International Journal of Theoretical Physics (2024) 63:308
https://doi.org/10.1007/s10773-024-05851-0

RESEARCH

Quantum Entanglement in Classical Systems: so what is the Subquantum Medium Made of?

Guillaume Attuel[1,2]

Received: 6 August 2024 / Accepted: 4 December 2024

Abstract
As astounding as it may still seem to many, Bell's theorems do not prove nonlocality. Non separable multipartite objects exist classically, meaning with local physics, the statistical state measurement of which violates the famous inequalities. Alleviating the almost century old confusion, the correct laws of statistics and logic pinpoint the true oddity of quantum objects: duality. As it is shown in the first part of this short essay, duality plus conservation laws allow the violation of Bell's inequalities for any spatio-temporal separation. To dig deeper into particle dualism, in the second part, a class of models is proposed as a working framework. It encompasses some chaotic excitable reaction-diffusion systems, whose generalized susceptibilities make them compatible with quantized fields and excitations, of any desired symmetry group including the renormalization semigroup. Particles are supposed topological in nature. Bohr-Sommerfeld quantization takes place thanks to topological invariants stemming from densely dispersed defects generated by a multifractal background. Entanglement phenomenology arises because latent variables exist that are carried away, along with the moving particles that have interacted, and by which correlations are preserved. Conservation is assumed to be born in the phase, just as momentum is for instance. In other words, all known phenomena in physics are deterministic, classical and real, in the sense that information does propagate locally and experiments conducted statistically do hide latent variables.

Keywords Locality · Duality · Latent variable · Entanglement · Multifractal · Characteristic classes

Mathematics Subject Classification (2010) 03.65.Ud · 03.65.Ta · 05.45.-a · 05.65.+b · 05.70.Ln · 03.70.+k

1 Introduction

At the dawn of the quantum era, the early developments of de Broglie's ideas were soon hampered by the Copenhagen interpretation: this competing idea of "fundamental and unsurpassable nondeterminism". It refers to an abstract vacuum comprising logical properties to

✉ Guillaume Attuel
guillaume.attuel@bits2beat.com

1 Research, Bits2Beat Predictive Analytics, 37 cours Clémenceau, Bordeaux 33000, France

2 GeoStat, Inria, 200 rue de la Vieille Tour, Bordeaux 33000, France

assigned objective entities, knowable to the physical world after the collapse of an abstract wave function bookkeeping track of all possibilities. The formulation as cumbersome and clumsy as it may be, is prone to magical thinking and it laid the ground to misled esoteric or new age interpretations, in which for instance the observer's mind influence is important.

Ever since, in the accepted dogma, fundamental particles are structureless point-like objects but endowed with macroscopic properties such as size and angular momentum. The bookkeeping wave function allows for the superposition of states because it is complex. It is not said to be simply tracing back probabilistically to our lack of knowledge, but to be fundamental and unsurpassable. This irrational mindset was challenged by a handful of physicists among whom de Broglie and Bohm are maybe the most well known.

Nonetheless, Madelung's analogy with fluid mechanics led to the definition of the so-called quantum potential [1–5] and, later, classes of media were sought for experimentally that could let some quantum phenomena emerge. Such a discovery was made by Couder and Fort recently with walking droplets [6–9]. From the ongoing efforts to describe them [10–16], we know that in this class of "corpuscle-wave mechanics" many quantum phenomena, sometimes very specific as superradiance, are exhibited apart from the indisputable demonstration of entanglement, so far, for physically separated composite states [17]. Note that memory held within the vibrating medium is an essential ingredient there, as the acceleration forcing reaches Faraday's critical threshold.

Complementarily, underlying randomness was soon recognized as an important trait, which opened up the debate on completeness [18]. Randomness was later shown to enable the recovery of some quantum mechanics postulates as the Schrödinger equation [19], or some phenomenology as antiparticle [20]. Brownian entanglement was even defined for strongly coupled springs in a thermal background, with the new insight that the definition could be based on the uncertainty principle [21, 22]. Quantum chaos is yet another related field of investigation where all phenomena occur in a semi-classical limit indistinctly, because it deals with the chaotic dynamics of distributions or waves, hence is described by operators and their spectra in a Hilbert space [23–25]. In fact, Anderson localization has been shown to halt droplet walkers diffusion as in the original quantum case with electrons [26].

Other classes or subclasses of systems, depending on which criteria are chosen, do let quantum-like phenomena emerge as well. It is well known for example that some chaotic or stochastic reaction-diffusion processes can be described by ladder operators in Fock space [27, 28, 31–36]. It was also remarked for instance by Anderson [37] that a plasma was like a kind of toy model for the massive photon in a superconductor, and in fact most of the phenomenology can be described with a Landau free energy. Also, Landau's resonance in plasmas [38] can be exactly accounted for using a direct mapping to scalar QED [39], renormalization well describing a Lamb shift found experimentally[1].

De Gennes realized in the seventies the deep connexion existing between long range order phase transitions[2] occurring in cholesteric liquid crystals and superconductors [40–42]. Spontaneous symmetry breaking in liquid crystals are a good example for the use of topology groups in classical systems [43].

[1] The calculation can be done using chaos theory [72] and microcanonical thermodynamics [73]. The QED result leads further to the practical conclusion (unpublished) that thermonuclear fusion is not achievable in such plasma configurations, in which a strong magnetic field confines the plasma, because de Broglie's and Debye's wavelengths become close and strong scattering prior to quantum tunnelling prevents thermodynamic equilibrium, therefore rendering Lawson's criterion for fusion invalid.

[2] The nematic to smectic A transition, for instance, while the nematic to uniform phase transition is analogous to the ferromagnetic phase transition.

In Section 2, we recall Jaynes's early observation [44] that the conclusion of nonlocality traditionally drawn from Bell's theorems is (not even) wrong logically. Unquestionably, the violation of Bell's inequalities (as CHSH[3]) shows that the joint probability of two measurements cannot be made independent [45]. However, this is readily achievable in classical systems of statistically mixed states, for which the law of measurement outcomes is like Malus's law, where the outcome depends on the relative angle of both analysers (for a singlet state). This result has been already rediscovered by Spreeuw [47] and demonstrated experimentally since [48, 49].

Unfortunately, the former authors speak wrongly of entanglement for ”local degrees of freedom” and agree that nonlocality is not achievable outside quantum experiments, showing strong psychological bias. What should have been underlined, and still must be, is that nonlocality is not a consequence of the inequality theorems since local experiments can violate the inequalities.

In the Copenhagen interpretation and alike, nonlocality means that the wave function of one single undetermined subsystem is “instantaneously” collapsed to a definite state by the collapse of the other entangled subsystem. The obvious clash with relativity leads to ideas such as “entanglement lies outside space-time”. But this is also not grounded on sound logic because any statistical experiment about multiple events lies obviously “outside space-time”, by checking if events are correlated or not, a posteriori, in a common future cone.

In Section 2, we firstly simply recall that combining duality along with a conservation law is sufficient to correlate the information stored within each entangled entity and carried away by well known local physics at finite speed.

In the following sections, reproducing all known quantum phenomena is derived step-by-step within reaction-diffusion systems, in a context thus much more realistic and mundane as even Bohmian pilot waves and closer to de Broglie's original views.

Out of equilibrium media, stationarily driven damped, are good examples as it is shown, those in which solitary excitations can emerge and a relaxation rate can be defined effectively as a random variable. One can then work out straightforwardly how the linear susceptibility acquires the status of a pre-Lagrangian operator (or equivalently a fluctuating effective pre-Hamiltonian). The effective Lagrangian itself is retrieved at annealed coarse-grained scales, guaranteeing conservation laws by Noether's theorem, and including any of the required global and local symmetries under scrutiny. The derivation of a free energy is straightforward following critical dynamics, whether or not dissipative [50], but in which an oscillatory behaviour is added.

Furthermore, Feynman paths integrals in space-time are constructed as projected Kac integrals, that is over $\mathbb{R}$ without requiring the inconvenient analytical continuation from a Wick rotation. This nicely comes in thanks to the definition of well regularized propagators, defined at any desired resolution, thanks to the Hahn-Jordan decomposition of complex measures.

Finally, the underlying necessary randomness can be made Lorentz invariant, for example, depending on the model nonlinearities and symmetries (sine-Gordon like nonlinearities are one well working example, confer Appendix A.1). In fact, if the background has to be scale invariant, it follows from the previous constraints that it is multifractal. It generates a specific dense set of topological defects, thanks to which phase singularities naturally lead to the Bohr-Sommerfeld rule for quantization of momentum and energy, at any scale above cutoff. In that respect, the hierarchy of Chern classes should be related to the multifractal cumulants.

[3] John Clauser, Michael Horne, Abner Shimony, and Richard Holt [46]

The demonstration of entanglement in this field theoretic context comes about in Subsections 8.4 and 8.5.

2 Entanglement

So, as it happens, Malus's law is all that's needed to break Bell's inequalities and the like, as Spreeuw discovered in the late 90s. The conclusion that was drawn from this work is however wrong and has unfortunately prevailed since: It appears that one distinguishes "local (or classical)" entanglement from "nonlocal (or quantum)" entanglement. This is based on belief and not on evidence or proof as it is shown in this section. Although he himself draws the wrong conclusion, what Spreeuw's finding [47] shows is that the theorem merely states an almost tautological property, as Jaynes pointed out already in the 80s : The premise that a set of hidden variables may predetermine mutually independent measurement events already breaks the internal symmetries of the electromagnetic object and the analysers law for outcomes.

Now, locality may be defined as the property that information propagates in time through space. Properties derived from symmetries go conserved within objects, themselves being propagated through space and time. Because entanglement can be shown to exist with local physics, the fact that Bell's inequalities are broken cannot prove nonlocality per se. Statistical mutual dependence is a notion orthogonal to the notion of space and time. It only shows that some latent variables (the proper term for statistical hidden variables) have propagated, carrying information about a conserved property to where and when state measurements take place.

2.1 Making Sense of Bell's Theorems

Even with Bohmian interpretations, EPR tests are said to prove that the wave function collapse is a nonlocal phenomenon because the violation of Bell's inequalities for a multipartite state seems not to depend on time and space. Actually, to demonstrate the contrary, let A and B be the following propositions:

- **A** "For any measurement of a state superposition implying at least two physical entities, Bell's inequalities are found to be respected"
- **B** "The collapse of the wave function is local."

We know they are not equivalent because of the existence of "classical entanglement", i.e. non separable state superposition of a purely statistical nature, violating Bell's inequalities. Whereas information propagates locally, in the original paper by Spreeuw, no spatio-temporal splitting of the entangled beam pair cebits[4] could be setup, please refer to [47]. If now one still insists on nonlocality, raising the hurdle height in proposition **A** by stating that (now in contraposition):

- **¬B** "The collapse of the wave function is nonlocal"
- **¬A** "There exists a measurement implying a state superposition of at least two physical entities that are *separated spatio-temporally*, such that Bell's inequalities are violated."

[4] A beam of light comprised of two vertically or horizontally polarized rays may define a complex vector in $\mathbb{C}^2$ for the photo-detection process. Mixed beams define entangled cebits.

Asserting that "**A** $\sim$ **B**" would be a logical mistake because of the logical equivalence "**A** implies **B**" is "(**B** and **A**) or ¬**A**," and because the augmented **A** does not provide any further information over the previous one, as we are about to see now.

Ultimately, the *spatio-temporal separation* to be considered is space-like. Since, from a local violation of Bell's inequalities, any other time-like separation can be reached just by a Lorentz boost. A classically prepared pair state (a statistical pair state) with one conserved property, shared among the members of the pair, can be split and each member sent to two different locations where the premeditated measurements at space-like separation are performed. Only *then* does the experiment apparatus recollect the data in a common future light-cone. Consider for that matter a measurement probability, the result of which would follow the law:

$$\begin{cases} P_2\,(\theta, +) \\ P_2\,(\theta, -) \end{cases} \tag{1}$$

where P_2 is the probability of finding a $+1$ value at analyser 2 and where θ is some specific variable carrying the shared property, while $+$ or $-$ are the results found at analyser 1 with probability P_1. By construction, the joint probability is made non separable:

$$P_{1,2} \neq P_1 P_2, \tag{2}$$

so that Bell's inequalities cannot be respected. We are told that only quantum systems may accomplish non separability at space-like distances. This is wrong, since one can think of a latent variable, call it $\epsilon \in [+1; -1]$, the value of which does not predetermine rigidly the definite results of any measurement but rather only its probability according to (1). Thus, the two classical members with one carrying ϵ_1 and the other carrying ϵ_2, with a conserved property such that

$$\epsilon_1 + \epsilon_2 = 0, \tag{3}$$

do transport away the law (1) such that the joint probability $P_{1,2} \neq P_1 P_2$ is also carried away anywhere, see figure Fig. 1. In fact, Bayes's theorem yields the true equality implying the conditional probabilities:

$$P_{1,2}\left(\theta_1, \theta_2\ ;\ \sum_i \epsilon_i = 0\right) \;=\; P_1(\theta_1\ ;\ \epsilon_1 \| \theta_2\ ;\ \epsilon_2) P_2(\theta_2\ ;\ \epsilon_2). \tag{4}$$

There are ways to achieve these considerations. Maybe, the most naïve one can imagine is analogous to a "Wigner's friend" type experiment: Concealing the results of "local entanglement" as Spreeuw's, for instance in two black-boxes, sending them far away then activate their opening via a secret key and recollect the data to check the correlations. Quantum mechanics would speak of state superposition (a Schrödinger's cat) as long as one box is not opened. Obviously, this seems here trivially wrong: The statistical superposition of states was imprinted and simply kept secret. The nuance, regarding hidden variables, lies within the non separable joint probability that was measured at first.

On this occasion, returning to spin measurements, such a modification of Spreeuw's experimental set up can be sketched, in which the four photodetectors (placed in the laboratory after the phase beam splitter and polarization rotator) would not reveal any outcome before receiving a special trigger, thereby mimicking some hidden physics. The apparently transparent measurement procedure is carefully followed by two famous imaginary physicists: Alice and Bob. They send the two pairs of unscathed photodetectors at distant locations then only trigger their openings with two laser beams hitting them at space-like distances. Some proper time later, they open the magic boxes and count occurrences. As we just argued, the outputs thus obtained violate Bell's inequalities by construction. Lack of knowledge is not responsible for the correlations but the non separable joint probability $P_{1,2}$.

To make it look less naïve and closer yet to quantum experiments, although Spreeuw's upper and lower rays in each beam had to be measured simultaneously in the lab, in order for the cebits to adequately represent artificially built qbits, it is still conceivable to send each one of the two mixed up beams to two different places. For example, this can be done as some very short pulses and calculating precisely when photodetection measurements must take place to recount simultaneity, thereby revealing the carefully built-in complex vector state correlations. Adding clocks to the detectors would do the right timing trick, see figure Fig. 1. This other kind of refined set up is no different than a cutting-edge EPR test set-up, like a "Wheeler's delayed choice" or a "quantum eraser" type experiment, where resynchronized detections are needed at each detector to sort out the outcomes *eventually*.

Whence, the mention *separated spatio-temporally* does not bring any new distinction to proposition **A**. We have therefore proven that **A** and **B** are not equivalent.

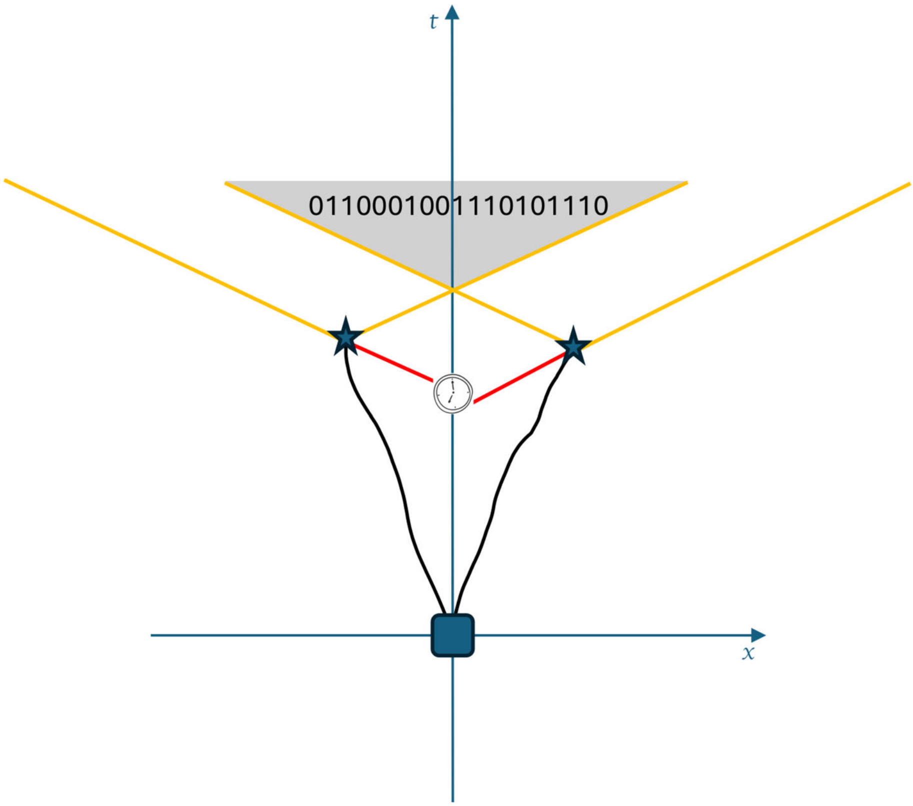

Fig. 1 Rightly timing in advance the triggering of information retrieval at distant locations allows for the classical set-up (the square blue box) to be moved around and split (the blue wiggling lines) at any spatio-temporal separation of the two classically entangled entities: $P_{1,2}\left(\theta_1, \theta_2\ ;\ \sum_i \epsilon_i = 0\right) \neq P_1(\theta_1\ ;\ \epsilon_1) P_2(\theta_2\ ;\ \epsilon_2)$. Within their common future cone is set the statistical counting and discovery of latent correlations (represented by binary bits in the common future cone). The precise calculation to resynchronize measurements is represented by a clock and the triggers are by red rays being radiated at different times, in the laboratory reference frame

Obviously and above all, the difficult part in the experimental set-up is the ability to send the non separable statistically entangled state directly as distinct pulses of light toward two different locations, before the first physical measurement takes place. This is precisely what duality achieves without tricks. With the help of duality and angular momentum conservation, Alice and Bob would send a statistical mixture, of one cebit to one place and the entangled statistical mixture of another cebit to another, and perform the detection there directly. Duality acts like wrapping up the tricked beam pair pulse set-up into a localized wave packet obeying the rules of $P_{1,2}$. This is what makes quantum systems so special, but certainly not unique as it is shown in the next sections.

In summary, nonlocality is not physically proven nor is it even properly conceived: Bell's theorems don't prove so called nonlocality. They are nonetheless compatible with wave-particle duality and conservation laws[5].

2.2 Explicit Entanglement Calculation

Let us now recall a step by step demonstration, basically Spreeuw's. Here, for the sake of illustration, the reasoning comes to the conclusion that conservation of angular momentum and latent polarized fields constitute the two essential classical ingredients required to violate Bell's inequalities thanks to Malus's law, in experiments in which measurements are realized independently at any given time or place.

In order to circumvent the designed principle of Bell's theorems, and according to the previous Subsection 2.1, let us see how it is sufficient to take into account the indeterminate initial conditions as latent information, for the measurement process to generate the appropriate probability distribution, as in (1). Again, this is given by Malus's law in the case of photons and beams. The probability that a linearly polarized photon, therefore the intensity of a large number of photons in a beam, passes through a polariser shifted by an angle $\Delta\Theta$ is:

$$P(\Delta\Theta) = \cos^2(\Delta\Theta) . \tag{5}$$

Take a pair state of zero total polarization and the direction along which one spin is pointing as $\vec{ez}$. One such photon is sent to the left, and a measurement gives either $(+, -)$ along this axis. The second photon sent to the right, reaching the other analyser, will pass through it along any axis following Malus's law, for instance according to (5) if a $+$ result was found at the first one. This is also true for the pair of upper and lower rays devised by Spreeuw[6].

Let us show that (5) combined with the conservation of angular momentum suffices to obtain CHSH's version. The assumption is that preparing a Bell state amounts to sharing information between two parties, which is then carried away by conservation, like angular momentum is, or polarization, etc. A photon like any other fundamental particle possesses a wavy nature and, in fact, a classical beam too as a statistical collection of such photons. In Jones notation, a product state of photon or beam pairs reads:

$$\Psi = \uparrow \otimes \downarrow , \tag{6}$$

[5] Quantum correlations are related to the non causal time ordered Feynman's propagator in QED. However, there is already no special reason why the Feynman propagator can't be used in classical systems, since the choice of propagators varies upon the question asked.

[6] For N spins, to account for the 2^N dimensional space, for each added spin, classical beams must be doubled in Spreeuw's complex amplitude set up, which is certainly impractical but not unfeasible.

where polarizations $|\uparrow\rangle = \begin{pmatrix}10\end{pmatrix}$ and $|\downarrow\rangle = \begin{pmatrix}01\end{pmatrix}$ in terms of the electric field components of the photons or of the upper and lower beam photodetections [47].

This notation would seem to make a strong case that the very difference between the classical and quantum realms lies in the superposition of states. But this is false, as Spreeuw already found out for this setup: if the upper and lower beams are polarized perpendicularly, one cannot write (6) in full compatibility with Malus's law, let us recall how.

In fact, if an experimental setting does not allow full retrieval of knowledge about the initial data, for instance here if it is unknown whether the first photon has spin up or down along $\vec{ez}$, then we have a statistical mixture of both cases. Now, let us avoid the a priori of deliberately using a separable product state for classical pairs, such as:

$$\Psi = \frac{1}{2}\left(\uparrow + \downarrow\right) \otimes \left(\downarrow + \uparrow\right), \tag{7}$$

and, instead, directly calculate the joint probability of experiment outputs and their correlation, as Alice and Bob can recount detections of the two polarisers A and B, with angles 0 and $\Delta\Theta$.

Recall that Malus's law, in Jones notations, can be rephrased using Pauli matrices, so let us write the analysers A and B polarizing light as operators σ_a and σ_b acting on polarization:

$$\sigma_b = \cos(2\Delta\Theta)\sigma_z + \sin(2\Delta\Theta)\sigma_x = \begin{pmatrix}\cos(2\Delta\Theta) & \sin(2\Delta\Theta)\\ \sin(2\Delta\Theta) & -\cos(2\Delta\Theta)\end{pmatrix}, \tag{8}$$

while $\sigma_a = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. Malus's law (5), accordingly with Born rule, thus reads as the quadratic norm of the scalar product between an eigenstate of the analyser and an eigenstate of light, for instance:

$$P(\Delta\Theta) = |\langle\begin{pmatrix}\cos(\Delta\Theta)\\ \sin(\Delta\Theta)\end{pmatrix}|\uparrow\rangle|^2. \tag{9}$$

The other eigenvector, in this case, $\begin{pmatrix}-\sin(\Delta\Theta)\\ \cos(\Delta\Theta)\end{pmatrix}$ takes account of the complementary probability of light going through.

We are now capable of computing the product $A \cdot B$ from the joint probabilities $P\left(a\pm\,; b\pm\right)$. As Jaynes noted back in the 80s [44] this cannot be written down as Bell's theorem premises would force us to consider:

$$\int P(A, a\lambda)P(B, b\lambda)\rho(\lambda)\mathrm{d}\lambda \tag{10}$$

for a set of some hidden variables λ, not because of nonlocality but because Malus's law is applied to both ends, via the electromagnetic field, inside the photons or of the rays macroscopically. Thereby probability theory does not allow the following factorization:

$$P(AB, ab\lambda) \neq P(A, a\lambda)P(B, b\lambda), \tag{11}$$

simply because both $P(A, a\lambda)$ and $P(B, b\lambda)$ are related a minima by $\lambda_A = -\lambda_B$, since they are latent variables accounting for the shared property:

$$\sum_i \lambda_i = 0, \tag{12}$$

here representing the initial total momentum state. In fact, Malus's law yields directly:

$$P\left(a+\,; b+\right) = P\left(a-\,; b-\right) = P\left(\Delta\Theta\right) \tag{13}$$

$$P\left(a-\,; b+\right) = P\left(a+\,; b-\right) = 1 - P\left(\Delta\Theta\right) \tag{14}$$

which is by construction all that is needed to obtain the quantum correlations. Obviously, the mathematical apparatus was exactly point-by-point the same in the classical and quantum cases. What quantum experiments probe is the correlation derived from (13), e.g.:

$$\langle A \cdot B \rangle = \cos\left(2\Delta\Theta\right), \tag{15}$$

where brackets mean statistical averaging.

To summarize what has been found here, a statistical averaging over a statistical collection of state superpositions (a beam of photon pairs) does not break the pair correlations. Therefrom, in the next sections, pursuing the thought experiment outlined in Subsection 2.1, we are going to ask reversely how one can get from a classical setting of complex oscillating fields, being classically entangled, down to the fundamental particle level. In other words, if spin is an internal symmetry property of the electromagnetic field inside particles and if we can safely assume that they are really wave packets carrying such local information unscathed by conservation, will be the question asked. This is pretty much what de Broglie stood for since the advent of quantum mechanics a long time ago [4] with rare discussions ever since, e.g. in [51].

3 The Known Bricks of Quantum Mechanics and Quantum Fields

Henceforth, the remarkable feature, with which Bell's theorems are nonetheless compatible, is the dualism of fundamental particles. Keeping within themselves the wavy nature of vector fields, along with their spin, they comply with the statistical laws for outputs of the field itself at the analyser. Nonetheless, in the accepted terminology of statistics, latent variables (again rather than hidden) may define subclasses within which the same results can be sorted and manifest subvariables as truly independent. In most studied cases however, the latent variable that determines independent classes has not been directly measurable. It is argued in the following that the internal oscillating vector field phase bears this latent information, from which conserved quantities are defined.

Despite this, fundamentally, quantum and classical mechanics seem to differ particularly by the absence of linear oscillations around equilibria in the former and, all the more so, the absence of thermodynamic relaxation despite the presence of intrinsic randomness. This peculiarity is historically at the root of its success in describing energy quantization, which solved the black body spectrum and also the stability of the hydrogen atom.

Nonetheless and paradoxically, quantum mechanics is a linear representation of the world, which hints at it being closely related to linear thermodynamics. However, Dirac remarked that hidden variable theories, whether local or nonlocal, are plagued with thermodynamic linear responses [57]. Hidden variables with randomness (or hidden random variables) imply in effect that a classically perturbed medium relaxes towards equilibrium, inescapably so that the hydrogen atom could not be stable. In contrast, the energy levels of the hydrogen atom in quantum mechanics arise from the discrete spectrum of the Hamiltonian for the electron trapped in the potential well of the more massive proton. Fundamental randomness manifests itself experimentally in the probabilistic nature of quantum state vectors, hence in the probabilistic account of experiments via Born rule.

That being said, a classically chaotic system receives a statistical description that may be generically indistinguishable from thermodynamics. Technically, this depends upon the spectrum of its Ruelle-Policott resonances, but for clarity we will assume finite time predictability. This is tied to the characteristic Lyapunov finite time scale, when it exists, for the loss of predictability of trajectories. A statistical description at much larger scales can then reveal an optimal situation on average, stationary or at equilibrium, the microscopic degrees of freedom being gathered macroscopically with a Taylor's series expansion in the manner of a Landau free energy, valid in the weak sense of distributions. In this context, it is well known that any such variational description of real scalar fields over a random background can be mapped over to a quantum description by a Wick rotation, going from Kac to Feynman paths integrals with analytical continuation.

Two profound differences still remain, for a complete equivalence, in the quantum state superposition and the quantum phase. As we saw in Section 2, quantum state superposition and statistical state superposition are in reality two indistinguishable notions. But, although it is unobservable directly by direct measurement of probabilities, the quantum phase is nonetheless at the root of the quantum realm weirdnesses, such as the Aharanov-Bohm effect.

Yet, a wave function is conceptually close to a partition function but computed with a complex weight. It is like a "square root" of the classical positive definite Gibbs probability measure or, better still, it is the generating function of moments of any observable. The emerging or forced-in complex phase is suited to track similarity transformations in the form of unitary exponential maps of differential operators, which correspond to volume preserving symmetries. This complex weight takes its roots in the wave-particle duality.

3.1 A Wish List of Quantum Oddities

In essence, let us go back to listing what seems different between the classical and quantum worlds, as we are used to, in these typical aspects:

- Quantum mechanics is a non-commutative "matrix mechanics" with the consequence of Ritz's composition law in spectroscopy.
- Conjugate variables like position and momentum are Fourier duals that follow Heisenberg's uncertainty principle.
- Particles are basically locally finite excitations of the fields. Antiparticles enable particle interactions to take place in any reference frame non ambiguously.
- Quantum particles follow non-differentiable paths, in other words random walks, the density measure of which does not exist unambiguously. They are described statistically by transition probability amplitudes by means of Feynman paths integrals, or wave functions, similar to generating functions.
- Annihilation and creation operators describe the discreteness of quanta of energies of the fields in Fock space.
- The non-commutative algebra of differential operators over a bounded physical space gets mapped over to the algebra of unitary compact operators. Inverse differential operators are propagators which appear in the phase of the complex weight as effective interaction potentials.
- Quantum fields are nonlocal, they possess a phase, which has been unobservable directly but leads to the superposition of states at the heart of quantum "weirdness," like the Aharanov-Bohm effect, superconductivity (phase stiffness).
- Forces are mediated between particles because perturbations generated by them are propagated. Propagation is related to force carrying excitations. Static forces between sources of perturbation are thus statistical in nature.
- Lie groups represent all the symmetries preserved in the Hamiltonian that the particles carry. Lie algebra of operators allow calculations.

- Quantum anomalies are found when the measure over paths breaks symmetries locally as the propagators are renormalized through scales.
- $\cdots$ what else?

So, as it would seem indeed, classical and quantum mechanics should be irreconcilable.

In the sequel, it is argued the exact contrary, namely that this no-go listing merely becomes a class definition, as one considers the classical mechanics of oscillating excitable media that generate non-differentiable trajectories of topological excitations, via critical and chaotic coupling. The list is completed with one item as such:

- The phase is also responsible for the "nonlocal weirdness."

3.2 Proposed Framework

In effect, for the proposed class of media, thermodynamic quantities of a dissipative excitable media, driven such as there exists a bath of finite size topological excitations spanning the whole medium, fluctuate as if the medium were locally randomly quenched in cycles. Those parametric fluctuations are deterministic, and their randomness inherently emanates from autonomous chaotic oscillations. At coarse-grained scales nonetheless, the non-equilibrium intrinsic randomness leads to the definition of a stochastic evolution operator.

The phase must help build an appropriate tool to distinguish between disordered cycles of local samples, that the one-point probability measure is not. We will define here the one-particle wave function through a wavelet convolution over the medium and further averaging over randomness, while the many particle wave functions are defined in a similar fashion via a tensor product of wavelets and averaging. In doing so, volume preserving symmetries are violated locally but restored weakly, and define linear differential operators that can be gathered within the linear generalized susceptibility, the average real part of which is identified with a Hamiltonian operator.

The spectrum of the Hamiltonian corresponds indeed to parametrically optimal (or anneal averaged) energy levels, once momentum is defined as usual: Kinetic energy derived from a Laplacian operator and combined to Parseval theorem, since the gradient gives the direction of motion just as for optical rays, canonical momentum $\frac{\partial}{\partial \dot{x}}\mathbb{L}$ is (h is defined in (17) in the next Subsection 3.3):

$$\vec{p} = -\hbar\frac{\vec{\partial}}{\partial x}. \tag{16}$$

It is straightforward to derive this formula from wave packets for which $\vec{p} = \frac{\partial}{\partial k}\omega(k)$, where $\omega(k)$ is the peak frequency in Fourier density, refer to [52] for example.

The imaginary part of the susceptibility is where hidden damping lies, being balanced by the assumed out-of-equilibrium drive of the medium cycles. Energy levels describe an averaged situation, but overall conservation can be guaranteed in a Hamiltonian closed system. One further assumption would require that the intrinsic randomness be a "1/f" noise. In particular, the instantaneous probability measure is singular and multifractal. Trajectories are non-differentiable as required for the non-commutativity of position and momentum operators. Complex multifractal processes over $\mathbb{C}^2$ or $\mathbb{H}$ are defined for the occasion.

By anneal averaging, linear propagators can hence be rescaled invariantly whence renormalized and, in general, their associated algebra is non commutative. Weak complex solutions are thus built as paths integrals that solve the annealed thermodynamic linear response to perturbations of the driven-damped (or only effectively driven-damped) autonomously disordered excitable medium.

The “wave function” is no longer ambiguous, as opposed to its status in standard quantum mechanics: There is no room for interpretation, since its definition is here both ontologic, as it is real, and epistemic, as it computes our lack of knowledge about the system’s microscopic degrees of freedom.

In this framework, entanglement arises naturally because two excitations, prepared simultaneously, carry shared information in their phase about the initial conditions without loss, and bare the same symmetries as the macroscopic field (e.g. spin), before any subsequent measurement process occurs. Lossless transport is guaranteed by topological robustness, despite the maybe worse case of a chaotically oscillating background. Because from its initial condition the phase evolves chaotically, the outcome of a measurement (e.g. transmission through a polariser) is unpredictable, however because of conservation laws and transmission laws, quantum correlations do emerge.

Mathematically speaking, the idea which is being developed has connexions with the finding of large scale approximations to multiscale solutions of systems of coupled real PDEs, whose short scale dynamics is chaotic and are considered as an effective stochastic background at larger scales, in the sense of a partition refinement of trajectories through Kolmogorov-Sinaï metric entropy. Such solutions become statistically exact in the lower range of the Fourier spectrum, thus exact in the weak sense, by means of linear propagators originating in all weakly conserved symmetries in the system. So this connects with and extends the treatment of stochastic Langevin equations in terms of paths integrals [53–55].

The system of coupled real PDEs can be viewed as an evolution matrix acting on real vectors, some components of which naturally oscillate in phase quadrature, which can thus be paired into complex components. Thus, such systems of coupled PDEs of real fields may already describe Lie groups over the complex field such as the relativistic spin group $SL\,(2, \mathbb{C})$. Subclasses are compatible with relativity depending solely on the dispersion relation, in which cases the real and static description of scalar statistical fields (and elliptic operators) in Euclidian space is swapped with the vaster causal realm of Minkowski space of complex vector fields (and hyperbolic operators).

3.3 Chaotic Susceptibility

Narrowing the scope on excitable media for clarity, it is now supposed that a linear response to small enough perturbations exists at a definite temperature T giving rise to natural pulsation responses at ω_0, hence an adiabatic invariant action characterizes any motion:

$$h = \frac{\kappa T}{\omega_0}. \tag{17}$$

Equation (17) stems from the Poincaré invariant for the microscopic Hamiltonian dynamics: $\oint p \; \mathrm{d}q$, where p and q are the generalized coordinates. Consider for illustration a proton orbiting a magnetic field line B, its angular momentum $\mathbf{L}$ is adiabatically conserved and $1/2\; mv^2/\mathrm{B} = q\mathbf{L}/m$, while $\kappa T = mv^2/2$ and $\omega = q\mathrm{B}/m$.

Moreover, the medium is supposed to generate nonlinear finite energy excitations, thought of as a bath of trains of “solitons”, which can bear oscillating coherence. Because of their finite scale and magnitude, they locally alter finitely the medium parameters. A typical example would be as the natural pulsation depends on a thermodynamic quantity like the finite density of matter in the medium: $\omega(\rho)$. The description is deliberately chosen so that the adiabatic invariant action is still by definition $h = \frac{\kappa T}{\omega_0}$, despite the finite excitations of null average that locally modulate the frequency around the value ω_0. We may call ω_0 an equilibrium value,

$$\omega = \omega_0(\overline{\rho}) + \delta\omega(\delta\rho), \tag{18}$$

where $\overline{\rho}$ is the average density, $\delta\rho$ the fluctuations due to the passage of finite magnitude and finite scale excitations.

Here, we might rush to conclude by analogy with e.g. the Klein-Gordon dispersion relation (leaving for later to return to it with a clean and auto-consistent derivation), following de Broglie's thermodynamic primary insight, that (17) bridges with the Compton wavelength, defining mass through temperature as an intrinsic rest oscillation (a mass gap in the spectrum):

$$m_0c^2 = \kappa T = \hbar\omega_0. \tag{19}$$

Besides, fluctuations are assumed to be stochastic at scales above the unpredictability scale, with regard to the particular path of any other locally defined topological excitations. So, at such scales, the granularity of the medium becomes parametric, looks random and non-differentiable. In that respect, any probabilistic transition path between two points is also non differentiable.

More generally, parametric fluctuations mean that the thermodynamic limit does not converge to stationary thermodynamic quantities, and instead to a distribution of values, dispersed spatio-temporally. In other words, there is a stochastic landscape of local equilibrium expectation values, as though there were local cycles of quenched equilibria.

The scenario may be coined "chaotic" in the sense that there exists deterministic dynamical oscillations with a fundamental cutoff scale $\tau > \omega_0^{-1}$, at least a few period cycles, above which the deterministic dynamics are completely unpredictable. But when a medium is perturbed close to local equilibrium, the typical kinetic linear response yields a frequency shift from the natural one and a damping rate because of the scattering of the initial perturbation, both are functions of the natural frequency and obey the Kramers-Kronig relation. The essential remark here is that, since the chaotic parametric fluctuations in the bath are not infinitesimal, the locally defined generalized linear complex susceptibility also varies finitely and stochastically.

Thus, an externally added driving source, a perturbing field ϕ, does not yield a simple relaxation equation, but an evolution equation according to a locally multiplicative stochastic process, above the cutoff scale, generically of the form:

$$\frac{\partial}{\partial t}\rho = i\chi_{\rho,\phi}(\rho,\phi)\,\phi. \tag{20}$$

Onsager relations allow us to reshape it into the more self-consistent and simpler form:

$$\frac{\partial}{\partial t}\rho = i\chi\ \rho. \tag{21}$$

With regard to paths, this equation is thus understood as infinitely many local stochastic ODEs, where the generalized susceptibility χ is a complex random variable and infinitesimal time increments are $\mathrm{d}t \gg \tau$. The density ρ may be replaced by any other thermodynamic real scalar or vector field, in which case χ becomes a random matrix.

It is shown in the sequel that the annealing limit of (21) requires at least the mathematics of quantum field theory to be described adequately.

3.4 Singular Background

One noteworthy assumption is made on χ, namely that it be a rough noise, or in other words a strictly "1/f" noise. This may be written as $\chi \doteq \lambda d\mathcal{B}$, with $\doteq$ meaning equality in law, λ a complex scalar (or matrix, see below), and with the random walk $\mathcal{B}$ having a Hurst exponent equal to zero: $H = 0$. A possible justification for this assumption traces back to the present scenario in which the chaotic autonomous system spontaneously generates parametric fluctuations, which are dynamically coupled to the field fluctuations. Noisy "1/f" signals are not infrequently observed in critical dynamical systems in nature. Furthermore, by integrating (21), the exponentiation of a "1/f" noise becomes a multifractal measure, and, provided the assumptions made in (21), this may be required for the medium to bear scale invariance and consequently propagators to be renormalizable.

Growth or damping rates may be only local and not global, and if so they imply transport, so that generalizing χ for it to incorporate linear differential propagators extending local relaxation rates to transport relaxation, such as translation or rotation at finite velocity, yields an effective stochastic evolution operator amending (21) into a stochastic PDE. Moreover, the case where ρ is a complex vector is envisioned and multifractal projections are made compatible with a compact group structure. The infinitesimal generators are incorporated in χ.

Just as in quantum or statistical field theory, ensemble averaging leads accordingly to the definition of an average evolution operator which describes weak conservation of symmetries (possibly violated locally but not on average) and correlations between excitations as effective forces between perturbation sources. Stated differently, the variational description of the average medium with this stochastic background bath exhibits static forces derivable from an average potential energy directly related to the inverse propagators. It comes in naturally that the average evolution operator bears a spectrum of eigenvalues indeed corresponding to the parametrically optimal energy levels so that it is identified with the Hamiltonian.

The situation bears strong similarities with geometric quantization and the correspondence between lattice disordered Ising spins and lattice QCD, where asymptotically the Lie group is compact and the homotopy class of the gauge connection is densely nowhere trivial, paving the way to the confinement transition. Here, the multifractal singular measure sets the "lattice" of local symmetry breaking since, as any singular measure, it is densely non differentiable. Consequently a dense set of phase singularities exists, compatible with quantization. It can be explicitly shown using geometrization how to relate the classical Hamiltonian with Poissonian dynamics to the Hamiltonian operator. Starting from pre-quantization (see e.g. [52, 56]), as we shall see in the remainder of this essay, the last quantization step is allowed without departing from this classical mechanics framework because of the reality of the phase, and the existence of topological invariants.

The program to avoid Dirac's comments [57] is to take literally the "complex square root" of a regular probabilistic description, of one or several interacting self-generated compact objects (particles), and in fact to consider it as a moment generating function, from the classical mechanics of a system with non-stationary local thermodynamics. In contrast to the extra dimension needed in Parisi-Wu's stochastic quantization scheme [58], here the hidden drive and damping with sustained chaotic oscillations yield quantum mechanics, as a slightly more subtle description than with a scalar statistical field of a disorderly coherently cyclic excitable medium. If no natural cutoff is introduced again, it is at the cost of no longer "giving an absolute meaning to size" related to "fundamental particles" [57]. Closer in principles are stochastic electrodynamics [59] or the Zitterbewegung theory of quantum mechanics [60], because both hold key the internal vibration of corpuscles at their Compton frequencies. Drawing conclusions from a different track of arguments, the present account probably combines some of those assumptions, such as the background fundamental ("zero point energy") vibration [59] and as the geometry of spin is directly encoded in the corpuscle's wave function phase factor [60].

4 Chaotic Excitable Systems as Described by Coupled PDEs

To illustrate the phenomenology sought, let us consider a disordered medium driven by periodic external perturbations, which make the medium go through cycles of phase transitions from a high temperature phase into a quenched phase. We assume that the system relaxes towards one randomly picked metastable minimum of the free energy landscape.

Quenching cycles update the random couplings, therefore the landscape is renewed and all local thermodynamics quantities are updated after each cycle. Averaging over many cycles in general yields unambiguous stationary quantities, if we can assume that the physical medium is self-averaging as it is coarse-grained. If the quenching drive is not achieved by external means, such parametric perturbations are dynamically coupled to the fields in a fully autonomous system.

Take as a typical example the relaxation process of density waves in a quenched state, which is describable by partial differential equations plus noise. By extension, if the noise is autonomously generated with feedback, what we are looking for may be modelled by a larger system of PDEs, the dynamics of which is chaotic.

In that respect, we consider the following system of partial differential equations:

$$\frac{\partial}{\partial t}\Psi = \mathbf{M}\left(\Psi, \frac{\vec{\partial}}{\partial x}\Psi\right) + \kappa \frac{\partial^2}{\partial x^2}\Psi, \tag{22}$$

with $\Psi \in \mathbb{R}^n$ a real vector of n physical components, and $\mathbf{M}$ is a $n \times n$ real matrix $\in \mathbf{GL}(n, \mathbb{R})$ nonlinearly coupling components.

Without the diffusion term, (22) is Hamilton-Jacobi equation. In that case, Ψ should be replaced by an action $\mathbf{S}$ along a path, so that rather replace $\Psi \to \ln\Psi$. $\Psi \sim \exp i\mathbf{S}$ would represent the distribution generating function of paths. Also, $\mathbf{M}$ may stand for the Hamiltonian operator of its symplectic flow.

More generally, the coefficient κ tells us that this system describes a physical medium already at a certain level of coarse graining, and the vector Ψ will represent generically macroscopic observable quantities. Together with the gradient term, they describe transport of information, either locally preserved or globally mixed and lost in the microscopic degrees of freedom.

Invoking linear kinetic theory, natural oscillations are likely to take place. If this is the case, one may replace $\mathbf{M}$ with a block diagonal matrix, each coupling two components in phase quadrature. In an appropriate basis of $\begin{pmatrix} \Psi \\ \frac{\vec{\partial}}{\partial x}\Psi \end{pmatrix}'$, this looks like

$$\mathbf{M}\begin{pmatrix} \Psi \\ \frac{\vec{\partial}}{\partial x}\Psi \end{pmatrix}' \approx \begin{pmatrix} \begin{bmatrix} 0 & -\alpha \\ \beta & 0 \end{bmatrix} & 0 \\ 0 & \ddots \end{pmatrix} \begin{pmatrix} \Psi \\ \frac{\vec{\partial}}{\partial x}\Psi \end{pmatrix}' \tag{23}$$

for any of the complex natural frequencies $\omega^2 = \alpha\beta \in \mathbb{R}$.

The parameters α and β may depend on the components of Ψ and on their gradient components $\frac{\vec{\partial}}{\partial x}\Psi$. The approximate equality designates the neglecting of nonlinear terms from direct linearization of matrix $\mathbf{M}$, terms which become dominant above a characteristic Lyapunov time scale. The matrix components are obtained typically by a linear analysis with kinetic theory for the system lying close to equilibrium.

For concreteness, let us name such components a and b, and assume we can completely split Ψ into vectors of those components A and B. By assumption $n = 2p$, so that A and B have dimension p, while generalizing to $n = 2p + 1$ is obvious. The matrix $\mathbf{M}$ is explicitly broken up into:

$$\begin{cases} \frac{\partial}{\partial t} A = \mathbf{F}(A,\ B) \ + \ \mathbf{H}\left(\frac{\vec{\partial}}{\partial x} A,\ \frac{\vec{\partial}}{\partial x} B\right) + \ \kappa_A \frac{\partial^2}{\partial x^2} A, \\ \frac{\partial}{\partial t} B = \mathbf{G}(A,\ B) \ + \ \mathbf{K}\left(\frac{\vec{\partial}}{\partial x} A,\ \frac{\vec{\partial}}{\partial x} B\right) + \ \kappa_B \frac{\partial^2}{\partial x^2} B, \end{cases} \tag{24}$$

in which the functions $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$, $\mathbf{K}$ nonlinearly couple the components a and b and their gradients. Systems described as (24) are reaction-diffusion media. Boundary conditions (Dirichlet, Neumann) often involve source terms, damping layer or free gradients. It is natural to group oscillating pair components into a complex vector $C = A + iB$ of components $c \equiv a + ib$. The evolution equation through time is now like (22) except for the hermitian unitary matrix of dimension $p \times p$. For instance, C is a (Weyl) spinor for $p = 2$.

In this regard and going back to Spreeuw's original idea, we just realized that four classically oscillating objects are required to mimic one quantum corpuscle with non zero spin. In order to connect with droplet walkers, it seems natural that two of the bonded objects in question are the droplet and its Faraday wave. Because the droplet effectively bounces off the fluid only vertically, while locally the wave amplitude defines an isophase contour line, the description resides in $\mathbb{R}^2$, not in $\mathbb{C}^2$. As such, to gain two extra dimensions, one requires either to bond two such wave-corpuscle objects together, or manage to create a secondary horizontal oscillation for the droplet, reminiscent of zitterbewegung [19, 20, 61–63], going along with a secondary oscillation of the isophase contour. And precisely, this seems to be what is happening at Faraday's forcing threshold [13].

4.1 Excitability

Reaction-diffusion systems are excitable when there exists finite size nonlinear robust solutions, with a specific topological signature, which generically grow exponentially fast (linear regime) from an intrinsic instability and saturate nonlinearly. Such solutions are called excitations, they are local and they transport information in a loss-less fashion at a finite velocity, typically in the order of

$$v \sim \sqrt{\omega_0 \kappa}, \tag{25}$$

if there exits one non zero lower pulsation ω_0.

One enlightening example is given by Berstein-Greene-Kruskal modes in one dimensional plasmas [64]. They are to be distinguished from normal modes since they solve nonlinear steady states of the Vlasov-Poisson problem (here, p=1). The plasma undergoes an instability because it is being self-trapped in self-generated electrostatic potential wells. At first infinitesimal, they grow exponentially fast as the process goes on before saturating nonlinearly. This connects them more or less to the droplet-walkers.

Now, many modes may mix chaotically, and in turn, that would manifest itself as a relaxation process, where modes decay exponentially fast towards a steady-state. It occurs despite the plasma being collisionless, and it is called Landau damping [38, 65]. The chaotic mixing of those modes can be thought of as giving way to a disordered (agitated) background, into which the injected energy from the drive has dissipated. Newly generated excitations, of maybe other types will live on top.

This scenario constitutes a thought paradigm upon which we rely in the sequel.

4.2 Hyperbolicity

In general, the system of (24) can be coined "hyperbolic" in the sense that its evolution through time can be made an ODE as one follows the trajectory of one excitation as long as it exists unambiguously. Such trajectories are similar to the characteristic curves of hyperbolic PDEs. This corresponds physically to moving to proper reference frames.

Solutions are sensitive to boundary conditions which determine the domain of definition of system (24), namely, the subset of allowed, weakly differentiable, functions.

We can think of the simplest case, a paradigmatic example for deterministic low dimensional chaos, as it happens, a fluid in an open tank heated from below and developing only one or two unstable modes into dissipative structures (Rayleigh-Bénard "rolls") [66–68]. Letting such boundary conditions go to infinity, even if boundary conditions are homogeneous at infinity, opens up a whole realm of possibilities since the effective local dynamics become infinite dimensional. This is the case too for the Vlasov-Poisson problem mentioned above, or for the origin of the hierarchical structures, the so-called Kolmogorov-Obukhov "cascade", observed in fully developed turbulence [69]. The framework to probabilistically describe such dynamics of local solutions at coarse grained scales with precise boundary conditions is a Hilbert space with norm $\mathbf{L}_2$.

Yet, the other notion of hyperbolicity in chaotic systems indeed allows us to define the important Lyapunov characteristic time scale τ for the temporal divergence of solutions along characteristics (since the system is also hyperbolic in the sense of PDEs). This characteristic scale is typically associated with phase mixing, when a phase can be defined, and sets a cutoff scale. Above this scale, we therefore assume that the dynamics is effectively generated on top of a stochastic background, in the sense of the partition refinement of any trajectory measured by the Kolmogorov-Sinaï metric entropy.

5 Nonlinear Excitation on Multifractal Background

As discussed briefly, the system of (24) simulates for instance a perturbed medium, the relaxation of which is assumed to be describable by kinetic theory with a linear response, a complex susceptibility, typically, as a natural pulsation and a damping rate.

For illustration and clarity purposes, let us consider the example of a polarized medium, a dielectric, so that here $p = 1$, at rest charge density $A \equiv \rho$, that has a natural finite pulsation ω_0 which is the limit towards long wavelengths $\forall k \to 0$ of a quadratic dispersion relation. The dispersion relation is found by linearizing (24) in Fourier space, mode wise, in which the electric field is $B \equiv \vec{E} = -\frac{\vec{\partial}}{\partial x}\Phi$ with ϕ the electric potential. Note that a more direct way to obtain dispersion relations is from near equilibrium thermodynamics, using equipartition and Gibbs relation [39], for example the Bohm and Gross relation in plasmas [70].

One shows that the microscopic evolution operator in the appropriate basis is $\mathbf{M}(\Psi) = \mathbf{L}\Psi$ and $\mathbf{L} = \begin{bmatrix} 0 & -\frac{\partial^2}{\partial x^2}\Phi \\ \frac{q^2}{m} & 0 \end{bmatrix}$, with electric charge unity q and mass unity m. Here, this natural plasma frequency, also called e.g. a mass gap in superconductivity, depends explicitly on the fluctuating field density ρ from the Poisson relation $-\frac{\partial^2}{\partial x^2}\Phi = \rho$.

Let us repeat that the feedback dependency makes the medium prone to develop nonlinear oscillations with finite amplitude density variations. Such a situation is found for instance here in plasmas, where BGK solitons grow nonlinearly stable [64].

The popular, albeit not quite exact, picture of a surfer, representing a small density perturbation riding the wave, who is henceforth accelerated or slowed down in the trough, illustrates nicely enough the typical phenomenon of trapping. Trapping further acts on the density variations within the wave and chaotic dynamics result in general, refer for example to the simplified Hamiltonian mean field model [71–73].

As a consequence, the UV cutoff scale for our scenario is of order $\tau \sim \delta\rho^{-1}$. In this context, Landau damping is closely connected to the divergence of trajectories, that is, closely related to phase mixing $\tau \sim \delta\omega^{-1}$ (e.g. van Kampen modes [74, 75]).

Let us recover this result, we can write that the natural pulsation depends on the relaxing field:

$$\frac{\partial}{\partial t}\rho = i\omega\left(\rho\right)\rho, \tag{26}$$

with $\omega \in \mathbb{R}$. There are local frequency shifts within any region of size a wavelength $\forall s \sim k^{-1}$:

$$\omega = \omega_0(\rho_0) + \delta\omega\left(\delta\rho\right), \tag{27}$$

where $\delta\omega \in \mathbb{R}$ is an implicit function of position at coarse grained scale s.

A deterministic scenario, on the one hand, with only one mode present for the ODE (26) and (27), could correspond to regimes in which low dimensional chaotic dynamics or routes toward chaos can occur, as for instance when the perturbation is external and resonant in which case the dynamics develop Arnold tongues [76].

On the other hand, (26) is understood as an autonomous ODE for the time evolution, indifferently along (moving with the flow $\frac{\mathrm{d}}{\mathrm{d}t}$) or "on" (fixed position with respect to the flow $\frac{\partial}{\partial t}$) characteristics. It becomes a stochastic ODE as one considers the ensemble of all possible chaotic evolutions along any of the chaotic trajectories within the extended system, with boundaries at infinity, at scales $s > k_0^{-1}$ and $t > \omega_0^{-1}$.

A broad spectrum of Fourier modes with an algebraic slope of their power density $\forall k > k_0$ is then observed, often named "cascades" by analogy with 3D incompressible fluid turbulence [69] and slight abuse of language. For example, k_0^{-1} may be as small as the inter-particle distance or as a characteristic length like the screening length, the Debye length, etc.

In the weak coupling scenario, perturbation theory shows self-consistently how, with those assumptions, the Green function

$$G_0 = \lim_{\alpha\to 0}\frac{1}{\omega - \omega_0 + i\alpha} \tag{28}$$

is renormalized according to Dyson's self-energy equation [77] $G = G_0 + G_0\delta\omega G_0 + G_0\delta\omega G_0\delta\omega G$, giving rise to a damping term and frequency Lamb shift where $\omega = \omega_0 + \omega_L + i\gamma_L$:

$$\omega_L + i\gamma_L = \lim_{\alpha\to 0}\sum_f \frac{1}{\omega_0 - \omega_f + i\alpha}|\delta\omega_{0,f}|^2, \tag{29}$$

where $|\delta\omega_{0,f}|^2$ is the power spectrum of the fluctuations centred at ω_f [78, 79]. When explicitly writing down the power spectrum, one retrieves Landau's damping formula [39].

In effect, it is assumed here non-perturbatively that ω bears the properties of a noise, at scales larger than the Lyapunov or Landau (or resonant) damping scale $\gamma_L^{-1} \sim \langle|\delta\omega|\rangle^{-1}$, so that it will be formally written as the differential of a random walk:

$$\Im(\delta\omega) \doteq \lambda \mathrm{d}\mathcal{B}, \tag{30}$$

with $\mathcal{B}$ characterized by a specific Hurst exponent H:

$$\left\langle |\delta_\tau \mathcal{B}|^2 \right\rangle \sim \tau^{2H}, \tag{31}$$

and λ is a real scalar.

Furthermore, if we impose that the modulus of $\delta\rho$ should scale, at large scales, according to a fractional exponent smaller than one $\eta < 1$, to make it Hölder continuous and non differentiable [81] as it is needed for consistency:

$$\|\delta\rho(at)\| \doteq a^\eta \|\delta\rho(t)\|, \tag{32}$$

putting together the integration (in the sense of distributions) of the stochastic version of (26) with the requirements (30)) and (32) yields:

$$\eta \log a \doteq \lambda \int_t^{at} \mathrm{d}\mathcal{B}(s), \tag{33}$$

implying that H must equal zero:

$$H = 0, \tag{34}$$

by definition of the random walk $\mathcal{B}$ in order for the integration to effectively yield the log in the r.h.s. of (33). This condition is not granted a priori and restricts the class of eligible systems of PDEs (24) for quantum phenomenology. The heuristic is that it's pretty ubiquitous in nature to find "1/f" noises, and one which would feed the relaxation inducer $\delta\omega$ would not be surprising. However, making this compatible with the dynamical formulation (24) may demand some tuning, the analysis of which is left for another work[7].

At this stage, η is not single valued based on generic conditions. On the contrary, it is assumed to be distributed according to some law which is determined based on context, typically Gaussian or Poissonian to give instances [80]. In other words, it makes the modulus of the variations of ρ a multifractal scale invariant density measure.

6 One-sample Solution

Owing to its multiplicative nature, the formal statistical resolution of (26) can be further casted into a Feynman-Kac paths integral formulation, stressing the fact that paths are not differentiable but absolutely continuous.

The system of equations for the previous example is of vector space dimension two (in spatial 1D), and the interesting field to be quantized is thus a complex scalar, namely the fluctuating quantity $c \equiv \rho + i\vec{E}\cdot\vec{e}_1$. The information content which remains in the correlation between both field components is contained in the phase term. For instance, basic oscillations enshrined in the phase quadrature between the two fields. In the sequel, we keep the more familiar notation ρ to denote any real or complex component as c (ρ not to be confused with a density matrix).

[7] The reader may refer to personal notes entitled "Building multifractal processes with and without long range covariance kernel" in which a peculiar example is worked out. To be filed on HAL

But, before we dig a bit further into some basic principles of paths integrals, let us take one more step into the formal integration of the stochastic version of (26) in which ω is a complex random scalar:

$$\rho(t) \doteq e^{i \int_0^t \mathrm{d}s\ \omega(s)} \rho(0), \tag{35}$$

and the exponentiation is understood as a limit of a multiplication series

$$e^{i \int_0^t \mathrm{d}s\ \omega(s)} \doteq \prod_{0}^{t=N\delta t} e^{i\delta t\ \omega(t)}, \tag{36}$$

for $N \to \infty$, $\delta t \to \tau \to 0$, which is a separation of scales or multiscale approach. The stochastic pulsation ω can be further understood as being the "Hamiltonian", or in other words the evolution operator.

In the positive real domain considering only $\Im(\omega)$, the one-sample multifractal density field generically reads:

$$\rho(t) \doteq \sigma \int_{-\infty}^{t} e^{\lambda\delta\mathcal{B}^0(t-s) - \lambda^2\mathbb{E}(\delta\mathcal{B}^0(t-s)^2)} \mathrm{d}\mathcal{B}^H(s), \tag{37}$$

in which additive noise was included, with variance σ and Hurst exponent H, within (26), in the sense of a generalized Ornstein-Uhlenbeck process. This form is what is known as Gaussian multiplicative chaos and was first described by Kahane [82]. More general multifractal processes can be defined, that loose nonlinear correlations but do keep the main one-point statistical ingredient (32)[8].

Signed and complex multifractal measures can be defined based on this solution (37), simply by using the Hahn-Jordan decomposition and projecting along the real axes. Physically, a signed or complex scalar is conveyed at some velocity along its characteristics where the scenario holds, that is to say, the evolution in time is multifractal, and consequently the spatial measure too. Therefore, what precedes remains also true for $\omega \in \mathbb{C}$ and for vectors C of dimension $p \geq 2$, conveyed as well along characteristics. We can think of Levy stable vectors as operators, the average evolution of which is decomposed into the infinitesimal generators of a Lie group, on top of the multifractal background enforcing a vectorial sum of its multifractal components, as we demonstrate in Section 8.3.

7 Paths Integral

7.1 Analysers and Probability

A statistical description can be given in two steps. The first one is by using local projection analysers, like wavelets. The second one is by averaging over many realizations. In standard quantum mechanics, unfortunately, the first is confused with the second.

Take a wavelet[9] Φ_l defined at projection scale l and centred at point x, so that the projection is a convolution, introducing the bra-ket Dirac notation constructively:

$$\Psi(x,t) = \int_0^L \mathrm{d}x'\ \Phi_l(x - x')\rho^*(x',t) = \langle x|\rho(t)\rangle\,, \tag{38}$$

[8] Ibid.

[9] A mother wavelet of the required dimension, in general either scalar or complex, but quaternionic wavelets exist.

for a one dimensional system of size L, where the vector $|\rho\rangle = \rho^*$ is the function conjugate to be acted on by convolution. The dual form is defined as the convolution $\langle x| \equiv \Phi_{l,x} \star$:

$$\langle x| \cdot \equiv \int_0^L \mathrm{d}x' \; \Phi_l(x - x') \cdot \tag{39}$$

while the vector is the analyzing wavelet conjugate, centred on the point of analysis $|x\rangle \equiv \Phi^*_{l,x}$ so that it normalizes to $\langle x|x\rangle = 1$ and we define the projection operator as the convolution with this kernel $|x\rangle\langle x| = \|\Phi_{l,x}\|^2 \star$, so that for instance:

$$|x\rangle\langle x|\, \rho(t)\rangle = \int \mathrm{d}x' \; \|\Phi_l\|^2(x - x')\rho(x', t), \tag{40}$$

giving the "energy" of ρ at position x over a scale l. The wavelet quadratic norm tends towards a Dirac as $l \to 0$. For consistency $\langle\rho| \equiv \rho$, hence $\langle\rho|x\rangle\langle x|\rho\rangle = \langle x|\rho\rangle\langle\rho|x\rangle$. This is how Born rule is recovered:

$$\|\Psi(x, t)\|^2 = \int \mathrm{d}x' \; \|\Phi_l\|^2(x - x')\|\rho\|^2(x', t). \tag{41}$$

Indeed, by definition of wavelets, the convolution (38) is non negligible if and only if the increment around x of the signal $\|\frac{1}{l}\int_x^{x+l} \delta\rho\|$ is also non-negligible at this scale. A localized excitation, such as a soliton with finite extension of size l, satisfies this condition if its phase oscillations are not of much higher frequency. In other words, the functional is a local focus on the activity of the whole medium at finite energy on position space x and at local frequency scale l. Note that it is also conceivable, though not used in common practice, to project with quaternionic wavelets in order to assess more structure within the excitations in higher dimensions.

A probability can thus be defined in the frequentist manner as a weight of the uncertain outcomes for the projection (38) evaluated against the whole system. By construction, $\mathbb{E}\left(\|\Psi\|^2(x, t)\right)$ is an empirical probability density function, counting the relative presence frequency of a localized excitation. By Parseval's theorem, the same is true in the Fourier domain (centred on momentum (16)).

7.2 Propagators

From the stochastic (35) and its evolution operator, a linear propagator G is constructed that works for Ψ. Put differently, we look for a linear hyperbolic PDE associated with G according to which Ψ evolves at longer time and length scales with respect to the Lyapunov scale.

From (35), inserted into (38), one has for a given time t' in the past:

$$\langle x|\rho(t)\rangle = \left\langle x|e^{i\int_{t'}^{t}\omega(s)}\rho(t')\right\rangle. \tag{42}$$

Therefrom, since the wavelet basis is normalized, we can project on x' and integrate over the whole domain:

$$\langle x|\rho(t)\rangle = \int \mathrm{d}x' \; \left\langle x|e^{i\int_{t'}^{t}\omega(s)}|x'\right\rangle\left\langle x'|\rho(t')\right\rangle. \tag{43}$$

Here, it is implicitly assumed that there is one excitation being tracked by the wavelet projection from position $x'(t')$ to $x(t)$. Those steps define a propagator for any given path leading to x at time t such that:

$$G\left(x, t; x', t'\right) = \left\langle x|e^{i\int_{t'}^{t}\omega(s)}|x'\right\rangle. \tag{44}$$

But in fact, this operator is contingent, because each path built from (40) is still randomly chosen at each realization of (35) from an arbitrary time and point in the past. In order to give an essential character to G, averaging over all those realizations of paths is required for a sound definition of transition probabilities, i.e. annealing. Quenched averaging would be considering only all possible paths for fixed randomness during the elapsed time. For very long time intervals, because randomness updates at each relaxation cycle, we may assume that the system self averages. Hence, we will no longer be distinguishing between both averaging in the subsequent arguments. Therefore, the definition (44) is replaced with the proper kernel:

$$G\left(x,t;x',t'\right) = \mathbb{E}\left(\left\langle x|e^{i\int_{t'}^{t}\omega(s)}|x'\right\rangle\right). \tag{45}$$

In so doing, the propagator G acts on the expectation of Ψ, and is suited for a statistical description over all possible realizations. In other words, the expectation of Ψ changes in space and time according to (45), which is monitoring its weak evolution (in the sense of distributions). Note that symmetries may be restored in this weak sense, whereas they are broken locally. For instance, ω may incorporate a translation operator $e^{-X\frac{\partial}{\partial x}}$ of size X, or a rotation operator with angle θ around axis e_j, $e^{-i\frac{1}{2}\theta\sigma_j}$. By assumed independence of G and Ψ, all this means precisely:

$$\mathbb{E}\left(\Psi(x,t)\right) = \int \mathrm{d}x'\; G\left(x,t;x',t'\right)\mathbb{E}\left(\Psi(x',t')\right), \tag{46}$$

Despite appearances, (45) is not yet a positive definite probabilistic relation, it may be signed, complex, quaternionic... etc. It has the quantum analogue meaning of a transition probability amplitude.

Quantum mechanics definition of the wave function does not give enough precision about the intrinsic randomness and definition of $\mathbb{E}(\Psi)$. This loophole leads to the whole problem of interpretation. Interpretations lie within two classes, either ontic or epistemic. Here, both are true by construction.

Moreover, from (46), it will be clear that the averaged Hamiltonian can bear symmetry invariance despite them being broken locally.

7.3 Lagrangian

Let us start with the classic low energy scenario, which in terms of linear dispersion relations amounts to:

$$\omega(k) = Dk^2 + \omega_0. \tag{47}$$

Let us show that one recovers the Hamiltonian $H = T + V$. Everything follows as in textbooks.

By construction, the Chapman-Kolmogorov relation is:

$$G\left(x,t;x'',t''\right) = \int \mathrm{d}x'\mathrm{d}t'\; G\left(x,t;x',t'\right)G\left(x',t';x'',t''\right).$$

Let us set units to $D = 1$ and shrink the time step down to zero $t'' - t' \equiv \delta t \to 0$ (see (56)). By (36), splitting the rapid oscillation component at the characteristic mass gap frequency ω_0 from the slower oscillations characteristic of the evolving envelop, we bring the evolution (45) and (46) to that of an excitation possibly at x evolving in an average potential

$V_l = \mathbb{E}(\omega(\rho) - \mathbb{E}(\omega_0))$, as described by the Feynman-Kac paths integral:

$$G(x, t; x_0, t_0) = \int \mathrm{D}x(s) \ e^{i \int_{t_0}^{t} \mathrm{d}s \ \frac{1}{2}|\frac{d}{ds}x(s)|^2 - V(x(s))}. \tag{48}$$

This separation of scales could be taken into account in an effective Hamiltonian anyway, but $\delta t > \omega_0^{-1}$ in (48). The kinetic energy term stems from the k^2 dispersive term (Laplacian differential operator) in (47). The measure over the space of paths is loosely defined by integration at the regularization limit, as in textbooks with the same limitations, where now $\delta t = \frac{t-t_0}{N}$: $\mathrm{D}x(s) = \lim_{N\to\infty} \delta t^{-\frac{N}{2}} \prod_{i=1}^{N-1} dx_i$. Consequently the propagator in (45) can now be decomposed using the set of eigenfunctions and eigenvalues of the Hamiltonian. For instance, if countable within a compact symmetry group, they are ϕ_n and E_n so that $G(x, t; x', t') = \sum_n \phi_n^*(x')\phi_n(x)e^{-iE_n(t-t')}$. Thereof, the Green function and resolvent calculations follow, which can be regarded as a justification of the use made of perturbation theory to derive (28) and (29).

All the quantities in (48) are defined at the scale l of the particular excitation under scrutiny, but for most practical cases the notation can be safely dropped, because we set the position $x(t)$ to be centred on the excitation of size scale $l \approx v\,\omega_0^{-1} = \sqrt{\frac{\kappa}{\omega_0}}$. In particular, the Lagrangian is:

$$\mathbb{L} = \frac{1}{2}|\frac{d}{ds}x(s)|^2 - V(x(s)). \tag{49}$$

This Lagrangian would describe the optimal equilibrium of a pinned elastic line living in an average potential landscape, and being randomly pulled an infinite number of times, if it were not for the effective Wick rotation $t \to it$.

From Section 5, we constrain the random pulls $\delta\omega$ to have logarithmic autocorrelation, similarly to a Gaussian free field in two dimensions. In one spatial dimension and one time direction, logarithmic correlations are still present within light cones (i.e. with Minkowski metric signature $r^2 = x^2 - t^2$), when Feynman propagator is used in replacement of the causal (advanced or retarded) propagators. This optimal equilibrium being averaged over many discontinuous realizations, recovers classical continuity weakly. It also describes for instance the optimal path of random pinning of density waves when $x(t)$ is replaced with the field of a Goldstone Boson gapless mode.

7.4 Density Matrix

Usually, paths integrals like (48) are justified by analytical continuation from Euclidian space into Minkowski's after Wick rotation, because there is no simple way to define regularized Lebesgue measures (the density of "paths") of complex functions in the Hilbert space. With regard to the Lebesgue-Radon-Nikodym decomposition, ρ is here understood as a random field, whose measure is given by the unique sum decomposition of an absolutely continuous density and a singular measure (with respect to the Lebesgue measure on the real axis $\mathbb{R}$).

From the definition (40) and (41), one may recognize the density matrix of a mixed state in $\mathbb{E}\left(\|\Psi\|^2(x, t)\right)$. As many realisations accumulate, an empirical probability density emerges for the various possible positions of the excitation, as taking the trace over the particular eigenstate centred on x:

$$\mathbb{E}\left(\|\Psi\|^2(x, t)\right) = p(x)\,\langle\rho(t)|\,x\rangle\,\langle x|\,\rho(t)\rangle. \tag{50}$$

A single excitation thus defines a pure state over the basis of positions:

$$1|x\rangle\langle x|, \tag{51}$$

in agreement with intuition, and indeed a mixed state defines probabilities:

$$p(x)|x\rangle\langle x|. \tag{52}$$

For such a single localized excitation, a mixed state telling that half of the time the excitation is found at x and the other half at y is well described by the density matrix:

$$\frac{1}{2}\left(|x\rangle\langle x| + |y\rangle\langle y|\right). \tag{53}$$

It is not yet manifest as to how a quantum superposition of states can be achievable statistically without focusing further on the physics. We have dug into that aspect though in Section 2, but one may forge some intuition thanks to the discovery of walkers by Couder and Fort. Those entities are localized but extend sufficiently so that they can interact with the two slits in that experiment, forming a law of transmission with interference fringes. Interference fringes mean that the probability of being at x is not independent of the probability of being at y, and the correct description requires non separable joint probabilities. Thus, one needs to generalize the description (50) to nondiagonal cases

$$p_{x,y}|x\rangle\langle y|, \tag{54}$$

in which the cross correlation term (or non diagonal term) from the joint probability:

$$p_{x,y} - p_x p_y \neq 0 \tag{55}$$

is not zero. Despite all, the hidden physics may remain unknown to the experimenter.

Let us return to the medium fluctuations, for any Hölder exponent smaller than one, $\eta < 1$, the density measure of a multifractal process is singular, and scales at any given coarse-grained scale l like: $\rho \sim l^{\eta-1}$. This means that the density matrix and trace over eigenstates scale similarly as one modifies the wavelet width. Because this choice is constrained from a cutoff scale below, one may probe how averaging over all expectations in (50) blurs or not this scale invariance. One already knows that at the localization transition, the multifractal signature is apparent in (50) [83, 84].

Multifractal measures can be defined on positive measures, signed measures, and complex measures, through Hahn-Jordan decomposition. The philosophy adopted here is therefore to not go through analytical continuation, but instead to understand paths integrals as a notation trick for the propagator (45), based on such decomposition. Consequently, (48) is also a unique decomposition into a sum of well defined paths integrals, as each one is now naturally regularized over $\mathbb{R}$. The dimension and symmetry group of ω are extended in Section 8.3.

Let us recall that the above derivation steps from (47) yields the Schrödinger equation. They suffice to guarantee the generalization to any dispersion relation, as one compatible with relativity, and they can be generalized obviously in standard ways to fields. For instance, to obtain Klein-Gordon's equation, the dispersion relation is $\omega^2 = c^2k^2 + \omega_0^2$, where $\hbar\omega_0/c^2$ is the rest mass. Let us also mention that quantum anomalies still naturally come forth as the propagators are renormalized[10].

[10] For instance in the historically first case discovered by Fujikawa, the chiral anomaly arises because of triangle Feynman diagrams. Interpretation with the Atiyah-Singer index theorem holds true, whatever the explicit construction of (48).

7.5 Non-differentiability Induced Non Commutativity

Let us recall a well known fact of the algebraic formulation of quantum mechanics: Non differentiable paths generate non commuting operators, as defined via paths integrals.

Because paths thus G are not differentiable, one needs to be cautious when acting with the displacement operator $e^{x\frac{\partial}{\partial x}}$. Evaluating the sequence of operators $\frac{\partial}{\partial x}$ and x from (48) shows that they are not commutable. This is because with (45) and non-differentiable paths where $x_{t+\Delta t} - x_t \neq x_t - x_{t-\Delta t}$ as $\Delta t \to 0$ in (46), we have the necessary non zero transition kernel:

$$G_{\pm} \; = \; \lim_{\Delta t \to 0^{\pm}} G\left(x_{t+\Delta t}, t + \Delta t; x_t, t\right) > 0, \tag{56}$$

only compatible with a non zero commutator between x and $\frac{\partial}{\partial x}$, where:

$$\int \mathrm{D}\mathbf{x}(s)\; e^{i\mathbf{S}(\mathbf{x})}\; [x,\; \frac{\partial}{\partial x}] \; = \; G_{\pm} \tag{57}$$

and $\mathbf{S}(\mathbf{x})$ is the action over a path $\mathbf{x}$ derived from (45) and (49).

In the classical setting of Brownian bodies [21], one can compute the velocity changes of their non differential paths at coarse grained time scales δt, as a regularized spatial derivative of the body position probability logarithm times the equilibrium temperature[11]. Let us recall on this occasion that Dirac's remarks obviously hold in this equilibrium situation. In the functional space of quantum systems and their analogues, the correspondence goes with the already known definition of momentum (16) that involves a spatial derivative ot the wave function (46) (or more precisely in this situation, of the Wigner distribution because it accounts for the instantaneous autocorrelations) times the quantum of action, defined in (17) as a Poincaré invariant in which the thermal temperature is combined with the frequency gap.

But (57) cannot be obtained per se in such a microscopic setting of Brownian bodies. Indeed, the displacement operator $e^{x\frac{\partial}{\partial x}}$ probes translational symmetry, and it is not meaningful within a statistical ensemble of translationally invariant microscopic states. The definition through uncertainty relations in [21] gives the interesting precision that fluctuations of microscopic sub-ensembles must show correlations, and this is not surprisingly achieved for strongly interacting subsystems. As the authors in [21] rightly point out, the uncertainty relations differ in this statistical equilibrium situation and in the quantum cases as, in the former, fluctuations involve a large number of bodies agitated by thermal noise, while in the latter a single body abides by the uncertainty relation without need of any external agitation.

Here, the displacement operator acts on wave packets, so that position and momenta are already conjugate variables via Fourier space and follow the same rules as in quantum systems. The displacement operator would still be meaningful, all the same, in the limit where interferences are blurred, by rescaling away the quantum of action $h \to 0$ as more energy is injected. The Wigner pseudo-distribution then converges towards a positive distribution of the position (e.g. known as Husini's) in this semi-classical limit. As it was mentioned in the introduction, dealing with the evolution of such localized distributions is the field of quantum chaos.

[11] Section 2, (26), written here with unit mass: $\delta v(x, \delta t) = -\delta t T \frac{\partial}{\partial x} \ln\left(P(x, t)\right)$.

We have thus obtained, up to this point, the paths integral quantum mechanical formulation, as a coarse-grained statistical description of a class of chaotically fluctuating classical excitable system, which self-generates a singular background.

8 Densely Dispersed Phase Defects

What precedes shows how ω acquires the status of a "pre-Lagrangian" operator (or equivalently there a fluctuating effective Hamiltonian), the Lagrangian itself, including any required symmetry groups within the process under scrutiny, emerging at annealed coarse-grained scales. We saw that excitations are typically endowed with a typical finite velocity, making it possible to define rest frames for appropriate dispersion relations. The main point, not surprising, is that the pre-Lagrangian remaining stochastic within its parameters (or random potential, a situation akin to optimal control), the final Lagrangian solves the optimal dynamical option. In other words, the classical path is the parametrically optimal path, according to the least action principle.

8.1 Bohr-Sommerfeld Quantization

As a result, let us jump to the optimal Lagrangian exposition of system (24). As in quantum field theory, there are sources, excitations, fields, and the random background. Gaussian integration over the stochastic fields reveals the residual interactions between excitations. These are all describable by symmetry (Lie groups) with a gauge connexion.

Generally speaking, all possible ground states (or vacua) are described by a manifold G/H of the cosets of the subgroup H to which the symmetry group G has spontaneously broken down to [52]. A topological defect is generated as one ground state binds to another, often via boundary conditions. Take the simplest pure gauge vector field for illustration purposes $A = \frac{\vec{\partial}}{\partial \mathbf{x}}\phi$, derived from ϕ the unknown phase, $G = U(1)$ and $H = 1$ the trivial group, for an ordered phase. The first homotopy group is $\pi_1(U(1)) = \mathbb{Z}$, each integer evaluating the charge of a vortex.

Now, the action of the random generator on this symmetry is to create many of those topological defects, as densely dispersed and compatible with a multifractal singular measure as can be. This non-perturbative case is akin to a BKT transition in two spatial dimensions, where the XY model has $U(1)$ symmetry becoming compact, or in the more general non commuting case illustrated by the confinement transition where $A = -ig\frac{\vec{\partial}}{\partial \mathbf{x}}g^{-1}$, with g defining the symmetry group of the Lagrangian. The topology of this simple vacuum manifold is characterized by (58), which is the first Chern-number, and can be generalized to any Chern number for non Abelian symmetries. Phrased another way, the gauge connexion is nowhere trivial, so that:

- For any contour around any point, circulating the gauge is quantized, there is a winding number:
$$\frac{1}{2\pi}\oint \frac{\vec{\partial}}{\partial \mathbf{x}}\phi \cdot \mathrm{d}\mathbf{x} \in \mathbb{Z}. \tag{58}$$
- For any continuous deformation of the contour, the integration (58) of $\frac{1}{2\pi}\mathrm{d}\phi$ generates random sampling from $\mathbb{Z}$, with a distribution centred on 0 and dominated by $\{-1\ \ 0\ \ 1\}$, since the medium has zero net topological charge.

By the action phase integral in Hamiltonian action-angle variables, hence Bohr-Sommerfeld's quantization follows for any closed orbit. It is unnecessary at this stage to assess whether an excitation be localized or delocalized over its orbit, such as one defined in spherical harmonics. Nonetheless, because the underlying gauge should be disordered, one might expect Anderson localization to occur. In any case, energy quantization follows too by the change of variable $\frac{\vec{\partial}}{\partial \mathbf{x}}\phi = \frac{\vec{\mathrm{d}}\mathbf{x}}{\mathrm{d}s}$, true for any localized excitation (of unit mass), in which the phase isochrone moves at group velocity. The kinetic energy is derived after annealing as in (49). Strictly speaking, we are dealing with a charged particle momentum, with respect to A.

8.2 Multifractal Branch Cut

The above assertions are justified mathematically because the contribution of the random phase generated by fluctuations contributes an integer. Intuitively, not only is the manifold not simply connected, the zeroth homotopy group is now also non-trivial, but even more the induced geometry is not smooth and non compact: In d+1 dimensions, there is certainly an uncountable number of homologously differing 1-cycles and disconnected parts due to the densely dispersed phase defects, so that, dually, de Rahm cohomology spaces generated by the closed form $\mathrm{d}\phi$ are also infinite dimensional. It is not exact because the phase ϕ is not a function, although its derivative is well defined and unique.

Take any path from time $t = 0$ to time t, the phase is:

$$\delta\phi = \phi(t) - \phi(0) = \int_0^t \omega(s)\mathrm{d}s, \tag{59}$$

where s is the time defined in the rest reference frame, following the maximum of the local excitation, the wavelet projection of which is (38), so that ϕ is defined along the path $x(t)$(or characteristic) taken by any local excitation. Equation (30) was $\Im(\omega) = \lambda \mathrm{d}\mathcal{B}^0 \in \mathbb{R}$, but let us extend the definition to the complex domain to include real angle arguments $\Re(\omega) \neq 0$ and put $\lambda \equiv 1$ for the simplicity of notation.

Thereby, globally at any given time, it is possible to contemplate the complex random phase field contribution as a holomorphic integration of a complex rough noise:

$$\delta\phi(x_0, x(t)) = i \int_{x_0}^{x(t)} \mathrm{d}\mathcal{B}^0(s) \in \mathbb{C}, \tag{60}$$

where s denotes now any virtual path of curvilinear coordinate traced within space. Now, when this path forms a closed loop in space $x \to x_0$, besides (33), an analytic continuation to the complex logarithm[12] has a branch cut

$$\log(x_0^{\pm}) - \log(x_0) = \pm 2i\pi,$$

($\pm$ meaning clockwise or counter-clockwise) which forces an integer contribution into the r.h.s. of (60), multiple of 2π, setting the real phase angle:

$$\Re\left(\delta\phi\left(x_0^{\pm}, x_0\right)\right) \in 2\pi\mathbb{Z}. \tag{61}$$

This gives the quantum consistence to the statistical quantization of ρ derived in Section 7.

12 There is here a transfer of property of the real logarithm to the complex logarithm because we assume that the complex rough noise behaves in scaling like the function $\frac{1}{z}$ at any coarse grained scale, where $z = r\exp(i\theta) \in \mathbb{C}$.

The branch cut exists in any spatial dimension since the integration (60) follows a one dimensional closed curve, which can be made small enough to stick to a tangential Euclidian surface, taking the background space to be smooth. This can be stated otherwise as, since ρ is not smooth, but only Hölder continuous, singularities fill up densely within any loop so that the topology imposes quantification of the phase (61). We have a situation equivalent in principle to Wilson or t'Hooft loops and Toulouse geometric frustration in spin glasses.

To summarize our finding, because momentum is defined as the directional derivative of the phase, the roughness of the noise (30), which is imposed onto the phase, yields Bohr-Sommerfeld's quantization (58). Note that any generalization follows naturally, like the classically chaotic paths that lead to the Gurtzwiller trace formulae of quantum chaos. The paths integral formulation (46) is generalizable to other dispersion relations and the argument follows also to other observables and fields. The annealed Lagrangian restores symmetries by Noether's theorem. Typically, the annealed Hamiltonian spectrum corresponds to the energy levels. Note that ladder operators populating Fock space already form the most natural algebra for stochastic reaction-diffusion processes (as already mentioned in the introduction and in Appendix A.2).

8.3 Vectors and Spin

Beyond quantization of momentum and energy, there so remains those two intricacies, the origin of which we should like to clarify once more: spin one half and entanglement.

Because we used a generic solution (37) on the real positive line, we need to further guarantee that the scenario still holds in higher internal dimensions. To our knowledge, it was unfortunately not investigated previously how to compute "multifractal" multiplicative processes such as (36) living in spaces with dimension greater than $\mathbb{R}^2$. When ρ is a complex vector at least living in $\mathbb{C}^2$, it seems nonetheless intuitive that basically the whole artillery of non-commutative Lie algebra and Lie groups known in quantum field theory are required. The previous subsections (8.1) and (8.2) dealt with some of its aspects.

Let us recall that multifractal statistical laws are fully characterized by the log-cumulant generating function, identified here with the logarithm of the mean expectation of the random part of the evolution operator in (35):

$$\zeta(q;t) = \log \mathbb{E}\left(e^{iq\int_0^t \mathrm{d}s\ \delta\omega(s)}\right), \tag{62}$$

namely, log-variance and log-correlations, considering only the low frequency chaotic variations. It can be expanded in powers of $iz \equiv i\int_0^t \mathrm{d}s\ \delta\omega(s)$ as $\zeta(q;t) = \sum_{n=0}^{\infty} C_n \frac{(iqz)^n}{n!}$. The relationship to the moments M_n of z are:

$$\begin{aligned} C_1 &= M_1 \\ C_2 &= M_2 - M_1^2 \\ C_3 &= M_3 - 3M_2M_1 + 2M1^3 \\ &\dots \end{aligned} \tag{63}$$

with the general relation $C_n = M_n - \sum_{k=1}^{k=n-1}\left(n-1l-1\right) C_l M_{n-l}$. The log-normal approximation for example consists in neglecting all cumulants for $n \geq 3$. Multifractal scaling is the condition that [80]

$$C_n(t) = (-1)^{n+1} c_n \log(t). \tag{64}$$

Now, if $\mathbf{M}(\Psi)$ in (23) is linearized as $\mathbf{L}\Psi$, with $\mathbf{L}$ a two by two complex unitary matrix, we could try to extend Hahn-Jordan decomposition (35) into the quaternions, as an extension from complex numbers, but the Clifford algebra is preferred as usual. Recall that 2×2 special unitary matrices are decomposable into Pauli matrices σ_i, the generator basis of the Lie algebra $\mathfrak{su}(2)$, so that $\mathbf{L} = \sum_{i=1}^{3} a_i \sigma_i + a_4 \mathbf{I}$, with coefficients $a_i \in \mathbb{R}$ constrained by the unitarity of $\mathbf{L}$, $\sum_i a_i^2 = 1$.

Because both Lie algebra are locally isomorphic $\mathfrak{su}(2) \sim \mathfrak{so}(3)$, an infinitesimal rotation in spatial dimension three can be associated to $\mathbf{L}$ with an angle $\delta\theta_{\vec{n}}$ being defined around some arbitrary axis $\vec{n}$ of norm unity: $\mathbf{L}_{\delta\theta} = 1 - i\delta\theta/2\, \vec{n} \cdot \vec{\sigma}$. This sets the formal integration in (35) to be a rotation written in exponential form:

$$\rho(t) = e^{-i \int_0^t \delta\theta_{\vec{n}}(s)/2\, \vec{\sigma}\cdot\vec{n}} \rho(0). \tag{65}$$

To find a track towards (64) one needs to recover (32), which implies that the diffusion of moments of ρ "freezes" logarithmically in time. Equation (65) reveals that the phase of any component of ρ may be influenced greatly by the multifractal background, an average evolution of which nonetheless being constrained by (61). In the rather intuitive signed and complex cases, it was argued in Section 6 that along characteristics $x(t)$, the evolution in time is multifractal for any projection on $\mathbb{R}^+$, $\mathbb{R}^-$ and $i\mathbb{R}$. Similarly, we can argue that it is true along projections on the three axes of the pseudo-quaternions in the case of a two dimensional complex vector.

This is achieved provided the evolution operator also belongs to the non-compact group of dilatations, or more generally of "boosts", with an imaginary angle $i\eta$, $\mathbf{L}_\eta = \mathbf{I} - \eta/2\, \vec{n} \cdot \vec{\sigma}$. The operator $\mathbf{L}_\eta$ is purposed to generate the multifractal measure (37) if η is rough, that is if

$$\int_0^t \eta(s)/2\, \vec{\sigma} \cdot \vec{n}\, \mathrm{d}s = \int_0^t \mathrm{d}\mathcal{B}^0(s), \tag{66}$$

with various rescaled "contracting/dilating" directions. The branch cut, for closed loops, forcing the phase singularities (61) must be recovered for the full evolution composition operator:

$$\mathbf{L} = \mathbf{L}_{\delta\theta} \mathbf{L}_\eta. \tag{67}$$

The global structure of real rotations $SO(3) \sim SU(2)$ is exactly preserved at any given time because the non-commuting generators of real and imaginary rotations are part of the wider group $SO(3, \mathbb{C}) \sim SO(3, 1)$, the Lie algebra of which is isomorphic to the direct sum: $\mathfrak{so}(3, 1) \sim \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Just as in the Abelian complex case, if the evolution is unpredictable, at least, the algebra $\mathfrak{su}(2)$ averaged over the noise is therefore preserved. In other words, an averaged Hamiltonian evolution acts only on the real angle, as rough random real and imaginary angle fluctuations are hidden.

This development reveals incidentally a possible connexion between Chern numbers and the multifractal cumulants in the expansion (62). The gauge connexion 2-form curvature defined as $\Omega = d\omega + \frac{1}{2}\omega \wedge \omega$ is indeed here non trivial because of singularities densely dispersed. Yet, Chern characters $Ch(n) = \frac{1}{n!} tr\{\left(\frac{i}{2\pi}\Omega\right)^n\}$ are the nth terms in the expansion of the generating function

$$tr\{\exp\left(\frac{i}{2\pi}\Omega\right)\} \tag{68}$$

and their integration over a compact region of space, which are the Chern numbers, can be understood as "moments", whereas Chern classes would correspond to cumulants (please compare the definitions in (63) and (68), both can be found e.g in [56] and in [80]). In that respect, the multifractal parameter c_2 should be related in such theories to the second Chern class.

8.4 Dealing with Internal Symmetries

Given the joint probabilistic machinery described in Section 2, the measurement process will live in a given algebra, e.g. $\mathfrak{su}(2)$. This is opposed to the over simplistic approach from Bell's theorems, in which only randomly picked a priori projection states along some predefined axis are supposed to be independently predetermined by a set of hidden variables, loosing contextuality.

The Lie algebra has been found to be bonded to the noisy evolution of excitations of ρ (67), in the previous Subsection 8.3. At the same time, it allows Noether's theorem to give rise on average to conservation laws, in the mathematical weak sense, via an optimal Lagrangian. So that, an initially shared property between two bodies can be conserved throughout time and space, before a substantial perturbation occurs.

This is directly related to the autonomous dynamics of an excitation within its reference frame, conserving the action phase integral, as long as the piece of medium crossed is adiabatically homogeneous and no external perturbation occurs. If we go back to our reasoning, because photon measurements (or electrons in Stern-Gerlach type of experiment) involve the electromagnetic field, we are entitled to postulate that initial conditions are hidden in the polarization vector's phase. The phase affects those experiments involving the vector and scalar potentials $\left(\vec{A}, \Phi\right)$.

Let us draw carefully the parallel with Section 2 for concreteness. Two isolated excitations generated on the medium are imagined, prepared simultaneously at the same location, such that the total angular momentum is null, for instance, something like the nonlinear scenario in which a soliton and an anti-soliton emerge simultaneously and counter-propagating, see Appendix A.1. As we saw previously, projections of angular momentum on any axis, for any of the two excitations, vary maybe randomly, but there is quantization of the action phase integral.

Determinism is here important for measurement outcomes, since measurements are physical processes between a macroscopic field (the apparatus) and the excitation internal fields. Typical experimental examples are:

- A polarizer, which acts as a layer of atoms dephasing the excitation phase plane as it crosses the layer.
- A spin (Stern-Gerlach) analyzer, basically a magnetic field, with the phase and the gauge connexion intertwined from Aharanov-Bohm effect.

Since the global action phase integral is conserved on average, there is a quantity embarked within each excitation with conserved null sum. So, can assume that the phase, if known, sets reproducibly the outcome of a measurement. Sampling this outcome at various realizations with time t_i generates a random series, because the phase is not known to the experimenter. It's already true in a Poissonian description of symplectic dynamics that one cannot simply project on fixed directions a specific outcome, if not knowing the initial phase, because of the Poisson brackets commutation relations, as is exemplified with the Larmor precession of a magnet around an applied magnetic field [56].

Thus, when one excitation goes through its "polarizer" or "magnetic field" and gets out with a given projection on some axis, the other one goes through its own "polarizer" or "magnetic field" and gets out deterministically, albeit unpredictably, with a projection statistically compatible with their shared property, on the one hand, and compatible with the probability law describing the interaction dynamics with the apparatus vector potentials. The sharing holds at any given time of a stationary environment until the measurement is done.

Again, this preservation necessitates that chaos arises only along individual particle paths but that nonlocal coherence is preserved in the background, here by topological constraints and hyperbolic (or ballistic) transport. Otherwise, it would be arguable that a Lyapunov time scale would erase memory, lost in microscopic degrees of freedom with elliptic (or diffusive) transport. That's one main difference with the setting of Brownian entanglement [21].

The subsequent evolution of the phase through the singular medium is chaotic but remains deterministic. As we argued, weak solutions preserve Lie groups symmetries, because their generators are part of the wider indefinite orthogonal group when including multifractal scaling generators in **L**. Thus, as quantization is a topological invariant, the locally preserved memory of initial conditions in the real part of the phase is not affected by the noise, because the noise does not affect averages. Consequently, initial conditions are well preserved throughout the two paths followed by the pair, until an event such as a measurement resets the phase in at least one of them. Hence, along the paths to the experimental apparatus we have:

$$f\left(\phi_i(x,t)\right) = f\left(\phi_i(t_0,x_0)\right), \tag{69}$$

for some function f.

In a yet more explicit form, consider a pair excitation, at time t_0 and position x_0, both corpuscles have opposite projections along a direction $\vec{n}$ of their angular momenta, for example define f as:

$$f\left(\phi_1(t_0,x_0)|\,n\right) = \theta_0/2\;\vec{\sigma}\cdot\vec{n} = -f\left(\phi_2(t_0,x_0)|n\right), \tag{70}$$

where the projection along $\vec{n}$ is made explicit as in Subsection (8.3). Therefrom, one can directly recover formally any case as, for instance, (15) in the case of photons with the three dimensional representation of $SU(2)$. This is possible if one can make sure that for such a medium, the excitations are really conveying those oscillations internally.

Wrapping up this mathematically, wavelet transforms express the locally deterministic and globally statistical state of the pair excitations as a tensor product of (46):

$$\Psi_{1,2} = \Psi_1 \otimes \Psi_2. \tag{71}$$

By construction, each local excitation experiences the self background noise along their characteristics, e.g. $\mathbf{L}_{\delta\theta}\mathbf{L}_{\eta}$. So, statistical summation into non separable states is clearly preparable, just as it is the case for the specific $SU(2)$ group, exemplified using Jones vectors in Section 2.

8.5 Entangled field excitations

By virtue of (46), we can now calculate the transition due to the unpredictable measurement at time t_a of one eigenstate, with spin aligned along an axis denoted $\vec{a}$, of one excitation passing through the analyser σ_a. It is expressed as:

$$\Psi_1\left(\vec{a}\cdot\vec{\sigma_a},t_a\right) = \langle\vec{a}\cdot\vec{\sigma_a}|\rho(t_a)\rangle = \left\langle\vec{a}\cdot\vec{\sigma_a}|e^{-i\int_{t_0}^{t_a}\,(\delta\theta_1(s)+i\eta(s))/2\,\vec{\sigma}\cdot\vec{n}}|\rho(t_0)\right\rangle, \tag{72}$$

and similarly along axis b for the other excitation, where $\vec{a}\cdot\vec{\sigma_a}|\rho(t_a)\rangle$ is employed to take eigenstates that are parallel or antiparallel to $\vec{a}$, as it was used in (6). A tensorial product $|a,b\rangle$ denotes eigenstates along a and b.

The square modulus of (72) converges to the probability density distribution of Born rule as it is sampled empirically. This probability density takes as argument the angle, as in (9). Now, the density matrix, derived from projections over those eigenstates, can also account for a mixture, such as a pair of outcomes along axis a:

$$\frac{1}{2}(a + a'). \tag{73}$$

Product states $a \otimes b$, with a tensorial product of solutions written above in (72), do account for both eigenvectors being distributed at random among the two excitations. But, as previously, a singlet superposition of the pair states:

$$\frac{1}{\sqrt{2}}(ab' + ba') \tag{74}$$

emerges when it is statistically impossible to determine the result of an operation $\vec{a} \cdot \vec{\sigma_a}$ with certainty, but only with respect to some specific law incorporating latent information shared with the other member of the pair acted on by σ_b, as in (9). Therefore, we have again that the joint probabilities is not separable:

$$p_{a,b} \neq p_a p_b. \tag{75}$$

This results from the interactions with both analysers σ_a and σ_b, exactly in the same way as in (11), but now easily generalizable to any symmetry algebra and apparatus law. It forces the density matrix to be non diagonalizable, see (55), which is the signature of a statistically entangled pair state like in (13), leading to cross correlations as in (15):

$$\sum_{a,b} \left(p_{a,b} - p_a p_b\right) |a, b\rangle\langle a, b| \neq 0. \tag{76}$$

9 Short Discussion and Conclusion

Two main results have been derived, the first one discards any nonlocal non-realistic interpretation of quantum mechanics, the second one investigates possible origins of the unique property of quantum systems that is duality, close in spirit to de Broglie's first insights. Because quantum systems are here shown to abide by the tenets of classical physics, the realm and overarching scope of the latter are broaden. The EPR paradox has lived as such, it is now upgraded to the founding remark which engulfs quantum physics within statistical physics.

Regarding the demonstration of realism, it was reminded that the only important assumption in Bell's theorem is the independence of measurements, as Jaynes was first to criticise in the late 80s and Khrennikov more recently. This constitutes altogether its major loophole, because obviously quantum mechanics breaks that assumption, but no one had ever proved that classical mechanics could evade such independence until Spreeuw's thought experiment in the late 90s. Consequently, in this essay, entanglement is presented as a natural statistical outcome when a non separable joint probability of some property is evaluated by joint measurements. The latent information is encoded in the phase of localized "dual objects." Just as in the Bohmian interpretation but without the need of a macroscopic nonlocal pilot-wave, because the phase interferes with the measuring apparatus, outputs are not independent. It constitutes an initial condition for the deterministic interaction dynamics at play within the apparatus, between the internal field state and the analyser's macroscopic field. For instance, conservation of angular momentum allows unaltered spin state information to be carried along any inertial particle track. In the case of a pair of elementary excitations in a singlet state, the latent statistical variable is encoded in the phase of the oscillating internal electromagnetic vector fields. That is the insight that connects the findings derived in this short essay to walking droplets, for example.

In that respect, the major added information, compared to a statistical description involving a partition function, of a quantum description, that builds on what was called naively at first the "square root" of a partition function, in fact simply a generating function for any observable moments, so resides in the phase, not surprisingly. Put again differently, phases of entangled entities interfere if we bring them back together. They do not confer to quantum phenomenology a more "ontic" attribute than to the "epistemic" phenomenology of any other system in the class of exact analogues about which knowledge is lacking. Quantum field theory was constructively introduced as an effective theory starting from the generalized kinetic susceptibility and averaging over high frequencies of chaotic oscillations, in an excitable medium. Showing that it is possible to probe beyond the quantum of action, it was defined as a Poincaré invariant, in a way measuring the amount of thermal energy within a period of spontaneous oscillations. The class of quantum analogues comprises those generating interacting self-focusing objects: Particles are localized topological waves emerging from the medium relaxation oscillations. Such entities are typically complex vector solitons keeping within themselves all properties based upon the symmetries of the background, all concealed within their phases. For instance, special relativity, that would govern the dynamics of their envelop, can be forced in via the appropriate dispersion relation. This has radical consequences for the interpretation of relativity: Because dispersion relations emerge at the thermodynamic level of coarse graining, for example via Gibbs relation and equipartition, the inferred internal structure of fundamental particles can be superluminal without any trouble, as Ohanian discussed in the 80s. The peculiar reconciliation between relativity and quantum fields can gain a description: Since internal symmetries are now understood, less abstractly than in the standard model, but as being imprinted topologically, it becomes calculable quite easily to show that a particle may be morphed into its antiparticle in a boosted reference frame, as emission and absorption are swapped.

The demonstration was given in an operational way, explicitly defining Dirac's bra and ket bilinear space acting on wave functions. Entanglement being the non diagonalizability property of density matrices, they must be derived also explicitly from first principles, which is given too. The operation goes quite straightforwardly by means of projections with wavelet transforms and annealing over the effective randomness of the background. Onsager relations and scale invariance let us write the effect of randomness as multiplicative chaos. The background is thus dominated by a singular measure which allows non-differential paths to induce non-abelian algebra of operators. The rationale for that assumption is heuristically that critical dynamics self-generate parametric fluctuations of the "1/f" type. Note that criticality is compatible for instance with the two important phenomena of long range static forces and confinement. It is also an important parameter for walking droplets. Quantization of momentum and energy emerges from the logarithmic branch cut as one integrates over the "1/f" fluctuations. Such topological invariants ensure also the necessary memory compatibility of the background. It is straightforward to infer that energy levels for the annealed dynamics are described by the spectrum of the annealed evolution operator, identified with a Hamiltonian in standard ways. Conservation laws are guaranteed, in the mathematically weak sense, by Noether's theorem. The multifractal background is essential at high enough energies, when second quantization is required and ladder operators become necessary descriptors. In any event, the multifractal background is also well consistent with the requirement of renormalization.

Madelung's "reduced" picture of phase and quantum potential is thus reversed and broaden. Dirac, Klein-Gordon, Schrödinger's equations, or any other relevant one, are seen as the normal forms, hiding nonlinearities, of more complicated coupled partial differential equations describing some underlying reaction-diffusion medium and lying maybe in one vaster universality class. As it seems, the approach could be coined a“stochastic zitterbewebung dynamics” but with more constraints, as we just listed. The compelling arguments were funnelled to achieve the appropriate algebraic structure of quantum phenomenology. For example, droplet walkers are localized Faraday waves with unpredictable paths in important instances, and quantization there arguably follows also by the topology of wave packets described in a Hilbert space.

A deep connection between topological invariants like Chern classes and the non-smooth geometry generated by the multifractal background is worth exploring in more detail. It seems to indicate, for instance, the emergence of spatial curvature via covariant derivatives changing with defect (matter) content, in large volumes and at annealing scales. A clue can be intuited thanks to the powerful theorems of algebraic and geometric topology, such as those relating characteristic classes of Chern and Euler, thus a step away from general relativity coming out as another effective field theory. Hopefully, newly gained insights about the microscopic underlying “dual” physics that can handle the dimensionality of Newton’s constant could help solving the quantization of gravity conundrum.

Another related connection between the multifractal spectrum, understood as the entropy of local scaling exponents within a range of scales, and entanglement entropy, or mutual information, also lays the ground for further work, since both derive from transition probabilities (46). Note that some quantum computing concepts may need a change of perspective since state superposition is only statistical, as now proven, however with an identical computing power, dimensionally speaking.

In short, the description of quantum physics was done in terms of emergence at coarse grained levels of nonlinear localized oscillatory excitations of finite energy living over some dynamical scale invariant “foam.” The preservation of information is encoded in the phase as symmetry operators and guaranteed topologically.

Appendix

A.1 Sine-Gordon Pair Excitation

Sine-Gordon equation can be derived as the normal form of some coupled PDEs. This equation preserves Lorentz invariance. Typically, pairs of kinks or soliton anti-soliton are excited, see e.g. Figure (2) from [39], please refer to it and the references contained therein for details.

A.2 Ladder Operators

Creation and annihilation operators are legitimate operators for a Fock space description of classical excitable systems. This works when differentiability of trajectories is lost, or when the involved chemicals (at the discrete level) react into products that diffuse [35, 36].

Textbook second quantization shows that (26) is obeyed by the ladder operators $a = \frac{1}{\sqrt{(2\omega)}}(\omega x + ip)\,0$ for a harmonic oscillator $\frac{d}{dt}a = -i\omega\, a$, which means that the field named “ρ” throughout the text is in fact, in the chosen terminology, the “pre-ladder” operator, as intuition would well tell us. More formally, ladder operators are a consequence of the non commutativity implied by non-differentiability of wave packets trajectories [52].

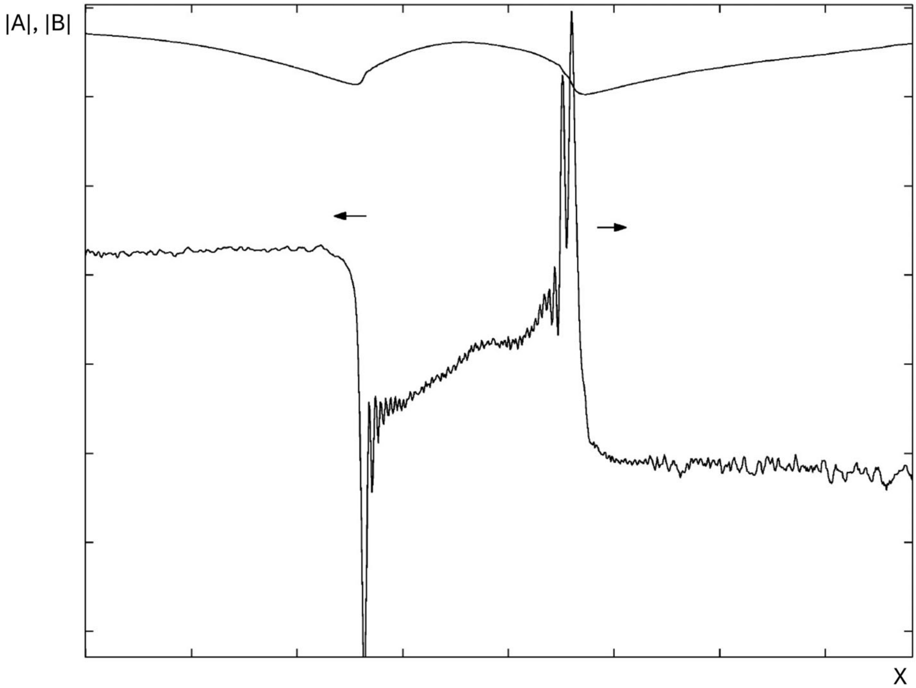

Fig. 2 Snapshot example of a nonlinear relaxation pair mode akin to a pair of kin-antikink in 1D simulation of an infinite dimensional (like Navier Stokes turbulence) instability. Both reactant $|A|$ and $|B|$ amplitudes are displayed spatially (X) at a given time in normalized units. The mode was generated spontaneously far from the driving source in a system like (24). Taken from [39]

Acknowledgements Dedicated to you my late dear friend*, you see, I won this bet. I know you wouldn't mind! *Mmanu Fleurence.

Author Contributions G.A. contributed entirely to the preparation and writing of the manuscript, including figures and reviewing.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing Interests The authors declare no competing interests.

References

1. Bohm, D.: A suggested interpretation of the quantum theory in terms of “hidden” variables. I. Phys. Rev. **85**, 166–179 (1952). https://doi.org/10.1103/PhysRev.85.1662
2. Bohm, D.: A suggested interpretation of the quantum theory in terms of “hidden” variables. II. Phys. Rev. **85**, 180–193 (1952). https://doi.org/10.1103/PhysRev.85.180
3. Bohm, D., Vigier, J.P.: Model of the causal interpretation of quantum theory in terms of a fluid with irregular fluctuations. Phys. Rev. **96**, 208–216 (1954). https://doi.org/10.1103/PhysRev.96.208
4. De Broglie, L.: La thermodynamique “cachée” des particules. Ann. Inst. Henri Poincaré,Vol. I, (1), 1–19, (1964), Section A : Physique théorique
5. De Broglie, L.: Interpretation of quantum mechanicsby the double solution theory Annales de la Fondation Louis de Broglie, Volume 12,(4) (1987)
6. Couder, Y., Fort, E., Gautier, C.-H., Boudaoud, A.: From Bouncing to Floating: Noncoalescence of Drops on a Fluid Bath. Phys. Rev. Lett. **94**, 177801,1 (2005)
7. Eddi, A., Fort, E., Moisy, F., Couder, Y.: Unpredictable Tunneling of a Classical Wave-Particle Association. Phys. Rev. Lett. **102**, 240401 (2009)
8. Couder, Y., Boudaoud, A., Protière, S., Fort, E.: Walking droplets, a form of wave-particle duality at macroscopic scale? Europhysics News **41**(1), 14–18 (2010)
9. Perrard, S., et al.: Self-organization into quantized eigenstates of aclassical wave-driven particle. Nat. Commun. **5**, 3219 (2014). https://doi.org/10.1038/ncomms4219
10. Harris, D.M., Moukhta, J., Fort, E., Couder, Y., Bush, J.W.M.: Wavelike statistics from pilot-wave dynamics in a circular corral. Phys. Rev. E **88**, 011001(R) (2013)
11. Bush, J.W.M., Oza, A.U., Moláček, J.: The wave-induced added mass of walkingdroplets. Fluid Mech. **755**, R7 (2014). https://doi.org/10.1017/jfm.2014.459
12. Dagan, Y., Bush, J.W.M.: Hydrodynamic quantum field theory: the freeparticle. Comptes Rendus Mécanique **348**(6–7), 555–571 (2020). https://doi.org/10.5802/crmeca.34
13. Bush, J.W.M., Oza, A.U.: Hydrodynamic quantum analogs. Rep. Prog. Phys. **84** 017001 (41pp) (2020). https://doi.org/10.1088/1361-6633/abc22c
14. Papatryfonos, K., Ruelle, M., Bourdiol, C., Nachbin, A., Bush, J.W.M., Labousse, M.: Hydrodynamic superradiance in wave-mediated cooperative tunneling. Commun. Phys. **5**, 142 (2022)
15. Frumkin, V., Bush, J.W.M., Papatryfonos, K.: Superradiant droplet emission from parametrically excited cavities. Phys. Rev. Lett. **130**, 064002 (2023)
16. Papatryfonos, K., Schroder, J.W., Frumkin, V.: Superradiant droplet emission from a single hydrodynamic cavity near a reflective boundary. (2024). arXiv:2408.02620
17. Papatryfonos, K., Vervoort, L., Nachbin, A., Labousse, M., Bush, J.W.M.: Bell test in a classical pilot-wave system, (2022). arXiv:2208.08940
18. Einstein, A., Podolsky, B., Rosen, N.: Can Quantum-Mechanical Description of Physical Reality Be Considered Complete? Physical Review **47**, 777 (1935)
19. Nelson, E.: Derivation of the Schrödinger Equation from Newtonian Mechanics. Phys. Rev. **150**(4), 1079–1085 (1966). https://doi.org/10.1103/PhysRev.150.1079
20. Feynman, R.P., Hibbs, A.R.: Quantum Mechanics and Integrals. McGraw-Hill, (1965)
21. Allahverdyan, A.E., Khrennikov, A., Nieuwenhuizen, T.M.: Brownian entanglement. Phys. Rev. A **72**, 032102 (2005)
22. Khrennikov, A.: Quantum epistemology from subquantum ontology: quantum mechanics from theory of classical random fields. Annal. Phys. **377**, 147 (2017). https://doi.org/10.1016/j.aop.2016.12.005
23. Sornette, D.: Anderson localization and quantum chaos in acoustics. Physica B: Condensed Matter **219–220**(1), 320–323 (1996)
24. Schaadt, K.: The Quantum Chaology of Acoustic Resonators. MSci Thesis University of Copenhagen July (1997)
25. Faure, F.: Semi-classical formula beyond the Ehrenfest time in quantum chaos. (I) Trace formula. Annales de l’Institut Fourier **57**(7), 2525—2599 (2007)
26. Abraham, A.J., Malkov, S., Ljubetic, F.A., Durey, M., Sáenz, P.J.: Anderson localization of walking droplets (2023). arXiv:2310.16000
27. Doi, M.: Second quantization representation for classical many- particle system. J. Phys. A: Math. Gen. **9**, 2465 (1976)
28. Doi, M.: Stochastic theory of diff usion-controlled reaction. J. Phys. A: Math. Gen. **9**, 1479 (1976)
29. Bouchaud, J.-P., Georges, A.: Anomalous diffusion in disordered media: Statistical mechanisms, models and physical applications. p127 vol195 Nos 4 & 5 Physics Reports, (1990)

30. Castillo, H.E., Chamon, C.d.C., Fradkin, E., Goldbart, P.M., Mudry, C.: Exact calculation of multifractal exponents of the critical wave function of Dirac fermions in a random magnetic field. Phys. Rev. **B56**, 10668 (1997)
31. Grassberger, P., Scheunert, M.: Fock-Space Methods for Identical Classical Objects. Fortschr. Phys. **28**, 547 (1980)
32. Peliti, L.: paths integral approach to birth-death processes on a lattice. J. Physique **46**, 1469 (1985)
33. Cardy, J.L., Täuber, U.C.: Field theory of branching and annihilating random walks. J. Stat. Phys. **90**, 1 (1998)
34. Cardy, J.L.: Scaling and Renormalization in Statistical Physics. Cambridge University Press (1996). https://doi.org/10.1017/CBO9781316036440
35. Schulz, M., Reineker, P.: Exact substitute processes for diffusion–reaction systems with local complete exclusion rules. New J. Phys. **7**, 31 (2005)
36. Cardy, J.L.: Reaction-diffusion processes. In: Field theory and non-equilibriumstatistical mechanics" lectures given by John Cardy at the LMS/EPSRC "methods of non-equilibrium statistical mechanics in turbulence" school, University of Warwick from 10-14 July (2006)
37. Anderson, P.W.: Plasmons, Gauge Invariance, and Mass. Phys. Rev. **130**, 439 (1963)
38. Landau, L.D.: On the vibrations of the electronic plasma. Zh. Eksp. Teor. Fiz. **16**, 574–86 (reprinted 1965 Collected Papers of Landau ed D ter Haar (Oxford: Pergamon) pp 445–60)
39. Attuel, G.: Aspects critiques des fluctuations d'un plasma magnétisé. Proposition de théorie cinétique stochastique. PhD thesis, École Polytechnique Paris, (2007). https://theses.hal.science/pastel-00004936/
40. Maier, W., Saupe, A.: Eine einfache molekulare Thoerie des nematischenkristallinflussigen Zustandes. Z. Naturforsch. **13a**, 564–566 (1958)
41. De Gennes, P.-G.: Phenomenology of short-range order effects in the isotropic phase of nematic materials. Phys. Lett. A **30**, 454–455 (1969)
42. De Gennes, P.-G.: An analogy between superconductors and smectics A. Solid State Commun. **10**, 753–756 (1972)
43. Pismen, L.M.: Vortices in Nonlinear Fields: From Liquid Crystals toSuperuids, from Non-equilibrium Patterns to Cosmic Strings. International Series of Monographs on Physics, Oxford Science Publications,ISBN: 9780198501671, (1999)
44. Jaynes, E.T.: Clearing up mysteries : The original goal. In: the Proceedings Volume, Maximum Entropy and Bayesian Methods, J. Skilling, Editor,Kluwer Academic Publishers, Dordrecht Holland , pp. 1–27 (1989)
45. Khrennikov, A.: Get Rid of Nonlocality from Quantum Physics. Entropy **21**, 806 (2009). https://doi.org/10.3390/e21080806
46. Clauser, J.F., Horne, M.A., Shimony, A., Holt, R.A.: Proposed experiment to test local hidden-variable theories. Phys. Rev. Lett. **23**(15), 880–4 (1969). https://doi.org/10.1103/PhysRevLett.23.880
47. Spreeuw, R.J.C.: A Classical Analogy of Entanglement. Found. Phys. **28**(3) (1998)
48. Qian, X.-F., Little, B., Howell, J.C., Eberly, J.H.: Shifting the quantum-classical boundary:theory and experiment for statistically classicaloptical fields. Optica **2**(7), 611 (2015)
49. Shen, Y., Rosales-Guzmán, C.: Nonseparable States of Light: From Quantum to Classical. Laser Photonics Rev. **16**, 2100533 (2022)
50. Hohenberg, P.C., Halperin, B.I.: Theory of dynamic critical phenomena. Rev. Modem Phys. **49**(3) (1977)
51. Ohanian, H.C.: What is spin?. Am. J. Phys. **54**(6) (1984)
52. Zee, A.: Quantum Field Theory in a Nutshell: Second Edition. Princeton University Press, 9780691140346 (2010)
53. Kardar, M., Parisi, G., Zhang, Y.-C.: Dynamic Scaling of Growing Interfaces. Phys. Rev. Lett. **56**(9), 889–892 (1986)
54. Martin, P.C., Siggia, E.D., Rose, H.A.: Statistical Dynamics of Classical Systems. Phys. Rev. A **8**, 423 (1973)
55. Forster, D., Nelson, D.R., Stephen, M.J.: Large-distance and long-time properties of a randomly stirred fluid. Phys. Rev. A **16**, 732 (1977)
56. Stone, M., Goldbart, P.: Mathematics for physics: A Guided Tour for Graduate Students. 9780521854030, Cambridge University Press (2009)
57. Dirac, P.: Principles of quantum mechanics. Oxford University Press, third edition (1947)
58. Parisi, G., Wu, Y.-S.: Perturbation theory without gauge fixing. Sci. Sinica. **24**, 483 (1981)
59. De la Peña, L., Cetto, A.M., Valdés-Hernández, A.: The emerging quantum: The physics behind quantum mechanics. Springer, Switzerland (2015). ISBN 978-3-319-07892-2. https://doi.org/10.1007/978-3-319-07893-9
60. Hestenes, D.: The Zitterbewegung interpretation of quantum mechanics. Found. Phys. **20**(10), 1213–1232 (1990)

61. Gersch, H.A.: Feynman's Relativistic Chessboard as an Ising Model. Int. J. Theoretical Phys. **20**, 491–501 (1981)
62. Jacobson, T., Schulman, L.S.: Quantum stochastics: the passage froma relativistic to a non-relativistic paths integral. J. Phys. A: Math. Gen. **17**, 375–383 (1984)
63. Kauffman, L.H., Noyes, H.P.: Discrete physics and the Dirac equation. Phys. Lett. A **218**, 139–146 (1996)
64. Bernstein, I.B., Greene, J.M., Kruskal, M.D.: Exact Nonlinear Plasma Oscillations. Phys. Rev. **108**(3), 546–550 (1957). https://doi.org/10.1103/PhysRev.108.546
65. Lynden-Bell, D.: The stability and vibrations of a gas of stars. Mon. Not. R. Astron. Soc. **124**(4), 279–296 (1962). https://doi.org/10.1093/mnras/124.4.279
66. Prigogine, I.: Non-equilibrium Statistical Mechanics. Interscience Publishers, (1962)
67. Prigogine, I.: Introduction à la thermodynamique des processus irréversibles. Dunod, (1965)
68. Manneville, P.: Rayleigh-Bénard Convection: Thirty Years of Experimental, Theoretical, and Modeling Work. In: Mutabazi, I., Wesfreid, J.E., Guyon, E. (eds) Dynamics of Spatio-Temporal Cellular Structures. Springer Tracts in Modern Physics, vol 207. Springer, New York, NY. (2006). https://doi.org/10.1007/978-0-387-25111-03
69. Frisch, U.: Turbulence: The Legacy of A.N. Kolmogorov. Cambridge University Press. ISBN: 9780521451031 (1995). https://doi.org/10.1017/CBO9781139170666 and references therein
70. Bohm, D., Gross, E.P.: Theory of plasma oscillations a and b. Phys. Rev. **75**(12), 1851–1876 (1949)
71. Dauxois, T., Latora, V., Rapisarda, A., Ruffo, S., Torcini, A.: The Hamiltonian Mean Field Model: From Dynamics to Statistical Mechanics and Back. Lecture Notes Phys. (2002). https://doi.org/10.1007/3-540-45835-216
72. Escande, D.F., Elskens, Y.: Microscopic dynamics of plasmas and chaos: the wave–particle interaction paradigm. Plasma Phys. Control. Fusion **45**A115 (2003). https://doi.org/10.1088/0741-3335/45/12A/008
73. Firpo, M.-C., Leyvraz, F., Attuel, G.: Equilibrium statistical mechanics for single waves and wave spectra in Langmuir wave-particle interaction. Phys. Plasmas **13**, 122302 (2006)
74. Van Kampen, N.G.: On the theory of stationary waves in plasmas. Physica **21**(6–10), 949–963 (1955)
75. Van Kampen, N.G.: The dispersion equation for plasma waves. Physica **23**, 641 (1957)
76. Arnold, V.I.: Geometrical methods in the theory of ordinary differential equations. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 250. Springer-Verlag, New York (1983)
77. Dalibard, J.: Mécanique quantique avancée, Cours de Master http://cel.archives-ouvertes.fr/cel-00092950/fr/ (1999)
78. Dawson, J.M.: Phys. Rev. **118**(2), 381 (1960)
79. Dupree, T.H.: A perturbation theory for strong plasma turbulence. Phys. Fluids **9**(9), 1773 (1966)
80. Arneodo, A., Bacry, E., Muzy, J.F.: Random cascades on wavelet dyadic trees. J. Math. Phys. **39**(8), 4142 (1998)
81. Mandelbrot, B.: A Multifractal Model of Asset Returns. Cowles Foundation Discussion Paper #1164 (1997)
82. Rhodes, R., Vargas, V.: Gaussian multiplicative chaos and applications: a review. (2013). arXiv:1305.6221
83. Wegner, F.J.: Inverse participation ratio in 2+ϵ dimensions. Zeitschrift für Physik B **36**, 209–214, Condensed Matter, (1980)
84. Castellani, C., Peliti, L.: Multifractal wavefunction at the localisation threshold. J. Phys. A: Math. Gen. **19**, L429 (1986)