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Covariant projective representation of symplectic group on discrete phase space

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Abstract. The phase point operator $\Delta(q,p)$ is the quantum mechanical counterpart of the classical phase point (q,p). The discrete form of $\Delta(q,p)$ was formulated for an odd number of lattice points by Cohendet et al. and for an even number of lattice points by Leonhardt. Both versions have symplectic covariance, which is of fundamental importance in quantum mechanics. However, an explicit form of the projective unitary representation of the symplectic group that appears in the covariance relation is not yet known. We show in this paper the existence and uniqueness of the representation, and describe a method to construct it using the Euclidean algorithm.

1. Introduction

The Wigner function was introduced by Wigner and utilized to study the quantum correction for thermodynamics in 1932 [1]. In recent years, its range of applications has extended to quantum optics, quantum chaos, quantum computing, and other fields, and it has again become a focus of interest for research in which quantum-classical correspondence is essential. The history of the Wigner function on discrete phase space is relatively young and marked in particular by its application to discrete phase space composed of a prime number of lattice points (prime-lattice phase space), formalized by Wootters in 1987 [2], and to discrete phase space composed of an odd number of lattice points (odd-lattice phase space) corresponding to integer spin, formalized by Cohendet et al. in the same year [3]. However, it was pointed out that its behavior on discrete phase space composed of an even number lattice points (even-lattice phase space) was found to differ substantially from that on odd-lattice phase space. In 1995, Leonhardt formulated the Wigner function on even-lattice phase space corresponding to half-integral spin, but found it necessary to incorporate a virtual degree of freedom (so-called ghost variable) [4, 5].

Symplectic transformation yields an invariant canonical commutation relation and is therefore an important symmetry in quantum mechanics. It is known that on continuous or odd-lattice phase space, if the phase point operator is sandwiched between a Fourier operator and its Hermitian conjugate, the argument of the operator rotates 90 degrees. It represents the simplest symplectic covariance (Fourier covariance) among linear canonical transformations. In the present article, we show that on odd- and even-lattice phase spaces the phase point operator derived from the Wigner function by Cohendet et al. and Leonhardt has symplectic covariance and that a projective unitary representation of such a symplectic transformation group exists and is unique.

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2. Wigner function on continuous phase space

2.1. Definition and marginal property

The Wigner function, which was originally a function on classical continuous phase space, is defined as

$$\mathcal{W}(q,p)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dre^{ipr/\hbar}\psi^*(q+\frac{r}{2})\psi(q-\frac{r}{2}). \tag{1}$$

Integration over momentum and position gives

$$\int_{-\infty}^{\infty}\mathcal{W}(q,p)dp = |\psi(q)|^2, \tag{2}$$

$$\int_{-\infty}^{\infty}\mathcal{W}(q,p)dq = |\tilde{\psi}(p)|^2, \tag{3}$$

respectively. It thus becomes a position distribution when integrated over momentum and a momentum distribution when integrated over position. The two wave functions $\psi(q)$ and $\tilde{\psi}(p)$ are interconvertible via Fourier transformation. As evident from this marginal property (we refer to it as marginality), the Wigner function is a kind of quantum-mechanical distribution function, but it yields negative values and is therefore referred to as a quasi-distribution function.

2.2. The phase point operator

The phase point operator $\Delta(q,p)$ is defined as the state independent part of the Wigner function,

$$\mathcal{W}(q,p)=\frac{1}{2\pi\hbar}\mathrm{Tr}[\rho\Delta(q,p)] \quad , \quad \rho=|\psi\rangle\langle\psi|. \tag{4}$$

In the position representation it is given by

$$\Delta(q,p)=\int_{-\infty}^{\infty}dre^{ipr/\hbar}|q+\frac{r}{2}\rangle\langle q-\frac{r}{2}|, \tag{5}$$

and in the momentum representation by

$$\Delta(q,p)=\int_{-\infty}^{\infty}dse^{-iqs/\hbar}|p+\frac{s}{2}\rangle\langle p-\frac{s}{2}|. \tag{6}$$

With the phase point operator, a classical Hamiltonian can be transformed to a quantized one $\hat{\mathcal{H}}$ with Weyl ordering:

$$\hat{\mathcal{H}}_{\mathrm{Weyl}}(\hat{q},\hat{p})=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dq\,dp\;\mathcal{H}(q,p)\Delta(q,p). \tag{7}$$

The phase point operator $\Delta(q,p)$ can therefore be regarded as a quantum operator corresponding to the classical phase point (q,p). From the marginality of the Wigner function, it has the following properties,

$$\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dp\Delta(q,p) = |q\rangle\langle q|, \tag{8}$$

$$\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dq\Delta(q,p) = |p\rangle\langle p|, \tag{9}$$

which are called the operator form of the marginality. An important advantage of considering the phase point operator is its capability for quantization of various geometrical objects [6-8] especially discrete systems [2-20]. We refer to Ref. [21-24] for quantum mechanics on finite abelian or Lie groups.

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3. Wigner function by Cohendet et al. and the phase point/Weyl operators on odd-lattice phase space

Beginning this section with the Weyl operator formulated by Cohendet et al., we briefly overview the phase point operator on discrete phase space composed of an odd number of lattice points (odd-lattice phase space).

3.1. Definition of the Weyl operator in odd-lattice case

Cohendet et al., in composing the Wigner function on odd-lattice phase space, first defined the Weyl operator as

$$({W^C}_{m,n}\psi)(k) = \exp\left(-\frac{4\pi imn}{N} + \frac{4\pi ink}{N}\right) \times \psi(k-2m), \tag{10}$$

where $m, n, k \in I = \{-\frac{N-1}{2}, -\frac{N-3}{2}, \cdots, \frac{N-3}{2}, \frac{N-1}{2}\}$ and N is an odd integer. The integer N is regarded as the modulus in I.

The phase Q, shift P, and inversion T operators are defined as follows for convenience in calculation,

$$Q = \sum_k |k\rangle\omega^k\langle k|, \tag{11}$$

$$P = \sum_k |k-1\rangle\langle k|, \tag{12}$$

$$T = \sum_k |-k\rangle\langle k|, \tag{13}$$

where ω is the primitive N-th root of unity:

$$\omega = \exp\left(\frac{2\pi i}{N}\right). \tag{14}$$

The commutation relation of Q and P is obtained as

$$PQ = \omega QP. \tag{15}$$

When the above Weyl operator is expressed using the phase and shift operators in I indexing, it is given by

$$ {W^C}_{m,n} = \omega^{-2mn}Q^{2n}P^{-2m}. \tag{16}$$

3.2. Definition of phase point operator in odd-lattice case

Cohendet et al. define the phase point operator as the T transformation of the Weyl operator,

$$ {\Delta^C}_{m,n} = {W^C}_{m,n}T = \omega^{-2mn}Q^{2n}P^{-2m}T, \tag{17}$$

with the following properties:

$$ {\Delta^{C\dagger}}_{m,n} = {\Delta^C}_{m,n}, \tag{18}$$

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$$\mathrm{Tr}({\Delta^C}_{m,n}) = 1, \tag{19}$$

$$\mathrm{Tr}({\Delta^{C\dagger}}_{m,n}{\Delta^C}_{m',n'}) = N\delta_{m,m'}\delta_{n,n'}, \tag{20}$$

$${W^{C\dagger}}_{m',n'}{\Delta^C}_{m,n}{W^C}_{m',n'} = {\Delta^C}_{m-2m',n-2n'}. \tag{21}$$

The Hermiticity Eq. (18), normalization Eq. (19), traciality Eq. (20) and covariance Eq. (21) are properties of the Stratonovich-Weyl kernel, which show the eligibility of the definition.

Using the phase point operator defined in Eq. (17), the Wigner function on odd-lattice phase space is defined in the same way as in the continuous case Eq. (4), i.e., as

$$\mathcal{W}_{m,n} = \frac{1}{N}\mathrm{Tr}\left[\rho{\Delta^C}_{m,n}\right] = \frac{1}{N}\langle\psi|{\Delta^C}_{m,n}|\psi\rangle. \tag{22}$$

The marginality of this Wigner function (discrete form of Eqs. (2) and (3)) and its operator form (discrete form of Eqs. (8) and (9)) can be confirmed in the same way as on continuous space.

4. Wigner function by Leonhardt and the phase point/Weyl operators on even-lattice phase space

In this section,we review the Wigner function on discrete phase space composed of an even number of lattice points (even-lattice phase space), as described by Leonhardt [4].

4.1. Definition of the Weyl operator in even-lattice case

The Wigner function composed by Leonhardt is established on both odd- and even-lattice phase space, but with incorporation of a virtual degree of freedom (ghost variable) between integral points for even-lattice phase space.

The 'characteristic function' is defined as

$$\tilde{W}^L_{m,n} \equiv \sum_{k=0}^{N-1} \exp\left[-\frac{2\pi i}{N}2m(k+n)\right]\langle k|\rho|k+2n\rangle. \tag{23}$$

Leonhardt defines the discrete Wigner function as a double-inverse Fourier transformation:

$$\mathcal{W}_{\mu,\nu} \equiv \frac{1}{D^2}\sum_{m,n}\exp\left[\frac{2\pi i}{N}2(m\nu+n\nu)\right]\tilde{W}^L_{m,n}. \tag{24}$$

Substitution of Eq. (23) into Eq. (24) then yields

$$\mathcal{W}_{\mu,\nu} = \frac{1}{D}\sum_{m}\exp\left(\frac{2\pi i}{N}2m\nu\right)\langle\mu-m|\rho|\mu+m\rangle. \tag{25}$$

Eq. (25) is the Wigner function that has the marginality on both odd- and even-lattice phase space, provided that for odd dimensions (μ,ν) is an integer phase space composed with $D = N$ and summed in the range $I = \{-\frac{N-1}{2}, -\frac{N-3}{2}, \cdots, \frac{N-3}{2}, \frac{N-1}{2}\}$ and that for even dimensions the phase space is composed with $D = 2N$ together with (μ,ν) integers and half-integers, summation is performed in the range $I' = \{0, \frac{1}{2}, 1, \cdots, \frac{2N-1}{2}\}$ (mod N). State vectors are set to zero on half-integer points. The Wigner function is real and normalized to unity, i.e., $\sum_{\mu,\nu}\mathcal{W}_{\mu,\nu} = 1$.

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As the characteristic function can be transformed to

$$\begin{aligned}\tilde{W}^L_{m,n} &= \sum_{k=0}^{N-1} \omega^{-2m(k+n)} \langle\psi|k+2n\rangle\langle k|\psi\rangle \\ &= \langle\psi|\omega^{2mn} Q^{-2m} P^{-2n}|\psi\rangle, \end{aligned} \tag{26}$$

and thus with $\tilde{W}^L_{m,n} = \mathrm{Tr}[\rho {W^L}_{m,n}]$ the Weyl operator is then defined as

$${W^L}_{m,n} = \omega^{2mn} Q^{-2m} P^{-2n}. \tag{27}$$

4.2. Definition of phase point operator in even-lattice case

The Leonhardt phase point operator is defined by the double-inverse Fourier transformation of $W^L_{m,n}$,

$${\Delta^L}_{m,n} = \frac{1}{N} \sum_{m',n' \in I'} \exp\left[\frac{2\pi i}{N} 2(mm' + nn')\right] {W^L}_{m',n'}. \tag{28}$$

With the phase, shift, and inversion operator defined by Eqs. (11), (12), and (13) indexed by $I'' = \{0, 1, \cdots, N-1\}$, Eq. (28) can then be expressed as

$${\Delta^L}_{m,n} = \omega^{-2mn} Q^{2n} P^{-2m} T \quad , \quad (m, n \in I'). \tag{29}$$

Eq. (29) reduce to Eq. (17) for odd N $(m, n \in I)$.

5. Group of symplectic transformations

In previous studies, it was shown that the continuous phase point operator is uniquely determined under certain symplectic covariance [25-27]. On the discrete phase space, the covariance is defined in a similar way.

5.1. Definition and its generator

We define the group Sp_N of symplectic transformations S on the discrete phase space $\mathbb{Z}_N \times \mathbb{Z}_N$ by analogy with the continuous case,

$$Sp_N = \left\{ S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; a,\ b,\ c,\ d \in \mathbb{Z}_N,\ \det S = 1 \in \mathbb{Z}_N \right\}, \tag{30}$$

where $\mathbb{Z}_N$ is a residue ring modulo N and its representatives are chosen from $\{0, \cdots, N-1\}$. The covariance relation becomes

$$U(S) \Delta_{m,n} U^\dagger(S) = \Delta_{S\cdot(m,n)}, \tag{31}$$

in this discrete case.

Here we show that the group Sp_N is generated from the two elements h_+ and h_-, which are defined as

$$h_+ \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in Sp_N, \qquad h_- \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in Sp_N. \tag{32}$$

We denote the group generated by h_+, h_- as Sp' and have $h_+^{-1} = h_+^{N-1}, h_-^{-1} = h_-^{N-1}$, as $h_+^N = h_-^N = I$.

Let S be an arbitrary element in Sp_N. Multiplying S by h_+ and h_-, we obtain

$$h_+^n S = \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix}, \tag{33}$$

$$S h_+^n = \begin{pmatrix} a & na+b \\ c & nc+d \end{pmatrix}, \tag{34}$$

$$h_-^n S = \begin{pmatrix} a & b \\ na+c & nb+d \end{pmatrix}, \tag{35}$$

$$S h_-^n = \begin{pmatrix} a+nb & b \\ c+nd & d \end{pmatrix}, \tag{36}$$

in which we perform the operation of multiplying a row (column) by an element in $\mathbb{Z}_N$ and then adding the result to the other row (column). We next define h_t as

$$h_t \equiv \begin{pmatrix} 0 & 1 \\ N-1 & 0 \end{pmatrix} \in Sp_N, \tag{37}$$

which can be represented in a form having h_+ on both sides of h_-^{N-1},

$$h_t = h_+ h_-^{N-1} h_+. \tag{38}$$

Multiplying S by h_t, we then have

$$h_t S = \begin{pmatrix} c & d \\ N-a & N-b \end{pmatrix}, \tag{39}$$

$$S h_t = \begin{pmatrix} N-b & a \\ N-d & c \end{pmatrix}. \tag{40}$$

We have thus performed the operation of interchanging rows and columns. Hence multiplying S by h_+ and h_- on the left and right appropriately, a given symplectic transformation $S \in Sp_N$ can be transformed into h_t. This means that S can be represented by h_+ and h_-, i.e.,

$$S = \prod_i h_{s_i}, \qquad s_i \in \{+, -\} \tag{41}$$

and

$$Sp_N = Sp'_N. \tag{42}$$

The explicit procedure is given as follows. We denote the Euclidean algorithm for b and d as

$$r_0 = \max(b, d), \qquad r_1 = \min(b, d), \tag{43}$$

$$r_i = k_i r_{i+1} + r_{i+2} \qquad (r_{i+2} < r_{i+1}; i = 0, \cdots, l-2; l \geq 2), \tag{44}$$

$$r_l = 0. \tag{45}$$

If $b = d$ the procedure stops at the first step with $l = 2$. Multiplying S by H defined as

$$H = h_+^{-k_{l-1}} \cdots h_-^{-k_1} h_+^{-k_0}, \qquad \text{for } b > d,\ l : \text{odd}, \tag{46}$$

$$H = h_t h_-^{-k_{l-1}} \cdots h_-^{-k_1} h_+^{-k_0}, \qquad \text{for } \ b > d, \ l : \text{even}, \tag{47}$$

$$H = h_+^{-k_{l-1}} \cdots h_-^{-k_1} h_+^{-k_0} h_t, \qquad \text{for } \ b < d, \ l : \text{odd}, \tag{48}$$

$$H = h_t h_-^{-k_{l-1}} \cdots h_-^{-k_1} h_+^{-k_0} h_t, \qquad \text{for } \ b < d, \ l : \text{even}, \tag{49}$$

from left, S can be transformed into

$$HS = \begin{pmatrix} \alpha & 0 \\ \gamma & \beta \end{pmatrix}, \qquad \alpha\beta = 1 \mod N. \tag{50}$$

Multiplying the left-hand side of Eq. (50) by $h_+^{\beta} h_-^{-\alpha}$ from the right and by $h_-^{-\beta-\beta\gamma}$ from the left, we have

$$h_-^{-\beta-\beta\gamma} \ HS \ h_+^{\beta} h_-^{-\alpha} = h_t. \tag{51}$$

This proves Eq. (42).

6. Uniqueness and explicit form of the projective unitary representation

Let us now show that the unitary representation $U(S)$ of Sp_N having the covariance Eq. (31) is determined up to a phase factor, hence its projective representation is unique. We also derive its explicit form using the Euclidean algorithm in this section.

6.1. Uniqueness and explicit form of the representation

Multiplying the both sides in Eq. (31) by a new $U(S')$ and its Hermitian conjugate from the left and right, respectively, we have

$$U(S')(U(S)\Delta_{m,n}U^\dagger(S))U^\dagger(S') = \Delta_{S'\cdot(S\cdot(m,n))}. \tag{52}$$

Taking $S'S = S''$, we have by definition

$$U(S")\Delta_{m,n}U^\dagger(S") = \Delta_{S"\cdot(m,n)}, \tag{53}$$

hence,

$$(U(S')U(S))\Delta_{m,n}(U(S')U(S))^\dagger = U(S")\Delta_{m,n}U^\dagger(S"). \tag{54}$$

From the traciality Eq. (20), the operators that commute with all phase point operators are phase factor multiples of unit operator, and we therefore have

$$U(SS') = \mathrm{e}^{i\theta} U(S)U(S'), \tag{55}$$

thus showing that $U(S)$ satisfying Eq. (31) is a projective unitary representation of Sp_N.

Let $U'(S)$ be another such representation. In Eq. (31), if we multiply the both sides by $U'^\dagger(S)$ from the left and by $U'(S)$ from the right, we get

$$(U'^\dagger(S)U(S))\Delta_{m,n}(U^\dagger(S)U'(S)) = \Delta_{S^{-1}\cdot(S\cdot(m,n))} = \Delta_{m,n}. \tag{56}$$

Hence, using the traciality again, we have

$$U'(S) = e^{i\theta} U(S). \tag{57}$$

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Thus, the projective unitary representation is unique.

Considering in conjunction with the uniqueness that a given symplectic transformation S is represented by h_+ and h_-, we find that a given $U(S)$ can be represented by $U(h_+)$ and $U(h_-)$ as

$$U(S) = \prod_i U(h_{s_i}), \qquad s_i \in \{+, -\}. \tag{58}$$

The sign factor s_i is determined from the Euclidean algorithm. One of such examples is given by

$$U(S) = U^{-1}(H)U^{-\beta-\beta\gamma}(h_-)U(h_t)U^{\alpha}(h_-)U^{-\beta}(h_+), \tag{59}$$

using Eqs. (50), (51), where $U(H)$ is the product of $U(h_+)$ and $U(h_-)$ according to Eqs. (46)-(49) in the previous section, and $U(h_t) = U(h_+)U^{-1}(h_-)U(h_+)$ using Eq. (38).

In Secs.VIII and IX, we consider $U(h_+)$ and $U(h_-)$ in more detail.

7. $U(h_+)$ and $U(h_-)$ on odd-lattice phase space

7.1. Derivation of the explicit form $U(h+)$ and $U(h-)$

Multiplying both sides of Eq. (31) by $U(S)$ from the right with $S = h_\pm$, we have

$$U(h_\pm){\Delta^C}_{m,n} = {\Delta^C}_{h_\pm\cdot(m,n)}U(h_\pm). \tag{60}$$

We use the phase point operators described earlier in Eq. (17) to find an explicit form of $U(h_+)$ and $U(h_-)$. We performed the actual calculation for lower dimensions (e.g. $N = 3, 5, 7$). For example, we have for $N = 7$

$$U(h_+) = \frac{1}{\sqrt{7}}\begin{pmatrix} 1 & \omega^4 & \omega^2 & \omega & \omega & \omega^2 & \omega^4 \\ \omega^4 & 1 & \omega^4 & \omega^2 & \omega & \omega & \omega^2 \\ \omega^2 & \omega^4 & 1 & \omega^4 & \omega^2 & \omega & \omega \\ \omega & \omega^2 & \omega^4 & 1 & \omega^4 & \omega^2 & \omega \\ \omega & \omega & \omega^2 & \omega^4 & 1 & \omega^4 & \omega^2 \\ \omega^2 & \omega & \omega & \omega^2 & \omega^4 & 1 & \omega^4 \\ \omega^4 & \omega^2 & \omega & \omega & \omega^2 & \omega^4 & 1 \end{pmatrix}, \tag{61}$$

$$U(h_-) = \begin{pmatrix} \omega & & & & & & \\ & \omega^2 & & & & & \\ & & \omega^4 & & & & \\ & & & 1 & & & \\ & & & & \omega^4 & & \\ & & & & & \omega^2 & \\ & & & & & & \omega \end{pmatrix}. \tag{62}$$

The obtained results suggest that the general form can be thus given for odd dimensions as

$$U(h_+) = \frac{1}{\sqrt{N}} \sum_{i,k\in I} |i\rangle \omega^{\frac{1}{2}(i-k)(i-k+N)} \langle k|, \tag{63}$$

$$U(h_-) = \sum_{i\in I} |i\rangle \omega^{\frac{1}{2}i(i+N)} \langle i|. \tag{64}$$

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Let us now confirm the symplectic covariance of the predicted $U(h_+)$ and $U(h_-)$ for a given phase point operator $\Delta^C_{m,n}$. In bra-ket notation, the phase point operator in Eq. (17) $\Delta^C_{m,n}$ is

$$\Delta^C{}_{m,n} = \sum_i \omega^{2n(-m+i)} |i\rangle\langle -i+2m|. \tag{65}$$

Accordingly,

$$U(h_+)\Delta^C{}_{m,n}U^\dagger(h_+) = \sum_i \omega^{2n(-(m+n)+i)} |i\rangle\langle -i+2(m+n)| = \Delta^C{}_{m+n,n} = \Delta^C{}_{h_+\cdot(m,n)}, \tag{66}$$

thus confirming the symplectic covariance. The general form of $U(h_+)$ for odd dimensions can therefore be regarded as Eq. (63). For $U(h_-)$, the symplectic covariance of Eq. (64) is similarly confirmed.

8. $U(h_+)$ and $U(h_-)$ on even-lattice phase space

8.1. Extension of the dimension of symplectic group

We derive $U(h_+)$ and $U(h_-)$ on even-lattice phase space in the same manner as the above derivation on odd-lattice phase space. More specifically, using Leonhardt's phase point operator with $N = 2$, $U(h_+)$ and $U(h_-)$ having symplectic covariance are

$$U(h_+) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \tag{67}$$

$$U(h_-) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. \tag{68}$$

However, these two explicit forms are not a representation of the symplectic group as defined by Eq. (30). To compose the Wigner function on discrete phase space in even dimensions, Leonhardt incorporated a virtual degree of freedom (ghost variable) and multiplied the number of variables by two. It is therefore also necessary to reconsider operation of the symplectic groups on even-lattice phase space.

In the following, we redefine the symplectic group in even dimensions. An even-lattice phase space is a $\mathbb{Z}_{2N} \times \mathbb{Z}_{2N}$ space taking into the ghost degree of freedom into account, and the symplectic group in this case is defined as

$$Sp_{2N} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a,\ b,\ c,\ d \in \mathbb{Z}_{2N},\ \det S = 1 \in \mathbb{Z}_{2N} \right\} \tag{69}$$

We consider the projective unitary representation based on this definition.

8.2. Derivation of the explicit form $U(h+)$ and $U(h-)$

From Eq. (67), we now have

$$\begin{aligned} \{U(h_+)\}^4 &= \frac{1}{4}\begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} \doteq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{(up to a phase factor)} \\ &= U(E) = U(h_+{}^4) \end{aligned} \tag{70}$$

where E is the unit element of Sp_{2N}, thereby confirming at least one projective unitary representation of $h_+ \in Sp_{2N}$. From $U(h_+)$ and $U(h_-)$ at $N = 2$ and from other lower dimensional examples, we predict

$$U(h_+) \;=\; \frac{1}{\sqrt{N}} \sum_{i,k\in I''} |i\rangle \tilde{\omega}^{(i-k)^2} \langle k|, \quad (71)$$

$$U(h_-) \;=\; \sum_{i\in I''} |i\rangle \tilde{\omega}^{i^2} \langle i|, \quad (72)$$

where $\tilde{\omega} = \omega^{\frac{1}{2}} = +\exp\left(\frac{2\pi i}{2N}\right)$ and $I'' = \{0, 1, \cdots, N-1\}$.

In the same manner as for odd dimensions, we next confirm the symplectic covariance by the predicted $U(h_+)$ and $U(h_-)$:

$$U(h_\pm){\Delta^L}_{m,n}U^\dagger(h_\pm) = {\Delta^L}_{h_\pm\cdot(m,n)}, \quad (73)$$

for the phase point operator $\Delta^L_{m,n}$ defined by Leonhardt (Eq. (29)):

$${\Delta^L}_{m,n} \;=\; \tilde{\omega}^{-4mn} Q^{2n} P^{-2m} T = \sum_{i\in I''} \tilde{\omega}^{4n(i-m)} |i\rangle\langle -i+2m|. \quad (74)$$

Note that $m, n \in I'$. For $U(h-)$, we obtain

$$U(h_-){\Delta^L}_{m,n}U^\dagger(h_-) = \sum_{i\in I''} \tilde{\omega}^{4(m+n)(i-m)} |i\rangle\langle -i+2m| = {\Delta^L}_{m,m+n} = {\Delta^L}_{h_-\cdot(m,n)}, \quad (75)$$

thereby confirming symplectic covariance for Sp_{2N} as the group of symplectic transformations S. Accordingly, the form of $U(h_-)$ in even dimensions in general can be regarded as in Eq. (72). For $U(h_+)$, we can similarly confirm that Eq. (71) has symplectic covariance with the Leonhardt's phase point operator.

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