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Review

Some Mathematical Aspects of $f(R)$-Gravity with Torsion: Cauchy Problem and Junction Conditions

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Abstract: We discuss the Cauchy problem and the junction conditions within the framework of $f(R)$-gravity with torsion. We derive sufficient conditions to ensure the well-posedness of the initial value problem, as well as general conditions to join together on a given hypersurface two different solutions of the field equations. The stated results can be useful to distinguish viable from nonviable $f(R)$-models with torsion.

Keywords: Cauchy problem; junction conditions; Einstein–Cartan gravity; $f(R)$-gravity with torsion

1. Introduction

Several open problems in modern physics at both ultraviolet and infrared scales seem to justify the need to enlarge or revise General Relativity (GR). For example, at astrophysical and cosmological scales, in order for observations to agree with the theoretical predictions of GR, it is necessary to assume the existence of the so-called dark matter and dark energy. But, up to now, at a fundamental level, no experimental evidence has been found to prove the existence of such unknown forms of matter and energy. This fact, together with other shortcomings of GR, represents the signal of a possible breakdown in our understanding of gravity; the possibility of developing extended or alternative theories of gravity is then to be seriously taken into account.

In the last thirty years [1,2], many extensions of GR have been actually proposed; among these, $f(R)$-gravity certainly remains one of the most direct and simplest [3–7]: it relies on the idea that the gravitational Lagrangian may depend on the Ricci scalar R in a more general way than the linear one as it happens in the Einstein-Hilbert action. Recently, $f(R)$-gravity has received great interest in view of its successes in accounting for both cosmic speed-up and missing matter at cosmological and astrophysical scales, respectively (see, for example, [8–10]).

At the same time, including the torsion tensor among the geometrical attributes of space-time is another way to extend GR. Cartan was the first to introduce torsion in the geometrical background; after him, Sciama and Kibble embodied it within the framework of Einstein gravity implementing the idea that spin can be source of torsion as energy does for curvature [11–13]. The resulting theory, known as Einstein–Cartan–Sciama–Kibble (ECSK) theory, has been the first generalization of GR trying to take the spin of elementary fields into account, and it still remains one of the most serious attempts in this direction [14–16].

Following this paradigm, $f(R)$-gravity with torsion consists in one of the simplest extensions of the ECSK theory, just as purely metric $f(R)$-gravity is with respect to GR. The key idea is again that of replacing the Einstein-Hilbert Lagrangian with a non-linear function of the scalar curvature. A remarkable consequence of the non-linearity of the gravitational Lagrangian is that torsion can be non-zero even without the presence of spin, as long as the trace of the matter stress–energy tensor is not constant [17–21]. This is a noticeable difference with respect to ECSK theory, where instead torsion can exist only coupled to spin. It is known that torsion may give rise to singularity-free and accelerated cosmological models [22], and a torsion arising from the non-linearity of the gravitational Lagrangian function could amplify this effects and make them possible even in the absence of spin. This is a feature that makes $f(R)$-gravity with torsion interesting enough to be studied in depth.

Of course, in order for any physical theory to be viable, it has to possess an associated initial value problem correctly formulated in such a way that the dynamical evolution is uniquely determined and consistent with causality requirements. More specifically, the following properties have to hold: (i) small perturbations of the initial data have to generate small perturbations in the subsequent dynamics; (ii) changes of the initial data have to preserve the causal structure of the theory. The initial value problem of the theory is well-posed if both these requests are satisfied.

It is well known that GR has a well-posed initial value problem, so resulting in a stable theory with a robust causal structure [23–26]. In order to be considered as a viable extension of the Einstein theory, $f(R)$-gravity should also have such a feature.

About this, by taking advantage of the dynamical equivalence with O'Hanlon theories [27], it is easily seen that purely metric $f(R)$-gravity possesses a well-posed Cauchy problem [28] regardless of the explicit form of the function $f(R)$.

As far as the theory with torsion is concerned, the issue is quite simple whenever the trace of the stress–energy tensor is constant: in this circumstance and in the absence of matter spin sources, in fact, the theory is equivalent to GR with or without a cosmological constant, depending on the explicit expression of the function $f(R)$. For instance, this is what happens in vacuo and in the case of coupling to electromagnetic or Yang–Mills fields. Instead, the coupling to other kinds of matter sources must be discussed carefully case by case. Here, we face the Cauchy problem in the presence of a perfect fluid or a Klein–Gordon scalar field. Making use of some different techniques, such as conformal transformations and dynamic equivalence with scalar-tensor theories, we formulate sufficient conditions to ensure that the related Cauchy problem is well-posed, also showing that there exist $f(R)$ functions that actually satisfy these requirements. The so-stated conditions can be adopted as a selection rule for viable $f(R)$-models with torsion.

Another important mathematical aspect concerning every theory of gravitation is related to the problem of matching different spacetimes like, for instance, joining together the interior with the exterior region of a relativistic stars. The requirements which have to be fulfilled to solder two different spacetimes are commonly known as junction conditions.

In GR, junction conditions have been investigated by different authors, including Lichnerowicz [29,30], Taub [31], Choquet–Bruhat [32] and Israel [33], and the solution of the problem is now very well known. In [34], the reader can find a very clear discussion about the topic.

On the contrary, at least in the authors' knowledge, very few works deal with junction conditions in ECSK theory: an analysis has been performed by Arkuszewski et al. [35], by means of the formalism of tensor–valued differential forms [36–38], while the same topic has been indirectly addressed by Bressange [39] following the same approach as in [34]. Concerning $f(R)$-gravity in purely metric formulation, a discussion of junction conditions has been proposed by Deruelle et al. [40] and Senovilla [41].

In this paper, we address the topic within the theory with torsion, analyzing the junction conditions for $f(R)$-gravity with torsion. Borrowing arguments and notations from [34], after formulating the junction conditions, we discuss their explicit form in the case of coupling to a Dirac field and a spin fluid. As we shall see, the resulting junction conditions are very similar to those existing in ECSK theory. However, this close similarity is only formal. Indeed, due to the contributions that the non linearity of the gravitational Lagrangian function $f(R)$ gives to the contortion tensor, the obtained junction conditions are seen to involve also the trace of the stress–energy tensor and its first derivatives evaluated on the separation hypersurface. This is a remarkable difference with respect to the ECSK theory, which translates into conditions also concerning the function $f(R)$. Therefore, as in the case of the Cauchy problem, the study of the junction conditions can help to distinguish viable from nonviable $f(R)$-models with torsion.

The layout of the paper is the following: In Section 2, we illustrate some generalities about $f(R)$-gravity with torsion. In Section 3, we address the Cauchy problem. In Section 4, we discuss the junction conditions. Finally, we devote Section 5 to conclusions. Throughout the paper, we use natural units ($\hbar = c = 8\pi G = 1$).

2. $f(R)$-Gravity with Torsion

In $f(R)$-gravity with torsion, the (gravitational) dynamical fields are given by a pseudo-Riemannian metric g and a metric compatible linear connection Γ, defined on the space-time manifold M. The covariant derivative induced by connection Γ is given by:

$$\nabla_{\partial_i}\partial_j = \Gamma_{ij}{}^h\,\partial_h. \tag{1}$$

The torsion and Riemann curvature tensors, induced by the dynamical connection Γ, are expressed as:

$$T_{ij}{}^h = \Gamma_{ij}{}^h - \Gamma_{ji}{}^h, \tag{2a}$$

$$R^h{}_{kij} = \partial_i\Gamma_{jk}{}^h - \partial_j\Gamma_{ik}{}^h + \Gamma_{ip}{}^h\Gamma_{jk}{}^p - \Gamma_{jp}{}^h\Gamma_{ik}{}^p. \tag{2b}$$

In view of the metric compatibility, the linear connection Γ can be decomposed as [14,15]:

$$\Gamma_{ij}{}^h = \tilde{\Gamma}_{ij}{}^h - K_{ij}{}^h, \tag{3}$$

where:

$$K_{ij}{}^h := \frac{1}{2}\left(-T_{ij}{}^h + T_j{}^h{}_i - T^h{}_{ij}\right) \tag{4}$$

is the so-called contorsion tensor, and $\tilde{\Gamma}_{ij}{}^h$ is the Levi–Civita connection induced by the metric g.

The field equations are obtained by varying an action functional of the form:

$$\mathcal{A}(g,\Gamma) = \int \left(\sqrt{|g|}f(R) + \mathcal{L}_m\right)\,ds, \tag{5}$$

where $R(g,\Gamma) = g^{ij}R_{ij}$ (with $R_{ij} := R^h{}_{ihj}$) denotes the scalar curvature associated with the connection Γ. The field equations result in [18–20]:

$$f'(R)R_{ij} - \frac{1}{2}f(R)g_{ij} = \mathcal{T}_{ij} \tag{6a}$$

and:

$$T_{ij}{}^h = \frac{1}{2f'}\left(\frac{\partial f'}{\partial x^p} + \mathcal{S}_{pq}{}^q\right)\left(\delta^p_j\delta^h_i - \delta^p_i\delta^h_j\right) + \frac{1}{f'}\mathcal{S}_{ij}{}^h, \tag{6b}$$

where $\mathcal{T}_{ij}$ and $\mathcal{S}_{ij}{}^h$ denote the stress–energy and the spin density tensors, respectively. In Equation (6a), attention must be paid to the order of the indexes, because the Ricci and stress–energy tensors R_{ij} and $\mathcal{T}_{ij}$ are not symmetric, in general.

It is worth noticing that, due to the independence between the metric tensor g_{ij} and the dynamical linear connection $\Gamma_{ij}{}^h$, the variation of the action functional (5) with respect to the metric tensor does not generate in Equation (6a) any term containing covariant derivatives of the scalar $f'(R)$ (for details, see [18]); this is a remarkable difference with respect to the purely metric formulation of $f(R)$-gravity [2], and it has important consequences: for instance, the theory with torsion is not of fourth derivative order, as is the purely metric $f(R)$-theory. Taking the trace of Equation (6a) into account, we get relation:

$$f'(R)R - 2f(R) = \mathcal{T} \tag{7}$$

between the curvature scalar R and the trace $\mathcal{T}$ of the stress–energy tensor.

From Equation (7), it is seen that if the trace $\mathcal{T}$ is constant, so R is. Of course, the same conclusion holds when $\mathcal{T}_{ij} = 0$. In such circumstances, the field equations of $f(R)$-gravity with torsion are seen to amount to the ones of Einstein–Cartan theory with (or without) cosmological constant if spin is present, or the ones of Einstein theory with (or without) cosmological constant in the absence of spin. This holds in general, with the exception of the particular case $\mathcal{T} = 0$ and $f(R) = \alpha R^2$. In such a case, indeed, Equation (7) is a trivial identity, and it does not impose any restriction on the scalar curvature R.

Therefore, from now on, we shall systematically suppose that $\mathcal{T}_{ij}$ is not zero and $\mathcal{T}$ is not constant, as well as that the relation (7) is invertible. In this way, the curvature scalar R can be thought as a suitable function of $\mathcal{T}$, namely:

$$R = R(\mathcal{T}). \tag{8}$$

The relation (8) plays a crucial role for the formulation of $f(R)$-gravity with torsion presented in this paper, as well as in our previous works. About this, it is worth noticing that the trace equation (7) gives rise to an algebraic or transcendental relation between the curvature scalar and the stress–energy tensor trace, but it is not a differential relation (unlike what happens in the purely metric formulation of $f(R)$-gravity). Therefore, the Dini theorem is generally applicable, and the relation (8) can be (almost always) supposed to locally exist. This allows us to express the torsion as a function of the matter fields and, therefore, to separate purely metric contributions from torsional ones within the Einstein-like equations, exactly as it happens in ECSK theory.

Defining the scalar field:

$$\varphi(\mathcal{T}) := f'(R(\mathcal{T})), \tag{9}$$

we can rewrite Equation (6a) in the equivalent form:

$$R_{ij} - \frac{1}{2}Rg_{ij} = \frac{1}{\varphi}\mathcal{T}_{ij} + \frac{1}{2\varphi}\left(f(R(\mathcal{T})) - f'(R(\mathcal{T}))R(\mathcal{T})\right)g_{ij}, \tag{10a}$$

$$T_{ij}{}^{h} = \frac{1}{2\varphi}\left(\frac{\partial\varphi}{\partial x^p} + \mathcal{S}_{pq}{}^{q}\right)\left(\delta^p_j\delta^h_i - \delta^p_i\delta^h_j\right) + \frac{1}{\varphi}\mathcal{S}_{ij}{}^{h}, \tag{10b}$$

which will be used in the following discussion. Making use of Equations (3), (4) and (10b), we can express the contorsion tensor as:

$$K_{ij}{}^{h} = \hat{K}_{ij}{}^{h} + \hat{S}_{ij}{}^{h}, \tag{11a}$$

$$\hat{S}_{ij}{}^{h} := \frac{1}{2\varphi}\left(-\mathcal{S}_{ij}{}^{h} + \mathcal{S}_{j}{}^{h}{}_{i} - \mathcal{S}^{h}{}_{ij}\right), \tag{11b}$$

$$\hat{K}_{ij}{}^{h} := -\hat{T}_j\delta^h_i + \hat{T}_p g^{ph} g_{ij}, \tag{11c}$$

$$\hat{T}_j := \frac{1}{2\varphi}\left(\frac{\partial\varphi}{\partial x^j} + \mathcal{S}_{jk}{}^{k}\right). \tag{11d}$$

Introducing the so-called torsion vector $T_i := T_{ij}{}^{j}$, we also mention the conservation laws [42]:

$$\nabla_a\mathcal{T}^{ai} + T_a\mathcal{T}^{ai} - \mathcal{T}_{ca}T^{ica} - \frac{1}{2}\mathcal{S}_{sta}R^{stai} = 0, \tag{12a}$$

$$\nabla_h\mathcal{S}^{ijh} + T_h\mathcal{S}^{ijh} + \mathcal{T}^{ij} - \mathcal{T}^{ji} = 0, \tag{12b}$$

which have to be satisfied by the stress–energy and spin density tensors of the matter fields. In particular, we recall that Equation (12b) amount to the antisymmetric part of the Einstein-like Equation (10a).

In the case that the spin density tensor is zero, separating the Levi–Civita terms from the torsional ones, we can rewrite the Einstein-like field Equation (10a) in the form [18]:

$$\tilde{R}_{ij} - \frac{1}{2}\tilde{R}g_{ij} = \frac{1}{\varphi}\mathcal{T}_{ij} + \frac{1}{\varphi^2}\left(-\frac{3}{2}\frac{\partial\varphi}{\partial x^i}\frac{\partial\varphi}{\partial x^j} + \varphi\tilde{\nabla}_j\frac{\partial\varphi}{\partial x^i} + \frac{3}{4}\frac{\partial\varphi}{\partial x^h}\frac{\partial\varphi}{\partial x^k}g^{hk}g_{ij} \right.$$
$$\left. -\varphi\tilde{\nabla}^h\frac{\partial\varphi}{\partial x^h}g_{ij} - V(\varphi)g_{ij},\right), \tag{13}$$

where the effective potential for the scalar field φ:

$$V(\varphi) := \frac{1}{4}\left[\varphi F^{-1}((f')^{-1}(\varphi)) + \varphi^2(f')^{-1}(\varphi)\right], \tag{14}$$

has been introduced. In Equation (13), $\tilde{R}_{ij}$, $\tilde{R}$, and $\tilde{\nabla}$ denote, respectively, the Ricci tensor, the scalar curvature, and the covariant derivative associated with the Levi–Civita connection of the dynamical metric g_{ij}.

The Einstein-like Equation (13) (together with Equation (9)) are deducible from a scalar-tensor theory with Brans–Dicke parameter $\omega_0 = -3/2$. This can be seen by recalling the action functional of a (purely metric) scalar-tensor theory:

$$\mathcal{A}(g,\varphi) = \int\left[\sqrt{|g|}\left(\varphi\tilde{R} - \frac{\omega_0}{\varphi}\varphi_i\varphi^i - U(\varphi)\right) + \mathcal{L}_m\right] ds, \tag{15}$$

where φ is the scalar field, $\varphi_i := \frac{\partial\varphi}{\partial x^i}$ and $U(\varphi)$ is the potential of φ, $\mathcal{L}_m(g_{ij},\psi)$ is the matter Lagrangian, function of the metric and some other matter fields ψ, and ω_0 is the so called Brans–Dicke parameter. By varying (15) with respect to the metric tensor and the scalar field, one gets the field equations:

$$\tilde{R}_{ij} - \frac{1}{2}\tilde{R}g_{ij} = \frac{1}{\varphi}\mathcal{T}_{ij} + \frac{\omega_0}{\varphi^2}\left(\varphi_i\varphi_j - \frac{1}{2}\varphi_h\varphi^h g_{ij}\right) + \frac{1}{\varphi}\left(\tilde{\nabla}_j\varphi_i - \tilde{\nabla}_h\varphi^h g_{ij}\right) - \frac{U}{2\varphi}g_{ij}, \tag{16}$$

and:

$$\frac{2\omega_0}{\varphi}\tilde{\nabla}_h\varphi^h + \tilde{R} - \frac{\omega_0}{\varphi^2}\varphi_h\varphi^h - U' = 0, \tag{17}$$

where $\mathcal{T}_{ij} := -\frac{1}{\sqrt{|g|}}\frac{\delta\mathcal{L}_m}{\delta g^{ij}}$ and $U' := \frac{dU}{d\varphi}$. By inserting the trace of Equation (16) into Equation (17), one gets the equation:

$$(2\omega_0 + 3)\,\tilde{\nabla}_h\varphi^h = \mathcal{T} + \varphi U' - 2U. \tag{18}$$

A direct comparison immediately shows that, for $\omega_0 := -\frac{3}{2}$ and $U(\varphi) := \frac{2}{\varphi}V(\varphi)$ (where $V(\varphi)$ is defined in Equation (14)), Equation (16) becomes formally identical to the Einstein-like Equation (13) for $f(R)$-gravity with torsion. Moreover, in such a circumstance, Equation (18) reduces to the algebraic equation:

$$\mathcal{T} + 2V'(\varphi) - \frac{6}{\varphi}V(\varphi) = 0, \tag{19}$$

relating the matter trace $\mathcal{T}$ to the scalar field φ. In particular, it is easily seen that, under the condition $f'' \neq 0$, Equation (19) represents exactly the inverse relation of (9), namely:

$$\mathcal{T} + 2V'(\varphi) - \frac{6}{\varphi}V(\varphi) = 0 \quad\Longleftrightarrow\quad \mathcal{T} = F^{-1}((f')^{-1}(\varphi)), \tag{20}$$

being $F^{-1}(X) = f'(X)X - 2f(X)$. The conclusion follows that, when the matter Lagrangian does not depend on the dynamical connection (the dynamical connection does not couple with matter), $f(R)$-gravity with torsion is dynamically equivalent to a scalar-tensor theory with a Brans–Dicke parameter $\omega_0 = -\frac{3}{2}$.

For later use, we also notice that field equations (13) can be simplified by rewriting them the Einstein frame. In fact, performing the conformal transformation:

$$\bar{g}_{ij} = \varphi g_{ij}. \tag{21}$$

Equation (13) assumes the simpler form (see for example [18,43]):

$$\bar{R}_{ij} - \frac{1}{2}\bar{R}\bar{g}_{ij} = \frac{1}{\varphi}\mathcal{T}_{ij} - \frac{1}{\varphi^3}V(\varphi)\bar{g}_{ij}, \tag{22}$$

where $\bar{R}_{ij}$ and $\bar{R}$ are, respectively, the Ricci tensor and the curvature scalar induced by the conformal metric $\bar{g}_{ij}$.

The relationships between the the conservation laws existing in the Jordan and those holding in the Einstein frame are clarified by the following results [44,45]:

Proposition 1. *Equations (13), (14), (19) imply the standard conservation laws* $\tilde{\nabla}^j \mathcal{T}_{ij} = 0$.

Proposition 2. *The condition* $\tilde{\nabla}^j \mathcal{T}_{ij} = 0$ *is equivalent to the condition* $\bar{\nabla}^j \bar{\mathcal{T}}_{ij} = 0$, *where* $\bar{\mathcal{T}}_{ij} := \frac{1}{\varphi}\mathcal{T}_{ij} - \frac{1}{\varphi^3}V(\varphi)\bar{g}_{ij}$ *and* $\bar{\nabla}$ *denotes the covariant derivative associated to the conformal metric* $\bar{g}_{ij}$.

3. The Cauchy Problem

First of all, we notice that, if the trace of the stress–energy tensor $\mathcal{T}$ is constant, $f(R)$-gravity with torsion reduces to GR with (or without) cosmological constant. Therefore, when this is the case, the Cauchy problem is well-formulated and well-posed [23,26,46]. For instance, this happens in vacuo and in the presence of electromagnetic (or also Yang–Mills) fields (if $f(R) \neq \alpha R^2$).

The situation is more complicated if $\mathcal{T} \neq$ const.: in such a circumstance, the theory no longer amounts to GR, so the classical Bruhat results [46] do not apply, and the well-formulation and well-posedness of the Cauchy problem is not automatically assured.

To overcome this issue, the Cauchy problem could be addressed by exploiting the dynamical equivalence with scalar-tensor theories with Brans–Dicke parameter $\omega_0 = -\frac{3}{2}$. Unfortunately, here a difficulty occurs: The d'Alembertian $g^{pq}\nabla_p\nabla_q\varphi$ disappears from Equation (18), and we no longer have the possibility of deriving the expression of the d'Alembertian as a function of the dynamical variables and their derivatives up to the first-order. In other words, we cannot eliminate the second-order derivatives of the scalar field φ from the Einstein-like Equation (16).

An alternative idea is to pass from the Jordan to the Einstein frame, making use of the conformal transformation technique. Following this approach, we can derive sufficient conditions for the well-posedness of the Cauchy problem for $f(R)$-gravity with torsion in the presence of a perfect fluid [44,47] or a Klein–Gordon scalar field [48]. Such conditions result in suitable requirements imposed on the function $f(R)$, so they can be assumed as a sort of selection rule for viable $f(R)$ models. In addition, we show that the function $f(R) = R + \alpha R^2$ satisfies the above mentioned conditions, in such a way that the set of viable $f(R)$ models is not empty.

3.1. The Cauchy Problem in Presence of a Perfect Fluid

We discuss the Cauchy problem in presence of a perfect fluid. As mentioned above, implementing a conformal transformation (21), we show that the initial value problem can be analyzed by applying the same results stated in [23,24,46] for GR.

To start with, given the metric g_{ij} of signature $(-+++)$ in the Jordan frame, let us consider a perfect fluid having stress–energy tensor of the form:

$$\mathcal{T}_{ij} = (\rho + p)\, U_i U_j + p\, g_{ij} \tag{23a}$$

and undergoing the usual conservation laws:

$$\tilde{\nabla}_j \mathcal{T}^{ij} = 0. \tag{23b}$$

In Equation (23a), ρ and p are the matter–energy density and the pressure of the fluid, respectively, while U_i denotes the components of the four velocity of the fluid (with $g^{ij}U_iU_j = -1$). Performing the conformal transformation (21), we rewrite the field equations in the Einstein frame as:

$$\bar{R}_{ij} - \frac{1}{2}\bar{R}\bar{g}_{ij} = \bar{\mathcal{T}}_{ij}, \tag{24a}$$

and:

$$\bar{\nabla}_j \bar{\mathcal{T}}^{ij} = 0, \tag{24b}$$

where:

$$\bar{\mathcal{T}}_{ij} = \frac{1}{\varphi}(\rho + p)\,U_iU_j + \left(\frac{p}{\varphi^2} - \frac{V(\varphi)}{\varphi^3}\right)\bar{g}_{ij}, \tag{25}$$

plays the role of the effective stress–energy tensor.

In view of Proposition 2, Equation (24b) is equivalent to Equation (23b). This is a crucial aspect for our discussion, allowing us to apply to the present case the results stated in [23,24,46]. To see this point, we first suppose that the scalar field φ is positive, that is $\varphi > 0$. The opposite case $\varphi < 0$ differs from the former only for some technical aspects, and it will be briefly outlined after. Of course, it is implicitly assumed that $\varphi \neq 0$ at least in a neighborhood of the initial space-like surface.

Under the assumed conditions, the stress–energy tensor (25) can be rewritten in the form:

$$\bar{\mathcal{T}}_{ij} = \frac{1}{\varphi^2}(\rho + p)\,\bar{U}_i\bar{U}_j + \left(\frac{p}{\varphi^2} - \frac{V(\varphi)}{\varphi^3}\right)\bar{g}_{ij}, \tag{26}$$

where $\bar{U}_i = \sqrt{\varphi}U_i$ is the four velocity of the fluid in the Einstein frame. Introducing the effective mass-energy density:

$$\bar{\rho} := \frac{\rho}{\varphi^2} + \frac{V(\varphi)}{\varphi^3}, \tag{27a}$$

and the effective pressure:

$$\bar{p} := \frac{p}{\varphi^2} - \frac{V(\varphi)}{\varphi^3}, \tag{27b}$$

we can express the stress–energy tensor (26) in the standard form:

$$\bar{\mathcal{T}}_{ij} = (\bar{\rho} + \bar{p})\,\bar{U}_i\bar{U}_j + \bar{p}\,\bar{g}_{ij}. \tag{28}$$

We notice that, given an equation of state of the form $\rho = \rho(p)$ and assuming that the relation (27b) is invertible $(p = p(\bar{p}))$, by composition with Equation (27a), we obtain an effective equation of state $\bar{\rho} = \bar{\rho}(\bar{p})$. Moreover, we recall that the explicit expressions of the scalar field φ and the potential $V(\varphi)$ depend on the specific form of the function $f(R)$. As a consequence, the request that the relation (27b) is invertible together with the condition $\varphi > 0$ (or, equivalently, $\varphi < 0$) can be assumed as criteria for the viability of the functions $f(R)$, providing us with suitable selection rules for admissible gravitational Lagrangian function $f(R)$ (see also [9]).

After that, in order to discuss the Cauchy problem, we can follow step-by-step the Bruhat's arguments [23,24,46]. In particular, we recall that the Cauchy problem for the system of differential equations (24), with stress–energy tensor given by Equation (28) and equation of state $\bar{\rho} = \bar{\rho}(\bar{p})$, is well-posed if the condition:

$$\frac{d\bar{\rho}}{d\bar{p}} \geq 1, \tag{29}$$

is satisfied. The requirement (29) is easily verified by means of the relation:

$$\frac{d\bar{\rho}}{d\bar{p}} = \frac{d\bar{\rho}/dp}{d\bar{p}/dp} \geq 1, \tag{30}$$

together with expressions (27) and the equation of state $\rho = \rho(p)$. Once again, condition (30) depends on the expressions of φ and $V(\varphi)$; thus, it is strictly related to the form of the function $f(R)$. Therefore, condition (30) represents a further criterion for the admissibility of $f(R)$-models.

For the sake of completeness, we conclude by outlining the case $\varphi < 0$. Supposing again that the signature of the metric in the Jordan frame is $(-+++)$, the signature of the conformal metric is now $(+---)$, and the components of the four velocity of the fluid in the Einstein frame are $\bar{U}_i = \sqrt{-\varphi}U_i$. The effective stress–energy tensor is expressed as:

$$\bar{\mathcal{T}}_{ij} = -\frac{1}{\varphi^2}(\rho + p)\,\bar{U}_i\bar{U}_j + \left(\frac{p}{\varphi^2} - \frac{V(\varphi)}{\varphi^3}\right)\bar{g}_{ij} = (\bar{\rho} + \bar{p})\,\bar{U}_i\bar{U}_j - \bar{p}\,\bar{g}_{ij}, \tag{31}$$

where, as above, the quantities:

$$\bar{\rho} := -\frac{\rho}{\varphi^2} - \frac{V(\varphi)}{\varphi^3}, \tag{32a}$$

and:

$$\bar{p} := -\frac{p}{\varphi^2} + \frac{V(\varphi)}{\varphi^3}, \tag{32b}$$

represent the effective mass-energy and the effective pressure.

After that, everything proceeds again as in [23,24,46], with the exception of a technical aspect: if ρ and p are positive, the quantity $r := \bar{\rho} + \bar{p} = -\dfrac{\rho + p}{\varphi^2}$ is now negative. About this, the reader can easily verify that, with the choice $\log(-f^{-2}r)$ instead of $\log(f^{-2}r)$ as in [23,24,46], the Bruhat's arguments apply equally well.

As a simple example, we consider the model $f(R) = R + \alpha R^2$ coupled with dust. In the Jordan frame, the matter stress–energy tensor is given by $\mathcal{T}_{ij} = \rho U_i U_j$, and the trace of the Einstein-like Equation (6a) yields the relation:

$$(1 + 2\alpha R)R - 2R - 2\alpha R^2 = -\rho \qquad \Longleftrightarrow \qquad R = \rho. \tag{33}$$

The scalar field (9) assumes the form:

$$\varphi(\rho) = f'(R(\rho)) = 1 + 2\alpha\rho. \tag{34}$$

Taking into account small values of the density $\rho << 1$ (for instance, the present cosmological baryonic matter density) and choosing values of $|\alpha|$ not comparable with $1/\rho$, we can reasonably suppose $\varphi > 0$, independently of the sign of the parameter α. We have to calculate the potential (14):

$$V(\varphi) = \frac{1}{4}\left[\varphi F^{-1}((f')^{-1}(\varphi)) + \varphi^2(f')^{-1}(\varphi)\right]. \tag{35}$$

To this end, since $(f')^{-1}(\varphi) = \rho$, from Equation (34) we get the relation:

$$\frac{1}{4}\varphi^2(f')^{-1}(\varphi) = \frac{1}{4}(1 + 2\alpha\rho)^2\rho\,, \tag{36}$$

and considering that $F^{-1}(Y) = f'(Y)Y - 2f(Y)$, we have the identities:

$$\frac{1}{4}F^{-1}((f')^{-1}(\varphi)) = \frac{1}{4}F^{-1}(\rho) = -\rho \tag{37}$$

and:
$$\frac{1}{4}\varphi F^{-1}((f')^{-1}(\varphi)) = -\frac{(1+2\alpha\rho)\rho}{4}\,. \tag{38}$$

We conclude that:
$$V(\varphi(\rho)) = \frac{\alpha\rho^2(1+2\alpha\rho)}{2}. \tag{39}$$

In the Einstein frame, the stress–energy tensor (26) is expressed as:
$$\bar{\mathcal{T}}_{ij} = \frac{\rho}{\varphi^2}\bar{U}_i\bar{U}_j - \frac{V(\varphi)}{\varphi^3}\bar{g}_{ij}. \tag{40}$$

Tensor (40) can be considered as the stress–energy tensor of a perfect fluid with density and pressure given, respectively, by:
$$\bar{\rho} := \frac{\rho}{\varphi^2} + \frac{V(\varphi)}{\varphi^3} = \frac{2\rho + \alpha\rho^2}{2(1+2\alpha\rho)^2} \tag{41a}$$

and:
$$\bar{p} := -\frac{V(\varphi)}{\varphi^3} = -\frac{\alpha\rho^2}{2(1+2\alpha\rho)^2}. \tag{41b}$$

It is an easy matter to verify that the function (41b) is invertible. Indeed, for $\rho > 0$, one has:
$$\frac{d\bar{p}}{d\rho} = -\frac{4\alpha\rho}{4(1+2\alpha\rho)^3} \neq 0. \tag{42}$$

In addition, we have:
$$\frac{d\bar{\rho}}{d\rho} = \frac{4-4\alpha\rho}{4(1+2\alpha\rho)^3}, \tag{43}$$

so that:
$$\frac{d\bar{\rho}}{d\bar{p}} = \frac{d\bar{\rho}/d\rho}{d\bar{p}/d\rho} = \frac{-1+\alpha\rho}{\alpha\rho} \geq 1 \quad \Longleftrightarrow \quad \alpha < 0 \tag{44}$$

With condition (29) satisfied, it is then proved that the model $f(R) = R + \alpha R^2$, with $\alpha < 0$, possesses a well-posed Cauchy problem when coupled with dust.

3.2. The Cauchy Problem in the Presence of a Scalar Field

We take the coupling with a Klein–Gordon scalar field into account. Again, we give sufficient conditions for the well-posedness of the related Cauchy problem. To this end, let us denote by ψ a Klein–Gordon scalar field with self-interacting potential $U(\psi) = \frac{1}{2}m^2\psi^2$. The corresponding stress–energy tensor is given by:
$$\mathcal{T}_{ij} = \frac{\partial\psi}{\partial x^i}\frac{\partial\psi}{\partial x^j} - \frac{1}{2}g^{ij}\left(\frac{\partial\psi}{\partial x^p}\frac{\partial\psi}{\partial x^q}g^{pq} + m^2\psi^2\right). \tag{45}$$

The associated Klein–Gordon equation is expressed as:
$$\tilde{\nabla}_j\frac{\partial\psi}{\partial x^i}g^{ij} = m^2\psi, \tag{46}$$

where $\tilde{\nabla}$ denotes the Levi–Civita covariant derivative induced by the metric g_{ij}. The trace of tensor (45) is:
$$\mathcal{T} := \mathcal{T}_{ij}g^{ij} = -\frac{\partial\psi}{\partial x^p}\frac{\partial\psi}{\partial x^q}g^{pq} - 2m^2\psi^2\,. \tag{47}$$

The trace (47) depends explicitly on the metric tensor g_{ij}. Because of this, the conformal transformation cannot be applied directly to the field equations (13), with the scalar field φ defined by (9). Indeed, if we proceed in this way, both the metric g_{ij} and $\bar{g}_{ij}$ would appear in the conformally transformed equations (22). This difficulty can be overcome making use of the already mentioned dynamical equivalence with $\omega_0 = -\frac{3}{2}$ Brans–Dicke gravity. The idea is then to discuss the Cauchy problem for a $\omega_0 = -\frac{3}{2}$ Brans–Dicke theory coupled with the given Klein–Gordon field ψ. The field equations of such a theory are the Einstein-like Equation (13), the Equation (19), and the Klein–Gordon Equation (46), where the scalar field φ is a dynamical variable related to the trace $\mathcal{T}$ through Equation (19). After implementing the conformal transformation (21), the Einstein-like Equation (13) assumes the simpler form (22). At the same time, recalling the relation:

$$\bar{\Gamma}_{ij}{}^{h} = \tilde{\Gamma}_{ij}{}^{h} + \frac{1}{2\varphi}\frac{\partial\varphi}{\partial x^j}\delta^h_i - \frac{1}{2\varphi}\frac{\partial\varphi}{\partial x^p}g^{ph}g_{ij} + \frac{1}{2\varphi}\frac{\partial\varphi}{\partial x^i}\delta^h_j, \tag{48}$$

linking the Levi–Civita connection $\tilde{\Gamma}_{ij}{}^{h}$ associated with the metric g_{ij} to the Levi–Civita connection $\bar{\Gamma}_{ij}{}^{h}$ induced by the conformal metric $\bar{g}_{ij}$, we can write the Klein–Gordon equation in terms of the conformal metric $\bar{g}_{ij}$ as:

$$-\frac{\partial\psi}{\partial x^i}\bar{g}^{ij}\frac{\partial\varphi}{\partial x^j} + \varphi\bar{\nabla}_j\frac{\partial\psi}{\partial x^i}\bar{g}^{ij} = m^2\psi, \tag{49}$$

where $\bar{\nabla}_j$ indicates the covariant derivative associated with the conformal metric $\bar{g}_{ij}$. Analogously, we can express the trace $\mathcal{T}$ as function of $\bar{g}_{ij}$, that is:

$$\mathcal{T} = -\frac{\partial\psi}{\partial x^p}\frac{\partial\psi}{\partial x^q}\varphi\bar{g}^{pq} - 2m^2\psi^2\,. \tag{50}$$

The relation corresponding to (19) now links the scalar field φ to the Klein–Gordon field ψ, its partial derivatives $\frac{\partial\psi}{\partial x^i}$, and the conformal metric $\bar{g}_{ij}$. Moreover, as it has been already pointed out, the quantity:

$$\bar{\mathcal{T}}_{ij} := \frac{1}{\varphi}\Sigma_{ij} - \frac{1}{\varphi^3}V(\varphi)\bar{g}_{ij}, \tag{51}$$

represents an effective stress–energy tensor. On the other hand, the Klein–Gordon equation (46) implies the conservation laws $\tilde{\nabla}^j\mathcal{T}_{ij} = 0$, thus also identifying $\bar{\nabla}^j\bar{\mathcal{T}}_{ij} = 0$ (see Proposition 2). This is a key point, allowing us to making use of harmonic coordinates and then to apply similar arguments as in [23,24,26].

More specifically, after rewriting the Einstein-like equations (22) in the equivalent form:

$$\bar{R}_{ij} = \bar{\mathcal{T}}_{ij} - \frac{1}{2}\bar{\mathcal{T}}\bar{g}_{ij}, \tag{52}$$

we adopt harmonic coordinates obeying the condition:

$$\bar{\nabla}_p\bar{\nabla}^p x^i = -\bar{g}^{pq}\bar{\Gamma}^i_{pq} = 0\,, \tag{53}$$

in such a way that equations (52) can be expresed as (see, for example, [23,26]):

$$\bar{g}^{pq}\frac{\partial^2\bar{g}_{ij}}{\partial x^p\partial x^q} = f_{ij}(\bar{g},\partial\bar{g},\psi,\partial\psi), \tag{54}$$

where f_{ij} are suitable functions depending only on the metric $\bar{g}$, the scalar field ψ, and their first order derivatives.

In addition to this, we suppose that Equation (19) is solvable with respect to the variable φ, and then to derive from Equation (19) itself a function of the form:

$$\varphi = \varphi(\bar{g}, \psi, \frac{\partial\psi}{\partial x^p}\frac{\partial\psi}{\partial x^q}\bar{g}^{pq}), \tag{55}$$

expressing the scalar field φ as a suitable function of the metric $\bar{g}$, the Klein–Gordon field ψ, and its first order derivatives. We notice that, in view of Equation (50), the dependence of φ on the derivatives of ψ is necessarily of the form indicated in Equation (55). Once again, the solvability with respect the scalar field φ to about Equation (19) depends on the explicit form of the potential $V(\varphi)$ which is defined in terms of the function $f(R)$ via the relation (14). Therefore, the possibility of solving Equation (19) with respect to φ can be taken as a rule to select viable $f(R)$-models. Moreover, from Equation (55), we obtain the identity:

$$\frac{\partial\varphi}{\partial x^i} = \frac{\partial\varphi}{\partial\left(\frac{\partial\psi}{\partial x^s}\frac{\partial\psi}{\partial x^t}\bar{g}^{st}\right)} 2\frac{\partial\psi}{\partial x^q}\bar{g}^{pq}\frac{\partial^2\psi}{\partial x^i\partial x^p} + f_i(\bar{g}, \partial\bar{g}, \psi, \partial\psi). \tag{56}$$

Inserting Equation (56) in Equation (49) and taking Equation (53) into account, we get the final form of the Klein–Gordon equation expressed as:

$$\left(\bar{g}^{ip} - \frac{2}{\varphi}\frac{\partial\varphi}{\partial\left(\frac{\partial\psi}{\partial x^s}\frac{\partial\psi}{\partial x^t}\bar{g}^{st}\right)}\frac{\partial\psi}{\partial x^j}\bar{g}^{ji}\frac{\partial\psi}{\partial x^q}\bar{g}^{pq}\right)\frac{\partial^2\psi}{\partial x^i\partial x^p} = f(\bar{g}, \partial\bar{g}, \psi, \partial\psi). \tag{57}$$

In Equations (56) and (57), f_i and f denote suitable functions of $\bar{g}_{ij}$, ψ, and their first order derivatives only.

Now, Equations (54) and (57) form a second order quasi-diagonal system of partial differential equations for the unknowns $\bar{g}_{ij}$ and ψ. The matrix of the principal parts of such a system is diagonal, and its elements are the differential operators:

$$\bar{g}^{pq}\frac{\partial^2}{\partial x^p\partial x^q}, \tag{58a}$$

and:

$$\left(\bar{g}^{ip} - \frac{2}{\varphi}\frac{\partial\varphi}{\partial\left(\frac{\partial\psi}{\partial x^s}\frac{\partial\psi}{\partial x^t}\bar{g}^{st}\right)}\frac{\partial\psi}{\partial x^j}\bar{g}^{ji}\frac{\partial\psi}{\partial x^q}\bar{g}^{pq}\right)\frac{\partial^2}{\partial x^i\partial x^p}. \tag{58b}$$

The operator (58a) is the wave operator associated with the metric $\bar{g}_{ij}$, while the operator (58b) is very similar to the sound wave operator involved in the analysis of the Cauchy problem for GR coupled with an irrotational perfect fluid [23,46]. It follows that the Cauchy problem associated with the system of Equations (54) and (57) can be discussed borrowing arguments and results from [23,46]. More in detail, we recall that if the quadratic form associated with (58b) is of Lorentzian signature and, if the characteristic cone of the operator (58b) is exterior to the metric cone, the system (54) and (57) is causal and Leray hyperbolic [49,50]. Under these conditions, the associated Cauchy problem is well-posed in suitable Sobolev spaces. Still borrowing from [23,46], if the signature of $\bar{g}_{ij}$ is $(+---)$, the above mentioned conditions are satisfied whenever the vector $\frac{\partial\psi}{\partial x^j}\bar{g}^{ij}$ is timelike and the inequality:

$$-\frac{2}{\varphi}\frac{\partial\varphi}{\partial\left(\frac{\partial\psi}{\partial x^s}\frac{\partial\psi}{\partial x^t}\bar{g}^{st}\right)} \geq 0, \tag{59}$$

holds. Of course, when the signature of the metric $\bar{g}_{ij}$ is $(-+++)$, the sign of inequality (59) has to be inverted. As it has been already remarked, the function (55) depends on the potential (14), which is determined by the explicit form of the function $f(R)$. Therefore, we can adopt requirement (59) as a criterion to single out viable $f(R)$-models with torsion.

As an illustrative example, we consider again the model $f(R) = R + \alpha R^2$. From the relation $F^{-1}(X) = f'(X)X - 2f(X) = -X$, the identity $(f')^{-1}(\varphi) = \dfrac{\varphi - 1}{2\alpha}$, and the expression (14), we easily obtain the effective potential:

$$V(\varphi) = \frac{1}{8\alpha}(\varphi - 1)^2\varphi. \tag{60}$$

Equation (60), together with Equations (19) and (50), yields:

$$\varphi = \frac{\left(\frac{1}{2\alpha} + 2m^2\psi^2\right)}{\left(\frac{1}{2\alpha} - \frac{\partial\psi}{\partial x^s}\frac{\partial\psi}{\partial x^t}\bar{g}^{st}\right)}, \tag{61}$$

which describes the scalar field φ as a function of the metric $\bar{g}_{ij}$, the Klein–Gordon field ψ, and its first order derivatives. By deriving (61), we have:

$$\frac{\partial\varphi}{\partial\left(\frac{\partial\psi}{\partial x^s}\frac{\partial\psi}{\partial x^t}\bar{g}^{st}\right)} = \frac{\varphi}{\left(\frac{1}{2\alpha} - \frac{\partial\psi}{\partial x^s}\frac{\partial\psi}{\partial x^t}\bar{g}^{st}\right)}. \tag{62}$$

If the metric $\bar{g}_{ij}$ has signature $(+ - - -)$, we see that the requirement (59) is fulfilled if $\alpha < 0$ and if $\bar{g}^{pq}\dfrac{\partial\psi}{\partial x^q}$ is a time-like vector field. On the contrary, when the signature of $\bar{g}_{ij}$ is $(- + + +)$, α has to be positive.

4. The Junction Conditions

In this section, we address the junction conditions issue within the framework of $f(R)$-gravity with torsion. As mentioned in the Introduction, the junction condition problem is crucial for any theory of gravitation; for instance, in order to join together the interior with the exterior region of a relativistic star, we need to know how matching different solutions of the field equations of the theory at a given hypersurface. After deriving general junction conditions, in order to highlight the main differences with respect to ECSK theory, we give two illustrative examples. For reasons of greater clarity and better readability, the proposed examples are presented in two separate subsections.

Let us consider a hypersurface Σ which separates two different regions $\mathcal{M}^+$ and $\mathcal{M}^-$ of spacetime. To begin with, let us deal with the case in which the hypersurface Σ is either timelike or spacelike; the case of null hypersurface will be discussed later. Let us denote by $\left(g^+_{ij}, \Gamma^{+\ h}_{ij}\right)$ and $\left(g^-_{ij}, \Gamma^{-\ h}_{ij}\right)$ two solutions of the field equations (10), defined in $\mathcal{M}^+$ and $\mathcal{M}^-$, respectively. We want to discuss how to solder together at Σ the two given Einstein–Cartan geometries, in order to obtain a unique solution of the field equations on the whole spacetime.

To this end, we refer Σ to local coordinates y^A $(A = 1, \ldots, 3)$, and we adopt a coordinate system x^i, locally overlapping both $\mathcal{M}^+$ and $\mathcal{M}^-$ in an open set containing Σ. After that, considering the arc length s between any point $p \in \mathcal{M}$ and Σ along the geodesic normal to Σ (with respect to one of the two given metric tensors) and passing through p itself, we define a function s which, without loss of generality, can be set negative in $\mathcal{M}^-$, positive in $\mathcal{M}^+$, and equal to zero at Σ. Indicating by n^i the unit normal (with respect to the chosen metric tensor) outgoing from Σ, one has the relations:

$$dx^i = n^i ds, \qquad n_i = \epsilon\partial_i s \qquad \text{and} \qquad n^i n_i = \epsilon, \tag{63}$$

where $\epsilon = 1$ if Σ is spacelike, and $\epsilon = -1$ if Σ is timelike. Moreover, given any geometric quantity W defined on both sides of the hypersurface Σ, we denote by:

$$[W] := W\left(\mathcal{M}^+\right)_{|\Sigma} - W\left(\mathcal{M}^-\right)_{|\Sigma} \tag{64}$$

the jump of W across Σ. The issue of matching different geometries at a given hypersurface Σ is usually discussed in the framework of distribution-valued tensors [29,30,32,51,52]. In this regard, denoting by $\Theta(s)$ (with $\Theta(0) := 1$) the Heaviside distribution, we introduce the following geometrical objects:

$$g_{ij} = \Theta(s) g^{+}_{ij} + (1 - \Theta(s))\, g^{-}_{ij}, \tag{65a}$$

$$\Gamma_{ij}{}^{h} = \Theta(s)\Gamma^{+}_{ij}{}^{h} + (1 - \Theta(s))\, \Gamma^{-}_{ij}{}^{h}, \tag{65b}$$

with the requirement that the quantities (65) define a solution of the field equations (10) in the distributional sense. To satisfy this request, the quantities (65) and all the the geometric quantities induced by them have to be well defined as distributions. In particular, this must apply to the Riemann and the Einstein tensors. Moreover, consistency between (65), (3) implies the identity:

$$\Gamma_{ij}{}^{h} = \Theta(s)\left(\tilde{\Gamma}^{+}_{ij}{}^{h} - K^{+}_{ij}{}^{h}\right) + [1 - \Theta(s)]\left(\tilde{\Gamma}^{-}_{ij}{}^{h} - K^{-}_{ij}{}^{h}\right), \tag{66}$$

where $\tilde{\Gamma}_{ij}{}^{h}$ are the Christoffel coefficients associated with the metric (65a). By differentiating (65), we get the relations:

$$\partial_k g_{ij} = \Theta(s)\partial_k g^{+}_{ij} + (1 - \Theta(s))\, \partial_k g^{-}_{ij} + \epsilon\delta(s)\, [g_{ij}]\, n_k, \tag{67a}$$

$$\partial_k \Gamma_{ij}{}^{h} = \Theta(s)\partial_k \Gamma^{+}_{ij}{}^{h} + (1 - \Theta(s))\, \partial_k \Gamma^{-}_{ij}{}^{h} + \epsilon\delta(s)\left[\Gamma_{ij}{}^{h}\right] n_k, \tag{67b}$$

where, referring the reader to [31,51,52] and references therein for the definition of the Dirac δ-function with support on the submanifold $\Sigma : s = 0$, we have used the identities $\frac{\partial s}{\partial x^i} = \epsilon n_i$ and $\frac{d\Theta(s)}{ds} = \delta(s)$.

Making use of Equation (67), as well as of the identities $\Theta^2(s) = \Theta(s)$ and $\Theta(s)\,(1 - \Theta(s)) = 0$, it is easily seen that the Levi–Civita contribution to the connection $\Gamma_{ij}{}^{h}$ contains a singular term having expression:

$$\frac{1}{2} g^{+hk}_{|\Sigma}\left([g_{ik}]\, n_j + \left[g_{jk}\right] n_i - [g_{ij}]\, n_k\right) \epsilon\delta(s). \tag{68}$$

Requirement (65b) implies then the vanishing of the term (68); thus,

$$[g_{ij}] = 0, \tag{69}$$

amounting to the fact that the two metrics have to coincide on the hypersurface Σ. In addition, from Equation (67b), we get the expression of the Riemann tensor of the the connection (65b):

$$R^{p}{}_{qij} = \Theta(s) R^{+p}{}_{qij} + (1 - \Theta(s))\, R^{-p}{}_{qij} + \delta(s) A^{p}{}_{qij}, \tag{70}$$

where we have denoted by:

$$A^{p}{}_{qij} := \epsilon\left(\left[\Gamma_{jq}{}^{p}\right] n_i - \left[\Gamma_{iq}{}^{p}\right] n_j\right) \tag{71}$$

the tensor connected with the presence of the δ-function term in the Riemann tensor (70). Once again, decomposition (3) can be used, so that we can rewrite the tensor (71) as the sum:

$$A^{p}{}_{qij} = \tilde{A}^{p}{}_{qij} + \bar{A}^{p}{}_{qij}, \tag{72}$$

where:

$$\tilde{A}^{p}{}_{qij} = \epsilon\left(\left[\tilde{\Gamma}_{jq}{}^{p}\right] n_i - \left[\tilde{\Gamma}_{iq}{}^{p}\right] n_j\right) \tag{73}$$

and:

$$\bar{A}^{p}{}_{qij} = \epsilon\left(\left[-K_{jq}{}^{p}\right] n_i + \left[K_{iq}{}^{p}\right] n_j\right) \tag{74}$$

are quantities related to the Levi–Civita and contortion, respectively.

The continuity of the metric tensor across the hypersurface Σ implies that its derivatives may have discontinuities only along the normal direction. Then, there exists a tensor field on Σ:

$$k_{ij} := \epsilon \left[\partial_h g_{ij}\right] n^h, \tag{75}$$

such that:

$$\left[\partial_h g_{ij}\right] = k_{ij} n_h. \tag{76}$$

From Equation (76), we get the expressions:

$$\left[\tilde{\Gamma}_{ij}{}^{h}\right] = \frac{1}{2}\left(k^h{}_j n_i + k^h{}_i n_j - k_{ij} n^h\right), \tag{77}$$

which, inserted into Equation (73), yield the explicit representation:

$$\tilde{A}^p{}_{qij} = \frac{\epsilon}{2}\left(k^p{}_j n_q n_i - k^p{}_i n_q n_j - k_{qj} n^p n_i + k_{qi} n^p n_j\right). \tag{78}$$

By contraction of Equation (78), we have:

$$\tilde{A}_{qj} := \tilde{A}^p{}_{qpj} = \frac{\epsilon}{2}\left(k^p{}_j n_q n_p - k n_q n_j - k_{qj}\epsilon + k_{qp} n^p n_j\right) \tag{79}$$

and:

$$\tilde{A} := \tilde{A}^q{}_q = \epsilon\left(k_{pq} n^p n^q - \epsilon k\right), \tag{80}$$

with $k := k_{ij} g^{ij}$. Making use of Equations (56), (80), we introduce the tensor:

$$\tilde{H}_{qj} = \tilde{A}_{qj} - \frac{1}{2}\tilde{A} g_{qj} = \frac{\epsilon}{2}\left(k^p{}_j n_q n_p - k n_q n_j - k_{qj}\epsilon + k_{qp} n^p n_j\right) - \frac{\epsilon}{2}\left(k_{st} n^s n^t - \epsilon k\right) g_{qj}, \tag{81}$$

which represents the δ-function part of the Einstein tensor, generated by Levi–Civita connection. Tensor (81) is symmetric and tangent to the hypersurface Σ. In fact, it is a straightforward matter to verify that $\tilde{H}_{qj} n^j = 0$. If we denote by $E^i_A := \frac{\partial x^i}{\partial y^A}$, the tensor $\tilde{H}_{qj}$ can be expressed as $\tilde{H}^{qj} = \tilde{H}^{AB} E^q_A E^j_B$, with [34]:

$$\tilde{H}_{AB} := \tilde{H}_{qj} E^q_A E^j_B = -\frac{1}{2} k_{qj} E^q_A E^j_B + \frac{1}{2} k_{pq} h^{pq} h_{AB}, \tag{82}$$

where $h^{pq} := g^{pq} - \epsilon n^p n^q$ and $h_{AB} := g_{ij} E^i_A E^j_B$ are the projection operator and the induced metric on the hypersurface Σ, respectively.

Analogously, we can single out the contributions given by contortion to the δ-function part of the Einstein tensor. By contraction, from Equation (74), we in fact:

$$\bar{A}_{qj} := \bar{A}^p{}_{qpj} = \epsilon\left(\left[-K_{jq}{}^{p}\right] n_p + \left[K_{pq}{}^{p}\right] n_j\right) \tag{83}$$

and:

$$\bar{A} := \bar{A}^q{}_q = 2\epsilon\left[K_{pq}{}^{p}\right] n^q. \tag{84}$$

By means of expressions (83), (84), we define the tensor:

$$\bar{H}_{qj} := \bar{A}_{qj} - \frac{1}{2}\bar{A} g_{qj} = \epsilon\left(\left[-K_{jq}{}^{p}\right] n_p + \left[K_{pq}{}^{p}\right] n_j\right) - \epsilon\left[K_{st}{}^{s}\right] n^t g_{qj}, \tag{85}$$

which, in general, is neither symmetric nor tangent to the hypersurface Σ. All the obtained results allow us to express the effective stress–energy tensor appearing on the right-hand side of Equation (10a) in the form:

$$\hat{\mathcal{T}}_{qj} = \Theta(s)\left[\frac{1}{\varphi}\mathcal{T}_{qj} + \frac{1}{\varphi}U(\mathcal{T})g_{qj}\right]^{+} + (1-\Theta(s))\left[\frac{1}{\varphi}\mathcal{T}_{qj} + \frac{1}{\varphi}U(\mathcal{T})g_{qj}\right]^{-} - \delta(s)H_{qj}, \tag{86}$$

where:

$$H_{qj} = \tilde{H}_{qj} + \bar{H}_{qj}, \tag{87}$$

and where, for simplicity, we have denoted by $U(\mathcal{T}) := \frac{1}{2}\left[f(R(\mathcal{T})) - f'(R(\mathcal{T}))R(\mathcal{T})\right]$.

From Equation (86), it follows that the request that the Einstein-like equations (10a) have a smooth transition across the hypersurface Σ is then equivalent to require that the tensor H_{qj} vanishes at Σ. Therefore, the remaining junction conditions can be obtained by imposing the vanishing of all projections of the tensor H_{qj} on Σ. About this, we have:

- the completely orthogonal projection of H_{qj} on Σ is automatically zero:

$$H_{qj}n^q n^j = \bar{H}_{qj}n^q n^j = -\epsilon\left[K_j{}^{qp}\right]n_p n_q n^j = 0 \tag{88}$$

  because $\tilde{H}_{qj}$ is tangent to Σ and the contorsion is antisymmetric in the last two indexes;
- the tangent–orthogonal projection of H_{qj} is:

$$H_{qj}E^q_A n^j = \bar{H}_{qj}E^q_A n^j = -\epsilon\left[K_{jq}{}^p\right]n_p E^q_A n^j + \left[K_{pq}{}^p\right]E^q_A. \tag{89}$$

  According to [35], the quantity in Equation (89) results in the jump of trace of the projection on Σ of the contorsion tensor. In fact, it is easily seen that the identity:

$$\left[K_{jq}{}^p h^j_i h^i_p E^q_A\right] = \left[K_{jq}{}^p\left(\delta^j_p - \epsilon n_p n^j\right)E^q_A\right] = -\epsilon\left[K_{jq}{}^p\right]n_p n^j E^q_A + \left[K_{pq}{}^p\right]E^q_A, \tag{90}$$

  holds.
- the orthogonal–tangent projection of H_{qj} is zero:

$$H_{qj}E^j_A n^q = \bar{H}_{qj}E^j_A n^q = \epsilon\left(-\left[K_{jq}{}^p\right]n_p n^q E^j_A + \left[K_{pq}{}^p\right]n_j E^j_A n^q\right) = 0, \tag{91}$$

  in view of the antisymmetry properties of the contorsion tensor and the orthogonality between the vectors n^i and E^i_A;
- the totally tangent projection of H_{qj} is given by:

$$H_{qj}E^q_A E^j_B = \tilde{H}_{qj}E^q_A E^j_B + \bar{H}_{qj}E^q_A E^j_B = \tilde{H}_{AB} + \epsilon\left(-\left[K_{jq}{}^p\right]n_p E^q_A E^j_B + \left[K_q{}^{qp}\right]n_p h_{AB}\right). \tag{92}$$

Summarizing everything, it is seen that the vanishing of the tensor H_{qj} needs the quantities (89), (92) to be zero at Σ. In particular, as it happens in GR, it can be shown that the condition $H_{qj}E^q_A E^j_B = 0$ is connected with the vanishing of the jump of the extrinsic curvature across Σ. To clarify this point, let us introduce the quantity:

$$Q_{AB} := \left(\nabla_i n_j\right)E^j_A E^i_B, \tag{93}$$

which generalizes the notion of extrinsic curvature for an arbitrary linear connection (3). From Equation (93) together with Equations (3) and (77), we get the relation:

$$[Q_{AB}] = \left[\nabla_i n_j E^j_A E^i_B\right] = \left[\nabla_i n_j\right]E^j_A E^i_B = \frac{\epsilon}{2}k_{ij}E^i_A E^j_B + \left[K_{ji}{}^h\right]n_h E^i_A E^j_B. \tag{94}$$

Comparing Equations (82), (92) and (94), it is then an easy matter to prove the identity:

$$H_{AB} := H_{kj} E^k_A E^j_B = -\epsilon \left([Q_{AB}] - [Q]\, h_{AB} \right), \tag{95}$$

where $[Q] := [Q_{AB}]\, h^{AB}$. It follows that the requirements $H_{AB} = 0$ and $[Q_{AB}] = 0$ at Σ are equivalent.

The request of vanishing of the quantities (89), (92) involves the Levi–Civita connection and the spin tensor (via the contorsion tensor) but also the trace of the energy–impulse tensor and its first derivatives. This is because of the torsional contributions given by the non-linearity of the function $f(R)$ and it represents a significant difference from the ECSK theory. In order to better clarify this last aspect, in the next subsections, we illustrate two examples dealing with the spin fluid and the Dirac field.

Before doing this, for the sake of completeness, we briefly outline also the case of null hypersurface. Then, let Σ be a null hypersurface described by an equation $\Phi(x^i) = 0$, where Φ is a smooth function. We suppose that $\mathcal{M}^+$ and $\mathcal{M}^-$ correspond to the domains where Φ is positive and negative, respectively. Again, we discuss the matching on Σ of two solutions of the field equations in the form:

$$g_{ij} = \Theta(\Phi) g^+_{ij} + (1 - \Theta(\Phi))\, g^-_{ij}, \tag{96a}$$

$$\Gamma_{ij}{}^h = \Theta(\Phi) \Gamma^+_{ij}{}^h + (1 - \Theta(\Phi))\, \Gamma^-_{ij}{}^h. \tag{96b}$$

The null normal vector is defined as $n_i = \alpha^{-1} \partial_i \Phi$, where α is a suitable non–zero function on Σ. By means of analogous arguments to those given above, it is immediately seen that the metric tensor (96a) has to be continuous across the hypersurface Σ, namely $[g_{ij}] = 0$. Following a usual procedure, let us then introduce a transverse vector field N^i satisfying the requirements $N^i n_i = 1$ and $N^i N_i = 0$. We have the relations $[n^i] = [N^i] = 0$. We also introduce the transverse metric:

$$h_{ij} = g_{ij} - n_i N_j - n_j N_i. \tag{97}$$

Due to the continuity of the metric tensor across Σ, its derivatives may have discontinuities only along the transverse direction. This implies the existence of a tensor field γ_{ij} on Σ, such that:

$$\gamma_{ij} = [\partial_s g_{ij}] N^s \qquad \Longleftrightarrow \qquad [\partial_s g_{ij}] = \gamma_{ij} n_s. \tag{98}$$

By Equation (98), we can express the jump of the Christoffel symbols as:

$$\left[\tilde{\Gamma}_{ij}{}^h \right] = \frac{1}{2} \left(\gamma^h{}_j n_i + \gamma^h{}_i n_j - \gamma_{ij} n^h \right). \tag{99}$$

Making use of Equation (99) and following the identical procedure illustrated above, it is easily seen that the δ-function part of the Einsein tensor is now given by the sum:

$$H^{ij} = \tilde{H}^{ij} + \bar{H}^{ij}, \tag{100}$$

where:

$$\tilde{H}^{ij} := \frac{\alpha}{2} \left(\gamma^i_h n^h n^j + \gamma^j_h n^h n^i - \gamma^h_h n^i n^j - \gamma_{hk} n^h n^k g^{ij} \right) \tag{101}$$

represents the contribution due to Levi–Civita terms, and:

$$\bar{H}^{ij} := \alpha \left(- \left[K^{jih} \right] n_h + \left[K_h{}^{ih} \right] n^j + \left[K_h{}^{hk} \right] n_k g^{ij} \right) \tag{102}$$

represents the contribution given by contorsion terms. As in the case of spacelike or timelike hypersurfaces, smooth transition across the hull hypersurface Σ at the level of Einstein-like equations requires the vanishing of the tensor (100).

4.1. The Coupling to a Spin Fluid

Let us consider a Weyssenhoff spin fluid with stress–energy and the spin tensors, respectively, given by [15,53,54]:

$$\mathcal{T}^{ij} = U^i P^j + p\left(U^i U^j - g^{ij}\right), \tag{103a}$$

and:

$$\mathcal{S}_{ij}{}^{h} = \mathcal{S}_{ij} U^h, \tag{103b}$$

where U^i is the 4-velocity, P^j denotes the 4-density of energy–momentum, $\mathcal{S}_{ij} = -\mathcal{S}_{ji}$ is the spin density, and p is the pressure of the fluid. By means of the conservation laws for the spin (12b), which are equivalent to the antisymmetric part of Einstein-like equations (10a), we can express the stress–energy tensor (103a) as [54]:

$$\mathcal{T}_{ij} = (\rho + p)\, U_i U_j - p g_{ij} - U_i \hat{T}_h \mathcal{S}^h{}_j - U_i \tilde{\nabla}_h \left(\mathcal{S}_{kj} U^h\right) U^k, \tag{104}$$

where $\rho := U^i P_i$ and $\tilde{\nabla}_h$ is the covariant derivative with respect to the Levi–Civita connection induced by the metric g_{ij}. In view of the usual convective condition $\mathcal{S}_{ij} U^j = 0$ [53,55] and the representation (11), it is easily seen that the vanishing at Σ of the quantities (89), (92) yields the explicit equations:

$$-\epsilon\left[K_{jq}{}^{p}\right] n_p E^q_A n^j + \left[K_{pq}{}^{p}\right] E^q_A = -\epsilon\left[\frac{1}{\varphi}\mathcal{S}_{qj} U_p\right] n^p n^j E^q_A - \left[\frac{1}{\varphi}\frac{\partial \varphi}{\partial x^q}\right] E^q_A = 0, \tag{105a}$$

$$\begin{aligned} &\tilde{H}_{AB} + \epsilon\left(-\left[K_{jq}{}^{p}\right] n_p E^q_A E^j_B + \left[K_q{}^{qp}\right] n_p h_{AB}\right) = \\ &\tilde{H}_{AB} + \epsilon\left(\left[\frac{1}{2\varphi}\left(\mathcal{S}_{jq} U^p + \mathcal{S}^p{}_q U_j + \mathcal{S}^p{}_j U_q\right)\right] n_p E^q_A E^j_B + \left[\frac{1}{\varphi}\frac{\partial \varphi}{\partial x^p}\right] n^p h_{AB}\right) = 0. \end{aligned} \tag{105b}$$

Equation (105b) can be decomposed into its symmetric and antisymmetric parts, thus giving rise to the further conditions:

$$\left[\frac{1}{2\varphi}\mathcal{S}_{jq} U^p\right] n_p E^q_A E^j_B = 0, \tag{106a}$$

$$\tilde{H}_{AB} + \epsilon\left(\left[\frac{1}{2\varphi}\left(\mathcal{S}^p{}_q U_j + \mathcal{S}^p{}_j U_q\right)\right] n_p E^q_A E^j_B + \left[\frac{1}{\varphi}\frac{\partial \varphi}{\partial x^p}\right] n^p h_{AB}\right) = 0. \tag{106b}$$

In order to illustrate a specific case, we imagine having to join together two static and spherically symmetric metrics:

$$ds^2_{\pm} = e^{\nu^{\pm}}\, dt^2 - e^{\lambda^{\pm}}\, dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{107}$$

solutions of Equation (10) coupled to a spin fluid. It is convenient to rename the spherical coordinates as $x^0 := t, x^1 := r, x^2 := \theta, x^3 := \phi$ in such a way that the 4-velocity of the fluid (supposed to be at rest in the chosen frame) is described by $U^i = U^0 \delta^i_0$, with $U^0 = e^{-\frac{\nu}{2}}$, and the unit normal to the hypersurface $\Sigma : x^1 = \text{const.}$ is given by $n^i = n^1 \delta^i_1$ with $n^1 = e^{-\frac{\lambda}{2}}$. The functions ν and λ, as well as all the involved matter fields, depend only on the radial variable r.

According to the convective condition $\mathcal{S}_{ij} U^j = 0$ and the stated spherical symmetry, we suppose that the spins of the particles composing the fluid are all aligned in the r direction; this means that only the components $\mathcal{S}_{23} = -\mathcal{S}_{32}$ of the spin density are non-zero [55]. Under these conditions, the stress–energy tensor of the spin fluid assumes the usual form:

$$\mathcal{T}_{ij} = (\rho + p)\, U_i U_j - p g_{ij}. \tag{108}$$

Using the above assumptions, it is easily seen that the constraints (105a), (106a) are automatically satisfied, while Equation (106b) reduces to:

$$\tilde{H}_{AB} + \epsilon \left[\frac{1}{\varphi} \frac{\partial \varphi}{\partial x^p} \right] n^p h_{AB} = 0. \tag{109}$$

Recalling the identity $\tilde{H}_{AB} = -\epsilon \left([\tilde{Q}_{AB}] - [\tilde{Q}]\, h_{AB} \right)$ [34], it is seen that Equation (109) relates the quantity $\left[\frac{1}{\varphi} \frac{\partial \varphi}{\partial x^p} \right]$ to the jump across Σ of the extrinsic curvature $\tilde{Q}_{AB}$ associated with the Levi–Civita connection of the metric (107). We note that, in the case of ECSK theory, condition (109) becomes $\tilde{H}_{AB} = 0$, which is the same condition holding in General Relativity [34].

Because of Equations (8) and (9), in general, the condition (109) involves the derivatives of matter fields. To see this point more in detail, we again take the model $f(R) = R + \alpha R^2$ into account. Due to Equation (108), from the trace equation (7) and the definition (9), we have the relations:

$$-R = \mathcal{T} = \rho - 3p, \tag{110}$$

and:

$$\varphi = 1 + 2\alpha \left(3p - \rho \right). \tag{111}$$

Moreover, it is easy to verify that:

$$\tilde{Q}_{00} = \frac{1}{2} \frac{\partial \nu}{\partial r} e^{\nu - \frac{\lambda}{2}}{}_{|r=r_0} \tag{112}$$

is the only non–vanishing component of the extrinsic curvature $\tilde{Q}_{AB}$ induced by the metric (107) on the hypersurface $\Sigma : \quad r = r_0 \quad$ const.. In view of this, requirement (109) is seen to reduce to the following two conditions:

$$\left[\frac{2\alpha \left(3\frac{\partial p}{\partial r} - \frac{\partial \rho}{\partial r} \right)}{1 + 2\alpha \left(3p - \rho \right)} \right] = 0, \tag{113a}$$

and:

$$\left[\frac{\partial \nu}{\partial r} \right] = 0. \tag{113b}$$

As an even more specific example, we suppose to have to joining together the interior spacetime $\mathcal{M}^-$ of a star with spin properties, with the exterior region $\mathcal{M}^+$ assumed empty. In such a circumstance, we have $\mathcal{T}^+_{ij} = 0$ and $\mathcal{S}_{ij}^{+\;h} = 0$, and in $\mathcal{M}^+$ the field equations (10) are identical to the Einstein equations (without cosmological constant) in vacuo; their unique solution $\left(g^+_{ij}, \Gamma_{ij}^{+\;h} \right)$ is then given by the Schwartzchild metric:

$$g^+_{ij}\, dx^i dx^j = \left(1 - \frac{2M}{r} \right) dt^2 - \left(1 - \frac{2M}{r} \right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right), \tag{114}$$

together with its Levi–Civita connection $\Gamma_{ij}^{+\;h} = \tilde{\Gamma}_{ij}^{+\;h}$. Consequently, the junction conditions (69), (113) assume the explicit form:

$$e^{\nu^-(r_0)} = \left(1 - \frac{2M}{r_0} \right), \qquad e^{\lambda^-(r_0)} = \left(1 - \frac{2M}{r_0} \right)^{-1}, \tag{115a}$$

$$\left(\frac{\partial \nu^-}{\partial r} \right)_{|r=r_0} = \frac{2M}{r_0 \left(r_0 - 2M \right)}, \qquad \left(3\frac{\partial p^-}{\partial r} - \frac{\partial \rho^-}{\partial r} \right)_{|r=r_0} = 0. \tag{115b}$$

On Equation (115), some comments are in order. Due to the second equation (115b), at Σ the spin fluid must behave like a sort of radiation, having a barotropic factor of the form $w = \left(\frac{\partial p^-}{\partial \rho^-} \right)_{|r=r_0} = 1/3$

at the boundary $r = r_0$. This fact is quite general: for all static and spherically symmetric solutions (107) of $f(R)$-gravity with torsion, the condition $\left(-3\frac{\partial p^-}{\partial r} + \frac{\partial \rho^-}{\partial r}\right)_{|r=r_0} = \frac{\partial \mathcal{T}^-}{\partial r}_{|r=r_0} = 0$ is always sufficient (together with (113b)) to fulfill the requirement (109), and it becomes necessary also whenever $\frac{\partial \varphi^-}{\partial \mathcal{T}}_{|r=r_0} \neq 0$ (like in the case $f(R) = R + \alpha R^2$, where $\frac{\partial \varphi^-}{\partial \mathcal{T}}_{|r=r_0} = -2\alpha$). On the other hand, whenever the condition $\frac{\partial \varphi^-}{\partial \mathcal{T}}_{|r=r_0} = 0$ is imposed, it yields a relation between density and pressure at the separation hypersurface, which constraints the equation of state [56].

4.2. The Coupling to a Dirac Field

Let ψ be a Dirac field with Lagrangian function given by:

$$\mathcal{L}_m = \left[\frac{i}{2}\left(\bar{\psi}\gamma^i D_i\psi - D_i\bar{\psi}\gamma^i\psi\right) - m\bar{\psi}\psi\right], \tag{116}$$

where $D_i\psi = \frac{\partial\psi}{\partial x^i} + \omega_i{}^{\mu\nu}\sigma_{\mu\nu}\psi$ and $D_i\bar{\psi} = \frac{\partial\bar{\psi}}{\partial x^i} - \bar{\psi}\omega_i{}^{\mu\nu}\sigma_{\mu\nu}$ are the covariant derivatives of the Dirac fields, $\sigma_{\mu\nu} = \frac{1}{8}\left[\gamma_\mu, \gamma_\nu\right]$, $\gamma^i = \gamma^\mu e^i_\mu$ with γ^μ denoting Dirac matrices and where m is the mass of the Dirac field. In what follows, the notation for which:

$$\gamma^\mu\gamma^\nu\gamma^\lambda = \gamma^\mu\eta^{\nu\lambda} - \gamma^\nu\eta^{\mu\lambda} + \gamma^\lambda\eta^{\mu\nu} + i\epsilon^{\mu\nu\lambda\tau}\gamma_5\gamma_\tau, \tag{117}$$

is used. From (116), we derive the Dirac equations:

$$i\gamma^h D_h\psi + \frac{i}{2}T_h\gamma^h\psi - m\psi = 0, \tag{118}$$

where, due to to the fact that torsion is no longer totally antisymmetric, the torsion vector $T_h := T_{hj}{}^j$ is present. The stress–energy and the spin density tensors are given by [15,42]:

$$\mathcal{T}_{ij} = \frac{i}{4}\left(\bar{\psi}\gamma_i D_j\psi - D_j\bar{\psi}\gamma_i\psi\right), \tag{119}$$

and:

$$\mathcal{S}_{ij}{}^h = -\frac{1}{4}\eta^{\mu\sigma}\epsilon_{\sigma\nu\lambda\tau}\left(\bar{\psi}\gamma_5\gamma^\tau\psi\right)e^h_\mu e^\nu_i e^\lambda_j. \tag{120}$$

In what follows, we can systematically assume that $\bar{\psi}\psi \neq 0$. Indeed, if $\bar{\psi}\psi = 0$, the trace of the stress–energy tensor would be constantly zero and the theory would amount to an ECSK-like theory for which the solution of the junction conditions problem is already known [35]. Therefore, without loss of generality, we can limit ourselves to dealing with spinor fields of type-1 and type-2 according to the Lounesto classification [57–59].

Making use of representation (11), it is seen that in this case the vanishing at Σ of the quantities (89), (92) yields the conditions:

$$-\epsilon\left[K_{jq}{}^p\right]n_pE^q_An^j + \left[K_{pq}{}^p\right]E^q_A = -\epsilon\left[\hat{K}_{jq}{}^p\right]n_pE^q_An^j + \left[\hat{K}_{pq}{}^p\right]E^q_A = -\left[\frac{1}{\varphi}\frac{\partial\varphi}{\partial x^q}\right]E^q_A = 0, \tag{121a}$$

$$\begin{aligned}\tilde{H}_{AB} + \epsilon\left(-\left[K_{jq}{}^p\right]n_pE^q_AE^j_B + \left[K_q{}^{qp}\right]n_ph_{AB}\right) &= \\ \tilde{H}_{AB} + \epsilon\left(-\left[\hat{K}_{jq}{}^p\right]n_pE^q_AE^j_B + \left[\hat{K}_q{}^{qp}\right]n_ph_{AB} - \left[\hat{S}_{jq}{}^p\right]n_pE^q_AE^j_B\right) &= \\ \tilde{H}_{AB} + \epsilon\left(\left[\frac{1}{\varphi}\mathcal{S}_{jq}{}^p\right]n_pE^q_AE^j_B + \left[\frac{1}{\varphi}\frac{\partial\varphi}{\partial x^p}\right]n^ph_{AB}\right) &= 0.\end{aligned} \tag{121b}$$

Splitting Equation (121b) in its symmetric and antisymmetric parts, we obtain the equations:

$$\tilde{H}_{AB} + \epsilon \left[\frac{1}{\varphi} \frac{\partial \varphi}{\partial x^p} \right] n^p h_{AB} = 0, \tag{122a}$$

$$\left[\frac{1}{\varphi} \mathcal{S}_{jq}{}^{p} \right] n_p E^q_A E^j_B = 0. \tag{122b}$$

As an illustrative example, we suppose joining two axially symmetric spacetimes, solutions of the field equations resulting again from the model $f(R) = R + \alpha R^2$. More in detail, we assume that the metric tensors in both the regions $\mathcal{M}^-$ and $\mathcal{M}^+$ are of Lewis–Papapetrou kind, expressed in spherical coordinates as:

$$g^{\pm}_{ij}\, dx^i dx^j = -B^2_{\pm}(r^2\, d\theta^2 + dr^2) - A^2_{\pm}(-W_{\pm}\, dt + d\phi)^2 + C^2_{\pm}\, dt^2, \tag{123}$$

where all functions $A_{\pm}(r,\theta)$, $B_{\pm}(r,\theta)$, $C_{\pm}(r,\theta)$, and $W_{\pm}(r,\theta)$ depend on the r and θ variables only. We assume that $\mathcal{M}^+$ is empty, while $\mathcal{M}^-$ is filled with a Dirac field. We also suppose that in $\mathcal{M}^+$ the metric is the Kerr one. This is consistent with the fact that $R + \alpha R^2$ gravity with torsion in vacuo is equivalent to GR and, therefore, admits the same solutions. In the Lewis–Papapetrou form (123), the coefficients of the Kerr metric are expressed as:

$$A^2_+(r,\theta) = \left[a^2 + \frac{\left(-a^2+m^2+2mr+r^2\right)^2}{4r^2} \right] \sin^2\theta + \frac{ma^2\left(-a^2+m^2+2mr+r^2\right)\sin^4\theta}{r\left(\frac{(-a^2+m^2+2mr+r^2)^2}{4r^2} + a^2\cos^2\theta\right)}, \tag{124a}$$

$$B^2_+(r,\theta) = \frac{a^2\cos^2\theta}{r^2} + \frac{1}{4} + \frac{m}{r} + \frac{3m^2-a^2}{2r^2} + \frac{m^3-a^2m}{r^3} + \frac{a^4-2a^2m^2+m^4}{4r^4}, \tag{124b}$$

$$\begin{aligned} C^2_+(r,\theta) = & \frac{m^2a^2\left(-a^2+m^2+2mr+r^2\right)^2\sin^4\theta}{\left(\left(a^2+\frac{(-a^2+m^2+2mr+r^2)^2}{4r^2}\right)\sin^2\theta + \frac{ma^2(-a^2+m^2+2mr+r^2)\sin^4\theta}{r\left(\frac{(-a^2+m^2+2mr+r^2)^2}{4r^2}+a^2\cos^2\theta\right)}\right)} \times \\ & \frac{1}{r^2\left(\frac{(-a^2+m^2+2mr+r^2)^2}{4r^2} + a^2\cos^2\theta\right)^2} + 1 - \frac{m\left(-a^2+m^2+2mr+r^2\right)}{r\left(\frac{(-a^2+m^2+2mr+r^2)^2}{4r^2} + a^2\cos^2\theta\right)}, \end{aligned} \tag{124c}$$

$$\begin{aligned} W_+(r,\theta) = & \frac{ma\left(-a^2+m^2+2mr+r^2\right)\sin^2\theta}{\left(a^2+\frac{(-a^2+m^2+2mr+r^2)^2}{4r^2}\right)\sin^2\theta + \frac{ma^2(-a^2+m^2+2mr+r^2)\sin^4\theta}{r\left(\frac{(-a^2+m^2+2mr+r^2)^2}{4r^2}+a^2\cos^2\theta\right)}} \times \\ & \frac{1}{r\left(\frac{(-a^2+m^2+2mr+r^2)^2}{4r^2} + a^2\cos^2\theta\right)}, \end{aligned} \tag{124d}$$

where a and m are the parameters entering the Kerr metric. We want to analyze the junction conditions at the hypersurface $\Sigma : \quad r = r_0 \quad$ const. To this end, by using Equations (118) and (119), we preliminarily notice that in the regions $\mathcal{M}^-$ and $\mathcal{M}^+$ we have, respectively:

$$\varphi^- = 1 + 2\alpha R = 1 - 2\alpha\mathcal{T} = 1 - \alpha m\bar{\psi}\psi, \tag{125a}$$

and:

$$\varphi^+ = 1. \tag{125b}$$

In view of Equation (125), the constraint (121a) implies that the scalar $\bar{\psi}\psi$ is forced to be constant on the hypersurface Σ. Moreover, it is easily seen that the requirement (122b) is equivalent to the conditions:

$$\bar{\psi}\gamma_5\gamma^0\psi_{|\Sigma} = 0, \qquad \bar{\psi}\gamma_5\gamma^2\psi_{|\Sigma} = 0, \qquad \bar{\psi}\gamma_5\gamma^3\psi_{|\Sigma} = 0, \tag{126}$$

which have to be satisfied at Σ by the spinor field ψ. The remaining condition (122a) can be discussed by rewriting it in the equivalent form:

$$\left[\tilde{Q}_{AB}\right] = -\frac{1}{2}\left[\frac{1}{\varphi}\frac{\partial\varphi}{\partial x^h}\right] n^h h_{AB}, \tag{127}$$

where $\left[\tilde{Q}_{AB}\right]$ indicates the jump across Σ of the extrinsic curvatures induced by the metrics (123). Denoting by $\tilde{A} := A^+(r_0,\theta) = A^-(r_0,\theta)$, $\tilde{B} := B^+(r_0,\theta) = B^-(r_0,\theta)$, $\tilde{C} := C^+(r_0,\theta) = C^-(r_0,\theta)$, and $\tilde{W} := W^+(r_0,\theta) = W^-(r_0,\theta)$ for simplicity, we have that the non–zero components of $\left[\tilde{Q}_{AB}\right]$ are:

$$\left[\tilde{Q}_{\theta\theta}\right] = -r_0^2\left[\partial_r B\right], \tag{128a}$$

$$\left[\tilde{Q}_{\phi\phi}\right] = -\frac{\tilde{A}}{\tilde{B}}\left[\partial_r A\right], \tag{128b}$$

$$\left[\tilde{Q}_{t\phi}\right] = \frac{\tilde{A}\left(2\tilde{W}\left[\partial_r A\right] + \tilde{A}\left[\partial_r W\right]\right)}{2\tilde{B}}, \tag{128c}$$

$$\left[\tilde{Q}_{tt}\right] = \frac{\tilde{C}\left[\partial_r C\right] - \tilde{A}\tilde{W}^2\left[\partial_r A\right] - \tilde{A}^2\tilde{W}\left[\partial_r W\right]}{\tilde{B}}. \tag{128d}$$

Due to Equations (125) and (128), the non-trivial equations of (127) result to have explicit expression:

$$\frac{\left[\partial_r B\right]}{\tilde{B}} = -\frac{\alpha m}{2(1-\alpha m\bar{\psi}\psi_{|\Sigma})}\partial_r\left(\bar{\psi}\psi\right)_{|\Sigma}, \tag{129a}$$

$$\frac{\left[\partial_r A\right]}{\tilde{A}} = -\frac{\alpha m}{2(1-\alpha m\bar{\psi}\psi_{|\Sigma})}\partial_r\left(\bar{\psi}\psi\right)_{|\Sigma}, \tag{129b}$$

$$\frac{2\left[\partial_r A\right]}{\tilde{A}} + \frac{\left[\partial_r W\right]}{\tilde{W}} = -\frac{\alpha m}{(1-\alpha m\bar{\psi}\psi_{|\Sigma})}\partial_r\left(\bar{\psi}\psi\right)_{|\Sigma}, \tag{129c}$$

$$\frac{\tilde{C}\left[\partial_r C\right] - \tilde{A}\tilde{W}^2\left[\partial_r A\right] - \tilde{A}^2\tilde{W}\left[\partial_r W\right]}{\tilde{C}^2 - \tilde{A}^2\tilde{W}^2} = -\frac{\alpha m}{2(1-\alpha m\bar{\psi}\psi_{|\Sigma})}\partial_r\left(\bar{\psi}\psi\right)_{|\Sigma}. \tag{129d}$$

From Equation (129), it is seen that the jumps of the r-derivatives of quantities $A^\pm$, $B^\pm$, and $C^\pm$ have to satisfy the relations:

$$\frac{\left[\partial_r A\right]}{\tilde{A}} = \frac{\left[\partial_r B\right]}{\tilde{B}} = \frac{\left[\partial_r C\right]}{\tilde{C}} = -\frac{\alpha m}{2(1-\alpha m\bar{\psi}\psi_{|\Sigma})}\partial_r\left(\bar{\psi}\psi\right)_{|\Sigma} = \frac{1}{2\varphi}\frac{\partial\varphi}{\partial(\bar{\psi}\psi)}\partial_r\left(\bar{\psi}\psi\right)_{|\Sigma}, \tag{130}$$

while the function $W(r,\theta)$ has to be of class $\mathcal{C}^1$.

In conclusion, it is shown that, in the non–linear case $f(R) \neq R + \lambda$, the scalar field $\bar{\psi}\psi$ is also involved in the characterization of the junction conditions. In particular, the derivatives of the metric components with respect to the coordinate r can have some jumps at the hypersurface Σ, connected with the r–derivative of the scalar quantity $\bar{\psi}\psi$. This is a difference from the linear case $f(R) = R + \lambda$ (ECSK theory), where, instead, the metric has to be at least of class $\mathcal{C}^1$.

5. Conclusions

The well-posedness of the Cauchy problem, as well as the well-formulation of the junction conditions, are crucial aspects of any theory of gravity. In fact, a well-posed initial value problem ensures uniqueness, continuity, and causality of solutions from initial data; at the same time, well-formulated junction conditions allow us to understand if and how two different space-times can be soldered at a given hypersurface, with obvious applications and consequences, for example, on an astrophysical level.

In this paper, we have discussed the Cauchy problem and the junction conditions within the framework of $f(R)$-gravity with torsion.

For what concerns the Cauchy problem, we have seen that the problem is always well-posed in vacuo and, in the absence of spin, every time the trace of the matter stress–energy tensor is constant; indeed, in such a circumstance, the theory amounts to an Einstein-like theory for which the well-posedness of the initial value problem is well-established: for instance, this is what happens in the case of coupling to an electromagnetic field or a Yang–Mills field. On the contrary, when the stress–energy tensor trace is not constant, the problem needs to be discussed case-by-case.

Here, we have faced the coupling to a perfect fluid and a Klein–Gordon scalar field. In both cases, we have derived sufficient conditions ensuring the well-posedness of the initial value problem. We have also proved that there exist $f(R)$ models with torsion, which actually satisfy the stated conditions: the model $f(R) = R + \alpha R^2$ does it. The key idea to achieve these results has been implementing a conformal transformation from the Jordan to the Einstein frame, proving that the conservation laws are formally preserved under such a transformation; this has allowed us to apply well-known Bruhat's results, holding in GR.

On the junction conditions, we have deduced the general requirements needed to solder at a given hypersurface two different solutions of $f(R)$-gravity with torsion. Despite a formal resemblance, junction conditions for $f(R)$-gravity with torsion differ from those holding in the ECSK theory because they involve the trace of the matter stress–energy tensor and its first derivatives; this is due to the contributions that the non-linearity of the gravitational function $f(R)$ gives to the contorsion tensor and, in general, it results in specific conditions that the matter fields have to satisfy at the separation hypersurface. In order to better clarify this aspect, we have given two illustrative examples, considering the model $f(R) = R + \alpha R^2$ coupled to a spin fluid and a Dirac field, respectively.

Finally, we have shown that the study of the initial value problem, as well as the junction conditions in the context of $f(R)$-gravity with torsion, singles out suitable conditions on the gravitational Lagrangian function $f(R)$ itself, which may be used as selection criteria for viable $f(R)$ models.

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