THEORY OF TRANSVERSE IONIZATION COOLING IN A LINEAR CHANNEL

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Abstract

Ionization cooling is the most hopeful method to reduce the emittance of muon beams, which plays an important role in neutrino factory and muon collider. Within the moment-equation approach, we present a way to derive the formulae of emittance in transverse under linear channel. All heating and coupling terms are reserved in the deriving process. From our formulae, it is a way to achieve a small emittance by designing the cooling channel compact to make the beta function changing sharply.

INTRODUCTION

The physics potentials of neutrino factories and muon colliders have stimulated worldwide studies of the feasibility of high-energy muon accelerators. Ionization cooling of particles proposed a long time ago in [1]. Ionization cooling theory is being studied for a very long time. And this cooling channel is developed to reduce the emittance of muon beam for envisioned neutrino factory and muon collider. Basic concept of this proposal is that the friction force acting to the particle moving through the absorber, directed against instant velocity of particle. As the longitudinal component of momentum lost in absorber could be restored by the longitudinal electric field in a RF cavity, the loss of transverse component is not, so this process resulting emittance reduction. This process is similar to the one with radiation losses; in some sense excitation of betatron oscillation while emitting the quanta in a channel with nonzero dispersion is similar to the (multiple) scattering in absorber.

Ionization cooling in a quadrupole channel has been discussed extensively by many authors, especially Neuffer's cooling formulae in [2]. Neuffer's formulae of transverse cooling theory is

$$\epsilon_{xn} = \frac{\beta_\perp E_s^2}{2\beta E_\mu L_{rad}|\mathrm{dE/ds}|} \tag{1}$$

where $\beta_\perp$ stands for the betatron function, E_μ is the particle energy, L_{rad} is radiation length.

And, Wang and Kim have developed coupled cooling equations including dispersion, wedges, solenoids, and symmetric focussing [3][4][5][6].

SINGLE PARTICLE DYNAMICS

We consider an idealized uncoupled quadrupole channel with quadrupole strength $K(s)$; horizontal bending radius ρ. Using the standard Frenet-Serret coordinates $\{x, y, s\}$, the Hamiltonian can be written as

$$\begin{aligned} H =& \frac{1}{2}\{P_x^2 + [K(s) + \frac{1}{\rho^2}]x^2\} + \frac{1}{2}[P_y^2 - K(s)y^2] - \frac{\delta x}{\rho} \\ &+ \frac{1}{2}[\frac{1}{\gamma_0^2}\delta^2 + V(s)z^2] + \eta[xP_x + yP_y] + \eta_x xz \\ &- \sqrt{\chi}(\xi_x x + \xi_y y) - \sqrt{\chi_\delta}\xi_z z \end{aligned} \tag{2}$$

where $\{x, P_x\}$, $\{y, P_y\}$ are the horizontal and vertical canonical variables, and $\{z, \delta\}$ are the longitudinal canonical variables. And γ_0 is the Lorentz factor of the reference particle, $K(s)$ is the quadrupole strength, $\rho(s)$ is the horizontal bending radius, $V(s)$ is the RF focusing strength, δ is momentum deviation from the nominal momentum, η is a positive quantity characterizing the cooling force from energy loss, χ is the projected mean-square angular deviation per unit length due to multiple scattering, $\xi_{(x,y,z)}$ is uncorrelated unit stochastic quantities describing the fluctuation forces due to multiple scattering and energy straggling.

From eq.(2), one could get the equations of motion directly. Because the forces are linear, we use the Fokker-Plank equations to study the phase-space distribution.

FORMULAE OF TRANSVERSE IONIZATION COOLING IN A LINEAR CHANNEL

According to eq.(2), horizontal and vertical motions in the two transverse planes are uncoupled. It is sufficient to treat only the x phase space dynamics, so the Hamiltonian simplifies into the form:

$$H = \frac{1}{2}[P_x^2 + K_x(s)x^2] + \eta x P_x - \sqrt{\chi}\xi_x x \tag{3}$$

where $K_x(s) = K(s) + \frac{1}{\rho^2}$.

Then the single-particle equations of motion according to eq.(3) are:

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\partial H}{\partial P_x} = P_x + \eta x \tag{4}$$

$$\frac{\mathrm{d}P_x}{\mathrm{d}s} = -\frac{\partial H}{\partial x} = -K_x(s)x - \eta P_x + \sqrt{\chi}\xi_x \tag{5}$$

After some algebra within eq.(4) and eq.(5), we obtained the second-order beam moments in x plane:

$$\langle x^2\rangle' = 2(\langle xP_x\rangle + \langle \eta x\rangle) \tag{6}$$

$$\langle xP_x\rangle' = \langle P_x^2\rangle - \langle K_x(s)x^2\rangle + \langle\sqrt{\chi}x\xi_x\rangle \tag{7}$$

$$\langle P_x^2\rangle' = -2(\langle K_x(s)xP_x\rangle + \langle \eta P_x^2\rangle - \langle\sqrt{\chi}P_x\xi_x\rangle) \tag{8}$$

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Obviously, there is a term correlated with η in eq.(4), which is coming from energy loss in absorber; and there are two more parts in eq.(5), the one correlated with η is from energy loss in absorber, and the other one correlated with $\sqrt{\chi}\xi_x$ is due to fluctuating force from multiple scattering and energy straggling. All these extra terms make the second-order beam moments more complicated.

Here we choose rms emittance to derive the formulae of ionization cooling in transverse plane. The rms emittance is define as:

$$\epsilon_{rms,x} = \sqrt{\langle x^2\rangle\langle P_x^2\rangle - \langle xP_x\rangle^2} \quad (9)$$

In the following derivation, we drop the subscript $_{rms}$ to simplify the notation to make the expression clear.

Similar to the electron ring theory, the beam size in horizontal plane is $\sigma_x = \sqrt{\beta_x\epsilon_x + D(s)^2\delta^2}$. As we just study the transverse plane, the $\delta = 0$ and $\langle x^2\rangle = \beta_x\epsilon_x$. So the envelope functions determined by the lattice functions could be expressed:

$$\begin{pmatrix} \langle x^2\rangle & \langle xP_x\rangle \\ \langle xP_x\rangle & \langle P_x^2\rangle \end{pmatrix} = \epsilon_x \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix} \quad (10)$$

Next, we will derive the formulae of transverse emittance in a linear ionization cooing channel. The normalized emittance in x plane is $\epsilon_{n,x} = \beta\gamma\epsilon_x$, so the emittance evolution along the cooling channel is

$$\frac{\mathrm{d}\epsilon_{xn}}{\mathrm{d}s} = \epsilon_x\frac{\mathrm{d}\beta\gamma}{\mathrm{d}s} + \beta\gamma\frac{\mathrm{d}\epsilon_x}{\mathrm{d}s} \quad (11)$$

In right of eq.(11), the first term relates with energy loss and contributes cooling effect. After some algebra, it is easy to get that:

$$\epsilon_x\frac{\mathrm{d}\beta\gamma}{\mathrm{d}s} = -\frac{1}{\beta^2}\frac{\epsilon_{xn}}{E}\left|\frac{\mathrm{d}E}{\mathrm{d}s}\right| \quad (12)$$

And combined with eq.(6) and eq.(7) and eq.(8), one could get:

$$\beta\gamma\frac{\mathrm{d}\epsilon_x}{\mathrm{d}s} = \frac{\beta\gamma}{\epsilon_x}(\langle x^2\rangle\langle\sqrt{\chi}P_x\xi_x\rangle - \langle xP_x\rangle\langle\sqrt{\chi}x\xi_x\rangle) \quad (13)$$

Taking the envelope functions eq.(10) into eq.(13), we have the expression:

$$\beta\gamma\frac{\mathrm{d}\epsilon_x}{\mathrm{d}s} = \beta\gamma\sqrt{\chi}\Big(\beta_\perp\langle P_x\xi_x\rangle + \alpha_\perp\langle x\xi_x\rangle\Big) \quad (14)$$

According to Cauchy-Schwarz theory in probability theory, there is a relationship between random variable ξ and η:

$$|\langle\xi\eta\rangle|^2 \le \langle\xi^2\rangle\langle\eta^2\rangle \quad (15)$$

If and only if the possibility $P\{\eta = t_0\xi\} = 1$, one has $|\langle\xi\eta\rangle|^2 = \langle\xi^2\rangle\langle\eta^2\rangle$. Here t_0 should be a constant.

As we have assumed that the stochastic quantities are dominated by standard Guassian white noise, this gives the properties $\langle\xi_x^2\rangle = 1$. And for a Guassian beam passing through an absorber, the stochastic quantities ξ should have connections with x and P_x respectively. It is reasonable to assume ξ and P_x are dependent and both have zero mean. From eq.(15), we rewrite eq.(14) into:

$$\beta\gamma\frac{\mathrm{d}\epsilon_x}{\mathrm{d}s} = \beta\gamma\sqrt{\chi}\Big(\beta_\perp\sqrt{\langle P_x^2\rangle}\sqrt{\langle\xi_x^2\rangle} + \alpha_\perp\sqrt{\langle x^2\rangle}\sqrt{\langle\xi_x^2\rangle}\Big) \quad (16)$$

By using Moliere scattering theory, one has $\langle P_x^2\rangle = \chi$. And $\chi = (\frac{E_s}{pc\beta})^2\frac{1}{L_{rad}}$, $\eta = \frac{|\mathrm{dE/ds}|}{pc\beta}$ are well defined in [1]. Taking all these expressions into eq.(16), we get

$$\beta\gamma\frac{\mathrm{d}\epsilon_x}{\mathrm{d}s} = \beta\gamma\sqrt{\chi}\Big(\beta_\perp\sqrt{\chi} + \alpha_\perp\sqrt{\frac{\beta_\perp}{\beta\gamma}\epsilon_{xn}}\Big) \quad (17)$$

Taking the eq.(12) and eq.(17) into eq.(11), the evolution of transverse normalized emittance along the cooling channel is:

$$\frac{\mathrm{d}\epsilon_{xn}}{\mathrm{d}s} = -2\eta\epsilon_{xn} + \beta\gamma\sqrt{\chi}\Big(\beta_\perp\sqrt{\chi} + \alpha_\perp\sqrt{\frac{\beta_\perp}{\beta\gamma}\epsilon_{xn}}\Big) \quad (18)$$

From eq.(18), one can get the equilibrium emittance by solving the equation:

$$2\eta\epsilon_{xn} - \alpha_\perp\kappa\sqrt{\epsilon_{xn}} - \kappa^2 = 0 \quad (19)$$

Here I introduce $\kappa^2 = \beta\gamma\beta_\perp\chi$ to make the expression clear. Within Viete's theorem, it is easy to get the root of the eq.(19):

$$\begin{aligned}\epsilon_{xn} &= \frac{4\kappa^2\eta + \alpha_\perp^2\kappa^2 \pm \alpha_\perp\kappa^2\sqrt{\alpha_\perp^2 + 8\eta}}{8\eta^2} \\ &= \frac{\kappa^2}{2\eta} + \frac{\alpha_\perp^2\kappa^2 \pm \alpha_\perp\kappa^2\sqrt{\alpha_\perp^2 + 8\eta}}{8\eta^2}\end{aligned} \quad (20)$$

When the muons travel to the waist of the lattice, the $\beta_\perp$ reaches the minimum and the $\alpha_\perp$ is zero, then the eq.(20) reduces to:

$$\begin{aligned}\epsilon_{xn,waist} &= \frac{\kappa^2}{2\eta} = \frac{\beta\gamma\beta_\perp}{2}(\frac{E_s}{pc\beta})^2\frac{1}{L_{rad}}\frac{pc\beta}{|\mathrm{dE/ds}|} \\ &= \frac{\beta_\perp E_s^2}{2\beta E_\mu L_{rad}|\mathrm{dE/ds}|}\end{aligned} \quad (21)$$

The eq.(21) is the same with Neuffer's transverse cooling formulae eq.(1). We derive this formulae without neglecting any terms, so it is more accurate to figure out the cooling process.

In our study of this linear cooling theory, it is natural to rewrite the eq.(20) into a new patten, which is clear for further analysis:

$$\begin{aligned}\epsilon_{xn} &= \frac{\kappa^2}{2\eta}\left(1+\frac{\alpha_\perp^2-\alpha_\perp\sqrt{\alpha_\perp^2+8\eta}}{4\eta}\right)\\ &= \frac{\kappa^2}{2\eta}\left(1-\frac{2}{1+\sqrt{1+\frac{8\eta}{\alpha_\perp^2}}}\right)\\ &= \frac{\beta_\perp E_s^2}{2\beta E_\mu L_{rad}|\mathrm{dE/ds}|}\left(1-\frac{2}{1+\sqrt{1+\frac{8\eta}{\alpha_\perp^2}}}\right)\\ &= \frac{E_s^2}{2\beta E_\mu L_{rad}|\mathrm{dE/ds}|}\Omega(\beta_\perp,\beta'_\perp) \qquad (22)\end{aligned}$$

where $\Omega(\beta_\perp,\beta'_\perp)=\beta_\perp\left(1-\frac{2}{1+\sqrt{1+\frac{8\eta}{\alpha_\perp^2}}}\right)$.

Here I introduce a function $\Omega(\beta_\perp,\beta'_\perp)$, which reflects the influence from the lattice design. According to eq.(22), the minimum of the emittance might not always be the waist of the lattice.

SUMMARY

In this paper, we have derived a new formulae eq.(22) in a linear ionization cooling channel. And the result goes well with Neuffer's theory in a waist of the lattice. From our theory, the minimum of the emittance might not always be the waist of the lattice and it should be affected by the function $\Omega(\beta_\perp,\beta'_\perp)$. Further, when it is hard to achieve a very small beta function, it is a way to keep the emittance by designing the beta function changing sharply. What's more, a G4beamline simulation would be carried out to check the theory and it might be a way to guide the cooling lattice design. A 6D formulae is under study.

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