will then discuss certain generalizations of the JT model and their corresponding Schwarzian-like actions.

6.1 The model

The JT model is defined by the choice of functions $U(X) = 0\,, V(X) = -X$ in the action (3.5) for dilaton gravity.[1] This leads to the second-order action

$$I_{JT} = -\frac{k}{4\pi}\int_{\mathcal{M}} d^2x\sqrt{g}\,X(R+2)\,, \tag{6.1}$$

where again $k = 1/(4G)$. From this we can see immediately that variation with respect to X yields $R = -2$. Thus, the solutions for the metric will be locally AdS_2. In total we find

$$\mathcal{E}_{\mu\nu} = g_{\mu\nu}\,X + \nabla_\mu\nabla_\nu X - g_{\mu\nu}\,\nabla^2 X = 0 \tag{6.2a}$$

$$\mathcal{E}_X = R + 2 = 0\;, \tag{6.2b}$$

as equations of motion.

Without additional boundary terms the action (6.1) is not well-defined. As is well-known, a Gibbons–Hawking–York boundary term is required if the metric is to be subject to Dirichlet boundary conditions only. Secondly, without specification of the boundary conditions the functional space on which the action is extremized is not even defined. In general, further counterterms have to be added to the action, so that the classical solution is indeed an extremum for all variations compatible with the boundary conditions. We will do so in section 6.3.

Our boundary conditions are as follows: In a coordinate system (ρ, τ) that is valid in the neighborhood of the boundary $\partial\mathcal{M}$, metric and dilaton field are of the form

$$ds^2 = d\rho^2 + (O(e^{2\rho}) + \ldots)\,d\tau^2 \qquad X = O(e^\rho)\,. \tag{6.3}$$

Notice that these are looser boundary conditions for the metric than

$$ds^2 = d\rho^2 + (\frac{1}{2}e^{2\rho} + O(1))\,d\tau^2 \qquad X = O(e^\rho)\,, \tag{6.4}$$

[1] A variant of the JT model in which the dilaton is shifted so that $V(X) = -X + a$, has been dubbed Almheiri–Polchinski model [163]. One of the properties of this model is that it allows for both linear and constant dilaton solutions, while the latter would be considered singular $X = 0$ in the JT model.

where the boundary metric is fixed, that led to the Fefferman-Graham form (4.6).[2] Note in particular that the boundary metric is not fixed by the conditions (6.3) and the dilaton is allowed to fluctuate to leading order. How can the boundary conditions (6.3) be consistent with the explanation surrounding equation (4.6)? In order to clarify this issue it is worthwhile to take an excursion into the conformal framework of Penrose [155].

A conformal excursion. Let us try to find a covariant formulation of what we mean by a two-dimensional (Euclidean) AdS_2 manifold $\mathcal{M}$. We assume: i) there exists a conformal factor $\Omega > 0$ such that the unphysical metric $\tilde{g} = \Omega^2 g$ has a smooth limit to $\partial\mathcal{M}$ with $\Omega = 0$ on $\partial\mathcal{M}$ and $\nabla_a\Omega$ nowhere vanishing on $\partial\mathcal{M}$; ii) the boundary has the topology of a circle (being at finite temperature)[3]; iii) the Ricci scalar $R = -2 + O(\Omega^2)$ at $\partial\mathcal{M}$.[4]

The derivative of the boundary defining function Ω yields the normal vector to the boundary $n_a = \nabla_a\Omega$. A conformal transformation of the Ricci scalar gives

$$\Omega^2\tilde{R} = (R - 2\Omega\tilde{\nabla}_a\tilde{n}^a + 2\tilde{n}^a\tilde{n}_a), \tag{6.5}$$

where tilded quantities are raised with the unphysical metric, $\tilde{\nabla}$ is the covariant derivative compatible with the unphysical metric and $\tilde{R}$ is the Ricci scalar of the unphysical metric. Since the unphysical metric is smooth near the boundary, the unphysical Ricci scalar is smooth as well, and taking the limit $\Omega \to 0$ we find

$$1 \,\widehat{=}\, \tilde{n}^a\tilde{n}_a\,, \tag{6.6}$$

where $\widehat{=}$ denotes evaluation at the boundary. If we were in Lorentzian signature, we would learn from this equation that the boundary is time-like.

The behavior of the dilaton near the boundary is obtained by checking the conformally transformed equation of motion (6.2) for smoothness. From this one finds that

$$\tilde{X} = \Omega X \tag{6.7}$$

is smooth at the boundary. The asymptotic structure of this system is therefore defined by the pair $(\underline{\tilde{g}_{ab}}, \underline{\tilde{X}})$ where the line denotes pullback to the boundary. But notice that this is not unique, as a change in the conformal factor $\Omega \to \omega\Omega$ with $\omega > 0$ induces a change

$$(\underline{\tilde{g}_{ab}}, \underline{\tilde{X}}) \to (\omega^2\underline{\tilde{g}_{ab}}, \omega\underline{\tilde{X}}) \tag{6.8}$$

on $\partial\mathcal{M}$. If we do not want to introduce further structure on the manifold, in the form of the conformal factor, then the two pairs should be equivalent. Thus, as part of the boundary conditions only the conformal class of the boundary metric $\underline{\tilde{g}_{ab}}$ is specified. Similarly, the boundary dilaton is only specified up to multiplication by an arbitrary function.

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[2] In the original work [2] even looser boundary conditions were considered. In order to reduce clutter and in an attempt to focus on particular results we will only discuss (6.3) and (6.4) in this thesis.

[3] We assume that the spacetime has only one boundary, i.e., it has the topology of a (possibly singular) disc.

[4] In higher dimension D condition (ii) is replaced by the requirement that the boundary is topologically $S^{D-2} \times S$ (or $S^{D-2} \times \mathbb{R}$ in Lorentzian signature) and (iii) is replaced by a condition on the asymptotic behavior of the stress-energy tensor and the requirement that Einstein's equations hold.

In this conformal framework, an asymptotic symmetry is a vector field ξ in $\mathcal{M}$ that has a smooth extension to the boundary $\partial\mathcal{M}$ and induces the change (6.8) in the boundary structure by acting on the physical metric with the usual Lie derivative. We have

$$\delta_\xi \tilde{g}_{ab} \equiv \Omega^2 \pounds_\xi g_{ab} = \pounds_\xi \tilde{g}_{ab} - 2\Omega^{-1}\xi^c n_c \tilde{g}_{ab} \tag{6.9a}$$

$$\delta_\xi \tilde{X} \equiv \Omega \pounds_\xi X = \pounds_\xi \tilde{X} - \Omega^{-1}\xi^c n_c \tilde{X}\,. \tag{6.9b}$$

Since this should have a smooth limit to $\partial\mathcal{M}$ we find that $\xi^c n_c = \Omega K$ for some arbitrary function K that is smooth at the boundary. It is important to stress that this only fixes a trivial symmetry generator, as this component actually vanishes at the boundary.[5] Setting $K = 0$ and calculating the pull-back of (6.9), we find that vector fields acting as conformal Killing vectors for the boundary metric $\underline{\tilde{g}_{ab}}$ produce the change (6.8). This is another derivation of the fact that the asymptotic symmetries of asymptotically AdS-spacetimes (note that nothing essential about our derivation assumed two dimensions) correspond to the conformal symmetries of the boundary metric, usually taken to be the flat metric.[6]

Two comments are in order. Firstly, it is important to notice that the conformal factor Ω was assumed to be part of the background structure in equation (6.9) since it was not varied under a symmetry transformation $\delta_\xi \Omega = 0$, in line with the general principle (1.2). Secondly, in most applications the coefficient of the subleading term K is chosen in such a way that the boundary metric does not transform $\delta_\xi \tilde{g}_{ab} = 0$.

We are now in the position to understand the two sets of boundary conditions (6.3) and (6.4). Starting with the latter, suppose we are given the boundary data $(\underline{\tilde{g}_{ab}}, \underline{\tilde{X}})$ being arbitrary functions. We can use the freedom (6.8) to set one of the two functions to any value, since the boundary metric is one-dimensional and thus conformal to any other one-dimensional metric. We decide to fix the boundary metric to $\underline{\tilde{g}^{(0)}_{ab}}$ to the value $\frac{1}{2}$ so that the boundary data for the dilaton remains arbitrary. Since the asymptotic symmetries act on $\underline{\tilde{g}^{(0)}_{ab}}$ as conformal Killing vectors we have

$$\delta_\xi \underline{\tilde{g}^{(0)}_{ab}} \,\widehat{=}\, \pounds_\xi \underline{\tilde{g}^{(0)}_{ab}} - 2K \underline{\tilde{g}^{(0)}_{ab}} = 2D_a\xi^a \underline{\tilde{g}^{(0)}_{ab}} - 2K \underline{\tilde{g}^{(0)}_{ab}} \tag{6.10}$$

where D_a denotes the covariant derivative with respect to the boundary metric, and the factor follows from taking the trace of the conformal Killing equation. Thus, we can choose $K = D_a\xi^a$ in order to preserve the above condition, $\delta_\xi \underline{\tilde{g}^{(0)}_{ab}} = 0$. The dilaton then transforms as

$$\delta_\xi \tilde{X} \,\widehat{=}\, \pounds_\xi \tilde{X} - D_a\xi^a \tilde{X} = \xi^a D_a \tilde{X} - D_a\xi^a \tilde{X}\,, \tag{6.11}$$

i.e., as a boundary vector. We will re-derive this result in coordinates below.

The above provides a nice geometrical interpretation for the more restrictive set of boundary conditions (6.4). What is the interpretation for the looser set of boundary conditions where the boundary metric is an arbitrary function? Everything on the above hinged on the assumption that Ω is arbitrary but non-dynamical. This led to equation (6.8) that allowed us to fix half of the boundary data. However, one can also think of Ω not as

[5]For a discussion of this issue in the case of four-dimensional asymptotically flat spacetimes see, e.g., [187, 188].

[6]Again, without a concrete theory or a symplectic structure we do not know whether these transformations are proper or improper gauge transformations using the terminology introduced at the end of section 3.5

background structure but as an additional dynamical field. Then fixing the boundary metric to a particular value by the transformation (6.8) is not possible anymore since Ω will take on a different value that is assumed to be physically distinguishable. Notice that by this we are leaving the realm of pure geometry since we are now assuming that the manifold $\mathcal{M}$ carries more structure than just the metric (and the dilaton). Furthermore, since the argument of (6.10) was based on having fixed the boundary metric using the transformations (6.8), there will be no way to fix the subleading component $\xi^c n_c = \Omega K$. Thus, we expect that this subleading component will become an independent parameter of transformations.

This concludes our digression into the conformal framework. What follows can be understood without the above but it provides a geometric point of view on the following results we will derive in coordinates.

6.2 Linear dilaton solutions

Let us now solve the equations of motion (6.2). From our boundary conditions we find the expansion

$$ds^2 = d\rho^2 + h(\rho,\tau)^2 \, d\tau^2 \tag{6.12}$$

with h being linear in e^ρ. The Ricci scalar is then $R = -2\,h^{-1}\,\partial_\rho{}^2 h$, so the equation of motion $\mathcal{E}_X = 0$ immediately gives

$$h(\rho,\tau) = e^\rho\,\mathcal{L}^+(\tau) - e^{-\rho}\,\mathcal{L}^-(\tau)\ . \tag{6.13}$$

Likewise, combining the different components of $\mathcal{E}_{\mu\nu} = 0$ yields the following equation for the dilaton

$$\partial_\rho^2 X = X\ , \tag{6.14}$$

which is solved by

$$X(\rho,\tau) = e^\rho\,\mathcal{X}^+(\tau) + e^{-\rho}\,\mathcal{X}^-(\tau)\ . \tag{6.15}$$

The equations $\mathcal{E}_{\rho\tau} = 0$ and $\mathcal{E}_{\rho\rho} = 0$ together yield the condition

$$\partial_\rho\left(\frac{-\partial_\tau X}{h}\right) = 0\ . \tag{6.16}$$

The quantity in parentheses must be a function of τ. It will be convenient to call this radial-independent quantity $\mathcal{X}^0$. As a result, the functions in (6.13) and (6.15) satisfy

$$(\mathcal{X}^\pm)' \pm \mathcal{L}^\pm\,\mathcal{X}^0 = 0\,, \tag{6.17}$$

where a prime indicates a derivative with respect to τ. Finally, if we evaluate $\mathcal{E}_{\rho\rho} = 0$ using (6.13), (6.15), and (6.17), we find one last condition

$$(\mathcal{X}^0)' + 2\big(\mathcal{L}^+\,\mathcal{X}^- - \mathcal{L}^-\,\mathcal{X}^+\big) = 0\ . \tag{6.18}$$

The three equations (6.17) and (6.18) comprise the on-shell conditions for the free functions appearing in (6.13) and (6.15). Notice that the three equations are not linearly independent: the combination

$$\mathcal{C} = \mathcal{X}^+ \mathcal{X}^- - \frac{1}{4}(\mathcal{X}^0)^2 \tag{6.19}$$

is preserved $\partial_\tau \mathcal{C} = 0$. This should come as no surprise since we immediately recognize (6.19) as the Casimir function available in every dilaton gravity model.

Some of the solutions derived above correspond to Euclidean versions of the two-dimensional black holes discussed in section 4.1. Suppose $h(\rho,\tau)$ vanishes for some (possibly τ dependent) value of ρ. We will denote this by $\rho = \rho_H(\tau)$. In Lorentzian signature this would correspond to a horizon.Then we have

$$\mathcal{L}^-(\tau)\, e^{-\rho_H(\tau)} = \mathcal{L}^+(\tau)\, e^{\rho_H(\tau)}\ . \tag{6.20}$$

To remove any conical singularities, we set $\mathcal{L}^+(\tau)\, e^{\rho_H(\tau)} = 2\pi/\beta$. Thus, we have black hole solutions with

$$ds^2 = d\rho^2 + \left(\frac{\pi}{\beta}\, e^{\rho-\rho_H(\tau)} - \frac{\pi}{\beta}\, e^{-\rho+\rho_H(\tau)}\right)^2 d\tau^2, \tag{6.21}$$

which can be rewritten as

$$ds^2 = d\rho^2 + \frac{4\pi^2}{\beta^2}\sinh^2\left(\rho - \rho_H(\tau)\right) d\tau^2. \tag{6.22}$$

6.3 Variational principle and asymptotic symmetries

We mentioned above that the action (6.1) is not well-defined without further boundary terms. In this section we want to show that a good variational principle is provided by the full action

$$\begin{aligned}\Gamma = & -\frac{k}{4\pi}\int_{\mathcal{M}} d^2x\sqrt{g}\, X\,(R+2) - \frac{k}{2\pi}\int_{\partial\mathcal{M}} dx\,\sqrt{\gamma}\, X\, K \\ & + \frac{k}{2\pi}\int_{\partial\mathcal{M}} dx\,\sqrt{\gamma}\left(\sqrt{X^2+c_0} + \frac{1}{2\,X}\,\gamma^{\mu\nu}\,\partial_\mu X \partial_\nu X\right)\,,\end{aligned} \tag{6.23}$$

where γ is the induced metric on $\partial\mathcal{M}$ and K is the trace of the extrinsic curvature of $\partial\mathcal{M}$ embedded in $\mathcal{M}$. The variation of this action yields

$$\begin{aligned}-\delta\Gamma = & \frac{k}{4\,\pi}\int_{\mathcal{M}} d^2x\,\sqrt{g}\left[\,\mathcal{E}^{\mu\nu}\,\delta g_{\mu\nu} + \mathcal{E}_X\,\delta X\,\right] \\ & + \frac{k}{2\pi}\int_{\partial\mathcal{M}} dx\,\sqrt{\gamma}\left[\,\left(\pi^{\mu\nu}+p^{\mu\nu}\right)\delta g_{\mu\nu} + \left(\pi_X + p_X\right)\delta X\,\right]\,.\end{aligned} \tag{6.24}$$

Setting to zero the bulk terms gives the equations of motion (6.2) while the coefficients of the field variations appearing in the boundary term are

$$\pi^{\mu\nu} = \frac{1}{2}\,\gamma^{\mu\nu}\,n^{\lambda}\nabla_{\lambda}X \tag{6.25}$$

$$p^{\mu\nu} = -\,\frac{1}{2}\,\gamma^{\mu\nu}\sqrt{X^2+c_0} + \frac{1}{2\,X}\,\Big(\gamma^{\mu\lambda}\,\gamma^{\nu\sigma} - \frac{1}{2}\,\gamma^{\mu\nu}\,\gamma^{\lambda\sigma}\Big)\,\partial_{\lambda}X\,\partial_{\sigma}X \tag{6.26}$$

$$\pi_X = K \tag{6.27}$$

$$p_X = -\,\frac{X}{\sqrt{X^2+c_0}} + \frac{1}{2\,X^2}\,\gamma^{\mu\nu}\,\partial_{\mu}X\,\partial_{\nu}X + D_{\mu}\Big(\frac{1}{X}\,D^{\mu}X\Big)\,. \tag{6.28}$$

The π's come from the variation of the terms in the first line of (6.23), while the p's come from the variation of the holographic counterterms in the second line.

The first holographic counterterm in the second line of (6.23) was obtained in [112] via variational arguments and the Hamilton–Jacobi approach to holographic renormalization. However, that derivation assumed that the boundary $\partial\mathcal{M}$ was an isosurface of the dilaton, which is not the case here. Solving the Hamilton–Jacobi equation order-by-order in a boundary derivative expansion yields the final term in (6.23). The first holographic counterterm contains a constant c_0. This constant was set to zero in [112], to preserve a stringy symmetry of the action (Buscher duality). We will not set it zero immediately. Indeed, we will find in section 6.4 that it has a natural interpretation in terms of the conformal quantum mechanics of dAFF [189].

Solutions of the equations of motion (6.2) have constant negative curvature, $R = -2$, and hence the bulk term in the action (6.23) vanishes. The non-zero contributions come from the boundary terms

$$\Gamma\,\big|_{R=-2} = -\frac{k}{2\pi}\int_{\partial\mathcal{M}} dx\,\sqrt{\gamma}\,\Big(X\,K - \sqrt{X^2+c_0} - \frac{1}{2\,X}\,\gamma^{\mu\nu}\,\partial_{\mu}X\partial_{\nu}X\Big)\,. \tag{6.29}$$

Taking the boundary $\partial\mathcal{M}$ as the $\rho_c\to\infty$ limit of the surface $\rho=\rho_c$, the on-shell value is

$$\Gamma\,\big|_{\text{pEOM}} = -\frac{k}{2\pi}\int d\tau\,\frac{\mathcal{L}^+}{2\,\mathcal{X}^+}\,\Big(4\,\mathcal{C} - c_0\Big)\,. \tag{6.30}$$

Obtaining this result involved integrating-by-parts, imposing the first equation of (6.17) and (6.18), and dropping total (boundary) derivatives. The fact that we did not have to impose all equations of motion will become important later. The subscript pEOM in (6.30) thus denotes being partially on-shell.

In the following it will be useful to define the quantity

$$Y = \frac{\mathcal{X}^+}{\mathcal{L}^+}\,, \tag{6.31}$$

and the zero-mode of its inverse

$$\overline{Y}^{-1} \equiv \frac{1}{\beta}\int_0^{\beta}\frac{d\tau}{Y} \tag{6.32}$$

that will play a leading role in this chapter and the next.

Evaluating the variation of the action (6.24) on a solution of the equations of motion, we have

$$\delta\Gamma\Big|_{\text{pEOM}} = \frac{k}{2\pi}\int d\tau\left[\frac{1}{4\,\mathcal{X}^+\mathcal{L}^+}\big(4\,\mathcal{C}+c_0\big)\,e^{-2\rho}\,\delta g_{\tau\tau} - \frac{\mathcal{L}^+}{2\,(\mathcal{X}^+)^2}\big(4\,\mathcal{C}+c_0\big)\,e^{-\rho}\,\delta X\right]. \tag{6.33}$$

From the powers of $e^{-\rho}$, we see that on-shell $\delta\Gamma$ vanishes for any variations of the fields that grow more slowly than the leading terms in $g_{\tau\tau}$ and X as $\rho\to\infty$. This is what one expects for an action that admits a variational principle with Dirichlet boundary conditions on the fields at $\rho\to\infty$. But if we consider variations of the *leading* terms in the fields, this becomes

$$\delta\Gamma\Big|_{\text{pEOM}} = \frac{k}{2\pi}\int d\tau\left[\frac{1}{2\,\mathcal{X}^+}\big(4\,\mathcal{C}+c_0\big)\,\delta\mathcal{L}^+ - \frac{\mathcal{L}^+}{2\,(\mathcal{X}^+)^2}\big(4\,\mathcal{C}+c_0\big)\,\delta\mathcal{X}^+\right]. \tag{6.34}$$

With the above definitions the on-shell variation reduces to

$$\delta\Gamma\Big|_{\text{pEOM}} = \frac{k}{4\pi}\int d\tau\,(4\,\mathcal{C}+c_0)\,\delta\Big(\frac{1}{Y}\Big). \tag{6.35}$$

Assuming now that we go fully on-shell, i.e., imposing all three equations (6.17) and (6.18) implies that the Casimir is constant. We have therefore

$$\delta\Gamma\Big|_{\text{EOM}} = \frac{k}{4\pi}\,(4\,\mathcal{C}+c_0)\int d\tau\,\delta\Big(\frac{1}{Y}\Big)\,. \tag{6.36}$$

Thus, the on-shell variation of the action is zero even for variations of the leading terms in the fields, provided the zero-mode of the ratio $\mathcal{L}^+/\mathcal{X}^+ = 1/Y$ is held fixed. While this variational principle might seem somewhat odd at first sight, we will see in section 6.4 that it comes with an interesting geometrical interpretation.

Since the free energy of a configuration is related to the on-shell action via $F=\beta^{-1}\,\Gamma|_{EOM}$ in the saddle-point approximation to the Euclidean path integral, the above yields

$$F = -\frac{k}{\pi\bar{Y}}\Big(\mathcal{C}+\frac{c_0}{4}\Big)\,, \tag{6.37}$$

since the Casimir is on-shell constant.

Asymptotic Symmetries

We will now turn to the asymptotic symmetries of our boundary conditions. Under a diffeomorphism $x^\mu\to x^\mu-\xi^\mu$, the bulk fields transform with the Lie derivative $\pounds_\xi$ along the vector field ξ.

$$\delta_\xi g_{\mu\nu} = \pounds_\xi g_{\mu\nu} = \xi^\alpha\partial_\alpha g_{\mu\nu} + g_{\mu\alpha}\partial_\nu\xi^\alpha + g_{\nu\alpha}\partial_\mu\xi^\alpha \qquad \delta_\xi X = \pounds_\xi X = \xi^\alpha\partial_\alpha X \tag{6.38}$$

In a neighborhood of $\partial\mathcal{M}$ ($\rho\to\infty$), the most general diffeomorphism that preserves the gauge $g_{\rho\rho}=1$ and the generalized Fefferman–Graham form of the fields (6.3) is given by

$$\xi^\tau = \sigma(\tau) + e^{-2\rho}\,\frac{\lambda'}{2(\mathcal{L}^+)^2} + \mathcal{O}(e^{-4\rho}) \tag{6.39a}$$

$$\xi^\rho = \lambda(\tau)\,. \tag{6.39b}$$

The action of this diffeomorphism on the fields is

$$\delta_\xi \mathcal{L}^+ = \lambda\, \mathcal{L}^+ + (\sigma\, \mathcal{L}^+)' \tag{6.40a}$$

$$\delta_\xi \mathcal{L}^- = -\lambda\, \mathcal{L}^- + (\sigma\, \mathcal{L}^- - \frac{\lambda'}{2\mathcal{L}^+})' \tag{6.40b}$$

$$\delta_\xi \mathcal{X}^+ = \lambda\, \mathcal{X}^+ - \sigma\, \mathcal{L}^+\, \mathcal{X}^0 \tag{6.40c}$$

$$\delta_\xi \mathcal{X}^0 = -2\,\sigma\, \mathcal{L}^+\, \mathcal{X}^- + 2\,(\sigma\, \mathcal{L}^- - \frac{\lambda'}{2\mathcal{L}^+})\, \mathcal{X}^+ \tag{6.40d}$$

$$\delta_\xi \mathcal{X}^- = -\lambda\, \mathcal{X}^- + (\sigma\, \mathcal{L}^- - \frac{\lambda'}{2\mathcal{L}^+})\, \mathcal{X}^0\ . \tag{6.40e}$$

For the stricter boundary conditions, the condition of preserving $\mathcal{L}^+ = \frac{1}{2}$ would give $\lambda = -2\sigma'$ thus fixing the subleading component ξ^ρ (to see that this indeed vanishes at the boundary, it is useful to do the change of coordinates $e^\rho = \Omega$). The pullback of X to the boundary, $\mathcal{X}^+$, then transforms as a vector. Since this is not possible anymore in the case of looser boundary conditions, the subleading component becomes an independent transformation parameter λ.

Under the diffeomorphism (6.39), the response of the on-shell action has the form (6.36). The ratio $\mathcal{L}^+/\mathcal{X}^+ = 1/Y$ transforms on-shell as a total derivative

$$\delta_\xi \Big(\frac{1}{Y}\Big)\Big|_{\text{pEOM}} = \Big(\frac{\sigma}{Y}\Big)'\ . \tag{6.41}$$

This means in particular that the zero mode of $1/Y$ is not changed, which is a non-trivial consistency check of our variational principle. Thus, the action (6.23) is invariant under diffeomorphisms that take the form (6.39) in a neighborhood of $\partial\mathcal{M}$.

6.4 Schwarzian action

We have shown that the variational principle is well-defined, if the zero-mode of the ratio $\mathcal{L}^+/\mathcal{X}^+ = 1/Y$ is fixed. In this section we clarify the interpretation of this variational principle and, provided with these results, show its relation to the Schwarzian action that rose to prominence recently in the context of SYK (-like) models.

Comments on the variational principle

As equation (6.41) shows, the quantity $1/Y$ transforms as a total derivative under an infinitesimal change of the boundary coordinate $\tau \mapsto \tau - \sigma(\tau)$. The quantity Y itself transforms as a vector on-shell

$$\delta_\xi Y\Big|_{\text{pEOM}} = Y'\sigma - \sigma' Y \tag{6.42}$$

under this infinitesimal change of coordinates and is a well-defined, nowhere vanishing vector field on $\partial\mathcal{M}$ for the following reasons. For consistency, $\mathcal{L}^+$ must be a nowhere vanishing positive function such that the induced metric on the cut-off surface ρ_c is Euclidean and non-singular in the limit $\rho_c \to \infty$. Similarly, the leading order component of the dilaton, $\mathcal{X}^+$, must be a non-zero (positive) function everywhere if we want to interpret the asymptotic region $\rho \to \infty$ as a weak coupling region $X \to \infty$. Consequently, the quantity $\overline{Y}$ that we keep fixed as part of our boundary conditions is finite and well-defined.

Furthermore, let us define the function $M(\tau)$,

$$M = \mathcal{T} - \mathcal{P}^2 - \mathcal{P}' \tag{6.43}$$

where

$$\mathcal{T} = \mathcal{L}^+\mathcal{L}^- \qquad\qquad \mathcal{P} = -\frac{(\mathcal{L}^+)'}{2\mathcal{L}^+}\,. \tag{6.44}$$

This can be regarded as a boundary stress tensor obtained by a (twisted) Sugawara construction (6.43) from $\mathcal{L}^\pm$. It transforms with an infinitesimal Schwarzian derivative,

$$\delta_\xi M = \sigma M' + 2\sigma' M + \frac{1}{2}\sigma''' \tag{6.45}$$

under infinitesimal reparametrizations of the boundary coordinate as can be checked straightforwardly using the relations (6.40). This is immediately recognized as the infinitesimal transformation of a coadjoint vector of the Virasoro group given in equation (4.39). Under finite transformations, $\tau \mapsto f(\tau)$, where $f(\tau)$ is a diffeomorphism on S^1 obeying

$$f'(\tau) > 0 \qquad f(\tau+\beta) = f(\tau) + \beta \tag{6.46}$$

we find the transformation law (4.38)

$$f\cdot M = \tilde{M}(f(\tau)) = \frac{1}{(f'(\tau))^2}\big(M(\tau) - \tfrac{1}{2}\,\mathrm{Sch}[f](\tau)\big). \tag{6.47}$$

In the following it will be convenient to evaluate the left hand side of (6.47) at τ instead of $f(\tau)$. Using the cocylce condition for the Schwarzian derivative with $g = f^{-1}$ yields

$$f\cdot M = \tilde{M}(\tau) = \big((f^{-1})'(\tau)\big)^2 M(f^{-1}(\tau)) + \tfrac{1}{2}\,\mathrm{Sch}[f^{-1}](\tau)\,. \tag{6.48}$$

Since a particular coadjoint orbit is a homogeneous space for the Virasoro group, the result (6.48) shows that any point on the orbit $\tilde{M}$ can be reached by acting with an appropriate diffeomorphism $f(\tau)$ on a chosen representative M. With the help of the quantity M, the on-shell conditions (6.17) and (6.18) are equivalent to the equation

$$\mathcal{C} = Y^2 M - \tfrac{1}{4}(Y')^2 + \tfrac{1}{2}YY''\,, \tag{6.49}$$

relating M, Y, and the Casimir function $\mathcal{C}$. Conservation of the Casimir $\mathcal{C}' = 0$ establishes

$$YM' + 2Y'M + \tfrac{1}{2}Y''' = 0\,. \tag{6.50}$$

We stress that only two of the three constraints are needed to derive equation (6.49), which implies that this equation is valid without assuming the conservation of the Casimir. By contrast, (6.50) is an immediate consequence of this conservation and is valid only if all three constraints are imposed. This distinction between “fully on-shell" and “partially on-shell" will be important for the following.

Since the (rescaled) leading order of the dilaton field transforms like a boundary vector and solves equation (6.50) it can be regarded as the *stabilizer* of the coadjoint orbit of the Virasoro group determined by M. If the on-shell condition of conservation of the Casimir function is not enforced, comparison between (6.50) and (6.45) suggests that the quantity Y generates infinitesimal diffeomorphisms under which M transforms anomalously.

By solving the equation

$$\frac{d\tau}{Y} = \frac{d\tilde{\tau}}{\overline{Y}} \tag{6.51}$$

one can always find a diffeomorphism $\tau \mapsto \tilde{\tau}$ to a new coordinate system $\tilde{\tau}$ in which Y takes the constant value $\overline{Y}$. In this coordinate system equation (6.49) yields

$$M = \mathcal{C}\overline{Y}^{-2}, \tag{6.52}$$

thus determining the constant representative of each orbit (cf. the discussion in section 4.4).[7] In this coordinate system the solution of equation (6.50) is straightforward. For generic values of M, $\overline{Y}$ will be the only periodic solution to this equation, and the stabilizer group is just $U(1)$. However, at the exceptional values

$$M = \frac{n^2\pi^2}{\beta^2} \tag{6.53}$$

one finds two additional solutions, and the stabilizer group is given by $\mathrm{PSL}^{(n)}(2,\mathbb{R})$, i.e., the n-fold cover of the Euclidean AdS_2 group $\mathrm{SO}(2,1) \simeq \mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2$. This singles out the Euclidean black hole configurations with $n = 1$ as smooth geometries. The relation between Casimir and temperature for smooth classical solutions is therefore given by

$$\mathcal{C} = \frac{\bar{Y}^2\pi^2}{\beta^2}. \tag{6.54}$$

The Schwarzian action

We can now make contact with the recent developments regarding a proposed duality between (nearly) AdS_2 gravity in the form of the JT model and the SYK model [46–48]. We will show that our on-shell action (6.30), deriving from an action with well-defined variational principle (6.23), can be naturally reformulated as a Schwarzian action.

Using the notation introduced in section 6.4, the on-shell action takes the form

$$\Gamma\big|_{\mathrm{pEOM}} = -\frac{k}{4\pi}\int_0^\beta \frac{d\tau}{Y}\Big(4\,\mathcal{C} - c_0\Big) = -\frac{k}{4\pi}\int_0^\beta \frac{d\tau}{Y}\big(4Y^2M - (Y')^2 - c_0\big), \tag{6.55}$$

where we used equation (6.49) in the second step and discarded a total derivative. The subscript pEOM denotes that we are only partially on-shell, i.e., we have only used two of three equations of motion and thus have not assumed on-shell constancy of the Casimir.

[7] Our conditions on the dilaton field, in particular the requirement that it is non-zero everywhere disallow orbits without constant representatives.

Otherwise the whole action would equal a constant. The mass function M must be an element of the Virasoro orbit with constant representative given by (6.53) with $n = 1$ since otherwise we would have a solution that is not smooth for given β.[8]

As a first observation note that (6.55) becomes the action of Euclidean conformal quantum mechanics discussed in [189, 190] coupled to the external source M upon replacing $Y \to q^2$

$$\Gamma\Big|_{\text{pEOM}} = -\frac{k}{\pi}\int_0^\beta d\tau\Big(q^2 M - (q')^2 - \frac{c_0}{4q^2}\Big). \tag{6.56}$$

This field redefinition is well-defined since we argued that Y is always positive. As mentioned above, the quantity c_0 becomes the coupling strength of the conformal quantum mechanics model. Consistently with Y transforming like a boundary vector under arbitrary reparametrizations, q transforms with conformal weight $-\frac{1}{2}$.

We return now to (6.55) and set $c_0 = 0$. This value is special since for string-related models of dilaton gravity it restores a stringy symmetry, Buscher duality [191], while for JT it restores homogeneity of the action in the dilaton field X. Let us define a diffeomorphism $g : S^1 \to S^1, \tau \mapsto u = g(\tau)$ by

$$g(\tau) = \bar{Y}\int_0^\tau \frac{d\eta}{Y(\eta)}\,. \tag{6.57}$$

Since Y is required to be positive and transforms as a vector under reparametrizations this diffeomorphism is well-defined. It defines a finite reparametrization of the boundary coordinate τ. We can therefore rewrite the action (6.55) as

$$\Gamma\Big|_{\text{pEOM}} = -\frac{k\bar{Y}}{\pi}\int_0^\beta du\,\Big((g^{-1})'(u)M + \tfrac{1}{2}\,\text{Sch}[g^{-1}](u)\Big)\,. \tag{6.58}$$

The Lagrangian in (6.58) is the coadjoint action of the Virasoro group (6.48) acting on the element M. This provides an effective action for the reparametrizations $g^{-1}(u)$.

Without loss of generality we assume M is a constant representative (since any element on the orbit can be reached from it), and setting $g^{-1}(u) \equiv \tau(u)$ we find

$$\Gamma\Big|_{\text{pEOM}} = -\frac{k\bar{Y}}{2\pi}\int_0^\beta du\,\bigg(\frac{1}{2}\Big(\frac{2\pi}{\beta}\Big)^2(\tau')^2 + \text{Sch}[\tau](u)\bigg)\,, \tag{6.59}$$

which is precisely the Schwarzian action at finite temperature β for finite reparametrizations of the circle τ [47, 192] (cf. equation (D.94) in appendix D). Comparing the expressions for the Schwarzian action on the two sides of the duality, one can relate the various parameters in front of the action. As in the usual holographic dictionary it is suggestive to set $k = \frac{1}{4G} \sim N$ so that large N leads to a weakly coupled gravity theory (remember that Newton's constant

[8] If one were considering contributions to the Euclidean path integral that allow for conical defects or other non-smooth features, other orbits than the one considered here are possible.

is dimensionless in two dimensions). For the coupling strength of the SYK model we find then $J \sim \bar{Y}^{-1}$.

The Schwarzian action (6.59) is invariant under PSL$(2,\mathbb{R})$ transformations. Redefining $f = \tan(\frac{\tau}{2})$ the Schwarzian action becomes

$$\Gamma\big|_{\text{pEOM}} = -\frac{k\bar{Y}}{2\pi}\int_0^\beta du\, \text{Sch}[f](u)\,, \tag{6.60}$$

which is invariant under the transformation $f \mapsto \frac{af+b}{cf+d}$. We can therefore derive the Schwarzian action by an effective field theory argument. In the SYK model the reparametrization symmetry of the circle is spontaneously broken to SL$(2,\mathbb{R})$ by the ground state. The corresponding Goldstone bosons are therefore elements of the quotient Diff(S^1)/SL$(2,\mathbb{R})$, which is the coadjoint orbit of the Virasoro group consistent with temperature β. The action with the lowest number of derivatives that is invariant under SL$(2,\mathbb{R})$ is the Schwarzian action. Notice that when deriving (6.59) we were always careful not to impose conservation of the Casimir. Otherwise we would have obtained only the ground-state contribution to the action.

In the next section we are going to see the breaking of conformal symmetry from a different point of view.

6.5 Charge algebra of the JT model

An expression for the canonical charges of dilaton gravity in the second order formalism was derived in (3.114). In the Fefferman–Graham like gauge that we are using in this section we have the following simplifications:

$$N = h(\rho,\tau), \qquad N^a = 0, \qquad h_{\rho\rho} = 1, \qquad K = 0. \tag{6.61}$$

Evaluating the charge for the asymptotic Killing vectors (6.39) one obtains after a straightforward calculation

$$\delta Q[\xi] = \frac{k}{2\pi}\Big(-\lambda\delta\mathcal{X}^0 + 2\sigma\mathcal{L}^+\delta\mathcal{X}^- - (\frac{\lambda'}{2\mathcal{L}^+} - \mathcal{L}^-\sigma)\delta\mathcal{X}^+\Big). \tag{6.62}$$

Stricter boundary conditions

We will first turn to the set of stricter boundary conditions (6.4) that was analyzed previously in [4,186] and that is more "typical" for asymptotically AdS$_2$ behavior since the leading order metric is not allowed to fluctuate, i.e., $\mathcal{L}^+$ is set to the convenient constant value $\mathcal{L}^+ = 1/2$.

The function $\mathcal{L}^-$ transforms with an infinitesimal Schwarzian derivative

$$\delta_\sigma\mathcal{L}^- = \sigma(\mathcal{L}^-)' + 2\sigma'\mathcal{L}^- + \sigma''' \tag{6.63}$$

and is related to the mass function by a factor $\frac{1}{2}$

$$M = \frac{1}{2}\,\mathcal{L}^-\,. \tag{6.64}$$

Therefore, also the mass function M again transforms with an infinitesimal Schwarzian derivative [as in (6.45)] under infinitesimal diffeomorphisms parametrized by σ.

The asymptotic symmetries for these boundary conditions were previously analyzed in [4, 186]. It was shown therein that the charges associated to these asymptotic symmetries are, in general, non-integrable. A prescription to calculate the Poisson brackets of non-integrable charges was proposed in [193]. The price one has to pay in that approach is a non-standard central extension, in the sense that it becomes field-dependent. In the present context we will follow a different way to deal with the non-integrability of the charges.

Setting $\mathcal{L}^+ = \frac{1}{2}$ and using the linearized equations of motion we arrive at

$$\delta Q[\sigma] = \frac{k}{2\pi}(-\sigma'\delta Y' + \sigma\delta Y'' + 2\sigma\delta(MY) + \sigma''\delta Y + 2\sigma M\delta Y)\,. \tag{6.65}$$

We mentioned above that this charge is not integrable. However, this conclusion depends crucially on the interpretation of the function $\mathcal{L}^-$. Since the variational principle does not require $\mathcal{L}^-$ to be fixed, one would consider it as a state-dependent function, i.e., a function that varies in phase space. However, if this is the case, then the charge is clearly non-integrable. If on the other hand $\mathcal{L}^-$ is thought of as an arbitrary but fixed function that takes everywhere on phase space the same value, i.e., as a *chemical potential*, then the variation can be pulled through and the charge is integrable. As a result we either obtain one or three resulting charges corresponding to the stabilizer group of the orbit determined by $\mathcal{L}^-$. This can be seen as follows. Since $\mathcal{L}^-$ is a fixed function we can set it to an arbitrary value, thus we will for simplicity assume that it is constant. Furthermore, only σ's that solve the stabilizer equation $\delta_\sigma\mathcal{L}^- = 0$ are then allowed. Given that $\mathcal{L}^-$ is not an element of the exceptional orbit (6.53), this equation has only the constant solution $\sigma = \bar{\sigma}$. Since the equation for the (rescaled) dilaton Y is of the same form [cf. equation (6.50)], the dilaton is also constant $Y = \bar{Y}$. Then there is only one non-zero term in (6.65) and we find $Q \sim \bar{Y}$. So the symmetry algebra is U(1). On the other hand if $\mathcal{L}^-$ is an element of the exceptional orbit, both the stabilizer equation determining σ and the equation for the dilaton (6.65) have three solutions. One can easily show that these produce three non-vanishing conserved charges forming an $\mathrm{SL}(2,\mathbb{R})$ algebra.

Is it possible to obtain an interpretation for the charge (6.65) also in the case when $\delta\mathcal{L}^- \neq 0$? First, let us define the averaged charges

$$\delta\hat{Q}[\sigma] = \frac{k}{2\pi\beta}\int_0^\beta d\tau(-\sigma'\delta Y' + \sigma\delta Y'' + \sigma\delta(\mathcal{L}Y) + \sigma''\,\delta Y + \sigma\mathcal{L}\delta Y)\,. \tag{6.66}$$

If the charge is independent of Euclidean time then the average (6.66) does not do anything. On the other hand, non-integrability is usually related to non-conservation of the charges (cf. e.g. asymptotically flat spacetimes with non-zero news [146, 193]). A charge that is dependent on Euclidean time is difficult to interpret in a thermodynamic context which suggests to look at the averaged expression (6.66).

The expression (6.66) is still non-integrable. In section 6.4 we saw that the quantity Y is a boundary vector that is related to infinitesimal reparametrizations of the boundary coordinate as suggested by equation (6.50). Taking this similarity serious we redefine the gauge parameter σ as

$$\sigma = \varepsilon Y \tag{6.67}$$

As we are going to show this leads to integrable charges with interesting properties. Inserting the redefinition (6.67) into the variation of the charges (6.66) we find that the charges become integrable

$$Q[\sigma] = \frac{k}{\pi\beta}\int_0^\beta d\tau\,\frac{\sigma}{Y}\left(Y^2 M - \frac{1}{4}Y'^2 + \frac{1}{2}YY''\right). \tag{6.68}$$

The quantity in parentheses is just the Casimir (6.49). However, let us not enforce the on-shell conservation of the Casimir for the moment. Then, following the same line of reasoning that led us to the Schwarzian action in section 6.4, we find that the charge (6.68) is given by

$$Q[\sigma] = \frac{k\bar{Y}}{2\pi\beta}\int_0^\beta du\,\sigma(u)\left(\frac{1}{2}\left(\frac{2\pi}{\beta}\right)^2(\tau')^2 + \text{Sch}[\tau|(u)\right). \tag{6.69}$$

By equation (6.48) the quantity in parentheses denotes a generic point $M(u)$ on the orbit of the constant representative $\text{Diff}(S^1)/\text{SL}(2,\mathbb{R})$, which leads to

$$Q[\sigma] = \frac{k\bar{Y}}{\pi\beta}\int_0^\beta du\,\sigma(u)M(u)\,. \tag{6.70}$$

These are just the usual charges on would expect on the phase space of a coadjoint orbit of the Virasoro group [158]. Indeed, one can check that

$$\delta_{\sigma_2}Q[\sigma_1] \equiv \{Q[\sigma_1], Q[\sigma_2]\} = \frac{k\bar{Y}}{\pi\beta}\int_0^\beta du\,\sigma_1(u)\big(\sigma_2 M' + 2\sigma_2' M + \tfrac{1}{2}\sigma_2'''\big)\,. \tag{6.71}$$

We therefore find a Virasoro algebra at central charge

$$c = \frac{6k\bar{Y}}{\pi}. \tag{6.72}$$

Thus, our requirement of a well-defined variational principle that led to the fixing of $\bar{Y}$ implies that the central charge (6.72) is state-independent.

However, the above derivation was based on the assumption that the strict on-shell conservation of the Casimir is not enforced. This is also clear from equation (6.71) since recalling the parametrization $\sigma_2 = \varepsilon_2 Y$ we obtain

$$\{Q[\sigma_1], Q[\sigma_2]\} = \frac{k\bar{Y}}{\pi\beta}\int_0^\beta du\,\sigma_1(u)\varepsilon_2\left(2Y'M + YM' + \frac{1}{2}Y'''\right) = 0, \tag{6.73}$$

due to equation (6.50) which was a consequence of the on-shell conservation of the Casimir.

In summary, the above discussion suggests the following general picture: Using the conformal boundary conditions we find that off-shell the averaged charges form a Virasoro algebra. The on-shell conservation law of the Casimir breaks this conformal symmetry. This pattern of on-shell breaking of conformal symmetry also is a distinctive feature of the SYK model [46, 47, 192, 194, 195].

Looser boundary conditions

Let us now consider the looser boundary conditions in which $\mathcal{L}^+$ is not fixed. While the transformation behavior of $\mathcal{L}^+$ and $\mathcal{L}^-$ was given in (6.40), the quantities $\mathcal{P}$ and $\mathcal{T}$ defined in (6.44) have a more interesting interpretation. We find

$$\delta_\varepsilon \mathcal{P} = -\frac{1}{2}\sigma'' + \sigma'\mathcal{P} + \sigma\mathcal{P}' - \frac{1}{2}\lambda' \tag{6.74a}$$

$$\delta_\varepsilon \mathcal{T} = \sigma\mathcal{T}' + 2\sigma'\mathcal{T} - \lambda'\mathcal{P} - \frac{1}{2}\lambda'' \,. \tag{6.74b}$$

Notice that $\mathcal{P}$ reduces to $\mathcal{P} = -\frac{1}{2}\partial_\tau \log \mathcal{L}^+$. This transformation behavior is characteristic of a warped conformal algebra [196] with twist term [197]. Our boundary conditions can thus be regarded as a two-dimensional analog of the AdS_3 boundary conditions introduced in [198].

Unfortunately, the construction of averaged charges that off-shell realize this charge algebra is not as clear as in the above case. We therefore refrain from presenting them here but refer to the original work [2].

Concluding remarks

This concludes our discussion of the JT model in the second order formulation. We saw that the well-defined variational principle (6.23) reduces to the Schwarzian action under the assumption that conservation of the Casimir is not imposed. The zero-mode that is fixed as part of the variational principle turned out to be related to the coupling constant of a possible SYK model on the boundary. The pattern of symmetry breaking that is distinctive for the SYK model can be reproduced, after a redefinition of the transformation parameter σ, using the canonical charges of the dilaton theory, assuming again non-conservation of the Casimir. In [2] a number of further checks of the action principle and the thermodynamics of the model are considered. Most interestingly, one finds that the entropy of black holes calculated using Wald's formula [199, 200] or using the on-shell action (6.37) coincides with the entropy calculated using the "off-shell" central charge (6.72) in the chiral Cardy formula.

In the next chapter we will rederive some of the above results using the PSM formulation of the JT model. This formulation lends itself to a number of interesting generalizations. Our aim will be the construction of boundary actions for these generalizations that exhibit properties similar to the Schwarzian action.

Chapter Seven

Boundary Actions for Generalized Jackiw–Teitelboim Models

We saw in the previous chapter that both the Schwarzian action and the symmetry breaking characteristic of the SYK model can be reproduced using the JT model as gravitational dual.

It is of interest to consider various extensions of SYK, since this enlarges the theory-space of possible holographic relationships and thus may allow to address relevant conceptual questions, for instance how general holography is and what are necessary ingredients for it to work.

While these are intriguing questions, our goals in the present chapter are more modest, namely to supply candidates on the gravity side that generalize the symmetry breaking mechanism in SYK. In a sense, our approach is complementary to recent work by Gross and Rosenhaus [201], who considered free Majorana fermions in the large N limit and conjectured that the bulk dual is some topological cousin of AdS_2 Vasiliev theory [186, 202–210]: they worked on the field theory side [deforming the free theory by a bi-local bi-linear interaction preserving $\text{SL}(2,\mathbb{R})$], while this chapter deals exclusively with the bulk side (not necessarily related to the Gross–Rosenhaus model).

More specifically, our focus is to extend the symmetry breaking mechanism summarized above to other infinite-dimensional symmetry groups that contain a Virasoro subgroup.

We are interested in two types of generalizations, one that has an interpretation in terms of dilaton gravity coupled to Yang–Mills and the other where Virasoro gets extended to W-symmetries, which arise in higher spin generalizations [186, 208, 209] of JT. Thus, we aim to provide the first few steps towards a higher spin (and Yang–Mills) generalization of SYK.

The holographic dual description of a finite temperature quantum field theory is generated by placing a Euclidean black hole in the bulk. Let us suppose the set of black hole solutions preserves a certain (in lower dimensions typically infinite-dimensional) asymptotic symmetry group $\mathcal{G}_\infty$. Demanding smoothness of the solutions yields a subset thereof that is invariant only under a subgroup $\mathcal{G} \subset \mathcal{G}_\infty$. In the SYK context this reproduces the symmetry breaking $\mathcal{G}_\infty \to \mathcal{G}$. The dynamics of the breaking is governed by a field belonging to the quotient space $\mathcal{G}_\infty/\mathcal{G}$. For instance, in the case of the Schwarzian action, the group $\mathcal{G}_\infty$ is $\text{Diff}(S^1)$, while $\mathcal{G} = \text{SL}(2,\mathbb{R})$. The field $\tau(u)$ of the Schwarzian in the previous section is a diffeomorphism associated to the orbit $\text{Diff}(S^1)/\text{SL}(2,\mathbb{R})$ [211].

In this chapter we consider generalized models of dilaton gravity based on a gauge group $\mathcal{G}$ [105,106]. The suitable extension of the Schwarzian dynamics is governed by one-dimensional actions located at the boundary of the space-time. In order to have a well-defined AdS_2 gravity interpretation we study cases where $\mathcal{G}$ contains an $SL(2,\mathbb{R})$ subgroup. More precisely, we are interested in two cases: Direct product groups, $SL(2,\mathbb{R})\times\mathcal{K}$, where $\mathcal{K}$ is a compact group representing (Yang–Mills) matter fields and higher rank groups $SL(N,\mathbb{R})$, where the spin two excitation is enhanced by the coupling of $N-1$ higher spin fields, analogous to the situation in three spacetime dimensions [212,213].

The outline of this chapter is as follows. In section 7.1 we demonstrate that the PSM formulation of the JT model gives rise to a BF theory based on the group $SL(2,\mathbb{R})$. We will use this result in section 7.2 as a motivation to consider generalized JT models in the form of BF theories based on arbitrary groups containing an $SL(2,\mathbb{R})$ subgroup. In section 7.3 we rederive the Schwarzian action as boundary action of the JT model using the formulation as a BF theory, and we present a generalization of this result to cases in which the JT model is coupled to Yang–Mills fields with gauge group $\mathcal{K}$. This derivation closely mimics the one of the last chapter. In order to consider generalizations to other gauge groups we propose in section 7.4 a boundary action for generic BF theories in two dimensions. Based on this boundary action we will then consider BF theories for the group $SL(N)$ in section 7.5, that can be regarded as higher spin extensions of the JT model. Using properties of differential equations that generalize Hill's equation (4.40) we reduce these actions to the gravitational sector and find corresponding higher spin generalizations of the Schwarzian action. Section 7.6 contains some considerations concerning the thermodynamics of our models.

7.1 The JT model as a gauge theory

In this section we want to reformulate the JT model of the last chapter as a PSM model along the lines of chapter 3. The choice of potentials U and V in the bulk action for the JT model (6.1) leads to the Poisson tensor

$$P^{Xb} = X^a \epsilon_a{}^b \qquad\qquad P^{ab} = X\epsilon^{ab}\,, \tag{7.1}$$

in the PSM action

$$I = -\frac{k}{2\pi}\int_{\mathcal{M}}\Big(X^I\,dA_I + \frac{1}{2}P^{IJ}A_I\wedge A_J\Big). \tag{7.2}$$

Introducing a metric $\eta_{IJ} = \mathrm{diag}(+1,+1,-1)$ on the target space, with volume form $\epsilon^{01X} = 1$, the Poisson tensor can be written as

$$P^{IJ} = \epsilon^{IJK}X_K\,. \tag{7.3}$$

Since ϵ^{IJK} are the structure functions of the Lie algebra $\mathfrak{so}(2,1)$ we can recognize equation (7.3) as a particular case of the Kirillov–Kostant bracket (2.64) [or (2.67)] that defines a Poisson structure on the dual of a Lie algebra parametrized by the coordinates X^I. However, since the Lie algebra $\mathfrak{so}(2,1)\simeq\mathfrak{sl}(2)$ is semi-simple we will use the existing non-degenerate, invariant metric to freely switch between the Lie algebra and its dual.

Using the above form of the Poisson tensor the (linear) gauge transformations for the JT model can be expressed as

$$\delta_\epsilon X^I = \epsilon^{IJK}\varepsilon_J X_K \qquad \delta_\epsilon A_I = -\,d\varepsilon_I - \epsilon_{IJK}A^J\varepsilon^K \tag{7.4}$$

with gauge parameter ε. Now choose $\mathfrak{so}(2,1)$ generators J_I satisfying the algebra

$$[J_0, J_1] = J_X \qquad [J_1, J_X] = -J_0 \qquad [J_X, J_0] = -J_1\,, \tag{7.5}$$

with invariant bilinear form given by

$$\langle J_I J_J \rangle = \frac{1}{2}\eta_{IJ}\,. \tag{7.6}$$

Then in terms of the Lie-algebra valued quantities $\mathcal{X} = X^I J_I$, $\mathcal{A} = A_I J^I$, and $\varepsilon = \varepsilon_I J^I$, the transformations are

$$\delta_\varepsilon \mathcal{X} = [\mathcal{X}, \varepsilon] \qquad \delta_\varepsilon \mathcal{A} = D\epsilon \equiv +(d\varepsilon + [\mathcal{A}, \varepsilon])\,. \tag{7.7}$$

As mentioned in section 4.3 the symmetry group of Euclidean AdS_2 space is $\mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{SO}^+(2,1)$. However, for practical purposes it will be more convenient to work with the double cover $\mathrm{SL}(2,\mathbb{R})$ of this group sharing the same Lie algebra. The transformation to $\mathfrak{sl}(2)$ generators is given by

$$L_0 = J_1 \qquad L_+ = J_0 + J_X \qquad L_- = J_X - J_0\,, \tag{7.8}$$

with inverse transformation

$$J_X = \frac{1}{2}(L_+ + L_-) \qquad J_0 = \frac{1}{2}(L_+ - L_-)\,. \tag{7.9}$$

The $\mathfrak{sl}(2)$ generators obey the commutation relations

$$[L_I, L_J] = (I - J)L_{I+J} \qquad I, J = +1, -1, 0\,, \tag{7.10}$$

with invariant bilinear form given by

$$\langle L_I L_J \rangle = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1/2 & 0 \\ -1 & 0 & 0 \end{pmatrix}_{IJ}\,. \tag{7.11}$$

From the above follows that the action (7.2) can also be written as

$$I = \frac{k}{\pi}\int_{\mathcal{M}} \mathrm{tr}(\mathcal{X}(d\mathcal{A} + \mathcal{A}\wedge\mathcal{A}))\,, \tag{7.12}$$

with equations of motion

$$D\mathcal{A} = d\mathcal{A} + \mathcal{A}\wedge\mathcal{A} = \mathcal{F} = 0 \tag{7.13a}$$
$$D\mathcal{X} = d\mathcal{X} + [A, \mathcal{X}] = 0\,. \tag{7.13b}$$

We have thus rephrased the Jackiw–Teitelboim model as a BF theory of the gauge group $\mathrm{SL}(2,\mathbb{R})$. This reformulation of the JT model was for the first time pointed out in [105, 106].

We saw in the previous section how the Schwarzian action emerges as an effective boundary theory for the JT model. We will show in section 7.3 how to obtain the Schwarzian action in the formulation (7.12). This form of the JT model suggests a number of straightforward generalizations. In particular, the gauge group $\mathrm{SL}(2,\mathbb{R})$ can be replaced by any other (semi-simple) gauge group $\mathcal{G}$.[1] Of course, not all of these groups will have a meaningful interpretation as a gravitational theory on AdS_2. We assume that a sufficient requirement for this is the existence of an $\mathfrak{sl}(2,\mathbb{R})$ sector in the gauge algebra, as we shall review below. Since this sector is linear, i.e., allows a simpler interpretation of the Poisson-σ model as non-Abelian BF-theory, we are going to consider exclusively extensions of JT that preserve linearity in the present chapter. The second ingredient for a satisfactory gravity interpretation, particularly in a holographic context, is the imposition of suitable boundary conditions on all fields. We will motivate our boundary conditions below.

7.2 Generalizations of the JT model

The gauge theoretic bulk action for generalizations of JT is then

$$I_0[\mathcal{X},\mathcal{A}] = \frac{k}{\pi}\int \langle \mathcal{X},\mathcal{F}\rangle \tag{7.14}$$

containing the coadjoint "dilaton" $\mathcal{X} = \mathcal{X}^A J_A$, the non-Abelian gauge field $\mathcal{A} = \mathcal{A}_\mu^A J_A\, dx^\mu$ and the associated curvature two-form $\mathcal{F} = d\mathcal{A} + \mathcal{A}\wedge\mathcal{A}$.[2] Both fields are valued in the Lie algebra $\mathfrak{g}$ with generators J_A satisfying $[J_A, J_B] = f_{AB}{}^C J_C$ where $f_{AB}{}^C$ are the structure constants of $\mathfrak{g}$. We raise and lower algebra indices with the invariant metric $h_{AB} = \langle J_A, J_B\rangle$. Throughout this chapter we consider fields that live in a space with the topology of a disk endowed with coordinates (τ,ρ) whose ranges are $0<\rho<\infty$ and $\tau\sim\tau+\beta$. For details see figure 7.1.

The BF-theory (7.14) is gauge invariant. Given a Lie algebra parameter ϵ, the fields transform as

$$\delta_\epsilon\mathcal{A} = d\epsilon + [\mathcal{A},\epsilon] \qquad \delta_\epsilon\mathcal{X} = [\mathcal{X},\epsilon] \tag{7.15}$$

and the infinitesimal variation of the bulk action (7.14) becomes a boundary term. The field equations that are obtained by varying (7.14) with respect to $\mathcal{X}$ and $\mathcal{A}$ are

$$\mathcal{F} = 0 \qquad d\mathcal{X} + [\mathcal{A},\mathcal{X}] = 0\,. \tag{7.16}$$

The first equation tells us that the on-shell connection is pure gauge, $\mathcal{A} = -(dG)G^{-1}$, with $G\in\mathcal{G}$ a not necessarily single-valued group element (that may account for non-trivial holonomies). The dynamics of the dilaton corresponds exactly to a gauge transformation that preserves the form of $\mathcal{A}$ or, in other words, $\mathcal{X}_{\text{on-shell}}$ is an element of the isotropy algebra of $\mathcal{A}$ (cf. section 2.3). This is precisely the behavior that we saw in the last chapter in equation (6.50) in the second order formulation.

By our assumption, $\mathfrak{g}$ must contain an $\mathfrak{sl}(2,\mathbb{R})$ subalgebra, which then allows an identification of this $\mathfrak{sl}(2)$-part with Cartan variables (zweibein and dualized Lorentz connection). The additional generators correspond to additional fields on the spacetime. This provides the first necessary ingredient for a gravity-interpretation of the non-Abelian BF-theory (7.14).

[1] Although it is certainly interesting to consider non-semisimple groups like ISO(2), i.e., the Euclidean two-dimensional Poincaré group, we restrict ourselves to semi-simple groups.

[2] Notice that we have changed the normalization of k by a factor of two with respect to the original work [1] in order to stay consistent with the normalization of the last section.

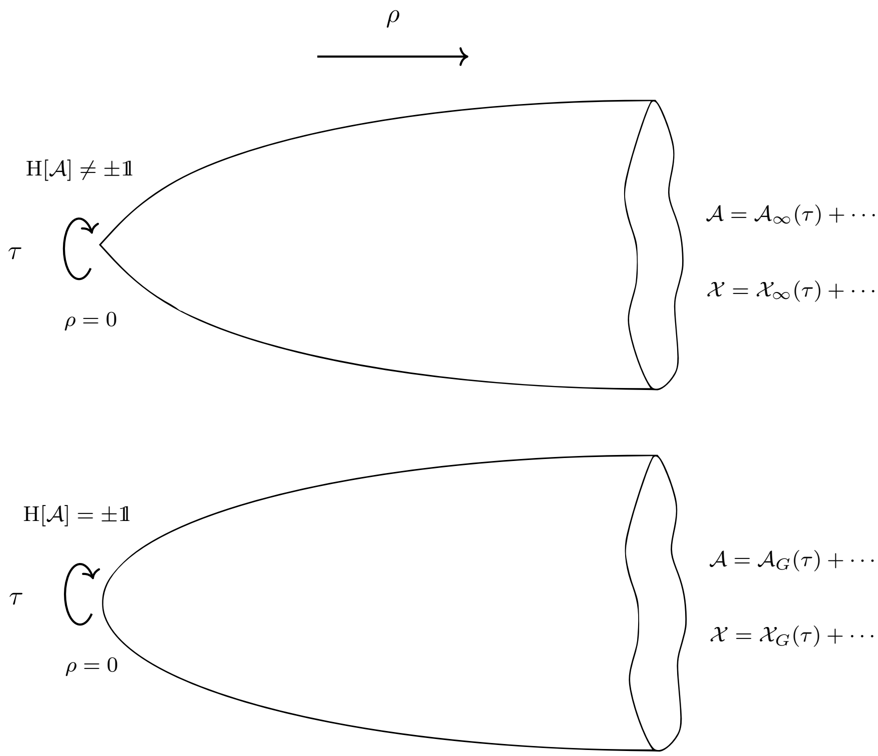


Figure 7.1: Finite temperature and asymptotic symmetry in the gauge theory formulation. Euclidean black holes are represented by fields $(\mathcal{A}, \mathcal{X})$ in a cigar-type geometry. The "Euclidean horizon" is located at $\rho = 0$. Demanding the black hole to be at Hawking temperature (absence of holonomies, $\mathrm{H}[\mathcal{A}] = \pm\mathbb{1}$) affects the asymptotic symmetries. The asymptotic fields $(\mathcal{A}_\infty(\tau), \mathcal{X}_\infty(\tau))$ become $(\mathcal{A}_G(\tau), \mathcal{X}_G(\tau))$ consistently with smoothness of the solutions.

The second ingredient, specific boundary conditions on the connection and the dilaton, are provided in section 7.3 below for the JT model and in later sections for generalizations thereof.

Observables

We can construct two types of observables for BF-theories (7.14): Wilson loops around the τ-cycle and Casimir functions.

The former are expressed as

$$\mathrm{H}[\mathcal{A}] = \mathcal{P} \exp \Big[- \oint \mathcal{A} \Big] \tag{7.17}$$

where $\mathcal{P}$ denotes path ordering and the integral is over the τ-cycle whose period is β. For pure gauge connections we have $dG + \mathcal{A}G = 0$, so solving G in terms of $\mathcal{A}$ yields $\mathrm{H} = G(\beta)G(0)^{-1}$. One is forced to demand that H belongs to the center of the group in order to single out smooth Euclidean solutions, yielding $G(\beta) = \mathcal{Z}G(0)$ where $\mathcal{Z}$ commutes with all the elements of $\mathcal{G}$. In the case of $\mathcal{G} = \mathrm{SL}(N, \mathbb{R})$ one chooses $\mathcal{Z} = (-1)^{N+1}\mathbb{1}$ as element of the center.

Another important class of gauge invariant observables are Casimir functions. Any semi-simple Lie-algebra $\mathfrak{g}$ admits invariant tensors $g_{A_1\cdots A_n}$, where n ranges from two to 1+rank of $\mathfrak{g}$ [for $\mathfrak{sl}(N)$ the range is from two to N]. The associated Casimirs are defined as

$$\mathcal{C}_n = -\frac{1}{n} g_{A_1\cdots A_n} \mathcal{X}^{A_1} \cdots \mathcal{X}^{A_n} . \tag{7.18}$$

Casimir functions play the role of conserved charges of the theory. Indeed, the dilaton equation of motion (7.16) establishes the conservation equations

$$\partial_\tau \mathcal{C}_n = 0 . \tag{7.19}$$

In the simplest case of $\mathfrak{sl}(2)$ the rank is 1 and the single Casimir reduces to the one existing in any dilaton gravity model; in particular, expression (6.19) in case of the JT model.

A variational principle for BF theories

Note that without boundary terms the action (7.14) does not have a well-defined variational principle: an infinitesimal variation yields

$$\delta I_0 = (\text{bulk equations of motions}) + \frac{k}{\pi} \int_{\rho=\infty} d\tau \, \langle \mathcal{X}, \delta\mathcal{A}_\tau \rangle . \tag{7.20}$$

The last term in (7.20) spoils the variational principle. However, we can get rid of the last term by adding a suitable boundary term I_B to the bulk action (7.14). Demanding

$$\delta I\big|_{\text{EOM}} = \delta I_0\big|_{\text{EOM}} + \delta I_B\big|_{\text{EOM}} \stackrel{!}{=} 0 \tag{7.21}$$

we find the following consistency condition

$$\delta I_B = -\frac{k}{\pi} \int_{\rho=\infty} d\tau \, \langle \mathcal{X}, \delta\mathcal{A}_\tau \rangle . \tag{7.22}$$

In order to find a local expression for I_B, we need pull the variation δ out of the integral. Without further assumption this cannot be done. To resolve this issue an integrability condition,

$$\mathcal{A}_\tau^\infty = f(\mathcal{X}^\infty) , \tag{7.23}$$

is needed. Here f is an arbitrary function of the dilaton $\mathcal{X}$ and the superscripts ∞ denote evaluation of the corresponding quantity in the limit $\rho \to \infty$. By means of the integrability condition (7.23) we can, in principle, find a local expression for I_B. In order to choose our boundary conditions we follow the ideas of [214]. This amounts to pick a connection satisfying certain asymptotic conditions associated with a group $\mathcal{G}_\infty$. The dilaton field is chosen to be the gauge parameter that preserves the form of the gauge field. This choice naturally selects an integrability condition that allows to define a well-defined variational principle. Let us apply this rationale to the JT model.

7.3 Jackiw–Teitelboim model and Schwarzian action (again)

The JT model is obtained as non-Abelian BF-theory described in section 7.1 by choosing as gauge group $\mathcal{G} = \mathrm{SL}(2,\mathbb{R})$. The invariant tensor is determined by the matrix trace $\langle L_m, L_n\rangle = \mathrm{tr}[L_n L_m]$, and the generators $L_m \in \mathfrak{sl}(2,\mathbb{R})$ with $m = \{-1,0,1\}$ satisfy the usual commutation relations $[L_m, L_n] = (m-n)L_{m+n}$. The fundamental representation for these generators is

$$L_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad L_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad L_{-1} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}. \tag{7.24}$$

To specify boundary condition for the dilaton and the gauge field we employ a convenient parametrization of the fields [2].

$$\mathcal{A} = b^{-1}(d+a)b \qquad \mathcal{X} = b^{-1}x(\tau)b \qquad a = a_\tau(\tau)\,d\tau \qquad b = \exp(\rho L_0) \tag{7.25}$$

We assume that this is always possible near the boundary.

Boundary conditions

In this chapter we consider only the analogue of the more restrictive boundary conditions (6.4), i.e., having set $\mathcal{L}^+ = 1$, since the interesting asymptotic behavior that is related to the SYK model already appears in that case. They are most conveniently represented in the so-called highest weight gauge for the field a_τ

$$a_\tau = L_1 + \mathcal{L}(\tau)\,L_{-1}\,. \tag{7.26}$$

In order to choose boundary conditions for the dilaton x we proceed as follows. First, let us study the gauge symmetries that preserve the form of the auxiliary connection (7.26). Solving

$$\delta_\Lambda a_\tau = \partial_\tau \Lambda + [a_\tau, \Lambda] = \mathcal{O}(a_\tau) \tag{7.27}$$

yields

$$\Lambda[\varepsilon; a_\tau] = \varepsilon L_1 - \varepsilon' L_0 + \big(\mathcal{L}\varepsilon + \tfrac{1}{2}\varepsilon''\big)L_{-1} \tag{7.28}$$

and implies transformation of the function $\mathcal{L}$ by an infinitesimal Schwarzian derivative.

$$\delta_\varepsilon \mathcal{L} = \varepsilon\mathcal{L}' + 2\varepsilon'\mathcal{L} + \tfrac{1}{2}\varepsilon''' \tag{7.29}$$

As we saw before, the dilaton field x is the stabilizer of a. Thus it satisfies $\delta_x a_\tau = 0$. We assume that x has the form of (7.28) with ε replaced by some y.

$$x = \Lambda[y; a_\tau] \tag{7.30}$$

The on-shell value of the dilaton satisfies the relation $\delta_y \mathcal{L} = 0$. This condition corresponds to the little group equation of a Virasoro coadjoint orbit for the representative $\mathcal{L}$ [cf. equation (2.57)].

The latter choice for x has some nice consequences. The first one is that y transforms as a one-dimensional vector field. From the dilaton transformation (7.15) we have $\delta_\epsilon x = [\Lambda[y], \Lambda[\varepsilon]]$.[3] The component along L_1 of this expression tells us

$$\delta_\varepsilon y = \varepsilon y' - y\varepsilon' \tag{7.31}$$

which is the announced vectorial transformation that we saw in (6.42) in the second order formulation. The second consequence is that the choice (7.30) gives us a suitable integrability condition. In fact, we can reexpress (7.30) in a simpler manner

$$x = y\left(a_\tau - u^{-1}\partial_\tau u\right), \tag{7.32}$$

with $u = \exp(-\frac{1}{2}y' L_{-1})\exp(\log(y)L_0)$. By inverting this relationship we can express the gauge field as

$$a_\tau = f_\tau x + u^{-1}\partial_\tau u, \tag{7.33}$$

where $f_\tau = 1/y$. From a more general perspective (7.33) can be used as an integrability condition (7.23) that relates the asymptotic connection a_τ with x in terms of two quantities: a one-form $f_\tau\, d\tau$ and a group element u. They are free boundary data.

Action principle and on-shell action

Inserting our boundary condition (7.33) into the variation of the boundary term (7.22) one obtains

$$\delta I_B = -\frac{k}{\pi}\int d\tau\big[\delta(f_\tau\, C) + C\delta f_\tau - \mathrm{tr}\big((\partial_\tau x + [u^{-1}\partial_\tau u, x])u^{-1}\delta u - \partial_\tau(x\, u^{-1}\delta u)\big)\big]\,. \tag{7.34}$$

Note that f_τ is a one-form component in one dimension, hence it can be written as $f_\tau = \frac{1}{\bar{y}}\partial_\tau f$, where

$$\frac{1}{\bar{y}} := \frac{1}{\beta}\oint\frac{d\tau}{y} \tag{7.35}$$

is the zero mode of the quantity $1/y$ introduced in (7.32). We assume, additionally, that $f(\tau)$ is a well-defined diffeomorphism respecting $f(\tau+\beta) = f(\tau)+\beta$. This ensures that the second term vanishes since on-shell the Casimir C is constant. The third term is zero on-shell and we can discard the last term by imposing that the fields are periodic on the τ cycle.

From (7.34) together with the parametrization for $\mathcal{X}$ and $\mathcal{A}$ in (7.25) we can infer that the bulk-plus-boundary action

$$I[\mathcal{X},\mathcal{A}] = I_0[\mathcal{X},\mathcal{A}] + \frac{k}{2\pi\bar{y}}\int d\tau\,\big(\partial_\tau f\big)\,\mathrm{tr}(\mathcal{X}^2) \tag{7.36}$$

has a well-defined action principle. (Note that the term $df = d\tau\,\big(\partial_\tau f\big)$ acts as a boundary volume form.) Moreover, since the field strength vanishes on-shell, the corresponding value of the on-shell action is

$$I_{\text{ on-shell}} = -\frac{k\beta}{\pi\bar{y}}\,C \tag{7.37}$$

[3] Notice that this is true only asymptotically, i.e., to leading order if the ρ dependence is reinstated.

The result for the on-shell action (7.37) agrees precisely with the result (6.37) in the second order formulation (if the factor β for free energy is reinstated and $c_0 = 0$ for which there seems to be no analogue in the first order formulation).

The relation (7.32) permits to express the Casimir in terms of $f_\tau = \frac{1}{\bar{y}} f'$ and $\mathcal{L}$.

$$C = \frac{\bar{y}^2}{f'^2} \left(\mathcal{L} - \tfrac{1}{2} \{ f\,; \tau \} \right) \tag{7.38}$$

Thus, the Casimir is determined from the coadjoint action of the Virasoro group.

Under the diffeomorphism $u = f(\tau)$ and renaming $f^{-1}(u) \equiv \tau(u)$, the corresponding on-shell action (7.37) reduces to the Schwarzian action

$$I_{\text{on-shell}}[\tau] = -\frac{k\bar{y}}{2\pi} \int_0^\beta du \left[2\tau'(u)^2 \mathcal{L} + \text{Sch}[\tau; u] \right] . \tag{7.39}$$

Imposing regularity on the connection implies $\mathrm{H}[a] = -\mathbb{1}$. In order to satisfy this condition we need to provide the general solution to the equation $(\partial_\tau + a_\tau) g = 0$ with anti-periodic boundary conditions $g(0) = -g(\beta)$ [Note that the relation between G and g is $G = b^{-1} g$].

Using the connection in highest weight gauge, (7.26), we find

$$g = \begin{pmatrix} -\psi_1' & -\psi_2' \\ \psi_1 & \psi_2 \end{pmatrix} \tag{7.40}$$

where ψ_1 and ψ_2 are two independent solutions to Hill's equation

$$(\partial_\tau^2 + \mathcal{L}) \psi = 0 \, , \tag{7.41}$$

that we met already in the classification of coadjoint orbits of the Virasoro algebra (4.40). If we make a diffeomorphism such that we go to the frame with constant $\mathcal{L}$ we find solutions of the form $\exp(i\sqrt{\mathcal{L}}\tau)$. In order to satisfy the anti-periodic boundary condition we find

$$\mathcal{L} = \pi^2 n^2 / \beta^2 \qquad n \in \mathbb{Z} \tag{7.42}$$

which corresponds to an element of the Virasoro coadjoint orbit associated with the Euclidean black hole. Thus, the value of $\mathcal{L}$ in (7.39) is restricted to be *any* element of the orbit $\text{Diff}(S^1)/\ \text{SL}(2,\mathbb{R})$. Choosing $\mathcal{L}$ to be the constant representative with $n = 1$ we recover precisely the form of the Schwarzian action (6.59).

Yang–Mills extensions of the JT model

The method of this section can be applied as well to BF-theories with gauge groups $\text{SL}(2,\mathbb{R}) \times \mathcal{K}$ where $\mathcal{K}$ is a compact group. These theories have the interpretation of dilaton gravity coupled to Yang–Mills. We will not go into the details of this construction but briefly quote the result of the original work [1]. Denote the gauge fields corresponding to the group $\mathcal{K}$ by $\alpha_\tau = \mathcal{P}^a I_a$, where I_a are the generators of the algebra of $\mathcal{K}$ and we have again gauged-away the radial dependence. Let λ be a group element of $\mathcal{K}$. Then we find a Schwarzian-like action of the form

$$I_{\text{ on-shell}}[\tau, \lambda] = -\frac{k\bar{y}}{\pi} \int_0^\beta du \left[(\tau')^2 \mathcal{L} + \frac{1}{2} \{ \tau; u \} - \text{tr}_{\mathfrak{k}} [2\mathcal{P} \lambda^{-1} \lambda' - (\lambda^{-1} \lambda')^2] \right] . \tag{7.43}$$

This is our desired generalization of the Schwarzian action. It is expected to be relevant for the low-energy description of SYK-like models with global symmetries [52, 57, 59, 215]. The Abelian case can be interpreted as the bosonic part of the $\mathcal{N} = 2$ super SYK model [56, 216].

7.4 Boundary actions of BF theories in two dimension

We saw in the last section a derivation of the Schwarzian action for the JT model that mimicked closely the discussion of chapter 6, i.e., starting from an action with a well-defined variational principle that essentially reduces to the Casimir, we expressed the Casimir in terms of free boundary data and reproduced the Schwarzian action. The dilaton acted as the stabilizer for the geometry-defining quantity $\mathcal{L}$. While this approach worked nicely for the JT model and gauge groups of the form $\mathrm{SL}(2,\mathbb{R}) \times \mathcal{K}$ it becomes less tractable for generalizations to other gauge groups containing an $\mathrm{SL}(2,\mathbb{R})$ subgroup, such as $\mathrm{SL}(N)$.

In this section we discuss therefore an alternative way to obtain a boundary action associated to BF theories in two dimensions, namely by providing a suitable Hamiltonian action.

Symplectic structure and geometric actions

The kinetic term of this Hamiltonian action is obtained from the symplectic form associated to (7.14). From (7.20), we know that a general variation of the Lagrangian gives

$$\delta L = (\text{equations of motions}) + \frac{k}{\pi} \int \langle \mathcal{X}, \delta \mathcal{A} \rangle \,. \tag{7.44}$$

We can therefore define a presymplectic potential θ on a time slice $\Sigma = \{\tau = \text{ constant}\}$

$$\theta = \frac{k}{\pi} \int_{\Sigma} \langle \mathcal{X}, \delta \mathcal{A} \rangle \,. \tag{7.45}$$

Now we restrict ourselves to the part of phase space on which the geometry equations of motion $\mathcal{F} = 0$ are imposed. This corresponds to the solutions of (7.16) given by

$$a = -\, dgg^{-1} \qquad\qquad x = g x_0 g^{-1} \qquad\qquad dx_0 = 0 \,, \tag{7.46}$$

where we have made use of the factorization similar to (7.25) in order to express everything in terms of ρ-independent quantities, i.e., we have assumed that it is possible to split-off the radial dependence near the boundary

$$G(\rho, \tau) = b^{-1}(\rho) g(\tau) \,. \tag{7.47}$$

by a suitable choice of group element b. Notice that this is, however, a non-trivial assumption considered to be part of our boundary conditions.

We find then that the presymplectic potential (7.45) reduces to a pure boundary term

$$\theta = -\frac{k}{\pi} \langle x_0, g^{-1} \delta g \rangle |_{\partial\Sigma} \,. \tag{7.48}$$

Following the definition (3.94), we find for the symplectic structure

$$\Omega = -\frac{k}{\pi}\,\delta_1\langle x_0, g^{-1}\,\delta_2 g\rangle|_{\partial\Sigma} - (\delta_1 \leftrightarrow \delta_2), \tag{7.49}$$

i.e., the symplectic structure is again given by a pure boundary term. In fact there is a nice geometric picture behind this symplectic structure that is worth exploring.

The Maurer–Cartan form Θ is a $\mathfrak{g}$-valued left-invariant one-form on a Lie group $\mathcal{G}$, defined by the condition $\Theta(e)(\xi) = \xi$ with $\xi \in T_e\mathcal{G}$. Assuming that $\mathcal{G}$ is a matrix Lie group we have due to left-invariance at any other point $g \in \mathcal{G}$

$$\Theta(g)(Y) = g^{-1}\cdot\xi \qquad \xi \in T_g G\,. \tag{7.50}$$

The Maurer–Cartan form obeys the Maurer–Cartan equation

$$d\Theta + [\Theta, \Theta] = 0 \tag{7.51}$$

where d is the differential on the Lie group (this should not be confused with the differential on the manifold $\mathcal{M}$).

Assume for the moment that $\delta x_0 = 0$, that is x_0 is a fixed coadjoint vector. Then the field configuration is specified by the value of g, and δg can be viewed as a tangent vector to the field configuration, i.e., an element of $T_g\mathcal{G}$. With this we find that Ω can be rewritten as

$$\Omega = \frac{k}{\pi}\,\langle x_0, [\Theta(\delta_1 g), \Theta(\delta_2 g)]\rangle\,. \tag{7.52}$$

This is precisely the Kirillov–Kostant–Souriau symplectic structure on a codadjoint orbit defined in equation (2.70). Using the Maurer–Cartan equation we can rewrite the symplectic structure (7.52) as

$$\Omega = -\frac{k}{\pi}\,d\langle x_0, \Theta\rangle\,, \tag{7.53}$$

where we have dropped the tangent vectors $\delta_1 g, \delta_2 g$, so Ω is now a two-form on the Lie group $\mathcal{G}$ (again, d is the differential on the Lie group $\mathcal{G}$).

Consider a surface $\mathcal{N}$ in the group manifold of $\mathcal{G}$, bounded by the curve $\Gamma = \partial\mathcal{N}$, that is parametrized as $\Gamma(s)$. Then Ω as given in (7.53) defines a *geometric action*

$$I[g] = \int_{\mathcal{N}} \Omega = -\frac{k}{\pi}\int_\Gamma ds\,\left\langle x_0, g^{-1}\frac{d}{ds}g\right\rangle, \tag{7.54}$$

where Stoke's theorem was used in the second step. For more details regarding geometric actions and applications to lower-dimensional gravity consult, e.g., [87, 217].[4]

However this is not yet the action we want. Remember that we assumed $\delta x_0 = 0$ which means that we are on a fixed coadjoint orbit. Having fixed x_0 corresponds to fixed Casimir and we saw in various places that on-shell conservation of the Casimir is in conflict with having a Schwarzian action. We therefore take (7.54) as our motivation for *defining* the action

$$I_{\text{geom}}[x, g] = -\frac{k}{\pi}\int_\Gamma ds\,\left\langle x, \frac{d}{ds}g\,g^{-1}\right\rangle, \tag{7.55}$$

[4] A similar action has been considered also in [218, 219] in a different context.

in which $x = gx_0g^{-1}$ is again introduced as a *dynamical variable*. Since the original symplectic form (7.49) was defined on the boundary of the spacetime $\mathcal{M}$, we can regard (7.55) as the kinetic term for a boundary action of a generic BF model. Notice that the symplectic form obtained from (7.55) coincides with the symplectic form (7.49) which means that the definition is consistent.

Boundary Hamiltonians

In order to give dynamics to this model we need to include a Hamiltonian preserving $\mathcal{G}$-invariance. The Casimir functions (7.18) naturally preserve the symmetry along Γ. The most general choice is

$$\mathcal{H} = \frac{k}{\pi} \sum_{i=2}^{N} \mu^{(i)}(s)\, C_i \tag{7.56}$$

where μ_i's are some arbitrary functions and N some integer depending on the gauge group (for SL(N, $\mathbb{R}$) this number is N). Then, the natural dynamical system for (generalized) dilaton gravity models follows from the reduced action principle

$$I^{\mathcal{H}}_{\text{geom}}[x, g] = -\frac{k}{\pi} \int_{\Gamma} ds \left(\left\langle x, \frac{d}{ds} g\, g^{-1} \right\rangle - \sum_{i=2}^{N} \mu^{(i)}\, C_i \right) . \tag{7.57}$$

Note that the on-shell action for this system is given by the sum of the Casimir functions. The equation of motion for x is given by

$$\left(\frac{d}{ds} g\, g^{-1} \right)_A = -\sum_{i=2}^{N} \mu^{(i)} g_{AA_2 \cdots A_i} x^{A_2} \cdots x^{A_i} \, . \tag{7.58}$$

Plugging this back in the action (7.57) we find

$$I^{\mathcal{H}}_{\text{geom}} = -\frac{k}{\pi} \sum_{i=2}^{N} (i-1) \int_{\Gamma} ds\, \mu^{(i)} C_i \, . \tag{7.59}$$

This acquires the same form for the previously known case with one Casimir function (7.37). In what follows we consider the more tractable case where $\mu^{(2)}$ is the only non-vanishing function. In that case,

$$\int ds\, \mathcal{H} = -\frac{k}{2\pi} \int ds\, \mu^{(2)}(s) \langle x, x \rangle \, . \tag{7.60}$$

From the above expression, we see that $\mu^{(2)}$ plays the role of an einbein. Defining $1/\bar{y}$ as the zero mode of $\mu^{(2)}$, we can always choose a new coordinate τ such that $\mu^{(2)}\, ds = \frac{1}{\bar{y}}\, d\tau$ where $\mu^{(2)} = \frac{1}{\bar{y}} \tau'(s)$. This allows us to relate the arbitrary curve parameter s in the Lie group to the boundary time τ of the dilaton gravitational theory.

It is convenient to express the action in the second order formulation. This is achieved by eliminating the momenta x using equation (7.58)

$$x = -\frac{\bar{y}}{\tau'} \partial_s g\, g^{-1} \quad \Longrightarrow \quad I^{\mathcal{H}}_{\text{geom}}[g] = \frac{k\bar{y}}{2\pi} \int d\tau \langle \partial_\tau g g^{-1}, \partial_\tau g g^{-1} \rangle \, . \tag{7.61}$$

We claim that this is the boundary action for any BF model based on a (semi-simple) Lie group $\mathcal{G}$.

Symmetries

The model (7.61) is invariant under multiplication by constant group elements both from the left and the right

$$g \mapsto a\, g\, b \qquad a, b \in \mathcal{G}\,. \tag{7.62}$$

It is straightforward to find the canonical charges that generate the symmetries (7.62). The symplectic potential

$$\theta(g, \delta g) = \frac{k\bar{y}}{\pi} \langle g^{-1}\partial_\tau g\,, g^{-1}\delta g\rangle \tag{7.63}$$

yields the symplectic structure

$$\Omega = \frac{k\bar{y}}{\pi} \langle g^{-1}\delta(\partial_\tau g), g^{-1}\delta g\rangle \tag{7.64}$$

for the model (7.61), where antisymmetrization in the variations is assumed. As expected, this is identical to the symplectic structure (7.49) after imposing the equation of motion for x.

The vectors in the tangent bundle of the group manifold ξ that are tangent to the flows generated by the symmetries (7.62) are given by

$$\xi_A = -Ag \qquad\qquad \xi_B = -gB \qquad A, B \in \mathfrak{g}\,, \tag{7.65}$$

respectively, where A and B are the Lie algebra elements corresponding to a and b via the exponential map. The canonical charge Q generating the flow tangent to a vector field ξ is given by Hamilton's equation

$$\delta Q = i_\xi \Omega\,. \tag{7.66}$$

In the present case one finds

$$Q^L_A = \frac{k\bar{y}}{\pi} \big\langle A, \partial_\tau g g^{-1}\big\rangle, \qquad Q^R_B = \big\langle B, g^{-1}\partial_\tau g\big\rangle \tag{7.67}$$

as generator of left and right symmetry, respectively. The Poisson brackets between the charges, read off from the symplectic structure,

$$\{Q^L_A, Q^L_{A'}\} = \frac{\pi}{k\bar{y}} Q^L_{[A',A]}, \quad \{Q^R_B, Q^R_{B'}\} = \frac{\pi}{k\bar{y}} Q^R_{[B',B]}, \quad \{Q^L_A, Q^R_B\} = 0 \tag{7.68}$$

show explicitly that the symmetry algebra consists of two commuting copies of $\mathfrak{g}$.

7.5 Higher Spin Schwarzian actions

Let us now specialize to the case $\mathcal{G} = \mathrm{SL}(N, \mathbb{R})$. Expression (7.61) corresponds then to the action of a particle on the $\mathrm{SL}(N, \mathbb{R})$ group manifold (for details related to $N = 2$ see [220, 221]). In the context of the JT model, this was already found in [222].

Reduction to the gravitational sector

The action (7.61) was derived using the gauge flatness condition (7.46) without assuming any particular form of a. However, in order to make contact with the previous sections the connection cannot be arbitrary but should fulfill two requirements:

- the geometry associated to a is asymptotically AdS_2 (with fluctuating dilaton);
- a is compatible with the temperature, i.e., has no conical deficits.

We will treat the second item later which means that we restrict ourselves to the *zero-temperature* case in the following.

Our approach is based on the parametrization of the $\mathrm{SL}(N)$ group element g as solution to a particular N-th order differential equation that generalizes Hill's equation (7.40). Another approach based on the Iwasawa decomposition of the group element g is presented in appendix C.

The interpretation of a as describing an asymptotically AdS_2 geometry is guaranteed if the connection is taken to be of the (highest-weight) form

$$a_\tau = L_1 + Q\,, \tag{7.69}$$

where $[L_{-1}, Q] = 0$. The connection (7.26) considered in the previous sections belong to this class. More generally, choosing the principal embedding $\mathfrak{sl}(2) \hookrightarrow \mathfrak{sl}(N)$, the adjoint representation of $\mathfrak{sl}(N)$ is decomposed in irreducible representations $\{W^s_m\}$, $s = 3, \dots, n$ of the 'spin-2 gravity' subalgebra (7.10) with commutators

$$[L_m, W^{(s)}_n] = ((s-1)m - n)W^{(s)}_{m+n}\,. \tag{7.70}$$

The above requirement thus restricts the connection (7.69) to be of the form

$$a_\tau = L_1 + \mathcal{L}L_{-1} + \sum_{i=3}^{N} \mathcal{W}_i\, W^{(i)}_{1-i}\,. \tag{7.71}$$

Note that these generators provide an orthogonal basis for the Lie algebra with respect to the Cartan-Killing metric.

From the form (7.71) and the flatness condition $a_\tau = -\partial_\tau g\, g^{-1}$ of (7.46) it follows that $\mathrm{SL}(N)$ element g therefore has to obey the equation

$$\Big(\partial_\tau + L_1 + \mathcal{L}\, L_{-1} + \sum_{i=3}^{n} \mathcal{W}_i\, W^{(i)}_{1-i}\Big)\, g = 0\,, \tag{7.72}$$

where the fundamental representation for the $\mathfrak{sl}(N)$ elements is assumed. We demonstrate now that this equation implies a parametrization of g in terms of the independent solutions to an N-th order differential equation, analogous to the case discussed in section 7.3 for $N = 2$.

In the fundamental representation the operator acting on g in equation (7.72) is of the form

$$\begin{pmatrix} \partial_\tau & -\sqrt{k_1}\mathcal{L} & \alpha_3\mathcal{W}_3 & \alpha_4\mathcal{W}_4 & \cdots & \alpha_{N-1}\mathcal{W}_{N-1} & \alpha_N\mathcal{W}_N \\ \sqrt{k_1} & \partial_\tau & -\sqrt{k_2}\mathcal{L} & \alpha_3\mathcal{W}_3 & \cdots & \alpha_{N-2}\mathcal{W}_{N-2} & \alpha_{N-1}\mathcal{W}_{N-1} \\ 0 & \sqrt{k_2} & \partial_\tau & -\sqrt{k_3}\mathcal{L} & \cdots & \alpha_{N-3}\mathcal{W}_{N-3} & \alpha_{N-2}\mathcal{W}_{N-2} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \sqrt{k_{N-2}} & \cdots & \partial_\tau & -\sqrt{k_{N-1}}\mathcal{L} \\ 0 & 0 & 0 & 0 & \cdots & \sqrt{k_{N-1}} & \partial_\tau \end{pmatrix}, \tag{7.73}$$

where $k_i = 2\sum_j (K^{-1})_{ij}$, K_{ij} is the Cartan matrix, and α_s is some normalization for the higher spin charges that, although straightforward to determine, will not be important in the following.

Writing the group element g in terms of n-dimensional row vectors

$$g = \begin{pmatrix} \Psi_N \\ \Psi_{N-1} \\ \vdots \\ \Psi_2 \\ \Psi_1, \end{pmatrix} \tag{7.74}$$

the structure of the operator (7.73) allows to express the vectors $\Psi_2, \ldots, \Psi_N$ in terms of $(N-1)$ derivatives of Ψ_1. Denoting the components of Ψ_1 by $\psi_1, \psi_2, \ldots, \psi_N$ the solution of (7.72) thus boils down to solving the n-th order differential equation

$$\psi_i^{(N)} + u_2\psi_i^{(N-2)} + u_3\psi_i^{(N-3)} + \cdots + u_{N-1}\psi_i' + u_N\psi_i = 0\,. \tag{7.75}$$

Notice the absence of a term proportional to $\psi_i^{(N-1)}$. This is related to the fact that g has determinant equal one, as it is an SL(N) element. The differential equation (7.75) generalizes Hill's equation (7.41) to which it reduces for $N = 2$.

The coefficient functions u_i are monomials of derivatives of $\mathcal{L}$ and $\mathcal{W}_i$. However, it is straightforward to show that the coefficient u_2 is always given by

$$u_2 = \frac{N(N^2-1)}{6}\,\mathcal{L}\,. \tag{7.76}$$

The differential equation (7.75) transforms covariantly under W_N transformations. While the transformation under arbitrary finite W transformations is not known apart from some specific cases (see e.g., [223]), under the subgroup of reparametrizations of τ, i.e., the Virasoro group, ψ_i transforms as

$$\psi_i(\tau) = \left(\frac{dt}{d\tau}\right)^{-\frac{N-1}{2}} \psi_i(t) \qquad \tau \to \tau(t). \tag{7.77}$$

For an infinitesimal transformation $\tau \mapsto \tau + \epsilon(\tau)$ one finds

$$\delta_\epsilon\psi_i = \epsilon\psi_i' - \frac{N-1}{2}\epsilon'\psi_i\,. \tag{7.78}$$

The coefficients u_i have a complicated transformation behavior under reparametrizations but it is straightforward to show that u_2 transforms as an anomalous two-tensor

$$u_2(t) = u_2(\tau)\left(\frac{d\tau}{dt}\right)^2 + \frac{N(N^2-1)}{12}\{\tau;t\}\,, \tag{7.79}$$

as suggested by the observation (7.76).

Further properties of the differential equation (7.75) have been intensively studied in the context of W-algebras and their relation to KdV flows and Gel'fand–Dikii Poisson structures. Here we do not go into the details of these interesting developments but refer to the ample literature, see e.g. [224] and references therein.[5]

We saw above that the general model (7.61) has a global $\mathrm{SL}(N)\times\mathrm{SL}(N)$ symmetry under multiplication from the left and from the right (7.62). However, it is clear from (7.46) or (7.72) that the global left symmetry of the model is broken if g in (7.61) is required to be a solution of that equation. On the other hand, multiplication of g on the right by an $\mathrm{SL}(N)$ element is still a symmetry. The action on the ψ_i of this symmetry is immediately clear from their representation as a row vector in (7.74), i.e., it is the natural action of $\mathrm{SL}(N)$ on an element of $\mathbb{R}^N$. It is convenient to introduce the following ratios

$$s_i = \frac{\psi_i}{\psi_N} \qquad 1\le i\le N-1\,, \tag{7.80}$$

since the determinant condition $\det g = 1$ allows then to express ψ_N as a function of the s_i's. Notice that the s_i's can be viewed as homogeneous coordinates on the $(N-1)$ dimensional real projective space $\mathbb{RP}^{N-1}$. The differential equation (7.75) is therefore associated to a curve $\gamma(\tau) = (s_1(\tau),...,s_{N-1}(\tau)) \in \mathbb{RP}^{N-1}$. The action of $\mathrm{SL}(N)$ on the ψ_i's then induces the action $\mathrm{PSL}(N)$ on the s_i. For instance in the case $N=2$ one finds the transformation

$$s_1 \mapsto \frac{as_1+b}{cs_1+d} \tag{7.81}$$

in accordance with (4.11).

Higher spin Schwarzian actions (zero temperature)

We are now ready to present the key point of the argument that allows us to construct analogues of the Schwarzian action in the higher spin cases: It is possible to construct $(N-1)$ *projective invariants* $I^{(r)}(s_i;\tau)$ with $r=2,\dots,N$ from the solutions s_i of the differential equation (7.75). They are invariant under the projective action of $\mathrm{SL}(N)$ on s_i and transform as r-tensors under reparametrizations of τ. In particular, for $r=2$ one finds

$$I^{(2)}(s_i;\tau) = u_2(\tau)\,, \tag{7.82}$$

with the same anomalous transformation law under diffeomorphisms. We do not present the general algorithm, which can be found, e.g., in [226, 227] but outline the calculation

[5] A recent interesting connection between Gel'fand–Dikii structures and three-dimensional gravity was shown in [225].

for $N = 2$. In this case (7.75) reduces to Hill's equation (7.41) with $u_2 = \mathcal{L}$. Plugging the parametrization (7.80) with $\psi_1 = s_1\psi_2$ in Hill's equation yields

$$s_1''\psi_2 + 2s_1'\psi_2' = 0\,. \tag{7.83}$$

Differentiating this equation and using Hill's equation again yields

$$(s_1''' - 2s_1'\mathcal{L})\psi_2 + 3s_1''\psi_2' = 0\,. \tag{7.84}$$

This set of two differential equation for ψ_2 and ψ_2' can have a non-trivial solution only if its determinant vanishes. We therefore find

$$\mathcal{L} = \frac{1}{2}\left(\frac{s_1'''}{s_1'} - \frac{3}{2}\left(\frac{s_1''}{s_1'}\right)^2\right) = \frac{1}{2}\{s_1;\tau\}\,. \tag{7.85}$$

Since the Schwarzian derivative is invariant under fractional linear transformations such as (7.81) and transforms anomalously under reparametrizations of τ, we have succeeded in finding $I^{(2)}(s_i;\tau)$ in the case $N = 2$.

By equations (7.76) and (7.82) it is always possible to write $\mathcal{L}$ in terms of $N-1$ functions s_i

$$\mathcal{L} = \mathcal{L}[s_1(\tau), \cdots, s_{N-1}(\tau)]\,. \tag{7.86}$$

Inserting $a = -\partial_\tau g g^{-1}$ in the action (7.61) and using the fact that our basis of sl(N) is orthogonal we find that the action can be rewritten as

$$I[g] = \kappa\frac{k\bar{y}}{\pi}\int d\tau\,\mathcal{L}, \tag{7.87}$$

where

$$\kappa = \mathrm{tr}(L_1L_{-1})\,. \tag{7.88}$$

By the arguments in the previous paragraphs this action is manifestly invariant under SL(N) transformations. For $N > 2$ it describes the appropriate higher-spin analogue of the Schwarzian action. Indeed, in the case $N = 2$ we reproduce

$$I[g] = -\frac{k\bar{y}}{2\pi}\int d\tau\left(\frac{s_1'''}{s_1'} - \frac{3}{2}\left(\frac{s_1''}{s_1'}\right)^2\right), \tag{7.89}$$

while for $N = 3$ we find

$$I[g] = -\frac{4k\bar{y}}{\pi}\int d\tau\left(\frac{f'''}{f'} - \frac{4}{3}\left(\frac{f''}{f'}\right)^2 + \frac{e'''}{e'} - \frac{4}{3}\left(\frac{e''}{e'}\right)^2 - \frac{1}{3}\frac{f''e''}{f'e'}\right), \tag{7.90}$$

where $e = s_1'/s_2'$ and $f = s_2$. The generalization of the Schwarzian appearing in this action and its relation to W_3 algebras is well-known, e.g., [228, 229]. The SL(3) invariance of this action, guaranteed by the above arguments, can be checked in a straightforward if tedious manner using the transformations

$$s_1 \mapsto \frac{a_{11}s_1 + a_{12}s_2 + a_{13}}{a_{31}s_1 + a_{32}s_2 + a_{33}} \qquad s_2 \mapsto \frac{a_{21}s_1 + a_{22}s_2 + a_{23}}{a_{31}s_1 + a_{32}s_2 + a_{33}}\,, \tag{7.91}$$

where a_{ij} denote components of an SL(3) matrix.

Higher spin Schwarzian actions (finite temperature)

We deal now with the second item of the list at the beginning of section 7.5, i.e., we demand the connection a to be compatible with the temperature given by the inverse of the periodicity of Euclidean time β. We set $\beta = 2\pi$ in this subsection.

As discussed in section 7.2, the absence of conical singularities is guaranteed if the connection (7.71) has trivial holonomy. This is equivalent to the condition

$$g(2\pi) = (-1)^{N+1}\, g(0) \tag{7.92}$$

on the group element (7.74). The solutions ψ_i we are looking for therefore have to obey (anti-) periodic boundary conditions for N (even) odd: $\psi_i(2\pi) = (-1)^{N+1}\psi_i(0)$. This is consistent with the fact that by equation (7.78) a solution ψ_i has (half-)integer spin for (even) odd N.

The holonomy condition on the group elements g (7.92) can be reformulated in the following neat geometric way. We mentioned above that a solution to (7.75) can be viewed as a curve in $\mathbb{RP}^{N-1}$ with Euclidean time τ as parameter. Since we are working at finite temperature β this corresponds to a map $\gamma : S^1 \to \mathbb{RP}^{N-1}$.

A generic solution γ is not closed but is shifted by an element $M \in \mathrm{SL}(N)$, called *monodromy*, after one period of Euclidean time

$$\gamma(\tau + 2\pi) = M\gamma(\tau) \qquad\qquad M \in \mathrm{SL}(N)\,. \tag{7.93}$$

The (conjugacy class of the) monodromy is an invariant of the differential equation (7.75) [230]. In other words, acting on the differential equation (7.75) with an arbitrary W transformation leads to a different solution curve albeit with the same monodromy as the solution curve of the original equation. In section 4.4 we discussed this in relation to the classification of coadjoint orbits of the Virasoro algebra.

The *homotopy class* of curves with a given monodromy, i.e., the (in-)ability to deform one into the other, provides another invariant for the differential equation (7.75). In fact, monodromy and homotopy class are the only invariants [230].

The holonomy condition (7.92) is thus translated to the following statement: To each solution curve $\gamma \in \mathbb{RP}^{N-1}$ one can associate its unique lift $\tilde{\gamma} = (\psi_1, \dots, \psi_N) \in \mathbb{R}^N$, as guaranteed by the determinant condition. The monodromy of the lift $\tilde{\gamma}$ has to obey

$$\widetilde{M} = (-1)^{N+1}\mathbb{1}\,. \tag{7.94}$$

This means that the monodromy of γ is

$$M = \mathbb{1}\,. \tag{7.95}$$

The number of homotopy classes for curves $\gamma \in \mathbb{RP}^{N-1}$ of monodromy $\mathbb{1}$ with the above properties have been determined in [231] (see also [232, 233]). They are $\mathbb{N}$ for $N = 2$, three for N odd, and two for $N > 2$ even. The infinitely many different homotopy classes in the case $N = 2$ correspond to the exceptional orbits (4.48) parametrized by the number n that we saw in the classification of Virasoro orbits.[6]

[6] The origin of this large number of homotopies is the inability to continuously deform a diffeomorphism that winds around the circle k times into one that winds around $k + 1$ times.

Based on the above, we propose below the finite temperature version of the SL(N)-invariant family of Schwarzian actions. To do so, we study in some detail the appearance of the Schwarzian theory in the $N = 2$ case and then propose a generalization for $N > 2$.

In the SL(2) case, equation (7.75) is identical to Hill's equation (7.41) with $\psi = \psi_{1,2}$. Suppose the parametrization τ is such that $\mathcal{L}$ is constant. The anti-periodic boundary conditions (7.92) force $\mathcal{L}$ to be $\mathcal{L} = 1/4$, where we chose the homotopy class associated to $n = 1$.[7] The two independent solutions read $\psi_1 = \sqrt{2}\cos(\tau/2)$ and $\psi_2 = \sqrt{2}\sin(\tau/2)$. For non-constant $\mathcal{L}$, we build up a one-parameter family of solutions that have the same monodromy as the previous solution. By applying a diffeomorphism $\theta(\tau)$ on ψ we find

$$\hat{\psi}_1(\tau) = \sqrt{\frac{2}{\theta'(\tau)}}\cos\left(\frac{1}{2}\theta(\tau)\right), \qquad \hat{\psi}_2(\tau) = \sqrt{\frac{2}{\theta'(\tau)}}\sin\left(\frac{1}{2}\theta(\tau)\right). \tag{7.96}$$

The corresponding $\mathcal{L}$ associated to this orbit of solutions is given by

$$\mathcal{L} = \left\{\cot\left(\frac{1}{2}\theta(\tau)\right); \tau\right\}. \tag{7.97}$$

Thus, using (7.87) to define the action, we conclude that the Schwarzian theory is recovered. An important observation that will be crucial for the generalization is that the argument of $\mathcal{L}$ in this approach is given by

$$\hat{s}(\tau) = \frac{\hat{\psi}_1}{\hat{\psi}_2} = \cot\left(\frac{1}{2}\theta(\tau)\right) \tag{7.98}$$

which is precisely the map that relates the projective line to the circle S^1. Note that while $(\hat{\psi}_1, \hat{\psi}_2) \in \mathbb{R}^2$ is anti-periodic on a 2π-period, the function $\hat{s}$ is periodic. This illustrates the relation between the monodromy (7.95) and the one associated to the lift (7.94).

Motivated by the previous analysis, we consider

$$\hat{s}_i : S^{N-1} \to \mathbb{RP}^{N-1} \tag{7.99}$$

which defines a projection of the coordinates s_i into the unit sphere S^{N-1} and satisfies the monodromy condition (7.95). We propose that

$$I[g] = \kappa\frac{k\bar{y}}{\pi}\int_0^{2\pi} d\tau\, \mathcal{L}[\hat{s}_1(\tau)\cdots\hat{s}_{N-1}(\tau)]\,, \tag{7.100}$$

is the Schwarzian action at finite temperature $\beta = 2\pi$ [with κ defined in (7.88)].

As an example, let us present the map (7.99) associated to the $N = 3$ case. This is given by the *central* projection of S^2 on $\mathbb{RP}^2$

$$\Big(\hat{s}_1, \hat{s}_2\Big) = \Big(\cot(\theta)\cos(\varphi), \cot(\theta)\sin(\varphi)\Big)\,. \tag{7.101}$$

The idea of this map is that we choose a point as the center of S^2 and a tangent plane to it representing $\mathbb{RP}^2$. Lines passing trough the center projects points $(\hat{s}_1, \hat{s}_2)$ on the two-sphere

[7] It would be interesting to see if the SYK model allows for an interpretation of different homotopy classes.

represented by (θ, φ). Provided $\theta(\tau + 2\pi) \sim \theta(\tau) + \pi$ and $\varphi(\tau + 2\pi) \sim \varphi(\tau) + 2\pi$, this map ensures that the monodromy condition is satisfied. An explicit expression for the SL(3) Schwarzian action at finite temperature is presented in appendix C, that was obtained using Iwasawa decomposition of the group element.

One-loop contribution from higher-spin fields

We would like to explore the effect of considering higher rank groups in the one-loop contribution to the free energy. Let us consider

$$Z[\beta] = \int d\mu[g]\, e^{-I[g]} \tag{7.102}$$

where $g \in$ SL$(N, \mathbb{R})$ and μ is a measure that we will leave unspecified for the moment. Action $I[g]$ in the partition function is given by

$$I[g] = \frac{\kappa}{\sigma^2} \int_0^{2\pi} d\varphi\, \mathcal{L}[\hat{s}_1(\varphi), \cdots, \hat{s}_{N-1}(\varphi)] \qquad\qquad \sigma^{-2} = \frac{2k\bar{y}}{\beta}\,. \tag{7.103}$$

where we have introduced the coordinate $\varphi = \frac{2\pi}{\beta}\tau$ [and again κ is defined in (7.88)]. The one-loop contribution can be computed from the second variations associated to (7.103). This can be expressed as

$$\delta^2 I[g] = \int_0^{2\pi} d\varphi\, \mathrm{tr}\big[(\partial_\varphi \epsilon - [\partial_\varphi g g^{-1}, \epsilon])\partial_\varphi \epsilon\big] \tag{7.104}$$

where we defined $\delta g g^{-1} = \sigma\, \epsilon(\varphi)$ and we have introduced the parameter σ in the definition to keep track of the perturbation expansion order. Evaluation in the gravitational sector amounts to use the condition $\partial_\varphi g g^{-1} = -a_\varphi^{\text{reg}}$, where a_φ^{reg} is the connection satisfying the regularity condition (7.92). Thus, the quadratic fluctuations around the saddle are controlled by the second-order operator

$$\Delta = -\partial_\varphi(\partial_\varphi + [a_\varphi^{reg}, \cdot])\,. \tag{7.105}$$

The operator Δ has $N^2 - 1$ zero modes corresponding to the SL$(N, \mathbb{R})$ isometries of a_φ^{reg}. Summing over inequivalent configurations in (7.102) implies that we should consider this modes as gauge symmetries. Following [47], the path integral measure should be corrected with the introduction of the product

$$\prod_{i=0}^{N^2-2} \delta(\epsilon^{(i)}(0))$$

which will remove the zero modes associated to Δ. To evaluate the quadratic contribution to (7.102), we must express the measure associated to g in terms of ϵ. This means that we need to trade every Fourier mode of $\delta g g^{-1}$ for a Fourier mode of ϵ, except for the $N^2 - 1$ zero modes that have been fixed. Extending the argument of [211] to the SL(N) case, the result for this determinant is given by σ^{1-N^2}. In turn, the one-loop contribution to the free energy is

$$F_{\text{1-loop}} = \frac{N^2-1}{2}\, T\, \log(2k\bar{y}T)\,. \tag{7.106}$$

In the next section we discuss consequences for the entropy of the leading and subleading terms, i.e., the zero- and one-loop contributions to the partition function for the extensions of the Schwarzian action (Yang-Mills, higher spins).

7.6 Entropy

It is of interest to calculate the entropy associated with thermal states in BF-theories, as this corresponds to the black hole entropy in cases where a gravitational interpretation exists. We focus first on the leading, classical, contributions to the AdS_2 black hole entropy.

One can derive the entropy in a variety of ways. In the present context perhaps the simplest derivation is from evaluating the Euclidean on-shell action (see [112] and references therein), multiplying by temperature to get free energy

$$F(T) = T\,\big(I_0 + I_B\big)_{\text{EOM}} = T\, I_B\big|_{\text{EOM}} \tag{7.107}$$

and then taking the T-derivative to get entropy [with I_B determined from (7.22) together with a suitable integrability condition (7.23)].

$$S = -\frac{dF}{dT} = -I_B\big|_{\text{EOM}} - T\,\frac{dI_B}{dT}\big|_{\text{EOM}} \tag{7.108}$$

In the spin-2 case we recover in this way from the on-shell action (7.37) the result of [2, 4, 192],[8]

$$S_{\text{JT}} = \frac{k}{\pi\bar{y}}\,\frac{dC}{dT} = 2k\pi\bar{y}\,T \tag{7.109}$$

where we have used the regularity condition (7.42) (setting $n = 1$) together with the relation (7.40) between Casimir C and mass function $\mathcal{L}$. Note that the result for entropy (7.109) is compatible with the third law of thermodynamics and shows the same temperature dependence as a Fermi-liquid (or -gas) with Sommerfeld constant[9] given by $\gamma = 2k\pi\bar{y}$.

The gravity result (7.109) for the entropy coincides qualitatively with the field theory result derived in [47], see their (G.241): the first term in their expansion is temperature-independent and captures the zero-temperature entropy S_0 that is not modeled by JT. The second term in their expansion

$$S - S_0 = 2a_3\frac{NT}{J} \stackrel{?}{=} S_{\text{JT}} \tag{7.110}$$

should then correspond to the entropy (7.109). Using the relation between N, β, J discussed below (6.59) shows that indeed these two expressions coincide for any N, J, T (subject to

[8] To compare with [192] we need to identify $\bar{\phi}_r$ in their (3.18) with our $\bar{y}$.

[9] The Sommerfeld constant is the ratio of specific heat and temperature in the limit $T \to 0$, which in the Fermi-liquid case reduces to the coefficient linear in T in the small-T expansion of the entropy.

$N \gg 1$ and $J \gg T$) for some value of the numerical coefficient a_3 that is independent from N, J, T.

The on-shell value of the boundary action (7.57) proposed in section 7.4 is given by the sum of Casimir (7.59)

$$F_{(N)} = -\frac{k}{\pi} \sum_{s=2}^{N} f_0^{(s)} C_s \tag{7.111}$$

where $f_0^{(s)}$ denotes the zero mode of the function $\mu^{(s)}$ associated with a field of spin s. Since we expect the scaling behavior $C_s \propto T^s$ [based on the scaling properties of the differential equation (7.75)] the entropy would be

$$S_{(N)} = \frac{k}{\pi} \sum_{s=2}^{N} f_0^{(s)} \frac{s\, C_s}{T} \sim \sum_{s=2}^{N} \hat{f}^{(s)} T^{s-1} \,. \tag{7.112}$$

In all cases above at low temperatures the entropy is dominated by the spin-2 contribution (as long as $f_0^{(2)} \neq 0$), which scales linearly in T. In particular, the JT-result for entropy (7.109) receives modifications from higher spin fields only at higher temperatures.

We consider now the 1-loop contribution to the entropy. The general expression (7.108) together with the classical (7.112) and 1-loop results (7.106) yields

$$S_{\text{1-loop}} = S_{(N)} - \frac{N^2 - 1}{2} \ln S_{(N)} + \mathcal{O}(1) \,. \tag{7.113}$$

For the SL(2) case the famous factor $-3/2$ (see e.g. [234] and refs. therein) in front of the log-term is recovered, while for general SL(N) this factor is instead $-(N^2 - 1)/2$. The result (7.113) implies that the dominant contribution from higher spin fields in the small temperature limit actually may come from the 1-loop contribution to the entropy, as the classical contribution is suppressed by T^{s-1}.

This concludes our discussion on possible generalization of the Schwarzian action from the gravitational side. In the conclusion we will point out various lines of research opened up by the results of this chapter.

Chapter Eight

Conclusions

Let us summarize what was achieved in this thesis and point out possible new avenues of investigation.

Summary

Following an introduction to symplectic structures and Poisson structures in chapter 2, we presented a reformulation of dilaton gravity theories as (non-linear) gauge theories in the form of PSM models in chapter 3. We extensively discussed geometry and interpretation of different AdS_2 solutions in chapter 4.

We started our search for AdS_2 holography in chapter 5 with a comprehensive study of all dilaton gravity models that allow for an AdS_2 solution with constant dilaton. By this we confirmed and extended previous claims in the literature, e.g., [43, 45, 163, 166] that this set-up does not allow for any physical states with finite energy. We confirmed this both classically, by showing that the canonical charges and the on-shell action vanish, and quantum mechanically by showing that the one-loop partition function is trivial. These results are completely independent of any choice of dilaton gravity model and are equally valid in the presence of additional matter fields such as Yang–Mills.

Chapter 6 was concerned with the simplest model of dilaton gravity that allows for linear dilaton solutions in AdS_2 space. We considered boundary conditions, similar to [198] in three dimensions for which the boundary metric in Fefferman–Graham gauge is not fixed, and provided a geometric interpretation using the conformal framework. Furthermore, the boundary value of the dilaton was not assumed to be fixed but rather transformed non-trivially under the asymptotic symmetry translations. In this way we circumvented the common lore that only constant dilaton solutions are consistent with all the isometries of AdS_2. Subsequently we showed that this action has a well-defined variational principle that required a new boundary term for the dilaton. We showed that this improved action principle gave rise to the Schwarzian action as a boundary action for JT model thus establishing a link to the SYK model. The asymptotic charges of the model were constructed and it was argued that they reproduce the pattern of symmetry breaking featured in the SYK model.

In chapter 7 we reformulated the JT model as BF theory and rediscovered the Schwarzian action using the fact that the dilaton behaved as the stabilizer of the asymptotic connection one-form a_τ. We then considered various generalizations. First to dilaton gravity-Yang–Mills theories where the gauge group of the BF action was taken to be $\mathrm{SL}(2,\mathbb{R}) \times \mathcal{K}$. We found the corresponding boundary action to consist of a gravitational Schwarzian part and a $\mathcal{K}$ Kac–Moody contribution. In order to generalize the Schwarzian action to higher rank groups $\mathcal{G}$ such as $\mathrm{SL}(N)$ we used the symplectic structure of the PSM model to define a boundary theory that was found to describe a particle moving on the group manifold of $\mathcal{G}$. Using the relation between flat $\mathfrak{sl}(N)$ connections and certain N-th order ordinary differential equations generalizing Hill's equation, we were able to construct *higher spin Schwarzian actions* at zero temperature and proposed a prescription to obtain finite temperature actions. We concluded with a discussion of one-loop partition function, obtained in analogy to [211], and discussed thermodynamic aspects of the action. An alternative approach to these higher spin Schwarzian actions is presented in appendix C.

Open issues and avenues for further research

CDV holography. The results of chapter 5 show that any dilaton gravity theory on AdS_2 in the constant dilaton sector necessarily contains only the vacuum. However, before delivering a final verdict over AdS_2 holography for constant dilaton, we should be careful enough to examine possible loopholes of our argument. In section 5.3 we calculated the one-loop contribution to the partition function using analytic continuation to the sphere. The reason for this was the presence of harmonic vector modes on AdS_2. It would be worthwhile to check if analytic continuation to the sphere can be avoided by treating these contributions explicitly (as we checked they do not contribute to the action) or using some of the methods developed in [235, 236]. Furthermore, all of our calculations were performed in the first order formulation of PSM models. We showed in chapter 3 that this is equivalent to the first order formulation of dilaton gravity and, after elimination of the connection and Lagrange multipliers enforcing the torsion constraint, to the second order formulation. However, this equivalence is only true up to boundary terms that might become important in holographic applications.

JT model. An immediate question that follows from the material presented in chapter 6 concerns the status of the looser boundary conditions with $\mathcal{L}^+ \neq \frac{1}{2}$. As mentioned in section 6.5 we were not able to integrate the corresponding charges even after performing similar tricks as in the stricter case treated in more detail in that section. Furthermore, the form of the boundary action following from (6.23) is always of the form (6.56) or (6.59), respectively since $\mathcal{L}^+$ only enters in the definition of the function M. It is therefore not clear what the physical significance of varying $\mathcal{L}^+$ is.

The variational principle we provided is somewhat unusual since it is valid only if integrated over the boundary $\partial\mathcal{M}$ having the topology of a circle. This is because the zero mode $\bar{Y}$ defined in (6.32) was required to be fixed. It is therefore valid only in Euclidean signature at finite temperature. It can be of interest to consider a Lorentzian version of (6.23), i.e., a variational principle where the boundary is not an isosurface of the dilaton. An application of our boundary conditions, in particular the fluctuating leading order of the dilaton, to dilaton gravity models giving rise to other geometries, would be of interest.

Although somewhat beyond the scope of the present work, let us mention the recent paper [237] in which the authors study the JT model in Lorentzian signature. For the solution of figure 4.1 coined wormhole solution, they construct the Hilbert space for the quantum

JT model in the presence of the two asymptotic regions. It is claimed that, while the JT model has a well-defined quantum theory, the Hilbert space does not factorize into Hilbert spaces defined on the two boundaries, from which follows that the JT model is not dual to any CFT on the boundary. They argue that a similar statement should be true for Einstein gravity on AdS_3, as well, in concordance with the result of [35]. It would be interesting to understand their argument in the PSM (or BF) formulation of the JT model that we have used in this work. In particular since this gauge theoretic formulation is very similar to the Chern–Simons formulation of three-dimensional gravity. This could yield valuable input on the pressing question of the existence of a CFT dual to Einstein gravity in three dimensions.

Generalizations of the Schwarzian action. An obvious question regarding our higher spin action is their interpretation on the boundary side. Just from naïve comparison with the SYK model one would expect that they arise from a model that exhibits an analogous spontaneous symmetry breaking $\mathcal{W}_N \to \mathrm{SL}(N)$ in the IR. Despite the existence of a large number of SYK-like models [49–70] none of these models shows the mentioned behavior, to the best of the author's knowledge. It is of interest to study in what way higher rank groups would change the thermodynamic behavior of the boundary theory in the low energy limit. Note in particular the entropy relation [192]

$$S_{\text{black}} \sim S_{\text{Schwarz}} \tag{8.1}$$

where $S_{\text{black}} = k\pi\bar{y}T$ is the JT black hole entropy and $S_{\text{Schwarz}} \propto NT/J$ is the field theory entropy in the small T limit with the $T = 0$ result subtracted. As shown in section 7.6 the same relation remains true at small temperatures after including higher spin fields. It is therefore not clear if there is a field theory generalization of SYK accessible in the regime $T \ll J$ that is sensitive to higher spin fields. However, even at small temperatures higher spin fields in principle are detectable semi-classically through a change of the numerical coefficient in the log-corrections to the entropy (7.113). It could be thus very interesting on the field theory side to generate SYK-like models where this coefficient in the log-corrections to the entropy can be tuned to $-(N^2-1)/2$, where N is some integer, in order to mimic the behavior of spin-N theories in AdS_2.

In our derivation of the Schwarzian action we assumed that the geometry-defining function $\mathcal{L}$ belonged to the exceptional orbit with $n = 1$ [cf., e.g., (6.53)] that has $\mathrm{PSL}(2,\mathbb{R})$ as its stabilizer group. It would be interesting to know if Schwarzian actions based on orbits of different homotopy class, i.e., having $n \neq 1$, have a meaningful interpretation in the boundary theory. The same goes for the higher spin Schwarzian actions that can be associated to orbits of either two or three different homotopy classes [see the discussion around equation (7.94)].

There are a number of immediate generalizations of our approach. While we assumed in the derivation of the boundary action (7.61) that the gauge group of the BF model is semi-simple this was just out of convenience and in no way essential. Therefore, a similar approach could be successful in the study of other groups with gravitational interpretation.

In particular, generalizing our result to the BF theory of the centrally extended Poincaré algebra (also known as Maxwell algebra, see e.g. [238]) [239, 240] would be an interesting first step. This dilaton gravity models describes the conformally transformed string black hole in two dimensions [108, 177, 178] (see also [111]). Apart from the centrally extended Poincaré algebra and the $\mathfrak{sl}(2,\mathbb{R})$ algebra associated to AdS_2 there exists a number of other *kinematical Lie algebras* with associated spacetimes in two dimensions (for a classification see [241]) that might be worthwhile to consider in our approach.

Generic linear dilaton vacua. While our discussion for constant dilaton vacua was valid for generic models of dilaton gravity, in our search for the asymptotic dynamics of dilaton gravity in AdS_2 we had to restrict ourselves to the JT model and similar extensions that allowed for a gauge theory of Lie-algebra type. It would be rewarding to generalize this discussion to arbitrary dilaton gravity models (perhaps subject to some condition on the potentials, e.g., asymptotic AdS_2 behavior) and to find possible quantum mechanics duals. This is quite an ambitious goal, since one can no longer rely on the simplicity of linear gauge algebras and instead has to deal with non-linear algebras and their associated groups. If successful, a large class of models can be described in this way, some of which emerge from dimensional reduction of gravity in arbitrary dimensions.

Part III

Appendix

Appendix A

ADM Split of Dilaton Gravity

In this appendix we present the ADM split for the second order dilaton gravity action (3.5) that we repeat here for convenience

$$I = \alpha \int_{\mathcal{M}} d^2x \sqrt{\sigma g} \Big(XR - \sigma U(X)(\nabla X)^2 - 2V(X) \Big). \tag{A.1}$$

where the prefactor was called α to reduce cluttering.

We foliate the spacetime $\mathcal{M}$ by a family of hypersurfaces Σ parametrized by a function t. Let t^a be a vector field such that $t^a \partial_a t = 1$. We are interested in a Hamiltonian split of (A.1) with respect to this vector field. Let n^a be the unit normal vector of Σ. Then the metric g_{ab} splits into

$$g_{ab} = \sigma n_a n_b + h_{ab} \,. \tag{A.2}$$

The sign σ is $\sigma = +1$ for Euclidean signature or a time-like surface in Lorentzian signature and $\sigma = -1$ for a spacelike foliation in Lorentzian signature. We define lapse N and shift N^a as

$$N = \sigma t^a n_a = (n^a \partial_a t) \qquad N_a = h_{ab} t^b \tag{A.3}$$

which yields the decomposition

$$t^a = N n^a + N^a \,. \tag{A.4}$$

Extrinsic curvature is defined as

$$K_{ab} = \frac{1}{2} h_a{}^c h_b{}^d \mathcal{L}_n h_{cd} = \frac{1}{2} \mathcal{L}_n h_{ab} \tag{A.5}$$

where the second equality sign follows from hypersurface orthogonality of n_a.

From the Gauss–Codazzi relations follows the decomposition

$$R = \bar{R} + \sigma (K^2 - K^{ab} K_{ab}) + 2\sigma \nabla_a (n^c \nabla_c n^a - n^a \nabla_c n^c) \tag{A.6}$$

for the Ricci scalar in terms of the Ricci scalar $\bar{R}$ intrinsic to the hypersurface and extrinsic curvature K_{ab} and its trace $K = K_{ab} h^{ab}$. Since the hypersurface Σ is one-dimensional we have $K_{ab} = h_{ab} K$ and $\bar{R} = 0$. We are then left with

$$R = 2\sigma \nabla_a (n^c \nabla_c n^a - n^a \nabla_c n^c) \,. \tag{A.7}$$

With the above definitions and the identity $\sqrt{\sigma g} = N\sqrt{h}$ the Lagrangian of the action (A.1) can be written in terms of variables intrinsic to the hypersurface Σ

$$L = \alpha \int_\Sigma dx\sqrt{h}\Big[2\sigma K\Big(\dot{X} - N^c D_c X\Big) - 2\sigma a^c D_c X - N^{-1}U(X)(\dot{X} - N^c D_c X)^2 \\ - N\sigma U(X)h^{ab}D_a X D_b X - 2NV(X)\Big] . \tag{A.8}$$

The vector a^a is the acceleration vector $a^a = n^c\nabla_c n^a$ that is normal to n^a; derivatives with respect to t^a are denoted by a dot, D_a is the covariant derivative compatible with the metric on the hypersurface h_{ab}, and boundary terms due to integration by parts have been dropped. From (A.8) we derive the momenta

$$\pi_X \equiv \frac{1}{\sqrt{h}}\frac{\delta L}{\delta \dot{X}} = 2\alpha\sigma K - 2U(X)(n^a\nabla_a X) \tag{A.9a}$$

$$\pi^{ab} \equiv \frac{1}{\sqrt{h}}\frac{\delta L}{\delta \dot{h}_{ab}} = \alpha\sigma(n^a\nabla_a X)h^{ab} \tag{A.9b}$$

where the form

$$K = \frac{1}{2N}\Big(\dot{h}_{ab} - (D_a N_b + D_b N_a)\Big)h^{ab} \tag{A.10}$$

of extrinsic curvature was used. A Legendre transform of (A.8) yields after some calculation the Hamiltonian quoted in section 3.5

$$H = \alpha \int_\Sigma \sqrt{h}\Big[\alpha^{-1}N^c(\pi_X D_c X - 2D_a\pi^a{}_b) + \\ N\big(\alpha^{-2}\sigma\pi^{ab}h_{ab}\pi_X + \alpha^{-2}U(X)\pi^{ab}\pi_{ab} + \sigma U(X)h^{ab}D_a X\, D_b X + 2V(X) - 2D^2 X\big)\Big]. \tag{A.11}$$

As expected, lapse and shift function act as Lagrange multipliers for Hamiltonian and diffeomorphism constraint, respectively.

Doing this calculation more carefully, one would have introduced momenta for N and N^a finding that both of them vanish. Calculating the bracket of these primary constraints with the Hamiltonian (A.11) one obtains Hamiltonian and diffeomorphism constraint as secondary constraints. The extended Hamiltonian is then given by adding all constraints with arbitrary coefficients to the canonical Hamiltonian. Since the net effect of this procedure is the addition of arbitrary functions to both lapse and shift, one can eliminate these two variables and their associated momenta completely from the system leaving only Hamiltonian and diffeomorphism constraint, multiplied by two Lagrange multipliers that for lack of better name (and to increase confusion in students) are denoted by the same letters N^c and N.

Appendix B

Coordinate Systems on AdS_2

The best starting point for constructing coordinate systems on AdS_2 is the three-dimensional ambient space with coordinates (Y_0, Y_1, Y_2) and Minkowski metric of signature $(-,-,+)$

$$ds^2 = -\,dY_0^2 - dY_2^2 + dY_1^2\,. \tag{B.1}$$

Two-dimensional AdS space is defined as the embedded surface

$$Y_0^2 + Y_2^2 - Y_1^2 = \ell^2\,, \tag{B.2}$$

where ℓ is the AdS-radius that is conventionally set to one. This surface can be parametrized by various coordinates. Introduce first

$$Y_0 = \ell \sec\nu \cos\tau \tag{B.3a}$$
$$Y_1 = \ell \tan\nu \tag{B.3b}$$
$$Y_2 = \ell \sec\nu \sin\tau \tag{B.3c}$$

with coordinate range $-\frac{\pi}{2} < \nu < \frac{\pi}{2}, -\pi < \tau < \pi$. This yields the induced metric

$$ds^2 = \frac{\ell^2}{\cos^2\nu}\big(-\,d\tau^2 + d\nu^2\big)\,. \tag{B.4}$$

Going to the universal cover by letting $-\infty < \tau < \infty$, this metric is AdS_2 in global coordinates. From this form of the metric it is clear that this spacetime is conformally flat (as every two-dimensional metric) and that it has two one-dimensional, timelike conformal boundaries at $\nu = \pm\pi$.

Another coordinate system for global AdS_2, that we will use more often is given by defining $r = \operatorname{arsinh}(\tan\nu)$ so that the metric becomes

$$ds^2 = \ell^2(-\cosh^2 r\, d\tau^2 + dr^2)\,. \tag{B.5}$$

The so-called Poincaré patch of AdS_2 is obtained by using the parametrization

$$Y_0 = \frac{1}{2z}(z^2 + \ell^2 - \hat{t}^2) \tag{B.6a}$$

$$Y_1 = \frac{1}{2z}(z^2 - \ell^2 - \hat{t}^2) \tag{B.6b}$$

$$Y_2 = \frac{\ell \hat{t}}{z} \tag{B.6c}$$

with coordinate range $0 < z < \infty, -\infty < \hat{t} < \infty$, from which the metric

$$ds^2 = \frac{\ell^2}{z^2}(dz^2 - d\hat{t}^2) \tag{B.7}$$

is obtained. Setting $z = e^{-\hat{\rho}}$ this is sometimes expressed as

$$ds^2 = \ell^2(d\hat{\rho}^2 - e^{2\hat{\rho}}\, d\hat{t}^2)\,. \tag{B.8}$$

Instead of the surface (B.2) in ambient space consider now

$$Y_0^2 + Y_2^2 - Y_1^2 = -\ell^2 \tag{B.9}$$

and define the coordinates

$$Y_0 = \ell \sinh\rho \cos\tau \tag{B.10a}$$

$$Y_1 = \ell \cosh\rho \tag{B.10b}$$

$$Y_2 = \ell \sinh\rho \sin\tau \tag{B.10c}$$

with coordinate range $0 < \rho < \infty$, $0 < \tau < 2\pi$. This leads to the induced metric

$$ds^2 = \ell^2\big(-\sinh^2\rho\, d\tau^2 + d\rho^2\big)\,, \tag{B.11}$$

in which the time coordinate τ is again unwrapped. Notice that this last metric (B.11) is special in some sense since the Killing vector with respect to which it is static has a horizon at $\rho = 0$.

All of the above metrics have constant negative curvature $R = -2/\ell^2$. Since there exists a unique, inextendible spacetime of constant negative curvature given by (B.4) or (B.5), respectively, the other two coordinate systems (B.7) and (B.11) can cover only a subset of the global spacetime. While the relation between global and Poincaré patch coordinates can be read off from (B.3) and (B.6) we add for completeness the transformation from (B.11) to (B.7)

$$\hat{t} = \frac{\sinh\tau}{\cosh\tau + \coth\rho} \qquad z = \frac{1}{\cosh\rho + \cosh\tau \sinh\rho}\,. \tag{B.12}$$

The Carter–Penrose diagram for global AdS with the various coordinate patches is presented in figure B.1.

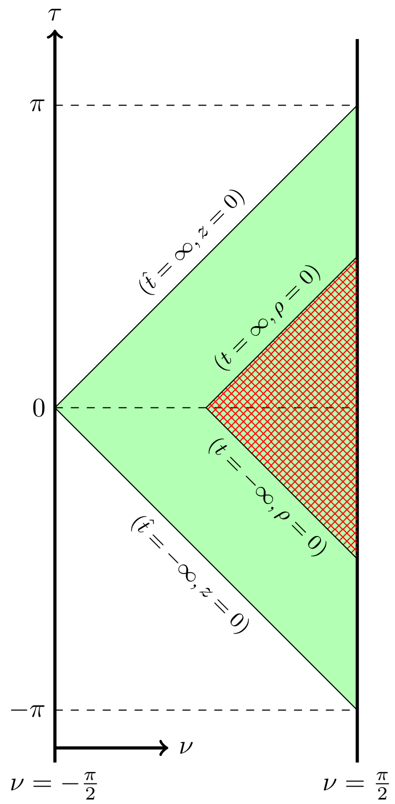


Figure B.1: Carter–Penrose diagram for global AdS_2 with coordinate patches for Poincaré patch coordinates (B.7) in green and black hole coordinates (B.11) in red.

Appendix C

Alternative Form of the SL(N) Schwarzian Action

In this appendix we want to present an alternative approach to constructing the finite temperature version of the generalized Schwarzian action. This approach is complementary to the one discussed in section 7.4. In contrast to the discussion presented there it will lead us to an explicit finite temperature version of the generalized Schwarzian action associated to SL(3). Our approach is closely related to the Hamiltonian reduction of WZW models. The main difference is our usage of the Iwasawa decomposition instead of the Gauss decomposition, since the former takes care of the correct holonomy condition.

Hamiltonian reduction

Our starting point is again the second order action (7.61) restricted to SL(N). In order to obtain a connection that is asymptotically AdS_2 we assume that a_τ is of the highest weight form, i.e., we will impose the constraint

$$-\partial_\tau g g^{-1} = L_1 + Q. \tag{C.1}$$

where $[L_{-1}, Q] = 0$. In order to parametrize this constraint we will now use the Iwasawa decomposition for the group element g. This consists of a product of three elements: R which belongs to S$O(N)$, D a diagonal matrix, and a nilpotent matrix N. Thus, we have

$$g = NDR,. \tag{C.2}$$

The matrices D and N are parametrized as

$$D = e^{-\frac{1}{2}\sum_i \varphi^i H_i}, \quad N = e^{\sum_a X^a E_a^+}. \tag{C.3}$$

Here X^a, φ^i are arbitrary functions of τ.[1]

[1]This procedure is well-known in the context of gauged WZW models (cf. [242] for a comprehensive review), which in some sense can be regarded as the "square" of the model we are discussing. Instead of the Iwasawa decomposition (C.2) the Gauss decomposition of SL(N) is used in these applications.

The Cartan basis of $\mathfrak{sl}(N,\mathbb{R})$ is represented by $N-1$ commuting generators $H_i = \mathcal{E}_{i,i} - \mathcal{E}_{i+1,i+1}$ where $(\mathcal{E}_{i,j})_{kl} = \delta_{ik}\delta_{jl}$ is the matrix with all entries zero except the ij-th being one. In addition we have the generators $E_i^{\pm}$ with the commutation relations

$$[H_i, H_j] = 0, \qquad [H_i, E_j^{\pm}] = \pm K_{ji} E_j^{\pm}, \qquad [E_i^+, E_j^-] = \delta_{ij} H_i\,. \tag{C.4}$$

The remaining generators E_α are obtained by repeated commutations of $E_i^{\pm}$. To both $E_\alpha^{\pm}$ and $E_i^{\pm}$ we refer collectively as $E_a^{\pm}$ labeled by the roots a. These are given by $E_a^{\pm} = \mathcal{E}_{(a\mp 1),a}$. The generators $E_i^{\pm}$ are associated to the *simple roots*.

The rotation group element SO(N) is generated by elements taken to be

$$J_a = E_a^+ - E_a^-\,. \tag{C.5}$$

In the case of SL(3) these generators satisfy $[J_a, J_b] = \epsilon_{abc} J^c$ with $\epsilon_{123} = 1$. On the other hand, for SL(2) we choose the single generator

$$J = E^+ - E^-\,. \tag{C.6}$$

The generators $H_i, E_a^{\pm}, J_a$ fulfill the following trace relations

$$\begin{aligned} \langle H_i H_j\rangle = &K_{ij}, \quad \langle E_a^{\pm} E_b^{\mp}\rangle = \delta_{ab}, \quad \langle E_a^{\pm} E_b^{\pm}\rangle = 0, \quad \langle E_a^{\pm} H_i\rangle = 0, \\ &\langle J_a J_b\rangle = -2\delta_{ab}, \quad \langle J_a E_b^{\pm}\rangle = \mp\delta_{ab}, \quad \langle J_a H_i\rangle = 0. \end{aligned} \tag{C.7}$$

where K_{ij} is the Cartan matrix of $\mathfrak{sl}(N,\mathbb{R})$.

We define the generator L_1 to be[2]

$$L_1 = -\sum_{i=1}^{N-1} E_i^-\,. \tag{C.8}$$

Condition (C.1) is now translated to the Cartan basis as

$$\langle E_i^+ \partial_\tau g\, g^{-1}\rangle = 1, \quad \langle E_\alpha^+ \partial_\tau g\, g^{-1}\rangle = 0, \quad \langle H_i \partial_\tau g\, g^{-1}\rangle = 0\,. \tag{C.9}$$

In what follows, we make use of the identities

$$\begin{aligned} D^{-1} E_a^+ D &= \sum_k e^{+\frac{1}{2}\sum_j K_{kj}\varphi^j} E_k^+ + \dots, \\ N^{-1} H_i N &= H_i + \sum_k K_{ki} X^k E_k^+ + \dots, \end{aligned} \tag{C.10}$$

where ellipsis stands for terms along E^+ that are not simple roots. These follow straightforwardly from the Baker–Campbell–Hausdorff formula.

[2] We choose this definition, differing from the one used in (7.73), in order to avoid carrying along factors of $\sqrt{k_i}$.

Using these identities and the trace relations (C.7), condition (C.9) can be written as

$$\partial_\tau R R^{-1} = \sum_k e^{-\frac{1}{2}\sum_j K_{kj}\varphi^j} J_k, \quad X^k = \frac{1}{2}\dot{\varphi}^k. \tag{C.11}$$

In terms of the parametrization (C.2), the lagrangian (7.61) reads

$$L = \frac{k\bar{y}}{2\pi}\langle g^{-1}\partial_\tau g g^{-1}\partial_\tau g\rangle = \frac{k\bar{y}}{2\pi}\langle (R^{-1}\dot{R})^2 + (D^{-1}\dot{D})^2 + 2N^{-1}\dot{N}D\dot{R}R^{-1}D^{-1}\rangle, \tag{C.12}$$

By means of relations (C.10) and (C.11) we find

$$\langle N^{-1}\dot{N}D\dot{R}R^{-1}D^{-1}\rangle = \sum_k \dot{X}^k = -\partial_\tau\langle L_0 D^{-1}\partial_\tau D\rangle, \tag{C.13}$$

where we have defined $L_0 = \sum_{i,j} K^{-1}_{ij} H_j$.

Imposing constraints (C.11) inside the action (7.61) we obtain

$$I[R] = \frac{k\bar{y}}{\pi}\int d\tau \left(\frac{1}{2}\langle (R^{-1}\dot{R})^2\rangle + T_D\right), \tag{C.14}$$

where $T_D = \langle \frac{1}{2}(D^{-1}\dot{D})^2 - L_0\partial_\tau(D^{-1}\partial_\tau D)\rangle$ is defined by expressing $\partial_\tau\varphi^i$ in terms of the components of $R^{-1}\dot{R}$ using (C.11).

We saw in section 7.4 that the original model (7.61) was invariant under both left and right multiplication by constant $SL(N,\mathbb{R})$ elements. As before, due to the constraint (C.1) only multiplication by the left remains. Thus, all actions (C.14) will have a global SL(N) invariance.

Let us discuss the two simplest examples:

SL$(2,\mathbb{R})$. The rotations of SO(2) are parametrized by $R = \exp\left[\frac{\theta}{2}J\right]$ with J being the single generator of the algebra. The normalization of θ is forced upon us by the requirement that the holonomy be minus one (cf. section 7.4) and we assumed $\beta = 2\pi$. The (one-dimensional) Cartan matrix is given by $K = 2$. The corresponding constraints (C.11) then translate to

$$\dot{\theta} = -2e^{-\varphi}, \qquad X = \frac{1}{2}\dot{\varphi}. \tag{C.15}$$

The action (C.14) is therefore

$$I = \frac{k\bar{y}}{2\pi}\int d\tau \left(-\frac{\dot{\theta}^2}{2} + \frac{1}{2}\dot{\varphi}^2 + \ddot{\varphi}\right). \tag{C.16}$$

Imposing the constraint (C.15) relating φ and θ leads to the expression $T_D = \frac{1}{4}(\partial_\tau\varphi)^2 + \frac{1}{2}\partial_\tau^2\varphi = -\frac{1}{2}\{\theta(\tau);\tau\}$. With this we recover the finite temperature Schwarzian action

$$I = -\frac{k\bar{y}}{2\pi}\int d\tau \left(\frac{1}{2}\dot{\theta}^2 + \mathrm{Sch}[\theta](\tau)\right) \tag{C.17}$$

for $\beta = 2\pi$ that we found already in different ways in the second and first order formulation. As we saw this action is SL(2) invariant as is required by the general argument given above.

Notice that the step from (C.16) to (C.17) involved the *choice* to solve the constraint (C.15) in terms of θ. Solving for φ and dropping the boundary term in (C.16) we find

$$I = \frac{k\bar{y}}{2\pi} \int d\tau \left(\frac{1}{2}\dot{\varphi}^2 - 2e^{-2\varphi} \right) \tag{C.18}$$

which is the action for *Liouville theory* in one dimension. There are two reasons why the action (C.17) is preferable over (C.18). First, the constraint (C.15) is not algebraic, thus care must be taken concerning the zero modes of various quantities. Solving the constraint in terms of θ guarantees this, while the other way round information about the zero mode of θ is lost. Related to that, the SL($2,\mathbb{R}$) symmetry is explicitly realized in the Schwarzian action (C.17), while it is realized only non-locally in (C.18).

It is tempting to muse that an analogue approach to the one presented here can be applied to three dimensional gravity. Also there the reduction of the asymptotic dynamics to Liouville theory [32] is valid only up to zero modes [243].

SL($3,\mathbb{R}$). Turning to SL(3) the Cartan matrix is given by

$$K_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{C.19}$$

We express an arbitrary element of the rotation group using Euler angles as $R = \exp(-\theta_3 J_3)\exp(\theta_2 J_2)\exp(\theta_1 J_3)$, then

$$\begin{aligned} \dot{R}R^{-1} = &\Big(-\dot{\theta}_3 + \dot{\theta}_1 \cos\theta_2\Big) J_3 \\ &+ \Big(\dot{\theta}_1 \sin\theta_2 \cos\theta_3 + \dot{\theta}_2 \sin\theta_3\Big) J_1 + \Big(-\dot{\theta}_1 \sin\theta_2 \sin\theta_3 + \dot{\theta}_2 \cos\theta_3\Big) J_2. \end{aligned} \tag{C.20}$$

Constraint (C.11) implies then

$$\begin{aligned} &\dot{\theta}_1 \cos\theta_2 - \dot{\theta}_3 = 0, \\ &\dot{\theta}_2 \sin\theta_3 + \dot{\theta}_1 \sin\theta_2 \cos\theta_3 = e^{-\varphi^1 + \frac{1}{2}\varphi^2}, \\ &\dot{\theta}_2 \cos\theta_3 - \dot{\theta}_1 \sin\theta_2 \sin\theta_3 = e^{-\varphi^2 + \frac{1}{2}\varphi^1}. \end{aligned} \tag{C.21}$$

And T_D is given by

$$\begin{aligned} T_D &= \frac{1}{4}\big((\dot{\varphi}_1)^2 - \dot{\varphi}_1\dot{\varphi}_2 + (\dot{\varphi}_2)^2 + 2\ddot{\varphi}_1 + 2\ddot{\varphi}_2\big), \\ &= \frac{1}{3}\Big((\dot{\psi}_1)^2 + \dot{\psi}_1\dot{\psi}_2 + (\dot{\psi}_2)^2\Big) + \ddot{\psi}_1 + \ddot{\psi}_2, \end{aligned} \tag{C.22}$$

where $\psi_1 = \varphi_1 - \frac{1}{2}\varphi_2$ and $\psi_2 = \varphi_2 - \frac{1}{2}\varphi_1$.

Due to our experience in SL(2) we want to solve the constraints keeping track of all the zero modes. In particular, one solves the first constraint of (C.21) for θ_2. Then one has

$$\theta_2 = \arccos(\frac{\dot{\theta}_3}{\dot{\theta}_1}) \equiv \arccos(x)\,., \tag{C.23}$$

where we defined the abbreviation x in the last equality. Plugging the solutions into the second and third equation of (C.21) one ends up with

$$-(1-x^2)^{-1/2}\dot{x}\sin\theta_3 + \dot{\theta}_1(1-x^2)^{1/2}\cos\theta_3 = e^{-\psi_1} \tag{C.24}$$
$$-(1-x^2)^{-1/2}\dot{x}\cos\theta_3 - \dot{\theta}_1(1-x^2)^{1/2}\sin\theta_3 = e^{-\psi_2}. \tag{C.25}$$

This can be rewritten as

$$\frac{\dot{\theta}_1}{\dot{\theta}_3}\partial_\tau\Big((1-x^2)^{1/2}\sin\theta_3\Big) = e^{-\psi_1} \tag{C.26}$$
$$\frac{\dot{\theta}_1}{\dot{\theta}_3}\partial_\tau\Big((1-x^2)^{1/2}\cos\theta_3\Big) = e^{-\psi_2}\,. \tag{C.27}$$

Defining the coordinates $\xi = (1-x^2)^{1/2}\cos\theta_3, \eta = (1-x^2)^{1/2}\sin\theta_3$ one obtains the relations:

$$\psi_1 = -\log\big(x^{-1}\dot{\eta}\big) \qquad \psi_2 = -\log\Big(x^{-1}\dot{\xi}\Big)\,. \tag{C.28}$$

Using these relations we can write the action (C.14) as

$$I = \frac{k\bar{y}}{\pi}\int d\tau\Big((x^4-x^2)^{-1}\Big(\dot{\eta}\xi - \dot{\xi}\eta + x^2\dot{x}^2\Big) + T_D\Big)\,. \tag{C.29}$$

The stress energy tensor T_D takes the form

$$T_D = -\frac{\dddot{\eta}}{\dot{\eta}} + \frac{4}{3}\frac{\ddot{\eta}^2}{\dot{\eta}^2} - \frac{\dddot{\xi}}{\dot{\xi}} + \frac{4}{3}\frac{\ddot{\xi}^2}{\dot{\xi}^2} + \frac{1}{3}\frac{\ddot{\eta}}{\dot{\eta}}\frac{\ddot{\xi}}{\dot{\xi}} - \frac{\dot{x}}{x}\left(\frac{\ddot{\eta}}{\dot{\eta}} + \frac{\ddot{\xi}}{\dot{\xi}}\right) - 2\left(\frac{1}{2}\frac{\dot{x}^2}{x^2} - \frac{\ddot{x}}{x}\right)\,. \tag{C.30}$$

While we have decided to express the action in terms of the variables ξ, η, x these are in fact fully determined by θ_1, θ_3 and their derivatives. Notice that the stress tensor is similar to the zero temperature Schwarzian action for SL(3) [cf. (7.91)]. In the SL(2) case finite and zero temperature Schwarzian action are related by the map (7.98) that projects the line to the circle. Using our proposal (7.100) one should be able to recover (C.29) with the map (7.101) from the zero temperature result (7.91).

Appendix D

The SYK Model

The roots of the Sachdev–Ye–Kitaev model (SYK model) go back to a model proposed by Sachdev and Ye to describe a spin fluid state [48]. The Hamiltonian they studied is of the form

$$H_{SY} = \frac{1}{\sqrt{NM}} \sum_{i<j} J_{ij} \vec{S}_i \vec{S}_j, \tag{D.1}$$

where the sum extends over N sites and $\vec{S}$ are spin-operators of the group $\mathrm{SU}(M)$. The coupling constants J_{ij} are randomly chosen from a Gaussian distribution which means that the system shows *quenched disorder*. The interesting physics for the present purpose occurs when taking the double limit $N \to \infty$ and $M \to \infty$ where the former should be thought of as the usual thermodynamic limit.

A possible holographic interpretation of the model in this limit was first suggested in [185]. Kitaev realized that the model could be simplified by replacing the two-point interaction of the spin-operators by a four-point interaction between Majorana fermions [46]. In this way only a single (thermodynamic) limit $N \to \infty$ has to be taken. Many aspects of the model were developed in more detail by Maldacena and Stanford [47].

Following a short section D.1 on properties of disordered systems, we will introduce the SYK model in section D.2. Our main aim is to understand the emergence of the Schwarzian action, that we saw already in the JT model (6.59), from the SYK model. Our plan is as follows: First, we want to study the large N behavior of the model. We will calculate the partition function and introduce variables in which the model becomes much simpler. These variables will turn out to be related to the holographic dual. In a complementary approach based on Feynman diagrams, we show that to leading order all correlation functions are dominated by so-called *melon diagrams*. From this we will see, secondly, that at strong coupling/low energies the theory shows an emerging reparametrization symmetry that is, however, spontaneously broken by the ground-state to $\mathrm{SL}(2,\mathbb{R})$. The arising pseudo-Goldstone bosons are governed by the Schwarzian action. This action will serve as our bridge to gravitational physics in the form of the JT model discussed in chapter 6.

Our discussion is largely based on [47].

D.1 Disordered systems

The SYK model is a quantum-mechanical model inspired by the analysis of disordered spin systems. Disorder in a physical system is usually studied by introducing random couplings in the Hamiltonian of the theory that follow a specific probability distribution. The use of these models was pioneered by Edwards and Anderson [244] in the study of spin-glass systems. The original Edwards–Anderson model takes on the form

$$H_{EA} = -\frac{1}{2}\sum_{\langle ij\rangle} J_{ij}\vec{S}_i\vec{S}_j. \tag{D.2}$$

Here $\vec{S}_i$ denotes a spin-operator at lattice site i and J_{ij} denote coupling parameters that are chosen randomly from a probability distribution $P(J)$. The sum extends over all nearest neighbor sites denoted by $\langle ij\rangle$. The Hamiltonian (D.2) describes impurities inserted at random locations in a metal that can interact both ferromagnetically and antiferromagnetically with each other. However, instead of assigning the impurities random lattice positions, the model (D.2) assigns fixed lattice positions to the impurities but lets the couplings change randomly. Physical systems of a type similar to (D.2) are said to exhibit *quenched disorder*. The impurities are thought of to be "frozen" into their position, i.e., the time-scales on which the J_{ij} are changing are much larger than the time-scales on which the spins interact with them. While physical quantities for a system with quenched disorder can be calculated from the partition function at a fixed value of the random coupling, Z_J, such a procedure does not yield much insight into the general behavior of the system. As in standard statistical mechanics one is interested in quantities, f_J, for which the fluctuations for different choices of J vanish in the thermodynamic limit $N \to \infty$,

$$\overline{f^2} - (\overline{f})^2 = O(N^{-a}) \qquad a > 0, \tag{D.3}$$

where by $\overline{f}$ we denote the average over J

$$\overline{f} = \int dJ\, P(J) f_J. \tag{D.4}$$

Quantities for which (D.3) holds, being extensive in general, are called *self-averaging*. It is well-known that fluctuations of free energy F around its thermal mean vanish like $O(N^{-1/2})$ in the thermodynamic limit. A general argument establishes that the same is true for the fluctuations due to different choices in the coupling constants J [245]. For a quenched system the free energy is consequently calculated according to

$$\overline{F} = \int dJ\, P(J) F_J = -\int dJ\, P(J)\beta^{-1}\ln Z_J. \tag{D.5}$$

Thermodynamics and correlations functions are then studied based on the averaged free energy (D.5).

Notice that this is not equivalent to first averaging the partition function and then calculating the free energy, i.e.

$$\overline{Z} = \int dJ\, P(J) Z_J \Rightarrow \bar{F} = -\beta^{-1}\ln\overline{Z}. \tag{D.6}$$

The reason for this is that in the case of (D.6) the couplings are treated on the same footing as the spin variables, i.e., they are not frozen but can interact with the spin system. A system of this type is known as *annealed*. The annealed case (D.6) is computationally much simpler to treat than the quenched case. For instance, if the distribution of random couplings $P(J)$ is Gaussian, the integral (D.6) can be usually calculated analytically.

An efficient method to calculate the free energy in the quenched case is the *replica method*. This method makes use of the identity

$$\ln Z = \lim_{n\to 0} \frac{Z^n - 1}{n}, \tag{D.7}$$

and the fact that $\overline{Z^n}$ can be calculated very efficiently. By calculating the n-th power of the partition function one has effectively replicated the system n times. Although the introduction of the n replicas appears to be a simple computational trick they acquire physical significance. In particular, one expects that the system is invariant under the replica symmetry, i.e., under permutations of the replicas among themselves. However, it turns out that this symmetry is broken at low temperatures in the Edwards-Anderson model (D.2) and many other models of quenched disorder. This breaking of replica symmetry is related to a phase transition to the magnetically ordered *spin-glass* state.

D.2 The SYK model

The SYK model is a quantum-mechanical model of N Majorana fermions with a four point interaction. In the following we want to be quite explicit about the model, mostly following the conventions of [47].

Consider the free Lagrangian for a theory of N Majorana fermions

$$L^{(0)} = \frac{i}{2}\psi_i\dot{\psi}_i\,. \tag{D.8}$$

Here ψ_i should be thought of as real, anticommuting Grassmann variables, i.e.

$$\psi_i^* = \psi_i \qquad \psi_i\psi_j = -\psi_j\psi_i. \tag{D.9}$$

Since we have $(AB)^* = B^*A^*$, the i in (D.8) is necessary for $L^{(0)}$ to be real. The momentum conjugate to ψ_i is

$$p_i = \frac{\partial L^{(0)}}{\partial\dot{\psi}_i} = -\frac{i}{2}\psi_i, \tag{D.10}$$

where we always use a derivative acting on the left, imposes a second class constraint that can be immediately eliminated by going to the Dirac brackets

$$\{\psi_i, \psi_j\}^* = -i\delta_{ij}. \tag{D.11}$$

The Hamiltonian

$$H = \dot{\psi}^i p_i - \frac{i}{2}\psi_i\dot{\psi}_i = 0 \tag{D.12}$$

vanishes identically.

Canonical quantization leads to the quantum-mechanical anticommutator between the operators $\hat{\psi}_i$ in the Schrödinger picture

$$\{\hat{\psi}_i, \hat{\psi}_j\} = \hbar\delta_{ij}, \tag{D.13}$$

where $\hbar$ was made explicit to stress the difference between (D.11) and (D.13). The operators $\hat{\psi}_i$ form an N-dimensional Clifford algebra and can consequently be represented in terms of Euclidean gamma matrices. While this observation is beneficial for numerical studies of the SYK model, we will not be interested in representations of (D.13) as we will be mostly concerned with the path-integral.

The SYK model is defined by adding to the free Lagrangian (D.8) an interaction term such that the interacting theory is

$$L = L^{(0)} + L^{(int)} = \frac{i}{2}\sum_{i=1}^{N}\psi_i\dot{\psi}_i - \frac{i^{q/2}}{q!}\sum_{i_1,i_2,..,i_q} j_{i_1 i_2 \ldots i_q}\psi_{i_1}\psi_{i_2}..\psi_{i_q}. \tag{D.14}$$

The crucial point in the definition of the SYK model is that the coupling constants $j_{i_1 i_2 \ldots i_q}$ are supposed to be randomly taken from a Gaussian distribution; from the Grassmann property of ψ_i it is clear that $j_{i_1 i_2 \ldots i_q}$ is totally antisymmetric. This random distribution is characterized by

$$\overline{j}_I = 0 \qquad \overline{j_I j_{I'}} = \delta_{II'}\frac{J^2(q-1)!}{N^{q-1}}, \tag{D.15}$$

where I is to be understood as a multi-index. More generally speaking, the system shows *quenched disorder* as the Edwards–Anderson model (D.2).

The Hamiltonian coming from the Lagrangian (D.14) is then given by the interaction piece only, i.e.,

$$H = \frac{i^{q/2}}{q!}\sum_{i_1,i_2,..,i_q} j_{i_1 i_2 \ldots i_q}\psi_{i_1}\psi_{i_2}..\psi_{i_q}\,. \tag{D.16}$$

In the following we will focus on the case $q = 4$ which is the case originally related to AdS_2 holography. Furthermore, we will consider the Euclidean version of this model. The Wick rotation $t = -i\tau$ leads to the definition of the Euclidean action

$$-S \equiv -\int d\tau\left(\frac{1}{2}\sum_{i=1}^{N}\psi_i\partial_\tau\psi_i + \frac{1}{4!}\sum_{i_1,i_2,i_3,i_4} j_{i_1 i_2 i_3 i_4}\psi_{i_1}\psi_{i_2}\psi_{i_3}\psi_{i_4}\right). \tag{D.17}$$

Let us pause for a moment for a dimensional analysis of (D.17). The Majorana fields are dimensionless $[\psi] = 0$, while the coupling constant has mass dimension 1, $[j] = 1$, and is therefore a *relevant coupling*. This means that the theory will be strongly/weakly coupled in the IR/UV. If the generic pattern of AdS/CFT as a strong/weak duality applies to the two-dimensional case, one can expect to see holography emerging in the low-energy regime. As we are going to see, this expectation will be met.

Feynman diagrammatics. The Euclidean propagator is defined as

$$G_{ij}(\tau) \equiv \langle T(\psi_i(\tau)\psi_j(0)\rangle = \langle \psi_i(\tau)\psi_j(0)\rangle\theta(\tau) - \langle\psi_i(0)\psi_j(\tau)\rangle\theta(-\tau)\,. \tag{D.18}$$

As the first step we calculate the free propagator of the theory, $G^{(0)}$. Its value in the free theory can be determined immediately from the Clifford algebra (D.13) and the fact that the Hamiltonian vanishes. Thus, in the Heisenberg picture we have $\psi_i(\tau) = \psi_i(0)$ and therefore

$$G^{(0)}(\tau)_{ij} = \langle T(\psi(\tau)_i\psi(0)_j\rangle_0 = \frac{1}{2}\mathrm{sgn}(\tau)\delta_{ij}, \tag{D.19}$$

where $\langle..\rangle_0$ denotes the expectation value in the free theory.

However, let us derive this result again using the Euclidean path integral as a useful exercise for the following. The Euclidean path integral of the free theory is given by

$$Z[\eta] = \prod_{i=1}^{N}\int \mathcal{D}\psi_i e^{-\int d\tau\left(\frac{1}{2}\psi_i\partial_\tau\psi_i + \eta_i\psi_i\right)}, \tag{D.20}$$

where we have added a classical source η_i in the form of a Grassmann number. The path integral can be evaluated using common techniques. First one can write

$$Z[\eta] = \prod_{i=1}^{N}\int \mathcal{D}\psi_i e^{-\int d\tau\left(\frac{1}{2}\tilde{\psi}_i\partial_\tau\tilde{\psi}_i\right)} e^{-\frac{1}{2}\int d\tau\, d\tau' \eta_i(\tau)G_{ij}(\tau-\tau')\eta_j(\tau')}, \tag{D.21}$$

where we redefined the fields $\tilde{\psi}$

$$\tilde{\psi}_j(\tau) \equiv \int \frac{d\omega}{2\pi} e^{-i\omega\tau}\left(\psi_j(\omega) + \frac{1}{i\omega}\eta_j(\omega)\right) \tag{D.22}$$

and defined the propagator

$$G_{ij}(\tau-\tau') \equiv \delta_{ij}\int \frac{d\omega}{2\pi} e^{-i\omega(\tau-\tau')}\frac{i}{\omega} = \frac{1}{2}\mathrm{sgn}(\tau-\tau')\delta_{ij}. \tag{D.23}$$

Assuming that the measure is invariant under the shift $\psi \to \tilde{\psi}$, we can integrate over the first term that we call $Z[0]$ and are left with

$$Z[\eta] = Z[0]\prod_{i=1}^{N} e^{-\frac{1}{2}\int d\tau\, d\tau' \eta_i(\tau)G_{ij}(\tau-\tau')\eta_j(\tau')}\,. \tag{D.24}$$

The time-ordered Euclidean propagator is equivalent to the two-point function in the free theory which can be obtained from the path integral (D.20), or equivalently (D.24), by

$$\langle T\psi_i(\tau)\psi_j(\tau')\rangle = Z[0]^{-1}\left(-\frac{\delta}{\delta\eta(\tau)}\right)\left(-\frac{\delta}{\delta\eta(\tau')}\right)Z[\eta]|_{\eta=0}. \tag{D.25}$$

Derivation with respect to the sources, observing the anticommutativity, we find indeed

$$\langle T\psi_i(\tau)\psi_j(\tau')\rangle = -\frac{1}{2}G_{ij}(\tau'-\tau) + \frac{1}{2}G_{ij}(\tau-\tau') = G_{ij}(\tau-\tau') = \frac{1}{2}\mathrm{sgn}(\tau-\tau')\delta_{ij}, \tag{D.26}$$

reproducing equation (D.19) at which we arrived using canonical techniques.

The above result for the free propagator (D.19) is true also in the finite temperature case, where we would impose periodicity $\tau \sim \tau + \beta$ in Euclidean time. If we are interested in the trace over the Hilbert space, as is the correct choice for the thermodynamics partition function, in contrast to the supertrace we have to impose anti-periodic boundary conditions on the fermion fields

$$\psi_i(0) = -\psi_i(\beta). \tag{D.27}$$

The propagator in position space trivially reflects this property. In momentum space, however, only discrete energy levels are allowed; the so-called *Matsubara frequencies*

$$\omega = \frac{2\pi}{\beta}(2n+1). \tag{D.28}$$

In the following calculations we will assume for simplicity that we are at zero temperature.

Let us now turn to the interacting theory based on the Euclidean action (D.17). Expanding the interaction term in the path integral to arbitrary order in the coupling constant one can straightforwardly set-up a perturbation theory. In addition to the path integral the average over the disorder with the Gaussian measure

$$d\mathcal{J} \equiv dJ_{ijkl} P(J_{ijkl}) \,, \tag{D.29}$$

where

$$P(J_{ijkl}) = \sqrt{\frac{N^3}{12\pi J^2}} \exp\left(-\frac{N^3 (J_{ijkl})^2}{12J^2}\right). \tag{D.30}$$

has to be taken. According to the discussion in section D.1 one should take the disorder average after calculating the path-integral. More precisely, let O denote an observable such as a two-point function, then the disorder average is defined as

$$\overline{\langle O \rangle} = \int d\mathcal{J} \, \frac{\int \mathcal{D}\psi_i \, O \, e^{-S}}{\int \mathcal{D}\psi_i \, e^{-S}}, \tag{D.31}$$

where S denotes the Euclidean action (D.17) of the SYK model. On the other hand if J_{ijkl} were treated as quantum fields on the same footing as the fermions one would compute

$$\overline{\langle O \rangle} = Z^{-1} \int \mathcal{D}\psi_i \; d\mathcal{J} P(J_{ijkl}) \, O \, e^{-S}. \tag{D.32}$$

We want to calculate the exact propagator of the theory in the large N limit. The propagator is the disorder-averaged two-point function

$$\begin{aligned} \overline{G}_{ij}(\tau - \tau') \equiv & \overline{\langle T \psi_i(\tau) \psi_j(\tau') \rangle} = \\ & \int d\mathcal{J} \, Z_J^{-1} \int \mathcal{D}\psi_i e^{-\int d\tau \left(\frac{1}{2} \psi_i \partial_\tau \psi_i + \frac{1}{4!} J_{ijkl} \psi_i \psi_j \psi_k \psi_l \right)} \psi_i(\tau) \psi_j(\tau'), \end{aligned} \tag{D.33}$$

with the partition function at fixed J being defined as

$$Z_J = \int \mathcal{D}\psi_i e^{-\int d\tau \left(\frac{1}{2} \psi_i \partial_\tau \psi_i + \frac{1}{4!} J_{ijkl} \psi_i \psi_j \psi_k \psi_l \right)}. \tag{D.34}$$

Using perturbation theory and Wick's theorem we can calculate this to arbitrary order in the coupling constant. To first order in J we find

$$\begin{aligned}&\overline{G}_{ij}(\tau-\tau') = \\ &\int d\mathcal{J}\, Z_J^{-1}\int \mathcal{D}\psi_i e^{-\int d\tau \frac{1}{2}\psi_k\partial_\tau\psi_k}\left(1-\frac{1}{4!}\int d\tau J_{l_1l_2l_3l_4}\psi_{l_1}\psi_{l_2}\psi_{l_3}\psi_{l_4}+...\right)\psi_i(\tau)\psi_j(\tau') \\ &= G^{(0)}_{ij}(\tau-\tau')+O(J^2),\end{aligned} \tag{D.35}$$

due to Wick's theorem and the antisymmetry of J_{ijkl}. Furthermore, the average of J vanishes by itself. To second order in perturbation theory we find the first non-vanishing contribution. Before performing the disorder average we find for the $O(J^2)$ term in (D.35)

$$\begin{aligned}&Z_J^{-1}\frac{1}{2(4!)^2}\int \mathcal{D}\psi_i e^{-\int d\tau\frac{1}{2}\psi_i\partial_\tau\psi_i} \\ &\qquad\int d\tau_1\, d\tau_2 J_{k_1k_2k_3k_4}J_{l_1l_2l_3l_4}(\psi_{k_1}\psi_{k_2}\psi_{k_3}\psi_{k_4})(\tau_1)(\psi_{l_1}\psi_{l_2}\psi_{l_3}\psi_{l_4})(\tau_2)\psi_i(\tau)\psi_j(\tau')\,,\end{aligned} \tag{D.36}$$

where $(\psi_{i_1}\psi_{i_2}...\psi_{i_n})(\tau_1)\equiv\psi_{i_1}(\tau_1)\psi_{i_2}(\tau_1)...\psi_{i_n}(\tau_1)$. Wick contraction yields one disconnected term that is canceled by the normalization factor Z_J^{-1}, i.e., contracting $\psi_i(\tau)$ and $\psi_j(\tau')$, and one non-vanishing contribution to the propagator. After the disorder average the propagator to second order is given by

$$\overline{G}_{ij}(\tau-\tau') = G^{(0)}_{ij}(\tau-\tau')+\delta_{ij}J^2\int dt\, dt' G^{(0)}(\tau-t)\Big(G^{(0)}(t-t')\Big)^3 G^{(0)}(t-\tau')+O(J^4). \tag{D.37}$$

Notice that this contribution is independent of N. In other words, the normalization in (D.30) was chosen such that this term is non-vanishing in the large N limit.

Equation (D.37) can be expanded in Feynman diagrams in the following way:

 $+\,O(J^4)$ (D.38)

The interpretation of the diagrams is clear from the context: a drawn line corresponds to the free propagator, every vertex is associated with a factor of J, and the broken line denotes the disorder average. The second diagram, or similarly the second term in (D.37) is called *melon diagram.* In the large N limit all Feynman diagrams of the SYK model are dominated by melon diagrams.

Since the J^3 contribution is again zero, let us demonstrate this for the J^4 term. All possible Feynman diagrams to fourth order (up to trivial permutations of internal propagators) with distinct choices of performing the disorder average are depicted in figure D.1. Determining the dependence on N of each diagram is simply a matter of labeling all internal legs and tracing through the Kronecker deltas imposed by propagators and disorder averages. From this procedure we find that the first and third diagram contribute to order $O(1)$, the fifth contributes to order $O(N^{-1})$ while the second, fourth and sixth are subsubleading to order

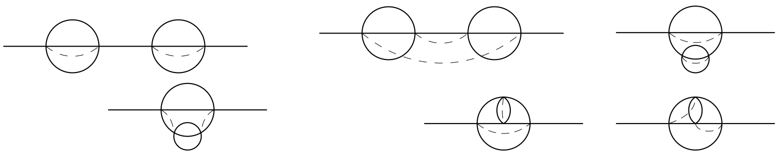


Figure D.1: Contributions to the propagator to order J^4

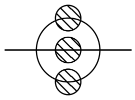


Figure D.2: The self-energy diagram

$O(N^{-2})$. We see therefore that the leading order corrections in N to the propagator to fourth order in perturbation theory come from iterated melon diagrams.

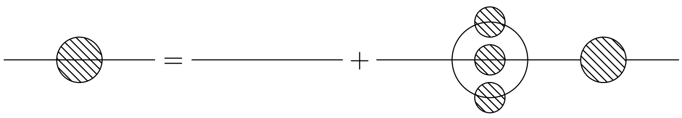
Thus, the large N expansion simplifies the problem to determine the sum of all melon diagrams. It is possible to write down an implicit equation for the exact propagator. Denote by $\Sigma(\tau-\tau')$ the melon diagram with the internal lines replaced by the exact propagator (cf. figure D.2). This is

$$\Sigma(\tau-\tau') = J^2\overline{G}(\tau-\tau')^3. \tag{D.39}$$

The exact propagator is obtained by concatenating this diagram with propagators, i.e.

$$\overline{G}(\tau-\tau') = G^{(0)}(\tau-\tau') + \int d\tau_1\, d\tau_2 G^{(0)}(\tau-\tau_1)\Sigma(\tau_1-\tau_2)\overline{G}(\tau_2-\tau'), \tag{D.40}$$

which is nothing but the Schwinger-Dyson equation for the exact propagator. In a diagrammatic notation this would be written as


(D.41)

By iterating equation (D.40) and going to Fourier space one obtains a geometric sum that yields the simple relation

$$\overline{G}(\omega) = \frac{1}{-i\omega - \Sigma(\omega)}. \tag{D.42}$$

The system (D.39) and (D.42) thus determines the exact propagator completely.

Effective Action. It is possible to obtain an effective Lagrangian for the fields $\overline{G}$ and Σ with equation (D.39) and (D.40) as equations of motion. The starting point for this is the replica trick (D.7). Defining n replicas of the Majorana fermions ψ_i^a, $a = 1, .., n$, the n-th power of the partition function at fixed J is given by

$$Z_J^n = \int \mathcal{D}\psi_i^a e^{-\int d\tau\left(\frac{1}{2}\psi_i^a\partial_\tau\psi_i^a + \frac{1}{4!}J_{ijkl}\psi_i^a\psi_j^a\psi_k^a\psi_l^a\right)}. \tag{D.43}$$

The disorder-average can be computed explicitly by completing the square after which one finds

$$\overline{Z^n} = \int \mathcal{D}\psi_i^a \exp\Bigg\{ - \int d\tau \left(\frac{1}{2}\psi_i^a \partial_\tau \psi_i^a \right) - \frac{J^2}{8N^3} \sum_{a,b=1}^{n} \int d\tau\, d\tau' \big(\psi_i^a(\tau)\psi_i^b(\tau')\big)^4 \Bigg\}. \tag{D.44}$$

Notice that the last term describes an interaction between different replicas of a single fermion. The action can be further simplified by inserting the functional delta function

$$1 = \int \mathcal{D}G_{ab}\, \delta\big(NG_{ab}(\tau,\tau') - \psi_i^a(\tau)\psi_i^b(\tau')\big), \tag{D.45}$$

where we introduced the bilocal field $G_{ab}(\tau,\tau')$. Introducing the field $\Sigma_{ab}(\tau,\tau')$ as Lagrange multiplier that enforces the constraint (D.45) equation (D.44) takes the form

$$\begin{aligned} \overline{Z^n} = \int \mathcal{D}\psi_i^a \mathcal{D}\Sigma_{ab} \mathcal{D}G_{ab} \exp\Bigg\{ &- \int d\tau \left(\frac{1}{2}\psi_i^a (\partial_\tau \delta_{ab} - \Sigma_{ab}) \psi_i^b \right) \\ &- \frac{N}{2} \sum_{a,b=1}^{n} \int d\tau\, d\tau' \Sigma(\tau,\tau') G_{ab}(\tau,\tau') - \frac{J^2}{4} G_{ab}(\tau,\tau')^4 \Bigg\}. \end{aligned} \tag{D.46}$$

The advantage of introducing the fields Σ and G is now apparent, since one can perform the integral over the fermion fields ψ_i^a explicitly

$$\begin{aligned} \overline{Z^n} = \int \mathcal{D}\Sigma_{ab} \mathcal{D}G_{ab} \exp\Bigg\{ &\sum_{a,b=1}^{n} \frac{N}{2} \log\det(\partial_\tau \delta_{ab} - \Sigma_{ab}) \\ &- \frac{N}{2} \int d\tau\, d\tau' \Sigma_{ab}(\tau,\tau') G_{ab}(\tau,\tau') - \frac{J^2}{4} G_{ab}(\tau,\tau')^4 \Bigg\}. \end{aligned} \tag{D.47}$$

In order to employ the identity (D.7) one needs to know $\overline{Z^n}$ as a function of replica number n. The path-integral can then be evaluated in the saddle point approximation in the large N limit. Practically, however, one exchanges the two limits and, assuming a specific ansatz for the fields in the path integral, evaluates the path integral in the saddle point approximation before performing the limit $n \to 0$. The difficulty lies in the choice of an appropriate ansatz for the fields appearing in $\overline{Z^n}$. Although the replicas of the system were originally introduced as a mere computational trick, in many instances an ansatz that is symmetric under permutations of the replicas leads to inconsistencies at low temperatures. The reason for this is the phenomenon of *replica symmetry breaking* that is associated with the emergence of magnetically ordered spin glass phases at low temperatures. It is widely assumed that the SYK model does not exhibit a spin glass phase at low temperatures. This is backed by numerical study performed in [246] which indicates that the model shows non-Fermi liquid behavior at low temperatures.[1] With this assumption it is possible to make

[1]Even without this result, one can show that interactions between different replicas do not contribute to the order in N we are interested in. The SYK model can therefore be treated to subleading order in N as an annealed system [66].

the replica symmetric ansatz

$$G_{ab}(\tau,\tau') = \delta_{ab} G(\tau,\tau') \qquad \Sigma_{ab}(\tau,\tau') = \delta_{ab}\Sigma(\tau,\tau') \tag{D.48}$$

which brings out an explicit factor of n such that the limit $n \to 0$ yields

$$e^{-\beta F} = \int \mathcal{D}\Sigma\mathcal{D}G \exp\left\{\frac{N}{2}\log\det(\partial_\tau - \Sigma) - \frac{N}{2}\int d\tau\, d\tau' \Sigma(\tau,\tau')G(\tau,\tau') - \frac{J^2}{4}G(\tau,\tau')^4\right\}. \tag{D.49}$$

This suggests to define the non-local effective action $\tilde{S}$ for the SYK model

$$\tilde{S} = -\frac{N}{2}\log\det(\partial_\tau - \Sigma) + \frac{N}{2}\int d\tau\, d\tau' \Sigma(\tau,\tau')G(\tau,\tau') - \frac{J^2}{4}G(\tau,\tau')^4. \tag{D.50}$$

Variation with respect to the bi-local fields $\Sigma(\tau,\tau')$ and $G(\tau,\tau')$ reproduces equations (D.39) and (D.42) under the assumption that the fields depend only on the difference $\tau - \tau'$. In the following we will denote the respective solutions to these equations as $G^*(\tau_1,\tau_2)$ and $\Sigma^*(\tau_1,\tau_2)$.

Conformal Solution. Although equations (D.39) and (D.42) look deceivingly simple a closed form for $G^*(\tau - \tau')$ is not known. Numerical solutions were obtained in [47]. In the low-energy regime, however, the equations are quite simple to solve. Remember that the theory becomes strongly coupled in the IR. This means that $\Sigma(\omega)$ containing a factor of J^2 will be the dominant term in the denominator of (D.42). Similarly, at finite temperature we are interested in the behavior at $\beta J \gg 1$. We can therefore solve

$$G(\omega)\Sigma(\omega) = -1, \tag{D.51}$$

or in configuration space

$$\int d\tau_1 G(\tau - \tau_1)\Sigma(\tau_1 - \tau') = -\delta(\tau - \tau'). \tag{D.52}$$

When the time-dependence of the variables is clear from the context we will sometimes write the above as

$$G * \Sigma = -\delta, \tag{D.53}$$

where $*$ is the convolution.

It is this equation that is the source of many interesting developments about the SYK model. In particular, equation (D.52) shows conformal invariance in one dimension, i.e., reparametrization invariance. Assume that $G(\tau - \tau_1)$ solves equation (D.52). If one reparametrizes

$$\tau = f(\sigma) \tag{D.54}$$

and simultaneously demands that $G(\tau - \tau')$ scales such that

$$G(\sigma - \sigma') = (f'(\sigma)f'(\sigma'))^{\Delta} G(\tau - \tau'), \tag{D.55}$$

then one obtains

$$\int d\sigma_1 G(\sigma - \sigma_1)\Sigma(\sigma_1 - \sigma') = -\delta(\sigma - \sigma'), \tag{D.56}$$

provided $\Delta = \frac{1}{4}$.

We have found that the equations of motion for the SYK model show an emergent conformal symmetry in the IR regime if the theory is expressed in terms of the quantities G and Σ. This is also directly apparent in the non-local action (D.50). In the IR regime one can neglect the derivative in the first term of the effective action which renders the action invariant under the reparametrizations (D.55).

Conformal symmetry suggests the following ansatz for the propagator in the conformal regime which we will denote by G_c^*

$$G_c^*(\tau) = \frac{b}{|\tau|^{2\Delta}} \mathrm{sgn}(\tau), \tag{D.57}$$

with coefficient b to be determined from (D.52) and conformal weight $\Delta = \frac{1}{4}$. From its Fourier transform

$$G_c^*(\omega) = i\sqrt{2\pi} \frac{b}{\sqrt{|\omega|}} \mathrm{sgn}(\omega) \tag{D.58}$$

and equation (D.51) one finds for the undetermined coefficient

$$b = \left(\frac{1}{4J^2\pi} \right)^{\frac{1}{4}}. \tag{D.59}$$

Due to the reparametrization invariance (D.55) it is straightforward to determine the finite temperature version of the conformal propagator (D.57). The function $f(\tau) = \tan(\frac{\tau\pi}{\beta})$ maps the circle with periodicity β to the real line. The conformal propagator for finite temperature β is therefore

$$G_c^*(\tau) = b \left(\frac{\pi}{\beta \sin \frac{\pi\tau}{\beta}} \right)^{2\Delta} \mathrm{sgn}(\tau). \tag{D.60}$$

We have seen that the SYK model exhibits an emergent conformal symmetry in the infrared regime. Both approaches, the non-local effective action (D.49) and the Schwinger-Dyson equations for the two-point function (D.39) and (D.42) show this explicitly. The interesting way in which conformal symmetry is broken can be studied using the four-point function.

D.3 Four-point function

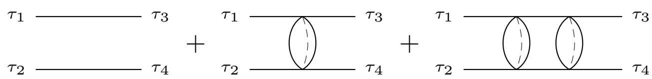
In this section we study the four-point function

$$\langle \psi_i(\tau_1)\psi_i(\tau_2)\psi_j(\tau_3)\psi_j(\tau_4) \rangle. \tag{D.61}$$

This is the most general four-point function since the disorder-average forces indices to be pair-wise equal. This four-point function consists of a disconnected term and a subleading contribution

$$\frac{1}{N^2} \sum_{i,j=1}^{N} \psi_i(\tau_1)\psi_i(\tau_2)\psi_j(\tau_3)\psi_j(\tau_4) \rangle = G^*(\tau_1 - \tau_2) G^*(\tau_3 - \tau_4) + \frac{1}{N}\mathcal{F} + O(N^{-2}). \tag{D.62}$$

The purpose of this section is to understand the structure of the subleading piece $\mathcal{F}$ in more detail. In the same way that the leading term in the two-point function came from melon diagrams the term $\mathcal{F}$ in (D.62) is obtained by summing *ladder diagrams.* More precisely, one has the following diagrammatic expansion



$$- (\tau_3 \leftrightarrow \tau_4) + \dots . \tag{D.63}$$

Due to the constraint (D.45) imposed in the transition to the non-local action (D.50) the four-point function (D.62) can be written as a two-point function in the non-local variables $G(\tau_1, \tau_2)$

$$\frac{1}{N^2}\sum_{i,j=1}^{N}\langle\psi_i(\tau_1)\psi_i(\tau_2)\psi_j(\tau_3)\psi_j(\tau_4)\rangle = Z^{-1}\int \mathcal{D}\Sigma\mathcal{D}G\, e^{-\tilde{S}}\, G(\tau_1,\tau_2)G(\tau_3,\tau_4). \tag{D.64}$$

Notice that we assume the time-ordering $\tau_1 > \tau_2 > \tau_3 > \tau_4$. In order to evaluate this integral it is convenient to perform an expansion around the solutions G^* and Σ^*,

$$G = G^* + |G^*|^{-1}g \qquad \Sigma = \Sigma^* + |G^*|\sigma, \tag{D.65}$$

where g, σ denote the fluctuations around G^* and Σ^*, respectively. The normalization of the fluctuations was chosen conveniently. The measure in the path integral changes to $\mathcal{D}\Sigma\mathcal{D}G = \mathcal{D}\sigma\mathcal{D}g$. The integrand independent of the fluctuations g in (D.64) yields the leading order term in (D.62). Since the terms linear in g vanish the subleading term $\mathcal{F}$ is given by

$$\frac{1}{N}\mathcal{F} = Z^{-1}\int \mathcal{D}\sigma\mathcal{D}g, e^{-\tilde{S}}\, |G^*(\tau_1,\tau_2)|^{-1}|G^*(\tau_3,\tau_4)|^{-1}g(\tau_1,\tau_2)g(\tau_3,\tau_4). \tag{D.66}$$

Expanding the action $\tilde{S}$ to quadratic order in the fluctuations one obtains

$$\begin{aligned}\tilde{S}(G,\Sigma) = \tilde{S}(G^*,\Sigma^*) - &\frac{N}{12J^2}\int d\tau_1 \dots d\tau_4 \sigma(\tau_1,\tau_2)K(\tau_1..\tau_4)\sigma(\tau_3,\tau_4)\\ &+ \frac{N}{2}\int d\tau_1\, d\tau_2\Big(g(\tau_1,\tau_2)\sigma(\tau_1,\tau_2) - \frac{3}{2}J^2 g(\tau_1,\tau_2)^2\Big),\end{aligned} \tag{D.67}$$

where $K(\tau_1,\tau_2;\tau_3,\tau_4)$ is defined by

$$K(\tau_1,\tau_2;\tau_3,\tau_4) = -3J^2|G^*(\tau_1,\tau_2)|G^*(\tau_1,\tau_3)G^*(\tau_2,\tau_4)|G^*(\tau_3,\tau_4)|. \tag{D.68}$$

K is symmetric under the permutation $(\tau_1,\tau_2) \leftrightarrow (\tau_3,\tau_4)$. One can think of K as a symmetric matrix in the space of antisymmetric functions of two time coordinates.

The action (D.64) is quadratic in the fluctuations σ. It is thus straightforward to perform the Gaussian integration over σ in (D.66). The right hand side is then given by

$$\frac{1}{N}\mathcal{F} = Z_2^{-1}\int \mathcal{D}g\, e^{-\tilde{S}_2}\, |G^*(\tau_1,\tau_2)|^{-1}|G^*(\tau_3,\tau_4)|^{-1}g(\tau_1,\tau_2)g(\tau_3,\tau_4), \tag{D.69}$$

where

$$\tilde{S}_2 = \frac{3J^2N}{4}\int d\tau_1...d\tau_4\, g(\tau_1,\tau_2)K^{-1}(1-K)(\tau_1,\tau_2;\tau_3,\tau_4)g(\tau_3,\tau_4). \tag{D.70}$$

The normalization in (D.69) is now provided by the partition function

$$Z_2 = \int \mathcal{D}g\, e^{-\tilde{S}_2}. \tag{D.71}$$

From equation (D.69) one therefore arrives at the relation between K and the four-point function $\mathcal{F}$

$$\frac{1}{3J^2}|G^*(\tau_1,\tau_2)||G^*(\tau_3,\tau_4)|\mathcal{F} = 2(1-K)^{-1}K \tag{D.72}$$

where we dropped the τ-dependence of K.

Although we arrived at equation (D.72) using the path integral of the non-local action the above relation can also be obtained from summing the ladder diagrams. There one notes that the ladder diagram $\mathcal{F}_{n+1}$ with $2(n+1)$ vertices is obtained from the diagram $\mathcal{F}_n$ with $2n$ vertices by acting with the kernel K', that is proportional to K

$$\mathcal{F}_{n+1} = K'\mathcal{F}_n \Rightarrow \mathcal{F} = \sum_{n=0}^{\infty} K'^n\mathcal{F}_0 = (1-K')^{-1}\mathcal{F}_0. \tag{D.73}$$

Since $\mathcal{F}_0$ is given by KI, where I is the identity matrix for antisymmetric functions of two variables, this reproduces equation (D.72) after reinstating the appropriate factors.

Suppose that one can define eigenfunctions $\Psi_\alpha(\tau_1,\tau_2)$ of K as

$$\int d\tau_3\, d\tau_4 K(\tau_1,\tau_2;\tau_3,\tau_4)\Psi_\alpha(\tau_3,\tau_4) = k(\alpha)\Psi_\alpha(\tau_1,\tau_2), \tag{D.74}$$

where α labels the set of eigenvalues associated to Ψ_α, that are normalized according to the standard inner product

$$\langle\Psi_\alpha,\Psi_\alpha\rangle = \int d\tau_1\, d\tau_2\, \Psi^*_\alpha(\tau_1,\tau_2)\Psi_\alpha(\tau_1,\tau_2) = 1. \tag{D.75}$$

The fundamental relation determining the four-point function $\mathcal{F}$ can therefore be written as

$$\frac{1}{3J^2}G^*(\tau_1,\tau_2)G^*(\tau_3,\tau_4)\mathcal{F}(\tau_1,\tau_2,\tau_3,\tau_4) = 2\sum_\alpha \frac{k(\alpha)}{1-k(\alpha)}\Psi_\alpha(\tau_1,\tau_2)\Psi^*_\alpha(\tau_3,\tau_4), \tag{D.76}$$

where we reinstated the dependence on τ and assumed that we have normal ordered time.

The calculation of the four-point function thus boils down to diagonalizing the kernel K. This is in principle a very difficult problem, in particular since the exact propagator G^* is only known numerically. However, in the conformal limit one has access to the exact propagator G^*_c and conformal invariance can be used to diagonalize the kernel K_c in this limit. Firstly, conformal invariance demands that K_c depends only on the SL(2) invariant cross ratio

$$\chi = \frac{(\tau_1-\tau_2)(\tau_3-\tau_4)}{(\tau_1-\tau_3)(\tau_2-\tau_4)}. \tag{D.77}$$

Secondly, one can show that K_c commutes with the Casimir. Since the eigenfunctions of the latter are non-degenerate they will be the eigenfunctions of K_c, labeled by the eigenvalues $h(h-1)$ of the Casimir. Equation (D.76) is valid also in the conformal limit up to a small but important issue which will be commented on below.

We will not go through this rather involved calculation in detail but just quote the results that we are going to need in the following. These were first obtained in [46] and then worked out later in more detail by [47, 247].

The eigenvalues $k_c(h)$ of K_c come in two sets: a discrete set with $h = 2, 4, 6, ..$ and a continuous set with $h = \frac{1}{2} + is$, $s \in \mathbb{R}^+$. They are explicitly given by

$$k_c(h) = -\frac{3}{2}\frac{\tan\frac{\pi(h-1/2)}{2}}{h-1/2}. \tag{D.78}$$

This gives rise to the following interesting observation. The four-point function (D.76) is well-defined unless $k(h) = 1$. However, by equation (D.78) it is clear that $k_c(h = 2) = 1$ which shows that the conformal four-point function has a divergence. Understanding the source for this divergence will lead to a deeper insight into the IR regime of the SYK model.

Consider a small reparametrization $\tau \to \tau + \epsilon(\tau)$ of the Euclidean time. According to equation (D.55) this changes the conformal solution G_c^* to $G_c^* + \delta_\epsilon G_c^*$ where

$$\delta_\epsilon G_c^* = (\Delta\epsilon'(\tau_1) + \Delta\epsilon'(\tau_2) + \epsilon(\tau_1)\partial_{\tau_1} + \epsilon(\tau_2)\partial_{\tau_2})G_c^*(\tau_1, \tau_2) \tag{D.79}$$

and similarly for Σ_c^*. Since the saddle-point equations, or equivalently the Schwinger-Dyson equations, (D.51) and (D.52) are reparametrization invariant in the conformal limit $G_c^* + \delta_\epsilon G_c^*$ will be another solution. To first order in the reparametrization (D.53) yields

$$\delta_\epsilon G_c^* * \Sigma_c^* + G_c^* * \delta_\epsilon \Sigma_c^* = 0. \tag{D.80}$$

Convoluting on the left with $-G_c^*$ and multiplying by $|G_c^*|$ we obtain

$$|G_c^*| \cdot \delta_\epsilon G_c^* - 3J^2 |G_c^*| \cdot \left(G_c^* * (G_c^*)^2 \delta_\epsilon G_c^* * G_c^*\right) = 0 \tag{D.81}$$

which can be rewritten as

$$|G_c^*(\tau_{12})|\delta_\epsilon G_c^*(\tau_{12}) + 3J^2 \int d\tau_3\, d\tau_4 |G_c^*(\tau_{12})| G_c^*(\tau_{13}) G_c^*(\tau_{24}) |G_c^*(\tau_{34})| |G_c^*(\tau_{34})| \delta_\epsilon G_c^*(\tau_{34}) = 0\,. \tag{D.82}$$

The second term of (D.82) is recognized to be the kernel K defined in (D.68) in the conformal limit such that the above can be condensed written as

$$(1 - K_c)(|G_c^*|\delta_\epsilon G_c^*) = 0. \tag{D.83}$$

This means that infinitesimal reparametrizations of the conformal solution (D.57) are eigenvectors of K with eigenvalue 1. The quadratic action (D.70) therefore vanishes on these fluctuations which produces the divergence in the four-point function.

However, notice that we have

$$\delta_\epsilon G_c^* = 0 \qquad \epsilon = 1, \tau, \tau^2, \tag{D.84}$$

i.e., the conformal solution is left invariant under $\mathfrak{sl}(2,\mathbb{R})$ transformations. One can also check directly in equation (D.57) that G_c^* is left invariant under the SL(2, ℝ) transformation

$$\tau = \frac{a\sigma + b}{c\sigma + d} \qquad ad - bc = 1 \qquad a, b, c, d \in \mathbb{R}. \tag{D.85}$$

The conformal solution G_c^* therefore spontaneously breaks the emerging conformal symmetry in the infrared to SL(2, ℝ). The reparametrizations that do not belong to SL(2, ℝ) become Nambu–Goldstone bosons that are zero-modes of the effective action (D.70) in the conformal limit. The divergence in the four-point function comes from these undamped directions in the functional integral (D.69). This suggests that the strict conformal/IR limit does not lead to a consistent theory by itself, rather for a consistent theory one needs an additional ingredient coming from the UV. The divergence comes from the $h = 2$ eigenvectors of the conformal kernel K which suggests that one has to treat this contribution away from the conformal limit. As we will see this will lead to a new action that has to be added to the conformal action in the IR limit.

D.4 Schwarzian action

As the first step, we will reconsider the reparametrizations $|G_c^*|\delta_\epsilon G_c^*$. Since we want to study the theory away from the conformal limit the Euclidean line and the Euclidean circle cannot be treated on the same footing anymore. In the following it is more convenient to work on the Euclidean circle at finite temperature β, $\tau \sim \tau + \beta$, in the coordinates $\theta = \frac{2\pi\tau}{\beta}$. Evaluating the reparametrizations (D.79) in this coordinate system for the Fourier modes $\epsilon = e^{-in\theta}$ and normalizing with respect to the inner product defined in (D.75) yields the eigenfunctions

$$\Psi_{2,n} = \left(\frac{3}{\pi^2|n|(n^2-1)}\right)^{1/2} e^{-i\frac{n}{2}(\theta_1+\theta_2)} \frac{\sin\frac{n(\theta_1-\theta_2)}{2}}{2\sin\frac{(\theta_1-\theta_2)}{2}} \left(\cot\frac{\theta_1-\theta_2}{2} - n\cot\frac{n(\theta_1-\theta_2)}{2}\right). \tag{D.86}$$

The $n = 0, -1, 1$ contributions associated to $\mathfrak{sl}(2,\mathbb{R})$ transformations vanish. Notice that in the conformal limit the $h = 2$ subspace of the kernel K is degenerate, since the $\Psi_{2,n}, n \geq 2$ eigenfunctions form a representation of $\mathfrak{sl}(2,\mathbb{R})$. This can be checked explicitly when one represents the generators as

$$P = e^{-i\theta}\left(\partial_\theta - \frac{i}{2}\right), \qquad K = -e^{i\theta}\left(\partial_\theta + \frac{i}{2}\right), \qquad D = i\partial_\theta. \tag{D.87}$$

The vector $\Psi_{2,2}$ is the highest weight vector in this representation. This degeneracy will be lifted when moving away from the conformal limit.

In the next step we determine the shift of the eigenvalues of K in the $h = 2$ subspace that is due to a small non-conformal contribution. More precisely, one can study the propagator G away from the conformal limit treating $(\beta J)^{-1}$ as expansion parameter

$$G = G_c + (\beta J)^{-1}\delta G_c + \ldots. \tag{D.88}$$

This will lead to a shift δK_c in the conformal kernel K_c. To first order in perturbation theory the eigenvalues of K_c in the $h = 2$ subspace will be shifted by

$$(\beta J)^{-1} \langle \Psi_{2,n}| \,\delta K_c \,|\Psi_{2,n}\rangle \,. \tag{D.89}$$

For the calculation of the shifted eigenvalue we refer to [47]. The result is

$$k(2,n) = 1 - \frac{\sqrt{2}\alpha_K}{\beta J}|n| + \dots , \tag{D.90}$$

where $\alpha_K \approx 2.85$ is a constant that can be determined numerically. As expected, the shift in the eigenvalues lifts the degeneracy in the $h = 2$ subspace and renders this contribution to the four-point function (D.76) finite.

However, notice the following: the right hand side of (D.76) will receive a contribution that is proportional to $\frac{\beta J}{\sqrt{2}\alpha_K |n|} + O((\beta J)^{-1})$ from the $\alpha = (2,n)$ sector. In the IR limit, i.e., at large βJ this term will be the dominant contribution. Due to the small breaking of conformal symmetry the former Nambu–Goldstone bosons will give an enhanced contribution.

Although the above discussion was based on the four-point function one can see that a parallel discussion can be carried on the level of the effective action (D.70). We saw above that in the conformal limit, when $K = K_c$, the reparametrizations of G_c are zero-modes of (D.69). In the same way as for the four-point function we want to study the contribution of the Nambu-Goldstone modes to the action when moving away from the conformal limit. Due to the normalization of the fluctuations g in (D.65) one should set

$$g = |G_c|\delta_\epsilon G_c \tag{D.91}$$

in (D.70). We know from the above that these are (up to normalization) equal to the eigenfunctions $\Psi_{2,n}$ of K_c. In particular, one finds

$$g_n = \frac{|n|(1-n^2)}{6J^2}\Psi_{2,n}. \tag{D.92}$$

Diagonalizing the operator in the action (D.69) only the $h = 2$ subspace will contribute. Using the shifted eigenvalue (D.90) we obtain an action that is proportional to $n^2(n^2-1)$. This can be written in position space as

$$\frac{S}{N} = \frac{\alpha_S\sqrt{2}}{J}\int_0^\beta \frac{1}{2}\left((\epsilon'')^2 - \left(\frac{2\pi}{\beta}\right)^2(\epsilon')^2\right), \qquad \alpha_S \equiv \frac{\alpha_K}{128\pi}. \tag{D.93}$$

This action vanishes for infinitesimal $\mathrm{SL}(2,\mathbb{R})$ transformations, as it must, since these were not part of the functional integral (D.71) in the first place. The action (D.93) is known as the infinitesimal *Schwarzian action*. Notice that it formally vanishes in the $J \to \infty$ limit. However, we saw above that the pure IR theory is not well-defined in this limit. Furthermore, the action is proportional to N which means that it will contribute at the same order as the conformal action in the large N limit.

Although we have studied only infinitesimal perturbations around the conformal solution (D.91) one can generalize the infinitesimal Schwarzian action to finite reparametrizations of the circle $\tau \to f(\tau)$. In this case one finds

$$S = -\frac{N}{\alpha_S\sqrt{2}J}\int_0^\beta d\tau\left(\mathrm{Sch}[f](\tau) + \left(\frac{2\pi}{\beta}\right)^2 (f')^2\right), \tag{D.94}$$

the Schwarzian action that we also obtained from the JT model on the gravity side (6.59).

The Schwarzian derivative vanishes for fractional linear transformations $f(\tau) = \frac{a\tau+b}{c\tau+d}$, which implies that the Schwarzian action (D.94) vanishes for finite $\mathrm{SL}(2,\mathbb{R})$ transformations.

One could have arrived at the form of the Schwarzian action also by simple effective field theory arguments. Since the ground-state spontaneously breaks the conformal symmetry consisting of all reparametrizations of the circle to $\mathrm{SL}(2,\mathbb{R})$ the Nambu–Goldstone bosons should correspond to the reparametrizations contained in $\mathrm{Diff}(S^1)/\mathrm{SL}(2,\mathbb{R})$. By the standard effective field theory argument the effective action governing these bosons must be the action compatible with these symmetries with the smallest number of derivatives. This leads directly to the Schwarzian action. The $1/J$ coefficient is fixed by dimensional analysis or the requirement that in the strict conformal limit the action (D.94) should vanish.

Bibliography

[1] H. A. González, D. Grumiller, and J. Salzer, "Towards a bulk description of higher spin SYK," *JHEP* **(to appear)** (2018) , arXiv:1802.01562 [hep-th].

[2] D. Grumiller, R. McNees, J. Salzer, C. Valcárcel, and D. Vassilevich, "Menagerie of AdS_2 boundary conditions," *JHEP* **10** (2017) 203, arXiv:1708.08471 [hep-th].

[3] D. Grumiller, J. Salzer, and D. Vassilevich, "Aspects of AdS_2 holography with non-constant dilaton," in *International Wokshop on Strong Field Problems in Quantum Theory Tomsk, Russia, June 6-11, 2016*. 2016. arXiv:1607.06974 [hep-th].

[4] D. Grumiller, J. Salzer, and D. Vassilevich, "AdS_2 holography is (non-)trivial for (non-)constant dilaton," *JHEP* **12** (2015) 015, arXiv:1509.08486 [hep-th].

[5] S. Prohazka, J. Salzer, and F. Schöller, "Linking Past and Future Null Infinity in Three Dimensions," *Phys. Rev.* **D95** no. 8, (2017) 086011, arXiv:1701.06573 [hep-th].

[6] A. Bagchi, D. Grumiller, J. Salzer, S. Sarkar, and F. Schöller, "Flat space cosmologies in two dimensions - Phase transitions and asymptotic mass-domination," *Phys.Rev.* **D90** no. 8, (2014) 084041, arXiv:1408.5337 [hep-th].

[7] D. Grumiller, R. McNees, and J. Salzer, "Cosmological constant as confining U(1) charge in two-dimensional dilaton gravity," *Phys.Rev.* **D90** no. 4, (2014) 044032, arXiv:1406.7007 [hep-th].

[8] D. Grumiller, R. McNees, and J. Salzer, "Black holes and thermodynamics - The first half century," *Fundam. Theor. Phys.* **178** (2015) 27–70, arXiv:1402.5127 [gr-qc].

[9] E. Wigner, "Nobel lecture: Events, laws of nature, and invariance principles." Nobel lecture, 1963.

[10] P. H. Mark A. Bedau, *Emergence - Contemporary Readings in Philosophy and Science*. Bradford Books. The MIT Press, 1 ed., 2008.

[11] A. Einstein, "Approximative Integration of the Field Equations of Gravitation," *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* (1916) 688–696.

[12] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, "Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems," *Proc. Roy. Soc. Lond.* **A269** (1962) 21–52.

[13] M. Henneaux and C. Teitelboim, "Asymptotically anti-De Sitter Spaces," *Commun.Math.Phys.* **98** (1985) 391–424.

[14] A. Ashtekar and A. Magnon, "Asymptotically anti-de Sitter space-times," *Class. Quant. Grav.* **1** (1984) L39–L44.

[15] J. Bekenstein, "Black holes and the second law," *Lett.Nuovo Cim.* **4** (1972) 737–740.

[16] J. D. Bekenstein, "Black holes and entropy," *Phys.Rev.* **D7** (1973) 2333–2346.

[17] S. Hawking, "Black hole explosions," *Nature* **248** (1974) 30–31.

[18] S. W. Hawking, "Particle creation by black holes," *Commun. Math. Phys.* **43** (1975) 199–220.

[19] R. M. Wald, "The thermodynamics of black holes," *Living Rev.Rel.* **4** (2001) 6, arXiv:gr-qc/9912119 [gr-qc].

[20] S. Hawking, "Breakdown of Predictability in Gravitational Collapse," *Phys.Rev.* **D14** (1976) 2460–2473.

[21] G. 't Hooft, "Dimensional reduction in quantum gravity," 1993.

[22] L. Susskind, "The World as a hologram," *J. Math. Phys.* **36** (1995) 6377–6396, hep-th/9409089.

[23] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Adv.Theor.Math.Phys.* **2** (1998) 231–252, arXiv:hep-th/9711200 [hep-th].

[24] S. Gubser, I. R. Klebanov, and A. M. Polyakov, "Gauge theory correlators from noncritical string theory," *Phys.Lett.* **B428** (1998) 105–114, arXiv:hep-th/9802109 [hep-th].

[25] E. Witten, "Anti-de Sitter space and holography," *Adv.Theor.Math.Phys.* **2** (1998) 253–291, arXiv:hep-th/9802150 [hep-th].

[26] J. D. Brown and M. Henneaux, "Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity," *Commun.Math.Phys.* **104** (1986) 207–226.

[27] M. Banados, C. Teitelboim, and J. Zanelli, "The Black hole in three-dimensional space-time," *Phys.Rev.Lett.* **69** (1992) 1849–1851, arXiv:hep-th/9204099 [hep-th].

[28] M. Banados, M. Henneaux, C. Teitelboim, and J. Zanelli, "Geometry of the (2+1) black hole," *Phys.Rev.* **D48** no. 6, (1993) 1506–1525, arXiv:gr-qc/9302012 [gr-qc].

[29] A. Strominger, "Black hole entropy from near horizon microstates," *JHEP* **9802** (1998) 009, arXiv:hep-th/9712251 [hep-th].

[30] J. L. Cardy, "Operator content of two-dimensional conformally invariant theories," *Nucl. Phys.* **B270** (1986) 186–204.

[31] H. W. J. Bloete, J. L. Cardy, and M. P. Nightingale, "Conformal invariance, the central charge, and universal finite size amplitudes at criticality," *Phys. Rev. Lett.* **56** (1986) 742–745.

[32] O. Coussaert, M. Henneaux, and P. van Driel, "The Asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant," *Class.Quant.Grav.* **12** (1995) 2961–2966, `arXiv:gr-qc/9506019 [gr-qc]`.

[33] S. Carlip, "What we don't know about BTZ black hole entropy," *Class. Quant. Grav.* **15** (1998) 3609–3625, `arXiv:hep-th/9806026 [hep-th]`.

[34] S. Carlip, "Conformal field theory, (2+1)-dimensional gravity, and the BTZ black hole," *Class.Quant.Grav.* **22** (2005) R85–R124, `arXiv:gr-qc/0503022 [gr-qc]`.

[35] A. Maloney and E. Witten, "Quantum Gravity Partition Functions in Three Dimensions," *JHEP* **02** (2010) 029, `arXiv:0712.0155 [hep-th]`.

[36] K. Jensen, S. Kachru, A. Karch, J. Polchinski, and E. Silverstein, "Towards a holographic marginal Fermi liquid," *Phys. Rev.* **D84** (2011) 126002, `arXiv:1105.1772 [hep-th]`.

[37] L. Gou, J. E. McClintock, R. A. Remillard, J. F. Steiner, M. J. Reid, J. A. Orosz, R. Narayan, M. Hanke, and J. García, "Confirmation Via the Continuum-Fitting Method that the Spin of the Black Hole in Cygnus X-1 is Extreme," *Astrophys. J.* **790** no. 1, (2014) 29, `arXiv:1308.4760 [astro-ph.HE]`.

[38] R. Fender, E. Gallo, and D. Russell, "No evidence for black hole spin powering of jets in X-ray binaries," *Mon. Not. Roy. Astron. Soc.* **406** (2010) 1425–1434, `arXiv:1003.5516 [astro-ph.HE]`.

[39] H. K. Kunduri and J. Lucietti, "Classification of near-horizon geometries of extremal black holes," *Living Rev. Rel.* **16** (2013) 8, `arXiv:1306.2517 [hep-th]`.

[40] M. Guica, T. Hartman, W. Song, and A. Strominger, "The Kerr/CFT Correspondence," *Phys. Rev.* **D80** (2009) 124008, `arXiv:0809.4266 [hep-th]`.

[41] A. Sen, "Black hole entropy function and the attractor mechanism in higher derivative gravity," *JHEP* **09** (2005) 038, `arXiv:hep-th/0506177 [hep-th]`.

[42] O. J. Dias, H. S. Reall, and J. E. Santos, "Kerr-CFT and gravitational perturbations," *JHEP* **0908** (2009) 101, `arXiv:0906.2380 [hep-th]`.

[43] V. Balasubramanian, J. de Boer, M. M. Sheikh-Jabbari, and J. Simón, "What is a chiral 2d CFT? And what does it have to do with extremal black holes?," *JHEP* **02** (2010) 017, `arXiv:0906.3272 [hep-th]`.

[44] A. J. Amsel, G. T. Horowitz, D. Marolf, and M. M. Roberts, "No Dynamics in the Extremal Kerr Throat," *JHEP* **09** (2009) 044, `arXiv:0906.2376 [hep-th]`.

[45] J. M. Maldacena, J. Michelson, and A. Strominger, "Anti-de Sitter fragmentation," *JHEP* **02** (1999) 011, `arXiv:hep-th/9812073 [hep-th]`.

[46] A. Kitaev, "A simple model of quantum holography." KITP strings seminar, 2015.

[47] J. Maldacena and D. Stanford, "Remarks on the Sachdev-Ye-Kitaev model," *Phys. Rev.* **D94** no. 10, (2016) 106002, arXiv:1604.07818 [hep-th].

[48] S. Sachdev and J. Ye, "Gapless spin fluid ground state in a random, quantum Heisenberg magnet," *Phys. Rev. Lett.* **70** (1993) 3339, arXiv:cond-mat/9212030 [cond-mat].

[49] Y.-Z. You, A. W. W. Ludwig, and C. Xu, "Sachdev-Ye-Kitaev Model and Thermalization on the Boundary of Many-Body Localized Fermionic Symmetry Protected Topological States," *Phys. Rev.* **B95** no. 11, (2017) 115150, arXiv:1602.06964 [cond-mat.str-el].

[50] A. Jevicki, K. Suzuki, and J. Yoon, "Bi-Local Holography in the SYK Model," *JHEP* **07** (2016) 007, arXiv:1603.06246 [hep-th].

[51] Y. Gu, X.-L. Qi, and D. Stanford, "Local criticality, diffusion and chaos in generalized Sachdev-Ye-Kitaev models," *JHEP* **05** (2017) 125, arXiv:1609.07832 [hep-th].

[52] D. J. Gross and V. Rosenhaus, "A Generalization of Sachdev-Ye-Kitaev," *JHEP* **02** (2017) 093, arXiv:1610.01569 [hep-th].

[53] M. Berkooz, P. Narayan, M. Rozali, and J. Simón, "Higher Dimensional Generalizations of the SYK Model," *JHEP* **01** (2017) 138, arXiv:1610.02422 [hep-th].

[54] A. M. García-García and J. J. M. Verbaarschot, "Spectral and thermodynamic properties of the Sachdev-Ye-Kitaev model," *Phys. Rev.* **D94** no. 12, (2016) 126010, arXiv:1610.03816 [hep-th].

[55] S. Banerjee and E. Altman, "Solvable model for a dynamical quantum phase transition from fast to slow scrambling," *Phys. Rev.* **B95** no. 13, (2017) 134302, arXiv:1610.04619 [cond-mat.str-el].

[56] W. Fu, D. Gaiotto, J. Maldacena, and S. Sachdev, "Supersymmetric Sachdev-Ye-Kitaev models," *Phys. Rev.* **D95** no. 2, (2017) 026009, arXiv:1610.08917 [hep-th]. [Addendum: Phys. Rev.D95,no.6,069904(2017)].

[57] E. Witten, "An SYK-Like Model Without Disorder," arXiv:1610.09758 [hep-th].

[58] J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, A. Streicher, and M. Tezuka, "Black Holes and Random Matrices," *JHEP* **05** (2017) 118, arXiv:1611.04650 [hep-th].

[59] I. R. Klebanov and G. Tarnopolsky, "Uncolored random tensors, melon diagrams, and the Sachdev-Ye-Kitaev models," *Phys. Rev.* **D95** no. 4, (2017) 046004, arXiv:1611.08915 [hep-th].

[60] R. A. Davison, W. Fu, A. Georges, Y. Gu, K. Jensen, and S. Sachdev, "Thermoelectric transport in disordered metals without quasiparticles: The Sachdev-Ye-Kitaev models and holography," *Phys. Rev.* **B95** no. 15, (2017) 155131, `arXiv:1612.00849 [cond-mat.str-el]`.

[61] C. Peng, M. Spradlin, and A. Volovich, "A Supersymmetric SYK-like Tensor Model," *JHEP* **05** (2017) 062, `arXiv:1612.03851 [hep-th]`.

[62] C. Krishnan, S. Sanyal, and P. N. Bala Subramanian, "Quantum Chaos and Holographic Tensor Models," *JHEP* **03** (2017) 056, `arXiv:1612.06330 [hep-th]`.

[63] G. Turiaci and H. Verlinde, "Towards a 2d QFT Analog of the SYK Model," *JHEP* **10** (2017) 167, `arXiv:1701.00528 [hep-th]`.

[64] F. Ferrari, "The Large D Limit of Planar Diagrams," `arXiv:1701.01171 [hep-th]`.

[65] T. Li, J. Liu, Y. Xin, and Y. Zhou, "Supersymmetric SYK model and random matrix theory," *JHEP* **06** (2017) 111, `arXiv:1702.01738 [hep-th]`.

[66] R. Gurau, "Quenched equals annealed at leading order in the colored SYK model," *EPL* **119** no. 3, (2017) 30003, `arXiv:1702.04228 [hep-th]`.

[67] G. Mandal, P. Nayak, and S. R. Wadia, "Coadjoint orbit action of Virasoro group and two-dimensional quantum gravity dual to SYK/tensor models," *JHEP* **11** (2017) 046, `arXiv:1702.04266 [hep-th]`.

[68] D. J. Gross and V. Rosenhaus, "The Bulk Dual of SYK: Cubic Couplings," *JHEP* **05** (2017) 092, `arXiv:1702.08016 [hep-th]`.

[69] C. Krishnan and K. V. P. Kumar, "Towards a Finite-N Hologram," *JHEP* **10** (2017) 099, `arXiv:1706.05364 [hep-th]`.

[70] M. Berkooz, P. Narayan, M. Rozali, and J. Simón, "Comments on the Random Thirring Model," *JHEP* **09** (2017) 057, `arXiv:1702.05105 [hep-th]`.

[71] A. Strominger, "The dS / CFT correspondence," *JHEP* **10** (2001) 034, `arXiv:hep-th/0106113 [hep-th]`.

[72] J. Polchinski, "S matrices from AdS space-time," `arXiv:hep-th/9901076 [hep-th]`.

[73] G. Barnich and C. Troessaert, "Aspects of the BMS/CFT correspondence," *JHEP* **05** (2010) 062, `arXiv:1001.1541 [hep-th]`.

[74] A. Bagchi, R. Basu, D. Grumiller, and M. Riegler, "Entanglement entropy in Galilean conformal field theories and flat holography," *Phys. Rev. Lett.* **114** no. 11, (2015) 111602, `arXiv:1410.4089 [hep-th]`.

[75] A. Bagchi, S. Detournay, R. Fareghbal, and J. Simón, "Holography of 3D Flat Cosmological Horizons," *Phys. Rev. Lett.* **110** no. 14, (2013) 141302, `arXiv:1208.4372 [hep-th]`.

[76] A. Strominger, "On BMS Invariance of Gravitational Scattering," *JHEP* **07** (2014) 152, arXiv:1312.2229 [hep-th].

[77] S. Weinberg, "Infrared photons and gravitons," *Phys. Rev.* **140** (1965) B516–B524.

[78] T. He, V. Lysov, P. Mitra, and A. Strominger, "BMS supertranslations and Weinberg's soft graviton theorem," *JHEP* **05** (2015) 151, arXiv:1401.7026 [hep-th].

[79] R. Jackiw, "Liouville field theory: A two-dimensional model for gravity?," in *Quantum Theory Of Gravity*, S. Christensen, ed., pp. 403–420. Adam Hilger, Bristol, 1984.

[80] C. Teitelboim, "The Hamiltonian structure of two-dimensional space-time and its relation with the conformal anomaly," in *Quantum Theory Of Gravity*, S. Christensen, ed., pp. 327–344. Adam Hilger, Bristol, 1984.

[81] B. Khesin and R. Wendt, *The geometry of infinite-dimensional groups*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge A Series of Modern Surveys in Mathematics. Springer, 1 ed., 2008.

[82] I. Vaisman, *Lectures on the Geometry of Poisson Manifolds*. Birkhäuser Basel, 1994.

[83] C. Laurent-Gengoux, A. Pichereau, and P. Vanhaecke, *Poisson Structures*. Grundlehren der mathematischen Wissenschaften. Springer, 2013 ed., 2012.

[84] J. Marsden and T. Ratiu, *Introduction to mechanics and symmetry*. Springer-Verlag, 1994.

[85] R. Abraham and J. Marsden, *Foundations of Mechanics*. Addison-Wesley Publishing Company, Inc., 2nd ed ed., 1987.

[86] L. Guieu and C. Roger, *Algèbre et le groupe de Virasoro : Aspects géométriques et algébriques, généralisations*. Les publications CRM, 2007.

[87] B. Oblak, *BMS Particles in Three Dimensions*. PhD thesis, Brussels U., 2016. arXiv:1610.08526 [hep-th].

[88] A. Weinstein, "The local structure of Poisson manifolds," *J. Differential Geom.* **18** no. 3, (1983) 523–557.

[89] J. Lee, *Introduction to smooth manifolds*, vol. 218 of *Graduate Texts in Mathematics*. Springer, 2012.

[90] N. M. J. Woodhouse, *Geometric quantization*. Oxford Mathematical Monographs. Clarendon Press, 1992.

[91] J. M. Lee, *Riemannian Manifolds: An Introduction to Curvature*. Graduate texts in mathematics. Springer, 1 ed., 1997.

[92] G. S. Birman and K. Nomizu, "The Gauss-Bonnet theorem for 2-dimensional spacetimes," *Michigan Math. J.* **31** no. 1, (1984) 77–81.

[93] P. R. Law, "Neutral Geometry and the Gauss-Bonnet Theorem for Two-dimensional Pseudo-Riemannian Manifolds," *Rocky Mountain J. Math.* **22** no. 4, (12, 1992) 1365–1383.

[94] C. Brans and R. H. Dicke, "Mach's principle and a relativistic theory of gravitation," *Phys. Rev.* **124** (1961) 925–935.

[95] J. M. Ezquiaga and M. Zumalacárregui, "Dark Energy After GW170817: Dead Ends and the Road Ahead," *Phys. Rev. Lett.* **119** no. 25, (2017) 251304, `arXiv:1710.05901 [astro-ph.CO]`.

[96] J. Sakstein and B. Jain, "Implications of the Neutron Star Merger GW170817 for Cosmological Scalar-Tensor Theories," *Phys. Rev. Lett.* **119** no. 25, (2017) 251303, `arXiv:1710.05893 [astro-ph.CO]`.

[97] P. Creminelli and F. Vernizzi, "Dark Energy after GW170817 and GRB170817A," *Phys. Rev. Lett.* **119** no. 25, (2017) 251302, `arXiv:1710.05877 [astro-ph.CO]`.

[98] T. Baker, E. Bellini, P. G. Ferreira, M. Lagos, J. Noller, and I. Sawicki, "Strong constraints on cosmological gravity from GW170817 and GRB 170817A," *Phys. Rev. Lett.* **119** no. 25, (2017) 251301, `arXiv:1710.06394 [astro-ph.CO]`.

[99] D. Grumiller, W. Kummer, and D. Vassilevich, "Dilaton gravity in two-dimensions," *Phys.Rept.* **369** (2002) 327–430, `arXiv:hep-th/0204253 [hep-th]`.

[100] J. Salzer, "The Cosmological Constant as a Thermodynamic Variable in 2d Dilaton Gravity," Master's thesis, Vienna University of Technology, 2013.

[101] H.-J. Schmidt, "Scale Invariant Gravity in Two-dimensions," *J. Math. Phys.* **32** (1991) 1562–1566.

[102] S. N. Solodukhin, "On higher derivative gravity in two-dimensions," *Phys. Rev.* **D51** (1995) 591–602, `arXiv:hep-th/9405132 [hep-th]`.

[103] T. Strobl, "Gravity in two space-time dimensions." Habilitation thesis, 1999.

[104] D. Cangemi and R. Jackiw, "Poincaré gauge theory for gravitational forces in (1+1)-dimensions," *Ann. Phys.* **225** (1993) 229–263, `hep-th/9302026`.

[105] K. Isler and C. A. Trugenberger, "A gauge theory of two-dimensional quantum gravity," *Phys. Rev. Lett.* **63** (1989) 834.

[106] A. H. Chamseddine and D. Wyler, "Gauge theory of topological gravity in (1+1)-dimensions," *Phys. Lett.* **B228** (1989) 75.

[107] J. Callan, Curtis G., E. Martinec, M. Perry, and D. Friedan, "Strings in Background Fields," *Nucl.Phys.* **B262** (1985) 593.

[108] E. Witten, "On string theory and black holes," *Phys.Rev.* **D44** (1991) 314–324.

[109] R. Dijkgraaf, H. L. Verlinde, and E. P. Verlinde, "String propagation in a black hole geometry," *Nucl. Phys.* **B371** (1992) 269–314.

[110] D. Grumiller, "An action for the exact string black hole," *JHEP* **05** (2005) 028, hep-th/0501208.

[111] J. Callan, Curtis G., S. B. Giddings, J. A. Harvey, and A. Strominger, "Evanescent black holes," *Phys.Rev.* **D45** (1992) 1005–1009, arXiv:hep-th/9111056 [hep-th].

[112] D. Grumiller and R. McNees, "Thermodynamics of black holes in two (and higher) dimensions," *JHEP* **0704** (2007) 074, arXiv:hep-th/0703230 [HEP-TH].

[113] D. Grumiller and R. Meyer, "Ramifications of lineland," *Turk.J.Phys.* **30** (2006) 349–378, arXiv:hep-th/0604049 [hep-th].

[114] A. Achucarro and P. Townsend, "A Chern-Simons Action for Three-Dimensional anti-De Sitter Supergravity Theories," *Phys.Lett.* **B180** (1986) 89.

[115] E. Witten, "(2+1)-Dimensional Gravity as an Exactly Soluble System," *Nucl. Phys.* **B311** (1988) 46.

[116] P. Schaller and T. Strobl, "Poisson structure induced (topological) field theories," *Mod.Phys.Lett.* **A9** (1994) 3129–3136, arXiv:hep-th/9405110 [hep-th].

[117] N. Ikeda, "Two-dimensional gravity and nonlinear gauge theory," *Annals Phys.* **235** (1994) 435–464, arXiv:hep-th/9312059 [hep-th].

[118] A. Medina and P. Revoy, "Algèbres de lie et produit scalaire invariant," *Annales scientifiques de l'École Normale Supérieure* **18** no. 3, (1985) 553–561.

[119] J. M. Figueroa-O'Farrill and S. Stanciu, "On the structure of symmetric selfdual Lie algebras," *J. Math. Phys.* **37** (1996) 4121–4134, arXiv:hep-th/9506152 [hep-th].

[120] S. Prohazka, *Chern-Simons Holography: Boundary Conditions, Contractions and Double Extensions for a Journey Beyond Anti-de Sitter*. PhD thesis, Vienna, Tech. U., 2017. arXiv:1710.11110 [hep-th].

[121] J. Zanelli, "Lecture notes on Chern-Simons (super-)gravities. Second edition (February 2008)," in *Proceedings, 7th Mexican Workshop on Particles and Fields (MWPF 1999): Merida, Mexico, November 10-17, 1999*. 2005. arXiv:hep-th/0502193 [hep-th].

[122] D. Birmingham, M. Blau, M. Rakowski, and G. Thompson, "Topological field theory," *Phys. Rept.* **209** (1991) 129–340.

[123] M. Blau and G. Thompson, "Topological Gauge Theories of Antisymmetric Tensor Fields," *Annals Phys.* **205** (1991) 130–172.

[124] G. Barnich and M. Henneaux, "Consistent couplings between fields with a gauge freedom and deformations of the master equation," *Phys. Lett.* **B311** (1993) 123–129, arXiv:hep-th/9304057 [hep-th].

[125] G. Barnich, F. Brandt, and M. Henneaux, "Local BRST cohomology in the antifield formalism. 1. General theorems," *Commun. Math. Phys.* **174** (1995) 57–92, arXiv:hep-th/9405109 [hep-th].

[126] G. Barnich, M. Henneaux, and R. Tatar, "Consistent interactions between gauge fields and the local BRST cohomology: The Example of Yang-Mills models," *Int. J. Mod. Phys.* **D3** (1994) 139–144, arXiv:hep-th/9307155 [hep-th].

[127] K. I. Izawa, "On nonlinear gauge theory from a deformation theory perspective," *Prog. Theor. Phys.* **103** (2000) 225–228, arXiv:hep-th/9910133 [hep-th].

[128] N. Ikeda, "Deformation of BF theories, topological open membrane and a generalization of the star deformation," *JHEP* **07** (2001) 037, arXiv:hep-th/0105286 [hep-th].

[129] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems.* Princeton University Press, Princeton, New Jersey, 1992.

[130] J. Gomis, J. Paris, and S. Samuel, "Antibracket, antifields and gauge theory quantization," *Phys. Rept.* **259** (1995) 1–145, arXiv:hep-th/9412228 [hep-th].

[131] L. Baulieu, A. S. Losev, and N. A. Nekrasov, "Target space symmetries in topological theories. 1.," *JHEP* **02** (2002) 021, arXiv:hep-th/0106042 [hep-th].

[132] M. Bojowald, A. Kotov, and T. Strobl, "Lie algebroid morphisms, Poisson sigma models, and off-shell closed gauge symmetries," *J. Geom. Phys.* **54** (2005) 400–426, arXiv:math/0406445 [math-dg].

[133] M. Kontsevich, "Deformation quantization of Poisson manifolds, I," *Lett. Math. Phys.* **66** (2003) 157–216, q-alg/9709040.

[134] A. S. Cattaneo and G. Felder, "A Path integral approach to the Kontsevich quantization formula," *Commun. Math. Phys.* **212** (2000) 591–611, arXiv:math/9902090 [math].

[135] S. Gukov and E. Witten, "Branes and Quantization," *Adv. Theor. Math. Phys.* **13** no. 5, (2009) 1445–1518, arXiv:0809.0305 [hep-th].

[136] M. Henneaux and C. Teitelboim, *Quantization of gauge systems.* Princeton University Press, Princeton, NJ, 1992.

[137] M. F. Ertl, M. O. Katanaev, and W. Kummer, "Generalized supergravity in two dimensions," *Nucl. Phys.* **B530** (1998) 457–486, hep-th/9710051.

[138] T. Strobl, "Target-superspace in 2d dilatonic supergravity," *Phys. Lett.* **B460** (1999) 87–93, arXiv:hep-th/9906230.

[139] T. Regge and C. Teitelboim, "Role of surface integrals in the hamiltonian formulation of general relativity," *Annals Phys.* **88** (1974) 286.

[140] G. Barnich and F. Brandt, "Covariant theory of asymptotic symmetries, conservation laws and central charges," *Nucl.Phys.* **B633** (2002) 3–82, arXiv:hep-th/0111246 [hep-th].

[141] G. Barnich and G. Compere, "Surface charge algebra in gauge theories and thermodynamic integrability," *J. Math. Phys.* **49** (2008) 042901, arXiv:0708.2378 [gr-qc].

[142] V. Balasubramanian and P. Kraus, "A Stress tensor for Anti-de Sitter gravity," *Commun.Math.Phys.* **208** (1999) 413–428, arXiv:hep-th/9902121 [hep-th].

[143] R. Aros, M. Contreras, R. Olea, R. Troncoso, and J. Zanelli, "Conserved charges for gravity with locally AdS asymptotics," *Phys. Rev. Lett.* **84** (2000) 1647–1650, arXiv:gr-qc/9909015 [gr-qc].

[144] J. Lee and R. M. Wald, "Local symmetries and constraints," *J. Math. Phys.* **31** (1990) 725–743.

[145] V. Iyer and R. M. Wald, "Some properties of Noether charge and a proposal for dynamical black hole entropy," *Phys. Rev.* **D50** (1994) 846–864, arXiv:gr-qc/9403028 [gr-qc].

[146] R. M. Wald and A. Zoupas, "A General definition of 'conserved quantities' in general relativity and other theories of gravity," *Phys. Rev.* **D61** (2000) 084027, arXiv:gr-qc/9911095 [gr-qc].

[147] F. Schöller. PhD thesis, TU Wien, 2018.

[148] V. Iyer and R. M. Wald, "A Comparison of Noether charge and Euclidean methods for computing the entropy of stationary black holes," *Phys.Rev.* **D52** (1995) 4430–4439, arXiv:gr-qc/9503052 [gr-qc].

[149] P. J. Olver, *Applications of Lie groups to differential equations*, vol. 107 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second ed., 1993.

[150] W. Kummer and D. J. Schwarz, "General analytic solution of R**2 gravity with dynamical torsion in two-dimensions," *Phys. Rev.* **D45** (1992) 3628–3635.

[151] T. Klosch and T. Strobl, "Classical and quantum gravity in (1+1)-dimensions. Part 2: The Universal coverings," *Class. Quant. Grav.* **13** (1996) 2395–2422, arXiv:gr-qc/9511081 [gr-qc].

[152] A. Achucarro and M. E. Ortiz, "Relating black holes in two-dimensions and three-dimensions," *Phys.Rev.* **D48** (1993) 3600–3605, arXiv:hep-th/9304068 [hep-th].

[153] M. Cadoni and S. Mignemi, "Dilatonic black holes in theories with moduli fields," *Phys. Rev.* **D48** (1993) 5536–5538, arXiv:hep-th/9305107 [hep-th].

[154] C. Fefferman and C. R. Graham, "The ambient metric," *Ann. Math. Stud.* **178** (2011) 1–128, arXiv:0710.0919 [math.DG].

[155] R. Penrose, "Zero rest mass fields including gravitation: Asymptotic behavior," *Proc. Roy. Soc. Lond.* **A284** (1965) 159.

[156] D. Borthwick, *Spectral Theory of Infinite-Area Hyperbolic Surfaces*. Progress in Mathematics. Springer International Publishing, 2016.

[157] W. Thurston and S. Levy, *Three-dimensional Geometry and Topology*, vol. 1. Princeton University Press, 1997.

[158] E. Witten, "Coadjoint Orbits of the Virasoro Group," *Commun. Math. Phys.* **114** (1988) 1.

[159] J. Balog, L. Feher, and L. Palla, "Coadjoint orbits of the Virasoro algebra and the global Liouville equation," *Int. J. Mod. Phys.* **A13** (1998) 315–362, arXiv:hep-th/9703045 [hep-th].

[160] J. A. de Azcárraga and J. M. Izquierdo, *Lie Groups, Lie Algebras, Cohomology and some Applications in Physics*. Cambridge University Press, 2011.

[161] A. Garbarz and M. Leston, "Classification of Boundary Gravitons in AdS_3 Gravity," *JHEP* **05** (2014) 141, arXiv:1403.3367 [hep-th].

[162] G. Barnich and B. Oblak, "Holographic positive energy theorems in three-dimensional gravity," *Class. Quant. Grav.* **31** (2014) 152001, arXiv:1403.3835 [hep-th].

[163] A. Almheiri and J. Polchinski, "Models of AdS_2 backreaction and holography," *JHEP* **11** (2015) 014, arXiv:1402.6334 [hep-th].

[164] T. Hartman and A. Strominger, "Central Charge for AdS(2) Quantum Gravity," *JHEP* **04** (2009) 026, arXiv:0803.3621 [hep-th].

[165] A. Castro, D. Grumiller, F. Larsen, and R. McNees, "Holographic Description of AdS(2) Black Holes," *JHEP* **0811** (2008) 052, arXiv:0809.4264 [hep-th].

[166] A. Castro and W. Song, "Comments on AdS_2 Gravity," arXiv:1411.1948 [hep-th].

[167] R. K. Gupta and A. Sen, "Ads(3)/CFT(2) to Ads(2)/CFT(1)," arXiv:0806.0053 [hep-th].

[168] M. Cvetič and I. Papadimitriou, "AdS_2 holographic dictionary," *JHEP* **12** (2016) 008, arXiv:1608.07018 [hep-th]. [Erratum: JHEP01,120(2017)].

[169] G. Gibbons and S. Hawking, "Action Integrals and Partition Functions in Quantum Gravity," *Phys.Rev.* **D15** (1977) 2752–2756.

[170] G. W. Gibbons and S. W. Hawking, eds., *Euclidean quantum gravity*. Singapore: World Scientific, 1993.

[171] W. Kummer, H. Liebl, and D. Vassilevich, "Exact path integral quantization of generic 2-D dilaton gravity," *Nucl.Phys.* **B493** (1997) 491–502, arXiv:gr-qc/9612012 [gr-qc].

[172] R. Camporesi and A. Higuchi, "Spectral functions and zeta functions in hyperbolic spaces," *J. Math. Phys.* **35** (1994) 4217–4246.

[173] F. Larsen and P. Lisbao, "Quantum Corrections to Supergravity on $AdS_2 \times S^2$," *Phys. Rev.* **D91** no. 8, (2015) 084056, arXiv:1411.7423 [hep-th].

[174] M. Bertin, S. Ertl, H. Ghorbani, D. Grumiller, N. Johansson, and D. Vassilevich, "Lobachevsky holography in conformal Chern-Simons gravity," *JHEP* **1306** (2013) 015, arXiv:1212.3335.

[175] D. V. Vassilevich and A. Zelnikov, "Discrete symmetries of functional determinants," *Nucl. Phys.* **B594** (2001) 501–517, hep-th/0009084.

[176] P. B. Gilkey, K. Kirsten, D. Vassilevich, and A. Zelnikov, "Duality symmetry of the p form effective action and supertrace of the twisted de Rham complex," *Nucl. Phys.* **B648** (2003) 542–556, arXiv:hep-th/0209125 [hep-th].

[177] G. Mandal, A. M. Sengupta, and S. R. Wadia, "Classical solutions of two-dimensional string theory," *Mod.Phys.Lett.* **A6** (1991) 1685–1692.

[178] S. Elitzur, A. Forge, and E. Rabinovici, "Some global aspects of string compactifications," *Nucl.Phys.* **B359** (1991) 581–610.

[179] S. D. Odintsov and I. L. Shapiro, "One loop renormalization of two-dimensional induced quantum gravity," *Phys. Lett.* **B263** (1991) 183–189.

[180] J. G. Russo and A. A. Tseytlin, "Scalar tensor quantum gravity in two-dimensions," *Nucl. Phys.* **B382** (1992) 259–275, arXiv:hep-th/9201021.

[181] J. Brown, *Lower Dimensional Gravity.* World Scientific, 1988.

[182] R. Emparan, C. V. Johnson, and R. C. Myers, "Surface terms as counterterms in the AdS / CFT correspondence," *Phys.Rev.* **D60** (1999) 104001, arXiv:hep-th/9903238 [hep-th].

[183] S. de Haro, S. N. Solodukhin, and K. Skenderis, "Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence," *Commun. Math. Phys.* **217** (2001) 595–622, arXiv:hep-th/0002230 [hep-th].

[184] I. Papadimitriou and K. Skenderis, "AdS / CFT correspondence and geometry," in *AdS/CFT correspondence: Einstein metrics and their conformal boundaries.*, O. Biquard, ed., pp. 73–101. 2005. arXiv:hep-th/0404176 [hep-th].

[185] S. Sachdev, "Holographic metals and the fractionalized Fermi liquid," *Phys. Rev. Lett.* **105** (2010) 151602, arXiv:1006.3794 [hep-th].

[186] D. Grumiller, M. Leston, and D. Vassilevich, "Anti-de Sitter holography for gravity and higher spin theories in two dimensions," *Phys. Rev.* **D89** no. 4, (2014) 044001, arXiv:1311.7413 [hep-th].

[187] R. P. Geroch and J. Winicour, "Linkages in general relativity," *J. Math. Phys.* **22** (1981) 803–812.

[188] F. Schöller, "Distinct Minkowski spaces from Bondi-Metzner-Sachs supertranslations," *Phys. Rev.* **D97** no. 4, (2018) 046009, arXiv:1711.02670 [gr-qc].

[189] V. de Alfaro, S. Fubini, and G. Furlan, "Conformal invariance in quantum mechanics," *Nuovo Cim.* **A34** (1976) 569.

[190] M. Astorino, S. Cacciatori, D. Klemm, and D. Zanon, "AdS(2) supergravity and superconformal quantum mechanics," *Ann. Phys.* **304** (2003) 128–144, arXiv:hep-th/0212096.

[191] T. H. Buscher, "A symmetry of the string background field equations," *Phys. Lett.* **B194** (1987) 59.

[192] J. Maldacena, D. Stanford, and Z. Yang, "Conformal symmetry and its breaking in two dimensional Nearly Anti-de-Sitter space," *PTEP* **2016** no. 12, (2016) 12C104, arXiv:1606.01857 [hep-th].

[193] G. Barnich and C. Troessaert, "BMS charge algebra," *JHEP* **12** (2011) 105, arXiv:1106.0213 [hep-th].

[194] J. Engelsöy, T. G. Mertens, and H. Verlinde, "An investigation of AdS_2 backreaction and holography," *JHEP* **07** (2016) 139, arXiv:1606.03438 [hep-th].

[195] K. Jensen, "Chaos in AdS_2 Holography," *Phys. Rev. Lett.* **117** no. 11, (2016) 111601, arXiv:1605.06098 [hep-th].

[196] S. Detournay, T. Hartman, and D. M. Hofman, "Warped Conformal Field Theory," *Phys. Rev.* **D86** (2012) 124018, arXiv:1210.0539 [hep-th].

[197] H. Afshar, S. Detournay, D. Grumiller, and B. Oblak, "Near-Horizon Geometry and Warped Conformal Symmetry," *JHEP* **03** (2016) 187, arXiv:1512.08233 [hep-th].

[198] C. Troessaert, "Enhanced asymptotic symmetry algebra of AdS_3," *JHEP* **08** (2013) 044, arXiv:1303.3296 [hep-th].

[199] R. M. Wald, "Black hole entropy is the Noether charge," *Phys.Rev.* **D48** (1993) 3427–3431, arXiv:gr-qc/9307038 [gr-qc].

[200] R. B. Mann, "Conservation laws and 2-d black holes in dilaton gravity," *Phys. Rev.* **D47** (1993) 4438–4442, hep-th/9206044.

[201] D. J. Gross and V. Rosenhaus, "A line of CFTs: from generalized free fields to SYK," *JHEP* **07** (2017) 086, arXiv:1706.07015 [hep-th].

[202] I. Bengtsson and P. Sandin, "Anti de Sitter space, squashed and stretched," *Class. Quant. Grav.* **23** (2006) 971–986, arXiv:gr-qc/0509076 [gr-qc].

[203] E. S. Fradkin and V. Ya. Linetsky, "Higher Spin Symmetry in One-dimension and Two-dimensions. 1.," *Mod. Phys. Lett.* **A4** (1989) 2635–2647.

[204] E. S. Fradkin and V. Ya. Linetsky, "Higher Spin Symmetry in One-dimension and Two-dimensions. 2.," *Mod. Phys. Lett.* **A4** (1989) 2649–2665.

[205] K. Li, "Construction of topological W(3) gravity," *Phys. Lett.* **B251** (1990) 54–60.

[206] K. Li, "Action for Topological W Gravity," *Nucl. Phys.* **B346** (1990) 329–348.

[207] M. A. Vasiliev, "Higher spin gauge interactions for matter fields in two-dimensions," *Phys. Lett.* **B363** (1995) 51–57, `arXiv:hep-th/9511063 [hep-th]`.

[208] K. B. Alkalaev, "On higher spin extension of the Jackiw-Teitelboim gravity model," *J. Phys.* **A47** (2014) 365401, `arXiv:1311.5119 [hep-th]`.

[209] K. B. Alkalaev, "Global and local properties of AdS_2 higher spin gravity," *JHEP* **10** (2014) 122, `arXiv:1404.5330 [hep-th]`.

[210] M. Mezei, S. S. Pufu, and Y. Wang, "A 2d/1d Holographic Duality," `arXiv:1703.08749 [hep-th]`.

[211] D. Stanford and E. Witten, "Fermionic Localization of the Schwarzian Theory," *JHEP* **10** (2017) 008, `arXiv:1703.04612 [hep-th]`.

[212] M. Henneaux, G. Lucena Gómez, J. Park, and S.-J. Rey, "Super- W(infinity) Asymptotic Symmetry of Higher-Spin AdS_3 Supergravity," *JHEP* **1206** (2012) 037, `arXiv:1203.5152 [hep-th]`.

[213] A. Campoleoni, S. Fredenhagen, S. Pfenninger, and S. Theisen, "Towards metric-like higher-spin gauge theories in three dimensions," *J.Phys.* **A46** (2013) 214017, `arXiv:1208.1851 [hep-th]`.

[214] M. Henneaux, A. Perez, D. Tempo, and R. Troncoso, "Chemical potentials in three-dimensional higher spin anti-de Sitter gravity," *JHEP* **1312** (2013) 048, `arXiv:1309.4362 [hep-th]`.

[215] J. Yoon, "SYK Models and SYK-like Tensor Models with Global Symmetry," *JHEP* **10** (2017) 183, `arXiv:1707.01740 [hep-th]`.

[216] T. G. Mertens, "The Schwarzian Theory - Origins," `arXiv:1801.09605 [hep-th]`.

[217] G. Barnich, H. A. Gonzalez, and P. Salgado-Rebolledo, "Geometric actions for three-dimensional gravity," *Class. Quant. Grav.* **35** no. 1, (2018) 014003, `arXiv:1707.08887 [hep-th]`.

[218] A. Alekseev and S. L. Shatashvili, "Path Integral Quantization of the Coadjoint Orbits of the Virasoro Group and 2D Gravity," *Nucl. Phys.* **B323** (1989) 719–733.

[219] A. Alekseev, L. D. Faddeev, and S. L. Shatashvili, "Quantization of symplectic orbits of compact Lie groups by means of the functional integral," *J. Geom. Phys.* **5** (1988) 391–406.

[220] G. Dzhordzhadze, L. O'Raifeartaigh, and I. Tsutsui, "Quantization of a relativistic particle on the SL(2,R) manifold based on Hamiltonian reduction," *Phys. Lett.* **B336** (1994) 388–394, `arXiv:hep-th/9407059 [hep-th]`.

[221] M. Heinze, B. Hoare, G. Jorjadze, and L. Megrelidze, "Orbit method quantization of the AdS_2 superparticle," *J. Phys.* **A48** no. 31, (2015) 315403, arXiv:1504.04175 [hep-th].

[222] M. Brigante, S. Cacciatori, D. Klemm, and D. Zanon, "The asymptotic dynamics of two-dimensional (anti-)de Sitter gravity," *JHEP* **03** (2002) 005, hep-th/0202073.

[223] J. Gomis, J. Herrero, K. Kamimura, and J. Roca, "Finite W(3) transformations in a multitime approach," *Phys. Lett.* **B339** (1994) 59–64, arXiv:hep-th/9409024 [hep-th].

[224] P. Di Francesco, C. Itzykson, and J. B. Zuber, "Classical W algebras," *Commun. Math. Phys.* **140** (1991) 543–568.

[225] A. Perez, D. Tempo, and R. Troncoso, "Boundary conditions for General Relativity on AdS_3 and the KdV hierarchy," *JHEP* **06** (2016) 103, arXiv:1605.04490 [hep-th].

[226] A. R. Forsyth, *Theory of differential equations. Ordinary linear equations*, vol. Volume 4, Part 3. Cambridge University Press, 1902.

[227] S. Govindarajan, "Higher dimensional uniformization and W geometry," *Nucl. Phys.* **B457** (1995) 357–374, arXiv:hep-th/9412078 [hep-th].

[228] A. Marshakov and A. Morozov, "A NOTE ON W(3) ALGEBRA," *Nucl. Phys.* **B339** (1990) 79–94. [Sov. Phys. JETP70,403(1990)].

[229] W. Li and S. Theisen, "Some aspects of holographic W-gravity," *JHEP* **08** (2015) 035, arXiv:1504.07799 [hep-th].

[230] V. Y. Ovsienko and B. A. Khesin, "Symplectic leaves of the gel'fand-dikii brackets and homotopy classes of nondegenerate curves," *Functional Analysis and Its Applications* **24** (01-03, 1990) .

[231] M. Z. Shapiro, "Topology of the space of nondegenerate curves," *Functional Analysis and Its Applications* **26** (07-09, 1992) .

[232] Z. Bajnok and D. Nogradi, "Geometry of W algebras from the affine Lie algebra point of view," *J. Phys.* **A34** (2001) 4811–4830, arXiv:hep-th/0012190 [hep-th].

[233] S. T. V. Ovsienko, *Projective Differential Geometry Old and New: From the Schwarzian Derivative to the Cohomology of Diffeomorphism Groups*. Cambridge Tracts in Mathematics. Cambridge University Press, 2004.

[234] A. Sen, "Logarithmic Corrections to Schwarzschild and Other Non-extremal Black Hole Entropy in Different Dimensions," *JHEP* **04** (2013) 156, arXiv:1205.0971 [hep-th].

[235] C. Keeler, P. Lisbao, and G. S. Ng, "Partition functions with spin in AdS_2 via quasinormal mode methods," *JHEP* **10** (2016) 060, arXiv:1601.04720 [hep-th].

[236] A. Castro, C. Keeler, and P. Szepietowski, "Tweaking one-loop determinants in AdS_3," *JHEP* **10** (2017) 070, arXiv:1707.06245 [hep-th].

[237] D. Harlow and D. Jafferis, "The Factorization Problem in Jackiw-Teitelboim Gravity," arXiv:1804.01081 [hep-th].

[238] S. Bonanos, J. Gomis, K. Kamimura, and J. Lukierski, "Maxwell Superalgebra and Superparticle in Constant Gauge Badkgrounds," *Phys. Rev. Lett.* **104** (2010) 090401, arXiv:0911.5072 [hep-th].

[239] H. L. Verlinde, "Black holes and strings in two-dimensions," in *Recent developments in theoretical and experimental general relativity, gravitation and relativistic field theories. Proceedings, 6th Marcel Grossmann Meeting, Kyoto, Japan, June 23-29, 1991. Pts. A, B*, pp. 178–207. 1991.

[240] D. Cangemi and R. Jackiw, "Gauge invariant formulations of lineal gravity," *Phys. Rev. Lett.* **69** (1992) 233–236, arXiv:hep-th/9203056 [hep-th].

[241] J. M. Figueroa-O'Farrill and S. Prohazka, "Spacetimes for kinematical lie algebras," *to appear* (2018) .

[242] L. Feher, L. O'Raifeartaigh, P. Ruelle, I. Tsutsui, and A. Wipf, "On Hamiltonian reductions of the Wess-Zumino-Novikov-Witten theories," *Phys. Rept.* **222** (1992) 1–64.

[243] M. Henneaux, L. Maoz, and A. Schwimmer, "Asymptotic dynamics and asymptotic symmetries of three-dimensional extended AdS supergravity," *Annals Phys.* **282** (2000) 31–66, arXiv:hep-th/9910013 [hep-th].

[244] S. Edwards and P. Anderson, "Theory of spin glasses," *Journal of Physics F Metal Physics* **5** (1975) .

[245] K. H. Fischer and J. A. Hertz, *Spin glasses*, vol. 33. Cambridge University Press, New York, 1993.

[246] W. Fu and S. Sachdev, "Numerical study of fermion and boson models with infinite-range random interactions," *Phys. Rev.* **B94** no. 3, (2016) 035135, arXiv:1603.05246 [cond-mat.str-el].

[247] J. Polchinski and V. Rosenhaus, "The Spectrum in the Sachdev-Ye-Kitaev Model," *JHEP* **04** (2016) 001, arXiv:1601.06768 [hep-th].

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Citizenship:	Austrian
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Education

since 03/2014	Doctoral Studies at TU Wien (advisor D. Grumiller) *Research Interests:* black holes, classical/quantum gravity, holography, lower-dimensional gravity
01/2017–02/2017	Research visit at Universidade do ABC, São Paulo, Brazil
10/2011–11/2013	Master Studies “Technische Physik” at TU Wien *graduated with distinction*
10/2010–03/2011	Erasmus exchange student at Universidad Autónoma Madrid, Spain
10/2007–10/2011	Bachelor Studies “Technische Physik” at TU Wien *graduated with distinction*
06/2006	Matura at BG XIX Gymnasiumstraße, Vienna; *passed with distinction*

Teaching and Work Experience

12/2013–03/2017	Project assistant at the Institute of Theoretical Physics, TU Wien
01–12/2014	Co-organizer of the Vienna Theory Lunch Seminar
10/2014–06/2015	Teaching assistant in *Electrodynamics I* and *Quantum Theory I*, TU Wien
09/2006–06/2007	Civil Service at Red Cross

Awards and Grants

03/2018	Erwin Schrödinger fellowship by the FWF
10/2014	Outstanding master thesis award of the Austrian Physical Society

Publication Record

5 publications in peer-reviewed journals (JHEP, Phys. Rev. D)

1 conference proceedings contribution

1 book contribution

Scientific Talks

- Conferences, Workshops
  - 'ESI Workshop on Quantum Physics and Gravity'
    June 2017, Erwin Schrödinger Institute, Vienna, Austria
  - 'ÖPG Annual Meeting 2016'
    Sep 27th-29th, 2016, Vienna, Austria

- Invited Seminars

2017 | 'Aspects of AdS_2 holography', Niels Bohr Institute, Copenhagen, Denmark

'Aspects of AdS_2 holography', University of Barcelona, Spain

'Holography in 2d Dilaton Gravity', ICTP-SAIFR São Paulo, Brazil

'Holography in 2d Dilaton Gravity', UFRJ, Rio de Janeiro, Brazil

2016 | 'Holography in 2d Dilaton Gravity', AEI-Golm, Germany

'Holography in 2d Dilaton Gravity', ULB Brussels, Belgium

2015 | 'Tangled Up in Two – 2d dilaton gravity', University of Vienna, Austria

Outreach

09/2016 | Invited talk in the public 'Young Minds' session at the annual meeting of the Austrian Physical Society

10–11/2015 | Guide through the exhibition "100 Years of General Relativity" at the Austrian Academy of Sciences

Attended Schools and Conferences

06/2017 | ESI Workshops on Quantum Physics and Gravity, Vienna, Austria

03/2017 | AdS3: theory and practice, GGI, Florence, Italy

09/2016 | ÖPG Annual Meeting 2016, Vienna Austria

11/2015 | Vienna Central European Seminar on Quantum and Gravity, Vienna, Austria

09/2015 | Saalburg Summer School on Theoretical Physics, Wolfersdorf, Germany

06/2015 | Strings 2015, Bangalore, India

02/2012 | Winter School on *Prospects of Particle Physics*, Schladming, Austria

09/2011 | European Conference on Complex Systems, Vienna, Austria

Publications

Research articles

[1] H. A. González, D. Grumiller, and J. Salzer, "Towards a bulk description of higher spin SYK," arXiv:1802.01562 [hep-th].

[2] D. Grumiller, R. McNees, J. Salzer, C. Valcárcel, and D. Vassilevich, "Menagerie of AdS_2 boundary conditions," *JHEP* **10** (2017) 203, arXiv:1708.08471 [hep-th].

[3] S. Prohazka, J. Salzer, and F. Schöller, "Linking Past and Future Null Infinity in Three Dimensions," *Phys. Rev.* **D95** no. 8, (2017) 086011, arXiv:1701.06573 [hep-th].

[4] D. Grumiller, J. Salzer, and D. Vassilevich, "AdS_2 holography is (non-)trivial for (non-)constant dilaton," *JHEP* **12** (2015) 015, arXiv:1509.08486 [hep-th].

[5] A. Bagchi, D. Grumiller, J. Salzer, S. Sarkar, and F. Schöller, "Flat space cosmologies in two dimensions - Phase transitions and asymptotic mass-domination," *Phys.Rev.* **D90** no. 8, (2014) 084041, arXiv:1408.5337 [hep-th].

[6] D. Grumiller, R. McNees, and J. Salzer, "Cosmological constant as confining U(1) charge in two-dimensional dilaton gravity," *Phys.Rev.* **D90** no. 4, (2014) 044032, arXiv:1406.7007 [hep-th].

Book contributions

[7] D. Grumiller, R. McNees, and J. Salzer, "Black holes and thermodynamics - The first half century," *Fundam. Theor. Phys.* **178** (2015) 27–70, arXiv:1402.5127 [gr-qc].

Conference proceedings

[8] D. Grumiller, J. Salzer, and D. Vassilevich, "Aspects of AdS_2 holography with non-constant dilaton," *Russ. Phys. J.* **59** no. 11, (2017) 1798–1803, arXiv:1607.06974 [hep-th].