universe

IMPACT FACTOR 2.9

CITESCORE 3.6

Article

Dyonic Black Holes in Kaluza–Klein Theory with a Gauss–Bonnet Action

Salvatore Mignemi

Special Issue
The Languages of Physics—A Themed Issue in Honor of Professor Richard Kerner on the Occasion of His 80th Birthday

Edited by
Prof. Dr. Giuseppe Dito, Prof. Dr. Júlio César Fabris and Prof. Marco Modugno

MDPI

https://doi.org/10.3390/universe9120509

universe

MDPI

Article

Dyonic Black Holes in Kaluza–Klein Theory with a Gauss–Bonnet Action

Salvatore Mignemi [1,2]

[1] Dipartimento di Matematica, Università di Cagliari, Via Ospedale 72, 09124 Cagliari, Italy; smignemi@unica.it
[2] INFN, Sezione di Cagliari, Cittadella Universitaria, 09042 Monserrato, Italy

Abstract: Kaluza–Klein theory attempts a unification of gravity and electromagnetism through the hypothesis that spacetime has five dimensions, of which only four are observed. The original model gives rise to the standard Einstein–Maxwell theory after dimensional reduction. However, in five dimensions, the Einstein–Hilbert action is not unique, and one can add to it a Gauss–Bonnet term, giving rise to nonlinear corrections in the dimensionally reduced action. We consider such a model, which reduces to Einstein gravity nonminimally coupled to nonlinear electrodynamics. The black hole solutions of the four-dimensional model modify the Reissner–Nordström solutions of general relativity. We show that in the modified solutions, the gravitational field presents the standard singularity at $r = 0$, while the electric field can be regular everywhere if the magnetic charge vanishes.

Keywords: Kaluza–Klein theory; Gauss–Bonnet Lagrangian; charged black holes

1. Introduction

In his career, Richard Kerner gave important contributions to the Kaluza–Klein (KK) framework, in particular showing that nonabelian gauge theories can also be obtained from the process of dimensional reduction by increasing the number of internal dimensions [1]. He was also one of the first to notice the relevance of Gauss–Bonnet (GB) terms in the action of higher-dimensional theories, showing that they give rise to nonlinear contributions to electrodynamics in the reduced theory [2].[1]

We recall that KK theory unifies gravity and electromagnetism through the hypothesis that spacetime has five dimensions, of which only four are observed [4,5]. After dimensional reduction, the original KK model gives rise to the standard Einstein–Maxwell theory but does not predict new observable effects. For this reason, it has been abandoned, although its higher-dimensional generalizations, implementing also weak and strong interactions, have been widely investigated in the 1980s [6].

However, it is well known that in dimensions higher than four, the Einstein–Hilbert action can be generalized by the introduction of the so-called Gauss–Bonnet terms, as first observed by Lovelock [7]. These terms represent the most general extensions of the Einstein–Hilbert action that give rise to second-order field equations in arbitrary dimensions. One of their most notable properties is that they do not introduce new degrees of freedom in the spectrum of the theory in addition to the graviton and therefore avoid the presence of ghosts or tachyons, in contrast with most higher-derivative actions [8]. In lower dimensions, they are total derivatives and do not contribute to the equations of motion.

Their dimensional reduction, however, gives rise to nonlinear corrections to the Maxwell action of the electromagnetic field. Nonlinear models of electrodynamics have a long history: they were first proposed by Born and Infeld [9] in the hope of avoiding the singularities related to pointlike charged sources and then by Heisenberg and Euler [10] to give an effective classical description of quantum electrodynamics in a suitable limit. A general formulation, which includes also the model studied in [2,3] as a special case, was given by Plebanski [11].

Citation: Mignemi, S. Dyonic Black Holes in Kaluza–Klein Theory with a Gauss–Bonnet Action. *Universe* **2023**, *9*, 509. https://doi.org/10.3390/universe9120509

Academic Editors: Giuseppe Dito, Júlio César Fabris and Marco Modugno

Received: 16 October 2023
Revised: 30 October 2023
Accepted: 5 December 2023
Published: 8 December 2023

However, the specific action obtained from the dimensional reduction of the GB action enjoys peculiar algebraic properties. For example, the purely electric or magnetic solutions of the Maxwell equations in flat spacetime are not modified, and hence, only dyonic solutions can be affected by these corrections [2]. Dyons are pointlike sources that present both electric and magnetic monopole charges. They were introduced in [12] and found applications in grand unified theories. Some solutions of this nonlinear electrodynamics model in flat space, hence neglecting gravity, have been discussed in [2,13,14].

The investigation of this model is also interesting in view of the no-hair and uniqueness theorems of general relativity, which state that for the standard Einstein–Maxwell theory, the only spherically symmetric solution is Reissner–Nordström (RN) [15]. However, several examples have been found where the presence of nonminimal couplings to gravity violate this statement [16–18]. As shown in [14], also in the present case, the theorem does not hold.

In fact, we have recently shown that a five-dimensional KK theory containing GB contributions admits exact solutions that modify the RN metric of general relativity [14]. In particular, when the nonminimal coupling between gravity and Maxwell fields arising from the dimensional reduction is neglected, its dyonic solutions display an everywhere regular electric field. These modifications could in principle give experimental evidence of the existence of extra dimensions, although the effects are extremely small and beyond our present observational ability.

Here, we give a more complete discussion of the dyonic solutions of the model, taking into account also the nonminimal interaction terms. It turns out that, contrary to the case where these terms are neglected, the electric field can be regular everywhere only for a vanishing magnetic field. [2] This may be considered as a positive feature of the model, since magnetic monopoles are not observed in nature.

2. Kaluza–Klein Theory

As it is well known, Kaluza–Klein theory was proposed by Kaluza [4] soon after the discover of general relativity, with the aim of unifying gravitation and electromagnetism. The basic idea was to consider a five-dimensional spacetime, where the fifth dimension is unobservable because the fields depend only on the four-dimensional coordinates, and to consider a metric of the form[3]

$$g_{\mu\nu} = \begin{pmatrix} g_{ij} + 4A_iA_j & 2A_i \\ 2A_j & 1 \end{pmatrix}, \tag{1}$$

where g_{ij} is the four-dimensional metric and A_i is the Maxwell potential, and we have chosen the normalization in order to simplify further calculations.

Substituting (1) in the five-dimensional Einstein–Hilbert action, this reduces to

$$I = \int \sqrt{-g}\, d^4x [R - F^{ij}F_{ij}], \tag{2}$$

where R is the Ricci scalar and $F_{ij} = \partial_i A_j - \partial_j A_i$ is the electromagnetic field strength. This is nothing but the standard four-dimensional Einstein–Maxwell action in our normalization, whence the inclusion of the electromagnetic field in the theory is obtained by just adding a dimension to spacetime.

The theory was later improved by Klein, who explained the unobservability of the fifth dimension by assuming that it is curled in a very small circle [5]. Further developments were introduced by Jordan and Thiry [20,21], who noticed that the size of the compactified dimension is not necessarily constant but can depend on the spacetime coordinates, so that g_{44} can vary, giving rise to a scalar field in the compactified theory.

Returning to (2), we recall that by the uniqueness theorems, the only spherically symmetric solution of the Einstein–Maxwell equations in the presence of a pointlike source with both electric and magnetic charge, Q and P, respectively, is given by the Reissner–Nordström metric, with electromagnetic potential

$$A = \frac{Q}{r^2}\,dt + P\cos\theta\, d\varphi \tag{3}$$

and metric

$$ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\theta^2 + \sin^2\theta d\varphi^2, \tag{4}$$

where

$$\Delta = 1 - \frac{2M}{r} + \frac{Q^2 + P^2}{r^2}. \tag{5}$$

A remarkable property of this solution is its duality under the exchange of Q and P. For $M^2 > P^2 + Q^2$, the metric (4) describes a black hole with two horizons at

$$r_\pm^* = M \pm \sqrt{M^2 - Q^2 - P^2}. \tag{6}$$

If $M^2 = P^2 + Q^2$, the two horizons coincide and the black hole is called extremal, while if $M^2 < P^2 + Q^2$, a naked singularity occurs.

Among the most interesting properties of black holes is the possibility to associate thermodynamical parameters to them. In particular, the temperature can be identified with the inverse of the periodicity of the time coordinate that makes the Euclidean section of the solution regular [22]. For spherically symmetric black holes, it can be calculated as [23]

$$T = \frac{1}{4\pi\sqrt{g_{00}g_{11}}}\frac{dg_{00}}{dt}\bigg|_{r=r_+^*}. \tag{7}$$

For the RN black hole, this formula gives

$$T = \frac{1}{4\pi}\frac{r_+^* - r_-^*}{r_+^{*2}}. \tag{8}$$

The entropy can instead be identified with the area of the horizon surface [24], which in RN is

$$S = 4\pi r_+^{*2}, \tag{9}$$

with r_+^* and r_-^* given above.

3. The Model

As mentioned above, Lovelock [7] noticed that in more than four dimensions, the Einstein–Hilbert action is not unique but can be extended by introducing higher-derivative terms that however give rise to second-order field equations (although nonlinear in the second derivatives). In lower dimensions, these terms reduce to topological invariants (whence the name Gauss–Bonnet) and do not contribute to the field equations.

In particular, in five dimensions, the only nontrivial extension of the gravitational Lagrangian of the Lovelock type is given by the quadratic GB invariant

$$S = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2. \tag{10}$$

The dimensional reduction of the generalized action obtained by adding the GB term to the Einstein–Hilbert action gives rise to models of gravity coupled to nonlinear electrodynamics, which, as a consequence of the properties of the higher-dimensional theory, contain only graviton and photon excitations [8] and are therefore relevant from a phenomenological perspective. Therefore, we consider a five-dimensional Einstein–Gauss–Bonnet theory, with action [2,3,14]

$$I = \int \sqrt{-g}\, d^5x(R + \alpha S), \tag{11}$$

where the coupling constant α has dimension $[L]^2$ and is usually assumed to be positive for stability reasons. Arguments based on quantum gravity or string theory fix it to be of the Planck scale, but in any case, observations set a very small upper limit on its value [19].

We adopt the ansatz (1), neglecting for the moment the possibility of introducing a scalar field, and insert it into (2). Discarding total derivatives, the action reduces to [2,3,25]

$$I = \int \sqrt{-g}\, d^4x \Big[R - F^{ij}F_{ij} + 3\alpha L_{NL} - 2\alpha L_{int}\Big], \tag{12}$$

where the nonlinear electrodynamics Lagrangian is given by

$$L_{NL} = (F^{ij}F_{ij})^2 - 2F^{ij}F_{jk}F^{kl}F_{li} \tag{13}$$

and the interaction term L_{int} is

$$L_{int} = F^{ij}F^{kl}(R_{ijkl} - 4R_{ik}\delta_{jl} + R\,\delta_{ik}\delta_{jl}). \tag{14}$$

Due to the algebraic properties of the GB action, the field equations derived from L_{NL} contain the electric and magnetic fields only in the combination $\mathbf{E} \cdot \mathbf{B}$ (in nonrelativistic notation), so that they do not modify purely electric or magnetic solutions [2], and only dyonic solutions can be affected.[4]

If one neglects L_{int}, the action describes a model of gravity minimally coupled to a specific form of nonlinear electrodynamics. Exact asymptotically flat spherically symmetric solutions of (12) in the absence of L_{int} have been investigated in [14], where it was shown that the Reissner–Nordström solution of general relativity is modified in the dyonic case.

We recall the results of [14]: it is known that the simplest way to obtain the field equations for a spherically symmetric solution is to substitute into the action an ansatz for the metric of the form

$$ds^2 = -e^{2\nu}dt^2 + e^{2\lambda-2\nu}dr^2 + e^{2\rho}d\Omega^2, \tag{15}$$

to which we add an ansatz for an electromagnetic potential of dyonic form:

$$A = a\,dt + P\cos\theta\, d\phi, \tag{16}$$

where ν, λ, ρ and a are functions of r and P is the magnetic charge.

Then, after integration by parts, the action in the absence of L_{int} becomes

$$I = 2\int dr\Big[(2\nu'\rho' + \rho'^2)e^{2\nu-\lambda+2\rho} + e^{\lambda} + a'^2e^{-\lambda+2\rho} - P^2e^{\lambda-2\rho} - 12\alpha P^2a'^2e^{-\lambda-2\rho}\Big], \tag{17}$$

where $' = d/dr$. One can now vary the action with respect to the fields ν, λ, ρ, and a and then choose a gauge. The standard choice is $e^{\rho} = r$.

In this way, one obtains three independent field equations, namely

$$\left[r^2e^{-\lambda}\left(1 + 12\alpha\,\frac{P^2}{r^4}\right)a'\right]' = 0. \tag{18}$$

$$\lambda' = 0, \tag{19}$$

$$(re^{2\nu})' = \left(1 - \frac{P^2}{r^2}\right)e^{2\lambda} - r^2 a'^2 - 12\alpha \frac{P^2}{r^2}, \tag{20}$$

Let us consider the solutions for a positive α [14]. Equation (18) gives for the radial electric field $E \equiv F_{01}$,

$$E = a' = \frac{Qr^2}{r^4 + 12\alpha P^2}, \tag{21}$$

with Q as an integration constant. A remarkable property of the solution is that, contrary to RN, for $P \neq 0$, E is regular everywhere, including the origin.

For the metric functions, Equation (18) gives $\lambda = 0$, as in RN. Finally, $e^{2\nu}$ is obtained from Equation (20) as

$$e^{2\nu} = 1 - \frac{2M}{r} + \frac{P^2}{r^2} + \frac{Q^2}{2\sqrt{2\tilde{\alpha}}r}\left[\pi + \arctan\left(1 - \frac{\sqrt{2}\,r}{\sqrt{\tilde{\alpha}}}\right) - \arctan\left(1 + \frac{\sqrt{2}\,r}{\sqrt{\tilde{\alpha}}}\right) + \frac{1}{2}\log\frac{r^2 - \sqrt{2\tilde{\alpha}}\,r + \tilde{\alpha}}{r^2 + \sqrt{2\tilde{\alpha}}\,r + \tilde{\alpha}}\right], \tag{22}$$

where we have set $\tilde{\alpha} = \sqrt{3\alpha P^2}$. The asymptotic behaviour of the solution is given by

$$e^{2\nu} = 1 - \frac{2M}{r} + \frac{Q^2 + P^2}{r^2} - \frac{3\tilde{\alpha}P^2Q^2}{5r^6} + o\left(\frac{1}{r^7}\right), \tag{23}$$

and has therefore the same form as for RN up to order $1/r^5$. It follows that the integration constants M and Q can be identified with the mass and the electric charge. Therefore, the solution still depends on three parameters, but it is no longer invariant for the exchange $Q \leftrightarrow P$.

The properties of the metric are analogous to those of the RN solution: a curvature singularity is present at the origin, which for M greater than its extremal value is shielded by two horizons.

If $\alpha < 0$, the electric field is still given by (21), but a singularity occurs at $r_0 = \sqrt{\tilde{\alpha}}$, where now $\tilde{\alpha} = \sqrt{-3\alpha P^2}$. The metric function λ still vanishes, while $e^{2\nu}$ takes instead a slightly different form:

$$e^{2\nu} = 1 - \frac{2M}{r} + \frac{P^2}{r^2} + \frac{Q^2}{2\sqrt{\tilde{\alpha}}\,r}\left[\frac{\pi}{2} - \arctan\frac{r}{\sqrt{\tilde{\alpha}}} - \frac{1}{2}\log\frac{r - \sqrt{\tilde{\alpha}}}{r + \sqrt{\tilde{\alpha}}}\right]. \tag{24}$$

A spherical curvature singularity occurs at $r = r_0$, and the solution can have one or two horizons depending on the specific values of the parameters, while the asymptotic behaviour is still given by (23).

It is evident that the effects of nonlinear electrodynamics are more relevant at small r and tend to vanish at infinity.

4. The Solution

We now consider the asymptotically flat spherically symmetric solution of the field equations stemming from the action (12), when also the term L_{int} is included. In this case, it is not possible to find an exact solution, and we must proceed perturbatively.

We look for spherically symmetric solutions with an electromagnetic field of the dyonic form (16) and a line element (15). The interaction term now leads to field equations containing both the invariants $\mathbf{E}\cdot\mathbf{B}$ and $\mathbf{E}^2 - \mathbf{B}^2$, allowing for the existence of deformed solutions even if $\mathbf{E}$ or $\mathbf{B}$ vanish.

Substituting (15) and (16) in (12), after integration by parts, the action becomes

$$\begin{aligned} I \;=\; & 2\int dr\Big[(2\nu'\rho' + \rho'^2)e^{2\nu-\lambda+2\rho} + e^{\lambda} + a'^2 e^{-\lambda+2\rho} - P^2 e^{\lambda-2\rho} \\ & +4\alpha\Big(3P^2 a'^2 e^{-\lambda-2\rho} + a'^2\rho'^2 e^{2\nu-3\lambda+2\rho} - a'^2 e^{-\lambda} + 2P^2\nu'\rho' e^{2\nu-\lambda-2\rho}\Big)\Big], \end{aligned} \tag{25}$$

where $' = d/dr$.

As before, we vary the action (25) with respect to the fields λ, ν, ρ, and a, and then choose the gauge $e^\rho = r$. The three independent field equations can be put in the form

$$\left[r^2 e^{-\lambda}\left(1 + 12\alpha\,\frac{P^2}{r^4} + 4\alpha\,\frac{e^{2\nu-2\lambda}-1}{r^2}\right)a'\right]' = 0. \tag{26}$$

$$\left(1 + \frac{4\alpha P^2}{r^4}\right)\lambda' = -\frac{4\alpha}{r}\left(a'^2 e^{-2\lambda} + \frac{3P^2}{r^4}\right), \tag{27}$$

$$(re^{2\nu})' + 4\alpha P^2\frac{(e^{2\nu})'}{r^3} = \left(1 - \frac{P^2}{r^2}\right)e^{2\lambda} - r^2 a'^2 + 4\alpha\, a'^2\left(1 - 3e^{2\nu-2\lambda} - \frac{3P^2}{r^2}\right), \tag{28}$$

An important effect of the nonminimal gravity–Maxwell coupling L_{int} is that the metric field λ no longer vanishes. This is a common feature in the presence of nonminimally coupled Maxwell fields [16–18]. Also, the radial electric field $E \equiv F_{01}$ is modified with respect to the solution (21), since (26) gives

$$E = a' = \frac{Q\,r^2 e^\lambda}{r^4 + 4\alpha\left(e^{2\nu-2\lambda}-1\right)r^2 + 12\alpha P^2}, \tag{29}$$

where Q is an integration constant that can be identified with the electric charge. Now, because of the term proportional to the metric function in the denominator, this is no longer necessarily positive definite even if $\alpha > 0$, and the electric field might be singular at its roots.

Equations (26)–(28) do not admit a solution in analytical form, so we perturb them in the small parameter α around the Reissner–Nordström background (3)–(5). The perturbative expansion will be valid for large values of r, namely $r \gg \sqrt{\alpha}$.

We therefore define the perturbations $\sigma(r)$, $\gamma(r)$, and $\phi(r)$ as

$$e^{2\nu} = \Delta + \alpha\sigma, \qquad \lambda = \alpha\gamma, \qquad E = \frac{Q}{r^2}(1 + \alpha\phi), \tag{30}$$

and substitute in the field equations, obtaining

$$\phi = \gamma - 4\left(\frac{\Delta - 1}{r^2} + \frac{3P^2}{r^4}\right), \tag{31}$$

$$\gamma' = -4\,\frac{Q^2 + 3P^2}{r^5}, \tag{32}$$

$$(r\sigma)' = -\frac{4P^2\Delta'}{r^3} + 2\left(1 - \frac{P^2}{r^2}\right)\gamma - \frac{2Q^2}{r^2}\phi + \frac{4Q^2}{r^4}\left(1 - 3\Delta - \frac{3P^2}{r^2}\right). \tag{33}$$

Integrating (32), we obtain at order α

$$\gamma = \frac{Q^2 + 3P^2}{r^4}. \tag{34}$$

Substitution in (31) gives

$$\phi = \frac{8M}{r^3} - \frac{3Q^2 + 13P^2}{r^4}. \tag{35}$$

Finally, inserting the previous results in (33) and integrating, one obtains

$$\sigma = \frac{2(Q^2 - P^2)}{r^4} - \frac{2M(Q^2 - P^2)}{r^5} + \frac{6Q^4 - 8P^2Q^2 - 2P^4}{5r^6}. \tag{36}$$

In all the solutions, we have chosen boundary conditions such that the corrections vanish at infinity. Hence, at order α,

$$e^{2\nu} \sim 1 - \frac{2M}{r} + \frac{Q^2+P^2}{r^2} + 2\alpha\left(\frac{Q^2-P^2}{r^4} - \frac{M(Q^2-P^2)}{r^5} + \frac{3Q^4-4P^2Q^2-P^4}{5r^6}\right), \quad (37)$$

$$e^{2\lambda} \sim 1 + 2\alpha\,\frac{Q^2+3P^2}{r^4}, \quad (38)$$

$$E \sim \frac{Q}{r^2}\left(1 + \frac{8\alpha M}{r^3} - \frac{\alpha(3Q^2+13P^2)}{r^4}\right). \quad (39)$$

Our approximation works well for $r \to \infty$. At leading orders in $1/r$, the asymptotic behaviour is the same as that in RN. Therefore, we can still identify M with the mass of the black hole and Q and P with its electric and magnetic charge. Notice that the corrections to the RN solutions are much larger than in the case where the L_{int} term is neglected, since they are now $o(1/r^4)$.

The horizons are displaced with respect to the RN solutions, where they are located at $r^*_\pm$; see (6). At the first order in α, one has $r_\pm = r^*_\pm + \alpha\Delta r_\pm$, where

$$\Delta r_\pm = -\frac{\sigma}{\Delta'}\Big|_{r=r^*_\pm}. \quad (40)$$

It follows that

$$\Delta r_\pm = \frac{5(P^2-Q^2){r^*_\pm}^2 - 3P^4 + 8P^2Q^2 - Q^4}{5{r^*_\pm}^3({r^*_\pm}^2 - P^2 - Q^2)}, \quad (41)$$

where to simplify the expression, we have chosen as independent parameters r^*_+ (r^*_-), Q, and P, writing the RN mass M in terms of the outer horizon and of the charges as

$$M = \frac{{r^*_+}^2 + Q^2 + P^2}{2r^*_+}. \quad (42)$$

It is apparent that the actual values of the displacement of the horizon strongly depend on the charges.

A calculation shows that the condition of extremality $r_+ = r_-$ is, at first order in α,

$$M^2 = P^2 + Q^2 + \frac{\alpha}{5}\,\frac{P^4 + 4Q^2P^2 - 3Q^4}{(P^2+Q^2)^2}. \quad (43)$$

Depending on the values of Q and P, the correction to the extremal value of the mass with respect to the RN case can be both positive or negative. This is clear from Figure 1, where the extremal values of Q in function of P for $M = 1$ and $\alpha = 0.1$ are compared for our solution and RN solution. For small P, the extremal value of Q is higher than in the RN case, while for a great P, it is smaller. We recall however that, while the value of r_+ obtained in this way is in general well approximated by (41), the value of r_- is reliable only for very small values of α.

In this approximation, the metric function $e^{2\nu}$ does not differ much from that of RN and the causal structure should therefore be analog. Hence, for an M greater than extremality, one has two horizon, while a naked singularity is present for M less than its extremal value. However, this is not necessarily true for greater values of α, where the approximation fails.

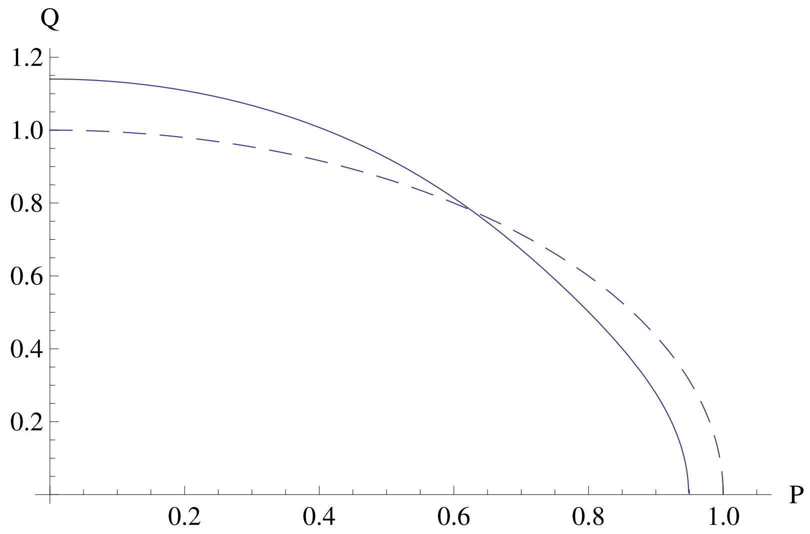

Figure 1. Extremal values of Q in function of P for $M = 1$ and $\alpha = 0.1$. Continuous curve: solution (37); dotted curve: Reissner–Nordström metric.

Using the standard definitions reported in Section 2, it is possible to derive the thermodynamical quantities associated to the black hole from the behaviour of the metric functions near the outer horizon. In the parametrization (15), the formula (7) gives

$$T = \frac{1}{4\pi} e^{-\lambda} (e^{2\nu})' \big|_{r=r_+} \sim \frac{1}{4\pi} \Big[\Delta'(1-\alpha\gamma) + \alpha\sigma'\Big]_{r=r_+}. \tag{44}$$

Hence,

$$T \sim \frac{1}{4\pi r_+^{*3}} \Bigg(r_+^{*2} - P^2 - Q^2 \\ -2\alpha \frac{7P^6 + 25P^4Q^2 + 17P^2Q^4 - Q^6 - 14P^4 r_+^{*2} - 36P^2Q^2 r_+^{*2} + 2Q^4 r_+^{*2} + 5(Q^2+P^2) r_+^{*4}}{5 r_+^{*4} (r_+^{*2} - P^2 - Q^2)} \Bigg) \tag{45}$$

The entropy S can be identified with the area of the horizon, namely,

$$S = 4\pi r_+^2 \sim 4\pi r_+^{*2} \left(1 - 2\alpha \frac{3P^4 - 8P^2Q^2 + Q^4 + 5(Q^2 - P^2) r_+^{*2}}{5 r_+^{*4} (r_+^{*2} - P^2 - Q^2)} \right). \tag{46}$$

It follows that the thermodynamical quantities also display a complicate dependence on the charges.

It is also interesting to investigate the behaviour of the solutions near the singularity. This can be calculated by an expansion in powers of r near $r = 0$. Setting

$$e^{\nu} \sim r^h, \qquad e^{\lambda} \sim r^l, \qquad E \sim r^k,$$

and substituting in the field Equations (26)–(28), one obtains $h = -2$, $l = -3$ and $k = -1$. It follows that the metric functions and the electric field diverge for $r = 0$, thus destroying the nice property of the solution in Section 2 of having a finite electric field at the origin.

The only exception is for $P = 0$. In this case, the electric field vanishes at the origin. In fact, now, $h = -1$, $l = 0$, and $k = 1$. The possible existence of electric solutions regular at the origin in the absence of a magnetic field has also been noticed in [19]. However, for small values of Q, numerical calculations show that the solutions become singular at a point $r_0 > 0$, presenting a spherical singularity, similarly to the $\alpha < 0$ solutions of Section 2.

Remarkably, the behaviour of the solutions is therefore opposite to those with $L_{int} = 0$, since regular solution can now exist only if $P = 0$.

5. Numerical Calculations

The solutions of the Equations (3)–(25) can also be obtained numerically. This is especially interesting in the regime $r \ll \alpha$, where the perturbative calculation of the previous section fails. In Figure 2 are reported the metric functions and the electric field calculated for $\alpha = 0.01$, $M = 1$, and several values of Q and P, such that $Q^2 + P^2 = 1/2$, so that $r_+ \sim 0.7$. The metric functions e^ν and e^λ do not change much for different values of the charges. In particular, $\epsilon^{2\nu}$ is similar to the RN solution, while $e^{-2\lambda} = 1$ for $r \gg \sqrt{\alpha}$ and then fades to 0 for $r \to 0$. The solutions are indistinguishable from RN for a large r. In general, a curvature singularity is present at $r = 0$ and the causal structure is essentially the same as that of the RN solution. Also, the electric field is singular at the origin as in the RN solution.

As mentioned before, an interesting special case is given by $P = 0$. In Figure 3 are depicted the metric functions $e^{2\nu}$, $e^{-2\lambda}$, and the electric field F for $M = 1$ and different values of the electric charge. For our choice of parameters, if $Q > 0.44$ the electric field is regular at the origin. The possibility of such a behaviour had been noticed in [19] using different methods. However, for a smaller Q, a singularity occurs for a finite value of r and the metric functions and the electric field diverge there.

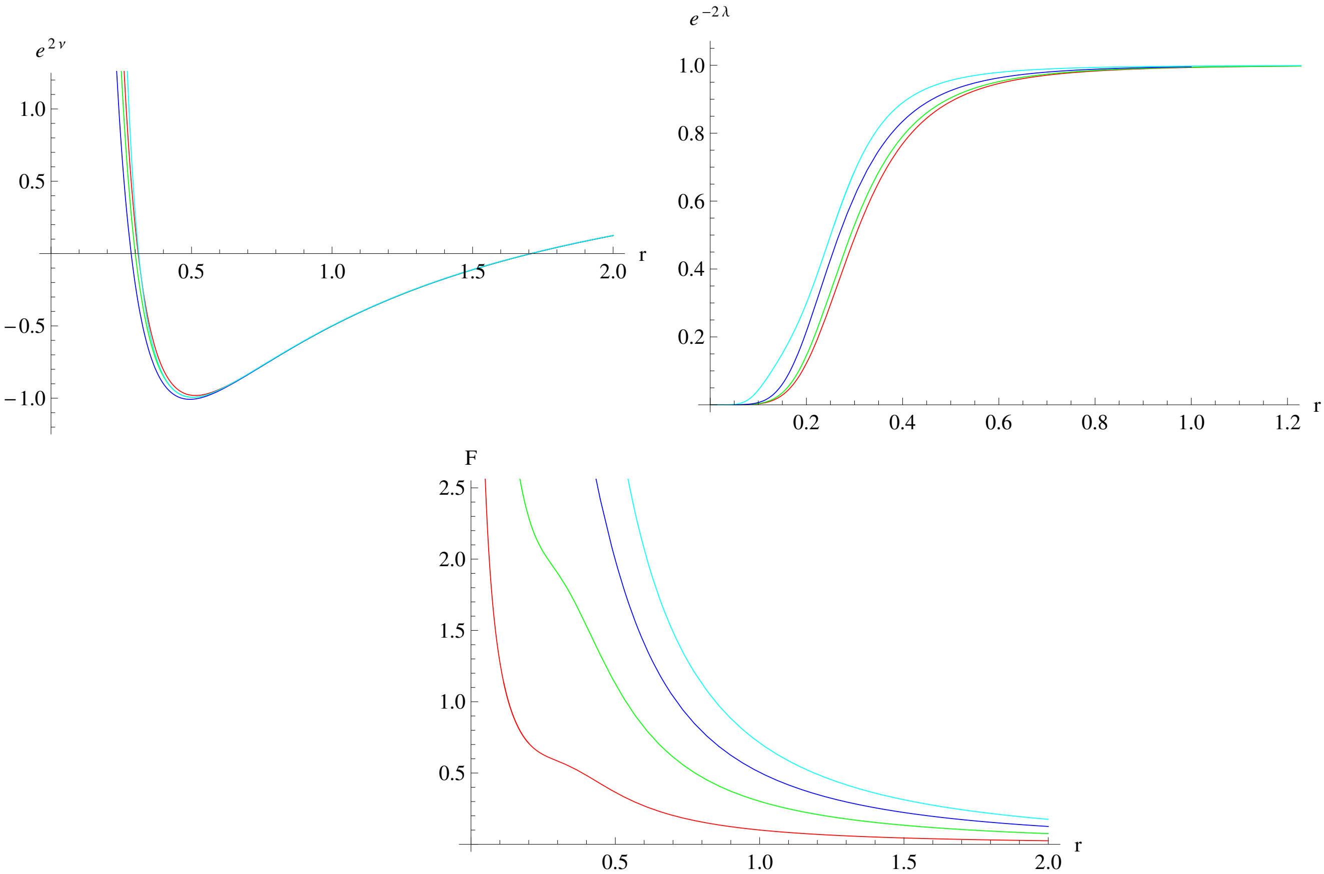

Figure 2. The metric functions $e^{2\nu}$ (**left** panel) and $e^{-2\lambda}$ (**right** panel) and the electric field F (**bottom** panel) for black holes with mass $M = 1$, $Q^2 + P^2 = \frac{1}{2}$, $Q = 0.1$ (in cyano), $Q = 0.3$ (in blue), $Q = 0.5$ (in green), and $Q = 0.7$ (in red).

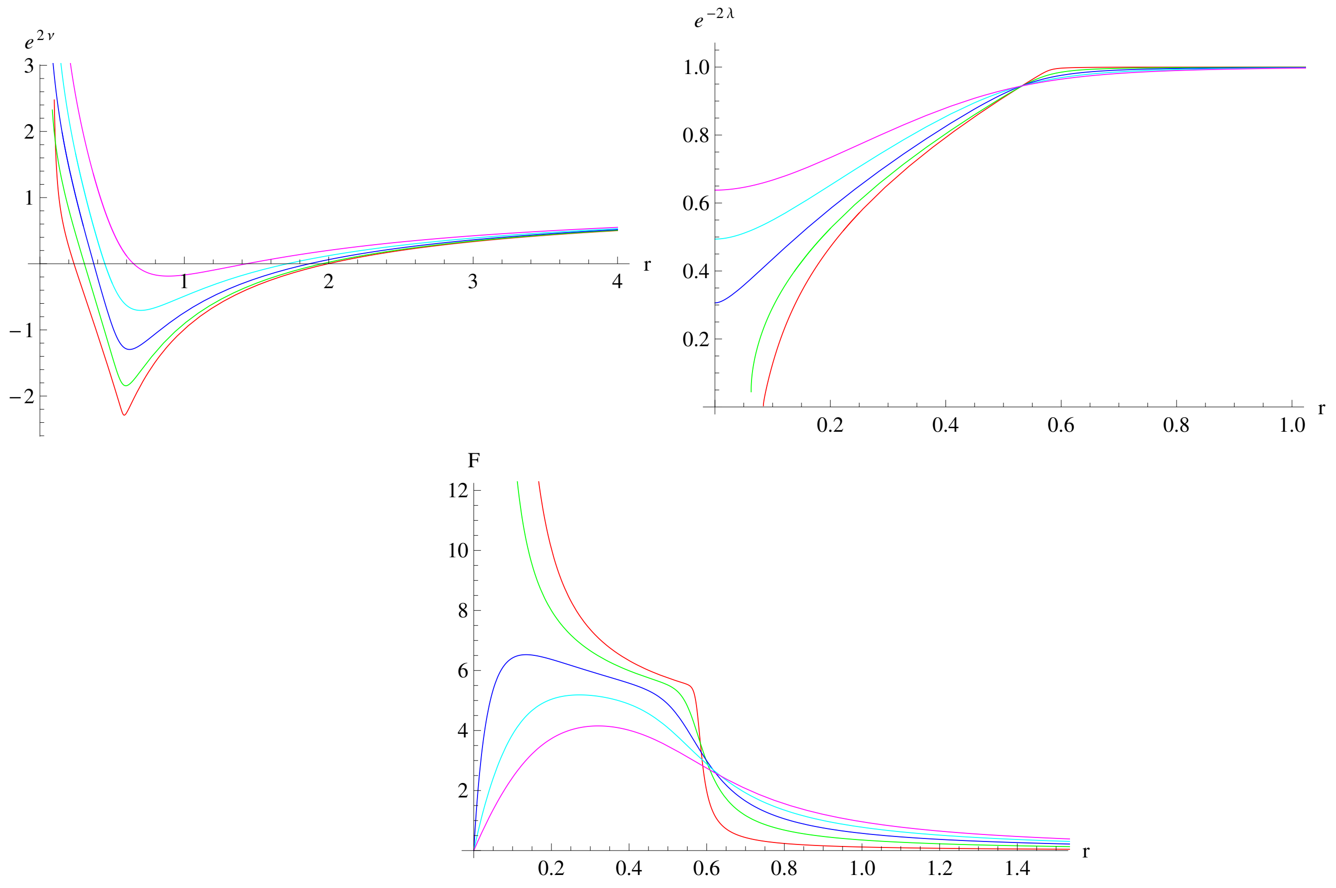

Figure 3. The metric function functions $e^{2\nu}$ (**left** panel) and $e^{-2\lambda}$ (**right** panel) and the electric field F (**bottom** panel) for black holes with mass $M = 1$, $P = 0$, $Q = 0.1$ (in cyano), $Q = 0.3$ (in blue), $Q = 0.5$ (in green), $Q = 0.7$ (in red), and $Q = 0.9$ (in magenta). As it is evident from the graphs, the curves with $Q = 0.1$ and $Q = 0.3$ display singularity at $r_0 = 0.08$ and $r_0 = 0.06$ respectively.

6. Conclusions

We have studied the solutions of the dimensionally reduced Kaluza–Klein theory with Einstein–Gauss–Bonnet action, including the interaction term that was disregarded in Ref. [14]. Since it was not possible to find analytic solutions of the field equations, we have used perturbative and numerical methods. The four-dimensional theory consists in general of relativity coupled to a particular form of nonlinear electrodynamics, and its solutions modify the RN solution of the Einstein–Maxwell theory but still depend on three parameters, identified with the mass and the electric and magnetic charges. They still present a black hole causal structure analogous to that of RN. A surprising consequence of the inclusion of L_{int} in the action is that dyonic solutions no longer display a regular electric field, like the solutions of sect. 2 [14], but instead, such behaviour can occur for purely electric solutions.

Our solutions also limit the validity of uniqueness theorems of general relativity, giving a further example of how nonlinear couplings of additional fields (in our case, electromagnetism) can give rise to solutions different from the standard ones, although depending on the same number of parameters.

The model investigated in this paper could be extended to higher dimensions, allowing for the possibility to introduce further terms of the GB type in the action. In such a case, nonabelian gauge fields are also present in the four-dimensional action [1,6], and new interesting phenomena could appear.

A different extension of our research would be the introduction of a scalar field into the ansatz of dimensional reduction [20,21]. In this case, the GB term is present also in the reduced four-dimensional theory, coupled to the scalar field. This fact may modify the structure of the solutions; see e.g., [18].

To conclude, we observed that, even if higher-dimensional KK theories have not obtained phenomenological success, mainly because of problems related to the inclusion of fermions into the theory, their investigation can still be useful and can produce interesting results, at least from a formal point of view.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The author declares no conflict of interest.

Notes

1 This was also independently observed by Buchdal [3].

2 The existence of solutions with regular electric fields was predicted also in [19], using different methods.

3 We set $\mu, \nu = 0, \ldots, 4$; $i, j = 0, \ldots, 3$ and use natural units.

4 In the general Plebanski models [11], the combination $\mathbf{E}^2 - \mathbf{B}^2$ can also occur in the field equations.

References

1. Kerner, R. Generalization of the Kaluza-Klein Theory for an Arbitrary Nonabelian Gauge Group. *Ann. Inst. Henri Poincaré Phys. Theor.* **1968**, *9*, 143.
2. Kerner, R. Électrodynamique non linéaire en théorie de Kaluza et Klein. *C. R. Acad. Sci. Paris* **1987**, *304*, 621.
3. Buchdal, H.A. On a Lagrangian for non-minimally coupled gravitational and electromagnetic fields. *J. Phys.* **1979**, *A12*, 1037. [CrossRef]
4. Kaluza, T. Zum Unitätsproblem in der Physik. *Sitz. Preuss. Akad. Wiss. Math. Phys.* **1921**, *1*, 966.
5. Klein, O. Quantentheorie und fünfdimensionale Relativitätstheorie. *Z. Phys.* **1926**, *37*, 895. [CrossRef]
6. Bailin, D.; Love, A. Kaluza-Klein theories. *Rep. Progr. Phys.* **1987**, *50*, 1087. [CrossRef]
7. Lovelock, D.The Einstein tensor and its generalizations. *Math. Phys.* **1971**, *12*, 498. [CrossRef]
8. Zumino, B. Gravity Theories in More Than Four-Dimensions. *Phys. Rep.* **1986**, *137*, 109. [CrossRef]
9. Born, M.; Infeld, L. Foundations of the new field theory. *Proc. R. Soc. Lond.* **1934**, *A144*, 435. [CrossRef]
10. Heisenberg, W.; Euler, H. Consequences of Dirac's theory of positrons. *Z. Phys.* **1936**, *98*, 714. [CrossRef]
11. Plebański, J. *Lectures on Non-Linear Electrodynamics*; NORDITA: Copenhagen, Denmark, 1968.
12. Schwinger, J. A Magnetic model of matter. *Science* **1969**, *165*, 757. [CrossRef] [PubMed]
13. Kerner, R. Non-Linear Electrodynamics Derived from the Kaluza-Klein Theory. In *Proceedings of the Varna Summer School, "Infinite Dimensional Lie Algebras and Quantum Field Theory"*; World Scientific: Singapore, 1987.
14. Mignemi, S. Dyonic black holes in nonlinear electrodynamics from Kaluza–Klein theory with a Gauss–Bonnet term. *Int. J. Mod. Phys.* **2022**, *A37*, 2250065. [CrossRef]
15. Black Hole Uniqueness Theorems. 2001. Available online: https://arxiv.org/pdf/hep-th/0101012.pdf (accessed on 29 October 2023).
16. Gibbons, G.W.; Maeda, K. Black Holes and Membranes in Higher Dimensional Theories with Dilaton Fields. *Nucl. Phys.* **1988**, *B298*, 741. [CrossRef]
17. Volkov, M.S.; Gal'tsov, D.V. NonAbelian Einstein Yang-Mills black holes. *JETP Lett.* **1989** *50* 346.
18. Mignemi, S.; Stewart, N.R. Charged black holes in effective string theory. *Phys. Rev.* **1993**, D47, 5259. [CrossRef] [PubMed]
19. Soleng, H.H.; Grøn, Ø. Modification of the Coulomb potential from Kaluza-Klein model with a Gauss-Bonnet term in the action. *Ann. Phys.* **1995**, 240, 432. [CrossRef]
20. Jordan, P. Relativistische Gravitationstheorie mit variabler Gravitationskonstante. *Naturwissenschaften* **1946**, *11*, 250. [CrossRef]
21. Thiry, M.Y. Les équations de la théorie unitaire de Kaluza. *Compt. Rend. Acad. Sci. Paris* **1948**, *226*, 216.
22. Gibbons, G.W.; Hawking, S.W. Action Integrals and Partition Functions in Quantum Gravity. *Phys. Rev.* **1977**, *D15*, 2752. [CrossRef]
23. York, J.W. Black hole in thermal equilibrium with a scalar field: The back-reaction. *Phys. Rev.* **1985**, *D31*, 775. [CrossRef]
24. Bekenstein, J. Black holes and the second law. *Lett. Nuovo Cimento* **1972**, *4*, 737. [CrossRef]
25. Müller-Hoissen, F. Non-minimal coupling from dimensional reduction of the Gauss-Bonnet action. *Phys. Lett.* **1998**, *B201*, 325. [CrossRef]