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ORGANISATION EUROPÉENNE POUR LA RECHERCHE NUCLÉAIRE
CERN EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

CAS CERN ACCELERATOR SCHOOL

**SYNCHROTRON RADIATION AND FREE ELECTRON
LASERS**

President Hotel, Grenoble, France
22-27 April 1996

PROCEEDINGS

Editor: S. Turner

GENEVA
1998

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will jointly organise a course on

SYNCHROTRON RADIATION & FREE-ELECTRON LASERS

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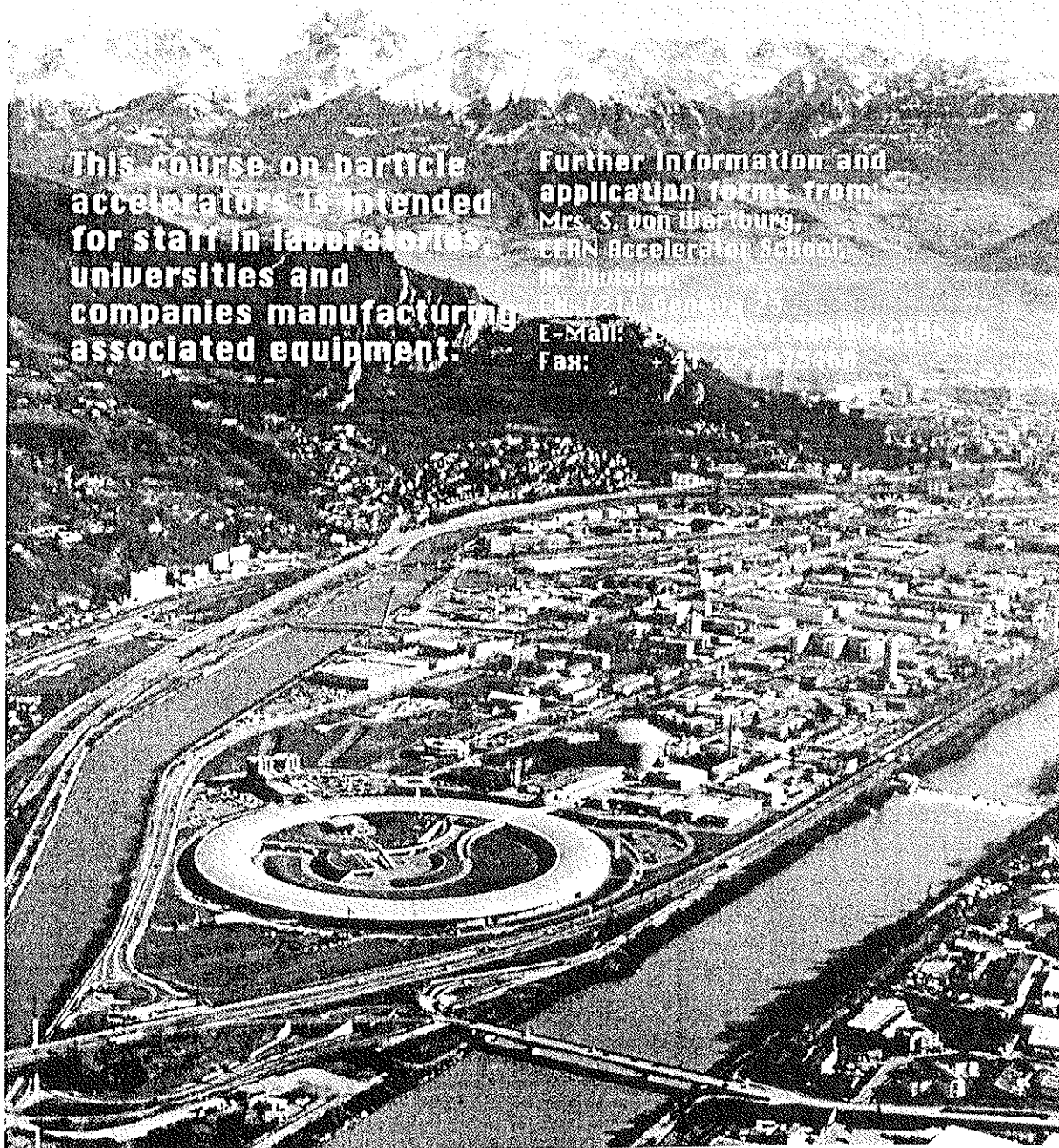
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PROGRAMME OF THE COURSE

SYNCHROTRON RADIATION AND FREE-ELECTRON LASERS

Hotel President, Grenoble, 22–27 April, 1996

Time	Monday 22 April	Tuesday 23 April	Wednesday 24 April	Thursday 25 April	Friday 26 April	Saturday 27 April
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	L U N C H					
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15.00	K. Wille					
	B R E A K			B R E A K		
15.10	Introduction to current and brightness limits	Lifetime and beam quality I		Lifetime and beam quality III	Poster session	
16.10	V. Suller	C. Bocchetta		C. Bocchetta		
	T E A			T E A		
16.30	S e m i n a r Scientific applications	Diagnostics I		Diagnostics II	S e m i n a r Practical aspects of beam stability	
17.30	D. Hauserman	A. Hofmann		A. Hofmann	P. Quinn	
	WELCOME COCKTAIL					
	EVENING MEAL		DINNER at the Château de Sassenages	EVENING MEAL	BANQUET	



FOREWORD

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With such enormous progress in the design of sources and in their range of applications, there was an urgent need to present an updated version of the course first presented in 1989. Not forgetting the basic theory of synchrotron radiation and electron synchrotrons, introductions are given to insertion devices and to the limitations to beam currents and radiation brightness. These topics are then developed before turning to linac and storage-ring free-electron lasers.

This course was only made possible by the generous support of several laboratories and many individuals. In particular, the help and encouragement of the ESRF management and staff, especially J.-L. Laclare, A. Ropert and T. Bouvet, was most invaluable. As always, the support of the CERN management, the guidance of the CAS Advisory and Programme Committees, and the attention to detail of the Local Organising Committee and the management and staff of the President Hotel ensured that the course was held under optimum conditions. Very special thanks must go to the lecturers at this course for the enormous task of preparing, presenting and writing up their topics. Finally, the enthusiasm of the participants, coming from so many parts of the world, was a convincing proof of the usefulness and success of this course.

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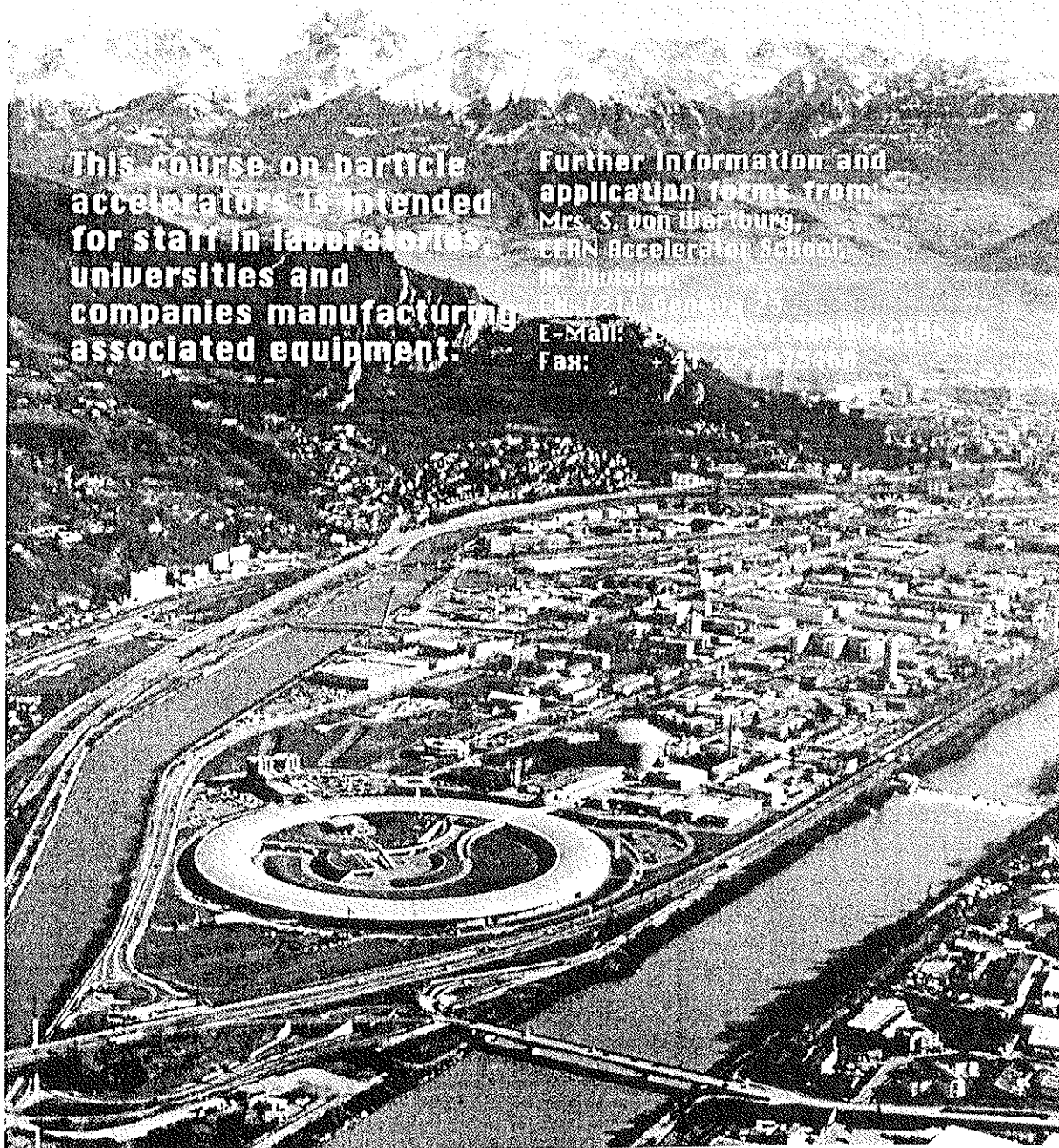
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A. Hofmann,
CERN, Geneva, Switzerland

Abstract

A qualitative discussion of synchrotron radiation is used first to obtain approximate expressions for the main properties such as opening angle, spectrum and polarization. Then the field of a moving charge is derived from the basic equation of electrodynamics, resulting in the Liénard-Wiechert equation. From the latter the radiated power is calculated for transverse and longitudinal acceleration. The most important application of the Liénard-Wiechert equation is the case of a charge moving uniformly on a circular orbit. The emitted radiation is called synchrotron radiation. Its properties, opening angle spectral density, polarization and photon distribution, are derived and discussed. Finally, a brief treatment of undulator and short magnet radiation is given in view of beam diagnostics application.

1 QUALITATIVE TREATMENT OF SYNCHROTRON RADIATION

1.1 Opening angle

We consider an electron moving in the laboratory frame F on a circular orbit of radius ρ emitting synchrotron radiation, Fig. 1. To estimate the opening angle of the radiation we go into a frame F' which moves at one instant with the same velocity, $\mathbf{v} = \beta\mathbf{c}$, as the electron. In this frame the trajectory of the electron has the form of a cycloid with a cusp where the electron undergoes an acceleration in the $-x'$ direction. Like any accelerated charge it will emit radiation which is in this frame F' approximately uniformly distributed. Going now back to the laboratory frame F , by applying a Lorentz transformation, this radiation will be peaked forward. A photon emitted along the x' -axis in the moving frame F' will appear at an angle $1/\gamma$ in the laboratory frame F . The typical opening angle of synchrotron radiation is therefore expected to be of the order of $1/\gamma$. For ultra-relativistic particles $\gamma \gg 1$ the radiation is confined to very small angles around the direction of the electron motion.

1.2 Typical frequency of the synchrotron radiation spectrum

Next we try to estimate the typical frequency of the emitted synchrotron radiation spectrum and consider an electron going through a long magnet where it emits radiation which reaches an observer P , Fig. 2. We ask ourselves how long the pulse of radiation will last. Due to the small opening angle this observer will see the light for a rather short time only. The radiation seen first is emitted at the point A , where the electron trajectory has an angle of $1/\gamma$ with respect to the direction towards the observer, and the last time at point A' where this angle is $-1/\gamma$. The length of the radiation pulse seen by the observer is just the difference in travel time between the electron and the photon in going from point A to point A'

$$\Delta t = t_e - t_\gamma = \frac{2\rho}{\beta\gamma c} - \frac{2\rho \sin(1/\gamma)}{c}.$$

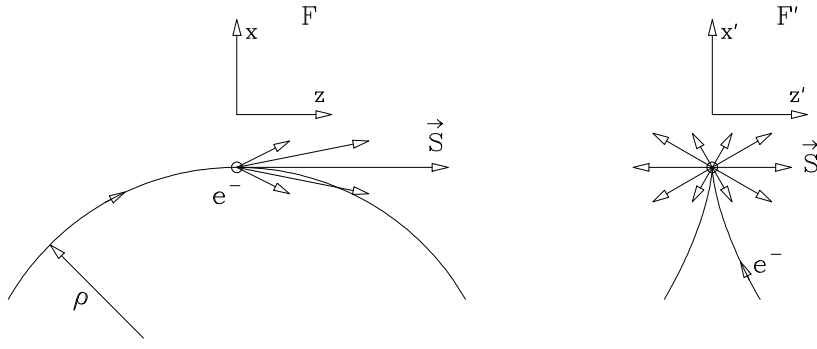


Figure 1: Opening angle of synchrotron radiation emitted by the moving source

For the ultra-relativistic case we consider here, $\gamma \gg 1$, we can expand the trigonometric function

$$\Delta t \sim \frac{2\rho}{\beta\gamma c} \left(1 - \beta + \frac{\beta}{6\gamma^2}\right) \sim \frac{2\rho}{\gamma c} \left(\frac{1}{2\gamma^2} + \frac{1}{6\gamma^2}\right) = \frac{4}{3} \frac{\rho}{c\gamma^3},$$

where we used the approximation

$$1 - \beta = \frac{1 - \beta^2}{1 + \beta} \sim \frac{1}{2\gamma^2}.$$

The typical frequency is approximately

$$\omega_{typ} \sim \frac{2\pi}{\Delta t} \sim \frac{3\pi c\gamma^3}{2\rho}.$$

The typical frequency is proportional to γ^3 , a factor γ^2 is due to the difference in velocity between electron and photon and a factor of γ is caused by the difference in trajectory length of the two particles in the magnet.

We consider now the radiation emitted in a short magnet having a length $L < 2\rho/\gamma$. An observer will receive the radiation emitted during the whole passage of the electron through this magnet, Fig. 3. The duration of the received pulse is now determined by the length L of the deflecting magnet. Again, the length of the radiated pulse is given by the difference in traveling time between the electron and photon going through the magnet

$$\Delta t_{sm} = \frac{L}{\beta c} - \frac{L}{c} = \frac{L(1 - \beta)}{\beta c} \approx \frac{L}{2c\gamma^2},$$

and the typical frequency is

$$\omega_{sm} \approx \frac{4\pi c\gamma^2}{L}.$$

This frequency contains only a factor γ^2 since the difference in trajectory length is small if the magnet is sufficiently short.

An undulator is an interesting source of synchrotron radiation. It consists of a spatially periodic magnetic field with period length λ_u in which the particle moves on a sinusoidal orbit, Fig. 4. Each of the periods represents a source of radiation. These

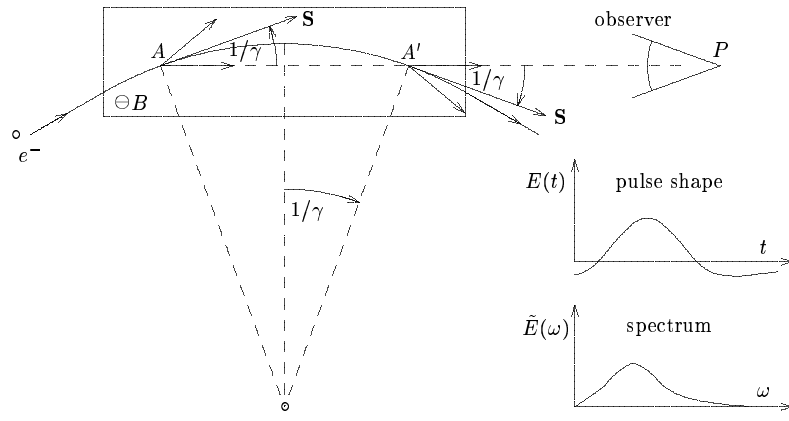


Figure 2: Typical frequency of the spectrum emitted in long magnets

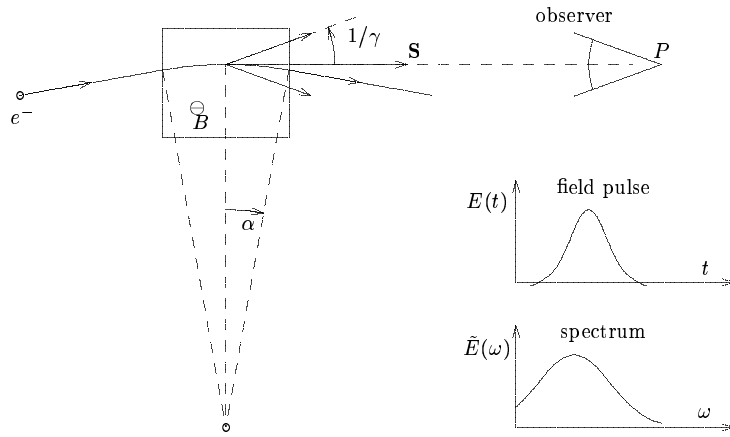


Figure 3: Typical frequency of the spectrum emitted in short magnet

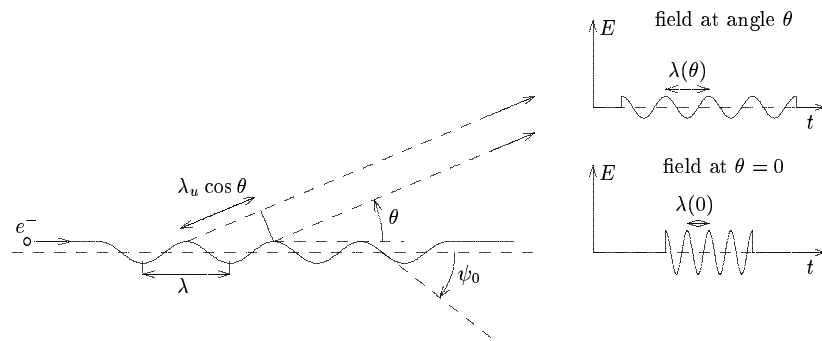


Figure 4: Spectrum emitted in undulators

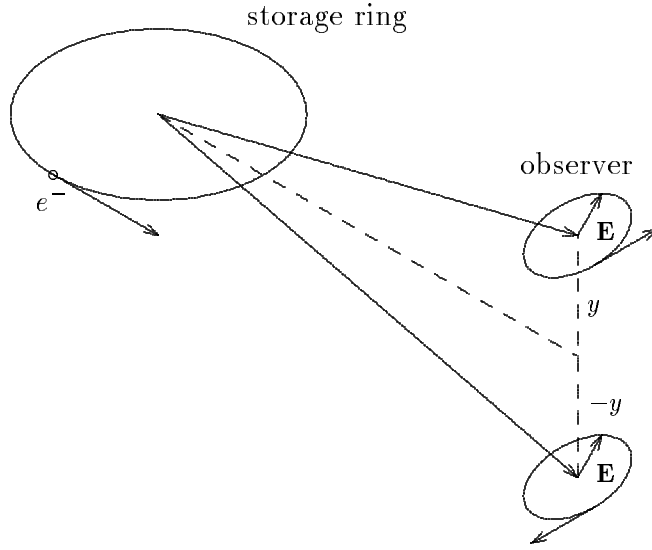


Figure 5: Elliptic polarization

contributions emitted towards an observer at an angle θ will interfere with each other. We get maximum intensity at a wavelength λ for which the contributions from different undulator periods are in phase. The time difference ΔT between the arrival of adjacent contributions is

$$\Delta T = \frac{\lambda_u}{\beta c} - \frac{\lambda_u \cos \theta}{c} = \frac{\lambda_u(1 - \beta \cos \theta)}{\beta c}.$$

For a relativistic particle the angle θ , where radiation of reasonable intensity can be observed, is small; $\theta \approx 1/\gamma$. We can approximate $\cos \theta \approx 1 - \theta^2/2$.

$$\Delta T \approx \frac{\lambda_u}{\beta c} \left(1 - \beta + \frac{\theta^2}{2} \right) \approx \frac{\lambda_u}{2c\gamma^2} (1 + \gamma^2 \theta^2).$$

The frequency for which we get constructive interference is just $\omega = 2\pi/\Delta T$.

$$\omega = \frac{4\pi c\gamma^2}{\lambda_u(1 + \gamma^2 \theta^2)}$$

Harmonics of this frequency might also be emitted.

1.3 Polarization

Since the acceleration of the electron is radial we expect the emitted radiation to be mostly polarized such that the electric field $\mathbf{E}$ lies in the median plane. This should be exact as long as the radiation is also observed in the (usually horizontal) median plane. For observations at a finite vertical angle a certain amount of circular polarization is expected as indicated in Fig. 5.

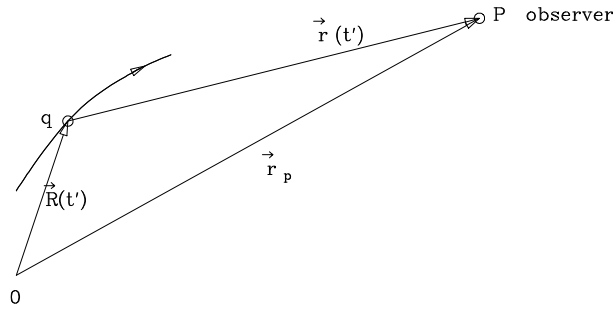


Figure 6: Particle trajectory and radiation geometry

2 POTENTIALS AND FIELDS OF A MOVING CHARGE

2.1 Relevant motion

We treat now the synchrotron radiation in a quantitative way. We follow basically the methods used in references [1, 2, 3, 4, 5] and repeat some of the derivations given in a previous paper [6]. The calculation of the potentials and fields of a moving charge is, unfortunately, rather formal involving lengthy derivations. However, the underlying physics is quite simple and is based on the fact that the radiation observed at a time t at a location $\mathbf{r}_p$ has to be emitted by a particle at a location $\mathbf{R}$ at an earlier time t' . The time difference $t - t'$ between the emission and observation of the radiation is simply the time this radiation takes to cover the spatial distance r between these two events.

This situation is illustrated in Fig. 6. An observer P whose location is given by the vector $\mathbf{r}_p$ observes the radiation field $\mathbf{E}(t)$ at the time t . This radiation was emitted at an earlier time t' by a moving charge q whose position is given by the vector $\mathbf{R}(t')$. The distance between the point of emission and the observer is given by the vector $\mathbf{r}$, with absolute value r , which fulfills the relation

$$\mathbf{R}(t') + \mathbf{r}(t') = \mathbf{r}_p. \quad (1)$$

The time difference between emission and observation is the time it takes the light to travel the distance r

$$t - t' = \frac{r(t')}{c}.$$

This relation between the two times looks very simple. However, in most applications it is rather difficult to evaluate since the distance $r(t')$ itself depends on t' .

The motion of the charge q and its velocity $\mathbf{v}$ are given by

$$\mathbf{R}(t') \text{ and } \mathbf{v}(t') = \boldsymbol{\beta}(t')c = \frac{d\mathbf{R}(t')}{dt'}.$$

The distance $\mathbf{r}$ between the particle and observer, its absolute value r and the unit vector $\mathbf{n}$ pointing towards the observer are

$$\mathbf{r}(t') , \quad r(t') = |\mathbf{r}(t')| \text{ and } \mathbf{n}(t') = \frac{\mathbf{r}(t')}{r(t')}.$$

In all our applications the observer does not move, $\mathbf{r}_p = \text{constant}$, and differentiating (1) gives

$$\frac{d\mathbf{R}(t')}{dt'} = \mathbf{v}(t') = c\boldsymbol{\beta} = -\frac{d\mathbf{r}(t')}{dt'}.$$

To get the time derivative of $r(t')$ we write

$$\mathbf{r} \frac{d\mathbf{r}}{dt'} = \frac{1}{2} \frac{d(\mathbf{r}^2)}{dt'} = r \frac{dr}{dt'} = -\mathbf{r} \cdot \mathbf{v}, \quad \frac{dr}{dt'} = -\mathbf{n} \cdot \mathbf{v}.$$

From this we get the important relation between the increments of the time t at the observer and the time t' of emission

$$dt = \left(1 + \frac{1}{c} \frac{dr}{dt'}\right) dt' = (1 - \mathbf{n} \cdot \boldsymbol{\beta}) dt'. \quad (2)$$

Usually we are concerned with the radiation emitted by a relativistic particle in the forward direction in which case two photons received by the observer within a very small time interval Δt have been emitted within a much longer time interval $\Delta t' = \Delta t / (1 - \mathbf{n} \cdot \boldsymbol{\beta})$.

2.2 Potentials of a moving charge (retarded potentials)

Our goal is to calculate the electric and magnetic fields $\mathbf{E}(t)$ and $\mathbf{B}(t)$ of the radiation received by an observer. These fields can be obtained from the scalar and vector potentials V and $\mathbf{A}$ through

$$\begin{aligned} \mathbf{B} &= \mu_0 \text{curl } \mathbf{A} = \mu_0 [\nabla \times \mathbf{A}] \\ \mathbf{E} &= -\text{grad } V - \mu_0 \frac{\partial \mathbf{A}}{\partial t} = -\nabla V - \mu_0 \frac{\partial \mathbf{A}}{\partial t}. \end{aligned}$$

We use here the Lorentz convention $\nabla V + \epsilon_0 \dot{V} = 0$.

For the case of a stationary charge and current distribution with charge density $\eta(x, y, z) = \eta(\mathbf{R})$ and velocity $\mathbf{v}(\mathbf{R})$, the resulting electrostatic potential V is simply given by Coulomb's law and the vector potential $\mathbf{A}$ by a similar expression

$$V = \frac{1}{4\pi\epsilon_0} \int \frac{\eta(\mathbf{R})}{r} dx dy dz, \quad \mathbf{A} = \frac{1}{4\pi} \int \frac{\eta(\mathbf{R})\mathbf{v}}{r} dx dy dz, \quad (3)$$

where r is the distance between the individual charges and the observer. If the charge distribution is not stationary, the density $\eta(t')$ and velocity $\mathbf{v}$ become functions of time. In carrying out the above integration we have to evaluate the charge density at the earlier time t' such that the potential created by the charges reaches the observer at the time t , i.e. $t' = t - r(t')/c$. The resulting potential V is called the retarded potential.

The integration

$$V(t) = \frac{1}{4\pi\epsilon_0} \int \frac{\eta(\mathbf{R}(t'))}{r(t')} dx dy dz$$

represents a thin sphere of radius r collapsing with the speed of light c towards the observer P counting all the charges on its way, Fig. 7. In this process the charges which move towards the observer are counted over a longer time and contribute more to the potential $V(t)$ and, vice versa, charges moving away from P contribute less.

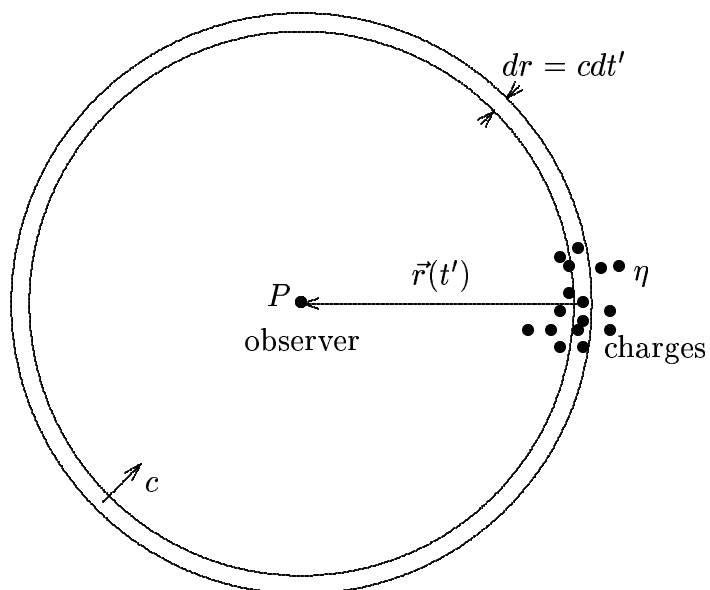


Figure 7: Collapsing sphere representing the integration over charges

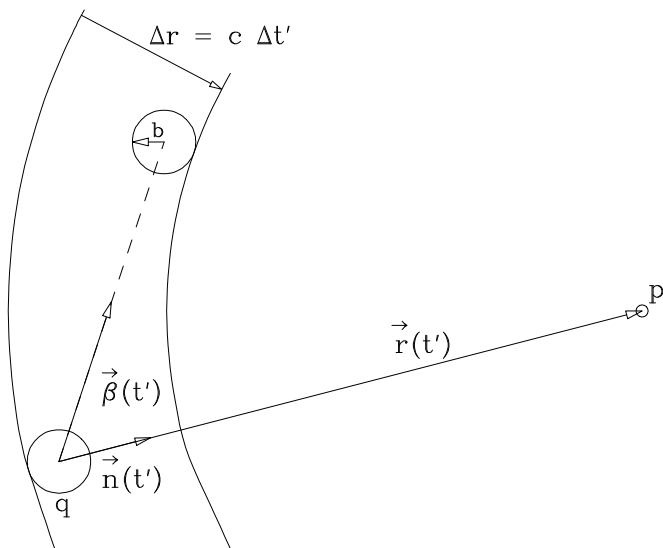


Figure 8: Contribution of a moving charge to the potential

Let us now consider a single charge of finite radius b and see for how long it contributes to the potential $V(t)$ at the observer, Fig. 8. First, we take the charge to be at rest and get for this time of contribution

$$\Delta t'_0 = \frac{2b}{c}.$$

Next we assume that the charge moves with velocity $\mathbf{v}$ which has the component $v_r = \mathbf{v} \cdot \mathbf{n}$ in the direction towards P . The time of contribution is now

$$\Delta t'_v = \frac{2b}{c - v_r} = \frac{2b}{c - \mathbf{n} \cdot \mathbf{v}} = \frac{2b}{c(1 - \mathbf{n} \cdot \boldsymbol{\beta})}.$$

The ratio between the times of contribution for the moving and the stationary charge is

$$\frac{\Delta t'_v}{\Delta t'_0} = \frac{1}{1 - \mathbf{n} \cdot \boldsymbol{\beta}}$$

which is independent of the radius b . We can let b go to zero and obtain for the electric potential of a moving point charge

$$V(t') = \frac{q}{4\pi\epsilon_0 r(t') (1 - \mathbf{n}(t') \cdot \boldsymbol{\beta}(t'))} = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r(1 - \mathbf{n} \cdot \boldsymbol{\beta})} \right)_{ret.}.$$

The index '*ret.*' indicates that the expression in the parenthesis has to be evaluated at the time t' in order to get the potential at the later time $t = t' + r(t')/c$. In the same way we obtain from (3) the vector potential $\mathbf{A}$ of a moving charge

$$\mathbf{A}(t) = \frac{q}{4\pi} \left(\frac{\mathbf{v}}{r(1 - \mathbf{n} \cdot \boldsymbol{\beta})} \right)_{ret.}.$$

These quantities $V(t)$ and $\mathbf{A}(t)$ are the retarded potentials of a moving charge and are also called Liénard-Wiechert potentials.

2.3 The fields of a moving charge

We obtained the potentials of a moving charge

$$V(t') = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{r(1 - \mathbf{n} \cdot \boldsymbol{\beta})} \right)_{ret.}, \quad \mathbf{A} = \frac{q}{4\pi} \left(\frac{\mathbf{v}}{r(1 - \mathbf{n} \cdot \boldsymbol{\beta})} \right)_{ret.}$$

from which we will now calculate the electric and magnetic fields using

$$\mathbf{E} = -\nabla V - \mu_0 \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \mu_0 \text{curl } \mathbf{A} = \mu_0 [\nabla \times \mathbf{A}]. \quad (4)$$

The difficulty in carrying out this operation lies in the fact that the above relation between potentials and fields requires differentiation with respect to the time t and the position $\mathbf{r}_p$ of the observer. Any change Δt or $\Delta \mathbf{r}_p$ in these coordinates will change the time t' the radiation was emitted through the relation $t = t' + r(t')/c$ and will change in turn also the quantities $\mathbf{r}(t')$, $\mathbf{R}(t')$ and $\mathbf{n}(t')$ in a rather complicated way.

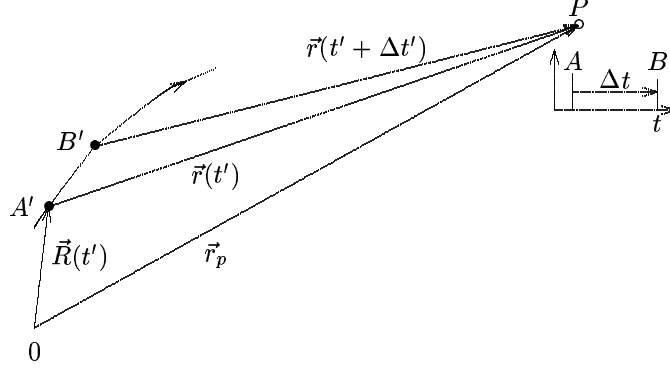


Figure 9: Calculating the derivation with respect to the time t

We start with the derivative with respect to t and consider two events (photons) A and B occurring at the observer P separated in time by Δt . These two photons have to be emitted at different times t' and, because the charge is moving, also at different locations A' and B' , Fig. 9. The time differences Δt and $\Delta t'$ are related by

$$\Delta t = \Delta t' + \frac{1}{c} \frac{dr}{dt'} = (1 - \mathbf{n} \cdot \boldsymbol{\beta}) \Delta t'$$

as we showed before (2). This gives for the time derivative

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{dt'}{dt} \frac{\partial \mathbf{A}}{\partial t'} = \frac{1}{1 - \mathbf{n} \cdot \boldsymbol{\beta}} \frac{\partial \mathbf{A}}{\partial t'}.$$

We calculate first the derivative with respect to t' of the expression $r - \mathbf{r} \cdot \boldsymbol{\beta}$ appearing in the denominator of the potentials

$$\frac{\partial}{\partial t'} (r - \mathbf{r} \cdot \boldsymbol{\beta}) = \frac{dr}{dt'} - \frac{d\mathbf{r}}{dt'} \cdot \boldsymbol{\beta} - \mathbf{r} \cdot \dot{\boldsymbol{\beta}} = -c(\mathbf{n} \cdot \boldsymbol{\beta}) + c\beta^2 - \mathbf{r} \cdot \dot{\boldsymbol{\beta}}.$$

This gives for the derivative of the vector potential $\mathbf{A}$ with respect to the observer time t

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{qc}{4\pi} \left[\frac{1}{1 - \mathbf{n} \cdot \boldsymbol{\beta}} \left(\frac{\dot{\boldsymbol{\beta}}}{r(1 - \mathbf{n} \cdot \boldsymbol{\beta})} - \boldsymbol{\beta} \frac{-c(\mathbf{n} \cdot \boldsymbol{\beta}) + c\beta^2 - \mathbf{r} \cdot \dot{\boldsymbol{\beta}}}{r^2(1 - \mathbf{n} \cdot \boldsymbol{\beta})^2} \right) \right]_{ret}.$$

Sorting this expression according to powers of r gives

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{qc}{4\pi} \left[\frac{1}{(1 - \mathbf{n} \cdot \boldsymbol{\beta})^3} \left(\frac{\dot{\boldsymbol{\beta}}(1 - \mathbf{n} \cdot \boldsymbol{\beta}) + \boldsymbol{\beta}(\mathbf{n} \cdot \dot{\boldsymbol{\beta}})}{r} + \frac{c(\boldsymbol{\beta}(\mathbf{n} \cdot \boldsymbol{\beta} - \beta^2))}{r^2} \right) \right]_{ret}. \quad (5)$$

Next we calculate the gradient which is the change of an expression for a small change in $\mathbf{r}_p$, i.e. of the position of the observer. We compare now two observers A and B separated by an infinitesimally small amount $\Delta \mathbf{r}_p$ in space and consider two photons arriving each at the *same* time t at the respective observers, Fig. 10. The vector $\mathbf{r}$ undergoes a change $\Delta \mathbf{r}$ when comparing the observer A to the observer B . One contribution $\Delta \mathbf{r}_1$ is directly caused by the change $\Delta \mathbf{r}_p$, namely $\Delta \mathbf{r}_1 = \Delta \mathbf{r}_p$, the other one $\Delta \mathbf{r}_2 = \Delta t' \partial \mathbf{r} / \partial t'$ is due to the difference $\Delta t'$ in time of emission for the two photons such that they arrive in the same time t at the respective observer A or B . Using $\partial \mathbf{r} / \partial t' = -\boldsymbol{\beta} c$ we have

$$\Delta \mathbf{r} = \Delta \mathbf{r}_p - c\boldsymbol{\beta} \Delta t' \quad \text{and} \quad \Delta r = \mathbf{n} \cdot \Delta \mathbf{r} = \mathbf{n} \cdot \Delta \mathbf{r}_p - c(\mathbf{n} \cdot \boldsymbol{\beta}) \Delta t'.$$

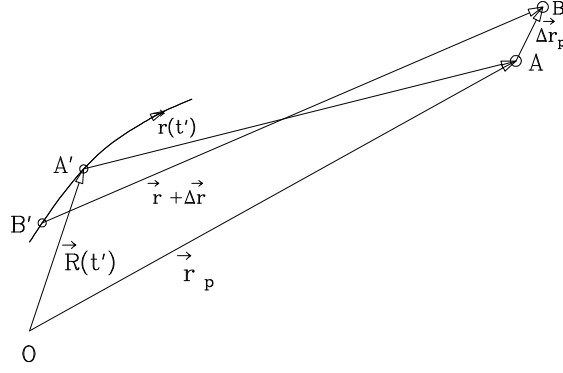


Figure 10: Calculation of the gradient

Since the two photons arrive at the same time at their respective observers the difference $\Delta t'$ in time of emission is given simply by the difference Δr in the traveling distances

$$\Delta r = -c\Delta t'$$

which gives

$$-c\Delta t' = \mathbf{n} \cdot \Delta \mathbf{r}_p - c(\mathbf{n} \cdot \boldsymbol{\beta})\Delta t' \text{ or } \Delta t' = -\frac{\mathbf{n} \cdot \Delta \mathbf{r}_p}{c(1 - \mathbf{n} \cdot \boldsymbol{\beta})}. \quad (6)$$

For the expression $r - \mathbf{r} \cdot \boldsymbol{\beta}$ appearing in the denominator of the potentials we get

$$\Delta(r - \mathbf{r} \cdot \boldsymbol{\beta}) = \Delta r - \Delta \mathbf{r} \cdot \boldsymbol{\beta} - \mathbf{r} \cdot \Delta \boldsymbol{\beta} = -c\Delta t' - \left(\Delta \mathbf{r}_p + \frac{d\mathbf{r}}{dt'}\Delta t' \right) \boldsymbol{\beta} - r \frac{\partial \boldsymbol{\beta}}{\partial t'}\Delta t'.$$

Using (6) gives

$$\Delta(r - \mathbf{r} \cdot \boldsymbol{\beta}) = \left(\frac{\mathbf{n}}{1 - \mathbf{n} \cdot \boldsymbol{\beta}} - \boldsymbol{\beta} - \frac{\beta^2 \mathbf{n}}{1 - \mathbf{n} \cdot \boldsymbol{\beta}} + \frac{(\mathbf{r} \cdot \dot{\boldsymbol{\beta}})\mathbf{n}}{c(1 - \mathbf{n} \cdot \boldsymbol{\beta})} \right) \Delta \mathbf{r}_p.$$

Comparing this with the equation

$$\Delta(r - \mathbf{r} \cdot \boldsymbol{\beta}) = \text{grad}(r - \mathbf{r} \cdot \boldsymbol{\beta}) \cdot \Delta \mathbf{r}_p$$

we can write

$$\text{grad}(r - \mathbf{r} \cdot \boldsymbol{\beta}) = \frac{(c(1 - \beta^2)\mathbf{n} - c(1 - \mathbf{n} \cdot \boldsymbol{\beta})\boldsymbol{\beta} + r(\mathbf{n} \cdot \dot{\boldsymbol{\beta}})\mathbf{n})}{c(1 - \mathbf{n} \cdot \boldsymbol{\beta})}.$$

The gradient of the scalar potential

$$V = \frac{q}{4\pi\epsilon_0} \frac{1}{(r - \mathbf{r} \cdot \boldsymbol{\beta})_{ret.}}$$

becomes

$$\text{grad } V = -\frac{q}{4\pi\epsilon_0} \frac{\text{grad}(r - \mathbf{r} \cdot \boldsymbol{\beta})}{r^2(1 - \mathbf{n} \cdot \boldsymbol{\beta})^2} = \frac{-q}{4\pi c\epsilon_0} \left(\frac{c(1 - \beta^2)\mathbf{n} - (1 - \mathbf{n} \cdot \boldsymbol{\beta})\boldsymbol{\beta} + r(\mathbf{n} \cdot \dot{\boldsymbol{\beta}})\mathbf{n}}{r^2(1 - \mathbf{n} \cdot \boldsymbol{\beta})^3} \right)_{ret.}.$$

This together with the time derivative of the vector potential (5) gives us for the electric field of a moving charge

$$\mathbf{E}(t) = -\text{grad } V - \mu_0 \frac{\partial \mathbf{A}}{\partial t} = \frac{q}{4\pi c \epsilon_0} \left(\frac{c(1 - \beta^2)(\mathbf{n} - \boldsymbol{\beta})}{r^2(1 - \mathbf{n} \cdot \boldsymbol{\beta})^3} + \frac{(\mathbf{n} \cdot \dot{\boldsymbol{\beta}})(\mathbf{n} - \boldsymbol{\beta}) + (1 - \mathbf{n} \cdot \boldsymbol{\beta})\dot{\boldsymbol{\beta}}}{r(1 - \mathbf{n} \cdot \boldsymbol{\beta})^3} \right)_{ret.}$$

Using the expression for the triple vector product

$$[\mathbf{A} \times [\mathbf{B} \times \mathbf{C}]] = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

we get

$$\mathbf{E}(t) = \frac{q}{4\pi \epsilon_0} \left(\frac{(1 - \beta^2)(\mathbf{n} - \boldsymbol{\beta})}{r^2(1 - \mathbf{n} \cdot \boldsymbol{\beta})^3} + \frac{[\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]]}{cr(1 - \mathbf{n} \cdot \boldsymbol{\beta})^3} \right)_{ret.} \quad (7)$$

Going through the corresponding calculation for the magnetic field $\mathbf{B}(t)$ results in the relation

$$\mathbf{B}(t) = \frac{[\mathbf{n} \times \mathbf{E}]}{c}. \quad (8)$$

The expressions obtained for $\mathbf{E}(t)$ and $\mathbf{B}(t)$ are called Liénard-Wiechert fields.

2.4 Discussion

The equations for the electric and magnetic fields of a moving point charge have to be discussed to appreciate their properties.

The fields $\mathbf{E}$ and $\mathbf{B}$ are perpendicular to each other.

For a stationary charge $\dot{\boldsymbol{\beta}} = 0$ we get Coulomb's law

$$\mathbf{E} = \frac{q\mathbf{n}}{4\pi \epsilon_0 r^2} = \text{const.}$$

For a charge moving with constant velocity $\dot{\boldsymbol{\beta}} = 0$; $\boldsymbol{\beta} = (0, 0, \beta) = \text{const.}$ we get

$$\mathbf{E}(t) = \frac{q}{4\pi \epsilon_0 \gamma^2} \left(\frac{\mathbf{n} - \boldsymbol{\beta}}{r^2(1 - \mathbf{n} \cdot \boldsymbol{\beta})^3} \right)_{ret.} \quad (9)$$

This is an unusual expression for the electric field of a charge moving with constant velocity since it refers to the position of the charge at the time t' the field was emitted to be observed at the later time $t = t' + r(t')/c$. It is worthwhile to discuss it in some detail to appreciate some of the properties of the retarded potentials. The situation is illustrated in Fig. 11. For simplicity we put the origin of the coordinate system to position $\mathbf{R}(t')$ of the charge at the time t' . The electric field emitted at this location travels to the point P covering the distance $r = c(t - t')$ where it is observed at the time t . This position is given by the vector $\mathbf{r}_p = r\mathbf{n}$. During the same time the charge q travels the distance $\beta c(t - t') = \beta r$ and arrives at the point A given by the vector $\mathbf{r}_A = \boldsymbol{\beta} c(t - t') = r\boldsymbol{\beta}$. The vector pointing from the position A of the charge at the time t of observation to the position P is given by $\mathbf{r}_{A,P} = r(\mathbf{n} - \boldsymbol{\beta})$. Comparing this with the direction of the observed electric field $\mathbf{E}$ given by (9) we find that the vector $\mathbf{r}_{A,P}$ is parallel to this direction of the electric field. In other words the field direction is as if it originated from the position A of the charge at the time t of observation. The situation of a charge moving with constant

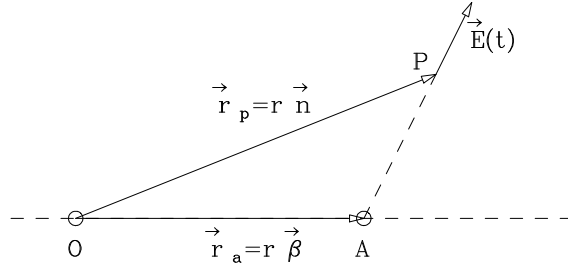


Figure 11: Field of a uniformly moving charge

velocity can be reduced to an electrostatic case by a Lorentz transformation into a frame in which the charge is at rest. From this fact we can deduce that the uniformly moving charge does not radiate any energy.

The general expression (7) for the field has two terms, one being proportional to $1/r^2$ and the other, to $1/r$. At large distances the second term will dominate and we can neglect the first one. The remaining field is often called the *far field* and is given by

$$\mathbf{E}_{far-field}(t) = \frac{q}{4\pi c\epsilon_0} \left(\frac{[\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]]}{r(1 - \mathbf{n} \cdot \boldsymbol{\beta})^3} \right)_{ret.}.$$

For this far field the electric field $\mathbf{E}$ and because of (8) also the magnetic field $\mathbf{B}$ are perpendicular to the direction of the unit vector $\mathbf{n}$ pointing from the charge to the observer. We will from now on concentrate on this “far field” which is also called radiation field and omit the corresponding index. This is fine as long as we use the field to discuss the polarization properties and to calculate the radiation power. It should however be pointed out that the ‘far-field’ alone does not satisfy Maxwell’s equations.

2.5 The power radiated by the particle

The power flux of this field is determined by the Poynting vector $\mathbf{S}$

$$\mathbf{S} = \frac{1}{\mu_0} [\mathbf{E} \times \mathbf{B}] = \frac{1}{\mu_0 c} [\mathbf{E} \times [\mathbf{n} \times \mathbf{E}]] = \frac{1}{\mu_0 c} (E^2 \mathbf{n} - (\mathbf{n} \cdot \mathbf{E}) \mathbf{E}).$$

For the far field we have $(\mathbf{n} \cdot \mathbf{E}) = 0$ and therefore $\mathbf{S} = E^2 \mathbf{n} / (\mu_0 c)$. This is the Energy passing through a unit area per unit time t of observation. To get the power P radiated by the particle we have to consider the energy W emitted by the charge per unit time t' of emission

$$P = \frac{dU}{dt'} = \int (\mathbf{n} \cdot \mathbf{S}) \frac{dt}{dt'} r^2 d\Omega = \int S(1 - \mathbf{n} \cdot \boldsymbol{\beta}) r^2 d\Omega,$$

where $S = |\mathbf{S}|$ is the absolute value of the Poynting vector and Ω is the solid angle. To calculate the power distribution we use a coordinate system (x, y, z) in which the particle is momentarily at the origin moving in the z -direction and we also use the angles θ and

ϕ of the corresponding spherical coordinate system. The unit vector $\mathbf{n}$ pointing from the particle to the observer and the normalized velocity vector $\boldsymbol{\beta}$ are

$$\mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \quad \text{and} \quad \boldsymbol{\beta} = \beta (0, 0, 1).$$

We take now the case with the acceleration perpendicular to the velocity and pointing in the $-x$ direction. This corresponds to the case of synchrotron radiation emitted by a particle going through a magnetic field $\mathbf{B}$ pointing in the y direction resulting in a curvature

$$\frac{1}{\rho} = \frac{eB}{\beta\gamma m_0 c}.$$

The normalized acceleration is

$$\dot{\boldsymbol{\beta}} = \frac{\beta^2 c}{\rho} (-1, 0, 0).$$

The distribution of the power radiated by the particle is

$$\frac{dP_T}{d\Omega} = \frac{q^2 [\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]]^2}{(4\pi)^2 \epsilon_0 c (1 - \mathbf{n} \cdot \boldsymbol{\beta})^5} = \frac{cr_0 m_0 c^2 \beta^4}{4\pi \rho^2} \left(\frac{(1 - \beta \cos \theta)^2 - (1 - \beta^2) \sin^2 \theta \cos^2 \phi}{(1 - \beta \cos \theta)^5} \right), \quad (10)$$

where we assumed a particle having the elementary charge $q = e$ and introduced the classical radius.

$$r_0 = \frac{e^2}{4\pi\epsilon_0 m_0 c^2} = \begin{array}{l} 2.818 \cdot 10^{-15} \text{ m for electrons} \\ 1.535 \cdot 10^{-18} \text{ m for protons} \end{array}.$$

Integrating (10) over the solid angle gives the total power radiated by the particle

$$P_{0T} = \frac{2r_0 c m_0 c^2 \beta^4 \gamma^4}{3\rho^2} = \frac{2r_0 m_0 c^2 \dot{\beta}_T \gamma^2}{3c} = \frac{2cr_0 \dot{p}_T^2 \gamma^2}{3m_0 c^2} = \frac{2r_0 c^3 e^2 \beta^2 \gamma^2 B^2}{3m_0 c^2}, \quad (11)$$

where we used the relation between the time derivative $\dot{p}_t$ of the momentum and the transverse acceleration

$$\dot{p}_T = m_0 c \dot{\beta}_T \gamma.$$

For the ultra-relativistic case $\beta \approx 1$, $\gamma \gg 1$ the radiation is peaked forward confined to a cone of opening angle $\theta \sim 1/\gamma$ and we get approximately

$$\frac{dP_T}{d\Omega} \approx \frac{2cr_0 m_0 c^2 \gamma^6}{\pi \rho^2} \left(\frac{1 + 2\gamma^2 \theta^2 (1 - 2 \cos^2 \phi) + \gamma^4 \theta^4}{(1 + \gamma^2 \theta^2)^5} \right) \quad \text{and} \quad P_{0T} \approx \frac{2r_0 c m_0 c^2 \gamma^4}{3\rho^2}. \quad (12)$$

It is interesting to also investigate the case where the acceleration is longitudinal, i.e. in the direction of the particle velocity, although this is of no practical importance. We get for the power distribution

$$\frac{dP_L}{d\Omega} = \frac{q^2}{(4\pi)^2 \epsilon_0 c} \frac{[\mathbf{n} \times [\mathbf{n} \times \dot{\boldsymbol{\beta}}]]^2}{(1 - \mathbf{n} \cdot \boldsymbol{\beta})^5} = \frac{r_0 m_0 c^2 \dot{\beta}_L^2}{4\pi c} \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5}.$$

The power is not emitted in the forward direction $\theta = 0$ but symmetrically around it with a typical angle of order $1/\gamma$ with respect to this direction. For the total power we find

$$P_{0L} = \frac{2r_0 m_0 c^2 \dot{\beta}_L^2 \gamma^6}{3c} = \frac{2cr_0 \dot{p}_L^2}{3m_0 c^2}, \quad (13)$$

where we used the relation between the time derivative of the momentum and the longitudinal acceleration which is different from the transverse case

$$\dot{p}_L = m_0 c (\beta_L \dot{\gamma} + \dot{\beta}_L \gamma) = m_0 c \dot{\beta}_L \gamma^3.$$

Comparing (13) with (11) we find that for the same absolute value of the time derivative of the momentum, i.e. the same force, the radiated power is smaller by γ^2 for a longitudinal force than for a transverse one. This is the main reason why linear colliders are, above a certain beam energy, advantageous compared to storage rings for colliding electrons and positrons.

2.6 Fourier transform of the radiation field

We calculated the electric and magnetic radiation fields emitted by a moving charge at the time t' and observed at the time t

$$\mathbf{E}(t) = \frac{q}{4\pi c \epsilon_0} \left(\frac{[\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]]}{r(1 - \mathbf{n} \cdot \boldsymbol{\beta})^3} \right)_{ret.}, \quad \mathbf{B}(t) = \frac{[\mathbf{n} \times \mathbf{E}]}{c}. \quad (14)$$

As we said before, the difficulty to evaluate these fields lies in the fact that the expressions involving the particle motion have to be evaluated at the earlier time t' which has, in general, a rather complicated relation to the time t of observation. For this reason it is in many cases easier to calculate directly the Fourier transform $\tilde{\mathbf{E}}$ of the electric field

$$\tilde{\mathbf{E}}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{E}(t) e^{-i\omega t} dt.$$

This integration involves the time t since we are interested in the spectrum of the radiation as seen by the observer. We can however make a formal transformation of the integration variable into the time $t' = t - r(t')/c$ and get

$$\tilde{\mathbf{E}}(\omega) = \frac{q}{4\pi c \epsilon_0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{[\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]]}{r(1 - \mathbf{n} \cdot \boldsymbol{\beta})^2} \right) e^{-i\omega(t' + r(t')/c)} dt'. \quad (15)$$

We omitted in the above equation the index ‘ret’ since the integration variable is anyway the time t' at which the expressions are evaluated. We take now the ultra-relativistic case $\gamma \gg 1$ which has the emitted radiation concentrated within a small opening angle of order $1/\gamma \ll 1$. The observer sees radiation originating only from a small part of the trajectory of length $\Delta l \sim 2\rho/\gamma$, where ρ is the radius of curvature as indicated on Fig. 2. In this case the vector $\mathbf{r}$ pointing from the particle to the observer will change little as long as the radiation is observed from a large distance $r \gg 2\rho/\gamma$. We neglect the variation of $\mathbf{r}$ in the triple vector product but not in the exponent as will be explained later. We can integrate (15) in parts

$$\int \frac{dU}{dt'} V dt' = UV - \int U \frac{dV}{dt'} dt'$$

with

$$U \approx \frac{[\mathbf{n} \times [\mathbf{n} \times \boldsymbol{\beta}]]}{1 - \mathbf{n} \cdot \boldsymbol{\beta}}, \quad \frac{dU}{dt'} = \frac{[\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]]}{(1 - \mathbf{n} \cdot \boldsymbol{\beta})^2}$$

$$V = e^{-i\omega(t' - r(t')/c)}, \quad \frac{dV}{dt'} = -i\omega(1 - \mathbf{n} \cdot \boldsymbol{\beta}) e^{-i\omega(t' - r(t')/c)},$$

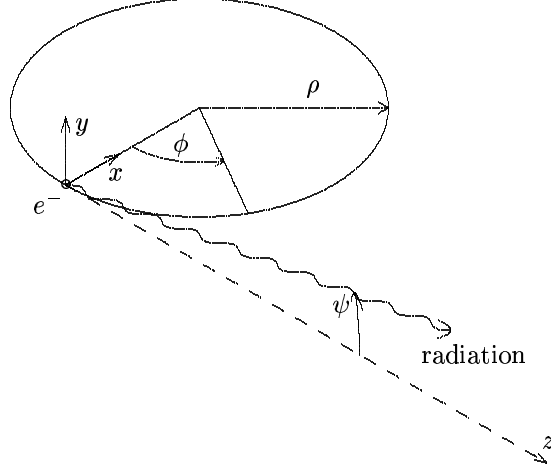


Figure 12: Geometry used to describe the synchrotron radiation

which gives

$$\begin{aligned} \tilde{\mathbf{E}}(\omega) = & \frac{e}{4\pi\sqrt{2\pi}\epsilon_0 cr} \left[\frac{[\mathbf{n} \times [\mathbf{n} \times \boldsymbol{\beta}]]}{1 - \mathbf{n} \cdot \boldsymbol{\beta}} e^{-i\omega(t' - r(t')/c)} \right]_{-\infty}^{\infty} \\ & + i\omega \int_{-\infty}^{\infty} [\mathbf{n} \times [\mathbf{n} \times \boldsymbol{\beta}]] e^{-i\omega(t' - r(t')/c)} dt' \end{aligned} \quad (16)$$

In many practical cases the first term on the right hand side of (16) can be neglected since it contains the values of the expression at $t' = \pm\infty$ which are not 'seen' by the observer. We arrive at the expression for the Fourier transformed radiation field

$$\tilde{\mathbf{E}}(\omega) = \frac{i\omega e}{4\pi\sqrt{2\pi} cr} \int_{-\infty}^{\infty} [\mathbf{n} \times [\mathbf{n} \times \boldsymbol{\beta}]] e^{-i\omega(t' - r(t')/c)} dt' \quad (17)$$

which is an easier equation to deal with than (14) giving the field in time domain. However, we should remember the frequency domain expression (17) is less accurate than (14). The Fourier transformed magnetic field $\tilde{\mathbf{B}}$ is still related to the electric field by

$$\tilde{\mathbf{B}}(\omega) = \frac{[\mathbf{n} \times \tilde{\mathbf{E}}(\omega)]}{c}.$$

3 RADIATION FROM A CHARGE MOVING ON A CIRCULAR ORBIT

3.1 The Fourier transformed electric field of the radiation

We consider now the radiation emitted by a charge which moves momentarily with a constant, ultra-relativistic velocity on a circular trajectory of bending radius ρ . This is just the case of ordinary synchrotron radiation emitted by a charged particle in a bending magnet where the curvature is

$$\frac{1}{\rho} = \frac{qB}{m_0 c \beta \gamma} \approx \frac{qBc}{m_0 c^2 \gamma}.$$

We assume a geometry as indicated in Fig. 12 and choose the origin of the coordinate system to be at the location of the particle at $t' = 0$. The position $\mathbf{R}$ and the normalized

velocity $\boldsymbol{\beta}$ are

$$\begin{aligned}\mathbf{R}(t') &= \rho \left((1 - \cos(\omega_0 t')) , 0 , \sin(\omega_0 t') \right) \\ \boldsymbol{\beta}(t') &= \beta \left(\sin(\omega_0 t') , 0 , \cos(\omega_0 t') \right) \\ \dot{\boldsymbol{\beta}}(t') &= \beta \omega_0 \left(\cos(\omega_0 t') , 0 , -\sin(\omega_0 t') \right)\end{aligned}\tag{18}$$

where we introduced the angular velocity $\omega_0 = \beta c / \rho$ of the particle. We chose the observation point P to lie in the yz -plane. This is no restriction since we assumed a circular motion with constant speed βc which is symmetric with respect to the angle ϕ . The vector pointing from the origin to the observer makes an angle ψ with respect to the xz -plane of the particle motion. We assume again observation from a large distance $r \gg 2\rho/\gamma$ and consider the vector $\mathbf{r}$ to be constant over the short time during which radiation is emitted which can be observed at P . We have for the unit vector $\mathbf{n}$ and for the vector product appearing in (17)

$$\begin{aligned}\mathbf{n} &= (0, \sin \psi, \cos \psi) \\ [\mathbf{n} \times [\mathbf{n} \times \boldsymbol{\beta}]] &= \beta \left(-\sin(\omega_0 t'), \cos \psi \sin \psi \cos(\omega_0 t'), -\sin^2 \psi \cos(\omega_0 t') \right).\end{aligned}$$

Since we assumed the ultra-relativistic case $\gamma \gg 1$ the vertical opening angle of the radiation is very small $\psi \ll 1$ and only a small part of the trajectory $\Delta\phi = \omega_0 t' \ll 1$ contributes to the radiation observed at P . We can therefore approximate the vector product by

$$[\mathbf{n} \times [\mathbf{n} \times \boldsymbol{\beta}]] \approx \beta (-\omega t', \psi, 0)$$

and the term in the exponent by

$$t' + \frac{r(t')}{c} = t' - \frac{\mathbf{n} \cdot \mathbf{R}}{c} + \frac{\mathbf{n} \cdot \mathbf{r}_p}{c} \approx \frac{r_p}{c} + \frac{t'}{2\gamma^2} (1 + \gamma^2 \psi^2) + \frac{c^2 t'^3}{6\rho^2}.$$

We are not interested in the constant r_p/c which represents just the traveling time of the light to go from the coordinate origin to the observer and introduce

$$t_p = t' - \frac{r(t') - r_p}{c} \approx \frac{t'}{2\gamma^2} (1 + \gamma^2 \psi^2) + \frac{c^2 t'^3}{6\rho^2}\tag{19}$$

This expression has a term of order t'^3 included since the first order term is divided by γ^2 and can become very small. With these approximations we get for the Fourier transformed electric field vector

$$\tilde{\mathbf{E}}(\omega) = \frac{-e\omega\omega_0}{4\pi\sqrt{2\pi}\epsilon_0 cr} \int_{-\infty}^{\infty} (-\omega_0 t', \psi, 0) \exp \left(-i\omega \left(\frac{t'}{2\gamma^2} (1 + \gamma^2 \psi^2) + \frac{c^2 t'^3}{6\rho^2} \right) \right) dt',$$

where we assume now a particle with elementary charge $q = e$. It should be noted that this expression gives the radiation field as a function of the frequency ω measured by the observer. The earlier time t' represents here a convenient variable of integration. We can split the field into the components in the x and y direction and express the exponential with trigonometric functions and make use of the symmetries involved

$$\begin{aligned}\tilde{E}_x(\omega) &= \frac{-e\omega\omega_0}{4\pi\sqrt{2\pi}\epsilon_0 cr} \int_{-\infty}^{\infty} t' \sin \left(\omega \left(\frac{t'}{2\gamma^2} (1 + \gamma^2 \psi^2) + \frac{c^2 t'^3}{6\rho^2} \right) \right) dt', \\ \tilde{E}_y(\omega) &= \frac{ie\omega\omega_0}{4\pi\sqrt{2\pi}\epsilon_0 cr} \int_{-\infty}^{\infty} \psi \cos \left(\omega \left(\frac{t'}{2\gamma^2} (1 + \gamma^2 \psi^2) + \frac{c^2 t'^3}{6\rho^2} \right) \right) dt' .\end{aligned}$$

We bring these expressions into a more standard form by a substitution

$$u = t' \left(\frac{\omega c^2}{2\rho^2} \right)^{1/3}, \quad dt' = \left(\frac{2\rho^2}{\omega c^2} \right)^{1/3} du, \quad (20)$$

and introduce the critical frequency ω_c

$$\omega_c = \frac{3c\gamma^3}{2\rho} \quad (21)$$

$$\begin{aligned} \tilde{E}_x(\omega) &= \frac{-e\omega}{4\pi\sqrt{2\pi}\epsilon_0 cr} \left(\frac{2\rho^2}{\omega c^2} \right)^{1/3} \int_{-\infty}^{\infty} u \sin \left(\left(\frac{3\omega}{4\omega_c} \right)^{2/3} (1 + \gamma^2\psi^2)u + \frac{u^3}{3} \right) du \\ \tilde{E}_y(\omega) &= \frac{ie\omega}{4\pi\sqrt{2\pi}\epsilon_0 cr} \left(\frac{2\rho^2}{\omega c^2} \right)^{1/3} \psi \int_{-\infty}^{\infty} \cos \left(\left(\frac{3\omega}{4\omega_c} \right)^{2/3} (1 + \gamma^2\psi^2)u + \frac{u^3}{3} \right) du. \end{aligned}$$

These integrals are expressions for the Airy function $\text{Ai}(v)$ and its derivative

$$\text{Ai}(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos \left(vt + \frac{t^3}{3} \right) dt \quad \text{and} \quad \text{Ai}'(v) = \frac{d\text{Ai}(v)}{dv} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} v \sin \left(vt + \frac{t^3}{3} \right) dt$$

which are closely related to the modified Bessel functions of fractional order $2/3$ and $1/3$

$$\text{Ai}(x) = \frac{1}{\pi} \sqrt{\frac{x}{3}} K_{1/3} \left(\frac{2x^{3/2}}{3} \right) \quad \text{and} \quad \text{Ai}'(x) = -\frac{1}{\pi} \frac{x}{\sqrt{3}} K_{2/3} \left(\frac{2x^{3/2}}{3} \right).$$

The Airy function and its derivative are further discussed in the Appendix. Using these expressions we get for the two field components of the radiation emitted by a charge moving uniformly on a circular trajectory with relativistic speed

$$\begin{aligned} \tilde{E}_x(\omega) &= \frac{e\gamma}{\sqrt{2\pi}\epsilon_0 cr} \left(\frac{3\omega}{4\omega_c} \right)^{1/3} \text{Ai}' \left(\left(\frac{3\omega}{4\omega_c} \right)^{2/3} (1 + \gamma^2\psi^2) \right) \\ \tilde{E}_y(\omega) &= \frac{ie\gamma}{\sqrt{2\pi}\epsilon_0 cr} \left(\frac{3\omega}{4\omega_c} \right)^{2/3} \gamma\psi \text{Ai} \left(\left(\frac{3\omega}{4\omega_c} \right)^{2/3} (1 + \gamma^2\psi^2) \right) \end{aligned} \quad (22)$$

or expressed in Bessel functions

$$\begin{aligned} \tilde{E}_x(\omega) &= \frac{-\sqrt{3}e\gamma}{4\pi\sqrt{2\pi}\epsilon_0 cr} \left(\frac{\omega}{2\omega_c} \right) (1 + \gamma^2\psi^2) K_{2/3} \left(\left(\frac{\omega}{2\omega_c} \right) (1 + \gamma^2\psi^2)^{3/2} \right) \\ \tilde{E}_y(\omega) &= \frac{i\sqrt{3}e\gamma}{4\pi\sqrt{2\pi}\epsilon_0 cr} \left(\frac{\omega}{2\omega_c} \right) \gamma\psi \sqrt{(1 + \gamma^2\psi^2)} K_{1/3} \left(\left(\frac{\omega}{2\omega_c} \right) (1 + \gamma^2\psi^2)^{3/2} \right). \end{aligned}$$

We will now discuss some of the properties of the radiation field given by (22). We have the two field components $\tilde{E}_x(\omega)$ and $\tilde{E}_y(\omega)$ which correspond usually to the horizontal and vertical polarization directions. The horizontal component $\tilde{E}_x(\omega)$ is symmetric in the angle ψ while the vertical component $\tilde{E}_y(\omega)$ is anti symmetric with respect to this angle. Going from above to below the median plane will change the sign of the vertical field $\tilde{E}_y(\omega)$. It will therefore vanish on the plane of the particle trajectory. This is, of course, directly evident from the above equations (22).

The fact that the expression for the vertical field $\tilde{E}_y(\omega)$ has an imaginary factor in front while this factor is real for the horizontal component indicates that the two fields are 90° out of phase for a given frequency ω . There is therefore some circular polarization present which changes sign when going through the median plane and vanishes on this plane itself where the polarization is purely horizontal.

3.2 The synchrotron radiation field in time domain

The radiation field emitted by a charge moving on a circular orbit can also be given in time domain. We use the Liénard-Wiechert expression (14)

$$\mathbf{E}(t) = \frac{q}{4\pi c\epsilon_0} \left(\frac{[\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]]}{r(1 - \mathbf{n} \cdot \boldsymbol{\beta})^3} \right)_{ret.} \quad (23)$$

From the expressions (18) for the unit vector, the velocity and its derivative we get for the triple vector product

$$\begin{aligned} & \left([\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]] \right)_{ret.} \\ &= \beta\omega_0 \left(-\cos(\omega_0 t') + \beta \cos \psi, \cos \psi \sin \psi \sin(\omega_0 t'), \sin^2 \psi \sin(\omega_0 t') \right) \end{aligned}$$

and using the ultra-relativistic approximation

$$\left([\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]] \right)_{ret} = \omega_0 \frac{(-(1 + \gamma^2 \psi^2 - (\gamma\omega_0 t')^2), \gamma\psi(\gamma\omega_0 t'), 0)}{2\gamma^2}.$$

For the expression in the denominator of (23) we get

$$1 - \mathbf{n} \cdot \boldsymbol{\beta} = 1 - \beta \cos \psi \cos(\omega_0 t') \approx \frac{1 + \gamma^2 \psi^2 + (\gamma\omega_0 t')^2}{2\gamma^2}.$$

Approximating also $r \approx r_p$ we get for the radiation field

$$\mathbf{E}(t) = \frac{q\omega_0\gamma^4}{\pi\epsilon_0 c r_p} \frac{(-(1 + \gamma^2 \psi^2 - (\gamma\omega_0 t')^2), \gamma\psi(\gamma\omega_0 t'), 0)}{(1 + \gamma^2 \psi^2 + (\gamma\omega_0 t')^2)^3}. \quad (24)$$

This expression still contains the emission time t' which should be replaced by the observation time t_p . The two time scales are related by (19)

$$t_p = \frac{t'}{2\gamma^2}(1 + \gamma^2 \psi^2) + \frac{c^2 t'^3}{6\rho^2}$$

which we have to solve for t' . Multiplying this equation with $\gamma^3\omega_0$, where $\omega_0 = c/\rho$, and using the critical frequency $\omega_c = 3\gamma^3\omega_0/2$ we can write this relation in a more compact form

$$(\gamma\omega_0 t')^3 + 3(1 + \gamma^2 \psi^2)\gamma\omega_0 t' - 4\omega_c t_p = 0. \quad (25)$$

The discriminant of this cubic equation is

$$D = (1 + \gamma^2 \psi^2)^3 + (2\omega_c t_p)^2 = (1 + \gamma^2 \psi^2)^3(1 + w^2) > 0$$

with

$$w = \frac{2\omega_c t_p}{(1 + \gamma^2 \psi^2)^{3/2}}.$$

Since $D > 0$ there exists one real solution of (25) which we obtain using the standard method of solving cubic equations

$$t' = \frac{\sqrt{1 + \gamma^2 \psi^2}}{\gamma\omega_0} \left((w + \sqrt{1 + w^2})^{1/3} + (w - \sqrt{1 + w^2})^{1/3} \right).$$

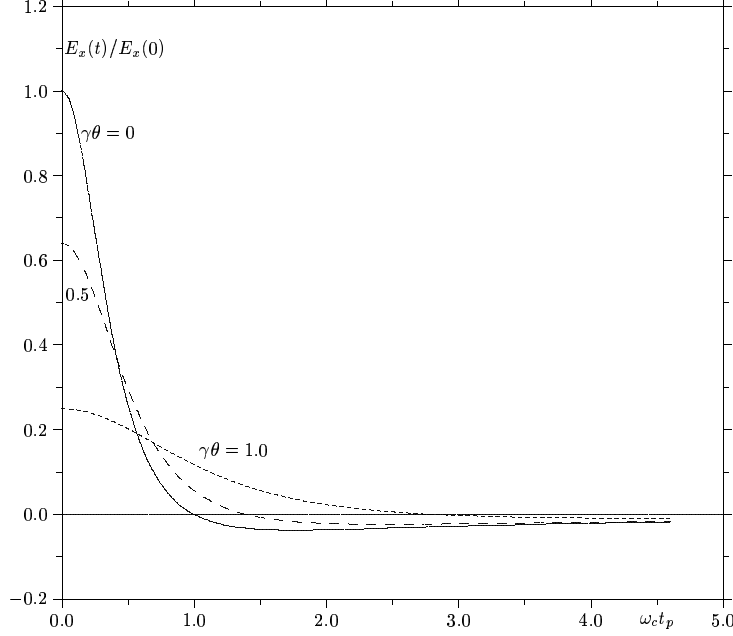


Figure 13: Horizontal radiation field versus observation time for $\gamma\theta = 0, 0.5, 1.0$

We use the relation

$$\text{Arcsinh}w = \ln(w + \sqrt{1 + w^2}) ; , \pm w + \sqrt{1 + w^2} = e^{\pm \text{Arcsinh}w}$$

and get within our third order approximation of the relation between the time scales

$$\frac{\gamma\omega_0 t'}{\sqrt{1 + \gamma^2\psi^2}} = \left(e^{\frac{1}{3}\text{Arcsinh}w} - e^{-\frac{1}{3}\text{Arcsinh}w} \right) = 2 \sinh\left(\frac{1}{3}\text{Arcsinh}w\right).$$

With this we get for the radiation field in time domain (24)

$$\begin{aligned} E_x(t_p) &= -\frac{q\omega_0\gamma^4}{\pi\epsilon_0 cr_p} \frac{(1 - 4 \sinh^2(\frac{1}{3}\text{Arcsinh}w))}{(1 + \gamma^2\psi^2)^2 (1 + 4 \sinh^2(\frac{1}{3}\text{Arcsinh}w))^3} \\ E_y(t_p) &= \frac{q\omega_0\gamma^4}{\pi\epsilon_0 cr_p} \frac{2\gamma\psi \sinh(\frac{1}{3}\text{Arcsinh}w)}{(1 + \gamma^2\psi^2)^{5/2} (1 + 4 \sinh^2(\frac{1}{3}\text{Arcsinh}w))^3}. \end{aligned}$$

These two field components are shown in Figs. 13 and 14. The observer receives a pulse which is symmetric in time for the horizontal field and antisymmetric for the vertical component. The width of the pulse is longer at larger observation angles ψ . The horizontal field $E_x(t_p)$ has a maximum at $t_p = 0$. For $\psi = 0$ it passes through zero at $t_p = 1/\omega_c$. The integral over the field pulse vanishes which is consistent with the fact that the radiation spectrum has no dc-component.

3.3 The spectral angular power distribution of the radiation

We calculate now the power radiated per unit solid angle by the particle moving on a circular orbit. As was pointed out in subsection 2.5 the power received by the observer per unit solid angle is

$$\frac{dP_{observer}}{d\Omega} = \frac{dW}{dtd\Omega} = (\mathbf{n} \cdot \mathbf{S})r^2 = \frac{r^2 E^2}{\mu_0 c},$$

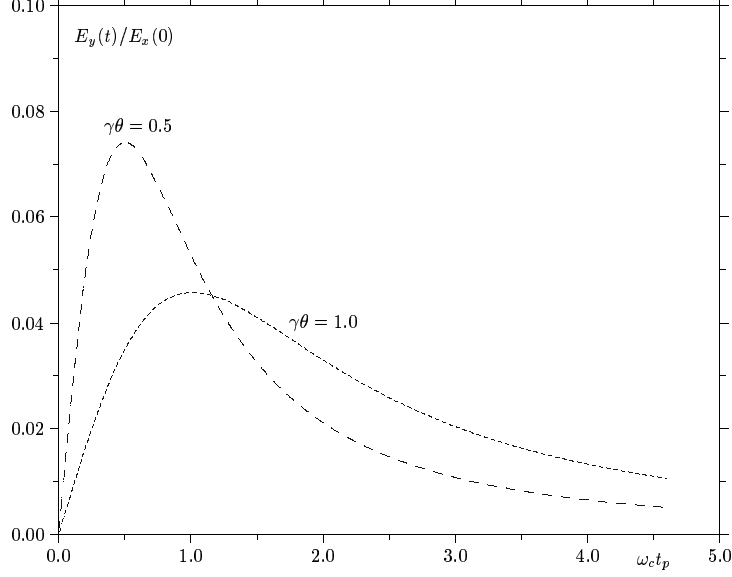


Figure 14: Vertical radiation field versus observation time for $\gamma\theta = 0, 5, 1.0$

while the power radiated by the particle is

$$\frac{dP}{d\Omega} = \frac{d^2W}{dt'd\Omega} = \frac{d^2W}{dt'd\Omega} \frac{dt}{dt'} = \frac{r^2 E^2}{\mu_0 c} (1 - \mathbf{n} \cdot \boldsymbol{\beta}).$$

The difference between the power observed and the power emitted by the particle is just a manifestation of the fact that the energy is received in a compressed time $\Delta t = \Delta t' (1 - \mathbf{n} \cdot \boldsymbol{\beta})$. We calculate first the received power $P_{obs.}$ which could directly be obtained from (22) if the field $E(t)$ were given. We calculated the Fourier transformed fields (22) and will use it now to calculate the spectral distribution of the power. The observer P receives a short flash of radiation which determines the form of the spectrum. We calculate the total energy W received during a single traversal of the particle

$$W = \frac{1}{\mu_0 c} \int_0^{4\pi} \int_{-\infty}^{\infty} E(t)^2 r^2 d\Omega dt.$$

The field $E(t)$ can be obtained from the Fourier transformed field $\tilde{E}(\omega)$

$$E(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{E}(\omega) e^{i\omega t} d\omega$$

which gives

$$\frac{r^2}{2\pi\mu_0 c} \int d\Omega \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{E}(\omega) \tilde{E}(\omega') e^{i(\omega+\omega')t} d\omega d\omega' dt.$$

Using the integral representation of the Dirac δ -functions

$$\int_{-\infty}^{\infty} e^{iat} dt = 2\pi \delta a$$

we get

$$W = \frac{r^2}{\mu_0 c} \int d\Omega \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{E}(\omega) \tilde{E}(\omega') \delta(\omega + \omega') d\omega d\omega'.$$

Since $E(t)$ is a real function its Fourier transform and its conjugate have the symmetry relation $\tilde{E}(\omega) = \tilde{E}^*(-\omega)$ and we get for the above integral

$$\begin{aligned} W &= \frac{r^2}{\mu_0 c} \int d\Omega \int_{-\infty}^{\infty} \tilde{E}(\omega) \tilde{E}(-\omega) d\omega = \frac{r^2}{\mu_0 c} \int d\Omega \int_{-\infty}^{\infty} |\tilde{E}(\omega)|^2 d\omega \\ &= \frac{2r^2}{\mu_0 c} \int d\Omega \int_0^{\infty} |\tilde{E}(\omega)|^2 d\omega. \end{aligned}$$

From this we obtain the spectral angular distribution of the radiated energy in one traversal, i.e. in one flash of radiation

$$\frac{d^2 W}{d\Omega d\omega} = \frac{2r^2 |\tilde{E}(\omega)|^2}{\mu_0 c}. \quad (26)$$

The factor 2 on the right hand side indicates that the spectral energy density is taken at positive frequencies only, contrary to the field which is taken at positive and negative frequencies. This is common practice since power can be measured directly but the sign of the frequency cannot be observed during such measurements. The field, however, is rarely accessible to direct measurements. If the particle circulates on a closed circular orbit of radius ρ with revolution frequency $\omega_0 = \beta c/\rho \approx c/\rho$ the observer receives $c/2\pi\rho$ such flashes per second from the particle and an *average* spectral angular power density

$$\frac{d^2 P}{d\Omega d\omega} = \frac{2r^2 |\tilde{E}(\omega)|^2}{2\pi\mu_0\rho}. \quad (27)$$

This expression (27) gives the average received power which is also the power radiated by the particle.

We use now (22) for the field components to calculate the square of the total field and use the index ‘ σ ’ for the horizontal and ‘ π ’ for the vertical polarization component

$$\begin{aligned} |\tilde{E}(\omega)|^2 &= \tilde{E}_x(\omega)^2 + \tilde{E}_y(\omega)^2 \\ &= \frac{e^2 \gamma^2}{9\epsilon_0^2 c^2 r^2} \left[\frac{9}{2\pi} \left(\frac{3\omega}{4\omega_c} \right)^{2/3} \text{Ai}'^2 \left(\left(\frac{3\omega}{4\omega_c} \right)^{2/3} (1 + \gamma^2 \psi^2) \right) \right. \\ &\quad \left. + \frac{9}{2\pi} \left(\frac{3\omega}{4\omega_c} \right)^{4/3} \gamma^2 \psi^2 \text{Ai}^2 \left(\left(\frac{3\omega}{4\omega_c} \right)^{2/3} (1 + \gamma^2 \psi^2) \right) \right] \\ |\tilde{E}(\omega)|^2 &= \frac{e^2 \gamma^2}{9\epsilon_0^2 c^2 r^2} [F_\sigma(\omega, \psi) + F_\pi(\omega, \psi)] = \frac{e^2 \gamma^2}{9\epsilon_0^2 c^2 r^2} [F_\pi(\omega, \psi)], \end{aligned} \quad (28)$$

with

$$\begin{aligned} F_\sigma(\omega, \psi) &= \frac{9}{2\pi} \left(\frac{3\omega}{4\omega_c} \right)^{2/3} \text{Ai}'^2 \left(\left(\frac{3\omega}{4\omega_c} \right)^{2/3} (1 + \gamma^2 \psi^2) \right) \\ F_\pi(\omega, \psi) &= \frac{9}{2\pi} \left(\frac{3\omega}{4\omega_c} \right)^{4/3} \gamma^2 \psi^2 \text{Ai}^2 \left(\left(\frac{3\omega}{4\omega_c} \right)^{2/3} (1 + \gamma^2 \psi^2) \right), \end{aligned} \quad (29)$$

or

$$\begin{aligned} F_\sigma(\omega, \psi) &= \left(\frac{3}{2\pi} \right)^3 \left(\frac{\omega}{2\omega_c} \right)^2 (1 + \gamma^2 \psi^2)^2 K_{2/3}^2 \left(\left(\frac{\omega}{2\omega_c} \right) (1 + \gamma^2 \psi^2)^{3/2} \right) \\ F_\pi(\omega, \psi) &= \left(\frac{3}{2\pi} \right)^3 \left(\frac{\omega}{2\omega_c} \right)^2 \gamma^2 \psi^2 (1 + \gamma^2 \psi^2) K_{1/3}^2 \left(\left(\frac{\omega}{2\omega_c} \right) (1 + \gamma^2 \psi^2)^{3/2} \right). \end{aligned}$$

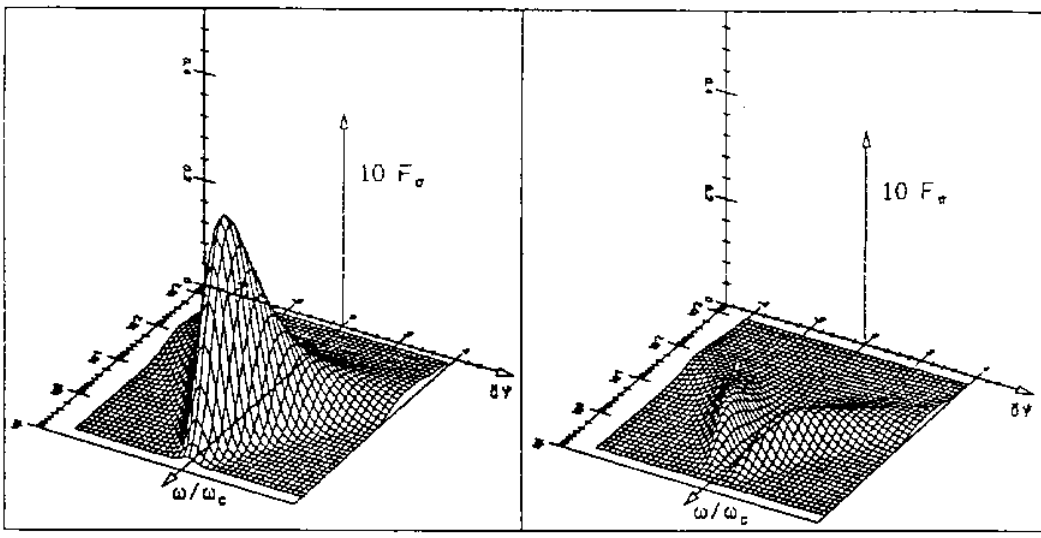


Figure 15: Spectral angular power density for the two polarization modes

Expressing the factors in the above equation (28) by the total radiated power P_0 (12) and the critical frequency ω_c (21)

$$P_0 = \frac{2r_0 c m_0 c^2 \gamma^4}{3\rho^2}, \quad \omega_c = \frac{3c\gamma^3}{2\rho} \quad (30)$$

we get for the angular spectral power density (27)

$$\frac{d^2 P}{d\Omega d\omega} = \frac{P_0 \gamma}{\omega_c} [F_\sigma(\omega, \psi) + F_\pi(\omega, \psi)],$$

We will see later that

$$\int_0^{2\pi} d\phi \int_{-\infty}^{\infty} d(\gamma\psi) \int_0^{\infty} F_\pi(\omega, \psi) d(\omega/\omega_c) = 1.$$

We discuss now this angular spectral power distribution. It is determined by the two functions $F_\sigma(\omega, \psi)$ and $F_\pi(\omega, \psi)$ which are plotted in Fig. 15 against the normalized vertical angle $\gamma\psi$ and against the logarithm of the normalized frequency ω/ω_c . The function F_σ referring to the horizontal polarization has a maximum in the median plane while F_π , giving the vertical polarization, vanishes there. Both components increase slowly with frequency reaching a maximum close to ω_c and decay quickly thereafter. The vertical opening angle decreases with frequency.

3.3.1 Behavior at low frequencies

We assume now that the frequency ω of the observed spectrum is much smaller than the critical frequency $\omega \ll \omega_c$. In this case the argument of the Airy-function is small except for large angles of observation where $\gamma^2\psi^2$ gets very large. We make, therefore, a small error by replacing $(1 + \gamma^2\psi^2)$ by $\gamma^2\psi^2$ in (29). Using the expression for the critical frequency and the total power (30) we get

$$\left(\frac{3\omega}{4\omega_c}\right)^{2/3} \gamma^2\psi^2 = \left(\frac{\omega\rho}{2c}\right)^{2/3} \psi^2 = \left(\frac{\omega}{2\omega_0}\right)^{2/3} \psi^2$$

and the total power distribution becomes

$$\frac{d^2P}{d\Omega d\omega} = \frac{2r_0m_0c^2}{\pi\rho} \left[\left(\frac{\omega}{2\omega_0} \right)^{2/3} \text{Ai}'^2 \left(\left(\frac{\omega}{2\omega_0} \right)^{2/3} \psi^2 \right) + \psi^2 \left(\frac{\omega}{2\omega_0} \right)^{4/3} \text{Ai}^2 \left(\left(\frac{\omega}{2\omega_0} \right)^{2/3} \psi^2 \right) \right]. \quad (31)$$

This expression does not contain γ indicating that for a given curvature the properties of the synchrotron radiation emitted at small frequencies $\omega \ll \omega_c$ are independent of the particle energy. This approximation can be applied to the case of beam diagnostics with synchrotron radiation. The radiation is used to form an image of the electron beam cross section to measure its dimensions. For technical reasons this is carried out with visible light while the critical frequency lies in the far ultra-violet or X-ray region justifying the approximation $\omega \ll \omega_c$.

At extremely low frequencies, lying in the range of micro-waves, the cut-off frequency of the beam surroundings will limit the emitted radiation. Furthermore some of the approximations used here are no longer valid in this region.

3.4 The spectral power density

For many applications one does not resolve the distribution of the radiation with respect to the vertical angle ψ and one is only interested in the spectral density of the radiation. We integrate the spectral angular power distribution (29) over the angle ψ

$$\frac{dP}{d\omega} = \int \frac{d^2P}{d\omega d\Omega} d\Omega = 2\pi \int \frac{d^2P}{d\omega d\psi} d\psi = \frac{P_0}{\omega_c} S \left(\frac{\omega}{\omega_0} \right) = \frac{P_0}{\omega_c} \left[S_\sigma \left(\frac{\omega}{\omega_c} \right) + S_\pi \left(\frac{\omega}{\omega_c} \right) \right].$$

The integrals are obtained from (48) and (49) by setting $a = b = (3\omega/4\omega_c)^{2/3}$

$$\begin{aligned} S_\sigma \left(\frac{\omega}{\omega_c} \right) &= \frac{27\omega}{16\omega_c} \left[-3 \frac{\text{Ai}'(z)}{z} - \frac{1}{3} + \int_0^z \text{Ai}(z') dz' \right], \\ S_\pi \left(\frac{\omega}{\omega_c} \right) &= \frac{27\omega}{16\omega_c} \left[-\frac{\text{Ai}'(z)}{z} - \frac{1}{3} + \int_0^z \text{Ai}(z') dz' \right], \\ S \left(\frac{\omega}{\omega_c} \right) &= \frac{27\omega}{8\omega_c} \left[-2 \frac{\text{Ai}'(z)}{z} - \frac{1}{3} + \int_0^z \text{Ai}(z') dz' \right], \\ \text{with } z &= \left(\frac{3\omega}{2\omega_c} \right)^{2/3}; \end{aligned} \quad (32)$$

or, expressed in Bessel functions

$$\begin{aligned} S_\sigma \left(\frac{\omega}{\omega_c} \right) &= \frac{9\sqrt{3}\omega}{16\pi\omega_c} \left[\int_{\omega/\omega_c}^\infty K_{5/3}(z') dz' + K_{2/3} \left(\frac{\omega}{\omega_c} \right) \right], \\ S_\pi \left(\frac{\omega}{\omega_c} \right) &= \frac{9\sqrt{3}\omega}{16\pi\omega_c} \left[\int_{\omega/\omega_c}^\infty K_{5/3}(z') dz' - K_{2/3} \left(\frac{\omega}{\omega_c} \right) \right], \\ S \left(\frac{\omega}{\omega_c} \right) &= \frac{9\sqrt{3}\omega}{8\pi\omega_c} \int_{\omega/\omega_c}^\infty K_{5/3}(z') dz'. \end{aligned}$$

The functions $S(\omega/\omega_c)$, $S_\sigma(\omega/\omega_c)$ and $S_\pi(\omega/\omega_c)$ which give the spectral power density of the total radiation and its horizontal and vertical polarization components are shown in Fig. 16. At the critical frequency $\omega_c = 3c\gamma^3/2\rho$ these functions have the following values

$$S(1) = 0.4040, \quad S_\sigma(1) = 0.3554, \quad S_\pi(1) = 0.0487.$$

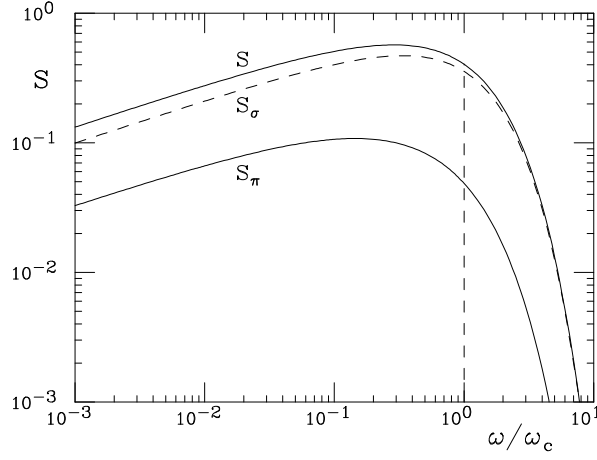


Figure 16: Normalized power spectrum

Using the approximations of the Airy function for small arguments (44) we get normalized spectral power density at low frequencies $\omega \ll \omega_c$ expressed here with the revolution frequency ω_0

$$\begin{aligned} S_\sigma \left(\frac{\omega}{\omega_c} \right) &\approx \frac{27 \cdot 2^{2/3}}{16 \Gamma(1/3)} \left(\frac{\omega}{\omega_c} \right)^{1/3} = 0.99993 \left(\frac{\omega}{\omega_c} \right)^{1/3}, \\ S_\pi \left(\frac{\omega}{\omega_c} \right) &\approx \frac{9 \cdot 2^{2/3}}{16 \Gamma(1/3)} \left(\frac{\omega}{\omega_c} \right)^{1/3} = 0.33331 \left(\frac{\omega}{\omega_c} \right)^{1/3}, \\ S \left(\frac{\omega}{\omega_c} \right) &\approx \frac{9 \cdot 2^{2/3}}{4 \Gamma(1/3)} \left(\frac{\omega}{\omega_c} \right)^{1/3} = 1.33323 \left(\frac{\omega}{\omega_c} \right)^{1/3}. \end{aligned}$$

At these low frequencies 3/4 of the radiation is horizontally and 1/4 is vertically polarized. Using the above equations, the expressions (30) for the total power and the critical frequency and the revolution frequency $\omega_0 = c/\rho$ we can give the spectral power density at low frequencies $\omega \ll \omega_c$

$$\frac{dP_\sigma}{d\omega} = \frac{3 \cdot 3^{2/3}}{6 \Gamma(1/3)} \frac{r_0 m_0 c^2}{\rho} \left(\frac{\omega}{\omega_0} \right)^{1/3}, \quad \frac{dP_\pi}{d\omega} = \frac{1 \cdot 3^{2/3}}{6 \Gamma(1/3)} \frac{r_0 m_0 c^2}{\rho} \left(\frac{\omega}{\omega_0} \right)^{1/3}$$

which is independent of γ as expected from (31).

We can also get the spectral power density at very high frequencies $\omega \gg \omega_c$ by using the approximations (45) for the Airy functions

$$\begin{aligned} S_\sigma \left(\frac{\omega}{\omega_c} \right) &\approx \frac{27\sqrt{2}}{16\sqrt{3}\pi} \sqrt{\frac{\omega}{\omega_c}} e^{-\frac{\omega}{\omega_c}} \left[1 + \frac{31}{72 (\omega/\omega_c)} - \frac{3503}{2 \cdot 72^2 (\omega/\omega_c)^2} \dots \right], \\ S_\pi \left(\frac{\omega}{\omega_c} \right) &\approx \frac{27\sqrt{2}}{16\sqrt{3}\pi} \sqrt{\frac{\omega}{\omega_c}} e^{-\frac{\omega}{\omega_c}} \left[\frac{24}{72 (\omega/\omega_c)} - \frac{4848}{2 \cdot 72^2 (\omega/\omega_c)^2} \dots \right], \\ S \left(\frac{\omega}{\omega_c} \right) &\approx \frac{27\sqrt{2}}{16\sqrt{3}\pi} \sqrt{\frac{\omega}{\omega_c}} e^{-\frac{\omega}{\omega_c}} \left[1 + \frac{55}{72 (\omega/\omega_c)} - \frac{10151}{2 \cdot 72^2 (\omega/\omega_c)^2} \dots \right]. \end{aligned}$$

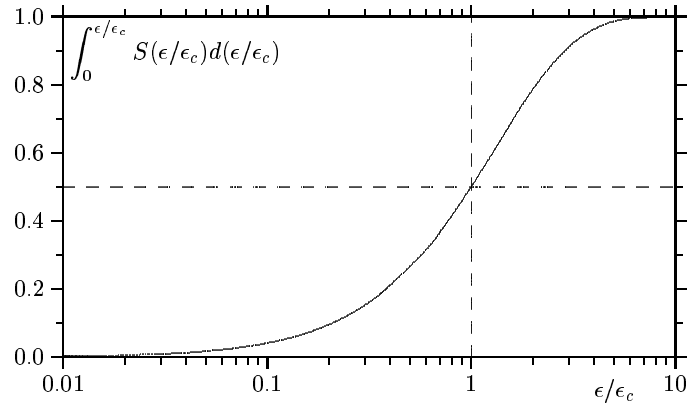


Figure 17: The normalized power spectrum after integration from 0 to ω_c

At very high frequencies the spectral power density decreases exponentially multiplied by a fractional power in frequency. Since the vertical polarization disappears with a higher power the radiation will become more and more horizontally polarized. It should be mentioned that the above expressions are good approximations only for frequencies which are many times larger than the critical frequency.

Sometimes the total power radiated below (or above) a certain frequency is of interest. The corresponding expressions are obtained by integrating over the normalized spectral power densities using the integrals (55,56)

$$\begin{aligned} \int_0^{\frac{\omega}{\omega_c}} S\left(\frac{\omega}{\omega_c}\right) d(\omega/\omega_c) &= 1 - \frac{3z^2}{4} \text{Ai}'(z) - 3z \text{Ai}(z) - \frac{12 + 3z^3}{4} \int_z^\infty \text{Ai}(z') dz', \\ \int_0^{\frac{\omega}{\omega_c}} S_\sigma\left(\frac{\omega}{\omega_c}\right) d(\omega/\omega_c) &= \frac{7}{8} - \frac{3z^2}{8} \text{Ai}'(z) - \frac{21}{8} z \text{Ai}(z) - \frac{21 + 3z^3}{8} \int_z^\infty \text{Ai}(z') dz', \\ \int_0^{\frac{\omega}{\omega_c}} S_\pi\left(\frac{\omega}{\omega_c}\right) d(\omega/\omega_c) &= \frac{1}{8} - \frac{3z^2}{8} \text{Ai}'(z) - \frac{3}{8} z \text{Ai}(z) - \frac{3 + 3z^3}{8} \int_z^\infty \text{Ai}(z') dz', \\ \text{with } z &= \left(\frac{3\omega}{2\omega_c}\right)^{2/3}. \end{aligned}$$

This integrated spectrum is shown in Fig. 17 for the total radiation.

If we carry the integration over all frequencies we get the total normalized power

$$\int_0^\infty S\left(\frac{\omega}{\omega_c}\right) d(\omega/\omega_c) = 1, \quad \int_0^\infty S_\sigma\left(\frac{\omega}{\omega_c}\right) d(\omega/\omega_c) = \frac{7}{8}, \quad \int_0^\infty S_\pi\left(\frac{\omega}{\omega_c}\right) d(\omega/\omega_c) = \frac{1}{8}.$$

Of the total radiated power a fraction 7/8 is horizontally polarized and a fraction 1/8 is vertically polarized. If we integrate the normalized spectral power density to the critical frequency

$$\omega_c = \frac{3c\gamma^3}{2\rho}$$

we get

$$\int_0^1 S\left(\frac{\omega}{\omega_c}\right) d(\omega/\omega_c) = 0.50, \quad \int_0^1 S_\sigma\left(\frac{\omega}{\omega_c}\right) d(\omega/\omega_c) = 0.42, \quad \int_0^1 S_\pi\left(\frac{\omega}{\omega_c}\right) d(\omega/\omega_c) = 0.08.$$

The critical frequency ω_c thus divides the total spectral power into two equal parts.

3.5 Angular distribution

Next we discuss the angular distribution of the radiation. It can of course be obtained from the general spectral angular distribution (29) by fixing the frequency ω . This is done for three different frequencies in Fig. 18. The vertical polarization component vanishes in the median plane $\psi = 0$ and increases with the angle ψ to a maximum and decreases again. The horizontal polarization (σ -mode) has a maximum in the median plane and decreases with increasing angle. As we already know from qualitative arguments, the angular distribution has a typical width of $1/\gamma$.

At very low frequencies we find the angular distribution from the expression (31) we derived before. It is plotted in Fig. 19 for the two polarization modes and the total radiation

We now calculate the rms opening angle of the radiation. The averaged square of the product between the angle and the Lorentz factor is

$$\langle \gamma^2 \theta_\sigma^2 \rangle = \frac{\int_0^\infty \gamma^2 \theta_\sigma^2 F_\sigma(\omega, \psi) d\psi}{\int_0^\infty F_\sigma(\omega, \psi) d\psi}, \quad \langle \gamma^2 \theta_\pi^2 \rangle = \frac{\int_0^\infty \gamma^2 \theta_\pi^2 F_\pi(\omega, \psi) d\psi}{\int_0^\infty F_\pi(\omega, \psi) d\psi}.$$

The integrals are obtained from (50) and (51) by setting $a = b = (3\omega/4\omega_c)^{2/3}$

$$\begin{aligned} \langle \gamma^2 \theta_\sigma^2 \rangle &= \frac{1}{4} \frac{5 \frac{Ai(z)}{z^2} + \frac{Ai'(z)}{z} + \int_z^\infty Ai(z') dz'}{-3 \frac{Ai'(z)}{z} - \int_z^\infty Ai(z') dz'} \\ \langle \gamma^2 \theta_\pi^2 \rangle &= \frac{3}{4} \frac{\frac{Ai(z)}{z^2} + \frac{Ai'(z)}{z} + \int_z^\infty Ai(z') dz'}{-\frac{Ai'(z)}{z} - \int_z^\infty Ai(z') dz'}. \end{aligned} \quad (33)$$

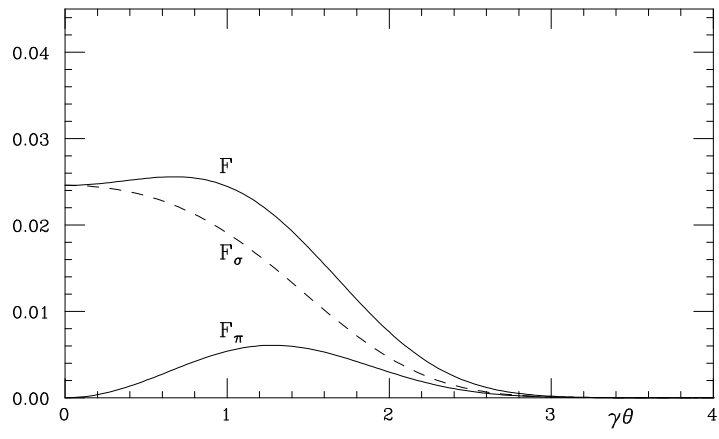
The rms. opening angles for the two polarization modes alone and for the total radiation are obtained from the square roots of the above expressions

$$\begin{aligned} \gamma \psi_{\sigma-rms} &= \sqrt{\langle \gamma^2 \theta_\sigma^2 \rangle}, \quad \gamma \psi_{\pi-rms} = \sqrt{\langle \gamma^2 \theta_\pi^2 \rangle}, \\ \gamma \psi_{rms} &= \sqrt{\frac{S_\sigma \langle \gamma^2 \theta_\sigma^2 \rangle + S_\pi \langle \gamma^2 \theta_\pi^2 \rangle}{S_\sigma + S_\pi}}. \end{aligned}$$

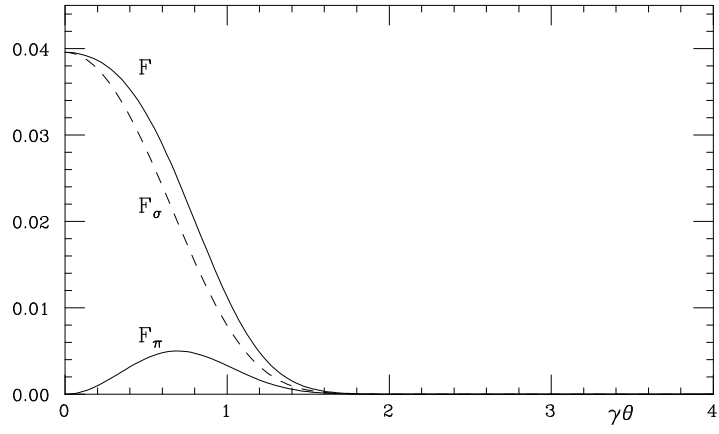
At the low frequency end of the spectrum $\omega \ll \omega_c$ we can find an approximate expression for the opening angles by using the expressions (31) and applying the integrals (48,49,50,51) with $a = 0$ and $b = (\omega/2\omega_0)^{2/3}$

$$\begin{aligned} \psi_{\sigma-rms} &\sim \sqrt{\frac{5}{12} \frac{Ai(0)}{-Ai'(0)}} \left(\frac{\omega_0}{\omega} \right)^{1/3} = 0.7560 \left(\frac{\omega_0}{\omega} \right)^{1/3} = 0.4097 \left(\frac{\lambda}{\rho} \right)^{1/3} \\ \psi_{\pi-rms} &\sim \sqrt{\frac{9}{12} \frac{Ai(0)}{-Ai'(0)}} \left(\frac{\omega_0}{\omega} \right)^{1/3} = 1.0143 \left(\frac{\omega_0}{\omega} \right)^{1/3} = 0.5497 \left(\frac{\lambda}{\rho} \right)^{1/3} \\ \psi_{rms} &\sim \sqrt{\frac{6}{12} \frac{Ai(0)}{-Ai'(0)}} \left(\frac{\omega_0}{\omega} \right)^{1/3} = 0.8282 \left(\frac{\omega_0}{\omega} \right)^{1/3} = 0.4488 \left(\frac{\lambda}{\rho} \right)^{1/3}. \end{aligned}$$

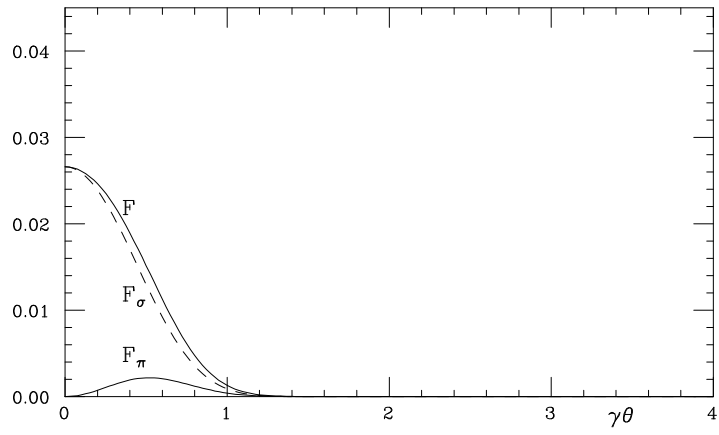
Next we investigate the angular distribution of the total radiation and integrate the spectral angular distribution (29) over the frequency ω . This integration is obtained from



$$\omega = 0.2\omega_c$$



$$\omega = \omega_c$$



$$\omega = 2\omega_c$$

Figure 18: Angular distribution at different frequencies ω

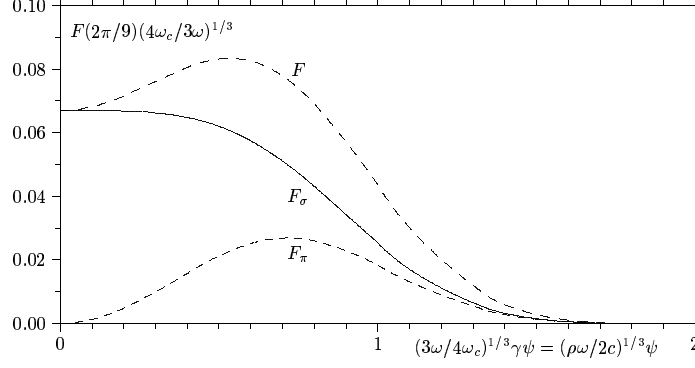


Figure 19: Vertical distribution of synchrotron radiation at low frequencies $\omega \ll \omega_c$

(54) by calling $p = 3\omega/4\omega_c$ and $b = 1 + \gamma^2\theta^2$.

$$\frac{dP}{d\Omega} = \frac{P_0\gamma}{\omega_c} \int_0^\infty (F_\sigma(\omega, \psi) + F_\pi(\omega, \psi)) d\omega = \frac{P_0}{2\pi} \frac{21}{32} \frac{\gamma}{(1 + \gamma^2\theta^2)^{5/2}} \left[1 + \frac{5}{7} \frac{\gamma^2\theta^2}{1 + \gamma^2\theta^2} \right]$$

and

$$\frac{dP}{d\psi} = P_0 \frac{21}{32} \frac{\gamma}{(1 + \gamma^2\theta^2)^{5/2}} \left[1 + \frac{5}{7} \frac{\gamma^2\theta^2}{1 + \gamma^2\theta^2} \right].$$

The first and second terms in the square brackets correspond to the horizontal (σ -mode) and vertical (π -mode) polarization respectively.

3.6 Photon distribution

So far we calculated and discussed synchrotron radiation as electromagnetic fields and corresponding power distributions. We know, however, that this radiation is emitted in quanta (photons) with energy $\epsilon = \hbar\omega$, with $\hbar = h/2\pi$ where $h = 6.6262 \cdot 10^{-34}$ Js, is Planck's constant. If $\dot{n}$ photons of energy ϵ are emitted per second, the power carried by them is $P = \dot{n}\epsilon$. By introducing the critical photon energy $\epsilon_c = \hbar\omega_c$ we can relate the spectral angular power density to the spectral angular photon flux

$$\frac{d^2\dot{n}}{d\Omega d\epsilon/\epsilon} = \frac{d^2P}{d\Omega \hbar d\omega} = \frac{P_0\gamma}{\epsilon_c} (F_\sigma(\omega, \psi) + F_\pi(\omega, \psi)).$$

This gives the number of photons radiated per second and per unit solid angle into a *relative* photon energy band width $\Delta\epsilon/\epsilon$. Integrating this over the solid angle gives the photon spectrum

$$\frac{d\dot{n}}{d\epsilon/\epsilon} = \frac{1}{\hbar} \frac{dP}{d\omega} = \frac{P_0}{\epsilon_0} S\left(\frac{\epsilon}{\epsilon_c}\right) = \frac{P_0}{\epsilon_0} \left[S_\sigma\left(\frac{\epsilon}{\epsilon_c}\right) + S_\pi\left(\frac{\epsilon}{\epsilon_c}\right) \right].$$

The form of the spectrum related to the relative band width is the same as the one of the power spectrum shown in Fig. 16. Sometimes the spectrum related to the absolute band width is more relevant

$$\frac{d\dot{n}}{d\epsilon} = \frac{P_0\gamma}{\epsilon_c^2} \frac{S_\sigma\left(\frac{\epsilon}{\epsilon_c}\right) + S_\pi\left(\frac{\epsilon}{\epsilon_c}\right)}{\left(\frac{\epsilon}{\epsilon_c}\right)}. \quad (34)$$

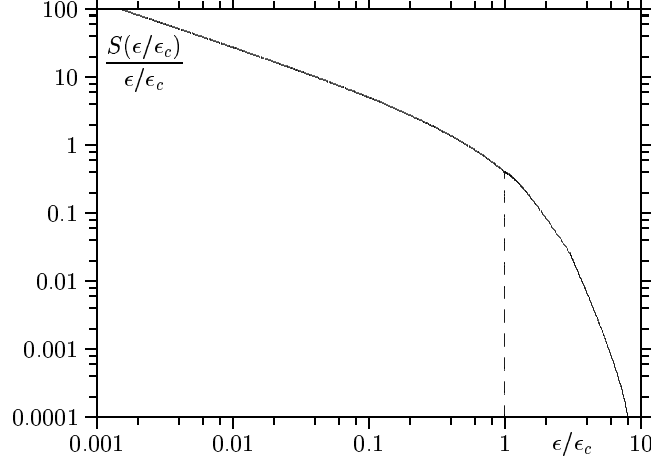


Figure 20: Normalized photon spectrum

The function $S(\epsilon/\epsilon_c)/(\epsilon/\epsilon_c)$ which determines the absolute photon spectrum is shown in Fig. 20.

By integrating (34) over all photon energies we get the number $\dot{n}$ of photons radiated by an electron per second

$$\dot{n} = \frac{P_0}{\epsilon_c} \int_0^\infty \frac{S_\sigma\left(\frac{\epsilon}{\epsilon_c}\right) + S_\pi\left(\frac{\epsilon}{\epsilon_c}\right)}{\left(\frac{\epsilon}{\epsilon_c}\right)} d(\epsilon/\epsilon_c).$$

Using the expressions (32) and the integrals (59,60) we get, for the total number of photons per second, and the partition into the two polarization modes

$$\dot{n} = \frac{15\sqrt{3}}{8} \frac{P_0}{\epsilon_0}, \quad \dot{n}_\sigma = \frac{12\sqrt{3}}{8} \frac{P_0}{\epsilon_0}, \quad \dot{n}_\pi = \frac{3\sqrt{3}}{8} \frac{P_0}{\epsilon_0}. \quad (35)$$

From this we get the average photon energies

$$\langle \epsilon \rangle = \frac{P_0}{\dot{n}} = \frac{8\sqrt{3}}{45} \epsilon_c, \quad \langle \epsilon_\sigma \rangle = \frac{P_\sigma}{\dot{n}_\sigma} = \frac{7\sqrt{3}}{36} \epsilon_c, \quad \langle \epsilon_\pi \rangle = \frac{P_\pi}{\dot{n}_\pi} = \frac{\sqrt{3}}{9} \epsilon_c.$$

For calculating the effect of quantum excitation one needs the variance $\langle \epsilon^2 \rangle$ of the photon energy. It is obtained in the same way using the integrals (61,62)

$$\langle \epsilon^2 \rangle = \frac{11}{27} \epsilon_c^2, \quad \langle \epsilon_\sigma^2 \rangle = \frac{25}{54} \epsilon_c^2, \quad \langle \epsilon_\pi^2 \rangle = \frac{5}{27} \epsilon_c^2.$$

Coming back to the number of photons radiated per unit time (35) we express the power P_0 and the critical energy ϵ_c by (30)

$$\frac{P_0}{\epsilon_c} = \frac{\frac{2r_0 c m_0 c^2 \gamma^4}{3\rho^2}}{\frac{3\hbar c \gamma^3}{2\rho}} = \frac{2ce^2 \gamma}{9\rho \epsilon_0} = \frac{4\omega_0 \alpha_f}{9},$$

where $\omega_0 = c/\rho$ is the revolution frequency and α_f is the fine structure constant

$$\alpha_f = \frac{e^2}{2ch\epsilon_0} = \frac{1}{137.036}$$

and obtain for the number of photons radiated per second

$$\dot{n} = \frac{5}{2\sqrt{3}}\gamma\omega_0\alpha_f.$$

In one revolution a single electron radiates the following number of photons

$$n/\text{rev.} = \frac{5\pi}{\sqrt{3}}\gamma\alpha_f = 0.0662\gamma.$$

4 UNDULATOR RADIATION

4.1 The undulator radiation field

An undulator is a spatially periodic magnetic structure designed to produce quasi-monochromatic radiation from relativistic particles. It has become an important radiation source for many experiments. The nature and properties of this device is treated extensively in lectures [7] at this school. We give here only a short summary in view of the lecture on beam diagnostics [8].

We consider a plane harmonic undulator with period length λ_u ; Fig. 21. It has in the median plane ($y = 0$) a magnetic field of the form

$$B(z) = B_y(z) = B_0 \cos(k_u z)$$

with $k_u = 2\pi/\lambda_u$. If the field is not too strong the trajectory of a particle going along the axis is of the form

$$x(z) = a \cos(k_u z), \quad a = \frac{eB_0}{m_0 c \gamma k_u^2}$$

and

$$\frac{dx}{dz} = -\psi_0 \sin(k_u z), \quad \psi_0 = \frac{eB_0}{m_0 c \gamma k_u} = \frac{K}{\gamma}.$$

Here, we introduced the undulator parameter

$$K = \frac{eB_0}{m_0 c k_u} = \gamma \psi_0$$

which gives the ratio between the maximum deflecting angle ψ_0 and the natural opening angle of the radiation $1/\gamma$. For the case $K < 1$ the emitted light is deflected by angle ψ_0 smaller than the natural opening angle. An observer will receive a weakly modulated field which is quasi-monochromatic. However, for $K > 1$ the deflection is larger than the natural opening angle and the observer will receive strongly modulated light containing harmonics of the basic modulation frequency.

We calculate now the radiation emitted by an ultra-relativistic charge e going along the axis through an undulator with N_u periods of length λ_u and parameter K shown in Fig. 21. The ‘far-field’ is given by the Liénard-Wiechert equation (14)

$$\mathbf{E}(t) = \frac{q}{4\pi\epsilon_0} \left(\frac{[\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]]}{cr(1 - \mathbf{n} \cdot \boldsymbol{\beta})^3} \right)_{ret.}, \quad \mathbf{B}(t) = \frac{[\mathbf{n} \times \mathbf{E}]}{c}. \quad (36)$$

The unit vector $\mathbf{n} = \mathbf{r}/r$ appearing in the above equation points from the charge to the observer. If he is located at a distance r_p from the undulator center much larger than

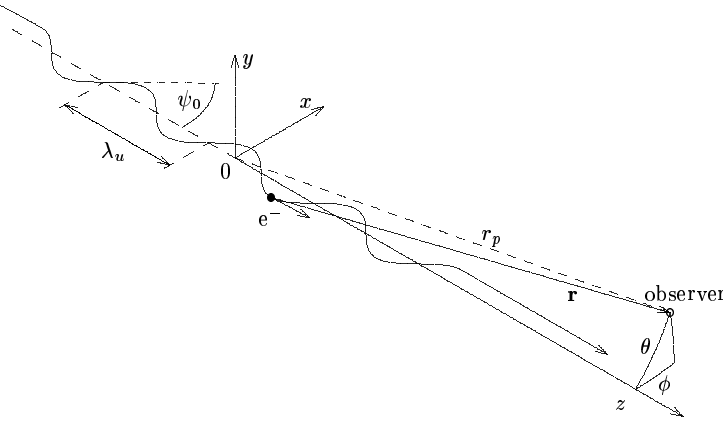


Figure 21: Geometry of undulator radiation

its length $L_u = N_u \lambda_u$ we can neglect the variation of r and the unit vector during the traversal of the particle in the above vector product but not in the relation between the two time scales. We have for $\gamma \gg 1$

$$r \approx r_p, \quad \mathbf{n} \approx (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \approx (\theta \cos \phi, \theta \sin \phi, 1 - \theta^2/2).$$

Furthermore, we treat only the case of weak field undulator $K < 1$ and make the corresponding approximations for the particle motion

$$\begin{aligned} x &= a \cos(k_u z) \approx a \cos(k_u \beta c t') = a \cos(\Omega_u t'), \quad z \approx \beta c t' \approx c t' \\ \text{with } a &= \frac{e B_0}{m_0 c \gamma k_u^2} = \frac{K}{k_u \gamma} \text{ and } \Omega_u = k_u \beta c \end{aligned}$$

or

$$\begin{aligned} \boldsymbol{\beta} &= \left(-\frac{\beta K}{\gamma} \sin(\Omega_u t'), 0, \beta \right) \\ \dot{\boldsymbol{\beta}} &= \left(-\frac{\beta^2 c k_u K}{\gamma} \cos(\Omega_u t'), 0, 0 \right). \end{aligned}$$

Using these expressions and the classical particle radius r_0 we get for (36)

$$\mathbf{E}(t) = \frac{4r_0 c B_0 \gamma^3}{r_p} \frac{(1 - \gamma^2 \theta^2 \cos(2\phi), \gamma^2 \theta^2 \sin(2\phi))}{(1 + \gamma^2 \theta^2)^3} \cos(\Omega_u t') \quad (37)$$

where we omit the vanishing z -component. Finally we would like to express the field as a function of the time t of the observer instead of the time t' of emission. According to Fig. 12 the two time scales are related by

$$t = t' + \frac{r}{c}.$$

Developing the expression for the distance r between particle and observer in terms of the distance r_p between the undulator center and the observer for $\gamma \gg 1$, $\theta \ll 1$ and $r_p \gg L_u$

$$r = \sqrt{r_p^2 + \beta^2 c^2 - 2r_p \beta c \cos \theta} \approx r_p \left(1 - \frac{\beta c}{r_p} + \frac{\beta c \theta^2}{2r_p} \right)$$

leads to

$$t \approx \frac{r_p}{c} + t'(1 - \beta) + t' \frac{\beta\theta^2}{2} \approx \frac{r_p}{c} + t' \frac{1 + \gamma^2\theta^2}{2\gamma^2}.$$

We get for the phase $\Omega_u t'$ in (37)

$$\Omega_u t' = k_u c t' = k_u c \frac{2\gamma^2}{1 + \gamma^2\theta^2} t = \omega_1 t.$$

With the frequency ω_1 and wave number k seen by the observer

$$\omega_1 = \Omega_u \frac{2\gamma^2}{1 + \gamma^2\theta^2}, \quad k = k_u \frac{2\gamma^2}{1 + \gamma^2\theta^2}$$

we get for the observed radiation field

$$\begin{aligned} \mathbf{E}(t) &= \frac{4r_0 c B_0 \gamma^3}{r_p} \frac{(1 - \gamma^2\theta^2 \cos(2\phi), \gamma^2\theta^2 \sin(2\phi))}{(1 + \gamma^2\theta^2)^3} \cos(\omega_1 t - k r_p) \\ &= \hat{\mathbf{E}} \cos(\omega_1 t - k r_p) \end{aligned}$$

with

$$\hat{\mathbf{E}} = \frac{4r_0 c B_0 \gamma^3}{r_p} \frac{(1 - \gamma^2\theta^2 \cos(2\phi), \gamma^2\theta^2 \sin(2\phi))}{(1 + \gamma^2\theta^2)^3}.$$

For a fixed distance r_p to the observer it is convenient define a time t_p which does not contain the uninteresting phase factor $\omega r_p/c$

$$t_p = t - \frac{r_p}{c} = \frac{1 + \gamma^2\theta^2}{2\gamma^2} t' \text{ giving } \omega t_p = k_u c t'. \quad (38)$$

The radiation emitted by the undulator represents a wave with a frequency ω_1 which is on the axis $2\gamma^2$ times larger than the frequency $\Omega_u = k_u c$ of the particle motion. It decreases with larger observation angle θ . Since the two field components $\tilde{E}_x$ and $\tilde{E}_y$ are in phase the radiation is linearly polarized in a direction which depends on the coordinates θ and ϕ of the observer.

The spectrum of the emitted radiation is obtained from the Fourier transform of the field

$$\tilde{\mathbf{E}}(\omega) = \frac{1}{\sqrt{2\pi}} \int \mathbf{E}(t) e^{-i\omega t} dt.$$

The limits of the integral corresponds to the emission times $t' = -L_u/2\beta c$ and $t' = L_u/2\beta c$ when the particle enters or exits the undulator

$$\begin{aligned} \tilde{\mathbf{E}}(\omega) &= \frac{\hat{\mathbf{E}}}{\sqrt{2\pi}} \int_{-\frac{N_u}{2\omega_1}}^{\frac{N_u}{2\omega_1}} \cos(\omega_1 t_p) \cos(\omega t_p) dt_p \\ &= \frac{\hat{\mathbf{E}}}{\sqrt{2\pi}} \left(\frac{\sin\left(\frac{(\omega - \omega_1)\pi N_u}{\omega_1}\right)}{(\omega_1 - \omega)} + \frac{\sin\left(\frac{(\omega + \omega_1)\pi N_u}{\omega_1}\right)}{(\omega + \omega_1)} \right). \end{aligned}$$

Next we assume that the undulator has many periods $N_u \gg 1$. In this case the first term in the bracket of the above equation has a maximum at $\omega = \omega_1$ while the second term is much smaller and can be neglected

$$\tilde{\mathbf{E}}(\omega) = \frac{4r_0 c B_0 \gamma^3}{\sqrt{2\pi} r_p} \frac{[1 - \gamma^2\theta^2 \cos(2\phi), \gamma^2\theta^2 \sin(2\phi)]}{(1 + \gamma^2\theta^2)^3} \frac{\pi N_u}{\omega_1} \frac{\sin\left(\frac{(\omega - \omega_1)\pi N_u}{\omega_1}\right)}{\frac{(\omega - \omega_1)\pi N_u}{\omega_1}}. \quad (39)$$

4.2 The angular spectral power density of undulator radiation

The angular spectral energy distribution of undulator radiation is obtained from the general treatment carried out earlier (26)

$$\frac{dW}{d\Omega d\omega} = \frac{2r_p^2 |\tilde{E}(\omega)|^2}{\mu_0 c}.$$

With the expression (39) for $\tilde{E}(\omega)$ we get

$$\begin{aligned} \frac{d^2 W_u}{d\Omega d\omega} &= \frac{2r_0 e^2 c^2 B_0^2 N_u \lambda_u \gamma^4 [(1 + \gamma^2 \theta^2 \cos(2\pi))^2 + (\gamma^2 \theta^2 \sin(2\phi))^2]}{\pi m_0 c^2 (1 + \gamma^2 \theta^2)^5} \\ &\quad \frac{N_u}{\omega_1} \left(\frac{\sin\left(\frac{(\omega - \omega_1)\pi N_u}{\omega_1}\right)}{\frac{(\omega - \omega_1)\pi N_u}{\omega_1}} \right)^2. \end{aligned}$$

Using the length of the undulator $L_u = N_u \lambda_u$ and integrating over the frequency ω and solid angle gives the total radiated energy

$$W_u = \frac{2r_0 e^2 c^2 \langle B^2 \rangle L_u \gamma^2}{3m_0 c^2} = \frac{2r_0 e^2 c^2 \langle B^2 \rangle (m_0 c^2 \gamma)^2 L_u}{3(m_0 c^2)^3},$$

where $\langle B^2 \rangle = B_0^2/2$ the variance of the field. The average power emitted during the passage through the undulator is related to this energy by $P_u = W_u c / L_u$ which can be used to convert the above expression into a angular spectral power distribution. We see that from (11) that for the same $\langle B^2 \rangle$ the radiated power is the same for the undulator as for a long magnet.

We can now express the spectral angular power distribution of a weak undulator in a more convenient form

$$\frac{d^2 P_u}{d\Omega d\omega} = P_u \gamma^2 (F_{u\sigma}(\theta, \phi) + F_{u\pi}(\theta, \phi)) f_N(\Delta\omega).$$

The two functions $F_{u\sigma}$ and $F_{u\pi}$ give the contributions of the two polarization modes

$$F_{u\sigma}(\theta, \phi) = \frac{3}{\pi} \frac{(1 - \gamma^2 \theta^2 \cos(2\phi))^2}{(1 + \gamma^2 \theta^2)^5}, \quad F_{u\pi}(\theta, \phi) = \frac{3}{\pi} \frac{(\gamma^2 \theta^2 \sin(2\phi))^2}{(1 + \gamma^2 \theta^2)^5}.$$

The function $f_N(\Delta\omega)$ gives the spectral distribution at a given angle θ which depends on the number N_u of undulator periods

$$f_N(\Delta\omega) = \frac{N_u}{\omega_1} \left(\frac{\sin\left(\frac{\Delta\omega\pi N_u}{\omega_1}\right)}{\frac{\Delta\omega\pi N_u}{\omega_1}} \right)^2 \quad \text{with } \Delta\omega = \omega - \omega_1.$$

This function is normalized and approaches the Dirac delta function for a large number N_u of periods

$$\int_{-\infty}^{\infty} f_N(\Delta\omega) d\omega = 1, \quad f_N(\Delta\omega) \rightarrow \delta(\Delta\omega) \text{ for } N_u \rightarrow \infty.$$

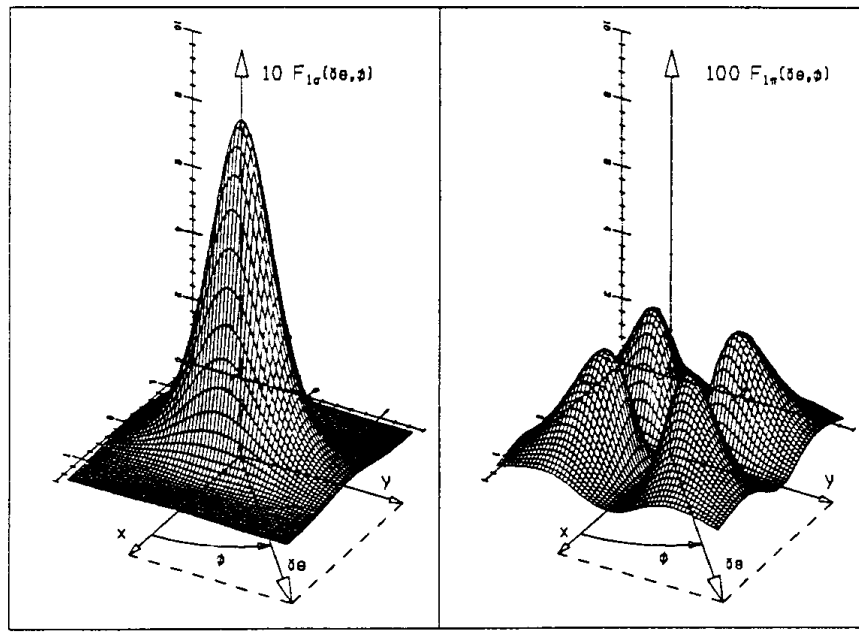


Figure 22: Angular distribution of undulator radiation for the two polarization modes

In this latter case the radiation is monochromatic at each observation angle θ with the frequency

$$\omega_1 = \frac{2\gamma^2 \Omega_u}{1 + \gamma^2 \theta^2} = \frac{\omega_{10}}{1 + \gamma^2 \theta^2}.$$

Integrating over $\Delta\omega$ gives the angular distribution of the emitted power

$$\frac{dP_u}{d\Omega} = P_u \gamma^2 (F_{1\sigma}(\theta, \phi) + F_{1\pi}(\theta, \phi)).$$

It is shown in Fig. 22 for the two polarization modes and in Fig. 23 in form of cuts through the x -plane ($\phi = 0$) and the y -plane ($\phi = \pi/2$).

4.3 The spectral power density

Integrating over the solid angle gives the spectral power density. For a very large N_u of undulator periods it is of the form

$$\begin{aligned} \frac{dP_{u\sigma}}{d\omega} &= \frac{3P_u}{\omega_{10}} \frac{\omega}{\omega_{10}} \left[\frac{1}{2} - \frac{\omega}{\omega_{10}} + \frac{3}{2} \left(\frac{\omega}{\omega_{10}} \right)^2 \right] = \frac{P_u}{\omega_{10}} S_{u\sigma} \\ \frac{dP_{u\pi}}{d\omega} &= \frac{3P_u}{\omega_{10}} \frac{\omega}{\omega_{10}} \left[\frac{1}{2} - \frac{\omega}{\omega_{10}} + \frac{1}{2} \left(\frac{\omega}{\omega_{10}} \right)^2 \right] = \frac{P_u}{\omega_{10}} S_{u\pi} \\ \frac{dP_u}{d\omega} &= \frac{3P_u}{\omega_{10}} \frac{\omega}{\omega_{10}} \left[1 - \frac{\omega}{\omega_{10}} + 2 \left(\frac{\omega}{\omega_{10}} \right)^2 \right] = \frac{P_u}{\omega_{10}} S_u \end{aligned} \tag{40}$$

which is shown in Fig. 24 for the two polarization modes and the total radiation. For a finite number of periods the sharp edge at $\omega = \omega_{10}$ will be smoothed out.

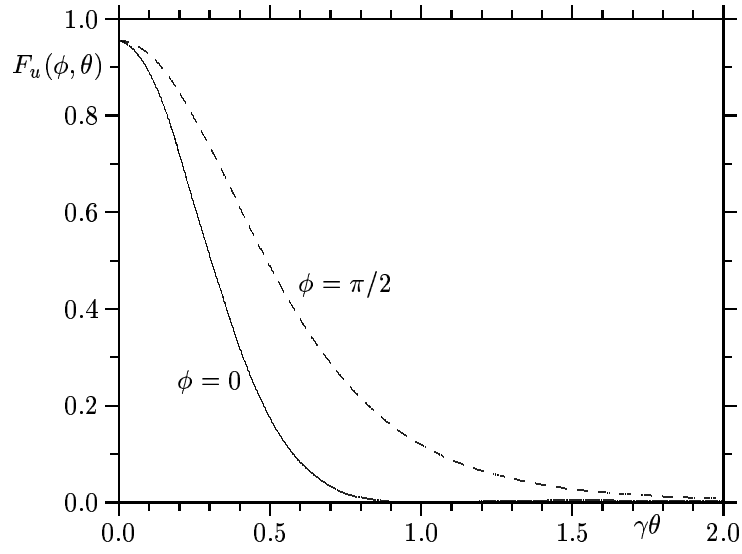


Figure 23: Cuts through the angular distribution of undulator radiation at $\phi = 0$ and $\phi = \pi/2$

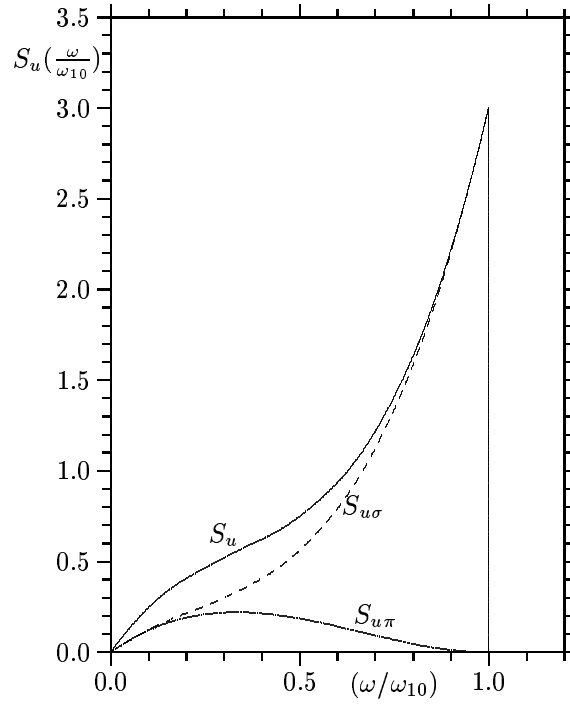


Figure 24: Normalized spectral power density $S = (dP/P_0)/(d\omega/\omega_{10})$ of undulator radiation

4.4 Strong field undulators

Most undulators operate with a parameter $K > 1$, [9, 6]. In this case higher harmonics m are produced and the above treatment is only an approximation for the lowest mode $m = 1$. The even modes have no radiation along the axis $\theta = 0$ and are of limited interest for diagnostics. For the odd modes we have the relation between frequency and angle, in the approximation of a large period number

$$\omega_m = mk_u c \frac{2\gamma^2}{1 + \gamma^2\theta^2 + K^2/2} = \frac{\omega_{m0}}{1 + \gamma^2\theta^2/(1 + K^2/2)} \quad \text{with} \quad \omega_{m0} = \frac{mk_u c}{1 + K^2/2}.$$

With increasing K the frequency of each mode m is reduced. This fact is often used to tune the undulator spectrum.

4.5 Short magnets — generalized undulators

We consider now a magnet which is sufficiently short and weak that the deflection it produces for the beam stays within an angle smaller than $1/\gamma$. We assume that the particle trajectory lies in the x, z -plane and follows closely the z -axis and give the magnetic field in the form

$$B_y = B_y(z).$$

Within this approximation and using the geometry shown in Fig. 21 the particle trajectory is determined by

$$\frac{1}{\rho} = \frac{eBc}{m_0 c^2 \gamma} \approx -\frac{1}{c^2} \frac{d^2 x}{dt^2} = -\frac{\dot{\beta}}{c}.$$

We use the Liénard-Wiechert expression for the radiation field

$$\mathbf{E}(t) = \frac{q}{4\pi\epsilon_0} \left(\frac{[\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]]}{cr(1 - \mathbf{n} \cdot \boldsymbol{\beta})^3} \right)_{ret.}$$

and take only the lowest order of the field strength

$$\begin{aligned} \mathbf{n} &= (\theta \cos \phi, \theta \sin \phi, 1 - \theta^2/2) \\ \boldsymbol{\beta} &= (0, 0, \beta) \\ \dot{\boldsymbol{\beta}} &= (\dot{\beta}, 0, 0) \end{aligned}$$

to get

$$\mathbf{E}(t) = \frac{4r_0 c \gamma^3}{r_p} \left(B(ct') \frac{[1 - \gamma^2 \theta^2 \cos(2\phi), \gamma^2 \theta^2 \sin(2\phi)]}{(1 + \gamma^2 \theta^2)^3} \right)_{ret.}.$$

Here we made the substitution $z = \beta ct' \approx ct'$ since the field seen by the particle at the time t' is relevant. We express this field in the time $t = t' + r/c$ or $t_p = t - r_p/c$ of the observer using the relation (38)

$$t_p = t - \frac{r_p}{c} = t' \frac{1 + \gamma^2 \theta^2}{2\gamma^2}$$

to get

$$\mathbf{E}(t_p) = \frac{4r_0 c \gamma^3}{r_p} B \left(\frac{2\gamma^2}{1 + \gamma^2 \theta^2} ct_p \right) \frac{[1 - \gamma^2 \theta^2 \cos(2\phi), \gamma^2 \theta^2 \sin(2\phi)]}{(1 + \gamma^2 \theta^2)^3}.$$

We get this field in frequency domain [10]

$$\begin{aligned}\tilde{\mathbf{E}}(\omega) &= \frac{4r_0c\gamma^3}{r_p} \frac{[1 - \gamma^2\theta^2 \cos(2\phi), \gamma^2\theta^2 \sin(2\phi)]}{(1 + \gamma^2\theta^2)^3} \\ &\cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} B \left(\frac{2\gamma^2}{1 + \gamma^2\theta^2} ct_p \right) e^{-i\omega t_p} dt_p.\end{aligned}$$

Using the substitutions

$$t_p = \frac{1 + \gamma^2\theta^2}{2\gamma^2} t' \quad , \quad z = ct' \quad \text{and} \quad k_{sm} = \frac{1 + \gamma^2\theta^2}{2\gamma^2 c} \omega$$

leads to $\omega t_p = k_{sm} z$ and to

$$\begin{aligned}\tilde{\mathbf{E}}(\omega) &= \frac{2r_0\gamma}{r_p} \frac{[1 - \gamma^2\theta^2 \cos(2\phi), \gamma^2\theta^2 \sin(2\phi)]}{(1 + \gamma^2\theta^2)^2} \\ &\cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} B(z) e^{-ik_{sm}z} dz.\end{aligned}$$

This expression contains the spacial Fourier transform of the magnetic field

$$\tilde{B}(k_{sm}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} B_y(z) e^{-ik_{sm}z} dz.$$

The radiation observed at frequency ω and angle θ is determined by the Fourier component $\tilde{B}(k_{sm})$ of the magnetic field. We get now for the radiation in frequency domain

$$\begin{aligned}\tilde{\mathbf{E}}(\omega) &= \frac{2r_0\gamma}{r_p} \frac{[1 - \gamma^2\theta^2 \cos(2\phi), \gamma^2\theta^2 \sin(2\phi)]}{(1 + \gamma^2\theta^2)^2} \tilde{B}_y(k_{sm}) \\ &= \frac{2r_0\gamma}{r_p} \frac{[1 - \gamma^2\theta^2 \cos(2\phi), \gamma^2\theta^2 \sin(2\phi)]}{(1 + \gamma^2\theta^2)^2} \tilde{B} \left(\frac{1 + \gamma^2\theta^2}{2c\gamma^2} \omega \right).\end{aligned}\tag{41}$$

It is interesting to consider the inverse Fourier transform of the field $B(z)$

$$B_y(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{B}_y(k_{sm}) e^{ik_{sm}z} dk_{sm}$$

which represents a decomposition of the $B_y(z)$ into infinitely long undulator fields of wave numbers

$$k_{sm} = \frac{2\pi}{\lambda_{sm}} = \frac{1 + \gamma^2\theta^2}{2c\gamma^2} \omega.\tag{42}$$

The above expression for $\tilde{\mathbf{E}}(\omega)$ is nothing else than the undulator radiation of each such component.

The angular spectral energy distribution is obtained from the relation

$$\frac{dW^2}{d\Omega d\omega} = \frac{2r_p^2 |\tilde{E}(\omega)|^2}{\mu_0 c}$$

giving

$$\frac{d^2W}{d\Omega d\omega} = \frac{2r_0ce^2(m_0c^2\gamma)^2}{\pi(m_0c^2)^3} \frac{[(1 - \gamma^2\theta^2 \cos(2\phi))^2 + (\gamma^2\theta^2 \sin(2\phi))^2]}{(1 + \gamma^2\theta^2)^4} \left| \tilde{B} \left(\frac{1 + \gamma^2\theta^2}{2c\gamma^2} \omega \right) \right|^2.$$

This expression gives the radiation from a general magnet provided that the deflection is weak and nowhere exceeds an angle of $1/\gamma$.

4.6 An undulator emitting radiation with a Gaussian angular distribution

We saw before that an undulator with an abrupt termination of the magnetic field at $\pm L_u/2$ is unphysical and leads to some problems when calculating the variance of the opening angle and of the Fraunhofer diffraction. We investigate now an undulator which is modulated by a Lorentz distribution [11] having a magnetic field $B(z)$ and

$$B(z) = B_0 \frac{\cos(k_u z)}{1 + (z/z_0)^2} = \frac{K_0 m_0 c k_u}{e} \frac{\cos(k_u z)}{1 + (z/z_0)^2}.$$

We assume that this undulator has many periods N_u within its characteristic length $2z_0$ such that $k_u z_0 = \pi N_u \gg 1$. The total energy radiated by an electron is

$$\begin{aligned} W_u &= \frac{2r_0 e^2 c^2 (m_0 c^2 \gamma)^2}{3(m_0 c^2)^3} \int_{-\infty}^{\infty} B(z)^2 dz = \frac{2r_0 e^2 c^2 (m_0 c^2 \gamma)^2}{3(m_0 c^2)^3} \frac{\pi z_0 B_0^2}{4} (1 + e^{2k_u z_0} (1 - 2k_u z_0)) \\ &\approx \frac{\pi r_0 e^2 c^2 (m_0 c^2 \gamma)^2 z_0 B_0^2}{6(m_0 c^2)^3} = \frac{\pi r_0 (m_0 c^2 \gamma)^2 K_0^2 k_u^2 z_0}{6m_0 c^2} \end{aligned}$$

with $K = eB_0/(m_0 c k_u)$. We treat this undulator as a short magnet and apply the formalism developed in section 4.5. We filter out the frequency component ω_{10} from the radiation

$$\omega = \omega_{10} = k_u c 2\gamma^2$$

which is given by the spacial Fourier component at (42)

$$k_{sm} = \omega_{10} \frac{1 + \gamma^2 \theta^2}{2\gamma^2} = k_u (1 + \gamma^2 \theta^2).$$

The Fourier transform of the magnetic field at this wave number is

$$\tilde{B} = \sqrt{\frac{\pi}{2}} B_0 z_0 (e^{-z_0 k_u \gamma^2 \theta^2} + e^{-z_0 k_u (2 + \gamma^2 \theta^2)}) \approx \sqrt{\frac{\pi}{2}} z_0 B_0 e^{-z_0 k_u \gamma^2 \theta^2}.$$

Using (41) we get for the radiation field

$$\begin{aligned} \tilde{\mathbf{E}}(\omega) &= \frac{\sqrt{2\pi} r_0 \gamma z_0 B_0}{r_p} \frac{[1 - \gamma^2 \theta^2 \cos(2\phi), \gamma^2 \theta^2 \sin(2\phi)]}{(1 + \gamma^2 \theta^2)^2} e^{-z_0 k_u \gamma^2 \theta^2} \\ &\approx \frac{\sqrt{2\pi} r_0 m_0 c^2 \gamma K_0 k_u z_0}{r_p e c} [1, 0] e^{-\pi N_u \gamma^2 \theta^2} \end{aligned}$$

where we have used $\gamma^2 \theta^2 \ll 1$ for $N_u \gg 1$ and are left with the horizontal polarization mode only. From this field and the relation (26) we obtain emitted angular spectral energy distribution

$$\frac{d^2 W_u}{d\Omega d\omega} = W_u \frac{6z_0}{\pi} e^{-2\pi N_u \gamma^2 \theta^2}.$$

The radiation from this Lorentz modulated undulator, filtered at ω_{10} , has a Gaussian angular distribution with rms values for the polar angle θ and the two Cartesian angles x' and y'

$$\theta_{rms} = \frac{1}{\gamma \sqrt{2\pi N_u}}, \quad x'_{rms} = y'_{rms} = \frac{1}{2\gamma \sqrt{\pi N_u}}.$$

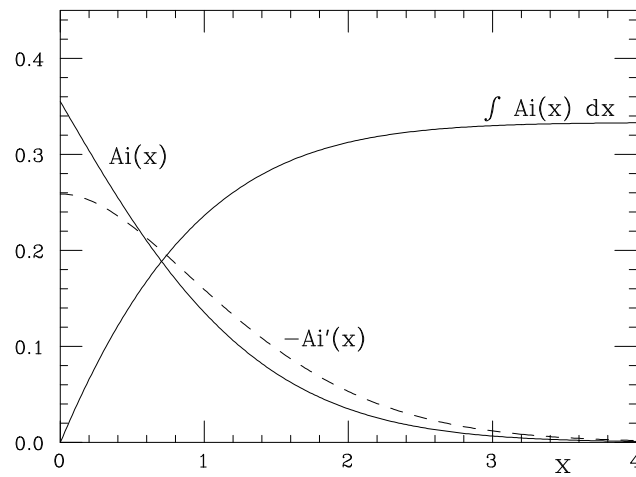


Figure 25: The Airy function $\text{Ai}(x)$, its derivative $\text{Ai}'(x)$ and its integral $\int_0^x \text{Ai}(x') dx'$

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PROPERTIES OF THE AIRY FUNCTIONS AND THEIR RELATION TO BESSEL FUNCTIONS We expressed the electric field of the synchrotron radiation with the Airy function and its derivative which can be defined by the integrals

$$\text{Ai}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos\left(\frac{u^3}{3} + xu\right) du, \quad \text{Ai}'(x) = \frac{d\text{Ai}(x)}{dx} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} u \sin\left(\frac{u^3}{3} + xu\right) du.$$

Detailed treatment of these functions can be found in the standard mathematical literature [12], we summarize here only the properties most relevant for the application to synchrotron radiation. For positive arguments the Airy function is rather smooth, as shown in Fig. 25, and satisfies the differential equation

$$\text{Ai}''(x) - x\text{Ai}(x) = 0. \quad (43)$$

For $x = 0$ the value of the functions are

$$\text{Ai}(0) = \frac{1}{3^{2/3} \Gamma(2/3)} = 0.35503 \quad \text{Ai}'(0) = -\frac{1}{3^{1/3} \Gamma(1/3)} = -0.25882.$$

For small positive arguments we can use a power expansion as an approximation for the Airy functions

$$\begin{aligned} \text{Ai}(x) &\approx \text{Ai}(0) \left[1 + \frac{1}{3!}x^3 + \frac{1 \cdot 4}{6!}x^6 \dots \right] + \text{Ai}'(0) \left[x + \frac{2}{4!}x^4 + \frac{2 \cdot 5}{7!}x^7 \dots \right] \\ \text{Ai}'(x) &\approx \text{Ai}(0) \left[\frac{1}{2!}x^2 + \frac{1 \cdot 4}{5!}x^5 \dots \right] + \text{Ai}'(0) \left[1 + \frac{1}{3!}x^3 + \frac{2 \cdot 5}{6!}x^6 \dots \right] \\ \int_0^x \text{Ai}(x) dx &\approx \text{Ai}(0) \left[x + \frac{1}{4!}x^4 + \frac{1 \cdot 4}{7!}x^7 \dots \right] + \text{Ai}'(0) \left[\frac{1}{2!}x^2 + \frac{1}{5!}x^5 + \frac{2 \cdot 5}{8!}x^8 \dots \right]. \end{aligned} \quad (44)$$

For large positive values of the argument the functions decrease exponentially

$$\begin{aligned} \text{Ai}(x) &\approx \frac{e^{-z}}{2\sqrt{\pi}x^{1/4}} \left[1 - \frac{5}{72z} + \frac{385}{2 \cdot 72^2 z^2} \dots \right] \\ \text{Ai}'(x) &\approx -\frac{x^{1/4}e^{-z}}{2\sqrt{\pi}} \left[1 + \frac{7}{72z} - \frac{455}{2 \cdot 72^2 z^2} \dots \right] \\ \int_z^{\infty} \text{Ai}(x) dx &\approx \frac{e^{-z}}{2\sqrt{\pi}x^{3/4}} \left[1 - \frac{41}{72z} + \frac{9241}{2 \cdot 72^2 z^2} \dots \right] \\ \text{with } z &= \frac{2}{3}x^{3/2} \gg 1. \end{aligned} \quad (45)$$

For later applications we need a few integrals involving Airy functions most of which were already given in [6].

First, we calculate the integral of the Airy function itself over all positive arguments

$$\int_0^{\infty} \text{Ai}(x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin(u^3/3 + xu)}{u} du \Big|_0^{\infty}.$$

We get for the upper limit

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin(u^3/3 + u(x \rightarrow \infty))}{u} du \sim \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin(xu)}{u} du = \frac{1}{2}, \text{ and}$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin(u^3/3)}{u} du = \frac{1}{2\pi} \frac{1}{3} \int_{-\infty}^{\infty} \frac{\sin(u^3/3)}{u^3/3} d(u^3/3) = \frac{1}{6}$$

for the lower limit which gives

$$\int_0^{\infty} \text{Ai}(x) dx = \frac{1}{3}. \quad (46)$$

For the spectral power density we need some integrals of the square of the Airy functions with the argument $x = a + by^2$. Using

$$\text{Ai}^2(x) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos\left(\frac{t^3}{3} + xt\right) \cos\left(\frac{s^3}{3} + xs\right) ds dt$$

and substituting $s + t = u$ and $s - t = v$ gives

$$\text{Ai}^2(x) = \frac{1}{16\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\cos\left(\frac{u^3}{12} + \frac{uv^2}{4} + xu\right) + \cos\left(\frac{v^3}{12} + \frac{u^2}{4} + xv\right) \right] dudv.$$

The expression under the integral is symmetric in u and v , furthermore both terms in the square bracket give the same value after integration

$$\text{Ai}^2(x) = \frac{1}{2\pi^2} \int_0^{\infty} \int_0^{\infty} \cos\left(\frac{u^3}{12} + \frac{uv^2}{4} + xu\right) dudv.$$

We replace now $x = a + by^2$ and integrate over y

$$\int_0^{\infty} \text{Ai}^2(a + by^2) dy = \frac{1}{2\pi^2} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \cos\left(\frac{u^3}{12} + u\left(a + \frac{v^2}{4} + by^2\right)\right) dudvdy.$$

Substituting $v = 2r \cos \phi$, $y = \frac{r \sin \phi}{\sqrt{b}}$, $dvdy = \frac{2r dr d\phi}{\sqrt{b}}$ and integrating over ϕ from 0 to $\pi/2$ gives

$$\int_0^{\infty} \text{Ai}^2(a + by^2) dy = \frac{1}{2\pi\sqrt{b}} \int_0^{\infty} \int_0^{\infty} \cos\left(\frac{u^3}{12} + u(a + r^2)\right) r dr du.$$

Making a further substitution $r^2 = w$ and $u = 2^{2/3}u'$ gives

$$\int_0^{\infty} \text{Ai}^2(a + by^2) dy = \frac{2^{2/3}}{4\pi\sqrt{b}} \int_0^{\infty} \int_0^{\infty} \cos\left(\frac{u'^3}{3} + u'2^{2/3}(a + w)\right) du' dw,$$

$$\int_0^{\infty} \text{Ai}^2(a + by^2) dy = \frac{2^{2/3}}{4\sqrt{b}} \int_0^{\infty} \text{Ai}(2^{2/3}(a + w)) dw.$$

Calling $2^{2/3}(a + w) = z'$ and $2^{2/3}a = z$ leads to the final form of our integral

$$\int_0^{\infty} \text{Ai}^2(a + by^2) dy = \frac{1}{4\sqrt{b}} \int_z^{\infty} \text{Ai}(z') dz' = \frac{1}{4\sqrt{b}} \left[\frac{1}{3} - \int_0^z \text{Ai}(z') dz' \right], \quad (47)$$

where the integral (46) over the Airy function has been used. Using the same method as above and applying (43) we can calculate the integral

$$\int_0^{\infty} y^2 \text{Ai}^2(a + by^2) dy = \frac{1}{8} \frac{1}{2^{2/3}b^{3/2}} \int_z^{\infty} (z' - z) \text{Ai}(z') dz' = \frac{a}{8} \frac{1}{b^{3/2}} \left[-\frac{\text{Ai}'(z)}{z} - \int_z^{\infty} \text{Ai}(z') dz' \right]. \quad (48)$$

Differentiating (47) twice with respect to a and using (43) gives

$$\int_0^\infty \left[Ai'^2(a + by^2) + (a + by^2) Ai^2(a + by^2) \right] dy = -\frac{1}{2 \cdot 2^{2/3} \sqrt{b}} Ai'(z)$$

which can be combined with (47) and (48) to give the integral

$$\int_0^\infty Ai'^2(a + by^2) dy = \frac{a}{8\sqrt{b}} \left[-3 \frac{Ai'(z)}{z} - \frac{1}{3} + \int_0^z Ai(z') dz' \right]. \quad (49)$$

Using the same method as for the derivation of (48) we get two integrals that we will use for the calculation of the rms. opening angle of the radiation

$$\begin{aligned} \int_0^\infty y^4 Ai^2(a + by^2) dy &= \frac{3}{32 \cdot 2^{4/3} b^{5/2}} \int_z^\infty z' (z' - z)^2 Ai(z') dz' \\ &= \frac{3a^2}{32 b^{5/2}} \left[\frac{Ai(z)}{z^2} + \frac{Ai'(z)}{z} + \int_z^\infty Ai(z') dz' \right]. \end{aligned} \quad (50)$$

Differentiating (47) once with respect to a and once with respect to b gives

$$\int_0^\infty y^2 \left[Ai'^2(a + by^2) + (a + by^2) Ai^2(a + by^2) \right] dy = \frac{1}{4 \cdot 2^{4/3} b^{3/2}} Ai(z)$$

combining this with (48) and (50) leads to

$$\int_0^\infty y^2 Ai'^2(a + by^2) dy = \frac{a^2}{32 b^{3/2}} \left[\frac{5Ai(z)}{z^2} + \frac{Ai'(z)}{z} + \int_z^\infty Ai(z') dz' \right]. \quad (51)$$

Setting $a = 0$ in (47) gives

$$\int_0^\infty Ai^2(by^2) dy = \frac{1}{12\sqrt{b}}.$$

Differentiating this twice with respect to b leads to the expression

$$\int_0^\infty y^4 \left[Ai'^2(by^2) + by^2 Ai^2(by^2) \right] dy = \frac{1}{16 b^{5/2}}. \quad (52)$$

We need one more integral which is derived the same way as (47)

$$\int_0^\infty y^6 Ai^2(by^2) dy = \frac{5}{256 b^{7/2}} \int_0^\infty z'^3 Ai(z') dz'$$

The integral appearing on the right hand side can easily be obtained from

$$I = \int_0^\infty Ai(gx) dx = \frac{1}{3g}$$

by differentiating it twice with respect to g

$$\frac{d^2 I}{dg^2} = \int_0^\infty x^2 Ai''(gx) dx = \int_0^\infty gx^3 Ai(gx) dx = \frac{2}{3g},$$

setting $g = 1$ gives

$$\int_0^\infty y^6 Ai^2(by^2) dy = \frac{5}{384 b^{7/2}}. \quad (53)$$

Combining this with (52) gives

$$\int_0^\infty y^4 \text{Ai}'^2(by^2)dy = \frac{7}{384 b^{5/2}}.$$

Substituting $y^3 = p$ in the last two integrals gives

$$\int_0^\infty p^{4/3} \text{Ai}^2(bp^{2/3})dp = \frac{5}{128 b^{7/2}}, \quad \int_0^\infty p^{2/3} \text{Ai}'^2(bp^{2/3})dp = \frac{7}{128 b^{5/2}}. \quad (54)$$

For the integrated power spectrum we need the expression which can be obtained by integration in parts

$$\int_0^z z \text{Ai}'(z)dz = z \text{Ai}(z)|_0^z - \int_0^z \text{Ai}(z)dz = z \text{Ai}(z) - \int_0^z \text{Ai}(z)dz \quad (55)$$

and

$$\begin{aligned} \int_0^z z^2 \int_z^\infty \text{Ai}(z')dz'dz &= \frac{z^3}{3} \int_z^\infty \text{Ai}(z)dz + \int_0^z \frac{z^3}{3} \text{Ai}(z)dz \\ &= \frac{z^3}{3} \int_z^\infty \text{Ai}(z)dz + \frac{1}{3} \int_0^z z^2 \text{Ai}''(z)dz \\ &= \frac{z^3}{3} \int_z^\infty \text{Ai}(z)dz + \frac{1}{3} \left[z^2 \text{Ai}'(z) - \frac{1}{2} \int_0^z z \text{Ai}'(z)dz \right]. \\ \int_0^z z^2 \int_z^\infty \text{Ai}(z')dz'dz &= \frac{z^3}{3} \int_z^\infty \text{Ai}(z)dz + \frac{1}{3} \left[z^2 \text{Ai}'(z) - \frac{1}{2} z \text{Ai}(z) + \frac{1}{2} \int_0^z \text{Ai}(z)dz \right]. \end{aligned} \quad (56)$$

For integrating over the photon spectrum we need

$$\begin{aligned} \int_0^\infty \frac{\text{Ai}'(z)}{\sqrt{z}}dz &= -\pi \int_0^\infty dz \int_0^\infty \frac{u \sin(u^3/3 + uz)}{\sqrt{z}}du = \\ &= -\frac{1}{\pi} \int_0^\infty du \int_0^\infty dz \left[u \sin(u^3/3) \frac{\cos(uz)}{\sqrt{z}} + u \cos(u^3/3) \frac{\sin(uz)}{\sqrt{z}} \right]. \end{aligned} \quad (57)$$

The integral over z can be found in reference [13]

$$\int_0^\infty \frac{\cos(uz)}{\sqrt{z}}dz = \int_0^\infty \frac{\sin(uz)}{\sqrt{z}}dz = \sqrt{\frac{\pi}{2u}}. \quad (58)$$

which gives

$$\int_0^\infty \frac{\text{Ai}'(z)}{\sqrt{z}}dz = -\frac{1}{\sqrt{2\pi}} \int_0^\infty \left[\sqrt{u} \sin(u^3/3) + u \cos(u^3/3) \right] du.$$

With the substitution $u^3 = 3v$ this can be brought into the form (58)

$$\int_0^\infty \frac{\text{Ai}'(z)}{\sqrt{z}}dz = -\frac{1}{\sqrt{6\pi}} \int_0^\infty \left[\frac{\sin v}{\sqrt{v}} + \frac{\cos v}{\sqrt{v}} \right] dv = -\frac{1}{\sqrt{3}}. \quad (59)$$

The following integral appears also when integrating over the photon spectrum and can be integrated in parts as (56) and brought to the form (59)

$$\int_0^\infty \sqrt{z} \int_z^\infty \text{Ai}(z')dz'dz = \frac{1}{3\sqrt{3}}. \quad (60)$$

Using the same method we can derive the two integrals needed for the variance of the photon energy

$$\int_0^\infty z^{5/2} \text{Ai}'(z) dz = -\frac{5}{4\sqrt{3}} \quad (61)$$

and

$$\int_0^\infty z^{7/2} \int_z^\infty \text{Ai}(z') dz' dz = \frac{35}{36\sqrt{3}}. \quad (62)$$

The Airy functions are closely related to the modified Bessel functions of the second kind $K_{1/3}$ and $K_{2/3}$

$$\text{Ai}(x) = \frac{1}{\pi} \sqrt{\frac{x}{3}} K_{1/3}\left(\frac{2x^{3/2}}{3}\right) \quad \text{and} \quad \text{Ai}'(x) = -\frac{1}{\pi} \frac{x}{\sqrt{3}} K_{2/3}\left(\frac{2x^{3/2}}{3}\right). \quad (63)$$

Which of the functions are used to express the properties of ordinary synchrotron radiation is only a matter of preference. We use here the Airy functions for derivations but give the important results also in Bessel functions. The functions $\text{Ai}(x)$, $\text{Ai}'(x)$ and $\int_0^x \text{Ai}(x') dx'$ are tabulated in [12].

INTRODUCTION TO DYNAMICS OF ELECTRONS IN RINGS IN THE PRESENCE OF RADIATION

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Abstract

Introduction to some basic ideas behind the workings of the present day electron storage rings with the emphasis on how the radiation process shapes the equilibrium properties of the electron beam.

1. INTRODUCTION

In the course of the past fifty years close to a hundred storage rings have been built around the world, first as instruments for the study of high energy particle physics, and, lately, in ever increasing numbers, as sources of synchrotron radiation. These devices are designed to store high current beams of electrons for periods of time (*beam lifetime*) that are typically greater than 10 hours. The electrons are ultrarelativistic, travelling with velocities very close to the speed of light, their trajectories bent into a closed path, thus covering in 10 hours the distance of the order of the diameter of the solar system.

Magnetic fields are used for bending the trajectories of the particles, as well as for focusing them to stay close to the *ideal orbit*. These linear magnetic fields, produced in *dipole* and *quadrupole* magnets determine the single particle behaviour in the storage ring in linear approximation. In general, particles execute transverse *betatron oscillations* around the ideal orbit.

The electron beam contains particles with energies that differ from the design ring energy. These *off-momentum* particles have a new ideal orbit that depends on the momentum deviation and is described by the *dispersion* function. They execute betatron oscillations around this new momentum-dependent closed orbit.

The first order optics exhibits severe chromatic aberrations (*chromaticity*). Non-linear fields are introduced via *sextupole* magnets to correct the chromaticity. The present generation of the synchrotron light sources uses very strong focusing lattices, and these non-linear fields lead to limitations on the size of the stable region around the ideal orbit, or limited *dynamic aperture*.

The dynamics of electrons in rings [1] is furthermore determined by the process of radiation. Steady loss of energy to synchrotron radiation and its replacement in the RF cavities results in *synchrotron oscillations*, where the particle energy oscillates around the design ring energy.

Moreover, this process leads to the *damping* of both betatron and synchrotron oscillation amplitudes. On the other hand, particle oscillations are excited due to the quantum nature of emitted radiation (*quantum fluctuations*). The balance between these two competing processes results in equilibrium beam distributions, with corresponding equilibrium beam sizes (*equilibrium emittances*).

2. RADIATION EFFECTS IN ELECTRON STORAGE RINGS

The most important features can be summarised as follows:

- Electrons steadily lose energy to synchrotron radiation, a small portion of the electron energy each turn. Moreover, to a good approximation, the loss occurs along a tangent to the electron trajectory, resulting in a reduction of both transverse and longitudinal components of the momentum.

- This loss is compensated in the RF cavities, but only the longitudinal component of the momentum is increased. This leads to steady reduction of the transverse component of the momentum, or to damping of betatron oscillations.
- Because the energy loss depends on the particle energy (it is proportional to E^2), synchrotron oscillations are also damped.
- On the other hand, synchrotron radiation is emitted in quanta (photons), leading to statistical fluctuations in the energy loss. These produce random excitation of betatron and synchrotron oscillations.
- The balance between the damping and the quantum fluctuations results in an equilibrium distribution of particles in the beam. It determines the equilibrium values of the beam emittance, energy spread and bunch length.

2.1 Energy loss due to synchrotron radiation

Charged particles radiate when they are deflected in the magnetic field [2, 3] (transverse acceleration). In the ultra-relativistic case, when the electron speed is very close to the speed of light, $\beta \approx c$, most of the radiation is emitted in the forward direction [2] and is concentrated in a cone with an opening angle of $1/\gamma$, where γ is the electron's Lorentz factor. That is to say, it is *emitted along the tangent to the electron trajectory* (since for a typical case of a few GeV electron, $\gamma \approx 1000$, the photon emission angles are within a milliradian of the tangent to the trajectory).

The *power* emitted by a particle is proportional to the square of its energy E and to the square of the magnetic field B :

$$P \propto E^2 B^2$$

and in terms of Lorentz factor γ and the local bending radius ρ can be written as follows:

$$P_{\text{SR}} = \frac{2}{3} \alpha \hbar c^2 \frac{\gamma^4}{\rho^2}$$

where α is the fine-structure constant and the Plank's constant is given in a convenient conversion constant:

$$\alpha = \frac{1}{137} \quad \text{and} \quad \hbar c = 197 \cdot 10^{-15} \text{ MeV} \cdot \text{m}$$

The emitted power is a very steep function of both the particle energy and particle mass, being proportional to the fourth power of γ . For the case of electrons, it is more common to use the following notation for the emitted power

$$P_{\text{SR}} = \frac{c C_\gamma}{2\pi} \cdot \frac{E^4}{\rho^2} \quad \text{where} \quad C_\gamma = \frac{4\pi}{3} \frac{r_e}{(m_e c^2)^3} = 8.858 \cdot 10^{-5} \left[\frac{\text{m}}{\text{GeV}^3} \right]$$

The amount of energy lost by an on-momentum particle that is following the ideal orbit plays an important role in the discussion below. Integrating the above expression around the machine:

$$U_0 = \oint P_{\text{SR}} dt = \oint P_{\text{SR}} \frac{ds}{c} = \frac{C_\gamma}{2\pi} E^4 \cdot \oint \frac{ds}{\rho^2}$$

The non-zero contributions to the integral above come from the curved parts of the ideal orbit (in the straight parts the bending radius is infinite), and, furthermore, assuming the same constant bending radius in all dipoles, we obtain for the energy loss per turn:

$$U_0 = C_\gamma \cdot \frac{E^4}{\rho} \quad \text{or, in terms of } \gamma, \quad U_0 = \frac{4\pi}{3} \alpha \hbar c \frac{\gamma^4}{\rho}$$

3. SYNCHROTRON OSCILLATIONS

3.1 Phase stability

The radiation losses are compensated in accelerating cavities that utilise time-dependent radio frequency fields, the RF system. The frequency is chosen so that it is an integer multiple of the revolution frequency. This integer is called a *harmonic number*, h

$$f_{RF} = h \cdot f_0$$

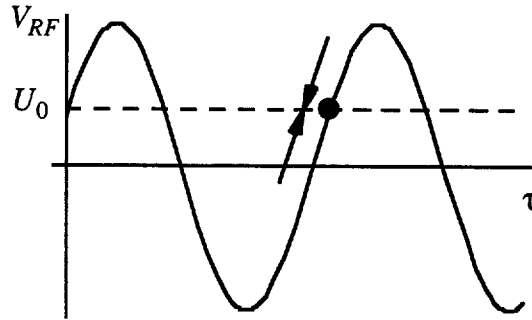
In other words, the length of the design orbit is an integer number of the RF wavelengths, and a particle that follows the design orbit remains phase locked with respect to the RF.

A so-called *synchronous* particle gains each turn just the right amount of energy from the RF cavities

$$U_{RF} = eV_{RF} = U_0$$

and will therefore be able to maintain the design energy. There are two phases with opposite RF voltage slopes for which the conditions above are satisfied. Only one of them provides stable equilibrium against small deviations in phase or arrival time at the cavity.

Let us consider what happens to a particle that arrives at the RF ahead of the synchronous one. For the case illustrated in the sketch below (where the horizontal axis represents *time delay* τ with respect to the synchronous particle)



- it will gain too much energy from the RF
- because its energy will be greater than the design energy, the magnetic field will not be strong enough to keep it on the design orbit, its trajectory will have greater average radius of curvature, thus requiring more time to complete the next turn
- it will come back to the RF cavity closer in time to the synchronous particle

Similarly, for a particle that arrives behind the synchronous one

- it will gain too little energy from the RF
- because its energy will be smaller than the design energy, the magnetic field will be too strong to keep it on the design orbit, its trajectory will have smaller average radius of curvature, thus requiring less time to complete the next turn
- it will come back to the RF cavity catching up with the synchronous particle

Particles with small deviations in phase are attracted towards the synchronous particle. Exactly the opposite happens around the phase with the opposite slope of the RF voltage. The points of stable phase thus provide h equidistant places around the machine circumference, separated by the RF wavelength, where bunches of electrons can be stored.

3.2 The time it takes to go around — momentum compaction factor

The time it takes a particle to go around the ring once obviously depends on the orbit length L and on the particle velocity. An off-energy particle will have different velocity and will follow an orbit of different length.

$$T = \frac{L}{v} \quad \text{or} \quad \frac{\Delta T}{T} = \frac{\Delta L}{L} - \frac{\Delta v}{v}$$

High energy electrons travel with speed very close to the speed of light, their velocity changes very little with energy. From relativity you may recall that

$$\frac{dv}{v} = \frac{1}{\gamma^2} \frac{dE}{E} \quad \Rightarrow \quad \frac{\Delta v}{v} = \frac{1}{\gamma^2} \cdot \delta$$

where δ is the relative energy deviation with respect to the design energy. In fact, the orbit length change in most cases turns out to be greater than the velocity change. A funny thing happens to a relativistic electron on the way around the ring: in spite of higher energy (and higher speed) it takes longer to go around.

Let us now look at the change in orbit length with the particle's energy. As was already mentioned before, the length of a small portion of the trajectory that lies in the horizontal plane:

$$dl = \left(1 + \frac{x}{\rho}\right) ds$$

depends on the horizontal deviation. This deviation can be written in two parts, the betatron oscillation and the energy dependent deviation. To first order the betatron oscillation does not change the orbit length, since its average displacement from the design orbit is zero. On the other hand, the energy dependent displacement generally has the same sign all around the ring. As a result, the orbit length of a particle with a relative energy deviation δ is

$$L_\delta = \oint dl = \oint \left(1 + \frac{x_\delta}{\rho}\right) ds = L + \Delta L$$

and, recalling the definition of the dispersion function, we can write the change in orbit length for an off-momentum particle as

$$\Delta L = \delta \cdot \oint \frac{D(s)}{\rho(s)} ds$$

The relative change in length of the off-momentum orbit is proportional to the energy deviation. The constant of proportionality is called the momentum compaction factor α , a constant that only depends on the properties of the magnetic guide field and is independent of energy:

$$\frac{\Delta L}{L} = \alpha \cdot \delta \quad \text{where} \quad \alpha \equiv \frac{1}{L} \oint \frac{D(s)}{\rho(s)} ds$$

Coming back to the expression for the change in revolution time with particle energy, we can re-write it in the following way:

$$\frac{\Delta T}{T} = \frac{\Delta L}{L} - \frac{\Delta v}{v} = \left(\alpha - \frac{1}{\gamma^2}\right) \cdot \delta$$

The term in parenthesis is called *phase-slip factor*, describing the change in particle arrival time at the RF cavities. For a certain design ring energy, so-called *transition energy*, this factor can become zero, corresponding to the isochronous mode of operation: all particles take the same time to complete one turn, independent of their energy. Writing the momentum compaction factor in terms of transition energy, the phase-slip factor is:

$$\eta \equiv \left(\frac{1}{\gamma_{tr}^2} - \frac{1}{\gamma^2}\right) \quad \text{where} \quad \alpha \equiv \frac{1}{\gamma_{tr}^2}$$

All the high energy electron storage rings operate above transition energy, i.e. $\gamma \gg \gamma_{tr}$. This regime, *above transition*, corresponds to the situation already mentioned above: higher energy electrons take longer time to complete one turn.

3.3 Damping of synchrotron or energy — time oscillations

The radiation loss and its compensation by the RF system, coupled with the dependence of the revolution time on particle energy deviation cause particles to execute energy-time oscillations, or so-called *synchrotron* oscillations. Let us now write down what happens to a particle in one turn:

- **energy gain from the RF system:**

the synchronous particle gains exactly U_0 ; a particle that arrives at the RF cavities with a time delay τ with respect to the synchronous one, receives the amount of energy that is proportional to the slope of the RF voltage (we consider here only small time delays, assuming that we are working in the linear part of the RF voltage waveform)

$$U_{RF} = eV_{RF}(\tau) = U_0 + e\dot{V}_{RF} \cdot \tau \quad \text{where} \quad \dot{V}_{RF} = \left. \frac{dV_{RF}}{d\tau} \right|_{\tau=0}$$

- **energy loss:**

every turn the particle loses on the average U_0 ; but recall that the radiation loss depends on the particle energy (proportional to the energy squared), so depending on the energy deviation ε ,

$$U_{rad} = U_0 + U' \cdot \varepsilon \quad \text{where} \quad U' \equiv \left. \frac{dU_{rad}}{dE} \right|_{E=E_0}$$

Let us delay for the moment the evaluation of the radiation loss derivative with energy. For now we will simply treat it as a constant in our derivation of the equation of motion for synchrotron oscillations.

- **energy balance:**

we can now write down the amount by which the particle energy changes during one turn

$$\Delta\varepsilon = (U_0 + e\dot{V}_{RF} \cdot \tau) - (U_0 + U' \cdot \varepsilon)$$

and since this change occurs during a time interval equal to the revolution period T_0 , we can turn this difference equation into a differential equation describing the time evolution of the particle energy:

$$\frac{d\varepsilon}{dt} = \frac{1}{T_0} (e\dot{V}_{RF} \cdot \tau - U' \cdot \varepsilon)$$

- **time lag with respect to the synchronous particle:**

similarly, from the above discussion on the momentum compaction factor, we can write down the differential equation for the other variable, describing the change with time in the particle's phase lag with respect to the RF. For a given particle energy deviation ε ,

$$\frac{d\tau}{dt} = -\alpha \cdot \delta = -\alpha \frac{\varepsilon}{E_0}$$

- **synchrotron oscillations:**

combining the two differential equations above, we can eliminate one of the variables obtaining a second-order differential equation for the evolution of the energy deviation in time:

$$\frac{d^2\varepsilon}{dt^2} + 2\alpha_\varepsilon \frac{d\varepsilon}{dt} + \Omega^2 \varepsilon = 0$$

a harmonic oscillator equation describing synchrotron oscillations with the frequency

$$\Omega^2 \equiv \frac{\alpha e \dot{V}_{RF}}{T_0 E_0}$$

In fact, it is a *damped* harmonic oscillator, with a velocity-like term and a damping decrement

$$\alpha_\epsilon \equiv \frac{U'}{2T_0} \quad \text{typically} \quad \alpha_\epsilon \ll \Omega$$

and since the damping occurs on a much slower scale than the synchrotron oscillations, the solution is a harmonic oscillation with an initial amplitude ϵ_0 that damps exponentially with the corresponding decrement

$$\epsilon(t) = \epsilon_0 e^{-\alpha_\epsilon t} \cos(\Omega t + \theta_\epsilon)$$

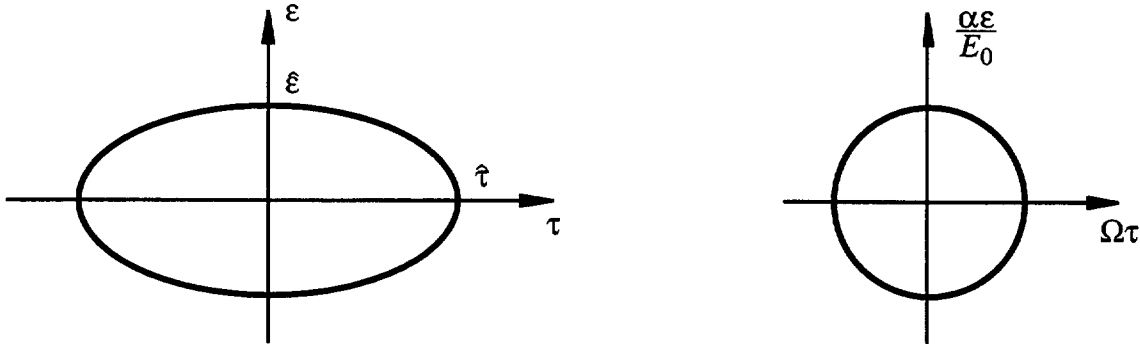
Similarly, we can write a damped harmonic oscillator equation describing the evolution of the time lag with respect to the synchronous particle. The solution again would be a harmonic oscillation with an exponentially decreasing amplitude:

$$\tau(t) = \tau_0 e^{-\alpha_\tau t} \cos(\Omega t + \theta_\tau)$$

The amplitudes and phases of the two solutions are of course closely related. At any instant:

$$\hat{\tau} = \frac{\alpha}{\Omega E_0} \hat{\epsilon} \quad \text{and} \quad \theta_\epsilon = \theta_\tau + \frac{\pi}{2}$$

The motion can be pictured in the phase space of these two conjugate variables $\{\epsilon, \tau\}$ as an ellipse or, in suitably normalised co-ordinates, as a circle



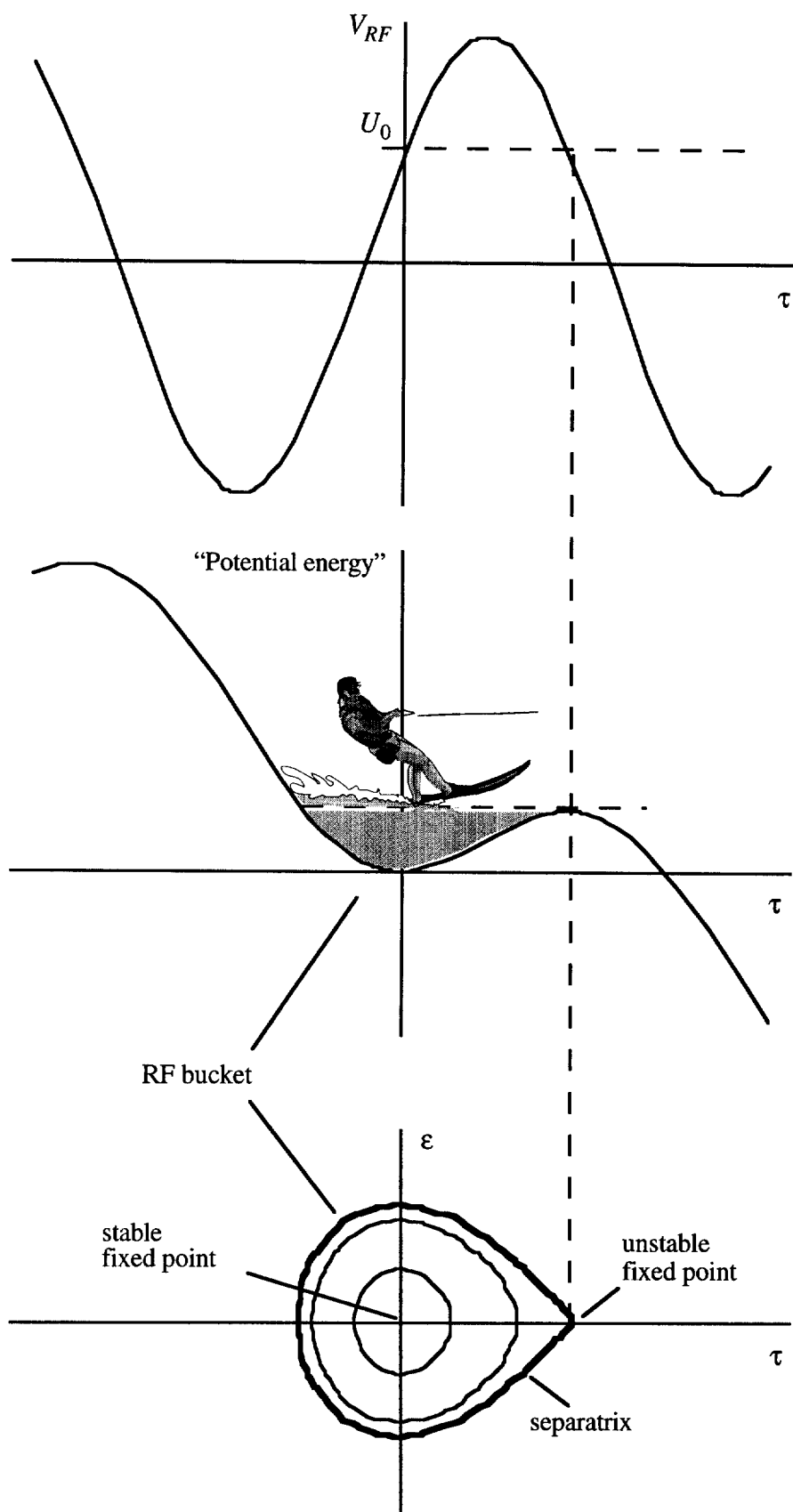
• synchrotron oscillations: phase space and RF buckets

The global picture is illustrated below. The RF accelerating voltage waveform is shown with two possible times of arrival or phases that provide exactly the average loss per turn. One of them is stable against small phase deviations, the other is unstable, as discussed above. This could be clearly seen from the plot of the "potential energy" function that corresponds to the "restoring force" seen by an out-of-phase particle sees due to the longitudinal dynamics.

In fact, the stable phase (stable fixed point in longitudinal phase space) corresponds to the minimum of a potential well. Particles with small oscillation amplitudes execute stable synchrotron oscillations in this well, their phase space portraits are ellipses, just like in the case of a harmonic oscillator. The height and the width of the ellipse correspond to the maximum energy and phase deviation of the oscillating particle. Similar to the harmonic oscillator picture, one can speak of the oscillation energy, proportional to the square of the oscillation amplitude.

There is a critical value of the oscillation amplitude, or oscillation energy, beyond which the particle motion becomes unbounded; it corresponds to the height of the potential-well barrier that can be seen in the middle plot. This energy level defines a separatrix in the phase space: a trajectory that encloses the region of stable synchrotron oscillation motion, shown as a bold curve in the phase space plot at the bottom. This area is usually referred to as the *RF bucket*, and its right edge corresponds to the unstable phase (unstable fixed point). There are h RF buckets around the ring, spaced by the RF wavelength, that can hold bunches of particles.

The height of the bucket corresponds to the maximum particle energy deviation that can be stored. It describes the *RF limited energy acceptance* of the storage ring.



- **damping of synchrotron oscillations: qualitative picture**

The above phase-space portraits of synchrotron oscillations are snapshots taken over time spans that are short compared to the damping time scale. Over a longer time period the amplitude of the oscillations, i.e. the size of the ellipse decreases exponentially and the phase space trajectory is an inward spiral that maintains the aspect ratio of the above snapshots.

Synchrotron oscillations are damped due to the fact that the energy loss per turn is a quadratic function of the particle's energy. While in the upper half of the ellipse, the particle's energy deviation is positive and its losses per turn are greater than the amount it receives from the RF system, its *energy deviation tends to decrease*. In the lower part of the ellipse, its energy deviation is negative and its losses per turn are smaller than the amount it receives from the RF, again leading to a *decrease in the energy deviation*.

4. DAMPING OF BETATRON OSCILLATIONS

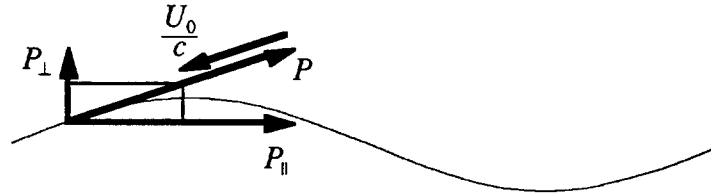
4.1 Qualitative picture

The steady loss of energy to synchrotron radiation is compensated in the RF cavities, where the particle receives each turn the average amount of energy lost.

Average energy loss. Every turn the electron loses a small portion of its energy, $U_0 \approx 10^{-3}$ of E_0 , in the form of synchrotron radiation, so that after one turn, on the average, its energy becomes:

$$E_1 = E_0 - U_0 = E_0 \left(1 - \frac{U_0}{E_0} \right)$$

Since the radiation is emitted along the tangent to the trajectory, only the *amplitude* of its momentum changes.



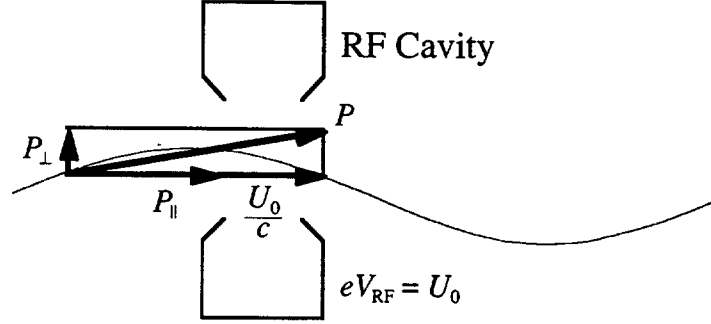
After one turn both the transverse and longitudinal components of the momentum vector are reduced by the same fraction:

$$P_1 = P_0 - \frac{U_0}{c} = P_0 \left(1 - \frac{U_0}{E_0} \right)$$

The transverse component of the momentum vector corresponds to the energy of the betatron oscillation (proportional to the square of the oscillation amplitude). Therefore, the amplitude of the betatron oscillation is reduced by a small amount every turn.

$$A_1^2 = A_0^2 \left(1 - \frac{U_0}{E_0} \right) \quad \text{or} \quad A_1 \cong A_0 \left(1 - \frac{U_0}{2E_0} \right)$$

Energy gain in the RF cavities. In the RF cavities only the longitudinal component of the momentum is increased by the amount corresponding to the average energy loss per turn. Over many turns, the energy of the particle is maintained at the design value.



Exponential damping of the betatron oscillations amplitudes. The relative change in the betatron amplitude that occurs every turn, i.e. every revolution time T_0 , is:

$$\frac{\Delta A}{A} = -\frac{U_0}{2E}$$

Thus the amplitudes are damped exponentially with the following damping decrement:

$$A = A_0 e^{-\frac{t}{\tau}} \quad \text{where} \quad \frac{1}{\tau} = \frac{U_0}{2ET_0}$$

Recall that the longitudinal damping time is a factor of two shorter. Typical damping time corresponds simply to *the number of turns it would take to lose the amount of energy equal to the particle energy*. The damping times are very fast, with values on the order of 10 ms.

Since the loss per turn, or radiation power are proportional to the fourth power of energy, the damping time is inversely proportional to the cube of the particle energy, a rather steep energy dependence. In storage rings where the particles are injected at an energy substantially lower than the design energy (e.g. LEP: 20 GeV), the damping times at injection can be very long indeed. In order to improve the stability of the beam at these low energies, measures are taken to shorten the damping times, e.g. at LEP by introducing wiggler magnets to increase the amount of loss per turn.

4.2 Comparison with adiabatic damping

It is interesting to compare the radiation damping to the adiabatic damping, e.g. in a linear accelerator. During the acceleration along the linac the longitudinal component of the momentum steadily increases, while the transverse momentum remains constant. Correspondingly, the slope of the particle's trajectory, being the ratio of transverse to longitudinal component of the momentum, decreases inversely proportional to the particle energy.

In a circular machine the particle returns every turn to the RF cavities, where similarly to the linac case, the slope of its trajectory is reduced. Every turn it radiates about the same amount of energy it has gained from the RF, but since the radiation is emitted tangentially to the trajectory, its slope remains unperturbed in the process. Thus the resulting damping of transverse oscillations occurs while the energy of the particle remains on the average the same.

4.3 Damping partition numbers. Robinson theorem

For particles that emit synchrotron radiation the dynamics is characterised by the damping of particle oscillations in all the three degrees of freedom. The corresponding damping times are of the same order, as we have seen from the qualitative picture above.

In fact, the total amount of damping (Robinson theorem [4, 5]), i.e. the sum of the damping times depends only on the particle energy and the emitted synchrotron radiation power:

$$\frac{1}{\tau_x} + \frac{1}{\tau_y} + \frac{1}{\tau_e} = \frac{2U_0}{ET_0} = \frac{U_0}{2ET_0}(J_x + J_y + J_e)$$

where we have introduced the usual notation of *damping partition numbers* that show how the total amount of damping in the system is distributed among the three degrees of freedom. So far we have seen the case of (1,1,2) and their sum is, according to the Robinson theorem, a constant.

$$J_x + J_y + J_z = 4$$

5. ADJUSTMENT OF DAMPING RATES

The partition numbers can differ from the above values, while their sum remains a constant. In fact, under certain circumstances, the motion can become “anti-damped”, i.e. the damping time can become negative, leading to an exponential growth of the oscillations amplitudes. In order to understand this situation we have to refine the simplified qualitative picture presented above.

Let us recall that the energy loss is proportional to the product of energy and the magnetic field squared:

$$P \propto E^2 B^2$$

Both the magnetic field and the particle energy can take values that are different from the design ones, while the particle executes transverse or longitudinal oscillations.

Let us first consider **betatron oscillations**. Following a betatron oscillation trajectory, a particle sees magnetic fields that differ from the design values (e.g. traversing the quadrupole magnets off-axis, the particle sees a dipole magnetic field). Nevertheless, because the average particle displacement as it travels around the ring is zero, *to first order in particle displacement*, there is no contribution to the energy loss. There is, of course, a non-zero term that is quadratic in displacement, but in our linearised analysis it can be neglected.

The situation is quite different for the case of **synchrotron oscillations**. A particle with an energy deviation follows a closed orbit that is defined by the *off-energy or dispersion* function, which typically has the same sign around the ring. The magnetic fields it sees differ from the design ones, and there is a non-zero contribution to the energy loss that is linear in particle displacement, that in turn is proportional to the energy deviation. In addition, because of the quadratic energy dependence, the energy loss also contains a term linear in energy deviation.

Evaluating the loss per turn to first order in energy deviation we recall that the synchrotron oscillations occur on a time scale that is much longer than the revolution time. The particle then follows a trajectory around the ring that is defined by the dispersion function, and we can write:

$$U_{rad} = U_0 + U' \cdot \delta, \quad \text{where} \quad U' \equiv \frac{dU_{rad}}{dE} = \frac{U_0}{E_0} (2 + \mathcal{D})$$

where we have introduced a new constant, $\mathcal{D}$ — an integral of the dispersion function and the magnetic guide field functions, i.e. bending radius and gradient around the ring.

$$\mathcal{D} \equiv \frac{\oint \frac{D}{\rho} \left(2k + \frac{1}{\rho^2} \right) ds}{\oint \frac{ds}{\rho^2}}$$

This constant is independent of the particle energy, and only deviates substantially from zero when a particle encounters combined function elements, i.e. where the product of the field gradient and the curvature is non-zero.

The synchrotron oscillations damping time, taking into account this analysis, becomes

$$\frac{1}{\tau_e} = \frac{U'}{2T_0} = \frac{U_0}{2E_0 T_0} (2 + \mathcal{D})$$

and a similar analysis of the horizontal betatron oscillations shows that the damping time in that case is:

$$\frac{1}{\tau_x} = \frac{U_0}{2E_0 T_0} (1 - \mathcal{D})$$

Correspondingly, the damping partition numbers can be written as follows:

$$J_x = 1 - \mathcal{D}, \quad J_\varepsilon = 2 + \mathcal{D}, \quad J_x + J_\varepsilon = 3$$

For the storage rings built in one plane (horizontal) the vertical dispersion function is zero and the vertical partition number remains unchanged.

The amount of damping can be repartitioned between the horizontal and energy-time oscillations by altering the value of the $\mathcal{D}$ constant [6]. This can be achieved by either using combined function magnetic elements in the lattice, or by introducing a special combined function wiggler magnet (so-called Robinson wiggler). Values of horizontal partition number as high as 2.5 have been obtained that way. Values of $\mathcal{D} > 1$ lead to anti-damping of horizontal betatron oscillations, while for $\mathcal{D} < -2$ the synchrotron oscillations become unstable.

6. QUANTUM FLUCTUATIONS — EQUILIBRIUM BEAM SIZES

If the radiation damping were the whole story, in a matter of seconds the size of the injected beam would shrink to microscopic dimensions. This would result in several problems that would make the construction and operation of electron storage rings utterly impractical. It suffices to mention the coherent synchrotron radiation production.

Typically, an ensemble of emitting electrons radiates incoherently, i.e. there is no definite phase relationship between the individual emitters. The total power is simply obtained by adding the individual emitters' power; it is linearly proportional to the number of electrons.

In a bunch of electrons that is small compared to the wavelength of the emitted synchrotron radiation the particles emit electromagnetic waves that are in phase, i.e. at a point of observation that is far away from the source the electric field vectors add. The total radiation power from N particles is then proportional to the square of their number! The corresponding increase in power required from the RF system to compensate for the losses would be unreasonably high.

6.1 Quantum nature of synchrotron radiation

In fact, the beam does not shrink to microscopic size, rather the effect of damping is counterbalanced by a kind of heating that is connected to the *quantum nature* of emission of radiation.

The synchrotron radiation is emitted as photons, the typical photon energy, corresponding to the critical frequency, being:

$$u_c = \hbar\omega_c = \frac{3}{2}\hbar c \frac{\gamma^3}{\rho}$$

and the number of photons emitted per second can be roughly approximated by dividing the total emitted power by the typical photon energy.

$$\mathcal{N} = \frac{P_{\text{SR}}}{u_c} = \frac{4}{9}\alpha c \frac{\gamma}{\rho}$$

The photon emission is a stochastic process, and in any time interval the expected number of photons emitted has a statistical uncertainty:

$$N \pm \sqrt{N}, \quad N = \mathcal{N} \cdot \Delta t$$

For example, the amount of energy loss in one turn fluctuates around the expected value $U_0 = N \cdot u_c$, where $N = \mathcal{N} T_0$, the typical spread being

$$\pm \sqrt{N} \cdot u_c$$

6.2 Equilibrium energy spread and bunch length

Over large time intervals the RF system maintains the average particle energy at the design value. Due to *quantum fluctuations*, i.e. statistical fluctuations in the number of photons emitted, the particle energy will be different from the design value, exciting energy-time oscillations. Eventually, the two competing processes of radiation damping and quantum fluctuations will strike a balance, resulting in a finite value of the energy spread. We can make a reasonable estimate of the steady state value of the energy spread by evaluating the typical energy deviation during a time period equal to the damping time (a characteristic time scale for this process):

$$\sigma_\varepsilon \approx \sqrt{\mathcal{N} \tau_\varepsilon} \cdot u_c$$

Since the damping time is the ratio of the particle energy to the radiation power, i.e. “the time it takes to radiate all the energy away”, the expected energy spread is approximately equal to the geometric mean of the particle energy and the typical photon energy:

$$\sigma_\varepsilon \approx \sqrt{E_0 \cdot u_c}$$

The relative energy spread is then approximately written as

$$\frac{\sigma_\varepsilon}{E_0} \approx \gamma \sqrt{\frac{\lambda_c}{\rho}}, \quad \lambda_c \approx 4 \cdot 10^{-13} \text{ m} \text{ — the electron Compton wavelength.}$$

The bending radius in the electron storage rings scales typically as square of the design energy, thus the relative energy spread is approximately the same in all the rings, roughly equal to 10^{-3} .

A more detailed calculation gives the following result for the equilibrium relative energy spread in a storage ring:

$$\left(\frac{\sigma_\varepsilon}{E}\right)^2 = \frac{C_q E^2}{J_\varepsilon \rho_0} \quad \text{where} \quad C_q = \frac{55}{32\sqrt{3}} \frac{\hbar c}{(m_e c^2)^3} = 1.468 \cdot 10^{-6} \left[\frac{\text{m}}{\text{GeV}^2} \right]$$

The energies of electrons in a bunch at equilibrium have a Gaussian distribution with a standard deviation equal to the above energy spread. A synchronous electron with an energy deviation equal to the above standard deviation, after a quarter of the period of synchrotron oscillation, will have zero energy deviation (see the discussion of synchrotron oscillations above), however it will be displaced from the synchronous particle in time by

$$\sigma_\tau = \frac{\alpha}{\Omega E_0} \sigma_\varepsilon$$

The distribution of electrons in time along the bunch is closely related to the distribution in energies, and for small bunch currents is also a Gaussian with an above standard deviation. The equilibrium bunch length (expressed in time units) and its energy spread are given by the above expressions.

6.3 Excitation of horizontal betatron oscillations

Let us now turn to the case of transverse oscillations, where a similar source of heating due to statistical fluctuations in the emission rate of photons ensures that the transverse beam size does not shrink to microscopic dimensions. We first look at a qualitative picture of how horizontal betatron oscillations get excited in the process of emission of photons.

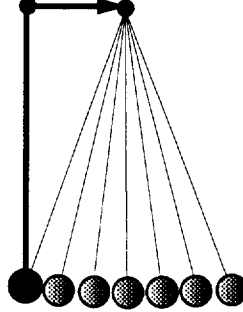
After an electron emits a photon, its energy is decreased and its new reference orbit will be determined by the dispersion function around the ring

$$x_{ref} = D \cdot \delta$$

where δ is the relative energy deviation. Since the emission is instantaneous and occurs along a tangent to the electron trajectory, neither the displacement nor the slope are affected. Moreover, because the reference orbit is shifted due to the change in energy, the electron starts to oscillate around this new reference orbit with an amplitude

$$x_{\beta} = D \cdot \delta$$

This situation is analogous to that of a pendulum at equilibrium, with its suspension point suddenly shifted sideways, while the pendulum itself remains at rest. After that the pendulum starts to oscillate around its new equilibrium (see the sketch below)



The emission of photons is a random process, so we have again a random walk, now in horizontal displacement. How far the particle will wander away from the ideal orbit is kept in check by the radiation damping. The balance will be achieved on a time scale of a horizontal damping time and, typically, the particle will have an equilibrium horizontal deviation of

$$\sigma_{x\beta} \approx \sqrt{\mathcal{N} \cdot \tau_x} \cdot D \cdot \delta = \sqrt{2} \cdot D \cdot \frac{\sigma_E}{E}$$

since the horizontal damping time is a factor of two longer than the longitudinal one. The equilibrium horizontal beam size is on the order of 1 mm, as the dispersion is typically on the order of 1 m.

Let us take a closer look at the amplitudes of the oscillations resulting from the emission of a photon. Horizontal displacement and slope can be written in terms of a betatron oscillation part and an energy dependent part (proportional to the dispersion function):

$$\begin{aligned} x &= x_{\beta} + x_{\epsilon} = x_{\beta} + D \cdot \delta \\ x' &= x'_{\beta} + x'_{\epsilon} = x'_{\beta} + D' \cdot \delta \end{aligned}$$

Neither position nor slope can change in the process of emission of a photon (the emission is quasi-instantaneous and occurs along a tangent to the trajectory). Clearly, the energy of the particle is reduced by the amount equal to the photon energy, i.e. the energy dependent contribution to the displacement must change. Therefore, the betatron oscillation contribution to the displacement must be equal and opposite to the energy contribution, as the total displacement remains constant. After the photon emission particle starts a betatron oscillation around the new closed orbit (determined by the dispersion function). The initial conditions read:

$$\begin{aligned} x_{\beta} &= -D \cdot \delta \\ x'_{\beta} &= -D' \cdot \delta \end{aligned}$$

Recall the form of the betatron oscillation and the corresponding amplitude, or Courant-Snyder invariant:

$$x_{\beta} = a\sqrt{\beta} \cos \phi(s) \quad \text{where} \quad a^2 \equiv \gamma x_{\beta}^2 + 2\alpha x_{\beta} x'_{\beta} + \beta x'^2_{\beta}$$

If, for simplicity's sake, we assume that before radiating a photon the particle was on the design trajectory, after emitting a photon that results in a relative particle energy change of δ , the amplitude of the betatron oscillation will be:

$$a^2 = \delta^2 \cdot (\gamma D^2 + 2\alpha D D' + \beta D'^2) = \delta^2 \cdot \mathcal{H}(s)$$

where we have introduced the $\mathcal{H}$ function that is very similar to the Courant-Snyder invariant, with displacement and slope of the trajectory replaced by the dispersion and its slope. It is a function of position around the ring, and its shape in the bending magnets (i.e. where the radiation occurs) determines the magnitude of the effect of quantum fluctuations on the beam size.

6.4 Equilibrium transverse beam size

A detailed calculation of the quantum fluctuations and damping effects yields the following result for the equilibrium beam emittance in the electron storage ring:

$$\varepsilon_{x0} \equiv \frac{\sigma_{x\beta}^2}{\beta} = \frac{C_q E^2}{J_x} \cdot \frac{\langle \mathcal{H} \rangle_{mag}}{\rho}$$

where angular brackets denote an average value of $\mathcal{H}$ in the bending magnets and where

$$C_q = \frac{55}{32\sqrt{3}} \frac{\hbar c}{(m_e c^2)^3} = 1.468 \cdot 10^{-6} \left[\frac{\text{m}}{\text{GeV}^2} \right]$$

The equilibrium beam size is linearly proportional to the ring energy. In the rings where the electrons are ramped in energy (e.g. LEP), the vacuum chamber dimensions must take this into account. The shape of the $\mathcal{H}$ function in the bends is optimised to obtain the desired value of the equilibrium emittance. The dependence of the emittance on the horizontal partition number can also be used to modify the emittance.

In almost all storage rings there is no bending of particles in the vertical plane; the design value of vertical dispersion is zero. Therefore, the process of photon emission would not excite vertical betatron oscillations and the vertical beam size would be very quickly damped to a very small value. Here we have to recall an important approximation we have made: the photons are emitted into a cone with an opening angle of $1/\gamma$, i.e. not exactly along the tangent to the particle's trajectory. The slope of the particle trajectory will therefore be changed by a small amount due to the recoil during the process of photon emission. This will excite a small amplitude vertical betatron oscillation. Again we have a source of heating due to quantum fluctuations. It can be shown that due to this finite opening angle effect the equilibrium vertical beam size is still very small [1, 7].

In reality, the vertical beam size is dominated by coupling effects: motion in the horizontal plane is coupled to the vertical plane. Thus horizontal betatron oscillations excited by quantum fluctuations get coupled to vertical betatron oscillations. Misalignments (roll) in dipole and quadrupole magnets as well as vertical closed orbit offsets in sextupole magnets are the main sources of coupling. In addition, the spurious vertical dispersion function in the dipole magnets that results from coupling of the horizontal dispersion in rotated quadrupoles and offset sextupoles provides another source of heating due to quantum fluctuations.

6.5 Summary of the stored beam properties

The equilibrium stored beam properties, determined by the process of synchrotron radiation, can be expressed in terms of so-called *synchrotron radiation integrals* [8]. These are integrals of the guide field functions (bending radius $\rho(s)$, quadrupole strength $k(s)$) and of functions of Twiss parameters, taken around the entire ring. They are evaluated by most of the

optics codes (e.g. MAD [9]). A brief summary of the main results for the case of a storage ring built in one (horizontal) plane is given here. It involves five radiation integrals defined below:

Radiation integrals:

$$\begin{aligned} I_1 &= \oint \frac{D}{\rho} ds \\ I_2 &= \oint \frac{ds}{\rho^2} \\ I_3 &= \oint \frac{ds}{|\rho^3|} \\ I_4 &= \oint \frac{D}{\rho} \left(2k + \frac{1}{\rho^2} \right) ds \\ I_5 &= \oint \frac{\mathcal{H}}{|\rho^3|} ds \end{aligned}$$

$$\mathcal{H} = \gamma D^2 + 2\alpha DD' + \beta D'^2$$

Momentum compaction factor:

$$\alpha = \frac{I_1}{2\pi R}$$

Energy loss per turn:

$$U_0 = \frac{1}{2\pi} C_\gamma E^4 \cdot I_2$$

$$C_\gamma = \frac{4\pi}{3} \frac{r_e}{(m_e c^2)^3} = 8.858 \cdot 10^{-5} \left[\frac{\text{m}}{\text{GeV}^3} \right]$$

Damping parameter:

$$\mathcal{D} = \frac{I_4}{I_2}$$

Damping times and partition numbers:

$$\begin{aligned} J_\epsilon &= 2 + \mathcal{D}, \quad J_x = 1 - \mathcal{D}, \quad J_y = 1 \\ \tau_i &= \frac{\tau_0}{J_i} \quad \text{where} \quad \tau_0 = \frac{2ET_0}{U_0} \end{aligned}$$

Equilibrium energy spread:

$$\left(\frac{\sigma_\epsilon}{E} \right)^2 = \frac{C_q E^2}{J_\epsilon} \cdot \frac{I_3}{I_2}$$

$$C_q = \frac{55}{32\sqrt{3}} \frac{\hbar c}{(m_e c^2)^3} = 1.468 \cdot 10^{-6} \left[\frac{\text{m}}{\text{GeV}^2} \right]$$

Equilibrium emittance:

$$\epsilon_{x0} = \frac{\sigma_{x\beta}^2}{\beta} = \frac{C_q E^2}{J_x} \cdot \frac{I_5}{I_2}$$

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In contrast to the situation a few years ago, there is now available quite a respectable choice of books that cover the basic physics of high energy particle accelerators. Some of them are listed below.

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INTRODUCTION TO INSERTION DEVICES

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Abstract

Synchrotron radiation with extremely high brilliance is emitted by wiggler and undulator magnets, so called “insertion devices”. They produce a periodically alternating field at the beam axis with well defined period length. The relativistic electron motion through the magnet is calculated. Because of the periodic transverse particle oscillation the resulting spontaneous radiation is mainly coherent with small width of spectral lines.

1. INTRODUCTION

At the very beginning the synchrotron radiation emitted by an electron beam passing through a bending magnet has been used for experiments. It is spread out over a wide horizontal fan as shown in Fig. 1. Since the probes are normally small, only a small fraction of the radiation is usable. Most of it is lost. In addition, the horizontal resolution is rather poor.

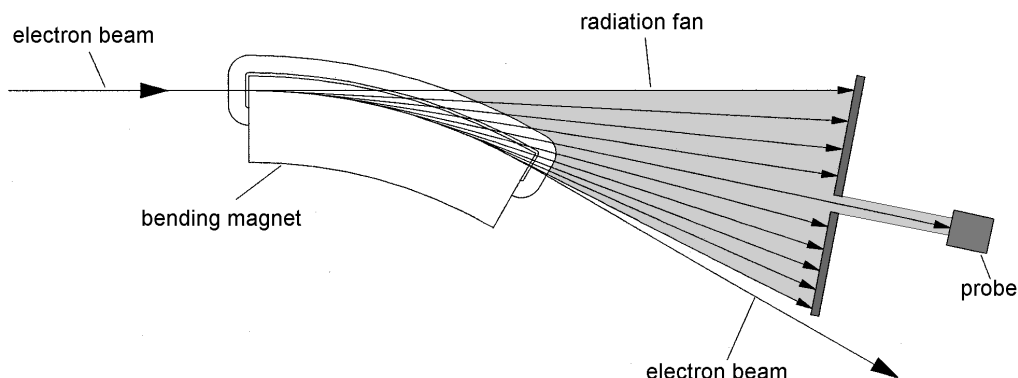


Fig. 1 Synchrotron radiation emitted by a bending magnet

Much better beam quality and significantly higher brilliance is provided by wiggler and undulator magnets (W/U magnets) [1] installed in special straight sections in electron storage rings. These types of magnets often are summarized under the name “insertion devices”. The general arrangement of such magnets is sketched in Fig. 2. It consists of a sequence of short bending magnets of constant length. Along the beam axis the resulting field can be described in good approximation by a sine curve with the period length λ_u . The overall bending angle of this device vanishes. Therefore, sufficiently long straight sections are required in modern storage rings for synchrotron radiation to install the W/U magnets.

The design principal of wiggler magnets is basically the same as of undulator magnets. The difference comes from the field strength. Wigglers provide a strong field resulting in a wide horizontal opening angle of the emitted radiation. The wide photon spectrum is very similar to that of a bending magnet. The opposite are undulators with rather weak fields and correspondingly small opening angles of synchrotron light. In this case the photons can interfere and the emitted radiation is mainly coherent with a wavelength determined by the period length of the undulator and the beam energy. The intensity of this coherent radiation is by orders of magnitude higher than achieved from simple bending magnets.

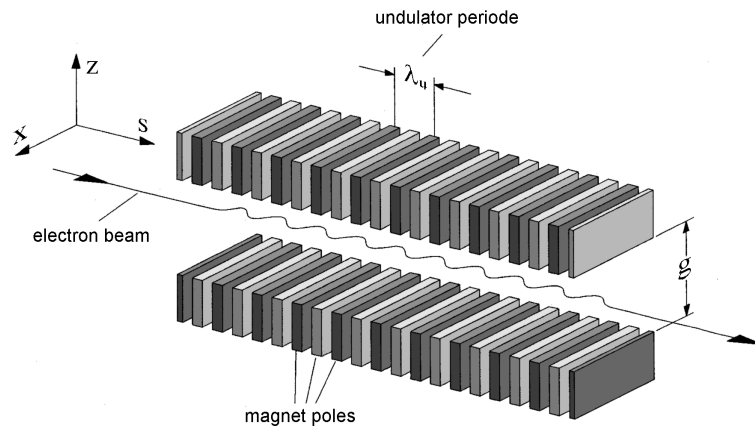


Fig. 2 General layout of a wiggler or undulator magnet

The synchrotron radiation from electron storage rings of the 3rd generation is essentially produced by wiggler and undulator magnets. A larger number of straight sections provide sufficient space for the insertion devices. The bending magnets only guide the beam along the circular orbit. The general layout of a modern synchrotron light source is sketched in Fig. 3. Some examples of third generation machines are listed in Table 1. More detailed informations are presented in [2].

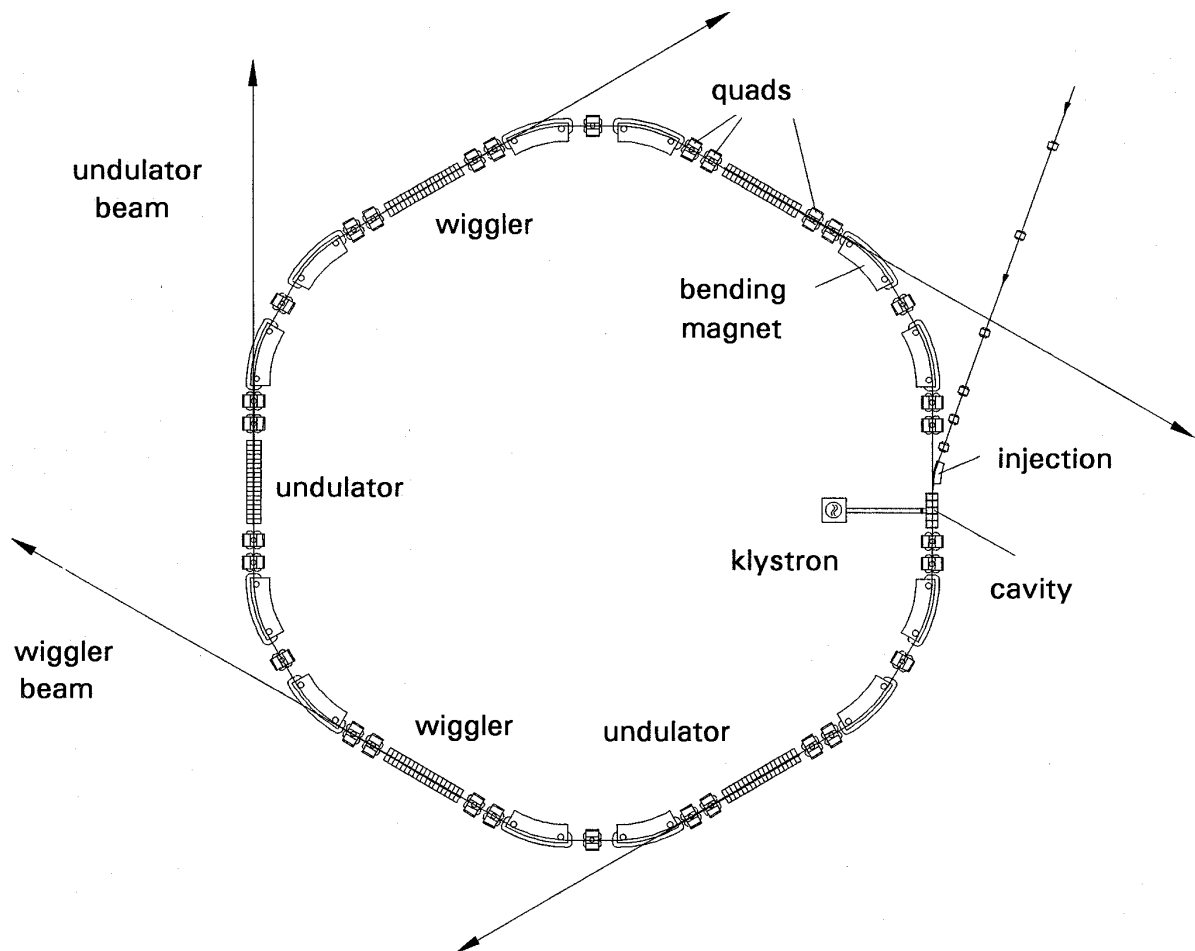


Fig. 3 General layout of a modern synchrotron light source of the 3rd generation

Table 1

List of some typical synchrotron light sources with insertion devices

Location	Ring	Energy [GeV]
Grenoble, France	ESRF	6.0
Argonne, USA	APS	7.0
Berkeley, USA	ALS	1.5
Trieste, Italy	ELETTRA	1.5 - 2.0
Dortmund, Germany	DELTA	1.5
Berlin, Germany	BESSY II	1.5 - 2.0

2. WIGGLER AND UNDULATOR FIELD

Wiggler and undulator magnets (W/U-magnets) produce in good approximation along the beam axis a periodic field with the scalar potential

$$\varphi(s, z) = f(z) \cos\left(2\pi \frac{s}{\lambda_u}\right) = f(z) \cos(k_u s). \quad (1)$$

Here a scalar potential is applied rather than a vector potential since in the interesting area around the beam no electrical current occurs. According to Fig. 2 s is the coordinate along the beam axis and z in the vertical direction. It is assumed that the magnet is unlimited in the horizontal (x coordinate) direction, providing a constant magnetic field along this coordinate. The vertical function $f(z)$ can be evaluated using Laplace equation

$$\nabla^2 \varphi(s, z) = 0. \quad (2)$$

Taking (1) gives

$$\frac{d^2 f(z)}{dz^2} - f(z) k_u^2 = 0 \quad (3)$$

with the solution

$$f(z) = A \sinh(k_u z). \quad (4)$$

Substitution of (4) into the potential equation (1) yields

$$\varphi(s, z) = A \sinh(k_u z) \cos(k_u s). \quad (5)$$

For the particle motion inside a W/U-magnet only the vertical field is important. We can derive it simply from the potential by

$$B_z(s, z) = \frac{\partial \varphi(s, z)}{\partial z} = k_u A \cosh(k_u z) \cos(k_u s). \quad (6)$$

The unknown constant A can be estimated with an acceptable accuracy applying the strength of the poletip field B_0 as defined in Fig. 4.

According to equation (6) the poletip field at $z = g/2$ (g = gap height) is

$$B_0 = B_z\left(0, \frac{g}{2}\right) = k_u A \cosh\left(k_u \frac{g}{2}\right) = k_u A \cosh\left(\pi \frac{g}{\lambda_u}\right) \quad (7)$$

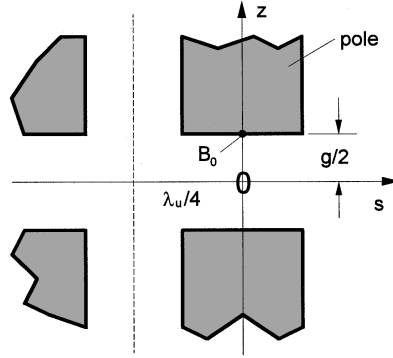


Fig. 4 Definition of the poletip field in a W/U-magnet

and the constant A becomes

$$A = \frac{B_0}{k_u \cosh\left(\pi \frac{g}{\lambda_u}\right)}. \quad (8)$$

With this result the W/U-field around the electron beam can finally be written in the form

$$B_z(s, z) = \tilde{B} \cosh(k_u z) \cos(k_u s) \quad (9)$$

with the peak field at the beam axis

$$\tilde{B} = \frac{B_0}{\cosh\left(\pi \frac{g}{\lambda_u}\right)}. \quad (10)$$

From this expression one can see that the peak field decreases very rapidly with increasing gap height g . This effect is also shown in Fig. 5.

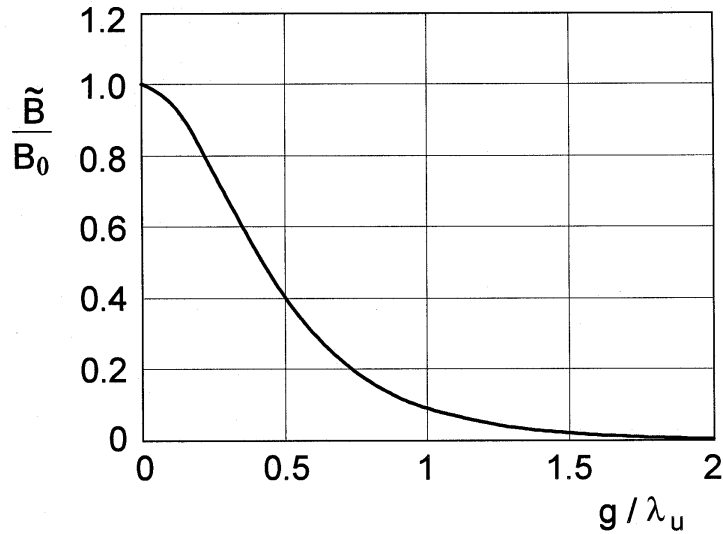


Fig. 5 Relative peak field at the beam axis as a function of the ratio g/λ_u

As a consequence of this behavior the gap height of a W/U-magnet has to be substantially smaller than the period length λ_u . This leads to very small gaps if short W/U-periods are required. The field calculation presented here is only a simple analytic

approximation, it is helpful for basic investigations. For an accurate design including tight tolerances more sophisticated methods and numerical calculations are required [1, 3, 4].

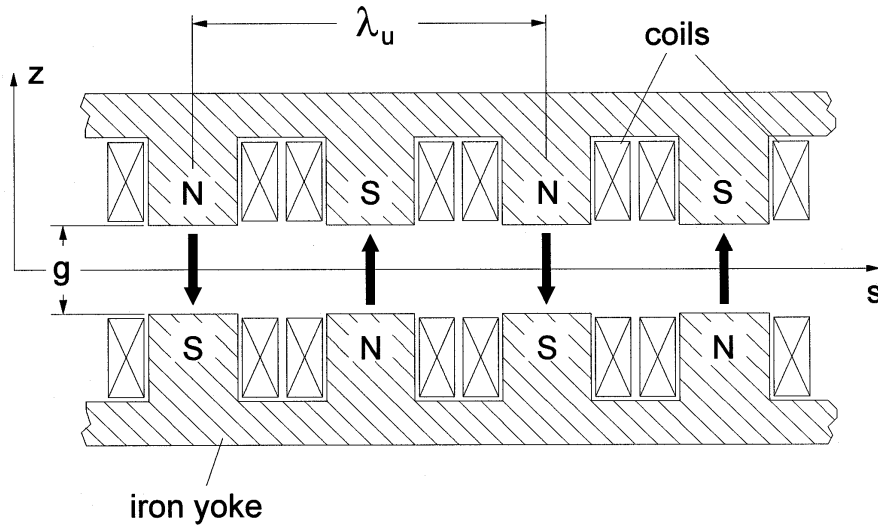


Fig. 6 Principal design of an electromagnetic wiggler or undulator

The very first W/U-magnets have been built with conventional iron yokes excited by magnet coils as shown in Fig. 6. The field strength can easily be changed by changing the current. This technique is used for period lengths not smaller than $\lambda_u \approx 25$ cm. For smaller periods the current density in the coils exceeds reasonable values. In this case one can take superconductive coils, but because of the cryostat and the expensive liquid helium technique the costs are disproportionate.

Therefore, another type of W/U-magnets has been developed utilizing small pieces of permanent magnets arranged as sketched in Fig. 7.

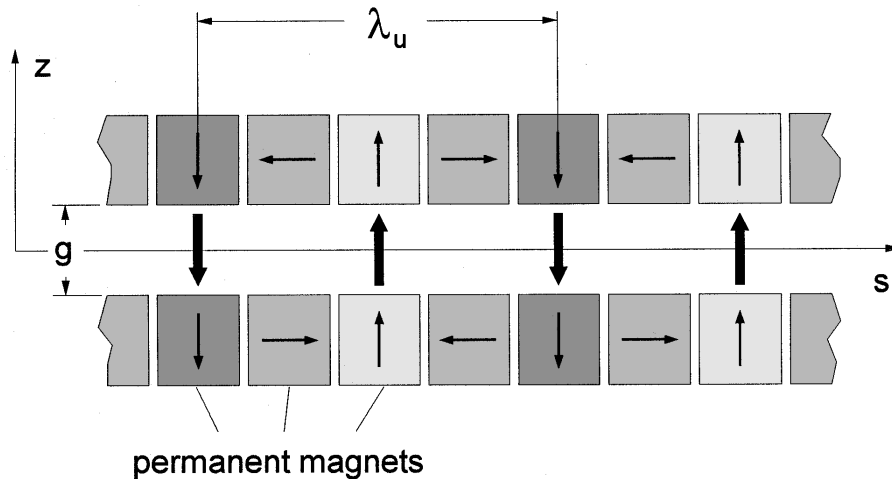


Fig. 7 W/U-magnet built with permanent magnets. The arrows indicate the orientation of the magnetic flux.

Since the flux density in the magnet pieces is constant, one can only vary the field strength at the beam axis by opening or closing the gap mechanically. The field changes according to equation (10). The poletip field has values which depend on the material used. For instance, samarium-cobalt (SmCo_5) provides flux densities about $B \approx 0.9 - 1.0$ T. If higher fields up to 2 T are required, one can build hybrid magnets (Fig. 8). They consist of poles made by soft iron and pieces of permanent magnets in between. The total flux of the magnet pieces is concentrated in the iron poles resulting in poletip fields about $B_0 = 2$ T.

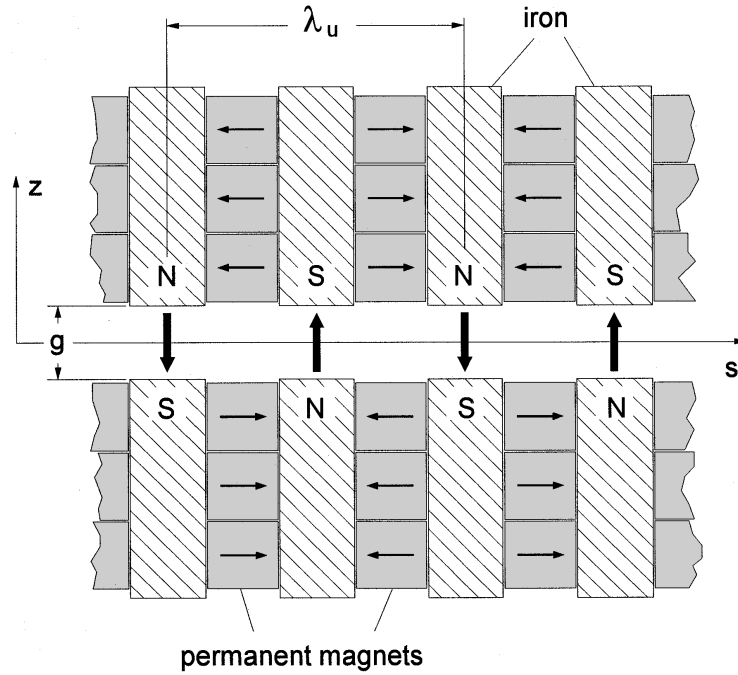


Fig. 8 Sketch of a hybrid magnet with iron poles exited by permanent magnet pieces

As mentioned above, the W/U-magnets are installed in straight sections of the storage ring. Thus, the total bending angle of the entire device must vanish (Fig. 9).

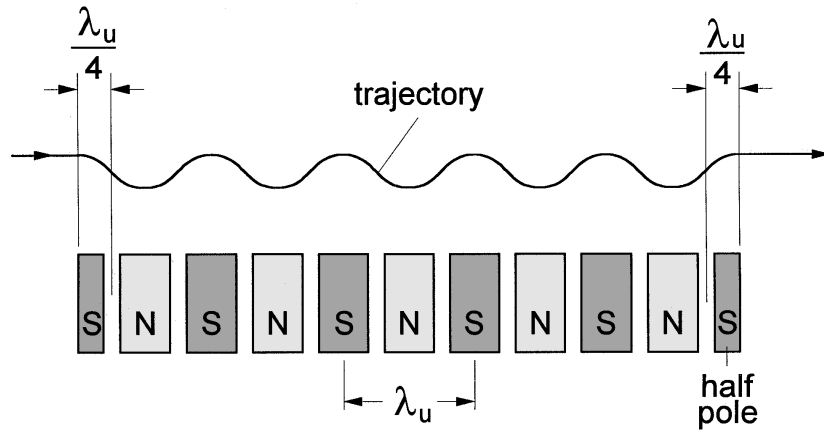


Fig. 9 Matching condition for the beam orbit over a W/U-magnet

This leads directly to the following matching condition:

$$\int_{W/U} B_z(s) ds = \tilde{B} \int_{s_1}^{s_2} \cos(k_u s) ds = 0. \quad (11)$$

It is satisfied if

$$s_1 = 0 \quad \text{and} \quad s_2 = n\lambda_u + \frac{\lambda_u}{2} \quad (n = 1, 2, \dots) \quad (12)$$

Very often this condition is realized by adding a magnet pole of half the length of a normal pole at the entrance and the exit of the magnet. The beam oscillation is then asymmetric with respect to the beam axis (Fig. 9). If symmetric oscillation is required, i.e. for free electron lasers, more sophisticated conditions can be applied.

3. EQUATION OF MOTION IN W/U-MAGNETS

3.1 The coupled set of equations of motion

Inside a W/U-magnet with the field $\mathbf{B}$ the Lorentz force

$$\mathbf{F} = \dot{\mathbf{p}} = m\gamma \dot{\mathbf{v}} = e \mathbf{v} \times \mathbf{B} \quad (13)$$

acts on the relativistic electron with the mass $m\gamma$ and the charge e . In the following we will only discuss the particle motion in the horizontal x-s-plane. The vertical motion is normally negligible. With this simplification we can write the field of the W/U-magnet and the particle velocity $\mathbf{v}$ in the form

$$\mathbf{B} = \begin{pmatrix} 0 \\ B_z \\ B_s \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} v_x \\ 0 \\ v_s \end{pmatrix}. \quad (14)$$

Inserting (14) into (13) directly provides

$$\dot{\mathbf{v}} = \frac{e}{m\gamma} \begin{pmatrix} -v_s B_z \\ -v_x B_s \\ v_x B_z \end{pmatrix}. \quad (15)$$

We again neglect the vertical particle motion and can finally write the result in the form of a coupled set of equations:

$$\boxed{\begin{aligned} \ddot{x} &= -\dot{s} \frac{e}{m\gamma} B_z(s) \\ \ddot{s} &= \dot{x} \frac{e}{m\gamma} B_z(s) \end{aligned}} \quad (16)$$

3.2 First-order solution of the equations of motion

Since $B_z(s)$ is a nonlinear function of the position s in the magnet (see eq. (9)), there exists no simple analytic solution. The longitudinal velocity v_s , however, is significantly larger than the horizontal component v_x . This fact allows us to write

$$\dot{x} = v_x \ll v_s \quad \text{and} \quad \dot{s} = v_s = \beta c = \text{const.} \quad (17)$$

Under these conditions, only the first equation of (16) is of interest and we get the relation

$$\ddot{x} = -\dot{s} \frac{e}{m\gamma} B_z(s) = -\frac{\beta c e \tilde{B}}{m\gamma} \cos(k_u s). \quad (18)$$

In storage rings it is often more convenient to describe the particle motion as a function of place s instead of time t . For transformation we use

$$\dot{x} = x' \beta c \quad \text{and} \quad \ddot{x} = x'' \beta^2 c^2 \quad (19)$$

and get from equation (18) the first order expression of the particle motion

$$x'' = -\frac{e \tilde{B}}{m\beta c \gamma} \cos(k_u s) = -\frac{e \tilde{B}}{m\beta c \gamma} \cos\left(2\pi \frac{s}{\lambda_u}\right). \quad (20)$$

It can be easily solved by integration. Since the electrons in storage rings for synchrotron radiation have extremely relativistic velocities, we can set $\beta = 1$ and obtain

$$\begin{aligned} x'(s) &= \frac{\lambda_u e \tilde{B}}{2\pi m \gamma c} \sin(k_u s) \\ x(s) &= \frac{\lambda_u^2 e \tilde{B}}{4\pi^2 m \gamma c} \cos(k_u s). \end{aligned} \quad (21)$$

The first equation gives the angle of the particle trajectory in a W/U-magnet with respect to the orbit (Fig. 10).

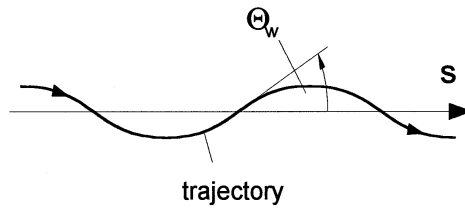


Fig. 10 Particle trajectory in a W/U-magnet

The maximum value of the trajectory angle is achieved at the crossing points with the orbit. It is

$$x'_{\max} = \Theta_w = \frac{1}{\gamma} \frac{\lambda_u e \tilde{B}}{2\pi m c}. \quad (22)$$

Now we define the dimensionless **wiggler** or **undulator parameter**

$$K = \frac{\lambda_u e \tilde{B}}{2\pi m c} \quad (23)$$

and can write the maximum trajectory angle in the form

$$\Theta_w = \frac{K}{\gamma}. \quad (24)$$

Especially for $K = 1$ the angle becomes

$$\Theta_w = \frac{1}{\gamma}, \quad (25)$$

which is identical to the natural opening angle of the synchrotron radiation cone. This relation suggests now the following differentiation between wiggler and undulator magnets:

Undulator	if	$K \leq 1$	i.e.	$\Theta_w \leq \frac{1}{\gamma}$
Wiggler	if	$K > 1$	i.e.	$\Theta_w > \frac{1}{\gamma}$

(26)

For very large K the horizontal opening angle of the radiation is correspondingly large and no interference of the photons takes place. The photon energy covers a dipole-like broad spectrum. For very small K the radiation from all undulator periods overlaps and strong interference effects occur, resulting in rather sharp coherent spectral lines.

3.3 Second-order solution of the equations of motion

With the simple first-order solution we can't describe for instance the Doppler effect observed in wiggler and undulator radiation. Therefore, we have to take additional conditions into account. But we are still not going to look for a general solution of the coupled set of equations of motions (16).

The longitudinal velocity $\dot{s}$ causes, because of the Lorentz force, a transverse oscillation $x(t)$, as described by equation (21). The resulting transverse velocity again causes a small longitudinal oscillation, which superimposes with the longitudinal velocity $\dot{s}$. The longitudinal velocity is actually not really constant. Thus, we can describe the particle motion in the form

$$\dot{s}(t) = \langle \dot{s} \rangle + \Delta \dot{s}(t) \quad (27)$$

with the average velocity $\langle \dot{s} \rangle$ and the longitudinal oscillation $\Delta \dot{s}(s)$. As shown in Fig. 11 the constant particle velocity βc can be written as

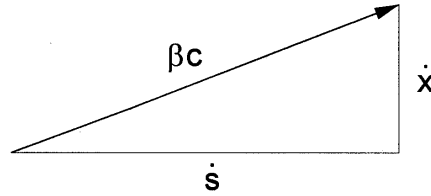


Fig. 11 Addition of the particle velocity

$$(\beta c)^2 = \dot{s}^2 + \dot{x}^2. \quad (28)$$

With

$$\beta^2 = 1 - \frac{1}{\gamma^2} \quad (29)$$

we find

$$\dot{s}(t) = c \sqrt{1 - \left(\frac{1}{\gamma^2} + \frac{\dot{x}^2}{c^2} \right)}. \quad (30)$$

Since $1/\gamma^2 \ll 1$ and $\dot{x} \ll c$ we can expand the root. The longitudinal has therefore the expression

$$\dot{s}(t) = c \left[1 - \frac{1}{2} \left(\frac{1}{\gamma^2} + \frac{\dot{x}^2(t)}{c^2} \right) \right] = c \left[1 - \frac{1}{2\gamma^2} \left(1 + \frac{\gamma^2}{c^2} \dot{x}^2(t) \right) \right]. \quad (31)$$

Taking the formula (21) and (23) gives

$$x'(s) = \frac{K}{\gamma} \sin(k_u s). \quad (32)$$

With the transformations

$$\dot{x} = \beta c x', \quad s = \beta c t \quad \text{and} \quad \omega_u = k_u \beta c \quad (33)$$

we get the time-dependent transverse velocity

$$\dot{x}(t) = \beta c \frac{K}{\gamma} \sin(\omega_u t). \quad (34)$$

We insert this relation into equation (31) and convert the sine function according to

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x)). \quad (35)$$

The longitudinal velocity becomes

$$\dot{s}(t) = c \left\{ 1 - \frac{1}{2\gamma^2} \left[1 + \frac{\beta^2 K^2}{2} (1 - \cos(2\omega_u t)) \right] \right\}. \quad (36)$$

It has the required form of equation (27) with the average velocity

$$\langle \dot{s} \rangle = c \left\{ 1 - \frac{1}{2\gamma^2} \left[1 + \frac{\beta^2 K^2}{2} \right] \right\} \quad (37)$$

and the time-dependent velocity oscillation

$$\Delta \dot{s}(t) = \frac{c \beta^2 K^2}{4\gamma^2} \cos(2\omega_u t). \quad (38)$$

For extremely relativistic particles we set $\beta = 1$. This condition is fulfilled in almost all storage rings for synchrotron radiation. The relative velocity of the particle along the undulator axis is

$$\beta^* = \frac{\langle \dot{s} \rangle}{c} = 1 - \frac{1}{2\gamma^2} \left[1 + \frac{K^2}{2} \right]. \quad (39)$$

Thus, we can write the transverse and longitudinal particle velocity in the form

$$\begin{aligned} \dot{x}(t) &= \frac{cK}{\gamma} \sin(\omega_u t) \\ \dot{s}(t) &= \beta^* c + \frac{cK^2}{4\gamma^2} \cos(2\omega_u t). \end{aligned} \quad (40)$$

The trajectory can finally be found simply by integrating these equations using the arbitrary initial conditions $x(0) = 0$ and $s(0) = 0$:

$$\begin{aligned} x(t) &= -\frac{K}{k_u \gamma} \cos(\omega_u t) \\ s(t) &= \beta^* ct + \frac{K^2}{8k_u \gamma^2} \sin(2\omega_u t). \end{aligned} \quad (41)$$

It is very impressive to observe the particle motion in the coordinate system $\mathbf{K}^*$, which travels along the undulator axis with the constant velocity β^* , rather than in the laboratory system $\mathbf{K}$. With the Lorentz transformations

$$x^* = x \quad \text{and} \quad s^* = \gamma(s - \beta t) \quad (42)$$

we find the trajectory functions in the moving system $\mathbf{K}^*$

$$\begin{aligned} x^*(t) &= -\frac{K}{k_u \gamma} \cos(\omega_u t) \\ s^*(t) &= \frac{K^2}{8k_u \gamma^2} \sin(2\omega_u t). \end{aligned} \quad (43)$$

In this system the trajectory has a shape like an “8” as shown in Fig. 12. For very small values of the undulator parameter K only the transverse motion dominates. The “8” is very narrow. For larger K the longitudinal amplitude grows quadratically, whereas the transverse amplitude is proportional to K . The growing longitudinal motion causes a Doppler effect of the emitted radiation.

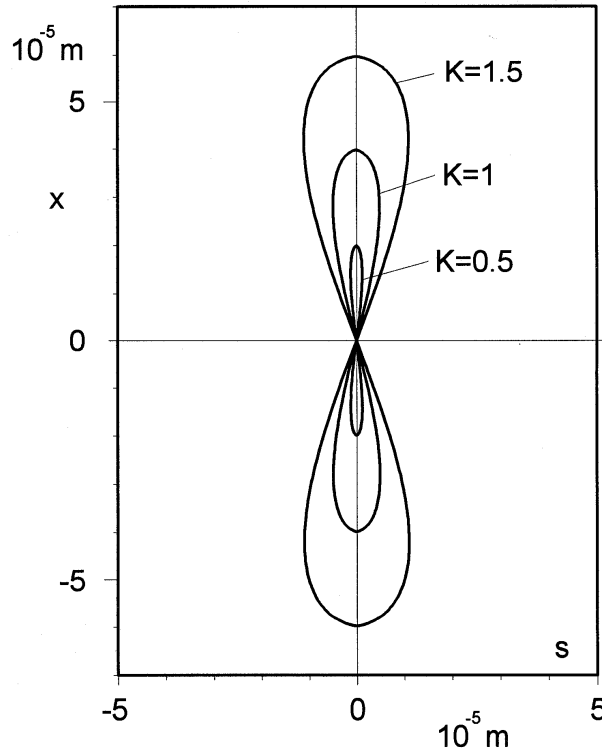


Fig. 12 Particle trajectory in an undulator magnet observed in the moving system $\mathbf{K}^*$

4. UNDULATOR RADIATION

The sinusoidal particle motion in an undulator causes coherent radiation in the laboratory system $\mathbf{K}$ with the frequency

$$\Omega_{rad} = \frac{2\pi\beta c}{\lambda_u} = k_u \beta c. \quad (44)$$

For very small undulator parameters $K \ll 1$ we can neglect the longitudinal oscillation $\Delta \dot{s}(t)$ and find in the moving system $\mathbf{K}^*$ the monochromatic radiation with the frequency

$$\omega_{rad}^* = \gamma^* \Omega_{rad}. \quad (45)$$

γ^* is the relative energy of the particle in $\mathbf{K}^*$. We transform this radiation from the moving system into the laboratory system. Here, we take a photon with the momentum p emitted under the angle Θ_0 with respect to the undulator axis (Fig. 13).

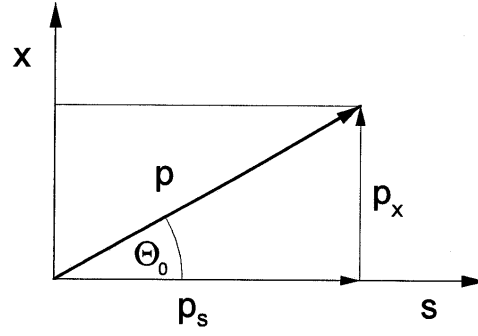


Fig. 13 Photon with the momentum p emitted in the laboratory system

The energy and the momentum of the photon are

$$E = \hbar\omega \quad \text{and} \quad p = \frac{\hbar\omega}{c} \quad (46)$$

and the 4-momentum becomes

$$P_\mu = \begin{pmatrix} E/c \\ p_x \\ p_z \\ p_s \end{pmatrix} = \begin{pmatrix} E/c \\ p \sin \Theta_0 \\ 0 \\ p \cos \Theta_0 \end{pmatrix}. \quad (47)$$

The transformation into the moving system $\mathbf{K}^*$ yields

$$\begin{pmatrix} E^*/c \\ p_x^* \\ p_z^* \\ p_s^* \end{pmatrix} = \begin{pmatrix} \gamma^* & 0 & 0 & -\beta^*\gamma^* \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta^*\gamma^* & 0 & 0 & \gamma^* \end{pmatrix} \begin{pmatrix} E/c \\ p \sin \Theta_0 \\ 0 \\ p \cos \Theta_0 \end{pmatrix}. \quad (48)$$

Considering equation (46), the resulting photon energy becomes

$$\frac{E^*}{c} = \gamma^* \frac{E}{c} - \beta^* \gamma^* p \cos \Theta_0 = \gamma^* \frac{\hbar\omega_{rad}}{c} (1 - \beta^* \cos \Theta_0). \quad (49)$$

With $E^* = \hbar\omega_{rad}^*$ we get

$$\frac{\hbar\omega_{rad}^*}{c} = \gamma^* \frac{\hbar\omega_{rad}}{c} (1 - \beta^* \cos \Theta_0). \quad (50)$$

From this equation we directly derive the frequency transformation of the relativistic Doppler effect

$$\omega_{rad} = \frac{\omega_{rad}^*}{\gamma^* (1 - \beta^* \cos \Theta_0)}. \quad (51)$$

Using equations (44) and (45) we have

$$\omega_{rad} = \frac{\Omega_{rad}}{1 - \beta^* \cos \Theta_0} \quad \Rightarrow \quad \frac{\omega_{rad}}{\Omega_{rad}} = \frac{\lambda_u}{\lambda_{rad}} = \frac{1}{1 - \beta^* \cos \Theta_0}. \quad (52)$$

The wavelength of the coherent undulator radiation emitted in the laboratory system under an angle Θ_0 becomes

$$\lambda_{rad} = \lambda_u (1 - \beta^* \cos \Theta_0). \quad (53)$$

We replace now the average velocity β^* by the expression (39). Since the angle $\Theta_0 \approx 1/\gamma$ is very small, we can expand the cosine function in terms of Θ_0 as $\cos \Theta_0 \approx 1 - \Theta_0^2/2$ and get the good approximation

$$\begin{aligned} \lambda_u (1 - \beta^* \cos \Theta_0) &= \lambda_u \left[1 - \left(1 - \frac{1}{2\gamma^2} \left[1 + \frac{K^2}{2} \right] \right) \cdot \left(1 - \frac{\Theta_0^2}{2} \right) \right] \\ &= \lambda_u \left[1 - \left(1 - \frac{\Theta_0^2}{2} - \frac{1 + K^2/2}{2\gamma^2} + \dots \right) \right] \\ &\approx \lambda_u \left(\frac{\Theta_0^2}{2} + \frac{1 + K^2/2}{2\gamma^2} \right). \end{aligned} \quad (54)$$

From this relation we finally get the very important **coherence condition** for the undulator radiation:

$$\boxed{\lambda_{rad} = \frac{\lambda_u}{2\gamma^2} \left(1 + \frac{K^2}{2} + \gamma^2 \Theta_0^2 \right)} \quad (55)$$

For many experiments with synchrotron radiation the width of the spectrum lines emitted from an undulator is an important parameter. Because of the limited length of the magnet

$$L_u = N_u \lambda_u \quad (56)$$

the radiation has the time duration

$$T = N_u \frac{\lambda_{rad}}{c} \quad (57)$$

where N_u is the number of periods. In Fig. 14 the emitted wave pulse from an undulator is sketched.

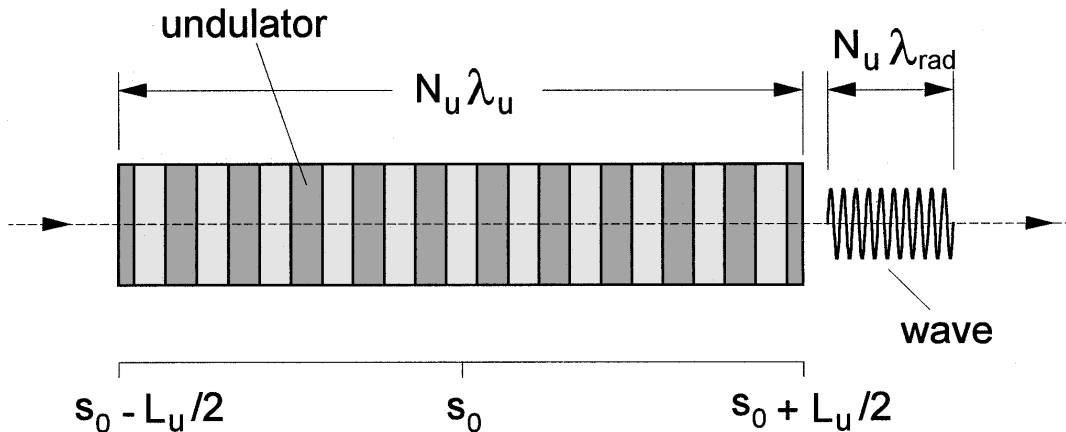


Fig. 14 Wave pulse emitted by a single electron passing through an undulator

The field strength of this undulator radiation can be expressed by

$$u(\omega_{rad}, t) = \begin{cases} a \exp(i \omega_{rad} t) & \text{if } -\frac{T}{2} \leq t \leq +\frac{T}{2} \\ 0 & \text{if } |t| > \frac{T}{2} \end{cases} . \quad (58)$$

The resulting continuous spectrum follows from the *Fourier transformation* of the wave function:

$$\begin{aligned} A(\omega) &= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{+\infty} u(\omega_{rad}, t) \exp(-i\omega t) dt \\ &= \frac{a}{\sqrt{2\pi T}} \int_{-T/2}^{+T/2} \exp[-i(\omega - \omega_{rad})t] dt \\ &= \frac{2a}{\sqrt{2\pi T}} \frac{\sin(\omega - \omega_{rad}) \frac{T}{2}}{\omega - \omega_{rad}} . \end{aligned} \quad (59)$$

With

$$\Delta \omega = \omega - \omega_{rad} \quad \text{and} \quad \omega_{rad} T = 2\pi N_u \quad (60)$$

the amplitude of the partial wave is

$$A(\omega) = \frac{a}{\sqrt{2\pi}} \frac{\sin\left(\pi N_u \frac{\Delta \omega}{\omega_{rad}}\right)}{\pi N_u \frac{\Delta \omega}{\omega_{rad}}} \quad (61)$$

and the intensity (see Fig. 15)

$$I(\Delta \omega) \propto \left[\frac{\sin\left(\pi N_u \frac{\Delta \omega}{\omega_{rad}}\right)}{\pi N_u \frac{\Delta \omega}{\omega_{rad}}} \right]^2 . \quad (62)$$

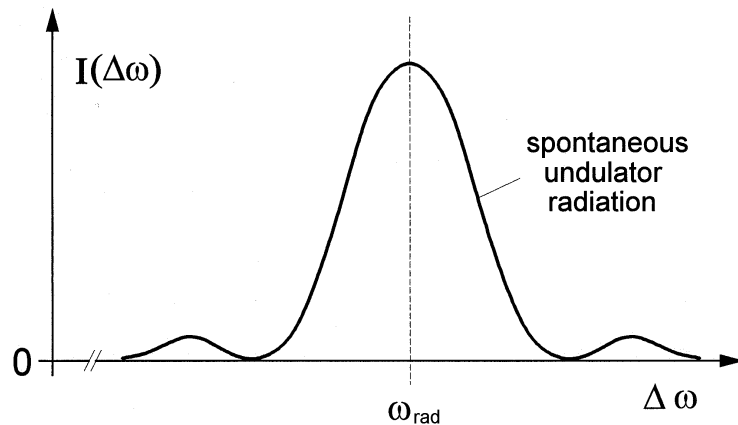


Fig. 15 Spectrum of the spontaneous undulator radiation

The full width half maximum (FWHM) of the spectral line can be found satisfying the condition

$$\left(\frac{\sin x}{x} \right) = \frac{1}{2} \quad \text{with} \quad x = \pi N_u \frac{\Delta \omega}{\omega_{rad}}. \quad (63)$$

The solution is $x = 1.392$ and we get

$$\frac{2\Delta \omega}{\omega_{rad}} = \frac{2x}{\pi N_u} = \frac{0.886}{N_u} \approx \frac{1}{N_u}. \quad (64)$$

The resolution of the undulator radiation is simply determined by the number of undulator periods. High resolution requires a sufficient large number N_u . Normally one can not observe these sharp spectral lines. There are higher harmonics and a frequency shift due to Doppler effect. A typical undulator spectrum is shown in Fig. 16.

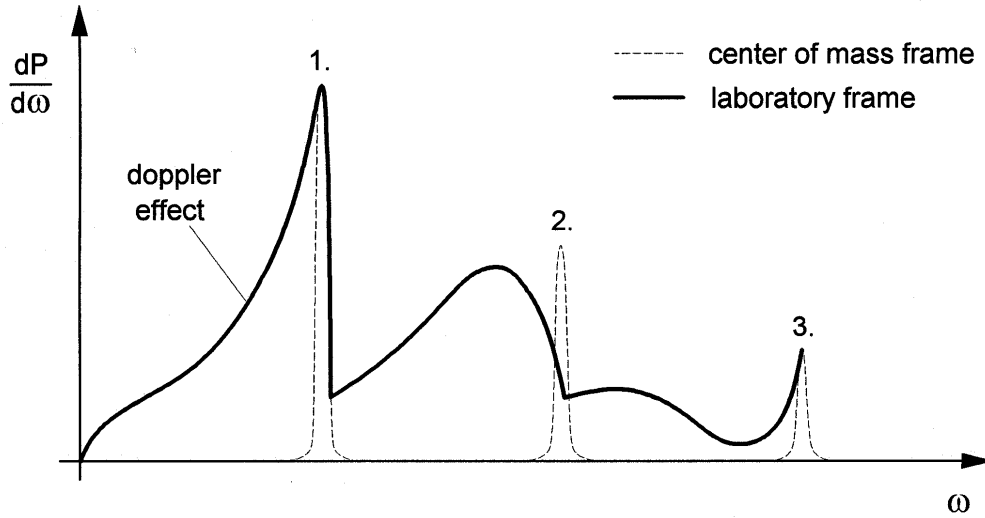


Fig. 16 A typical undulator spectrum

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INTRODUCTION TO CURRENT AND BRIGHTNESS LIMITS

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Abstract

Fundamental features of the measurement and description of beam current in synchrotron radiation sources are presented and examples given for a range of different accelerators. From a simple treatment of the electric field surrounding a moving charge the concept of the impedance of beam vacuum chamber is introduced. The interaction of the beam current with this impedance is described, as are some mechanisms which can lead to current limitations. The brightness of a source of synchrotron radiation is defined and worked examples are given for the ESRF. Brightness limits are briefly described in outline.

1. INTRODUCTION

Current and brightness limits are potentially serious concerns for synchrotron light sources because in effect they can impose a limit on the usefulness of the source and the experimental science programmes they can cover. As will be seen, the expression for brightness contains the beam current so any limit in current automatically limits the brightness.

We will present some very simple ideas relating to beam current in electron storage rings and give an impression of the magnitudes involved by taking some examples from a range of different storage rings. Using simple expressions it will be demonstrated how the electro-magnetic fields associated with beam currents can significantly interact with the structures surrounding the beam to produce effects which can limit the current. Then the basic equations for the brightness of synchrotron light will be stated and the factors which may limit the brightness will be explained. The brightness of two different types of source will be evaluated (for ESRF dipoles and undulators) to illustrate these factors.

2. BEAM CURRENT MEASUREMENT AND TYPICAL VALUES

The beam current in an accelerator is defined in the same way as a normal electric current, that is, the rate at which charge passes a fixed point. It can be measured by two different techniques, as shown in Fig. 1. Notice that the two methods will give greatly differing results for the same number of particles (see below).

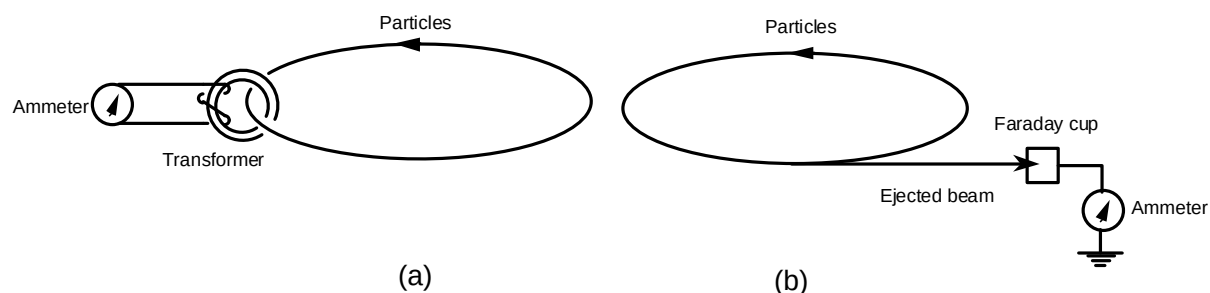


Fig. 1

- (a) DC current transformer (this measures the magnetic field produced by the current) [1]
- (b) Faraday cup (this measures the total beam charge and needs the beam to be ejected) [2]

If there are N Particles of charge q moving with velocity v in a circular accelerator of circumference $2\pi R$, the current transformer will indicate a circulating beam current I_0

$$I_0 = \frac{Nqv}{2\pi R}$$

If we restrict our view to synchrotron radiation sources with electrons at relativistic ($v = c$) velocity this expression becomes

$$I_0 = \frac{Nec}{2\pi R} \text{ or } = Nef_0$$

where e is the electronic charge and f_0 is the orbit frequency.

Note that if the same beam were ejected into a Faraday Cup it would indicate

$$I_{fc} = Nef_{rep}$$

where f_{rep} is the repetition rate of the accelerator.

$$\frac{I_0}{I_{fc}} = \frac{f_0}{f_{rep}}$$

This ratio is typically a very large number since f_0 is usually MHz and f_{rep} is Hz, so that in the example of the ISIS [3] spallation source accelerator where a high current of 800 MeV protons is ejected at 50 Hz, the Faraday cup indicates an average current of 200 μ A whereas a current transformer would show 6 A for the circulating beam current.

The circulating beam current is determined both by the number of electrons and by the circumference, as is seen in these examples:-

	$2\pi R(\text{m})$	$I_0 (\text{mA})$	N
HELIOS [4]	9.6	300	$6.0 \cdot 10^{10}$
ESRF [5]	844	200	$3.5 \cdot 10^{12}$
LEP [6]	$26.7 \cdot 10^3$	$6 (8 \times \frac{3}{4})$	$3.3 \cdot 10^{12}$

The small circumference of HELIOS produces a relatively high current with few electrons, whilst even the large number of electrons in the large circumference of LEP does not produce a big current.

The beam current transformer measures the average current as if the electrons were uniformly spread around the circumference. In reality they are in a bunch train imposed by the Radio Frequency accelerating system. The harmonic number h is the maximum number of bunches around the circumference.

$$h = \frac{2\pi R}{\lambda_{rf}} = \frac{f_{rf}}{f_0}$$

The bunches typically have a gaussian structure in time, as shown in Fig. 2, where the length of the bunches is often described by the full width at half maximum (fwhm) of the peak. This may be conveniently expressed in terms of a fraction k of the period of the RF [7]

$$\text{fwhm} = 2.36\sigma_t = k \frac{1}{f_{\text{rf}}}$$

Normally k will have a value between 0.01 and 0.1, depending on the storage ring design.

From the well known properties of gaussians we can relate the peak current I_{pk} to the average I_0

$$\sqrt{2\pi}\sigma_t I_{\text{pk}} = \frac{I_0}{f_{\text{rf}}} = 0.94 \frac{I_0}{k}$$

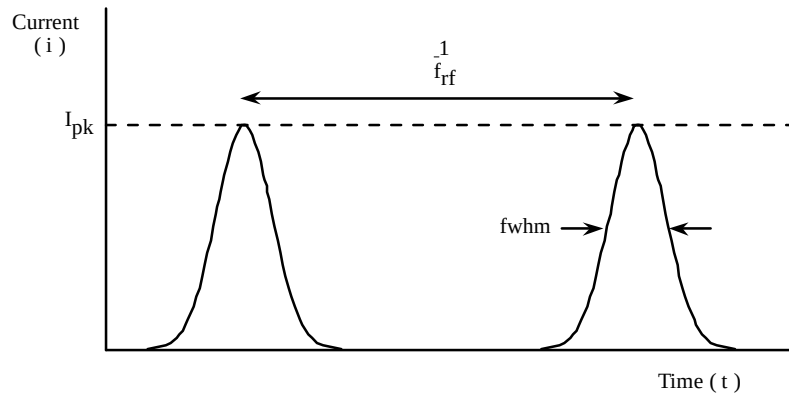


Fig. 2 Time structure of the bunches

With the value for k in storage rings typically being between 0.1 and 0.01 it can be easily seen that the peak current can be 10 to 100 times higher than the average current. Even higher peak currents are produced when electron storage rings operate, as is often the case, with the beam current concentrated in a single (or few) bunches. The current transformer will still register a reading $I_{0(\text{sb})}$ as though the electrons are spread around the circumference, but the peak current is increased by a factor of the harmonic number h because now:

$$\sqrt{2\pi}\sigma_t I_{\text{pk}} = \frac{I_{0(\text{sb})}}{f_0}$$

$$I_{\text{pk}} = 0.94h \frac{I_{0(\text{sb})}}{k}$$

Consider these examples of peak currents in storage rings operated in single or few bunch mode, from which it can be seen that peak currents can easily reach values of hundreds of amperes.

	$I_{0(\text{sb})}(\text{mA})$	k	f_0	h	N	$I_{\text{pk}}(\text{A})$
SRS [8]	100	0.11	$3.1 \cdot 10^6$	160	$2.0 \cdot 10^{11}$	136
ESRF [9]	10	0.03	$3.5 \cdot 10^5$	992	$1.8 \cdot 10^{11}$	310
LEP [10]	$\frac{3}{4}$	0.025	$1.1 \cdot 10^4$	31360	$4.3 \cdot 10^{11}$	885

It is not surprising that the electro-magnetic fields associated with these very high peak currents can produce effects which destabilise the electron bunches and produce current limitations.

3. FOURIER COMPONENTS OF THE BEAM CURRENT

Consider the time structure of the beam current in more detail. The train of bunches in the beam current, as shown in Fig. 2, can be expressed as a fourier series:

$$i(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

To allow the coefficients of this series to be estimated by inspection, a simplifying assumption can be made that the bunches are rectangular in time with length k/f_{rf} and height I_{pk} and by setting the origin of the time axis at the centre of a bunch, all the coefficients b_n are zero. The interval between bunches $1/f$ is (yet) an unspecified number of rf periods.

The coefficients of the series are evaluated as

$$a_n = \frac{1}{\pi} \int_0^{2\pi} i(t) \cos n\omega t. d(\omega t)$$

and describe the amplitudes of the different frequencies present in the beam current. We evaluate a_0 with $n = 0$ thus

$$\begin{aligned} a_0 &\approx \frac{1}{\pi} I_{pk} 2\pi f \frac{k}{f_{rf}} \\ &\approx 2I_{pk} k \frac{f}{f_{rf}} \\ &\approx 2I_0 \end{aligned}$$

and $a_0/2$ gives the expected amplitude for the steady component of the beam current.

The question "Which value of n gives a zero for the coefficient a_n ?" can be answered approximately by inspecting the product of the beam current function and the cosine term in the integral. It will give an indication of the highest frequency component present. The coefficient will be zero when

$$\begin{aligned} n\omega \frac{k}{2f_{rf}} &= \pi \\ n &= \frac{1}{k} \frac{f_{rf}}{f} \end{aligned}$$

The value of f is determined by the interbunch spacing and we can consider two limiting situations.

In multibunch	$f = f_{rf}$
In singlebunch	$f = f_0$

But in both these cases the actual frequency nf at which the coefficient goes to zero is the same, namely

$$= \frac{f_{rf}}{k}$$

The beam current spectrum is therefore a series extending to at least 10 to 100 times the radio frequency. In multibunch the series ($n = 1/k$) is at intervals of f_{rf} but in singlebunch there are many more harmonics ($n = h/k$), at intervals of the orbit frequency f_0 .

Some limitations of beam current and brightness arise from the interaction between the components of this spectrum and the accelerator environment in which the beam current circulates.

4. FIELDS OF RELATIVISTIC ELECTRONS

Consider the fields surrounding an electron. If the electron is stationary there is only a uniformly symmetric electric field shown in Fig. 3. At a radius r the magnitude of the electric field can quickly be calculated using Gauss' Law

$$\oint E \cdot ds = \frac{e}{\epsilon_0} \quad \epsilon_0 \text{ is the permittivity of free space}$$

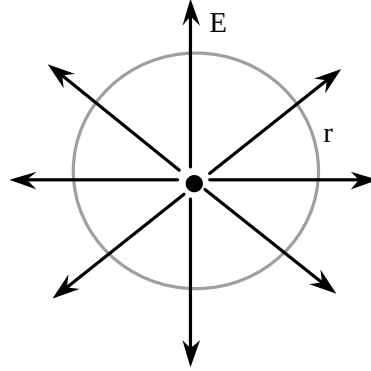


Fig. 3 Electric field surrounding a stationary electron

The integral is made over the surface of a sphere at radius r , thus

$$E \cdot 4\pi r^2 = \frac{e}{\epsilon_0}$$

$$E = \frac{1}{4\pi\epsilon_0} \frac{e}{r^2}$$

If the electron is moving relativistically with respect to an observer the electric field suffers a relativistic contraction along the line of motion. For an electron moving with velocity βc with $\gamma = \frac{1}{\sqrt{1 - \beta^2}}$ the electric field seen by the observer at rest is as shown in Fig. 4.

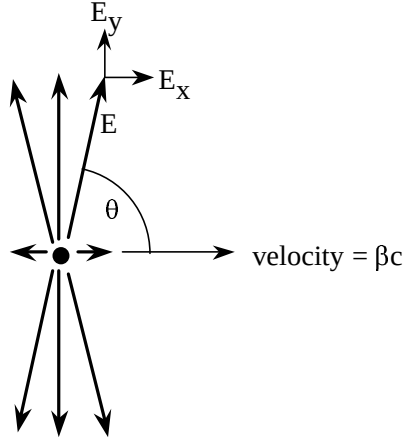


Fig. 4 Electric field of a relativistic electron

The x direction is along the axis of the electron's motion, while the y direction is perpendicular to it. The general expressions for the electric field components are [11]

$$E_x = \frac{\gamma \cos \theta}{(1 + (\gamma^2 - 1) \cos^2 \theta)^{3/2}} \cdot \frac{e}{4\pi\epsilon_0 r^2}$$

$$E_y = \frac{\gamma \sin \theta}{(1 + (\gamma^2 - 1) \cos^2 \theta)^{3/2}} \cdot \frac{e}{4\pi\epsilon_0 r^2}$$

If these expressions are examined for two extreme values of θ

- Along the axis of motion at $\theta = 0$

$$E_y = 0, \quad E_x = \frac{1}{\gamma^2} \frac{e}{4\pi\epsilon_0 r^2}$$

- Within a small angle to the normal to the axis of motion at $\theta = \frac{\pi}{2} - \frac{1}{\gamma}$

$$E_y = \frac{\gamma}{2^{3/2}} \cdot \frac{e}{4\pi\epsilon_0 r^2} \quad E_x = \frac{1}{2^{3/2}} \cdot \frac{e}{4\pi\epsilon_0 r^2}$$

Because γ has a large value for relativistic electrons, it can be seen that the only significant field component is within $\pm \frac{1}{\gamma}$ of the normal to the electron motion. Thus the electric field is concentrated into a disc of angular thickness $\pm \frac{1}{\gamma}$ at right angles to the electron's motion as shown in Fig. 5

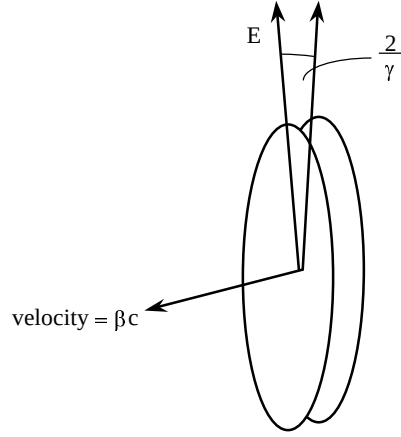


Fig. 5 The electric field of a relativistic electron is concentrated into a disc at right angles to the velocity

Using this concept of the field concentrated into a disc we can quickly confirm (approximately) the result above by using Gauss' Law over the edge of the disc

$$\oint E ds \approx E_y 2\pi \frac{2r}{\gamma} = \frac{e}{\epsilon_0}$$

$$E_y = \gamma \frac{e}{4\pi\epsilon_0 r^2} \quad (\text{compare to the exact treatment above}).$$

The moving charge also appears as a current i to the stationary observer. If we consider this current to be the passage of charge e in the time $\frac{2r}{\gamma c}$ which the disc edge takes to pass the observer, then

$$i = \frac{e\gamma c}{2r}$$

Using Ampère's law for the magnetic field surrounding a current

$$\oint B dl = \mu_0 i \quad \mu_0 \text{ is the permeability of free space}$$

$$B 2\pi r = \mu_0 \frac{e\gamma c}{2r}$$

$$B = \frac{\gamma\mu_0}{4\pi r^2} ec$$

Since $\epsilon_0\mu_0 = \frac{1}{c^2}$, we have the expected relation between the magnetic and electric fields;

$$B = E/c$$

5. EFFECTS DUE TO THE VACUUM CHAMBER WALLS

It has been shown that a relativistic electron has its electric field concentrated into a disc. When the electron is moving within the conducting vacuum chamber of an accelerator

the disc must terminate at the chamber wall on an opposite sign image charge moving along with the electron.

If the walls are not perfectly conducting the image charge will dissipate energy in the walls; this is a resistive wall effect. However, the energy dissipation is usually negligible. For example, a 1000-m circumference accelerator with a vacuum chamber made of stainless steel would have a resistance of about $1\ \Omega$. Applying Ohm's Law to a 300 mA image current flowing in this resistance gives a dissipation of only 0.1 W. The high frequency components of the beam current see a higher resistance due to the skin effect but again the dissipation is usually not important.

The resistive loss in the chamber walls can be pictured as a longitudinal drag force on an electron bunch and as such does not cause any stability problems for the bunch. If the bunch is not aligned transversely with the axis of the vacuum chamber the transverse asymmetry of the electric field disc can be expected to produce a transverse force component and this can lead to an instability. This is called the transverse resistive wall instability [12].

In addition to resistance the vacuum chamber walls can show other [13] electrical impedance properties, that is, capacitance or inductance. These arise from changes in the geometrical shape of the vacuum chamber, as seen by the fields surrounding the beam. As the beam moves at relativistic velocity from a region with one shape of the chamber to another, both the fields and the image currents become distorted and can even be left in these shape changes 'ringing' behind the beam. These residual fields can then interact with the tail of the bunch, or with beam bunches which pass later (even with the same bunch on a later orbit), and are the principal reason for the existence of beam current and brightness limitations.

Even though the capacitances and inductances resulting from shape changes in the vacuum chamber are quite small (a 10-mm step in a 100-mm diameter pipe contributes an inductance of order $0.01\ \mu\text{H}$ [14]) it must be remembered that the peak current in the bunch may exceed hundreds of amperes. In the chamber impedance such a bunch can produce fields equivalent to hundreds of volts and this is sufficient to affect the stability of the bunches.

When the field from the head of a bunch affects the tail of the same bunch this is called a single-bunch effect. [15]

When the field of a bunch affects later bunches this is called a multibunch effect [15].

The observable effects may be:

1. Bunch position oscillations (transverse or longitudinal)
2. Bunch shape oscillations (“ ”)
3. Bunch size changes (“ ”)

All these effects, which originate in the electromagnetic fields generated by the beam in the vacuum chamber, may be described by a general impedance function [13]. This is usually expressed as an impedance containing two different contributions:

- I. A broadband impedance (hence low Q -factor) representing the total effect of the wall resistance (including skin effects) together with capacitive and inductive terms arising from all the geometrical effects of dimension changes, branches, tees, slots, etc.
- II. Several (perhaps many) narrow band (hence high Q -factor) impedances due to resonant structures in the chamber such as RF cavities.

The broadband resonator impedance is a shunt arrangement shown in Fig. 6 representing the longitudinal chamber impedance Z_{II} .

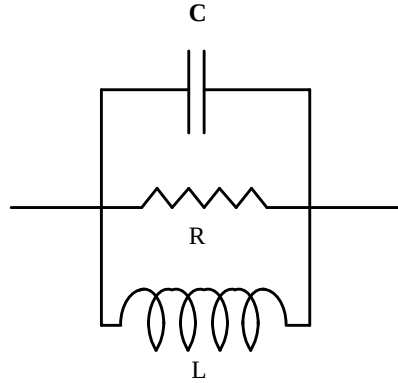


Fig. 6 Components of the broadband longitudinal chamber impedance Z_{II}

The most useful formula used is:

$$Z_{II} = \frac{R}{1 + jQ \left(\frac{\omega_{res}}{\omega} - \frac{\omega}{\omega_{res}} \right)}$$

$$\omega_{res} = \frac{1}{\sqrt{LC}}, \quad Q = R\sqrt{\frac{C}{L}}$$

Remembering that the beam spectrum extends up to many times the RF, and that at a frequency of several GHz most vacuum chambers start to propagate like a waveguide, it is to be expected that ω_{res} will be in the GHz region.

To obtain a suitably broadband response over this range Q must be of the order ~ 1 .

For $Q = 1$, Z_{II} has the following Real and Imaginary Parts, as shown in Fig. 7.

The important impedance parameter in single bunch instabilities is

$$\frac{Z_{II}}{n}, \quad \text{where } n = \frac{\omega}{\omega_o}$$

This effectively normalises between accelerators of different circumference. For modern light source accelerators which need to operate with good beam quality at several hundred mA it is generally accepted that the magnitude [16]

$$\frac{Z_{II}}{n} \leq 1 \, \Omega$$

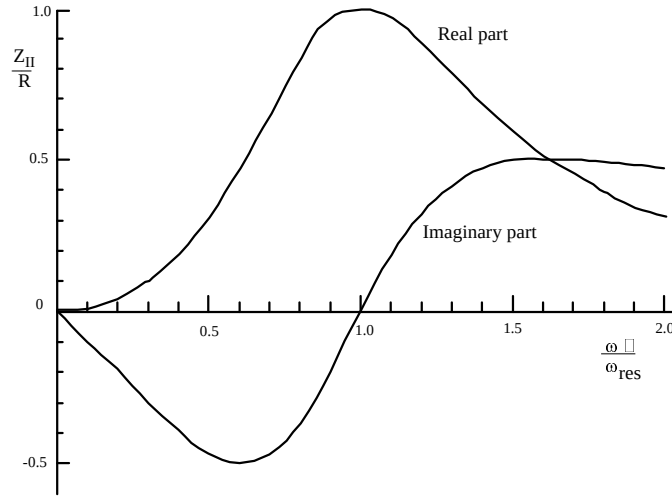


Fig. 7 Real and Imaginary components of the broadband impedance

Examples of measured $\frac{Z_{II}}{n}$ in light sources:

SRS $1.8 \pm 0.6 \cdot$	[8]
ALS $0.2 \cdot$	[17]
LEP $0.25-0.5 \cdot$	[18]

It is intuitively apparent that a beam moving off centre down the vacuum chamber will generate different fields than a beam which is centred. This must imply that there is a transverse component $Z_{\perp}$ to the chamber impedance, although it is closely related to the longitudinal impedance Z_{II} .

The generally used approximate relationship between them is [19]:

$$Z_{\perp} = \frac{Z_{II}}{n} \frac{2R}{b^2}$$

where R is the accelerator average radius and b is the radius of the vacuum chamber aperture. Note that the dimensions of $Z_{\perp}$ are Ω/m .

The mathematical treatment of instabilities is well beyond the scope of this paper. The concepts which have already been introduced, however, can be appreciated for the role that they play in the behaviour of instabilities.

An expression for the growth rate of some coordinate of a theoretical test Particle within the beam can be derived [19], which is related to the beam current and the chamber impedance. An instability will not necessarily develop on all occasions because there are often damping mechanisms (by synchrotron radiation emission [20] or by Landau damping [21], for example) which stifle the instability because the damping rate exceeds the growth rate. But since the growth rate is related to the beam current, above some threshold current the instability will grow. Even then the growth may stabilise at a new equilibrium because of frequency shifts with amplitude and Landau damping.

Thus below the instability threshold current the beam will exhibit its theoretical dimensions, whereas above the threshold the beam may grow to an increased size, or oscillate, or suffer an intensity reduction until it restabilises below the threshold. All these behaviours are obviously important to users of synchrotron radiation facilities.

6. BRIGHTNESS OF A SYNCHROTRON RADIATION SOURCE

The usefulness of a synchrotron radiation source may be judged by an experimenter primarily in terms of how many photons per second can be directed onto the sample. The relevant factors which influence this are the emission properties of the source and the acceptance parameters of the beamline, the optics, the detectors and the sample itself. Ideally there should be a good match between the emission and the acceptance.

The long established radiometric parameter which describes this emission property of a source of radiation is called the Brightness, although the term Radiance or Illuminance may sometimes be used [22]. Brightness is defined as the radiated flux per unit area of the source and per unit solid angle of emission.

$$\text{Brightness} = \frac{d^4 F}{dx dz d\theta d\phi} \text{ photons/sec/mm}^2/\text{mr}^2/0.1\% \text{ bandwidth}$$

where F is the radiated photon flux in a narrow 0.1% bandwidth, over which it can be assumed that the flux does not vary. The coordinates are as defined in Fig. 8.

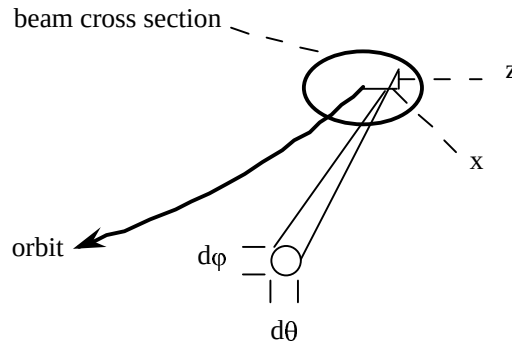


Fig. 8 Coordinates used in defining source properties

Notice that Brightness, as defined above, in Europe is sometimes referred to as Brilliance [23], with an accompanying incorrect use of the term brightness for the Spectral Flux Intensity. The author prefers to avoid Confusion by using the established radiometric definitions as given above, which is also the practice in the USA.

If F is integrated over all wavelengths a simple expression gives the total power in the synchrotron radiation spectrum:-

$$\text{Power(kW)} = \frac{88.5 E^4(\text{GeV}) I_0(\text{A})}{R(\text{m})}$$

where E is the electron energy, I_0 the beam current (average) and R the bending radius.

Examples:

	E (GeV)	I_0 (A)	R (m)	Power (kW)
HELIOS [24]	0.7	0.3	0.52	12.3
SRS [25]	2.0	0.3	5.55	76.5
ESRF [26]	6.0	0.2	23.3	985
LEP [10]	50.0	0.006	3096	1072

The synchrotron radiation spectrum is described with reference to a characteristic (often called 'critical') wavelength λ_c , or photon energy ϵ_c

$$\lambda_c(\text{\AA}) = \frac{5.59 R(\text{m})}{E^3(\text{GeV})} = \frac{18.6}{B(\text{Tesla})E^2(\text{GeV})}$$

$$\varepsilon_c(\text{keV}) = \frac{12.39}{\lambda_c(\text{\AA})}$$

where B is the bending magnetic field.

Examples:

	E (GeV)	R (m)	B (T)	λ_c (Å)	ε_c (keV)
BESSY-I [27]	0.8	1.78	1.5	19.4	0.64
HELIOS [28]	0.7	0.52	4.5	8.5	1.5
SRS [25]	2.0	5.55	1.2	3.9	3.2
ESRF [29]	6.0	23.3	0.86	0.60	20.7
LEP [30]	50.0	3096	0.054	0.14	88.5

When the radiation at a given wavelength is integrated over all angles of vertical emission the resultant Spectral Flux Density is given by

$$\frac{dF}{d\theta} = 2.46 \cdot 10^{13} I_0(\text{A}) E(\text{GeV}) \left(\frac{\lambda}{\lambda_c} \right)^2 G \left(\frac{\lambda}{\lambda_c} \right) \text{ photons/sec/mr/0.1\% bandwidth}$$

where $G \left(\frac{\lambda}{\lambda_c} \right)$ [31] is a numerical factor which essentially governs the shape of the spectrum.

Note that source Brightness as defined above is a function whose value depends on the source density distribution and on the observation angle. It is often more convenient to use, as a figure of merit, an average brightness which for dipole sources is defined [32]:

$$\text{Average Brightness} = \frac{\frac{dF}{d\theta}}{2.36\sigma_x 2.36\sigma_z 2.36\sigma'_\gamma}$$

where $dF/d\theta$ is the vertically integrated flux, $2.36\sigma_x$ is the fwhm of the horizontal electron beam size, $2.36\sigma_z$ is the fwhm of the vertical electron beam size, and $2.36\sigma'_\gamma$ is the fwhm of the photon emission angle in the vertical plane. The latter is a combination of the electron beam vertical divergence and the photon emission angle thus

$$\sigma'_\gamma = \sqrt{\sigma_z'^2 + 0.41 \left(\frac{\lambda}{\lambda_c} \right) \frac{1}{\gamma^2}}$$

As an example let us calculate the Average Brightness of a dipole magnet in the ESRF [23]

General data	Source point data	Electron beam data	At $\lambda = \lambda_c$
$E = 6 \text{ GeV}$ $I_0 = 200 \text{ mA}$ $\varepsilon_x = 4 \cdot 10^{-9} \text{ m.rads}$ $\varepsilon_z = 4 \cdot 10^{-11} \text{ m.rads}$	$\beta_x = 1.0 \text{ m}$ $\beta_z = 25.0 \text{ m}$ $\eta = 0.03 \text{ m}$	$\sigma_x = 0.07 \text{ mm}$ $\sigma_z = 0.032 \text{ mm}$ $\sigma'_z = 0.0013 \text{ mr}$	$\lambda_c = 0.6 \text{ \AA}$ $\sigma'_\gamma = 0.055 \text{ mr}$

$\sigma_E = 10^{-3}$			

$$\begin{aligned}\frac{dF}{d\theta} &= 2.46 \cdot 10^{13} \cdot 0.2 \cdot 6 \cdot 0.65 \quad \text{at } \lambda = \lambda_c \\ &= 1.9 \cdot 10^{13} \text{ photons/sec/mr/0.1\% band}\end{aligned}$$

$$\begin{aligned}\text{Dipole Average Brightness} &= \frac{1.9 \cdot 10^{13}}{2.36^3 \cdot 0.07 \cdot 0.032 \cdot 0.055} \\ &= 1.2 \cdot 10^{16} \text{ photons/sec/mm}^2/\text{mr}^2/0.1\% \text{ band.}\end{aligned}$$

The brightness performance of an undulator is calculated slightly differently. The flux in the central cone of an undulator F_n at a specified wavelength is averaged over the emission angle of that cone to give the Average On-axis Brightness. Because of the usually very small source size and divergence in an undulator diffraction effects must be taken into account [33].

$$\text{Average On-axis Brightness} = \frac{F_n}{2.36^4 \sigma_{\gamma x} \sigma_{\gamma x}' \sigma_{\gamma z} \sigma_{\gamma z}'} \text{ photons/sec/mm}^2/\text{mr}^2/0.1\% \text{ band}$$

where $\sigma_{\gamma z}$, $\sigma_{\gamma z}'$ are the photon source sizes in both planes and $\sigma_{\gamma x}'$, $\sigma_{\gamma x}'$ are the photon source divergence in both planes, taking into account diffraction effects.

$$\begin{aligned}\sigma_{\gamma x,z} &= \sqrt{\sigma_{x,z}^2 + \sigma_{\gamma}^2} & \sigma_{\gamma} &= \frac{1}{4\pi} \sqrt{\lambda_n L} \\ \sigma_{\gamma x,z}' &= \sqrt{\sigma_{x,z}^{/2} + \sigma_{\gamma}^{/2}} & \sigma_{\gamma}' &= \sqrt{\frac{\lambda_n}{L}}\end{aligned}$$

where L = undulator length, λ_u = undulator period.

The standard expression for the radiation produced in the n^{th} harmonic in an undulator is

$$\lambda_n = \frac{\lambda_u}{n^2 \gamma^2} \left(1 + \frac{k^2}{2} \right)$$

where the deflection parameter $k = 93.4 \lambda_u(\text{m}) B_o(\text{Tesla})$ and the flux in the central cone is [34]

$$F_n = 1.43 \cdot 10^{14} \frac{L}{\lambda_u} I_0 Q_n(k) \text{ photons/sec/0.1\% bandwidth}$$

where

$$Q_n(k) = \left(1 + \frac{k^2}{2} \right) \frac{f_n}{n}$$

and f_n is a numerical factor, related to k [35].

As an example let us calculate the On-axis Brightness of an ESRF undulator [36]

General data	Source point data	Electron beam data	Photon data
$\lambda_u = 46 \text{ mm}$ $N = 36$	$\beta_x = 0.5 \text{ m}$ $\beta_z = 4.0 \text{ m}$	$\sigma_x = 0.06 \text{ mm}$ $\sigma_x' = 0.12 \text{ mr}$	$\sigma_{\gamma} = 0.0016 \text{ mm}$ $\sigma_{\gamma}' = 0.012 \text{ mr}$ $\sigma_{\gamma x} = 0.06 \text{ mm}$

$L = 1.66 \text{ m}$ $\gamma = 1.17 \cdot 10^4$	$\eta = 0.04 \text{ m}$	$\sigma_z = 0.013 \text{ mm}$ $\sigma'_z = 0.003 \text{ mr}$	$\sigma'_{\gamma_x} = 0.12 \text{ mr}$ $\sigma_{\gamma_z} = 0.013 \text{ mm}$ $\sigma'_{\gamma_z} = 0.012 \text{ mr}$

Consider $k = 1$, then the wavelength of the fundamental is

$$\lambda = \frac{46 \cdot 10^{-3}}{2(1.17 \cdot 10^4)^2} \left(1 + \frac{1}{2}\right) = 2.5 \cdot 10^{-10} \text{ m}$$

$$F_n = 1.43 \cdot 10^{14} \cdot 36 \cdot 0.2 \cdot \frac{3}{2} \cdot 0.37$$

$$= 5.71 \cdot 10^{14} \text{ photons / sec / 0.1\% band}$$

$$\text{On-axis Brightness} = \frac{5.71 \cdot 10^{14}}{2.36^4 \cdot 0.06 \cdot 0.013 \cdot 0.12 \cdot 0.012}$$

$$= 1.6 \cdot 10^{19} \text{ photons/sec/mm}^2/\text{mr}^2/0.1\% \text{ band} .$$

7. BRIGHTNESS LIMITATIONS

It has been shown that the brightness of a synchrotron radiation source depends on the beam current and the cross sectional dimensions of the beam itself. Both these quantities may be degraded by effects arising from the impedance behaviour of the beam vacuum chamber; the beam may be limited to a threshold current; or above it the beam dimensions may increase. The end result is to limit the brightness properties of the source.

These are what might be called technological limitations, which may be overcome by appropriate designs to ensure a suitably low impedance. Alternatively, feedback systems may be employed to counteract specific instabilities.

But as the technological design of synchrotron sources moves steadily towards the achievement of yet smaller beam dimensions, the ultimate limit will then be reached when, no matter how small the beam is, diffraction effects predominate. Until that happy day for experimentalists is reached, there remains much to be done by accelerator theorists, scientists and technicians.

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LATTICES AND EMITTANCES

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Abstract

The new generation of synchrotron light sources is characterised by an extremely high brilliance synchrotron radiation achieved in low-emittance storage rings specially designed to accommodate insertion devices. The basic principles guiding the minimisation of the emittance are recalled. The magnet lattice configurations that are currently studied as candidates for these storage rings (DFA, TBA, FODO...) are described and the problems associated with these low-emittance lattices and the required strong focusing discussed. Finally the effects of insertion devices on beam dynamics as well as on emittances and energy spread are recalled.

1. FROM HIGH BRILLIANCE TO LOW EMITTANCE

High-brilliance synchrotron radiation can be obtained by utilising insertion devices in low-emittance storage rings. A small beam emittance is also very important for storage ring based FELs where additional requirements must also be fulfilled. In this lecture, emphasis will be placed on storage-ring lattices specially designed to maximise the brilliance and achieve the full potential from insertion devices.

Brilliance of X-rays has increased by many orders of magnitude since the advent of synchrotron radiation sources as shown in Fig. 1.

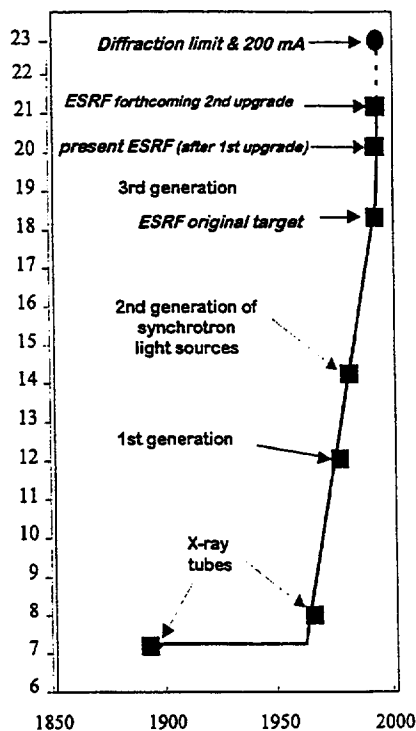


Fig. 1 Increase of brilliance over the last decades

The photon flux is an important source characteristic. Nevertheless, for a number of experiments in which the beam is focused at the sample, the true figure of merit is the spectral brilliance B (called, in short, brilliance) which is defined as the photon flux per unit solid angle and per unit source area, emitted in a relative bandwidth. Brilliance may be expressed as:

$$B \propto \frac{I}{\sigma_x \sigma'_x \sigma_y \sigma'_y} f(E, gap) \quad (1)$$

with:

I = beam intensity

E = machine energy

σ, σ' = photon beam size and divergence (x stands for the horizontal plane and y for the vertical plane)

Brilliance is generally quoted in units of:

$$\frac{\text{photons per second}}{(\text{mrad})^2 (\text{mm})^2 0.1\% \text{ bandwidth}}$$

It can be seen from Eq. (1) that the brilliance is inversely proportional to the product of transverse photon beam sizes and divergences resulting from the convolution of two contributions:

i) the electron beam emittances $\varepsilon_{E,x,y}$

$$\sigma_{E,x,y} = \sqrt{\beta_{E,x,y} \varepsilon_{E,x,y}} \quad \sigma'_{E,x,y} = \sqrt{\frac{\varepsilon_{E,x,y}}{\beta_{E,x,y}}} \quad (2)$$

where $\beta_{E,x,y}$ are the betatron functions

ii) the photon beam emittance associated with the single electron emission $\varepsilon_R = \frac{\lambda}{2\pi}$

$$\sigma_R = \frac{1}{\pi} \sqrt{\frac{\lambda L}{2}} \quad \sigma'_R = \sqrt{\frac{\lambda}{2L}} \quad (3)$$

where λ is the wavelength of the radiation and L the undulator length.

The total photon beam sizes and divergences are given by:

$$\sigma_{x,y} = \sqrt{\sigma_{E,x,y}^2 + \sigma_R^2} \quad \sigma'_{x,y} = \sqrt{\sigma_{E,x,y}'^2 + \sigma_R'^2} \quad (4)$$

Figure 2 illustrates the sharing of the two contributions in the case of the ESRF for two different values of the vertical emittance. The smaller the emittance of the electron beam, the higher the brilliance of the produced radiation. As soon as the brilliance is no longer dictated by the emittances of the electron beam, a better matching of the electron beam with respect to the photon beam must also be achieved. This calls for a high flexibility of the lattice design, enabling the β functions to be varied in a large range.

Most existing rings have emittances of 100 nanometer-radians. The new generation of rings (third generation light sources) under early operation or in the construction or design stage has design values between 5 and 10 nanometer-radians. Yet the maximum level of brilliance is reached only at the so-called diffraction limit when the beam emittance is about equal to a fraction of the wavelength of the radiation. This sets the scale for the ideally desirable beam emittance as shown in Table 1.

Table 1
Diffraction limit emittances

Source type	Energy (GeV)	λ (Å)	E_C (keV)	ε_R (nm.rad)
Hard X-ray	6	0.88	14	0.014
VUV	2	78	1.6	0.2

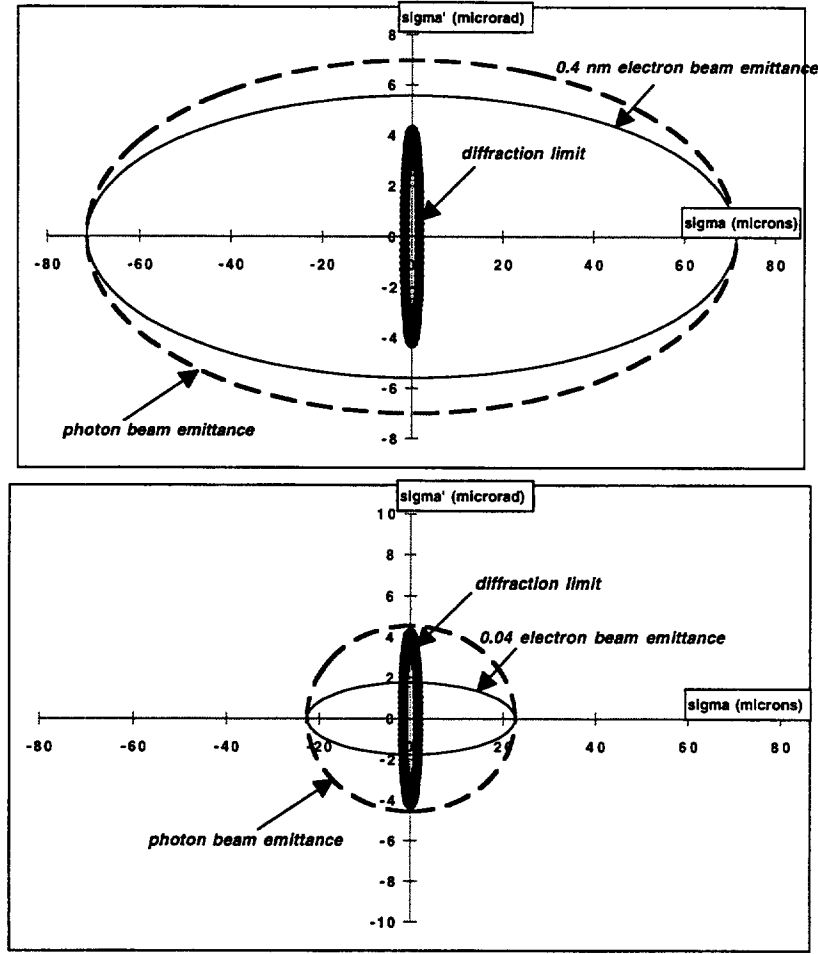


Fig. 2 Effective photon-beam emittances

2. LOW-EMITTANCE LATTICES

2.1 Equilibrium emittance

The particle beam emittance in an electron storage ring is determined by an equilibrium between quantum excitation that causes individual particles to oscillate transversally and damping of the betatron oscillations.

In an undistorted machine, radiation is produced mainly in the normal bending magnets. As a particle emits a photon, it loses energy and this results in a sudden change of its equilibrium orbit which causes a corresponding increase of the betatron oscillation amplitude about the new equilibrium orbit (Fig. 3). This process of continuous increase of the beam emittance is compensated by the damping provided by the RF accelerating system which restores the energy lost by synchrotron radiation only in the longitudinal direction.

The resulting horizontal equilibrium emittance is given by the expression [1]:

$$\epsilon_{x0} = \frac{C_q \gamma^2 \langle H / \rho^3 \rangle}{J_x \langle 1 / \rho^2 \rangle} \quad (5)$$

where $\langle \rangle$ means the average around the storage ring and with:

ρ = bending radius

γ = total energy in mc^2 units

J_x is the horizontal damping partition number

$$C_q = \frac{55}{32\sqrt{3}} \frac{h}{2\pi mc} = 3.8410^{-13} m$$

and H is the Courant-Snyder dispersion invariant which is specified by the properties of the guide field. It is given by:

$$H = \gamma_x \eta^2 + 2\alpha_x \eta \eta' + \beta_x \eta'^2 \quad (6)$$

where α_x , β_x , γ_x are the Twiss coefficients and η , η' are the dispersion function and first derivative respectively.

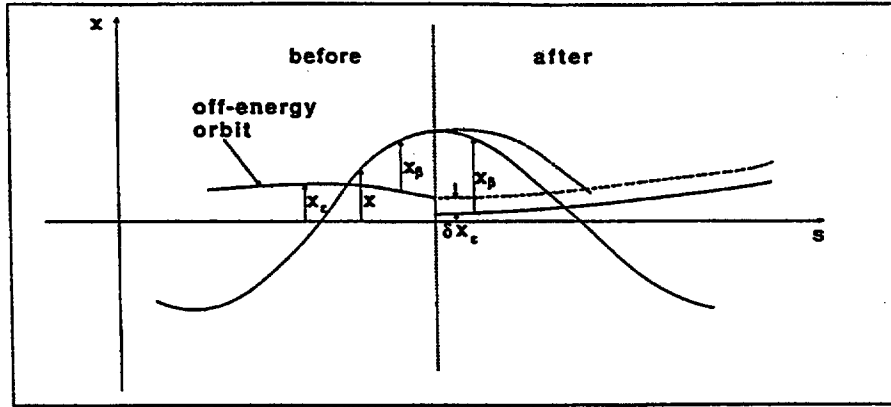


Fig. 3 Effect of energy loss on betatron oscillations

For an isomagnetic guide field ($\rho = \rho_0 = \text{Constant}$ in magnets, $\rho = \infty$ elsewhere), Eq. (5) becomes:

$$\varepsilon_{x_0} = \frac{C_q \gamma^2 \langle H \rangle_{mag}}{J_x \rho_0} \quad (7)$$

where $\langle H \rangle_{mag}$ is the average of H taken only in the magnets. That is:

$$\langle H \rangle_{mag} = \frac{1}{2\pi\rho_0} \int [\gamma_x \eta^2 + 2\alpha_x \eta \eta' + \beta_x \eta'^2] ds \quad (8)$$

In the vertical plane, there is radiation damping but ideally no dispersion and hence no quantum excitation, so that in a perfect machine the vertical emittance is zero. The vertical emittance is determined only by coupling between the horizontal and vertical planes due to magnet imperfections and misalignments. The coupling is generally expressed by a constant κ which describes the sharing of the natural emittance ε_{x_0} between the two planes:

$$\varepsilon_x = \frac{1}{1 + \kappa} \varepsilon_{x_0} \quad (9)$$

$$\varepsilon_y = \frac{\kappa}{1 + \kappa} \varepsilon_{x_0}$$

where ε_x and ε_y are the effective emittances. Typically the natural coupling is of the order of a few %. It can be controlled by means of skew quadrupoles. Correction algorithms are usually applied to reduce the initial figure and increase the brilliance. As an example, the ESRF is routinely operated with a 0.4 % coupling, as compared to the design 10 % value.

2.2 Minimisation of emittance

From Eq. (7), it is clear that minimisation of emittance, assuming in a first approach J_x to be constant, is achieved by the minimisation of the dispersion invariant function. This means that in the bending magnets, where the photons are emitted, the dispersion function η must be low and the β function optimised.

For a single magnet with length L and bending radius ρ , the integral for deriving the average value of H is given by:

$$I = \int_0^L [\gamma\eta^2 + 2\alpha\eta\eta' + \beta\eta'^2] ds \quad (10)$$

Knowing the optical functions at the entrance of the bending magnet (referred by the index 0) and their transfer through the magnet, I can easily be analytically developed. A simplified expression is obtained by approximating the resulting expression to second order in L/ρ . One gets:

$$I \approx (\gamma_0\eta_0^2 + 2\alpha_0\eta_0\eta'_0 + \eta_0'^2)L + (\alpha_0\eta_0 + \beta_0\eta'_0)\frac{L^2}{\rho} - (\gamma_0\eta_0 + \alpha_0\eta'_0)\frac{L^3}{3\rho} + \left[\frac{\beta_0 L}{3} - \frac{\alpha_0 L^2}{4} + \frac{\gamma_0 L^3}{20} \right] \frac{L^2}{\rho^2} \quad (11)$$

Optimum values of the optical functions at the beginning of the magnet can then be derived from Eq. (11), in order to achieve the minimum emittance. Figure 4 shows how the emittance changes with variations of α_0 and β_0 away from their optimum in the particular case of zero dispersion at one end of the magnet.

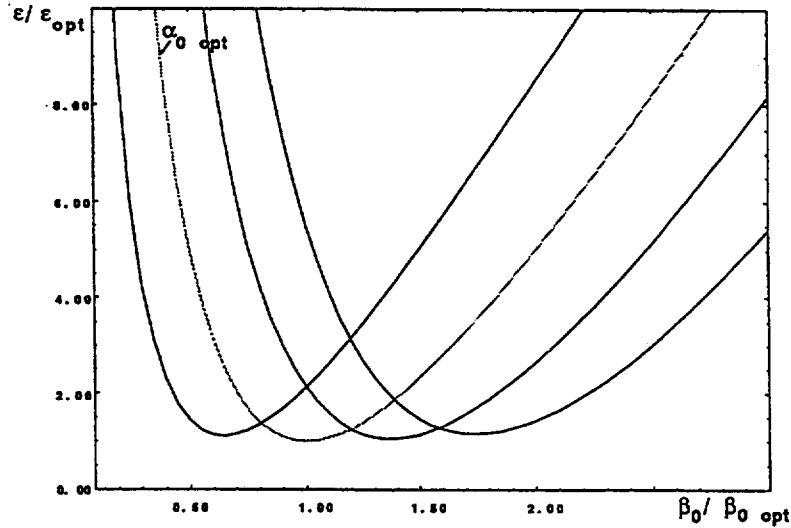


Fig. 4 Beam emittance as a function of initial betatron values

To minimise ε_{x0} , one can also take into account the variation of J_x . The horizontal partition number is given by [2]:

$$J_x = 1 - \frac{1}{2\pi\rho^2} \int_{mag} \eta(1-2n)ds \quad (12)$$

where the magnetic field index is defined by: $n = -\frac{\rho}{B} \frac{dB}{dx}$

In a machine with separated functions (i.e. zero field gradient in the dipoles), the quantity J_x is approximately equal to 1. J_x can be made larger by introducing vertical focusing in the dipoles but its maximum value is limited by the rule:

$$\sum J_i = J_x + J_y + J_z = 4 \quad (13)$$

which means, in the particular case of no vertical bend, $J_x + J_z = 3$, hence $J_x \leq 3$. In practice the maximum tolerable value is $J_x \leq 2$, which results potentially in an emittance reduction of a factor two.

Another possible way to achieve very low emittances is to install damping wigglers in dispersion-free straight sections. For instance, this method would reduce the emittance in PEP by almost one order of magnitude [3] with a total length of installed wiggler magnets of 200 m. This method is applicable to any storage ring but since the damping effect scales proportionally with the bending radius of the ring magnets, it is more efficient in large machines.

3. LATTICE TYPES

3.1 Generalities

The basic structure of low-emittance lattices consists of straight sections designed for the installation of insertion devices separated by arc sections. In the arcs, the focusing is chosen so as to minimise the chromatic invariant. Minima for H can be found for various types of lattices, involving proper choices for η_x , β_x , and their derivatives in the bending magnet.

A variety of types of lattices are available to design low emittance lattices. They can be divided into two classes:

- i) structures without bending magnets in the dispersive section: double focusing achromat (DFA) commonly known as the Chasman-Green lattice [4], expanded Chasman-Green, empty FODO, triplet achromat lattice (TAL).

The double focusing achromat lattice has been used for the NSLS rings in Brookhaven [5]. SUPERACO [6] is already operated with an expanded Chasman-Green structure. The expanded Chasman-Green structure is the basis of the conceptual designs of several third generation synchrotron radiation sources in operation : ESRF [7], APS [8], ELETTRA [9] or under construction or design stage: SPRING8 [10], BESSY2 [11], SOLEIL [12]. The triplet achromat lattice was used in ACO at Orsay [13]; it has the advantage of requiring the shortest circumference.

- ii) structures with additional bending magnets in the dispersive region: triplet bend achromat (TBA), FODO.

Triple bend structures are utilised in both the ALADDIN [14] and BESSY [15] synchrotron radiation sources. ALADDIN is a machine optimised primarily to use radiation from the dipole magnet and does not provide zero dispersion in the long straight sections. In the BESSY structure, significant focusing in the vertical plane is provided by the edge focusing of the dipoles. This focusing is necessary in order to limit the amplitude of the vertical beta function.

The triple bend achromat lattice is the logical extension of the DFA. Adding a third bending magnet within the arc section allows extra flexibility. This type of lattice is run at ALS [16], SRRRC [17], PLS [18]. It is also proposed for the DIAMOND project [19].

The FODO lattice is the most commonly used lattice for high energy physics storage rings because of its very compact structure. In contrast to the other types of lattices, the FODO does not naturally provide a space for insertion devices; special lattice sections must be introduced to achieve dispersion-free insertions. The damping rings at Stanford [20] and the SXRL storage ring [21] are good examples of the use of this structure.

3.2 The double focusing achromat

The double focusing achromat lattice or basic Chasman-Green represents the most compact of the structures used in low emittance storage rings. The basic scheme uses two dipole magnets with a focusing quadrupole between them (Fig. 5). The strength of the quadrupole is adjusted so that the dispersion generated by the first dipole is cancelled by passing through the second dipole.

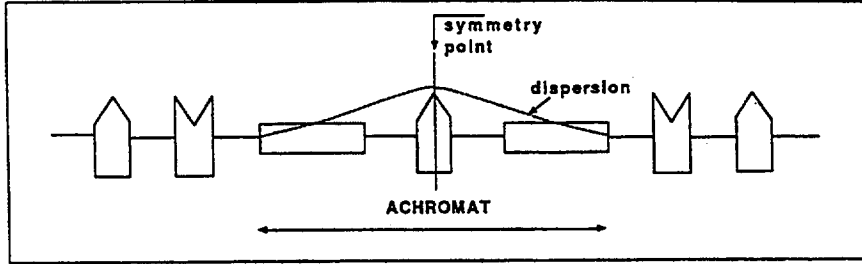


Fig. 5 Basic DFA structure

In this form, the structure is rather inflexible since the quadrupole does not provide focusing in both planes. Therefore, defocusing quadrupoles can be added upstream and downstream of the focusing quadrupole to restore focusing in both planes and to provide more flexibility in the adjustment of transverse dimensions (i.e. the ESRF with four quadrupoles or ELETTRA with three quadrupoles). This is the so-called expanded Chasman-Green achromat. The optical functions of these two lattices are shown in Figs. 6 and 7.

The emittance of a lattice built up of such cells has been computed by various authors [22, 23]. If the dispersion and its derivative are zero at the entrance of the bending magnet, the emittance is given by:

$$\varepsilon_x = \frac{C_q \gamma^2}{J_x} \theta^3 \left[\gamma_0 \frac{L}{20} + \beta_0 \frac{1}{3L} - \alpha_0 \frac{L}{4} \right] \quad (14)$$

where β_0 , γ_0 , α_0 are the Twiss parameters at the entrance of the bending magnet, L is the length of the magnet and θ is the bending angle.

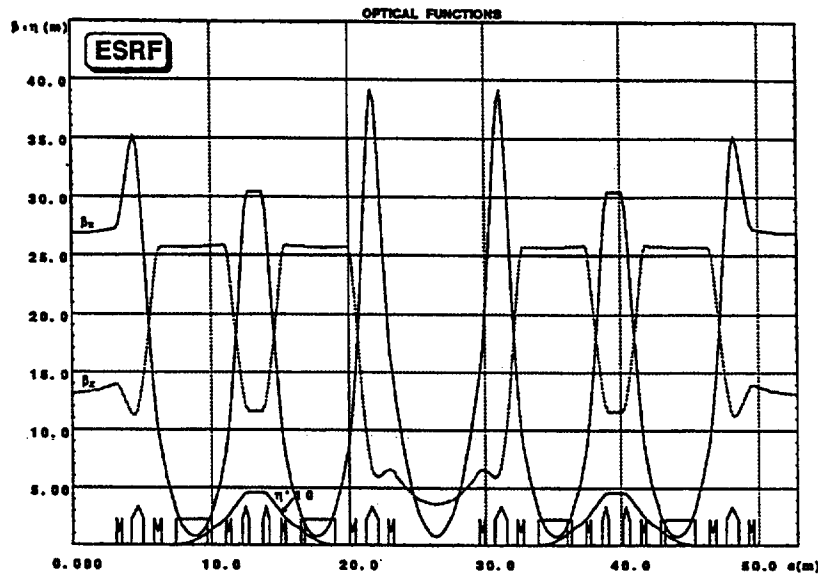


Fig. 6 ESRF lattice functions

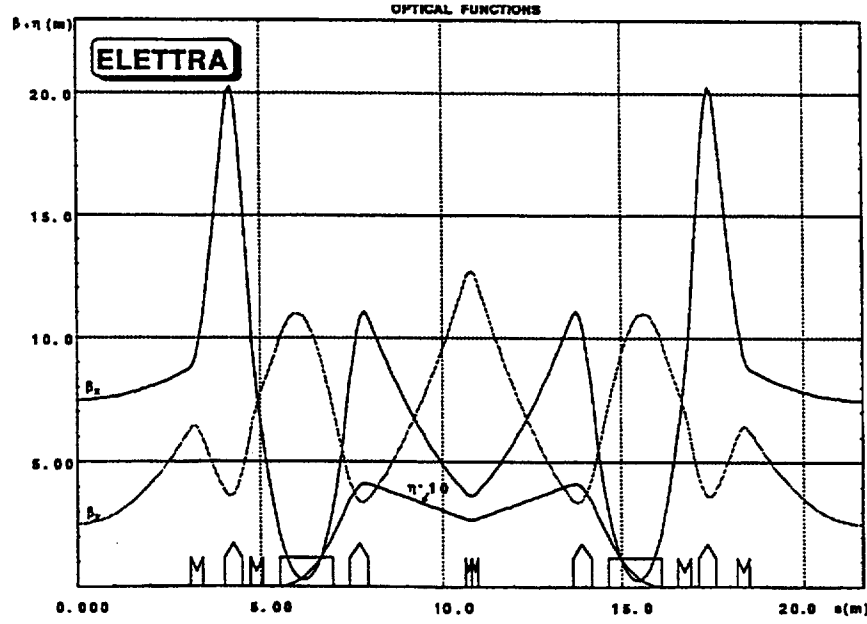


Fig. 7 ELETTRA lattice functions

The emittance can be minimised by a proper adjustment of the optical parameters at the entrance of the bending magnet. The minimum value is achieved when the minimum of the horizontal beta function within the dipole occurs at a distance $s = 3/8 L$ from the beginning of the magnet and the value of the minimum betatron function is :

$$\beta = L \frac{\sqrt{3}}{8\sqrt{5}} \quad (15)$$

It yields an emittance of:

$$\varepsilon_x = \frac{C_q \gamma^2}{J_x} \theta^3 \frac{1}{4\sqrt{15}} \quad (16)$$

It is difficult to reach the minimum emittance because the very low β value in the bending magnet ($\beta_{min} = 19$ cm for $L = 2$ m) generates unacceptably high β values in the focusing structure outside the bending magnet and leads to large chromaticities. Therefore, actual lattices result from a compromise with the needs of chromaticity correction and a more realistic estimate would be to increase the emittance by a factor two to three over the ideal minimum value.

The advantage of the DFA structure is to provide a large dispersion between the two bending magnets. As a consequence, sextupole strengths required for chromaticity correction, which are inversely proportional to the dispersion, are reduced. This is the reason why the expanded Chasman-Green is more suitable for large rings where the dispersion in the achromat becomes small. On the other hand, a clear disadvantage of such lattices is the fixed phase advance across the achromat which is approximately π . This places severe constraints on the horizontal betatron tune and limits the flexibility of the lattice since the β values in the insertions are coupled to the choice of tune.

Given the fact that the emittance is determined by the balance between radiation excitation and damping, it is possible to decrease the amplitude of the excitation by changing the distribution of the dispersion and optical functions inside the bending magnets [24]. Figure 8 schematically shows how this can be achieved by breaking the double achromatic condition. The hatched area indicates the relative magnitude of the excitation which can be made smaller in the non-achromatic case. Running a DBA with a distributed dispersion is now being applied on several machines. In the ESRF case, this strategy enabled almost a factor of 2 to be gained on the horizontal emittance from 7 nanometer-radians down to 4 nanometer-radians.

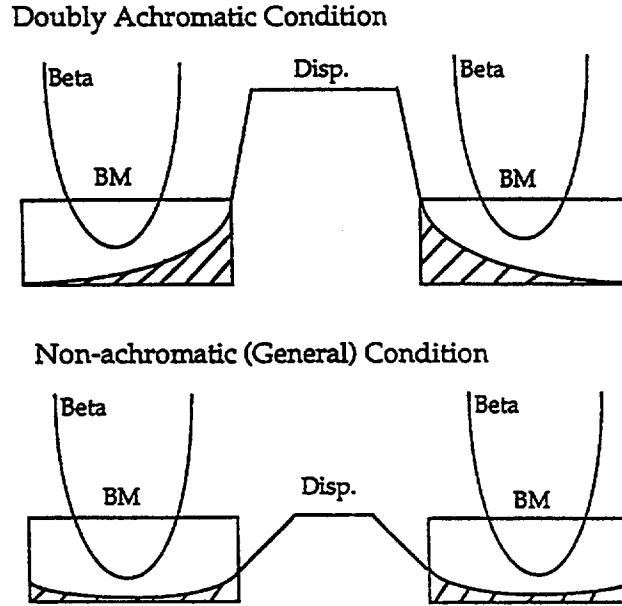


Fig. 8 Comparison of the doubly- and non-achromatic condition

3.3 Triplet achromat lattice

This lattice can be made very compact since there are no quadrupoles in the insertion straight sections, as shown in Fig. 9.

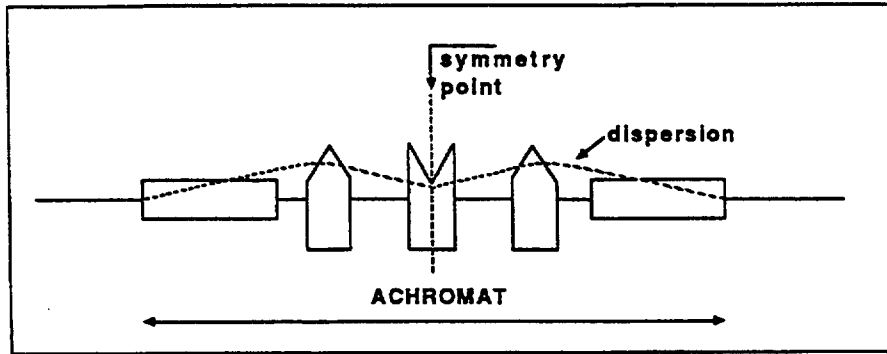


Fig. 9 Basic TAL structure

Using the same derivation as in the previous section, one can express the minimum emittance of the lattice in the form:

$$\varepsilon_x = \frac{C_q \gamma^2}{J_x} \theta^3 \frac{2}{3} \left(\frac{\beta}{L} \right)_{opt} \quad (17)$$

using the optimum value of β in the middle of an insertion section of length $2 L_i$ given by:

$$\left(\frac{\beta}{L} \right)_{opt}^2 = \frac{3}{4} \left[\frac{1}{5} + \frac{L_i}{L} + \frac{4}{3} \left(\frac{L_i}{L} \right)^2 \right] \quad (18)$$

The main disadvantage of the lattice is that the emittance depends on the value of the horizontal betatron function in the insertion region. Figure 10 shows the ACO lattice functions as an example of a triplet achromat structure.

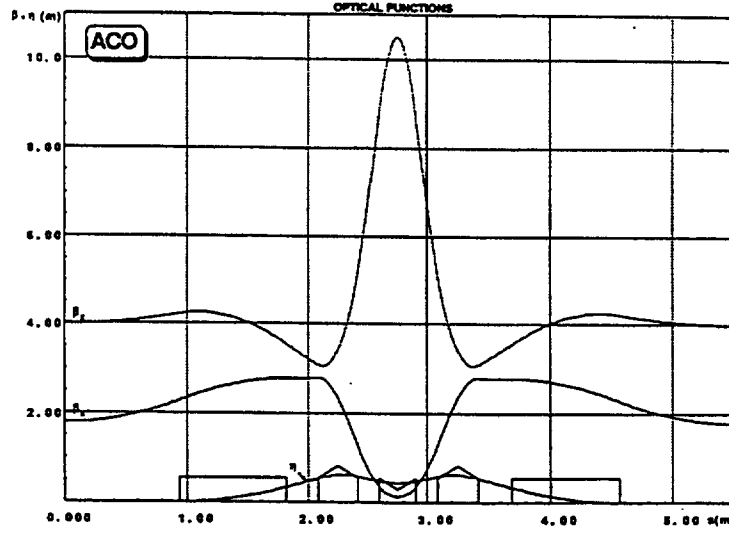


Fig. 10 ACO lattice functions

3.4 Triple-bend achromat

A unit cell of the basic TBA lattice with one horizontally focusing quadrupole in-between the bending magnets is shown in Fig. 11. The addition of an extra bending magnet within the achromat gives more flexibility for adjusting the phase advance across the achromat. The phase advance can be varied from π to 2π by changing the positions of the achromat quadrupoles and/or the length of the central magnet compared to the outer magnets [25]. A more practical way to obtain a "movable" quadrupole with fixed magnet geometry is to include a second family of quadrupoles in the achromat.

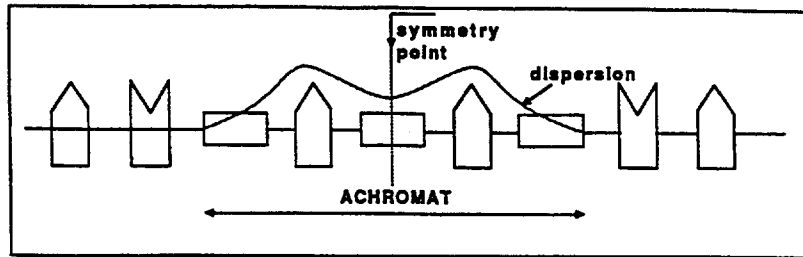


Fig. 11 Basic TBA structure

The emittance of the TBA lattice is obtained by summing the values of the driving integral $I = \int H ds$ over the two types of magnets. It can be expressed in the following form:

$$\epsilon_x = \frac{C_q \gamma^2}{J_x} \frac{\frac{I_1}{\rho_1^3} + \frac{2I_0}{\rho_0^3}}{\frac{\theta_1}{\rho_1} + \frac{2\theta_0}{\rho_0}} \quad (19)$$

where the index 0 refers to the outer magnet (radius ρ_0 and deflection angle θ_0) and the index 1 to the inner magnet.

Assuming equal bending magnet deflection, the minimum emittance is given by:

$$\epsilon_x = \frac{C_q \gamma^2}{J_x} \theta^3 \frac{7}{36\sqrt{15}} \quad (20)$$

It is obtained for β values in the outer bending magnets identical to those of the double focusing achromat structure and for $\beta_x = L/\sqrt{15}$ in the inner magnet. Even lower emittance values can be obtained by optimising the bending angle distribution. The optimum yields a structure in which the centre deflection angle is 1.5 times larger than the outer deflection angle.

To control the vertical beta function in the bending magnets, vertical focusing can be introduced either in the form of lumped quadrupoles adjacent to the bending magnet or by introducing a gradient into the bending magnet fields, as proposed in [26]. This also changes the damping partition numbers in such a way as to reduce the emittance. Such a solution with combined function magnets has been adopted in the ALS design (Fig. 12).

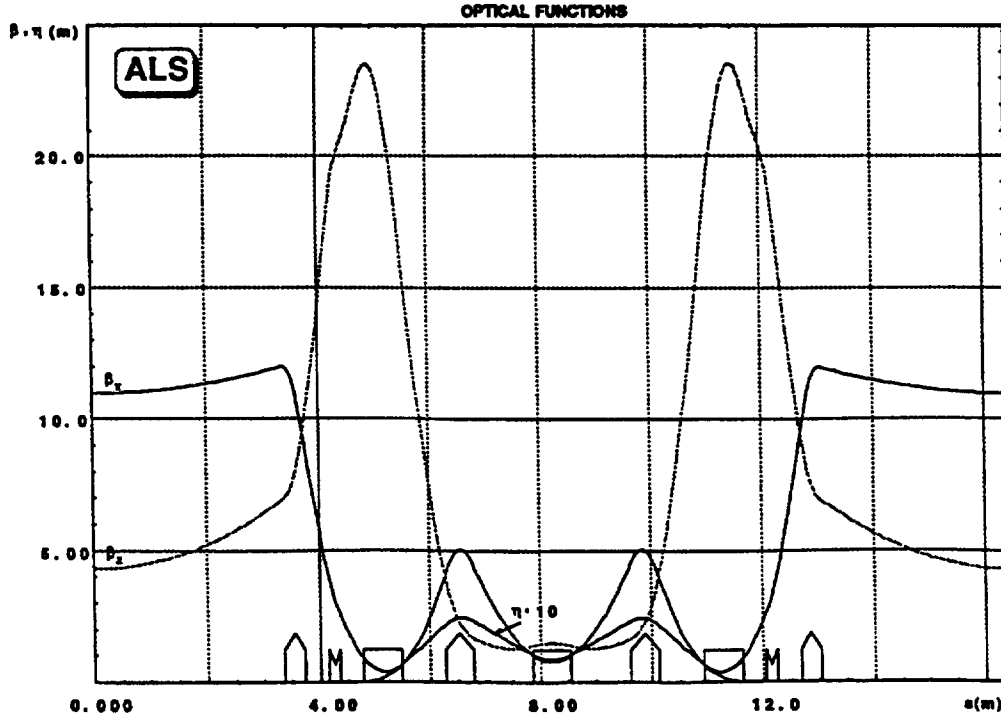


Fig. 12 ALS lattice functions

However, to keep the emittance small, the dispersion in the inner magnet must be reduced to small values which leads to large correcting sextupoles. When compared to the case of a double focusing achromat lattice with comparable emittance, these sextupoles need to be about two or three times stronger. Such structures are more suitable for small rings where the deflection angle per bending magnet, and correspondingly the dispersion, is larger.

3.5 FODO structures

As used in storage rings constructed for high energy physics, a FODO lattice consists of a sequence of alternating focusing and defocusing quadrupoles separated by bending magnets. The emittance of a regular FODO structure scales as [27]:

$$\epsilon_x = 4 \frac{C_q \gamma^2}{J_x} \theta^3 F(\mu_c) \quad (21)$$

where $F(\mu_c)$ is a function of the betatron phase advance per cell μ_c . The variation of $F(\mu_c)$ is plotted in Fig. 13. The minimum is reached at about $\mu_c = 135^\circ$ and is rather flat in the range $100^\circ \leq \mu_c \leq 160^\circ$. The large phase advance required to achieve a small emittance implies very strong focusing.

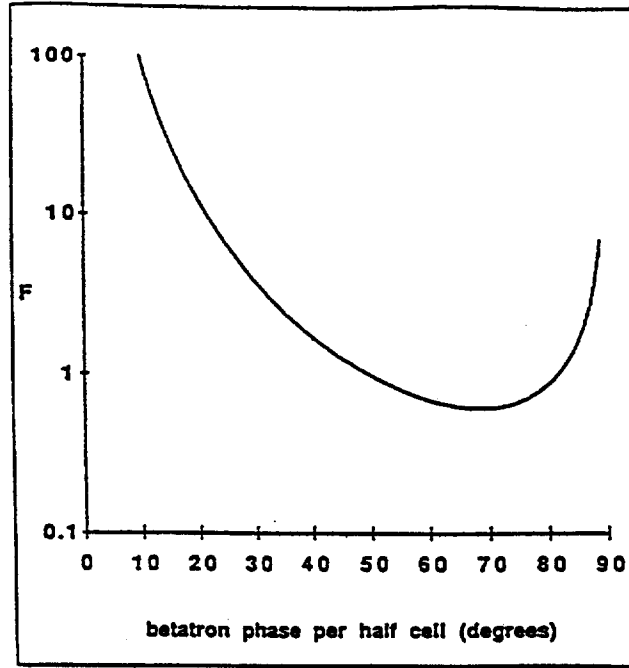


Fig. 13 Emittance behaviour as a function of the betatron phase per half cell

The FODO achromat is composed of a regular FODO structure followed by a dispersion suppresser, as shown in Fig. 14. The overall emittance is produced by the contribution from the nominal FODO cell bending magnets and the dispersion suppresser bending magnet. It can be expressed as [28]:

$$\epsilon_x = \frac{C_q \gamma^2}{J_x} \frac{\frac{n_0 I_0}{\rho_0^3} + \frac{2I_1}{\rho^3}}{\frac{n_0 \theta_0}{\rho_0} + \frac{2\theta_1}{\rho}} \quad (22)$$

where n_0 is the number of FODO half cells per achromat and $\theta = L / \rho$.

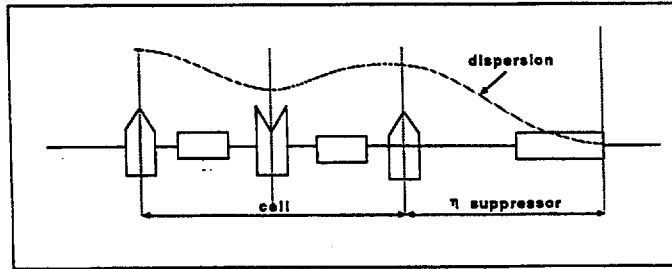


Fig. 14 Basic FODO achromat

Once the cell geometry and the phase advance are chosen, the emittance can be minimised by optimising the length of the dispersion suppresser magnet and the bending angle distribution. Figure 15 (taken from [28]) shows the variation of emittance with these parameters.

The FODO has the advantage of large flexibility. Also, since the average betatron function is reduced, the vacuum requirements can be relaxed. On the other hand, it is characterised by an extremely dense packing of magnetic elements which could lead to layout

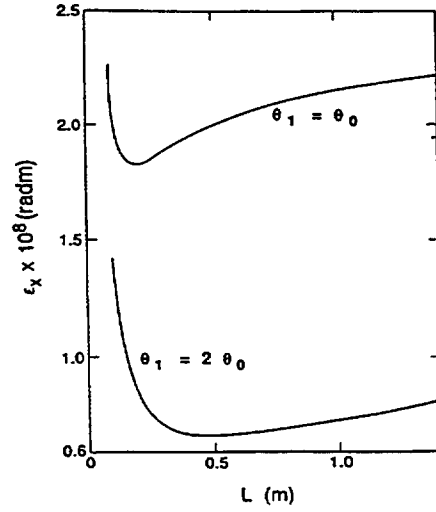


Fig. 15 Variation of emittance with length of dispersion suppresser

and engineering problems. Also it suffers from intrinsic problems similar to those of the triplet bend achromat, because of the small dispersion which is required to achieve a low emittance and leads to large sextupole strengths. The lattice functions of a FODO type structure are given in Fig. 16.

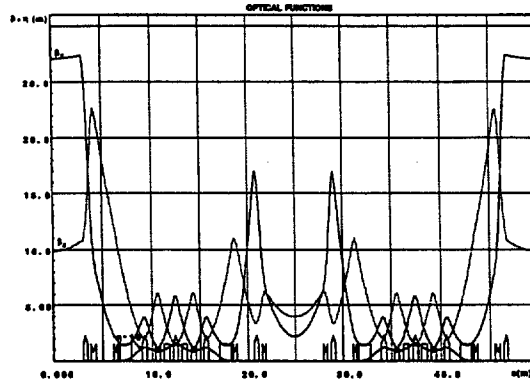


Fig. 16 Lattice functions of a FODO type structure

3.6 RACETRACK lattices

The lattices presented in the previous sections are built-up from identical cells and are characterised by a high degree of symmetry. However, the increasing demand for very long straight sections to accommodate FELs or long undulators has led to designing RACETRACK lattices. Such designs are being considered for a number of projects [29]. They raise a number of challenging issues due to the breaking of the symmetry of the lattice, the limited range of optical functions, and the need to match these long straight sections to the rest of the lattice whilst keeping a sufficient dynamic aperture. Different technical solutions are proposed: regular cells with missing bending magnets (SPRING8), normal cells replaced by RACETRACK ones and extra quadrupoles to match β values (SOLEIL / DIAMOND). This is sketched in Figure 17.

3.7 Comparison of lattice performances

It can be seen from the previous sections that the beam emittance of all these lattices scales as $E^2 \theta^3$ where E is the beam energy and θ the bending angle. This means that, in order to achieve a low emittance, the bending angle must be decreased and a large number of cells is needed. In order to make a comparison of the various types of lattices, it is useful to rewrite the emittance in the following form:

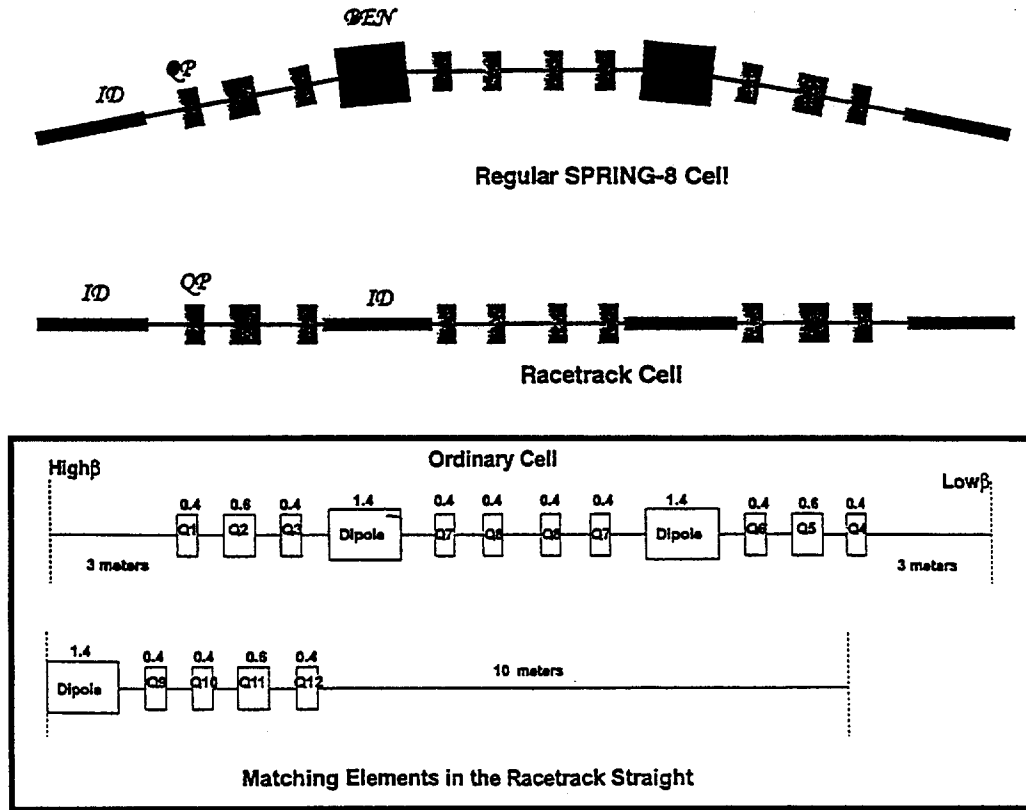


Fig. 17 A few RACETRACK schemes

$$\varepsilon_x = \frac{E^2}{J_x N_d^3} F(\mu_x, lattice) \quad (23)$$

where N_d is the number of bending magnets and E the beam energy (expressed in GeV).

Table 2 lists the minimum values of F which are achievable for the various lattices and gives an example of each lattice case. F is varying over a range of roughly a factor ten from the FODO lattice down to the DFA lattice. But because of the cubic dependence on N_d , even a factor ten can be compensated by a factor of about two in N_d . Thus, to get the same emittance with a FODO, one has to double the number of bending magnets in the ring. Design considerations imply in most cases that the emittance is larger than the minimum value.

Table 2
Minimum values of F

	F_{min}	Examples	Energy (GeV)	N_d	F
DFA	$2.35 \cdot 10^{-5}$	ESRF	6	64	$5.02 \cdot 10^{-5}$
TAL	$2.45 \cdot 10^{-5} (*)$	ACO	0.536	8	
TBA	$1.83 \cdot 10^{-5}$	ALS (**)	1.5	36	$1.14 \cdot 10^{-4}$
FODO (***)	$4.52 \cdot 10^{-4}$				

(*) to be multiplied by $(\beta/L)_{opt}$

(**) with gradient in the dipoles

(***) regular cells without dispersion suppresser

The emittance is an important issue when choosing a lattice. However criteria for evaluation of lattices include other considerations, such as chromatic correction and dynamic aperture, sensitivity to errors, flexibility, beam lifetime.... Some of these problems will be discussed in Section 4.

3.8 Future directions for low emittance lattices

The next generation of lattices will aim at being diffraction limited. In order to achieve this requirement, several ideas can be explored:

- i) extrapolation of present designs. As discussed in Section 3.7, present lattices are characterised by a large number of cells to obtain a low emittance. Further increase of the machine size [30] could provide the required gain (Table 3).

Table 3
Scaling of existing machines

Source type	Energy (GeV)	Circumf. (m)	ϵ (nm.rad)
Hard X-ray (ESRF)	6	850	4
Hard X-ray	6	1700	0.5
VUV	2	850	0.2

- ii) design of new models. Several concepts have been proposed [31], [32]. They are based on the design of regular cells for the minimisation of emittance, followed by matching sections and straight sections for insertion devices. An example of such design achieving an emittance of 0.5 nanometer-radians for a 3-GeV, 400-m-long machine is shown in Fig. 18.

However, the most challenging issue for these future machines lies more in intrinsic limitations such as intrabeam scattering [33], which could spoil the emittances well before the diffraction limit is reached, rather than in the design of a super low emittance lattice.

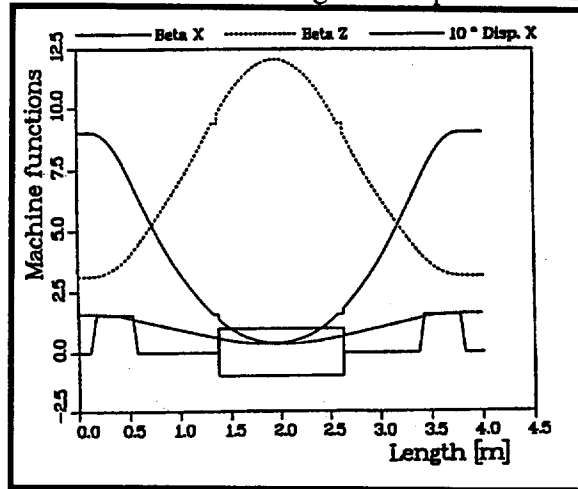


Fig. 18 "Multiple-Bend Achromat"

4. PROBLEMS ASSOCIATED WITH LOW-EMITTANCE LATTICES

4.1 Chromaticity correction and dynamic aperture

Achieving such low emittances means that the storage ring will be highly sensitive to non-linear effects. One of the main problems raised is the correction of chromaticity. The

chromaticity is the tune shift experienced by individual particles due to momentum errors $\Delta p/p$. It is defined as:

$$\zeta_{x,y} = \frac{\Delta v_{x,y}}{(\Delta p / p)} = -\frac{1}{4\pi} \int \beta_{x,y}(s) K_{x,y}(s) ds \quad (24)$$

where $K_{x,y}$ is the quadrupole gradient.

The strong focusing required to obtain a low emittance implies strong gradients and also large values of betatron functions so that the chromaticity tends to be large. Typical values are of the order of -50 to -100. This results in large tune spreads within the beam (in the ESRF case for instance, $\xi_x = -114$, $\xi_y = -35$ which leads to $\sigma_{v_x} = -0.12$, $\sigma_{v_y} = -0.04$ for $\sigma_{\Delta p/p} = 1.10^{-3}$). As a consequence, large momentum excursions enhanced by quantum fluctuations or Touschek scattering push the tunes of particles to nearby resonances, resulting in a beam loss or a limited lifetime. In addition, to prevent instabilities (such as the resistive wall instability in multibunch mode or the so-called "head-tail" effect in single bunch mode), the chromaticity must be corrected to zero or adjusted to a slightly positive value. This is performed by means of sextupoles located in the dispersive region. The momentum error produces an orbit shift which generates a quadrupole field at each sextupole location so that the correcting effect is given by:

$$\zeta_s = \frac{1}{4\pi} \int \beta_{x,y}(s) \eta(s) m(s) ds \quad (25)$$

where $m(s)$ is the sextupole gradient.

As a matter of fact, these sextupoles need to be very strong since the dispersion generated by the bending magnets is rather small in all lattices. Unfortunately, these sextupoles introduce various kinds of chromatic and geometric aberrations, such as betatron-amplitude-dependent and momentum-dependent tune shifts, change of the betatron functions and dispersion with momentum. These aberrations limit the maximum stable amplitudes of oscillations. The dynamic aperture, defined as the boundary of stable motion in the x - y plane, may be considerably smaller than the physical aperture. Therefore, the issue is to ensure that the motion remains bounded for a time long enough with respect to time scales of interest (damping times, synchrotron periods, i.e. typically several hundred of turns).

Adequate dynamic aperture is an important figure of merit. The transverse aperture must be large enough to accommodate the oscillations of the injected beam and Coulomb scattered particles. Without this, the injection efficiency would be very poor and the beam lifetime very short. A large chromatic aperture is required to improve the lifetime due to Touschek and bremsstrahlung effects. General guidelines can be put forward in terms of rms beam size σ . For quantum lifetime considerations, 6σ is the usual figure. Injection requires about 20σ . Ultimately the designer goal is aiming at obtaining 100σ . For the ESRF, these requirements correspond to ± 2.4 mm, ± 8 mm, ± 40 mm in high- β straight sections where $\sigma_x = 0.4$ mm.

In order to obtain reasonably large dynamic apertures and minimise all detrimental effects, complicated arrangements of sextupoles have to be found for each lattice. The common strategy consists in using two sextupole families in the dispersive part for chromaticity correction and minimisation of the tune dependence on energy. Their location results from a compromise between the maximum decoupling of the horizontal and the vertical planes and the best efficiency (i.e. maximum β and η values). As shown in Fig. 19, this usually gives a small dynamic aperture. Therefore, additional sextupoles located in the dispersion-free straight sections have to be used to minimise the driving terms of third-order resonances, amplitude-dependent tune shifts and enlarge the dynamic aperture [34]. With regard to dynamic aperture, all lattices can be brought to comparable performance as shown in Fig. 20 for the ESRF.

The dynamic aperture is usually experimentally approached by closing a scraper jaw and measuring the corresponding lifetime evolution. The beneficial effects of harmonic sextupoles are highlighted in Fig. 21 [35].

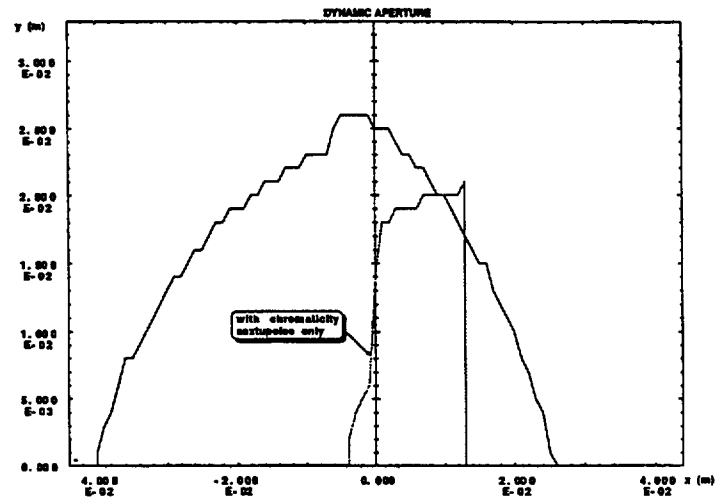


Fig. 19 Effects of sextupoles on the dynamic aperture

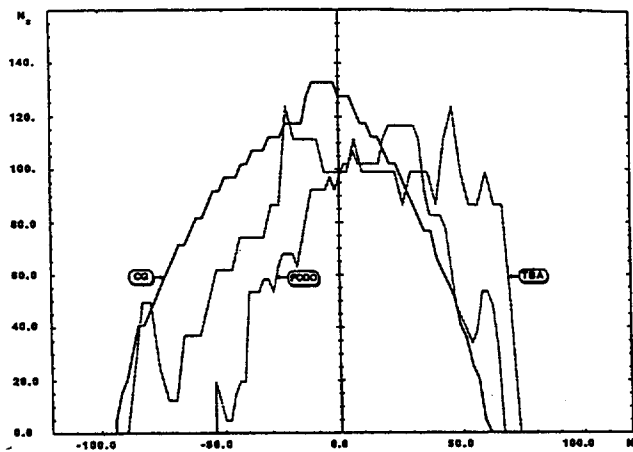


Fig. 20 Dynamic aperture of CG, TBA, FODO lattices for the ESRF

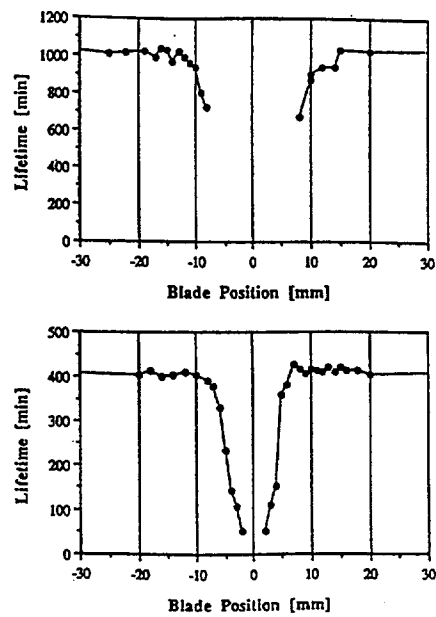


Fig. 21 ELLETRA dynamic aperture reduction when switching off the harmonic sextupoles

4.2 Sensitivity to errors

Common to all low-emittance lattices is a strong sensitivity to errors that may affect machine performance and have a significant impact on practical design. After having established a lattice that gives adequate dynamic aperture, it is essential to check the sensitivity to errors of all kinds (magnet construction imperfections generating systematic and random field errors, magnet misalignments, effects of insertion devices). Unacceptable distortions of the linear optics, reduction of dynamic aperture, beam emittance blow-up or even unstable machines could be generated. The severity of this problem is illustrated in Fig. 22 which shows that a significant reduction of dynamic aperture is induced by very small quadrupole positioning errors (in that case the effects of a misalignment of 0.025 mm have been computed for different sample machines).

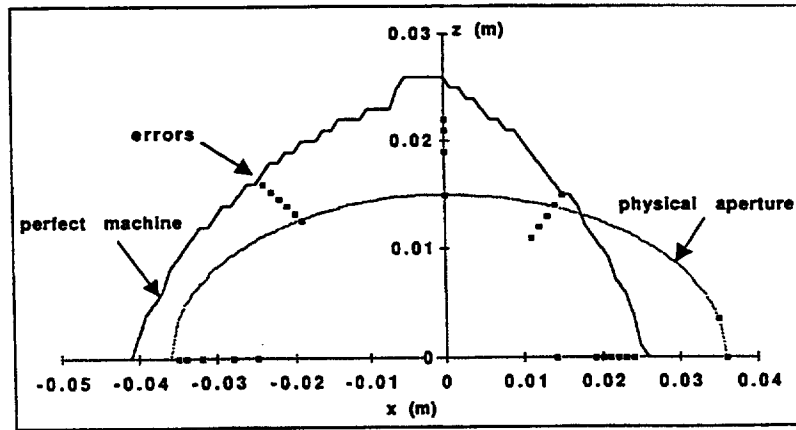


Fig. 22 Effects of quadrupole positioning errors on ESRF dynamic aperture

4.2.1 Closed-orbit distortions

Given the strong focusing required to achieve low emittances, the amplification of quadrupole misalignments is very large and is the dominant contribution to closed-orbit distortions. The sensitivity of a given lattice can be quantified via the magnitude of the amplification factor. This quantity is the ratio of the rms orbit at the observation point to the rms amplitude of the error. Typical values stand between 50 and 100. As an example, Fig. 23 shows the distribution of the rms orbit distortions for all errors at the ALS. As a consequence, state-of-the-art alignment and construction capabilities (1/10 mm positioning errors for quadrupoles and sextupoles, tilts of magnetic elements of a few 10^{-4} rad) are required.

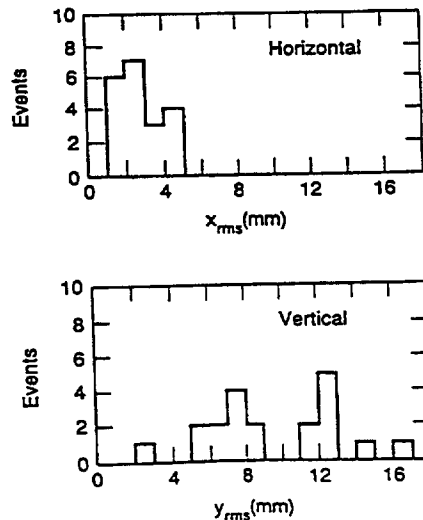


Fig. 23 Predicted rms orbit distortions produced by 20 sets of random errors at the ALS

Since a quite perfect closed orbit (i.e. 0.1 to 0.2 mm rms residual distortions) is necessary to achieve correct performance, a large number of beam position monitors with an accuracy of 0.1 to 0.2 mm and sophisticated closed orbit correction schemes, including dynamic feedback systems to control beam stability, must be implemented. Figure 24 shows the residual closed orbit achieved at ELETTRA [35] by a global correction using the most effective corrector and bump methods. At the ESRF, rms orbits in the 80 μm range are routinely obtained with a SVD correction scheme [36].

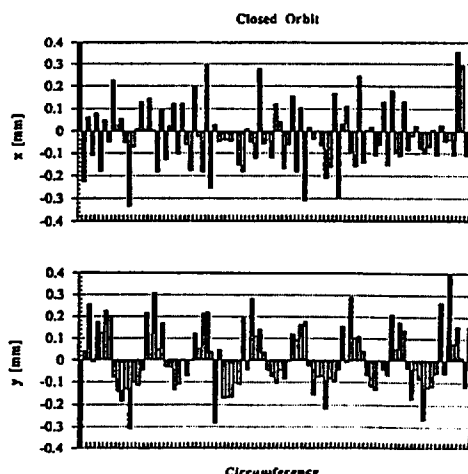


Fig. 24 Closed orbit after correction at ELETTRA

4.2.2 Focusing errors

These are mainly induced by random gradient errors in quadrupoles and horizontal mispositioning of sextupoles. They result in:

- i) the excitation of resonances and stop-bands, limiting the allowed region in the tune diagram and resulting in a reduction of the lifetime
- ii) a modulation of β -functions, thus spoiling the horizontal emittance
- iii) a reduction of the dynamic aperture (Fig. 25)

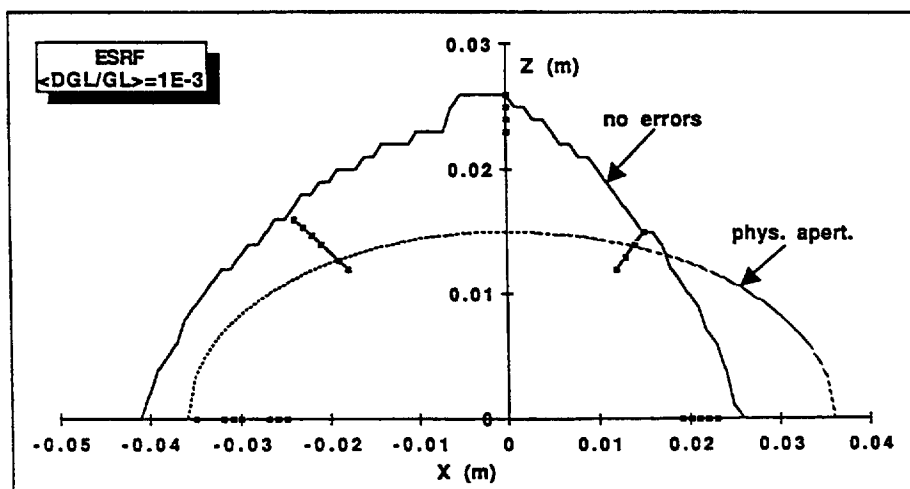


Fig. 25 Dynamic aperture reduction with random gradient errors

As a consequence, tolerances on quadrupole gradients in the $1.0 \cdot 10^{-3}$ range are generally required, which implies tight mechanical tolerances on bore radius and pole assembly, accurate magnetic measurements, and sorting procedures before installation in the ring. In addition, the correction of quadrupolar resonances proves to be very valuable for optimising machine performance. At least one pair of correctors per resonance is needed. Figure 26 illustrates the

beneficial effects of the correction of the half-integer stop-bands close to the tune in the ESRF case, with the subsequent enlargement of the longitudinal acceptance which in turn contributes to the increase of lifetime. This feature is of prime importance for low energy machines for which the Touschek effects, linked to an insufficient acceptance in energy, plays a major role.

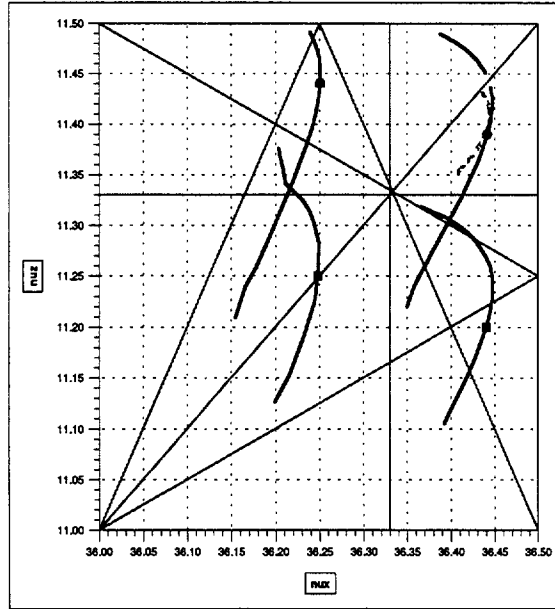


Fig. 26 Chromatic tune variation around a few ESRF working points

4.2.3 Emittances

Low emittances are one of the main figures of merit of the third generation of synchrotron light sources. However the achievement of the expected emittances can be spoilt by many factors.

- i) Blow up of the horizontal emittance due to the proximity of quadrupolar resonances and/or an imperfect closed orbit. Around integer values of ν_x , the emittance is sensitive to both closed orbit errors and focusing errors, whilst, around half integer values of ν_x , the effect of closed orbit errors is negligible. The influence of these two sources is shown for the ESRF in Figs. 27 and 28.

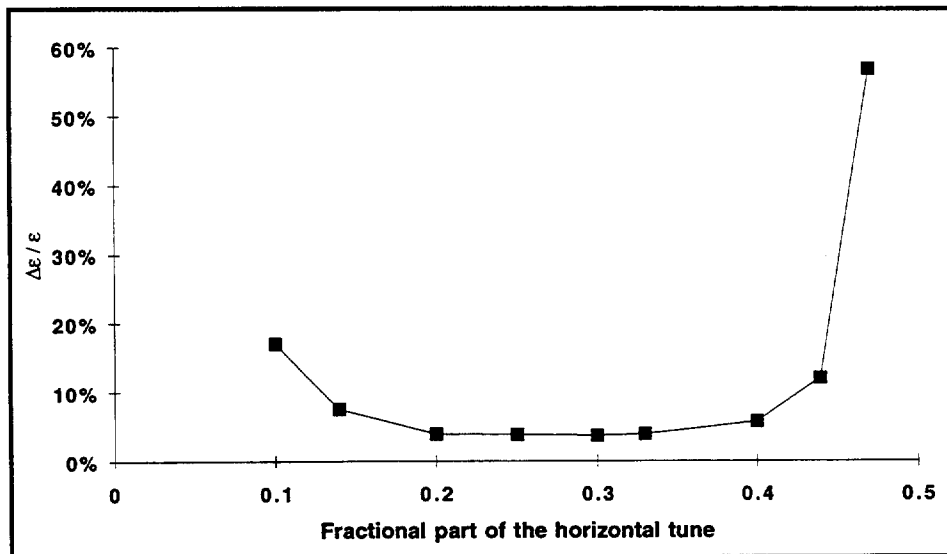


Fig. 27 Emittance evolution versus tune

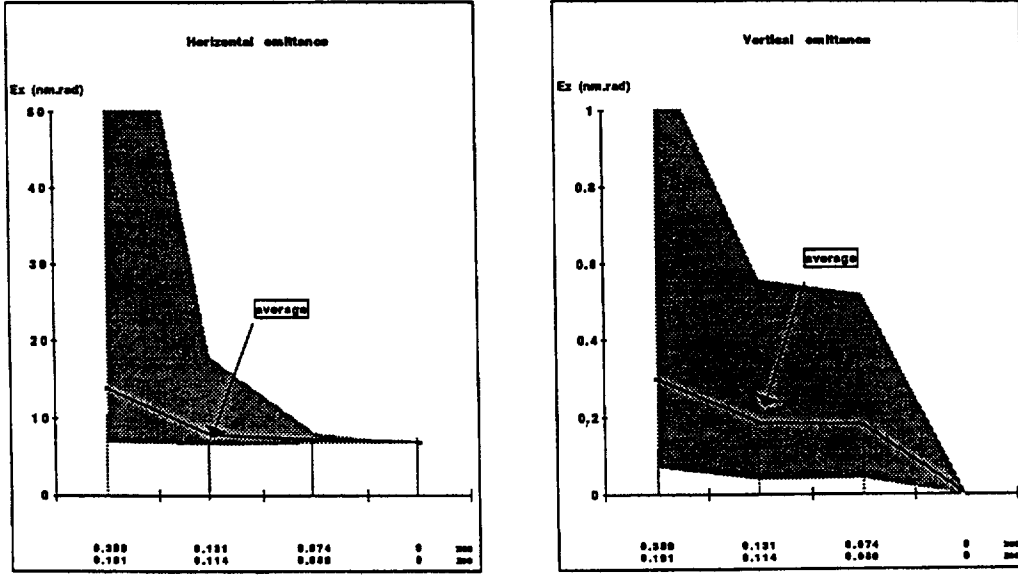


Fig. 28 Emittance blow-up with imperfect closed orbit

- ii) The vertical emittance, which is null in an ideal machine, is generated by several sources of imperfections in a real machine: radiation in the presence of spurious vertical dispersion due to dipole and quadrupole rotation errors or closed orbit in quadrupoles and sextupoles, coupling of horizontal and vertical betatron oscillations due to skew quadrupole field errors. As already mentioned in Section 2.1, the vertical emittance is generally characterised via the coupling:

$$\kappa = \frac{\varepsilon_y}{\varepsilon_x} \quad (27)$$

Most target specifications aim at a natural figure of 10 % which could be decreased below 1 % after correction. The correction strategy consists in powering skew quadrupoles to correct the coupling resonances close to the working point and/or the spurious vertical dispersion, then in deducing the beam emittances from beam size measurements by imaging the radiation. Depending on the lattice, the influence of the spurious dispersion on the emittance is more or less important as shown in Eq. 27 and Fig. 29 [35].

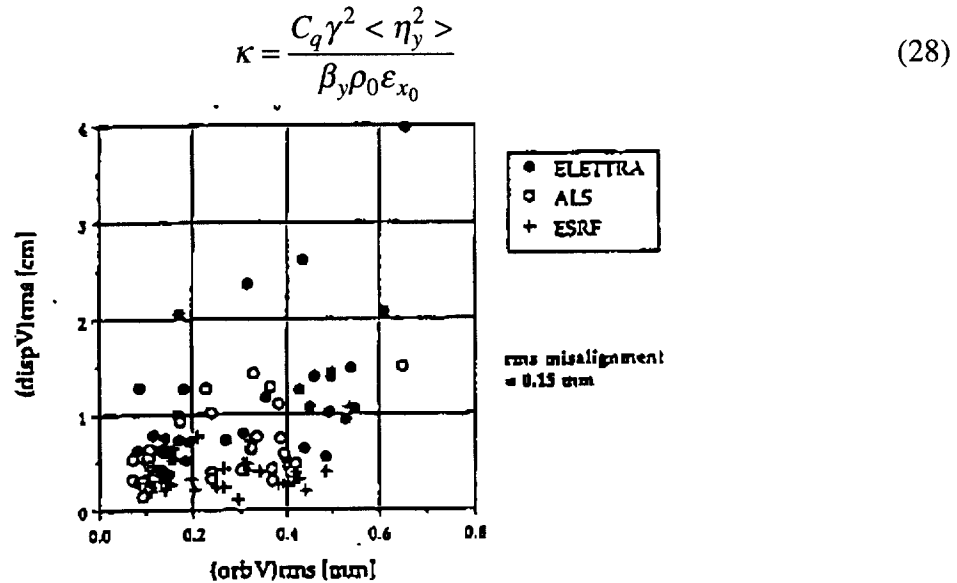


Fig. 29 Comparison of several machines in terms of spurious dispersion

4.2.4 Intrabeam scattering

Multiple Coulomb scattering within the bunches causes emittance growth in both transverse and longitudinal spaces. The addition of small emittances and high currents in a small bunch volume tends to increase the mechanism. As shown in Fig. 30, the induced emittance blow-up could annihilate the trends towards diffraction limited emittances.

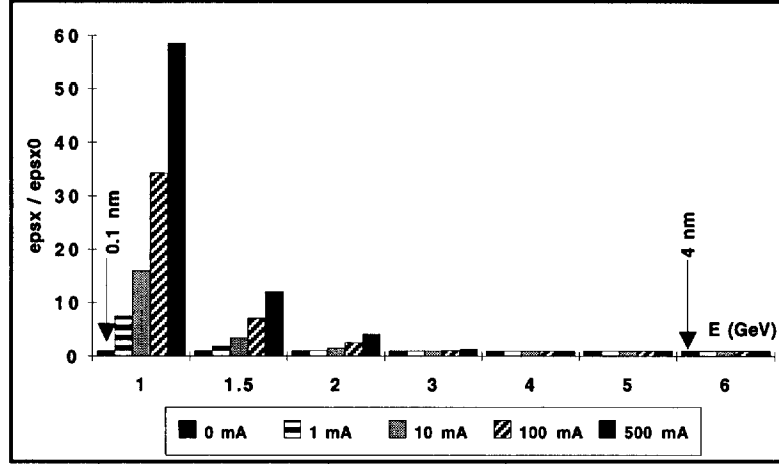


Fig. 30 IBS emittance blow-up when running the ESRF at low energy

4.3 Small momentum compaction

The achievement of a low horizontal emittance is obtained by minimising the radial dispersion of trajectories with energy at locations where particles radiate, thus leading to small orbit-length deviations as a function of energy deviation and therefore a momentum compaction factor α_1 , as indicated in Eq. (28):

$$\frac{dL}{L} = \alpha_1 \frac{dp}{p} \quad \alpha_1 = \frac{1}{L} \int_0^{2\pi R} \frac{\eta(s)}{\rho(s)} ds \quad (28)$$

Typical values are :

$$\alpha \sim 3.0 \cdot 10^{-4} \quad \text{for APS, ESRF}$$

$$\alpha \sim 1.5 \cdot 10^{-3} \quad \text{for ALS, ELETTRA}$$

Equation (29) shows that, the smaller α_1 , the shorter the bunch length σ_L . In the ESRF case, the zero-current bunch length is of the order of 15 ps.

$$\sigma_L = \frac{\alpha_1 R}{v_s} \sigma_{\frac{\Delta E}{E}} \quad (29)$$

where R is the machine radius, v_s the synchrotron tune and $\sigma_{\frac{\Delta E}{E}}$ the energy spread.

However, some adverse effects have to be taken into account [37], [38], [39]. When α_1 , becomes very small, non-linear contributions become important and can affect the longitudinal stability. In particular the α_2 term (see Eq. (30)) must be compensated.

$$\frac{dL}{L} = \alpha_1 \frac{dp}{p} + \alpha_2 \left(\frac{dp}{p} \right)^2 \quad (30)$$

Also, the wake field induced by the interaction of the bunch with the impedance of the vacuum chamber has a strong defocusing effect on the bunch which lengthens with increasing current. The production of intense and short bunches (in the picosecond range) looks therefore questionable for future storage rings.

4.4 Orbit stability

Orbit stability has more severe requirements than for other accelerator applications because of the very small vertical beam size at the source point (a typical figure is $\sigma_y < 100 \mu\text{m}$ at undulator locations). Any change of the beam position with time will lead to an enlargement of the source and a macroscopic emittance growth, as sketched in Fig. 31. To avoid spoiling the source emittance, the beam centre of mass must be kept within a few μm and a few $1/10 \mu\text{rad}$ all around the machine.

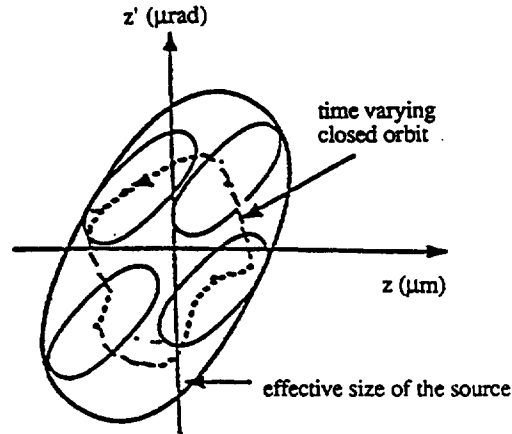


Fig. 31 Macroscopic emittance growth

The tolerances on the stability of the beam position concern both the long-term stability, therefore deal with ground settlement, and the short-term motion induced by vibrations transmitted through the ground [40].

As already pointed out, the closed orbit is extremely sensitive to quadrupole centre displacements. Therefore, the buildings, infrastructure, slab and magnet supports must be carefully designed to minimise transmission of vibrations to the magnetic elements. The standard closed-orbit position monitors are not able to guarantee the required stability. Therefore, it is necessary to use feedback systems based on the detection of the photon beam and local steering magnets.

5. EFFECTS OF INSERTION DEVICES ON THE BEAM

5.1 Introduction

With the advent of the new generation of all-ID machines, the development of insertion devices has been growing exponentially over the last few years (for instance, there are more insertion devices operated at the ESRF than installed around the world before 1990). Therefore, it is essential to thoroughly investigate how seriously will the particle motion be influenced by these additional elements.

The electron beam properties can be affected by insertion devices because these devices are intrinsically non-linear and will also provide focusing in one or both transverse planes, thus destroying the periodicity of the lattice and causing distortion of the particle motion. The perturbations induced by the presence of insertion devices may be divided into two groups. Firstly there are effects due to the magnetic field of the device which do not depend on radiation emission. They result in distortion of the linear optics, tune shifts, excitation of resonances, and reduction of the dynamic aperture. There are also effects due to the additional radiation emitted by the beam in the insertion device. The radiation damping and quantum excitation processes can be affected, causing changes in the emittance and energy spread of the beam.

These effects are produced mainly by wigglers and to a lesser extent by undulators. Some of them can be used to advantage, for instance to reduce the emittance far below that obtainable with other known methods. However, most of the effects are detrimental to machine performance and must be compensated. This minimisation implies some consequences on the machine design.

5.2 Matching to insertion requirements

Before installing insertion devices in a ring, the question of determining the values of the β functions that maximise the performance of the devices has to be considered. Generally the choice of β is a compromise between the machine and users' requirements. If one concentrates on the machine side, small β values are preferable for the following reasons:

- i) minimisation of closed orbit displacements when varying the gap;
- ii) minimisation of the volume occupied by the beam in the straight sections, allowing therefore to further decrease the insertion device gap without detrimental effects on the lifetime as illustrated in Fig. 32.
- iii) maximizing the brilliance with matching the electron beam emittances to the photon beam.

The users' requirements are somewhat mixed. In the undulator straight section, one generally wants to minimise the inhomogeneous broadening which occurs mainly through the angular spread. Ideally one would like to have:

$$\beta_{x,y} \geq \epsilon_{x,y} \frac{L_{ID}}{\lambda} \quad (30)$$

where λ is the radiation wavelength and L_{ID} is the length of the undulator. This corresponds to rather large β values. For wigglers, the figure of merit is the obtention of maximum brilliance. This calls for low values of β .

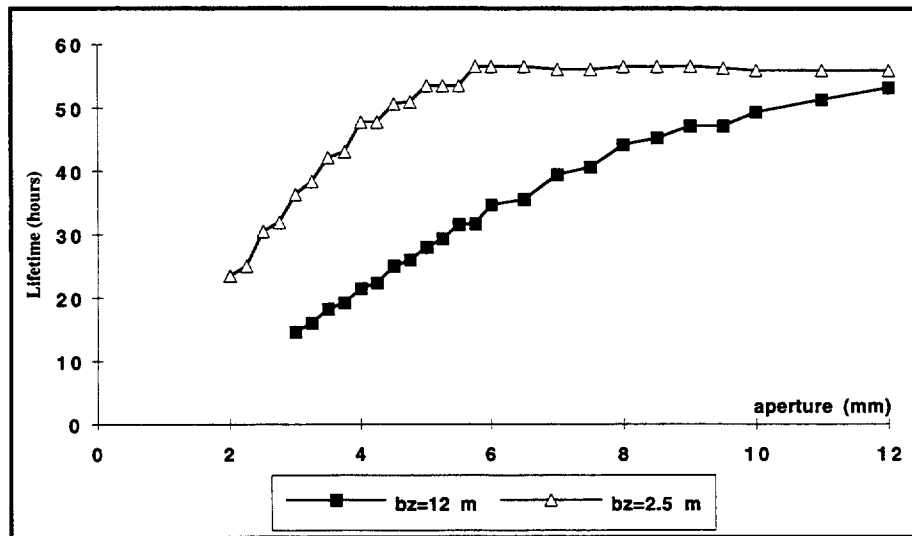


Fig. 32 Comparison of lifetime versus gap for two β_y values

In any case, lattice flexibility is an extremely important issue. A high periodicity configuration (with intermediate β values in all straight sections) is probably easier for the designer to achieve. However, the lattice should be able to accommodate a variable number of undulators and wigglers and provide β adjustment capability.

5.3 Magnetic-field effects

The basic requirement of an insertion device is that, after passing through it, the beam should return to its nominal orbit. If this is not the case, a closed orbit will be set up, giving beam displacements around the machine and thus perturbing other beam lines. To first order, this requirement is fulfilled by making the field distribution symmetrical about the midpoint of the device and by adjusting the integrated field length to zero. Taking the ESRF as an example, maintaining the closed orbit rms deviation below 1/10 of the beam size would require that the integral of the vertical (horizontal) magnetic field along the axis of propagation be smaller than $8 \cdot 10^{-5}$ T.m ($4 \cdot 10^{-5}$ T.m) for a single insertion device. Very precise magnetic field measurements are necessary to fulfil these specifications. The alternative solution is to provide independent

adjustment of the integral of the field by end coils. These tight tolerances can be fulfilled in practice as shown in Fig. 33 [41] where the closed orbit distortions induced by ESRF ID gap closing are interpreted in terms of rms size or divergence growth normalised to the electron beam size or divergence.

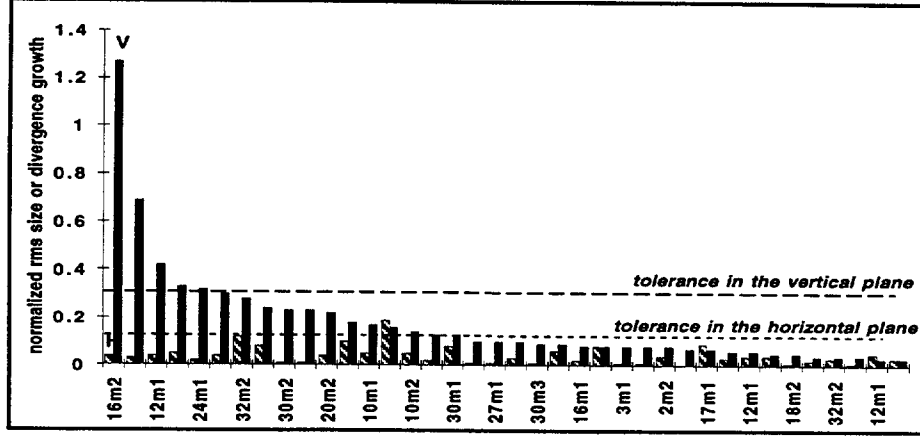


Fig. 33 Closed-orbit distortions induced by IDs

5.4 Linear and non-linear effects on beam dynamics

5.4.1 Equations of motion

Assuming a sinusoidal field variation, the analytical treatment of the effects of insertion devices on beam dynamics has been developed [42, 43]. We recall here the main steps of the procedure based on the use of Hamiltonian formalism.

The components of the insertion device magnetic field used for the derivation of equations of motion are as follows:

$$\begin{aligned}
 B_x &= \frac{k_x}{k_y} B_0 \sinh(k_x x) \sinh(k_y y) \cos(kz) \\
 B_y &= B_0 \cosh(k_x x) \cosh(k_y y) \cos(kz) \\
 B_z &= -\frac{k}{k_y} B_0 \cosh(k_x x) \sinh(k_y y) \sin(kz)
 \end{aligned} \tag{31}$$

with

$$k_x^2 + k_y^2 = k^2 = \left(\frac{2\pi}{\lambda}\right)^2 \tag{32}$$

where λ is the period length of the insertion device and B_0 is its magnetic field. The parameter k_x measures the transverse variation of the field due to the limited pole width or curved pole faces. Here x is the horizontal direction, y the vertical one and z is the beam direction. The above formula which keep only the first harmonics of the field variation in the z -direction, satisfy the Maxwell equations. A more accurate field treatment would be required for large deviations from the axis.

The Hamiltonian of the motion can be written in the following form:

$$H = \frac{1}{2} \left\{ p_z^2 + [p_x - A_x \sin(kz)]^2 + [p_y - A_y \sin(kz)]^2 \right\} \tag{33}$$

where

$$A_x = \frac{1}{k\rho} \cosh(k_x x) \cosh(k_y y) \tag{34}$$

$$A_y = -\frac{k_x}{k_y} \frac{\sinh(k_x x) \sinh(k_y y)}{k\rho}$$

and ρ is the radius of curvature in the field B_0 .

After a canonical transformation to change to betatron variables, the Hamiltonian is averaged over a period of the insertion device and the hyperbolic functions are expanded to fourth order in x and y . One gets:

$$H = \frac{1}{2}[p_x^2 + p_y^2] + \frac{1}{4k^2\rho^2}[k_x^2 x^2 + k_y^2 y^2] + \frac{1}{12k^2\rho^2}[k_x^4 x^4 + k_y^4 y^4 + 3k_x^2 k_y^2 x^2 y^2] - \frac{\sin(ks)}{2k\rho}[p_x(k_x^2 x^2 + k_y^2 y^2) - 2k_x^2 p_y x y] \quad (35)$$

The equations of motion can then be derived and written as:

$$x' = -\frac{k_x^2}{2k^2\rho^2}\left[x + \frac{1}{6}k_x^2 x^3 + \frac{1}{2}k_y^2 x y^2\right] \quad (36)$$

$$-\frac{\cos(ks)}{\rho}\left[\frac{1}{2}(k_x^2 x^2 + k_y^2 y^2) + \frac{1}{4}k_x^2 k_y^2 x^2 y^2 + \frac{1}{24}(k_x^4 x^4 + k_y^4 y^4)\right] - \frac{\sin(ks)}{\rho}k_y y' \left[1 + \frac{1}{2}k_x^2 x^2 + \frac{1}{6}k_y^2 y^2\right]$$

$$y' = -\frac{k_y^2}{2k^2\rho^2}\left[y + \frac{1}{6}k_y^2 y^3 + \frac{1}{2}k_x^2 x^2 y\right] \quad (37)$$

$$+\frac{\cos(ks)}{\rho}\left[k_x^2 x y + \frac{1}{6}k_x^2 k_y^2 x y^3 + \frac{1}{6}k_x^4 x^3 y\right] + \frac{\sin(ks)}{\rho}k_y x' \left[1 + \frac{1}{2}k_x^2 x^2 + \frac{1}{6}k_y^2 y^2\right]$$

As shown by the equations of motion, there is a linear effect which is equivalent to that of a horizontally and vertically focusing quadrupole whose strengths are given by:

$$K_{Q_{x,y}} = \frac{1}{2}\left(\frac{k_{x,y}}{\rho k}\right)^2 \quad (38)$$

There is also a non-linear part whose major effect comes from an octupole-like term. The dependence of the focusing strength on B^2/E^2 exhibited by Eqs. (36) and (37) shows that these effects are more severe for high-field wigglers in low energy machines. Since undulators usually have low fields, their linear effect should be smaller.

For insertion devices built with plane pole faces, i.e. $k_x = 0$, the focusing effect is present in the vertical plane only, having its source in the edge focusing of the magnet. Also the non-linear term vanishes, apart from a fourth-order term which is oscillating over one period of the device. By shaping the pole pieces of the magnets, vertical focusing can be transferred to the horizontal plane. However, we shall restrict the following discussion to the case $k_x = 0$.

5.4.2 Linear optics distortions

The vertical tune shift and the distortions of the β functions can be estimated from the standard perturbation theory. Making some simplifying approximations, one gets:

$$\Delta v_y \approx \frac{\beta_y L_{ID}}{8\pi\rho^2} \quad (39)$$

where L_{ID} is the insertion device length and β_y the average vertical lattice function. It has to be noted that the tune shift depends on the total length of the device and not on the period length. For devices of the same length, the tune shift is proportional to the square of the magnetic field.

Thus, high-field devices are expected to present stronger linear distortions than those with lower fields and shorter periods. This is illustrated in Fig. 34.

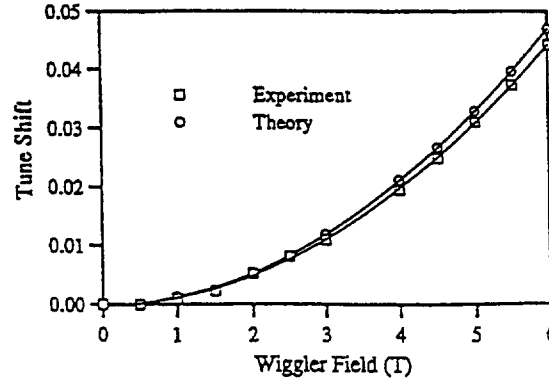


Fig. 34 Tune shifts induced by the 5 T SCW at SRS [44]

A consequence of this change in tune is a shift of the working point which might approach intrinsic resonances and be troublesome. From Eq. (39), one sees that the tune shift is directly proportional to the vertical β value at the insertion device. Hence the tune shift can be minimised by locating wigglers in a straight section where the vertical β is small. It was mentioned previously that it is preferable to have high β for undulators. Therefore the lattice should be able to accommodate these two different kinds of insertion straight sections.

The distortions of the β functions are given by:

$$\left(\frac{\Delta\beta_y}{\beta_y} \right)_{\max} \approx \frac{L_{ID}\beta_y}{4\rho^2 \sin(2\pi\nu_y)} \quad (40)$$

Table 4 gives some numerical examples computed for various machines. An example of β -functions modulation is shown in Fig. 35 [45].

Table 4
Linear optics distortions generated by insertion devices

Machine	Energy (GeV)	Type of device	B (T)	L_{ID} (m)	λ (cm)	β_y (m)	$\Delta\nu_y$	$\Delta\beta/\beta(\%)$
VEPP-3	0.35	U	0.6	7	10	5	0.35	-
VEPP-2M	0.7	W	8	0.6	12	1	0.5	-
SUPERACO	0.8	U	0.49	3.2	12.8	4.8	0.04	-
ALS	1.5	W	5	1.96	14	4	0.35	390
		U	1.15	4.8	9	4	0.042	
ELETTRA	2	W	5	3	30	2.6	0.15	80
		U	1.2	5.5	5.5	2.6	0.035	
SRS	2	W	5			7.3	0.05	20
		U	0.33	1	10	4.8	0.005	
PF	2.5	W	1.5	3.18	10	12.8	0.052	41
		U	0.45	3.63	6	12.8	0.005	4
ESRF	6	W	1.25	2	10	3.5	0.001	1.3
		U	0.63	6	5.5	13	0.003	1.8
APS	7	U	0.65	5	2.2	10	0.0015	-

As seen from Table 4, the linear effects are negligible for high energy machines. On the other hand, for wigglers in low energy machines, it is evident that such strong distortions lead to a large increase of the half-integer resonance stop-band and reduce the available tune space for machine operation. Compensation schemes for these focusing effects will be described in the next section.

It has also to be mentioned that this β -beat around the ring will change the vertical beam size. At some positions the size will increase and at others it will decrease. However the vertical emittance is not changed by this effect.

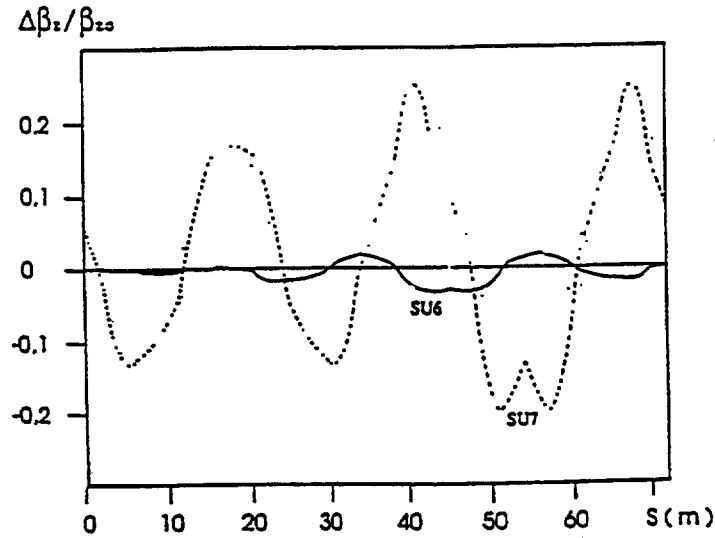


Fig. 35 Comparison of the β -beat induced by undulators at SUPERACO
($B = 0.3$ T for SU6 and $B = 0.5$ T for SU7)

5.4.3 Matching techniques

In order to compensate fully for the focusing effects of the insertion devices, six parameters (α_x , β_x , α_y , β_y , $\Delta\mu_x$, $\Delta\mu_y$) in the insertion straight section should be matched to the values existing in the bare lattice. This would require the adjustment of six independent parameters. Several strategies can be envisaged to perform the compensation:

- i) eliminate the β -beat outside the insertions by rematching the insertion device into the lattice with a pair of quadrupoles at each end of the straight section. The simplest way consists in tuning a quadrupole doublet to obtain zero β slope at the symmetry point (known as the " α matching"). A more elaborate compensation [46] can be performed by tuning the vertical β function in the insertion device to its characteristic value given by $\beta_y = \rho\sqrt{2}$.

After readjusting the β values, the tunes still deviate from their original values; therefore, in a second step, the quadrupoles of all achromats can be used to recover the original tune values. For machines having triplets in the insertion straight sections, since only two quadrupoles are required for the matching, the third quadrupole can be adjusted so that the sum of the tune changes in both planes is minimised.

- ii) another correction method adopted at the Photon Factory storage ring [47] consists in a simultaneous correction of the tune shifts and distortions of the β functions. To perform the correction, an adequate set of quadrupoles in the ring is selected (in principle all the quadrupoles in the ring can be used but, in practice, only a few of them are efficient) and retuned. The quadrupole strengths are determined by the least square method so as to minimise the distortions. In the Photon Factory case, priority is given to the

compensation of tune shifts which is a more stringent requirement than the correction of β 's for machine operation. However distortions of betatron functions can be reduced from 40% to less than 5%.

However, whatever the rematching procedure, in most cases the change in quadrupole strengths could be rather large (more than 10% of the nominal values). In addition the quadrupoles should be individually powered to take care of the different types of insertion devices in the various straight sections.

5.4.4 Effects on dynamic aperture

One of the major detrimental aspects of the installation of insertion devices is the resulting reduction of dynamic aperture. There are two effects of the insertion device that cause this deterioration:

i) introduction of non-linearities leading to enhancement of the amplitude-dependent tune shifts and distortion of phase space. It can be seen from the Hamiltonian given in Eq. (35) that, due to the non-linear field components, resonances can occur at integer values of $3 \nu_x$, $\nu_x + 2\nu_y$, $4 \nu_x$, $4 \nu_y$ and $2 \nu_x + 2 \nu_y$ (if $k_x \neq 0$). Analytical calculations of the amplitude-dependent tune shifts, stop-band widths and island widths have been performed [51]. The non-linear tune shifts (at half physical aperture) have been computed in Table 5 for some insertion devices. From these results, it can be expected that the non-linear fields will produce significant effects at large betatron amplitudes, more especially for low energy machines. These effects are also illustrated in Fig. 36 [48] and in Fig. 37 [45].

Table 5
Non-linear tune shifts

	LBL U9.0	BNL-VUV wiggler	SPEAR 54 poles	FRASCATI undulator	APS
Energy (GeV)	1.5	0.75	3	0.6	7
$\Delta\nu$	0.005	0.007	< 0.001	0.01	0.0007

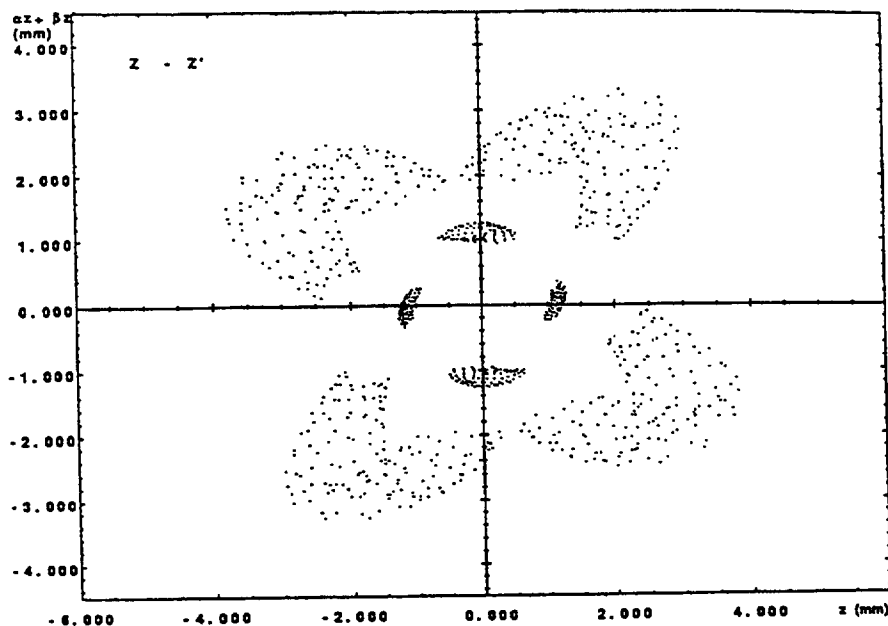


Fig. 36 Vertical phase space trajectories showing the distortion of phase space and occurrence of islands

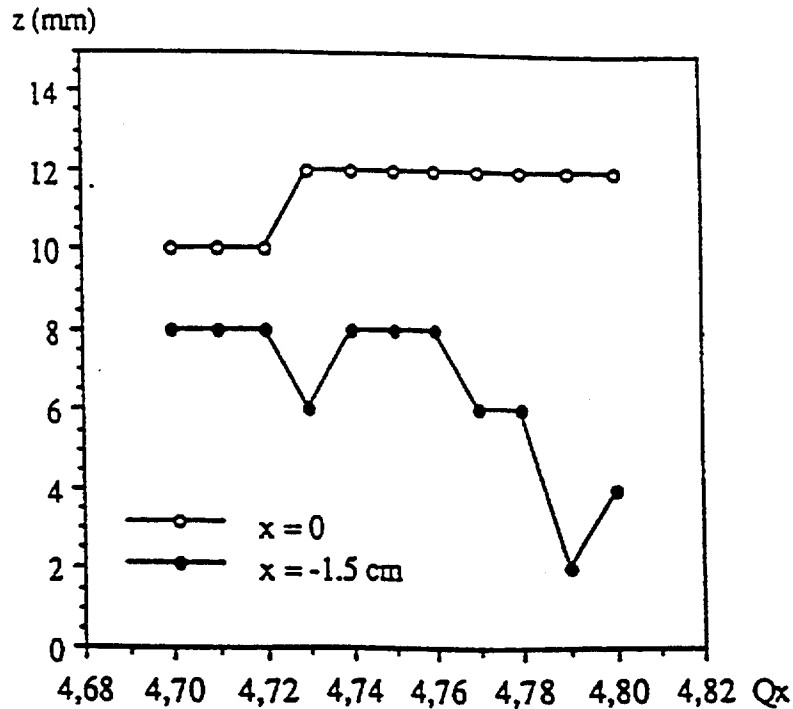


Fig. 37 Variation of the vertical aperture as a function of tune for the SU7 undulator at SUPERACO

ii) reduction of the machine periodicity. Although the matching techniques previously mentioned tend to reduce the linear perturbation, the insertion device introduces a change in betatron phase advance through the lattice which leads to breaking of the superperiodicity for the sextupole distribution and to the excitation of additional sextupole resonances. In principle, these betatron phase distortions could be compensated locally by utilising four pairs of quadrupoles located symmetrically about the device. Beyond the fact that this technique is space demanding, it is effective only if there are no sextupoles in the matching section, which limits its application in most cases.

The effects on the dynamic aperture can be investigated by tracking particles through the lattice elements and the insertion device field described above. In codes such as RACETRACK [49] or BETA [50], the insertion device is sliced into several segments per period with the non-linear terms averaged over each slice and applied as kicks.

Figure 38 shows the resulting reduction of dynamic aperture for two of the typical ALS devices [51]. For the ESRF [7], the effects of one undulator and of fifteen identical undulators on the dynamic aperture are plotted in Fig. 39. Again, it turns out that high energy machines are less affected than low energy ones; this is mainly due to the fact that the non-linear terms scale as p^{-2} . One can also conclude that the reduction of dynamic aperture could be more pronounced with undulators (which usually have shorter periods) than with wigglers, since the non-linearities increase in an inversely proportional way with the square of the period length.

When more insertion devices are included, the dynamic aperture shrinks further. This is illustrated for a high energy machine in Fig. 39 and for a low energy machine [9] in Fig. 40. Although the observed reduction in dynamic aperture appears to be alarming, it should be noticed that in most cases the dynamic aperture is still greater than the physical aperture. To ease the situation, one could think of scenarios reducing the effects by opening up the gap during injection when the aperture requirements are more stringent. This issue is one of the challenging problems confronting the next generation of all-insertion device synchrotron radiation sources.

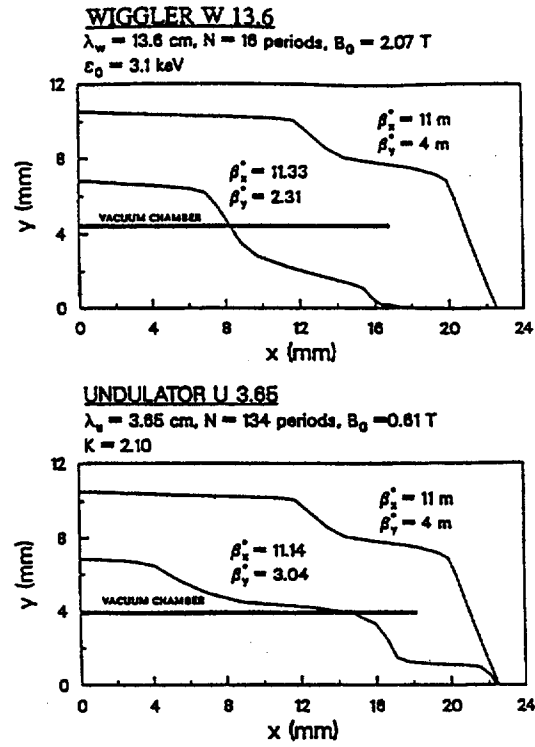


Fig. 38 Effects of insertion devices on ALS dynamic aperture

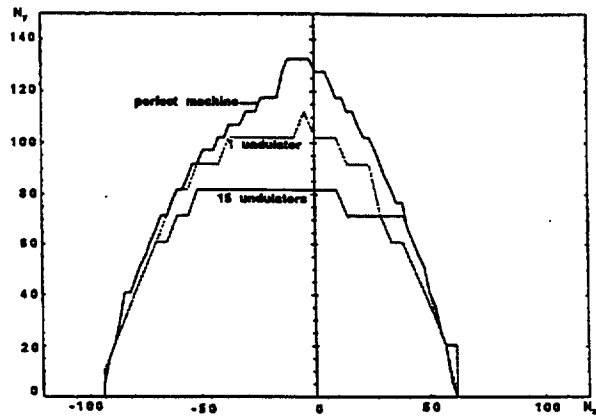


Fig. 39 Effects of undulators on ESRF dynamic aperture

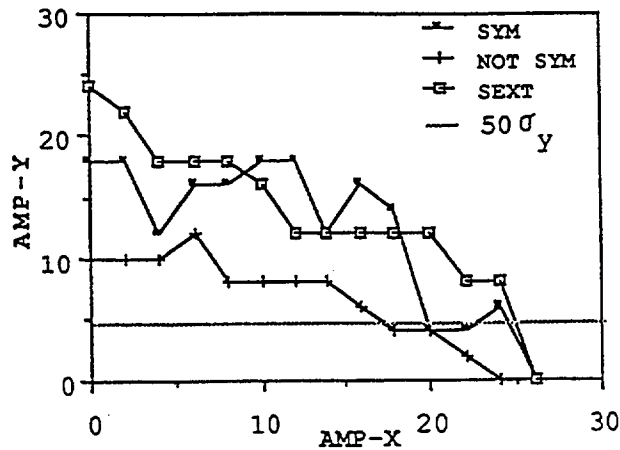


Fig. 40 Effects of undulators on ELETTRA dynamic aperture

5.5 Radiation effects

5.5.1 Energy loss

The primary effect of the additional quantum excitation in the insertion devices is to increase the beam energy loss. The immediate consequence is that the r.f. system must have the capability of compensating for this increased energy loss. Table 6 shows that this effect is quite significant. Using the formula giving the energy loss in an isomagnetic guide field [1]:

$$U_0 = C_\gamma \frac{E^4}{\rho_0} \quad (41)$$

with ρ_0 the bending radius, E the energy and $C_\gamma = 8.85 \cdot 10^{-5} \text{ m (GeV)}^{-3}$, one can express the total energy loss in the presence of insertion devices as:

$$U = U_0 \left[1 + \frac{1}{2} \frac{L_{ID}}{2\pi\rho_0} \left(\frac{\rho_0}{\rho} \right)^2 \right] \quad (42)$$

where L_{ID} is the total length of insertion devices and ρ is the radius of curvature in the field B_0 of the insertion device. Notice that the factor $1/2$ comes from the definition of ρ which differs from the usual "average" radius of curvature by a factor $\sqrt{2}$.

Table 6
Energy loss due to synchrotron radiation

	Energy loss in dipoles U_0 (MeV)	Energy loss in IDs U_0 (MeV)
ELETTRA	0.258	0.326
ESRF	4.75	1.5

5.5.2 Effects on emittance and momentum spread

The additional radiation produced in the insertion device will tend to increase the quantum fluctuations and lower the damping time. Depending on the value of the dispersion at the insertion device location (including the self-generated dispersion), one or the other effect will tend to dominate.

The emittance and energy spread in the bare machine can be expressed in terms of the Synchrotron Radiation Integrals [52]. One gets:

$$\varepsilon_x = C_q \gamma^2 \frac{I_5}{I_2 - I_4} ; \quad \sigma_\varepsilon = \left[C_q \gamma^2 \frac{I_5}{I_2 - I_4} \right]^{\frac{1}{2}} \quad (43)$$

with

$$I_2 = \int \frac{1}{\rho^2} ds ; \quad I_3 = \int \frac{1}{|\rho^3|} ds \quad (44)$$

$$I_4 = \int \frac{(1-2n)}{\rho^3} \eta ds ; \quad I_5 = \int \frac{H}{|\rho^3|} ds$$

The introduction of insertion devices will contribute extra terms to these integrals and hence change the beam emittance and energy spread, according to:

$$\varepsilon_x = \varepsilon_x^0 \frac{1 + \frac{I_5^{ID}}{I_5^0}}{1 + \frac{I_2^{ID} - I_4^{ID}}{I_2^0 - I_4^0}}; \quad \sigma_\varepsilon = \sigma_\varepsilon^0 \left[\frac{1 + \frac{I_3^{ID}}{I_3^0}}{1 + \frac{2I_2^{ID} + I_4^{ID}}{2I_2^0 + I_4^0}} \right]^{\frac{1}{2}} \quad (45)$$

where the index 0 indicates the contribution of the storage ring without insertion devices and the index ID the contribution from insertion devices alone.

The calculations of the insertion device integrals are performed, assuming a sinusoidal variation of the field along the axis of propagation z , for an insertion device characterised by N periods of wavelength λ and bending radius ρ . The i_4 and i_5 terms correspond to additional contributions arising from a non-vanishing dispersion function in the straight sections. The i_4 adds directly to I_4 . For i_5 , the combination is more complicated; when the self-generated dispersion from the insertion device is negligible, i_5 will be the dominant term.

$$I_2^{ID} = \frac{\lambda N}{2\rho^2}; \quad I_3^{ID} = \frac{4\lambda N}{3\pi\rho^3} \quad (46)$$

$$I_4^{ID} = -\frac{3\lambda^3 N}{32\pi^2\rho^4}; \quad i_4^{ID} = \frac{7\lambda^2\eta_0'}{9\pi^2\rho^3}$$

$$I_5^{ID} = \frac{\lambda^4}{4\pi^4\rho^5} \left[\frac{3}{5\pi} + \frac{3}{16} \right] \langle \gamma \rangle L_{ID} - \frac{9\lambda^3}{40\pi^4\rho^5} \langle \alpha \rangle L_{ID} + \frac{\lambda^2}{15\pi^3\rho^5} \langle \beta \rangle L_{ID}$$

$$i_5^{ID} = \frac{4\lambda N}{3\pi\rho^3} [\gamma_0\eta_0^2 + \beta_0\eta_0'^2]$$

where $\langle \rangle$ stands for the average of α, β, γ in the device of length L_{ID} and the index 0 refers to the entrance of the device.

The emittance behaviour can be rewritten in a simplified way, due to the fact that the I_4 contribution from the insertion device is usually small and the third term in I_5 is the dominant part. Furthermore, we will assume that the I_4 contribution of the bare machine can be neglected when compared to I_2 . We get now:

$$\varepsilon_x = \varepsilon_x^0 \frac{1 + \frac{\lambda^2 \langle \beta \rangle L_{ID}}{15\pi^3\rho^5} \frac{I_5^0}{I_5^{ID}}}{1 + \frac{1}{2} \frac{L_{ID}}{2\pi\rho_0} \left[\frac{\rho_0}{\rho} \right]^2}; \quad \sigma_\varepsilon = \sigma_\varepsilon^0 \left\{ \frac{1 + \frac{4}{3\pi} \frac{L_{ID}}{2\pi\rho_0} \left[\frac{\rho_0}{\rho} \right]^3}{1 + \frac{1}{2} \frac{L_{ID}}{2\pi\rho_0} \left[\frac{\rho_0}{\rho} \right]^2} \right\}^{\frac{1}{2}} \quad (47)$$

The insertion device radiation always contributes to the damping term in the denominator of Eq. (47). The change in the horizontal emittance depends on the relative values of the chromatic invariant. Since the dispersion generated by undulators is usually very small, quantum excitation will be much smaller than the damping effect and the cancellation of energy damping by the increase in quantum fluctuations would occur at high fields. Therefore for all cases of interest, there is a reduction of emittance due to the inclusion of undulators. On the other hand, in low energy machines containing high field wigglers, the self-generated dispersion is dominant and leads to an increase in emittance which could be prohibitive. These conclusions are illustrated in Fig. 41 which shows the emittance behaviour as a function of the insertion device field in two storage rings for typical sets of insertion devices:

- ESRF 8 wigglers ($\lambda = 10\text{cm}$, $N = 20$, nominal field $B_0 = 1.25\text{ T}$)
 16 undulators ($\lambda = 5.5\text{cm}$, $N = 100$, nominal field $B_0 = 0.63\text{ T}$)
- ELETTRA 3 wigglers ($\lambda = 50\text{cm}$, $N = 10$, nominal field $B_0 = 5.0\text{ T}$)
 10 undulators ($\lambda = 5.5\text{cm}$, $N = 100$, nominal field $B_0 = 0.5\text{ T}$)

The situation might deteriorate if the dispersion function has a non-zero value in the insertion device straight section due, for instance, to an imperfect closed orbit correction. In that case, quantum excitation takes over and leads to an increase in emittance.

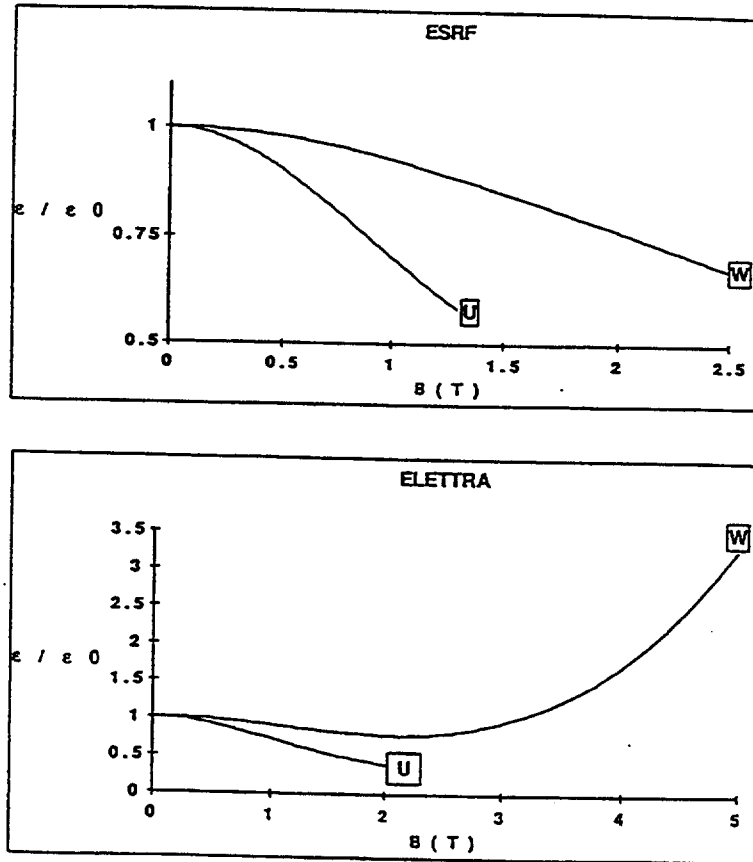


Fig. 41 Effects of insertion devices on emittance

The effect on the energy spread is similar to the effect on the beam emittance. The change in the energy spread depends on the ratio of the magnetic field in the insertion device to that in the bending magnets. Except for low fields, there is an increase in the beam energy spread as shown in Fig. 42 for the same insertion devices, which can cause serious stability and acceptance problems if the energy spread is allowed to increase too much. Again the effect is more pronounced for wigglers than for undulators.

The effects of wigglers can even serve to reduce the emittance. From Eq. (47) one can derive the condition under which the beam emittance is unperturbed or reduced:

$$\lambda^2 \leq 5.8710^9 \frac{E \epsilon_{x_0}}{B^3 \langle \beta \rangle} \quad (48)$$

with E the beam energy (in GeV) and B the insertion device field (in T)

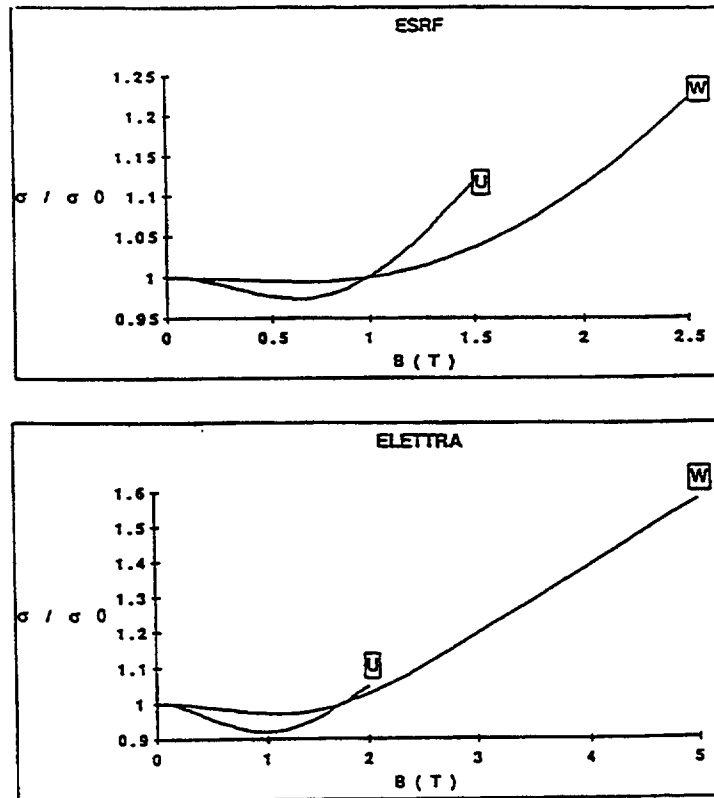


Fig. 42 Effects of insertion devices on momentum spread

The period of the device I must not exceed the value determined by Eq. (48) in order to obtain a reduced emittance. The installation of appropriate damping wigglers was proposed recently at PEP [3]. It would require the use of wigglers with a maximum field of 1.26 T and a period length of 12 cm. The resulting theoretical emittance is shown in Fig. 43 as a function of the total installed wiggler length.

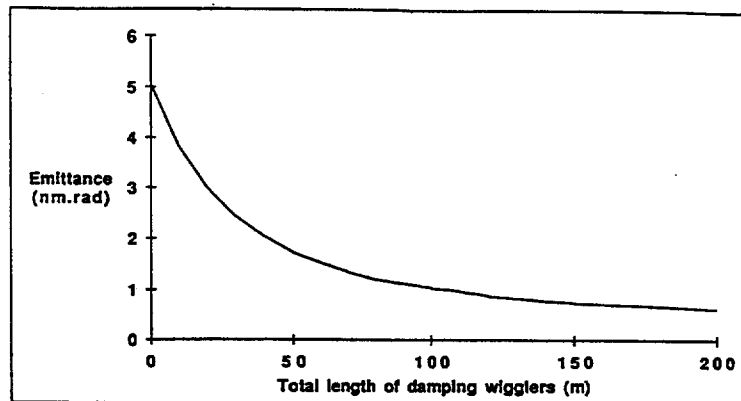


Fig. 43 PEP emittance as a function of the total wiggler length

Similar results could be obtained for the new synchrotron light sources under construction like the ESRF or the APS. Using the proposed insertion devices for the ESRF (for instance sixteen wigglers, 2 m in length and with a 2 T field), an emittance reduction by a factor of about two could be achieved.

This method is applicable to any storage ring. However, since the damping effect scales proportionally with the bending radius of the ring magnets, it is easier to reduce the emittance

with damping wigglers in large storage rings than in small ones. Such a method of reducing the emittance looks promising; but some adverse effects (heat load due to the intense radiation, r.f. system capability..) have still to be solved. Obviously, the space required for installing these devices creates severe constraints. The success of the emittance reduction also depends on further investigations on the effects on dynamic aperture, the limitations due to non-zero dispersion at the wiggler location, Touschek effect and intrabeam scattering.

6. CONCLUSIONS

Lattice design for synchrotron light sources has followed the dramatic evolution of performances of these machines. A few years ago, it was feared that the lattices proposed for third generation light sources would be oversensitive and very difficult to commission. Since then, several sources have been commissioned with their design lattice and have reached or even gone beyond their target performances. Many others, with different lattices, are in the process of being tested or are in the construction phase. The next problem to address to designers is that of diffraction limited lattices.

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INSERTION DEVICES: UNDULATORS AND WIGGLERS

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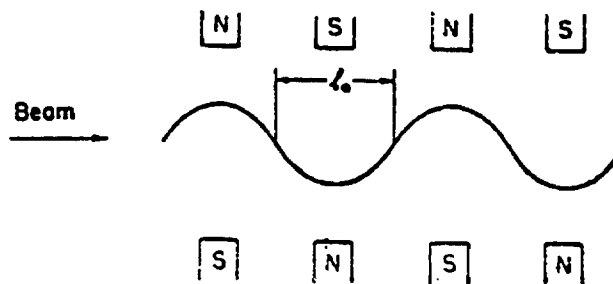
Abstract

The main features of the design and performance of insertion devices are discussed, including the radiation characteristics, magnetic design aspects, and performance optimisation for both standard devices and those producing circularly polarized radiation. The special requirements of undulators used for free-electron lasers are also considered.

1. INTRODUCTION

Undulators and wigglers are magnetic devices producing a spatially periodic field variation that cause a charged particle beam, usually electrons or positrons, to emit electromagnetic radiation with special properties. They are also commonly called "insertion devices", particularly in the context of storage rings, since they can be installed and operated more or less independently of the operation of the ring itself. A large number of insertion devices are in operation in synchrotron radiation facilities world-wide providing radiation with enhanced features compared to that from the bending magnets: higher photon energies, higher flux and brightness, and different polarization characteristics. Periodic magnets of this type are also at the heart of devices which generate coherent radiation, called free-electron lasers (FELs).

Although insertion devices have only been regularly employed as sources of synchrotron radiation in storage rings for less than two decades, their history goes back much further. The first discussion of the use of artificial periodic structures for the generation of microwaves using energetic charged particles was made 50 years ago [1]. A little later, Motz independently proposed that a periodic magnetic structure, which he termed an "undulator" (Fig. 1), could be used to generate quasi-monochromatic electromagnetic radiation from microwaves up to hard X-rays using electron beams with energies in the 1 MeV to 1 GeV range [2]. The initial interest in these ideas was as a source of intense coherent microwave radiation using a tightly bunched electron beam, and experiments were later carried out [3] in which radiation with a wavelength around 1.9 mm was produced using a 3-5 MeV beam; visible light was also observed when using a 100 MeV beam. Similar experiments were also carried out in the Soviet Union [4]. Little further progress was made at that time however because of the difficulties of producing electron beams with the required characteristics to produce coherent radiation as well as the lack of higher energy beams for producing shorter wavelength radiation. In related work to that of Motz, Phillips in 1957 devised a device called a ubitron which uses an undulator and a low energy electron beam to achieve amplification of microwave radiation [5]. The relativistic analogue of this proposed by Madey in 1971 [6] became known as a "free-electron laser" and was first demonstrated in 1976.



Schematic arrangement of undulator magnets.

Fig. 1 Origins of the undulator: from the first page of Motz's article [2]

The possibilities for using undulators in high energy synchrotrons and storage rings as a source of intense, quasi-monochromatic VUV, XUV and X-ray radiation was discussed by several authors in the late 60's and early 70's [7-10]. The first experimental investigations of the properties of undulator radiation at higher energies were carried out in the Soviet Union, using various synchrotrons. The first studies in 1971 used an extracted 3.6 GeV beam from the Arus synchrotron at Erevan to generate X-rays [11]. Subsequently in 1977-79 undulators were installed in the Pakhra (Lebedev Institute, Moscow) [12] and Sirius (Tomsk Polytechnical Institute) [13] synchrotrons. At the same time a large amount of theoretical work was carried out. The first complete and accessible analysis of the spectral/angular distribution in both planar and helical devices was given in Ref. [14]. In the West, the first detailed study of the radiation properties of the helical undulator was made by Kincaid [15]. The first undulators to be installed in storage rings were at the VEPP-3 ring at INP, Novosibirsk in 1979 [16] (for FEL experiments) closely followed by SSRL, Stanford in 1980 [17] (for generation of synchrotron radiation).

The use of periodic magnets in a regime in which interference effects can be neglected is also of interest. In this case the resulting spectrum at high photon energies is smooth, similar to that of a bending magnet, however the radiation intensity can be much higher as a result of the increased number of emitting poles, and higher magnetic field which generates radiation with a higher critical energy. The development of these devices, generally called "wigglers", went on in parallel to that of undulators, see for example Refs. [18,19]. The idea was an extension of the concept in which a single high field pole generates a harder spectrum than the usual bending magnets. The first device of this kind, called a "wavelength shifter", was built for the Tantalus storage ring in 1971 [20]. The first "multipole wiggler" devices were electromagnetic. For example, a 1.8 T, 5-pole device was installed at SSRL in 1979 [21], followed by a similar device on ADONE [22]. Since then permanent magnet devices have been developed with many more poles, for example the 55-pole device installed at SSRL in 1983 [23]. Superconducting technology has also been applied in order to reach higher field strengths. For example, a 3.5 T device was installed on VEPP3 in 1979 [24], and a 5 T wiggler on the SRS in 1982 [25]. Superconducting wigglers are currently operating in several synchrotron radiation facilities: SRS (England), DCI and ESRF (France), UVSOR and Photon Factory (Japan) and NSLS Xray Ring (USA).

Another reason for installing insertion devices in electron rings is to deliberately change the electron beam properties such as damping times, energy spread, and emittance. A periodic device with a gradient field was first described by Robinson in 1958 [26] and later used to convert the C.E.A. from a synchrotron to a storage ring [27]. The use of standard wiggler magnets to change beam sizes and polarization times was discussed in Ref. [28]. The effect of insertion devices on beam properties was discussed in detail in another CERN Accelerator School Proceedings [29] and so will not be repeated here.

In this report we consider the main features of the design and performance of insertion devices, including the radiation characteristics, magnetic design aspects, and performance optimisation. Both standard devices and those producing circularly polarized radiation are treated and the special requirements of undulators used for free-electron lasers are also examined. For further details the reader can refer to the many Conference and Workshop Proceedings [18,30-35], text books [36-42], Proceedings of Schools [43-45] and review articles [46-56].

2. BASIC FEATURES OF THE RADIATION FROM STANDARD INSERTION DEVICES

2.1 Electron motion

In order to understand the properties of the radiation emitted in insertion devices we must begin by firstly calculating the electron motion. We consider a standard device (Fig. 2) as one with median plane symmetry which produces a sinusoidal vertical field component along the magnet axis: $B_y = B_0 \sin(kz)$, where $k = 2\pi/\lambda_0$ and λ_0 is the insertion device period length.

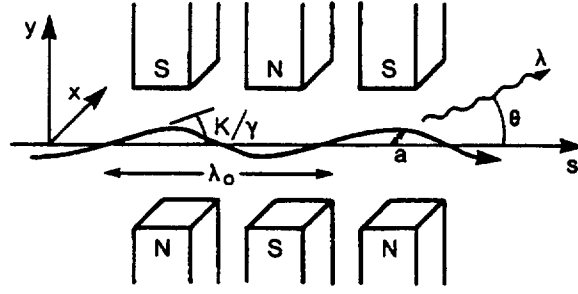


Fig. 2 Schematic of planar undulator and coordinate system

In this analysis we can ignore any transverse (x) variation in the field. The simplified equation of motion is then as follows:

$$\ddot{x} = \frac{e}{\gamma m} (-\dot{z} B_y) \quad \ddot{z} = \frac{e}{\gamma m} (\dot{x} B_y) \quad (1)$$

The first equation can be integrated directly to give:

$$\dot{x} = \frac{e B_o}{\gamma m} \frac{\cos(kz)}{k}$$

and hence

$$\beta_x = \dot{x}/c = \frac{K}{\gamma} \cos(kz) \quad (2)$$

where the dimensionless undulator or deflection parameter is defined as follows:

$$K = \frac{e B_o \lambda_o}{2\pi m c} = 93.4 B_o [T] \lambda_o [m]$$

The horizontal motion of the electron causes the electron velocity along the z axis to vary also, since the electron energy, and hence total speed remain unaltered:

$$\beta_x^2 + \beta_z^2 = \beta^2 \quad (= \text{constant})$$

Inserting Eq. (2) into the above we obtain:

$$\beta_z \cong \beta \left(1 - \frac{K^2}{4\gamma^2} - \frac{K^2}{4\gamma^2} \cos 2kz \right)$$

The average velocity along the z -axis is thus:

$$\bar{\beta} \cong \beta \left(1 - \frac{K^2}{4\gamma^2} \right) \cong 1 - \frac{1}{2\gamma^2} - \frac{K^2}{4\gamma^2}$$

We will only consider cases in which $K/\gamma \ll 1$ and so we can write to a good approximation that $z = \bar{\beta} c t$ and $kz = \Omega t$ where $\Omega = 2\pi \bar{\beta} c / \lambda_o$. We have then:

$$\dot{x} = \frac{K}{\gamma} c \cos(\Omega t) \quad \dot{z} = \bar{\beta} c - \frac{K^2}{4\gamma^2} c \cos(2\Omega t)$$

which can be integrated directly to give:

$$x = \frac{K}{\gamma} \frac{c}{\Omega} \sin(\Omega t) \quad z = \bar{\beta} c t - \frac{K^2}{4\gamma^2} \frac{c}{2\Omega} \sin(2\Omega t)$$

In most cases the actual motion of the particle is quite small: for example, a realistic device with a 50 mm period and $K = 2$ in a 2 GeV ring has a maximum deflection angle (x') of 0.5 mrad and oscillation amplitude of 4 μm . The z-motion is even smaller with an amplitude of only 2.6 Å. Although the modulation in the z-motion is relatively small, it is significant in terms of the emitted wavelength (the fundamental has a wavelength of 49 Å in this example) and so has an important effect on the radiation characteristics, as will be seen later.

2.2 Interference

The radiation properties from an insertion device can most simply be understood in terms of an interference of wavefronts emitted by *the same electron* at different points in the magnet (see Fig. 3). In the time it takes the electron to move through one period length from point A to an equivalent point B ($\lambda_o/\bar{\beta}c$) the wavefront from A has advanced by a distance $\lambda_o/\bar{\beta}$ and hence is ahead of the radiation emitted at point B by a distance d where:

$$d = \frac{\lambda_o}{\bar{\beta}} - \lambda_o \cos \theta$$

and where θ is the angle of emission with respect to the electron beam axis. When this distance is equal to an integral number, n , of radiation wavelengths there is constructive interference of the radiation from successive poles:

$$\frac{\lambda_o}{\bar{\beta}} - \lambda_o \cos \theta = n \lambda \quad (3)$$

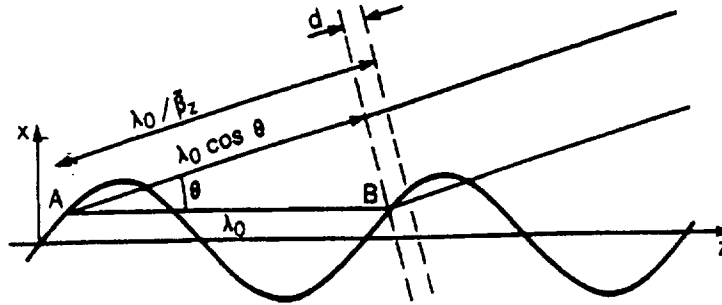


Fig. 3 Interference in an undulator

Inserting the expression for the average electron velocity:

$$\frac{1}{\bar{\beta}} \cong 1 + \frac{1}{2\gamma^2} + \frac{K^2}{4\gamma^2}$$

results in the following interference condition:

$$\lambda = \frac{\lambda_o}{2\gamma^2} \frac{1}{n} \left(1 + \frac{K^2}{2} + \gamma^2 \theta^2 \right) \quad (4)$$

where $n = 1, 2, 3 \dots$ etc. is the harmonic number. From this we can obtain the following expressions for the radiation wavelength and photon energy in practical units:

$$\lambda[\text{\AA}] = 1305.6 \frac{\lambda_o}{E_{[\text{GeV}]}} \frac{1}{n} \left(1 + \frac{K^2}{2} + \gamma^2 \theta^2 \right) \quad \text{and} \quad \varepsilon[\text{eV}] = 9.498 n \frac{E_{[\text{GeV}]}}{\lambda_o (1 + K^2/2 + \gamma^2 \theta^2)}$$

The interference condition tells us immediately several important facts about the radiation from insertion devices:

- The fundamental wavelength of the radiation is very much shorter than the period length of the device, because of the large γ^2 term (for electrons, $\gamma = 1957 E [\text{GeV}]$)
- The wavelength of the harmonics can be varied either by changing the electron beam energy (γ) or the insertion device field strength, and hence K value.
- The wavelength varies with observation angle. Overall therefore the spectrum covers a wide range of wavelength. However, if the range of observation angles is restricted using a "pinhole" aperture, the spectrum will show a series of lines at harmonic frequencies.

The interference model can supply further information about the spread in wavelength and angles as follows. If the insertion device consists of N periods in a length L the condition for constructive interference over the entire length becomes:

$$\frac{L}{\beta} - L \cos \theta = nN\lambda \quad (5)$$

Now we ask at what value of wavelength (λ') the interference becomes destructive. This is obtained when there is one complete extra wavelength of wavefront separation, i.e. 2π phase advance, over the length of the device:

$$\frac{L}{\beta} - L \cos \theta = nN\lambda' + \lambda' \quad (6)$$

Subtracting Eqs. (5) and (6) yields the range of wavelengths at fixed angle θ :

$$\frac{\Delta\lambda}{\lambda} = \frac{1}{nN}$$

Similarly, changing angle θ at fixed wavelength results in destructive interference when:

$$\frac{L}{\beta} - L \cos \theta' = nN\lambda + \lambda \quad (7)$$

Subtracting Eqs. (7) and (5) gives $\Delta(\theta^2) = 2\lambda/L$. In the special case of radiation with the on-axis wavelength (i.e. $\theta = 0$) the width therefore becomes:

$$\Delta\theta = \sqrt{\frac{2\lambda}{L}} = \frac{1}{\gamma} \sqrt{\frac{1 + K^2/2}{iN}}$$

The cone of radiation at this wavelength can therefore be significantly smaller in opening angle compared to that of conventional synchrotron radiation ($\sim 1/\gamma$), a first indication of the high brightness of such sources.

2.2 Electron rest frame

Another model that gives a simple understanding of the properties of undulator radiation is to consider the electron motion in the frame that moves with the electron's average velocity along the axis, $\bar{\beta}c$. The electron sees the undulator approaching with relativistic factor $\bar{\gamma}$ where:

$$\bar{\gamma} = \frac{1}{(1 - \bar{\beta}^2)^{1/2}} \cong \frac{\gamma}{(1 + K^2/2)^{1/2}}$$

The undulator period appears Lorentz contracted with period length $\lambda_o/\bar{\gamma}$. In the moving frame the electron is therefore caused to oscillate with this wavelength, emitting a dipole radiation pattern. The transformation to the lab. frame involves a relativistic Doppler shift:

$$\lambda = \lambda' \bar{\gamma} (1 - \bar{\beta} \cos \theta) = \frac{\lambda'}{2\bar{\gamma}} (1 + \bar{\gamma}^2 \theta^2)$$

Inserting the expression above for $\bar{\gamma}$ gives the same expression for the wavelength, Eq. (4).

In the moving frame the z oscillation amplitude is multiplied by a factor $\bar{\gamma}$ and so the ratio between the oscillation amplitudes in the two planes is given as follows:

$$\frac{z' \text{ amplitude}}{x' \text{ amplitude}} = \frac{\bar{\gamma} \left(\frac{K^2 \lambda_o}{8\gamma^2 2\pi} \right)}{\left(\frac{K \lambda_o}{\gamma 2\pi} \right)} = \frac{K}{8(1 + K^2/2)^{1/2}}$$

Thus, when K is small the oscillation is almost purely simple harmonic motion in the x-direction and as a result the radiation consists of a single harmonic. As K increases however the electron performs a "figure-of-eight" motion in the x'-z' plane (see Fig. 4) implying an increased emission at higher harmonic frequencies. Calculating the power emitted in the dipole mode [57] and comparing it to the total power emitted one obtains the ratio:

$$\frac{\text{power in the fundamental}}{\text{total power}} = \frac{1}{(1 + K^2/2)^{1/2}}$$

which shows that the relative power in the fundamental decreases rapidly with increasing K value.

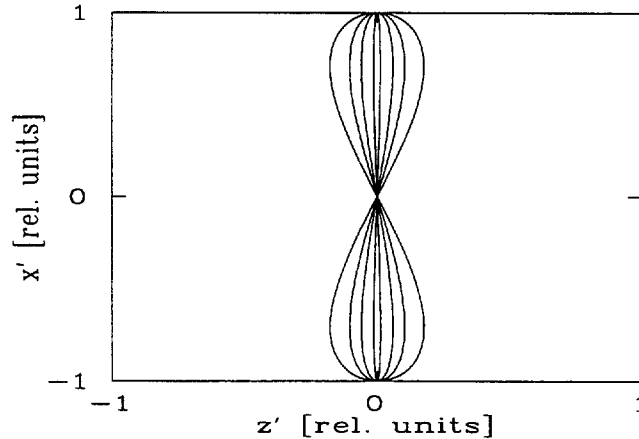


Fig. 4 Figure-of-eight motion in the frame of reference moving with the electron's average velocity, normalized to the same x' amplitude, for $K = 0.1$ (inner), 0.5, 1.0, 10.0 (outer)

2.3 Electric field

Another way of deducing some of the characteristics of the radiation is to consider in a qualitative way the electric field emitted by the electron. We imagine the electron moving on an oscillating trajectory with a maximum angular excursion of K/γ while emitting radiation into a cone with the typical opening angle of synchrotron radiation, $\sim 1/\gamma$. It is reasonable to suppose therefore that for $K \leq 1$ an observer "sees" a continuous sinusoidal emission from the device.

Correspondingly the radiation spectrum, which is the Fourier Transform of the electric field, consists of a single harmonic (see Fig. 5). At large K on the other hand the observer only detects radiation near the peaks in the trajectory, corresponding to the position of the magnet poles. In this case the electric field is strongly peaked and so the radiation spectrum contains a large number of harmonics.

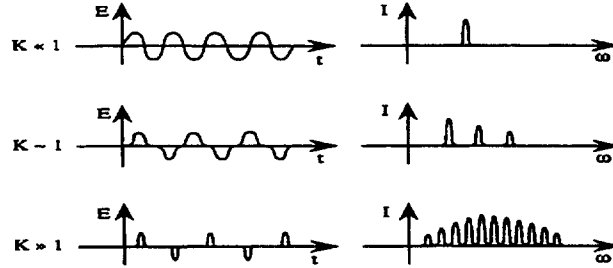


Fig. 5 Schematic diagram showing the electric field and corresponding radiation spectrum in devices with various K values

We can also deduce from this picture the spectral width of the harmonics, as follows. The observed signal can be considered a continuous periodic signal multiplied by a "top-hat" function i.e. a function that is equal to unity within the range $\pm 2/T_o$, i.e. $\pm N/2 v_1$ where v_1 is fundamental frequency, and zero outside. According to the convolution theorem the Fourier Transform is the transform of the continuous signal convoluted with the transform for the top-hat function. The former consists of a series of lines at the harmonic frequencies, while the latter is a sinc function $(\sin(N\pi v/v_1)/\pi v_1)$ with a width in frequency that is inversely proportional to the length of the impulse: $\Delta v = 1/T_o = v_1/N$. After convolution each line has this same width, and so the relative width becomes $\Delta v/v = 1/nN$, in agreement with the result of section 2.1.

2.4 The high K limit

It is evident from the above that the number of harmonics in the spectrum increases as a function of the K value. The number can be estimated by comparing the equation for the wavelength of the harmonics, Eq. (4), with that of the critical wavelength for conventional synchrotron radiation corresponding to the peak field amplitude [58]:

$$\lambda_c = \frac{4\pi mc}{3\gamma^2 e B_o}$$

The harmonic number corresponding to a given wavelength is then as follows:

$$n = \frac{(3/4)K(1+K^2/2)}{(\lambda/\lambda_c)}$$

Table 1

Harmonic numbers giving a wavelength equal to the critical wavelength ($\lambda = \lambda_c$) as a function of K

K	n
1	1
2	5
3	12
5	51
10	383
20	3015

We can get an idea of the number of harmonics in the spectrum by calculating the harmonic number corresponding to the critical wavelength, $\lambda = \lambda_c$. Table 1 shows that the number increases very rapidly with the K value. Since the harmonics are equally spaced in frequency ($\omega_1, 2\omega_1, 3\omega_1, 4\omega_1 \dots$) at high harmonic numbers they are relatively closer together ($\Delta\omega/\omega = 1/n$) making it easier for the interference structure to become smoothed-out, resulting in a spectrum similar to that of a bending magnet. Interference effects will always remain however around the fundamental and low order harmonics. Thus, a device with relatively large K value ($K \geq 5$) can exhibit both strong interference effects near the fundamental as well as a quasi-smooth spectrum at high harmonic numbers. The degree of smoothing that occurs will be discussed in more detail in Section 3.7.

A note on nomenclature:

In the context of synchrotron radiation sources devices with low K values are traditionally called "undulators" while those with high K values are generally called "multipole wigglers" or simply "wigglers". It should be clear from the descriptions above however that there is no fundamental distinction between them. Unfortunately there is no universal agreement over nomenclature, which is further confused by the fact that magnets employed in free-electron lasers are often (particularly in the USA) called "wigglers", rather than the more historically accurate term "undulators".

3. RADIATION FROM INSERTION DEVICES: DETAILED ANALYSIS

3.1 Spectral/angular distribution

The energy radiated by a single electron during one passage through an undulator, per unit solid angle ($d\Omega$), per unit frequency interval ($d\omega$) is given by the Fourier Transform of the electric field seen by the observer [59]:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{c}{4\pi^2} \left| \int_{-\infty}^{\infty} R E(t) e^{i\omega t} dt \right|^2 \quad (8)$$

The electric field seen by the observer at time t is related to the electron acceleration at the earlier emission time, conventionally referred to as the "retarded time", $t_{ret.}$, as follows:

$$E(t) = \frac{e}{\sqrt{4\pi\epsilon_0 c}} \left[\frac{\hat{n} \wedge \{(\hat{n} - \beta) \wedge \dot{\beta}\}}{(1 - \hat{n} \cdot \beta)^3 R} \right]_{t_{ret.}} \quad (9)$$

where $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is the unit vector from the point of emission to the observer (see Fig. 6). The observer and emission times are related by: $t = t_{ret.} + R/c$ where R is the distance between the emission and observer points, and hence:

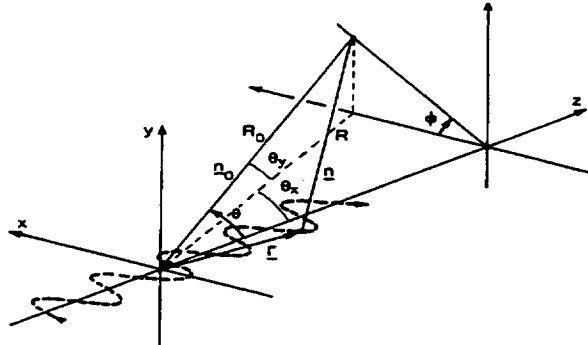


Fig. 6 Geometry for the analysis of undulator radiation

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2 c}{(4\pi\epsilon_o)4\pi^2} \left| \int_{-\infty}^{\infty} \left[\frac{\hat{n} \wedge \{(\hat{n} - \beta) \wedge \dot{\beta}\}}{(1 - \hat{n} \cdot \beta)^3} \right]_{t_{ret.}} e^{i\omega t} dt \right|^2 \quad (10)$$

In cases in which the observer is not very far from the source, the observation angle $\hat{n}$ must be considered to be a variable in the equation above, leading to additional "near-field" effects [60]. For most cases however it is possible to consider only the "far-field" case with $\hat{n}$ constant. In this case we can write $R = R_o - \hat{n}_o \cdot r$ and hence, ignoring a constant phase term:

$$t = t_{ret.} - \hat{n}_o \cdot r/c$$

from which it also follows that

$$\frac{dt}{dt_{ret.}} = 1 - \hat{n}_o \cdot \beta \quad (11)$$

At this point we make a brief aside to examine the effect of the difference between the emission and observer time on the electric field distribution. Figure 7 (left) shows the calculated electric field for insertion devices with various K values. As already anticipated in section 2.3, the field is essentially sinusoidal for small K values and becomes sharply peaked for large K values. This is even more evident when expressed in terms of the observer time, as shown in Fig. 7 (right). Here the time axis has been normalized to a unit time period because of the large difference between the three cases. At small K the pattern remains sinusoidal, but at large K it becomes extremely sharply peaked. The reason for this behaviour is the varying "time compression" factor [43], Eq. (11), relating the emission and observer times, which for small angles between the direction of motion and the angle of observation can be expressed as:

$$\frac{dt_{ret.}}{dt} = \frac{2\gamma^2}{(1 + \gamma^2 \theta^2)}$$

Thus, for small K and for large K in the region of the poles when the angle θ is close to zero, the time scales are compressed by the large factor $2\gamma^2$. For most of the trajectory however in the case of large K the factor is much smaller since the electrons are not travelling in the direction of observation. On an absolute scale therefore the electric field emission close to the peaks remains the same as in Fig. 7 (left). The difference in Fig. 7 (right) is due to the much longer time *between* the peaks (in terms of the observer time) for large K than for small K , which is reflected in the longer wavelength of the radiation.

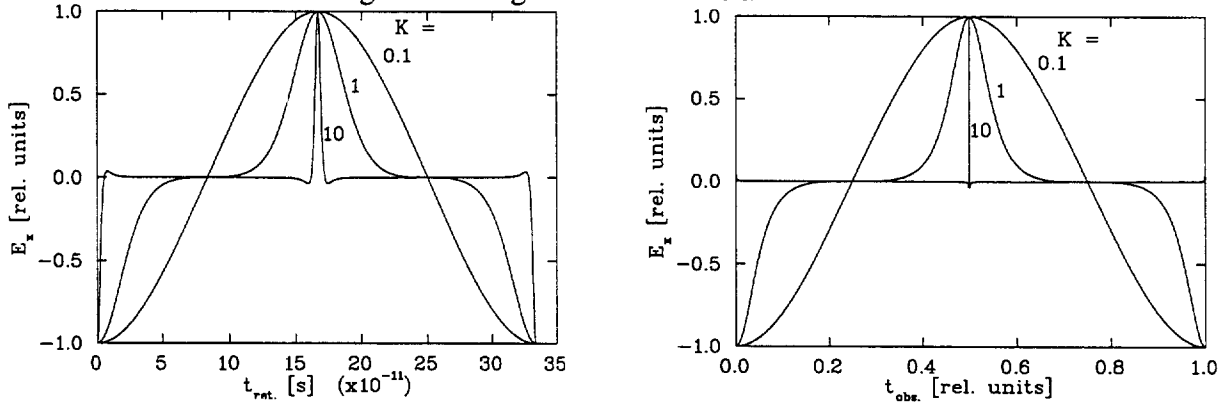


Fig. 7 Electric field calculated using Eq. (9) for one period in an undulator ($\lambda_o = 0.1$ m, $E = 2$ GeV) expressed in terms of the emission or retarded time (left) and observer time (right), normalized to the same time period

Returning to the calculation of the spectral and angular distributions, the result expressed in terms of the emitter time is then as follows:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2}{(4\pi\epsilon_0)4\pi^2 c} \left| \int_{-\infty}^{\infty} \frac{\hat{n} \wedge \{(\hat{n} - \beta) \wedge \beta\}}{(1 - \hat{n} \cdot \beta)^2} e^{i\omega(t - \hat{n} \cdot r/c)} dt \right|^2 \quad (12)$$

where from now on it is also to be understood that $\hat{n} = \hat{n}_o$. This equation, or the more general Eq. (10), can be used directly for numerical calculation: having specified the magnetic field of the device, the electron motion $(r, \beta, \dot{\beta})$ can be computed and then the integration above can be performed.

A further simplification can generally be made, by integrating the above expression by parts, after which one obtains [59]:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2 \omega^2}{(4\pi\epsilon_0)4\pi^2 c} \left| \int_{-\infty}^{\infty} \{\hat{n} \wedge (\hat{n} \wedge \beta)\} e^{i\omega(t - \hat{n} \cdot r/c)} dt \right|^2 \quad (13)$$

In the case of a periodic magnet the integral for each period is the same, apart from a phase factor and hence we can write:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2 \omega^2}{(4\pi\epsilon_0)4\pi^2 c} \left| \int_{-\lambda_o/2\beta c}^{\lambda_o/2\beta c} \{\hat{n} \wedge (\hat{n} \wedge \beta)\} e^{i\omega(t - \hat{n} \cdot r/c)} dt \right|^2 \left| (1 + e^{i\delta} + e^{i2\delta} + e^{i3\delta} + \dots e^{i(N-1)\delta}) \right|^2$$

where the phase increase between one period and the next is given by:

$$\delta = \omega \left(T - \frac{\lambda_o \cos \theta}{c} \right) = \omega \left(\frac{\lambda_o}{\beta c} - \frac{\lambda_o \cos \theta}{c} \right)$$

which can be expressed in terms of the fundamental frequency, using Eq. (4), as follows:

$$\delta = \frac{2\pi\omega}{\omega_1(\theta)}$$

The geometric series of phase factors can be simplified as follows:

$$\left| (1 + e^{i\delta} + \dots e^{i(N-1)\delta}) \right|^2 = \frac{\sin^2 N\delta/2}{\sin^2 \delta/2} = \frac{\sin^2 N\pi\omega/\omega_1}{\sin^2 \pi\omega/\omega_1} \quad (14)$$

This function, sometimes called the "grating function" in analogy with diffraction from a grating or crystal, represents the interference between successive periods [61,62], and selects a narrow range of frequencies and angles near the harmonics, i.e. $\omega = n\omega_1(\theta)$, as shown in Fig. 8 (left). The lineshape for each harmonic is the same which we can express as follows, normalized to unit amplitude:

$$L(N\Delta\omega/\omega_1(\theta)) = \frac{\sin^2(N\pi\Delta\omega/\omega_1(\theta))}{N^2 \sin^2(\pi\Delta\omega/\omega_1(\theta))} \quad (15)$$

where $\Delta\omega = \omega - n\omega_1(\theta)$. For reasonably large values of N (≥ 10) the shape becomes independent of N and can be expressed in terms of a single variable $N\Delta\omega/\omega_1(\theta)$, as shown in Fig. 8 (right).

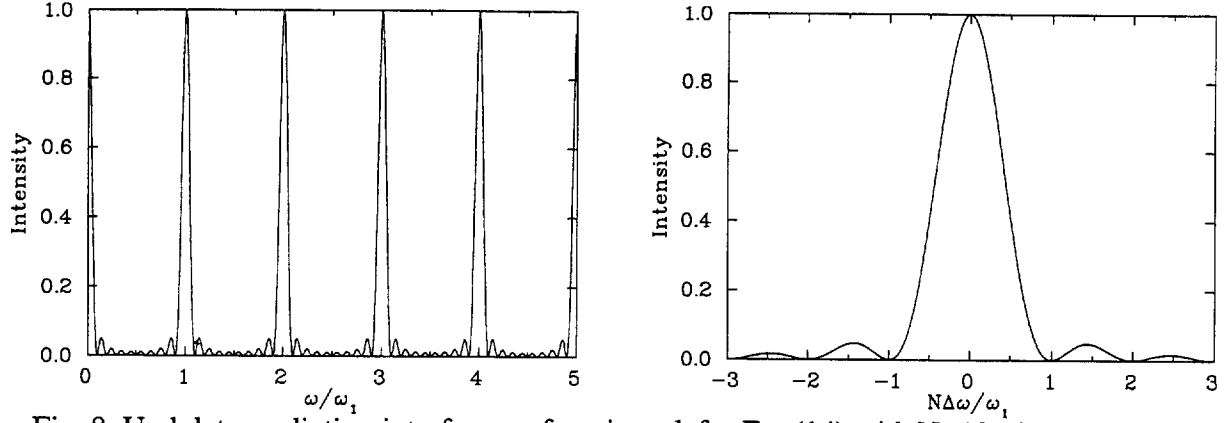


Fig. 8 Undulator radiation interference functions: left - Eq. (14) with $N=10$, right - Eq. (15)

The remaining part of the integral is a more smoothly varying function and so need only be evaluated at the harmonic frequencies, and thus becomes a function only of observation angle, $\hat{n}$, and harmonic number. As in the case of Eq. (12) the integration can of course be carried out numerically, in which case any form of magnetic field distribution can be used. One can also proceed analytically in the case of a pure sinusoidal motion [14, 63] (see Appendix). The result can be expressed in the following form:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2 \gamma^2 N^2}{(4\pi\epsilon_0)c} L(N\Delta\omega/\omega_1(\theta)) F_n(K, \theta, \phi) \quad (16)$$

The function $F_n(K, \theta, \phi)$ is smoothly varying in the vertical plane, but is an oscillatory function with n peaks in the horizontal plane, as shown in Fig. 9. The complex structure in the horizontal plane arises from the interference of the emission from the two poles within each period [64].

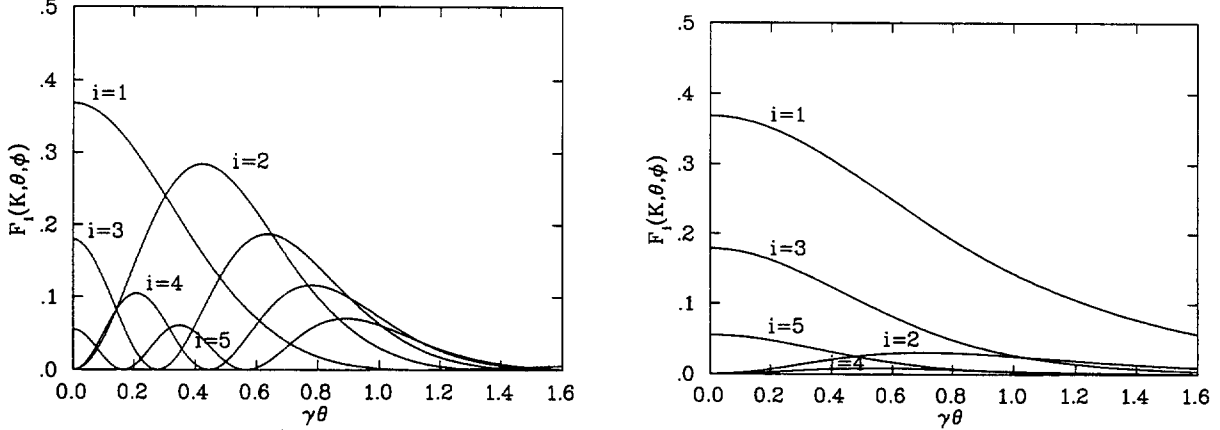


Fig. 9 Angular flux density function in the horizontal (left) and vertical (right) planes for the case $K = 1$

On-axis the expression simplifies to the following:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{N^2 e^2 \gamma^2}{(4\pi\epsilon_0)c} L(N\Delta\omega/\omega_1(\theta)) F_n(K)$$

where:

$$F_n(K) = \frac{n^2 K^2}{(1 + K^2/2)^2} \left[J_{\frac{n+1}{2}}(Z) - J_{\frac{n-1}{2}}(Z) \right]^2 \quad \text{and} \quad Z = \frac{n K^2}{4(1 + K^2/2)}$$

It is interesting to note that if we neglect the longitudinal oscillation, i.e. $Z = 0$, the result would be only a single harmonic on-axis, $n = 1$. We can say therefore that the higher harmonics on-axis arise as a direct result of the z -motion, whose amplitude increases as K increases. Figure 10 shows the on-axis angular flux density function $F_n(K)$, which again indicates that the higher harmonics become more important as K increases.

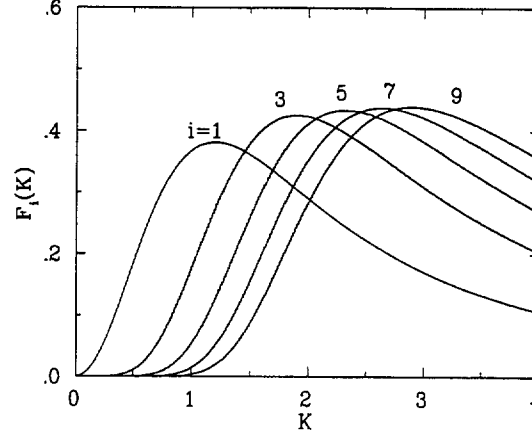


Fig. 10 On-axis angular flux density function

The expression above gives the energy radiated (Joules) by the passage of a single electron through the undulator. Multiplying by the number of electrons passing per second, I_b/e , where I_b is the average beam current (Amps), gives the radiated power (Watts). It is more usual however to express the intensity in terms of the photon flux, the number of photons per second, $\dot{n}$: dividing by the photon energy $\hbar\omega$ gives the photon flux per unit bandwidth, or alternatively dividing by $\hbar$ gives the photon flux per unit relative bandwidth. Finally we obtain in practical units of photons/s/mrad²/0.1% bandwidth:

$$\left. \frac{d^2 \dot{n}}{d\omega/\omega \, d\Omega} \right|_{\theta=0} = 1.744 \cdot 10^{14} \, N^2 \, E_{[GeV]}^2 \, F_n(K) \, I_b$$

3.2 Total flux

It is also of interest to know how much total flux is available at a given wavelength. We can obtain an approximate expression by assuming that the intensity variation off-axis is determined only by the lineshape function, Eq. (15), which is a function of both frequency and angle:

$$L\left(N \frac{\Delta\omega}{\omega_1(\theta)}\right) = L\left(N \frac{\Delta\omega}{\omega_1(0)} + \frac{L\theta^2}{2\lambda}\right) \quad (17)$$

where the "detuning parameter" $\Delta\omega/\omega_1(0) = (\omega - n\omega_1(0))/\omega_1(0)$ is the difference between the frequency and the on-axis value. Integrating over angle we obtain:

$$\int_0^\infty \int_0^{2\pi} L(N \Delta\omega/\omega_1(0)) \, \theta \, d\theta \, d\phi = \frac{2\pi\lambda}{L} f(N \Delta\omega/\omega_1(0))$$

We obtain therefore in practical units of photons/s/0.1% bandwidth:

$$\frac{d\dot{n}}{d\omega/\omega} = 1.431 \cdot 10^{14} \, N \, Q_n(K) f(N \Delta\omega/\omega_1(0)) \, I_b \quad (18)$$

where $Q_n(K) = (1 + K^2/2)F_n(K)/n$. The flux function $Q_n(K)$ and the detuning function $f(N\Delta\omega/\omega_1(0))$ are shown in Fig. 11. It can be seen that for zero detuning (i.e. $\omega = \omega_n(0)$) the flux is very close to half of the usually quoted result. Nearly twice as much flux can be obtained however by a small detuning to lower frequency by approximately $\Delta\omega/\omega_1(0) \cong -1/N$.

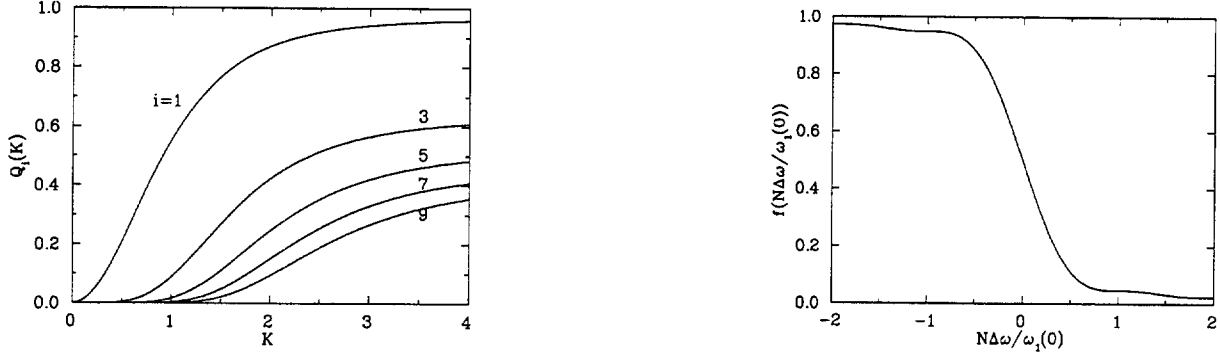


Fig. 11 Undulator flux function (left) and flux as a function of detuning (right)

It is interesting to compare these results with that of a simple bending magnet source. The peak bending magnet angular flux density is given approximately by $1.95 \cdot 10^{13} E_{[GeV]}^2 I_b$, in the same units as above. The ratio between the undulator and the peak bending magnet angular flux density is therefore $8.9 N^2 F_n(K)$. For example, the angular flux density of an undulator with 50 periods and $K=1$ exceeds that of a bending magnet by nearly 4 orders of magnitude. The peak bending magnet flux, integrated over all vertical angles, is given approximately by $2.23 \cdot 10^{13} E_{[GeV]} I_b$ per mrad horizontal. The ratio in this case is therefore $6.4 N F_n(K)(1 + K^2/2)/n E_{[GeV]}$. The same undulator as above in a 2 GeV ring gives approximately an order of magnitude higher flux on its first harmonic than the maximum flux per mrad from a bending magnet source. We can begin to appreciate therefore not only the higher flux, but particularly the much higher brightness of the undulator source compared to a bending magnet.

3.3 Brightness and the effect of electron beam emittance

We start by considering the effect of the electron beam divergence on the angular flux density. In general a numerical approach is required in order to carry out the 2-dimensional convolution of the complex angular distribution of the radiation with the Gaussian electron beam. We can however derive an approximate result by considering that the radiation also has a Gaussian distribution. Figure 12 (left) shows the approximate form of the angular distribution of the radiation as determined by the lineshape function, for zero detuning and for a detuning of $\Delta\omega/\omega_1(0) \cong -1/N$. It is clear that a Gaussian approximation is only valid at zero-detuning, and even then is not very accurate. The dashed line shows a Gaussian approximation for the zero-detuning case with standard deviation, $\sigma'_R = \sqrt{\lambda/2L}$. The significance of this value is that it gives the correct ratio between integrated flux and peak angular flux density:

$$\frac{d\dot{n}/(d\omega/\omega)}{d^2\dot{n}/(d\omega/\omega)d\Omega|_{\theta=0}} = 2\pi\sigma_R'^2 = \pi\frac{\lambda}{L}$$

Using the Gaussian approximation above the effective divergences of the source become simply:

$$\Sigma'_x = \sqrt{\sigma_x'^2 + \sigma_R'^2} \quad \Sigma'_y = \sqrt{\sigma_y'^2 + \sigma_R'^2}$$

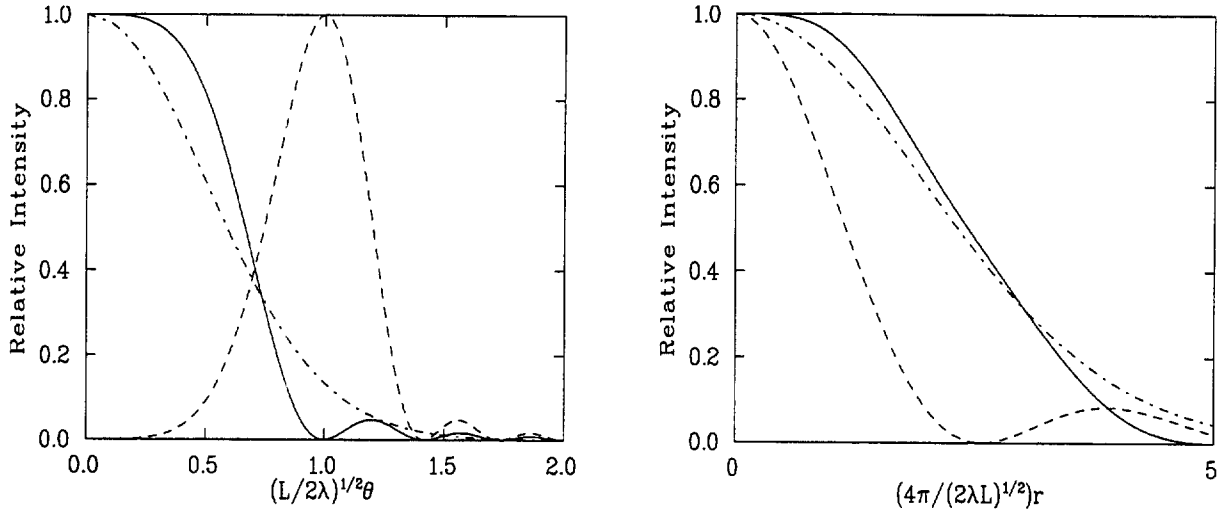


Fig. 12 Intensity as a function of angle (left) and position (right).
Solid lines - zero detuning, dashed lines - detuning of $\Delta\omega/\omega_1(0) \cong -1/N$,
dot-dashed lines - Gaussian approximations for zero detuning (see text)

where σ_x, σ_y are the rms angular divergences of the electron beam. The total flux remains constant and so we can write the expression for the modified on-axis angular flux density including electron beam divergence as follows:

$$\left. \frac{d^2 \dot{n}}{d\omega/\omega \, d\Omega} \right|_{\theta=0} = \frac{d\dot{n}/(d\omega/\omega)}{2\pi \Sigma'_x \Sigma'_y}$$

The radiation brightness includes not only the angular divergence of the source but also the emitting source size. The standard unit of measurement is thus photons/s/mm²/mrad²/0.1% bandwidth. The most simple view of how the size of the source comes about, is to consider projecting the angular cone of the radiation to the centre of the undulator. We might expect therefore a spatial distribution at the centre of size $\sigma_R \sim \sigma'_R L/2$, and hence with the above expression for σ'_R , we obtain $\sigma_R \sim \sqrt{2\lambda L}/4$. The correct approach however is to integrate the wave amplitudes, in the same way as one calculates diffraction from an aperture. The spatial distribution is thus given by:

$$I(r) = \left| \frac{1}{\lambda} \int_0^\infty \int_0^{2\pi} A(\theta, \phi) e^{-i2\pi(\hat{n} \cdot r)/\lambda_o} \theta \, d\theta \, d\phi \right|^2$$

Carrying out this calculation with the angular distribution given by Eq. (15) results in the functions shown in Fig. 12 (right). The zero-detuning case can be approximated well by a Gaussian with standard deviation $\sigma_R = \sqrt{2\lambda L}/2\pi$, i.e. very similar to the simple result above. The effective source sizes are then given by the convolution with the electron beam dimensions, σ_x, σ_y , i.e.:

$$\Sigma_x = \sqrt{\sigma_x^2 + \sigma_R^2} \quad \Sigma_y = \sqrt{\sigma_y^2 + \sigma_R^2}$$

Combining the information about source size with the angular flux density allows us to estimate the brightness, as follows:

$$B = \frac{d\dot{n}/(d\omega/\omega)}{4\pi^2 \Sigma_x \Sigma_y \Sigma'_x \Sigma'_y}$$

The limiting cases of the above equation are:

- i/ $\sigma_R > \sigma_{x,y}$ and $\sigma'_R > \sigma'_{x,y}$. At long wavelength and with small electron beam dimensions and divergences the natural radiation dimensions dominate; this is often referred to as the "diffraction limit".
- ii/ $\sigma_R < \sigma_{x,y}$ and $\sigma'_R < \sigma'_{x,y}$. At the other extreme case the brightness is dominated by the electron beam emittance, and we can write in the usual case (insertion device at a symmetry point with zero dispersion) $B = Flux/4\pi^2 \varepsilon_x \varepsilon_y$, where $\varepsilon_x = \sigma_x \sigma'_x$, $\varepsilon_y = \sigma_y \sigma'_y$ are the electron beam emittances in the two planes.

Both extremes are indicated in Fig. 13 which shows the variation of the denominator in the above expression, $\Sigma_x \Sigma_y \Sigma'_x \Sigma'_y$, as a function of photon energy using the parameters of ELETTRA as representative of "third-generation" synchrotron radiation facilities, compared to the result for a zero emittance electron beam. At low photon energies therefore the radiation is close to being diffraction limited, while at high energies the electron beam emittance is clearly the limiting factor.

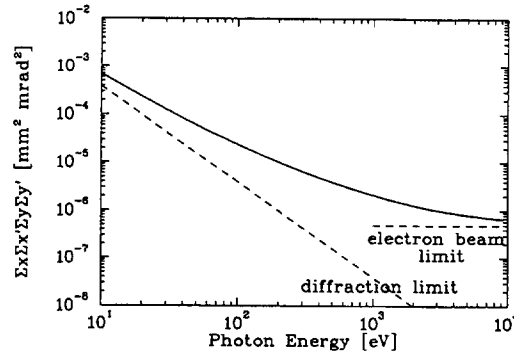


Fig. 13 Product of photon beam emittances in the case of ELETTRA

It must be remembered that for various reasons the expression for the brightness is only an approximation. It has already been shown that away from the condition of zero-detuning, the angular distribution function becomes strongly non-Gaussian, as shown in Fig. 12 (left). Under these conditions it is difficult to arrive at a sensible definition of the effective standard deviation which gives the correct convolution with the electron beam divergence. The calculation of the source size also should take into account the effect of detuning. A further problem is that, as already noted, no account is taken of the off-axis angular distribution function $F_n(K, \theta, \phi)$ which can have a significant effect on the overall angular distribution particularly when n is large and N is small. Finally, another potentially important effect is not included, namely the electron beam energy spread. A better approximation can therefore be made by making a numerical calculation of the on-axis angular flux density (see section 3.6), including the true angular distribution function and the effects of emittance and energy spread, and then applying an approximate calculation of the source size, i.e. defining the brightness as:

$$B = \frac{d^2 \dot{n} / (d\omega/\omega) d\Omega}{2\pi \Sigma_x \Sigma_y}$$

To go beyond these approximate treatments requires a more fundamental definition of the brightness in terms of the electric field distribution [65] which however is outside the scope of the present discussion.

3.4 Coherence

The degree of coherence of the radiation beam is an increasingly important parameter for various synchrotron radiation experiments. Basically, there are two measures of coherence - temporal (or longitudinal) coherence and spatial (or transverse) coherence. Temporal coherence

determines the extent to which two parts of a beam, one delayed with respect to the other, can mutually interfere. Spatial coherence on the other hand determines the extent to which two spatially separated (but not delayed) parts can interfere. Temporal coherence is determined by the degree of monochromaticity of the radiation, and is described by the longitudinal coherence length over which waves of different wavelength remain in phase:

$$l_s = \frac{\lambda^2}{2 \Delta \lambda}$$

In most cases this is determined by the monochromator bandwidth, which is usually considerably smaller than the linewidth of the undulator radiation.

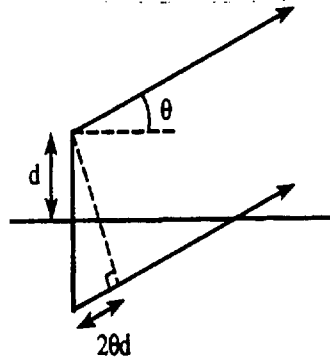


Fig. 14 Definition of spatial coherence in terms of phase-space area

Spatial coherence is determined by the phase-space area of the source. Simple geometry is sufficient to show that only waves which are emitted from an area with half-size d and half angle θ where $2d\theta \leq \lambda/4$ are in phase, and hence coherent (Fig. 14). It follows therefore from the definitions above that the amount of spatially coherent flux is directly proportional to the brightness. The minimum obtainable product of size and divergence is defined by the Uncertainty Principle and is represented by the fundamental (Gaussian) mode of a laser resonator, for which $\sigma_R \sigma'_R = \lambda/4\pi$. Such a beam has therefore complete spatial coherence. According to the definitions above, undulator radiation, in the absence of electron beam emittance, has a product of approximately $\sigma_R \sigma'_R = \lambda/2\pi$. The radiation is nevertheless completely spatial coherent, since it is a single radiation pattern from a single electron. Partial coherence results only when a summation is taken over many electrons in an electron beam. We can therefore write that the coherent flux is given approximately as follows:

$$F_{coh.} = B \left(\frac{\lambda}{2\pi} \right)^2$$

and hence also that the fraction of coherent flux is:

$$\frac{F_{coh.}}{F_{tot.}} = \frac{(\lambda/2\pi)^2}{4\pi^2 \Sigma_x \Sigma_y \Sigma'_x \Sigma'_y}$$

A more detailed treatment of brightness and coherence aspects can be found in Refs. [45, 65].

3.5 Power and power density

The instantaneous power radiated per unit solid angle in the direction $\hat{n}$ by a single electron in arbitrary motion in a magnetic field (i.e. with no acceleration) is given as follows [66]:

$$\frac{dP}{d\Omega} = \frac{e^2}{(4\pi\epsilon_0)4\pi c} \left[\frac{\dot{\beta}^2}{(1 - \hat{n} \cdot \beta)^3} - \frac{(\hat{n} \cdot \dot{\beta})^2}{\gamma^2(1 - \hat{n} \cdot \beta)^5} \right]$$

After integrating over the length of the insertion device to get the energy radiated per unit solid angle, and multiplying by I_b/e to get the power radiated by a beam of electrons in the insertion device, the result can be expressed in the following form in practical units [67]:

$$\frac{dP}{d\Omega} [\text{W/mrad}^2] = 13.44 \cdot 10^{-3} E_{[\text{GeV}]}^4 I_b \int \left[\frac{v_x'^2 + v_y'^2}{D^3} - \frac{(v_x^2)' + (v_y^2)'}{D^5} \right] dz \quad (19)$$

where $v_x(z) = \gamma(\beta_x - \theta_x)$, $v_y(z) = \gamma(\beta_y - \theta_y)$, $D = 1 + v_x^2 + v_y^2$ and where the prime indicates the derivative with respect to z .

In the case of a plane sinusoidal device $v_x = K \sin \alpha - \gamma \theta_x$ and $v_y = -\gamma \theta_y$, where $\alpha = 2\pi z/\lambda_o$. After some simplification the result becomes:

$$\frac{dP}{d\Omega} [\text{W/mrad}^2] = 13.44 \cdot 10^{-3} E_{[\text{GeV}]}^4 I_b \frac{2\pi}{\lambda_o} N K^2 \int_{-\pi}^{\pi} \cos^2 \alpha \left[\frac{1}{D^3} - \frac{4(\gamma \theta_x - K \sin \alpha)^2}{D^5} \right] d\alpha$$

where $D = 1 + (\gamma \theta_y)^2 + (K \sin \alpha - \gamma \theta_x)^2$. This can be expressed in the following form:

$$\frac{dP}{d\Omega} [\text{W/mrad}^2] = 10.84 E_{[\text{GeV}]}^4 B_o N I_b G(K) f_K(\theta_x, \theta_y)$$

where $G(K) = \frac{K(K^6 + \frac{24}{7}K^4 + 4K^2 + \frac{16}{7})}{(1 + K^2)^{7/2}}$ and

$$f_K(\theta_x, \theta_y) = \frac{16K}{7\pi G(K)} \int_{-\pi}^{\pi} \sin^2 \alpha \left[\frac{1}{D^3} - \frac{4(\gamma \theta_x - K \cos \alpha)^2}{D^5} \right] d\alpha, \text{ as obtained by Kim [68].}$$

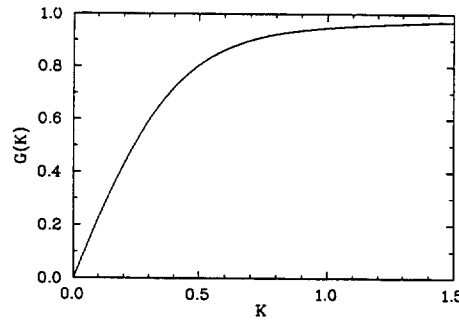


Fig. 15 The function $G(K)$

The function $G(K)$ is illustrated in Fig. 15 and is close to unity for $K > 1$. The function $f_K(\theta_x, \theta_y)$ has a peak value of unity on-axis and is shown in Fig. 16 in the horizontal and vertical planes for various K values. It can be seen that in the vertical plane the angular distribution is insensitive to the value of K and in the limit $K \rightarrow \infty$ is equal to that of a bending magnet. For large K in the horizontal plane the distribution becomes semi-circular, since the power density is proportional to the field at the corresponding tangent point: at an angle $\theta_x = (K/\gamma) \cos(kz)$ the field at the tangent point is $B = B_o \sin(kz)$ and hence $(B/B_o)^2 + (\gamma \theta_x/K)^2 = 1$.

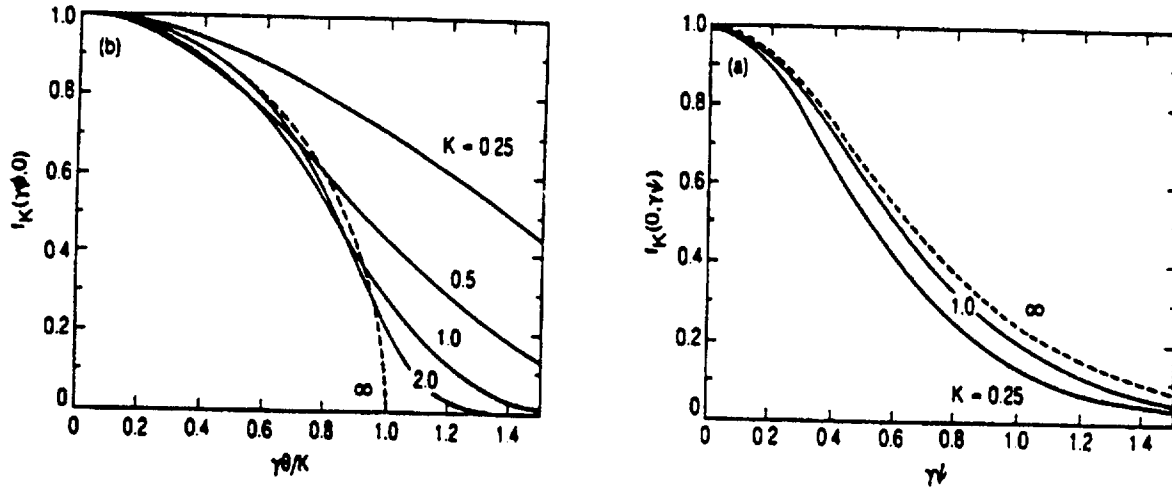


Fig. 16 Power density as a function of angle in the horizontal (left) and vertical (right) planes

The instantaneous rate of total power emitted by a single electron is given by the Liénard result (1898) [59]:

$$P = \frac{2}{3} \frac{e^2 c}{4\pi\epsilon_0 \rho^2} \beta^2 \gamma^4$$

where ρ is the radius of curvature of the trajectory. Integrating over the insertion device length and converting to practical units we obtain:

$$P_{\text{tot}} [\text{W}] = 1265 \cdot E_{[\text{GeV}]}^2 \int B^2 ds I_b \quad (20)$$

in which B is the magnitude of the magnetic field, irrespective of its direction. For an insertion device of length L with sinusoidal field amplitude B_o we have then:

$$P_{\text{tot}} [\text{W}] = 633 \cdot E_{[\text{GeV}]}^2 B_o^2 L I_b$$

Table 2

Total power P_{tot} (kW) and peak power density $d^2P/d\Omega$ (kW/mrad²) for typical undulator and multipole wiggler parameters in storage rings of various energies; insertion device length = 3 m, beam current = 200 mA.

$E(\text{GeV})$	$\lambda_o = 0.05 \text{ m}, B_o = 0.5 \text{ T}$		$\lambda_o = 0.15 \text{ m}, B_o = 1.6 \text{ T}$	
	$P_{\text{tot.}}$	$d^2P/d\Omega$	$P_{\text{tot.}}$	$d^2P/d\Omega$
0.8	0.06	0.03	0.62	0.03
2.0	0.38	1.0	3.9	1.1
6.0	3.4	84.3	35.0	89.9

Table 2 gives the power and power density emitted by a "typical" undulator and wiggler in rings of three different energies. It can be seen that the numbers increase rapidly with energy, particularly for the power density ($\sim E^4$). Multipole wigglers can emit significantly more total power than undulators (for the same length), however the peak power densities are similar because of the approximate inverse scaling of field strength with period length. Handling the very high power and power densities requires special designs for beamline components. Care must also be taken to prevent damage to the electron beam vacuum chamber in the event of beam mis-steering. Automatic interlock systems are therefore required to detect any unwanted electron beam movement and to dump the beam within a sufficiently short time interval.

3.6 Practical computation of radiation spectra

Various computer programs have been written that can calculate the spectral and angular distribution of undulator radiation using a number of different approaches:

- i/ Direct numerical integration of Eqs. (10), (12) or (13) e.g. [69-71]
- ii/ Calculation of the electric field in the time domain using Eq. (9) followed by Fourier Transformation to obtain the radiation spectrum e.g. [72,73]
- iii/ Use of the analytic approximations for a pure sinusoidal field involving series of Bessel functions (see Appendix) e.g. [74, 75]
- iv/ Other techniques e.g. [76-78]

Methods i/ and ii/ are applicable to the case of a real magnetic field and also for the near-field case, whereas method iii/ assumes an ideal sinusoidal magnetic field in the far-field case. The effect of electron beam emittance can be included either by numerical convolution or by Monte-Carlo methods. Each type of code has its particular advantages and disadvantages. Method iii/ for example is useful for a rapid calculation of the main effects of angular acceptance and electron beam effects, whereas i/ and ii/ are usually more time consuming but provide more detailed information on the effects of magnetic field errors etc. Methods i/ and iii/ can calculate also the angular distributions at a given frequency, which is not possible with method ii/. Some special techniques have been developed for the cases where the K value is large [76,77]. Here we will not discuss further the various numerical approaches, but rather present some results using one code [75] that illustrate the main features of the effects of angular acceptance and electron beam emittance and energy spread.

Figure 17 shows an example of the effect of electron beam emittance on the on-axis spectrum for an undulator installed on ELETTRA. It can be seen that with the nominal emittances there is a relatively small effect on the first harmonic (17% reduction), but this is much larger on the 11th harmonic (factor of 4 reduction). The effect of a ten-fold increase in emittance is also shown, which is sufficient to reduce the first harmonic by a factor of 3. Figure 18 illustrates the effect of electron beam energy spread, with zero emittance. The nominal energy spread gives rise to a negligible reduction in peak intensity on the first harmonic, but quite significant reduction (factor of 4) on the 11th harmonic. It should be noted that the effect of emittance is not symmetric, broadening the lineshape mainly on the long wavelength side, in accordance with Eq. (4), whereas the effect of energy spread is symmetric.

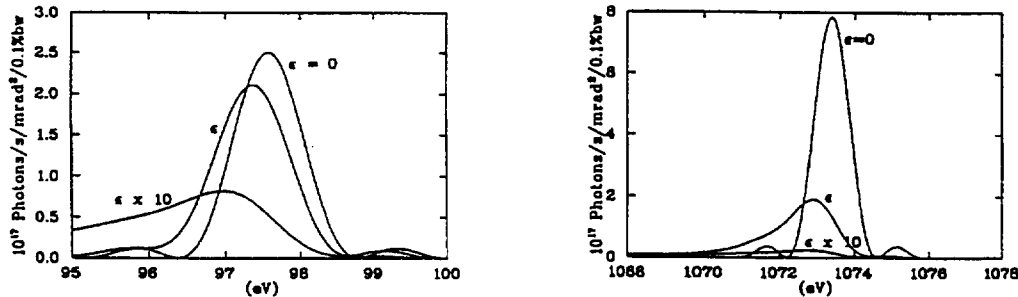


Fig. 17 Effect of electron beam emittance on the 1st (left) and 11th (right) harmonics for an undulator in ELETTRA; $E = 2$ GeV, $\lambda_0 = 56$ mm, $N = 81$, $K = 3.45$

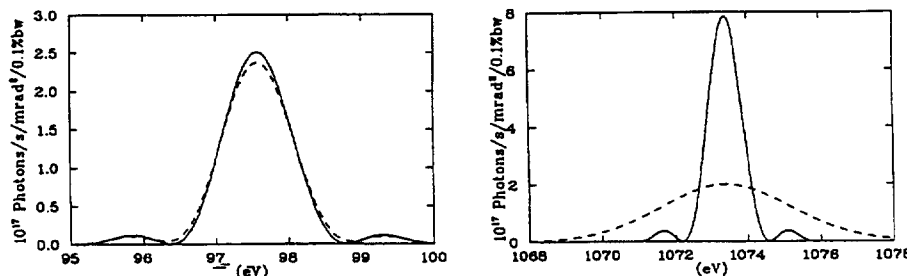


Fig. 18 Effect of electron beam energy spread on the 1st (left) and 11th (right) harmonics for an undulator in ELETTRA; parameters as Fig. 17

Finally, Fig. 19 (left) shows the effect of the size of the acceptance, both with and without electron beam emittance and energy spread included. As the pinhole size increases the peak flux increases up to a maximum value, whereas the line continues to widen. The calculated first harmonic on-axis photon energy is 97.6 eV and it can be seen that the flux at this energy is roughly one half of the maximum value which occurs with a detuning $\Delta E/E$ of approximately $1/N$, which in this case corresponds to about 1.2 eV, in agreement with the observations made in section 3.2. The same calculation is shown also for the 5th harmonic. In this case a smaller pinhole is needed to collect the same fraction of flux due to the smaller angular divergence of the source. With a large acceptance an extra peak appears in the spectrum due to the intensity maxima that occur off-axis in the horizontal plane, as shown in Fig. 9.

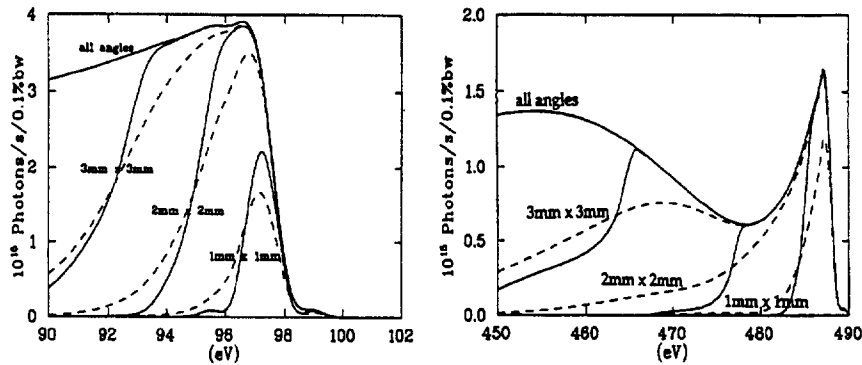


Fig. 19 Effect of "pinhole" size on the 1st (left) and 5th (right) harmonics for an undulator in ELETTRA; parameters as Fig. 17, distance from source 10 m

3.7 Wiggler regime

In this section we return in more detail to the question of when the interference effects which characterize an undulator spectrum become sufficiently "smoothed out" that the result is what may be termed the "wiggler" mode of operation. Generally this requires that there are a sufficient number of harmonics in the spectrum ($K \geq 5$) and that the frequency is not too close to the fundamental ($n \geq 10$). The possible effects that can help to smooth out the spectrum are as follows:

- wavelength acceptance: the relative spacing of the harmonics is $\Delta\omega/\omega = 1/n$ and so if the wavelength acceptance exceeds this value then the harmonics become smoothed out.
- angular acceptance: if the angular acceptance is sufficiently large the $n+1$ 'th harmonic off-axis can reach the wavelength of the n 'th harmonic on-axis. It is easy to show that this occurs when $\gamma^2\theta^2 \geq (1 + K^2/2)/n$.
- electron beam divergence: the same effect as the above occurs if the electron beam divergence becomes too large.
- electron beam energy spread: since the change in frequency is related to the electron energy by $\Delta\omega/\omega = 2\Delta E/E$, it follows that the spectrum is smoothed if $\Delta E/E \geq 1/2n$.
- magnetic field errors: random errors in the magnetic field distribution lead to an imperfect constructive interference, particularly for higher harmonics, which can also contribute to the transition between undulator and wiggler characteristics.

In most cases the bandpass of the monochromator is too small to have a significant effect, and the most dominant is the angular acceptance, followed by electron beam divergence and energy spread and magnetic field errors. It should be noted that the effects of angular acceptance and beam divergence are both much bigger in a higher energy ring whereas the other effects are energy independent.

The resulting spectrum of a high K-device is therefore quite complex and can be described in terms of three regions:

- the fundamental and first few harmonics where there are strong interference effects
- the high photon energy region where the spectrum is essentially smooth
- an intermediate zone where the spectrum is modulated.

As an example Fig. 20 shows a calculated spectrum for a device with a relatively large K value but with a small angular acceptance. At sufficiently high photon energies the spectrum becomes smooth and the intensity equals $2N$ times that of a bending magnet - shown by the solid line in the figure.

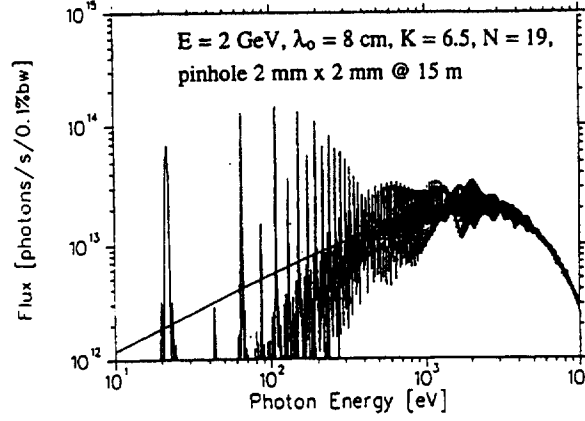


Fig. 20 Calculated spectrum for a high K device in ELETTRA

In the absence of interference effects the radiation spectrum can be calculated as for a bending magnet but with a critical energy that varies with the angle of emission in the horizontal plane:

$$\varepsilon_c/\varepsilon_{co} = \sqrt{1 - (\gamma\theta_x/K)^2}$$

The brightness can be approximated by the following expression [79]:

$$B = \frac{d\dot{n}/d\theta_x}{(2\pi)^{3/2} \left[(\sigma_x^2 + a^2 + \sigma_x'^2 L^2/12) (\sigma_y^2 + \sigma_y'^2 L^2/12) (\sigma_y'^2 + \sigma_R'^2) \right]^{1/2}}$$

where $d\dot{n}/d\theta_x$ is the total flux per unit horizontal angle, integrated over the vertical angle, and σ_R' is the vertical opening of the radiation, as for a bending magnet source; a is the amplitude of the sinusoidal motion ($=K\lambda_o/2\pi\gamma$). The above expression assumes that the magnet length is not too great ($L < 4\beta_x$, $L < 4\beta_y$), as well as a relatively small amplitude of motion ($a < \sigma_x$) and electron beam divergence ($\sigma_y' < \sigma_R'/2$), otherwise a summation must be made over every pole. The brightness of a wiggler is usually several orders of magnitude less than that of an undulator (see 5.2). To obtain significant flux the radiation is usually collected over a wide range of angles, which results in an increased effective source size which also needs to be considered in the beamline optics design. Further details of the geometrical properties of the radiation from multipole wiggler sources is given in Refs. [80,81].

4. INSERTION DEVICE TECHNOLOGY: INTRODUCTION

4.1 Electromagnets

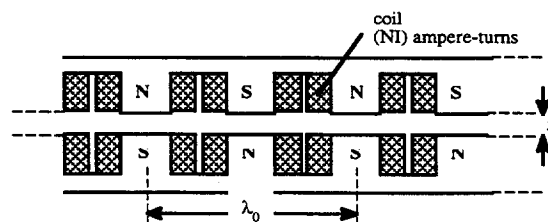


Fig. 21 Schematic of an electromagnetic undulator

To a first approximation an electromagnetic insertion device can be thought of as a series of dipole magnets. We can therefore estimate the field strength by applying Ampère's Law to a circuit as for a simple dipole magnet (see for example Ref. [82]):

$$\mu_o(NI) = B_o g/2$$

where (NI) is the number of Ampère-turns per coil. Using the definition of K this can be re-written as follows:

$$\frac{(NI)}{K} = \frac{4260}{(\lambda_o/g)} \quad (21)$$

It can be seen therefore that in order to maintain a particular value of K , the number of Ampère-turns must increase rapidly as the ratio of period to gap decreases. There is a problem however to reach small period lengths even if the gap can be reduced correspondingly, since the space available for the coils decreases and hence the current density rapidly becomes too big for conventional (i.e. non superconducting) coils.

The above discussion assumed that the field is constant in the vertical direction, however this is not in general a good approximation since a sinusoidal field variation along the axis implies a variation of the field in the vertical direction in order to satisfy Maxwell's Equations. Using a 2D approximation i.e. no variation in the x-direction, the field can be written as follows:

$$B_y = \sum_{n=1}^{\infty} B_n \cosh(nky) \cos(nkz)$$

$$B_z = \sum_{n=1}^{\infty} B_n \sinh(nky) \sin(nkz)$$

Assuming a geometry with pole lengths equal to one-quarter of a period length, and that the B_z field component at the height of the poles ($y = g/2$) is either zero at the poles or $\pm 8\mu_o(NI)/\lambda_o$ at the coils (from Ampère's Law) allows the coefficients B_n to be determined [83]. Comparisons with numerical calculations show that two terms in the series are sufficient to give accurate results and hence one obtains [84]:

$$B_y = \frac{32\mu_o(NI)}{\sqrt{2}\pi\lambda_o} \left[\frac{\cosh(ky)}{\sinh(kg/2)} - \frac{\cosh(3ky)}{3\sinh(3kg/2)} \right] \quad (22)$$

From this can be derived the following expression for the ratio between the field on-axis (B_o) and at the pole (B_p), shown in Fig. 22 (left):

$$\frac{B_o}{B_p} = \frac{1}{\cosh(kg/2)} \left[\frac{1 - (\sinh(kg/2)/3\sinh(3kg/2))}{1 - (\tanh(kg/2)/3\tanh(3kg/2))} \right]$$

The number of Ampère-turns required for a given K value can be expressed as follows:

$$\frac{NI}{K} = 1.18 \cdot 10^3 \sinh(kg/2) \left[1 - \frac{\sinh(kg/2)}{3\sinh(3kg/2)} \right]^{-1}$$

which is shown in Fig. 22 (right) compared to the result for a simple dipole, Eq. (21).

In conclusion, it can be seen that for period-to-gap ratios greater than about 4 the magnet poles can be considered independent. At smaller values the field on-axis reduces and the excitation required increases significantly compared to the simple dipole model. As a rule-of-thumb a ratio $\lambda_o/g \geq 2$ is needed to obtain significant magnetic field on-axis, with not too high an excitation.

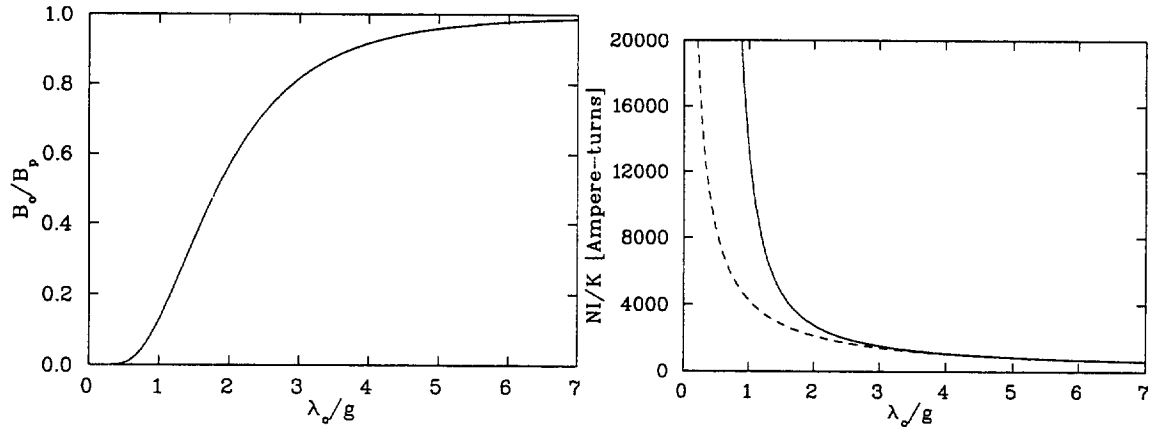


Fig. 22 Ratio of field on axis to the field at the pole-tip (left) and number of Ampère-turns required (right) as a function of period/gap ratio in an electromagnet

4.2 Permanent magnets

The use of permanent magnets for undulator and wiggler construction dates back to the very first undulator described by Motz in 1953 [3]. Permanent magnets were also employed in the ubitron developed by Phillips in the late 50's and early 60's [5]. Subsequently pioneering work on the development of permanent magnet undulators and their implementation in storage rings was carried out in the late 1970's and early 80's at INP, Novosibirsk (USSR) [16,85,86] and LBL, Berkeley (USA) [17,87,88]. The two schemes most commonly used are the "pure permanent magnet" and the "hybrid" schemes shown in Fig. 23.

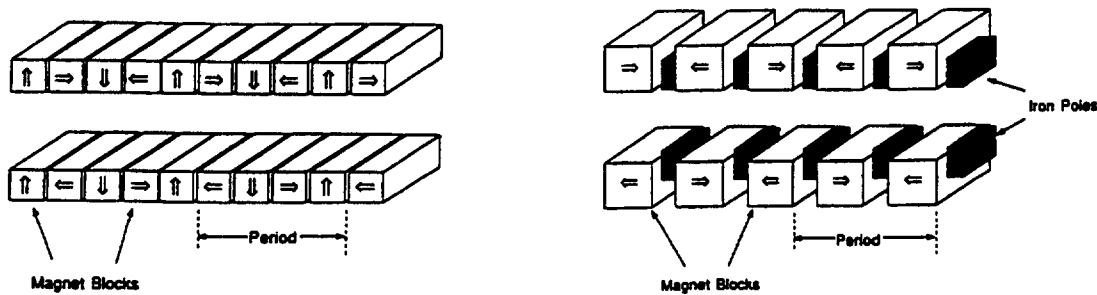


Fig. 23 Schematic diagrams of the "pure-permanent magnet" (left) and "hybrid" (right) types of insertion device

The two types of permanent magnet material in common use are samarium cobalt (SmCo_5 or $\text{Sm}_2\text{Co}_{17}$) and the more recently developed neodymium-iron-boron (NdFeB). The latter is the usual choice since it has a higher remanent field strength (1.1-1.3 T, compared to 0.9-1.0 T for SmCo) as well as a lower cost. The main disadvantages are that it has less radiation resistance and also a larger variation with temperature ($0.11\%^\circ\text{C}^{-1}$ compared to $0.04\%^\circ\text{C}^{-1}$ for SmCo). Figure 24 shows a typical de-magnetization curve for such hard permanent magnet materials. Manufacturer's tables usually quote the following 3 quantities: the remanent field (B_r) where $H=0$, the coercive force ($\mu_0 H_{c,B}$) where the induction becomes zero and the intrinsic coercive force ($\mu_0 H_{c,J}$) where the magnetization becomes zero. A large value of $\mu_0 H_{c,J}$ is an advantage to avoid demagnetization of the pieces during assembly. If it necessary to take demagnetization effects into account, manufacturers can supply permanent magnet blocks that have been thermally and/or magnetically stabilized - heated to a higher temperature, or subjected to a higher reverse magnetic field than the blocks will experience during assembly or operation.

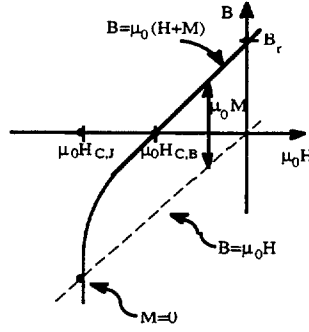


Fig. 24 De-magnetization curve of a permanent magnet

The slope of the curve in the linear region ($\mu = B/\mu_0 H$) is typically 1.05-1.06. The fact that this is close to unity has important consequences [87,89]. Unit permeability implies constant magnetization and so the equivalent charges $\rho = \nabla \cdot \mathbf{M}$ and equivalent currents $\mathbf{J} = \nabla \times \mathbf{M}$ are zero except at the permanent magnet block surfaces. The material therefore behaves like vacuum surrounded by charge or current sheets, hence the expression "charge (or current) sheet equivalent material" or CSEM. This fact means that the field from different pieces can be superimposed linearly. It also leads to a simple calculation of the fields in the case of simple geometries, in the absence of ferromagnetic material. It is interesting to note that the equivalent current is very large, 8 kA per cm of the block dimension for a 1 T remanent field, which explains why permanent magnets are so efficient in generating magnet field. Another advantage is the fact that, unlike electromagnets, the field remains the same if all of the dimensions are scaled. For the most part the small effects due to the non-unit permeability can be neglected. At the level of detail however, this is no longer true; some of the small effects that can arise due to non-linear superposition in pure permanent magnet systems are discussed in Ref. [90].

The field achievable in a 2D approximation from a pure permanent magnet geometry can be calculated analytically [87] and is given as follows:

$$B_o = 2 B_r \frac{\sin(\pi/M)}{(\pi/M)} (1 - e^{-2\pi h/\lambda_o}) e^{-\pi g/\lambda_o} \quad (23)$$

where M is the number of blocks per period in each array, usually 4 as shown in Fig. 23, and h is the block height.

In the hybrid case the field cannot be calculated analytically. Instead the following type of empirical relationship is often used to estimate the achievable field amplitude, based on a series of 2D field calculations with optimized pole and magnet dimensions:

$$B_o = a \exp \left(-b \frac{g}{\lambda_o} + c \left(\frac{g}{\lambda_o} \right)^2 \right) \quad (24)$$

In the above, the values appropriate to SmCo_5 material with $B_r = 0.9$ T have been found to be: $a = 3.33$, $b = 5.47$, $c = 1.8$ [88] whereas for NdFeB with $B_r = 1.1$ T the values are $a = 3.44$, $b = 5.08$, $c = 1.54$ [91]. The validity of the above expression is usually quoted as $0.07 < g/\lambda_o < 0.7$, giving a peak field of 2.3 T (SmCo_5) or 2.4 T (NdFeB) for $g/\lambda_o = 0.07$. Sometimes the above expression is multiplied by a factor 0.95 to account for 3D effects, non-optimized dimensions etc.

4.3 Performance comparison

Figure 25 compares the performance of electromagnet, pure permanent magnet (PPM) and hybrid designs, using the above formulae in the case of a 20 mm gap. The maximum field strength achievable is limited in the case of the electromagnet and hybrid designs by steel

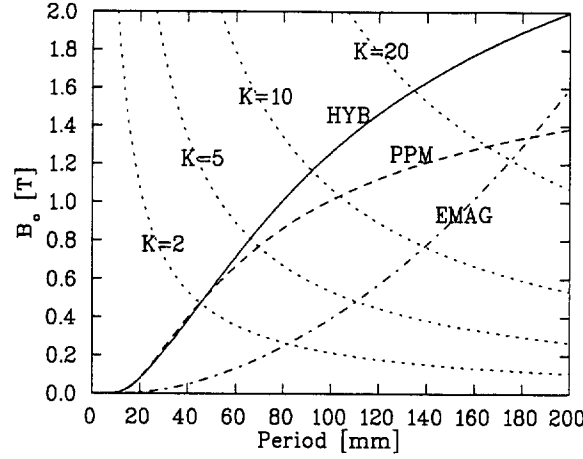


Fig. 25 Comparison of performance of electromagnetic (EMAG), pure permanent magnet (PPM) and hybrid (HYB) insertion devices

EMAG - Eq. (22) with rectangular coils of area $\lambda_o/8 \times \lambda_o/4$ with $J=10$ A/mm²

PPM - Eq. (23) with $M = 4$, $h = \lambda_o/2$, $B_r = 1.2$ T.

HYB - Eq. (24) with $a = 3.27$, $b = 5.08$, $c = 1.54$

saturation, not included in the formulae, and in the case of the PPM by the remanent field of the material. It is clear that the electromagnet is only competitive in the case of long period lengths, or if a reduced field strength is acceptable. For fields in excess of about 0.6 T the hybrid design exceeds the PPM performance. However, in the undulator regime, $K \leq 5$, the performance of the PPM and hybrid types are very similar.

5. INSERTION DEVICE PERFORMANCE LIMITS AND PARAMETER OPTIMIZATION

5.1 Undulator performance limits

Following the previous general discussions of the properties of the radiation emission (Section 3) and the magnetic performance of various insertion device types (Section 4) we will now put these two together to examine what determines the practical performance limits and how the main parameters are chosen in a number of practical cases, based on the more usual choice of permanent magnet rather than electromagnet technology. To do this it will be sufficient to use simple formulae for the radiation flux, Eq. (18), and magnet performance, Eqs. (23) and (24), since they are generally sufficiently accurate to enable the main parameters to be determined on the basis of the required photon energy tuning range. In practise, the next stage would be to revise the parameters after carrying out a detailed magnet design.

We note first of all that optimization of the brightness and flux at a given photon energy are equivalent since the ratio between brightness and flux involves factors depending only on photon energy, electron beam emittance and undulator length which are fixed quantities. Given the magnet period and gap, we can therefore calculate the field amplitude using either Eq. (23) or (24), and hence K and the flux per unit length of undulator as follows:

$$\text{Flux}[\text{photons} / \text{s} / 0.1\% \text{bandwidth} / \text{m} / \text{A}] = 1.43 \cdot 10^{14} \frac{F_n(K)(1 + K^2/2)}{\lambda_o n}$$

The photon energy can also be expressed in terms of known quantities as follows:

$$\frac{\varepsilon}{E^2}[\text{eV}/\text{GeV}^2] = \frac{9.498 n}{\lambda_o (1 + K^2/2)}$$

We can therefore plot one quantity against another i.e. photon flux as a function of normalized photon energy, for different magnet periods and gaps. Figure 26 (left) shows the resulting curves of both fixed gap (variable period) and fixed period (variable gap), assuming in this case a pure permanent magnet geometry, Eq. (23) with $B_r = 1.2$ T and $h = \lambda_o/2$. Figure 25 shows that in the present range there is little difference between the performance of the two permanent magnet configurations. From the dashed curves in Fig. 26 (left) (fixed gap) it is clear that at any photon energy the smallest gap always gives the highest flux, especially at the highest photon energies where flux is critically dependent on magnet gap. The solid curves (fixed period) show the tuning curves achievable in a practical case with fixed period length. It can be seen that longer periods give a wider tunability, because of the larger K value, whereas the short periods which are required to reach the highest photon energies result in much reduced tunability.

The results above refer only to the fundamental ($n = 1$) of the radiation. Higher harmonics can however be used in order to extend the tuning range. This is illustrated in Fig. 26 (right), where the tuning curves for the 1st, 3rd and 5th harmonics are shown for various period lengths, with a minimum gap of 20 mm. Longer periods and hence larger K values result in an

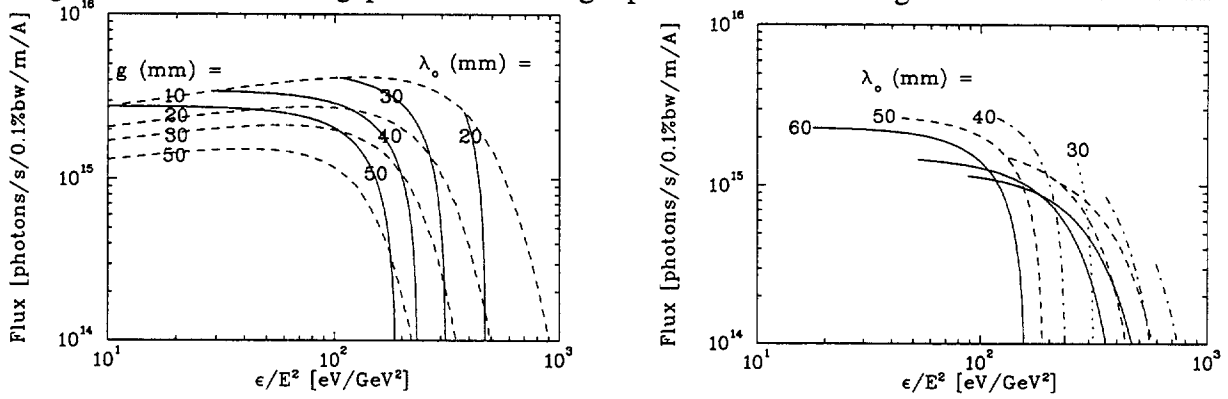


Fig. 26 Undulator flux as a function of normalized photon energy;
left - first harmonic as a function of gap and period
right - first, third and fifth harmonics as a function of period, gap = 20 mm

overlap of the harmonics, allowing a wide tuning range to be covered. As the period and hence K value reduces so also does the overlap of the harmonics, eventually leaving a gap between them. For a "reasonable" overlap between the 1st and 3rd harmonics a K value ≥ 2.2 is required, which also guarantees overlap of the successive harmonics. If one is willing to give up some tunability in favour of reaching higher photon energies, a K value ≥ 1.3 is a reasonable choice, since it allows tuning through the 3rd, 5th and successive harmonics. Figure 26 illustrates the fact that there is an optimum range of photon energies that can be covered with a machine of a given energy, and with a given minimum gap. At low photon energies the limitation is the large K value and hence the high radiation power density, whereas at high photon energies the limitation is the rapidly reducing flux. Table 3 indicates what might be considered the optimum range of photon energies for 3 "standard" machine energies, 0.8 GeV (e.g. SuperACO), 2 GeV (e.g. ELETTRA) and 6 GeV (e.g. ESRF). It can be seen that although there is significant overlap between the ranges, the different machines are nevertheless optimized for different spectral regions, namely the VUV, Soft X-ray and X-ray ranges respectively.

Table 3
"Optimum" photon energy ranges as a function of
storage ring energy, based on a minimum undulator gap of 20 mm

Ring Energy (GeV)	Photon energy range
0.8	6 eV - 450 eV
2.0	40 eV - 2.8 keV
6.0	360 eV - 25 keV

It should be emphasized that the "optimum" photon energy ranges shown in the Table are only indicative. For example, as we have already seen, the performance is strongly determined by the minimum gap that can be allowed. Special insertion devices can also be built with even smaller gaps to extend the performance at higher energy. The lower photon energy limit is also rather arbitrary, and depends in practise on the acceptable power loading. Overall however it remains true that undulator performance is one of the most important factors determining the choice of energy for a synchrotron radiation facility, and that this is essentially limited (at the high photon energy limit) by the minimum gap that can be allowed for successful machine operation.

5.2 Undulator parameter optimization

As a practical example we will consider the design of undulators for a 1.5 GeV ring based on a minimum gap of 20 mm; the principles involved however remain valid for any other choice of parameters. Figure 27 shows the flux and brightness achievable from a range of different undulators, together with that from a multipole wiggler and a bending magnet for comparison. In the latter cases the flux is integrated vertically, and per mrad of horizontal angle. Although the multipole wiggler can provide a reasonably high flux, clearly the brightness is several orders of magnitude inferior to that of an undulator. The parameters of the devices are given in Table 4 which includes also the total power and peak power density of the radiation.

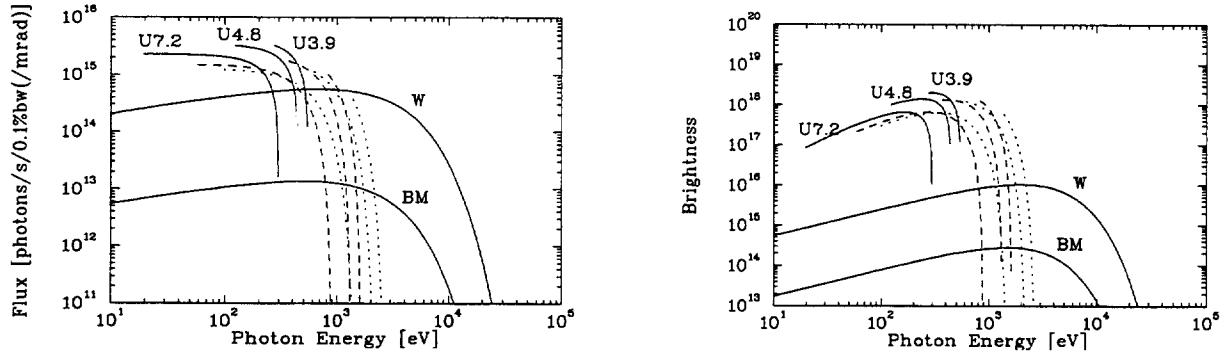


Fig. 27 Flux (left) and brightness (right) for various sources in a 1.5 GeV ring:
 U - 1st (solid), 3rd (dashed) and 5th (dotted) harmonics; W - multipole wiggler;
 BM - bending magnet. ($E=1.5$ GeV, $I_b = 0.4$ A, $\varepsilon_x=10^{-8}$ m rad, $\varepsilon_y = 10^{-9}$ m rad,
 $\rho_x = 5$ m, $\rho_y = 2$ m, ID length = 3 m, BM field = 1.2 T)

Table 4

Parameters of various undulators shown in Fig. 27; peak power density units W/mrad²

λ_o (mm)	g (mm)	B_o (T)	K	$P_{tot.}$ (W)	$d^2P/d\Omega$
39	20	0.38	1.40	247	615
26	10	0.57	1.40	554	1381
18.5	5	0.81	1.40	1120	2787
48	20	0.51	2.29	443	686
72	20	0.78	5.26	1029	702
100	≥ 20	0.47	4.40	379	309
150	≥ 20	0.25	3.50	107	109
200	≥ 20	0.11	2.56	21	29

As stated in the Section above, at the low photon energy end of the spectrum the radiation power density can become a limiting factor. Figure 28 (a) shows a selection of different devices all with a fundamental of 20 eV. As the period length increases the power and power density decrease rapidly, as shown in Table 4, whereas the flux decreases more slowly. The choice therefore is a compromise between power, flux and tuning range.

In the medium and high energy parts of the spectrum the choice of undulator parameters is often dictated by the required tuning range rather than simply the energy of the fundamental. For example, Fig. 28 (b) shows the performance for 3 undulators with similar period lengths in the range 43–48 mm. The device which gives the peak flux at 200 eV for example has a period of 43 mm, however in this case the K value is too small (1.8) to allow a continuous coverage between the fundamental and 3rd harmonic at 600 eV. A slightly larger period length of 46 mm ($K=2.1$) or 48 mm ($K=2.3$) results in only a small reduction in flux at 200 eV but provides a much wider tuning range.

At the high energy extreme the performance reduces rapidly, even if higher harmonics are used, as shown in Fig. 28 (c). A lower gap is clearly advantageous in this case, and the figure shows the result for a 5 mm and 10 mm gap. In each case the period has been adjusted to maintain a K value of 1.4 which allows tunability from the 3rd harmonic upwards. Table 4 gives the parameters for these devices.

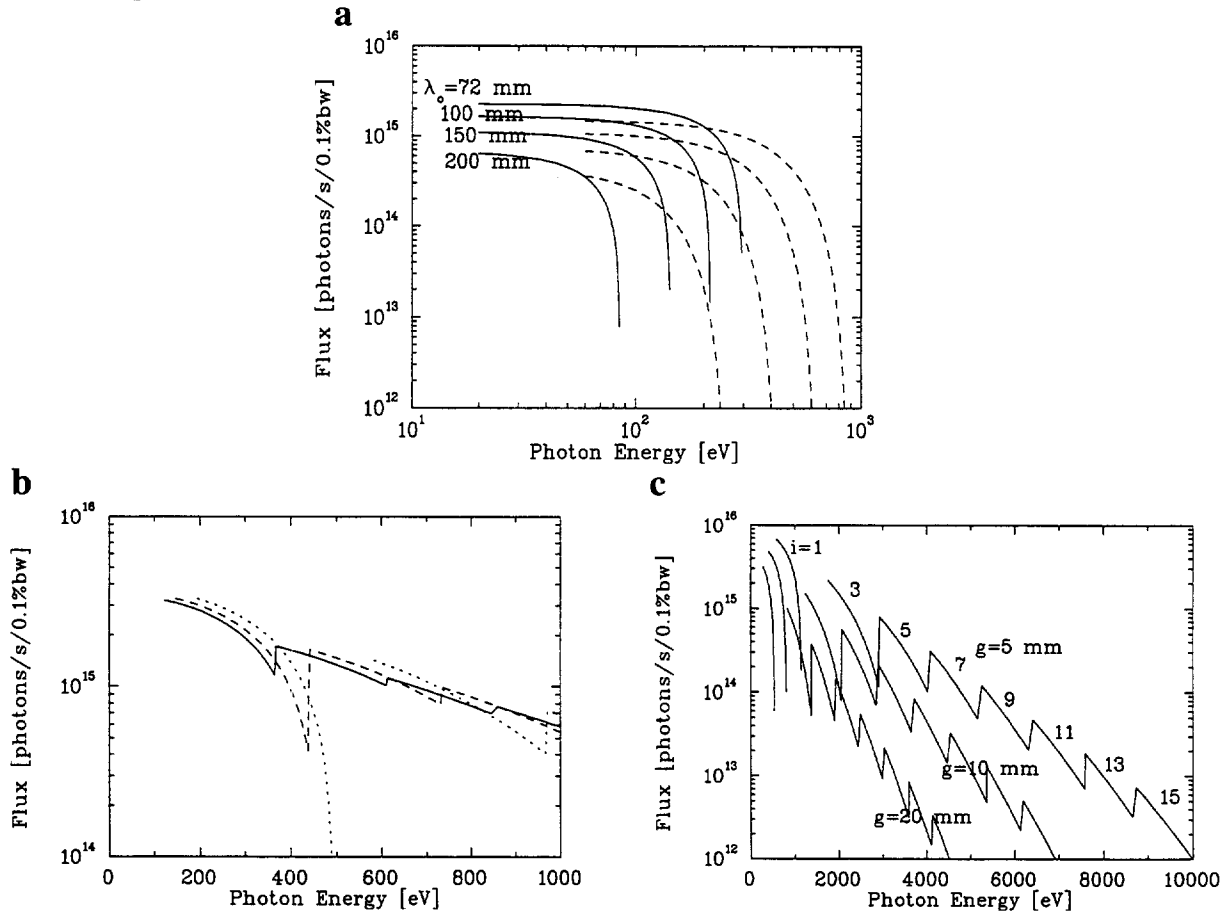


Fig. 28 Undulator performance in low, medium and high energy spectral regions. (a) $\lambda_o = 72 - 200$ mm; (b) $\lambda_o = 43$ mm (dotted), 46 mm (dashed) and 48 mm (solid); (c) undulators optimized for gaps of 5 mm ($\lambda_o = 18.5$ mm), 10 mm ($\lambda_o = 26.2$ mm) and 20 mm ($\lambda_o = 39.2$ mm). Ring parameters as Fig. 27.

6. INSERTION DEVICE TECHNOLOGY: DETAILED MAGNETIC DESIGN

6.1 Detailed magnetic design: periodic part

Having first established the main parameters of the device on the basis of the required characteristics of the radiation (Section 5) the next step in the design process is generally to carry out the detailed design, which usually starts with the periodic part of the structure. In the case of the pure permanent magnet case 3D analytic formulae exist for the field due to a

parallelepiped block in the approximation of unit permeability, which is generally a sufficiently good approximation. Adding the field from a sufficient number of blocks then allows the field in the centre of the device to be calculated. Many laboratories have codes which will perform these calculations, e.g. Ref. [92].

Using such a program it is then a relatively simple matter to optimize the remaining parameters, namely the magnet height and width, see Fig. 29. The space between the blocks is usually sufficiently small to have a negligible effect. Fig. 30 (left) shows the variation of field amplitude with block height for various block widths in one particular case. To reach sufficiently close to the maximum achievable value a height of 26 mm i.e. $h = \lambda_o/2$ was chosen. A width of 85 mm was chosen, not because of the field strength - for which a width of about 40 mm would be sufficient - but in order to give a satisfactory field homogeneity. The transverse field variation is shown in Fig. 30 (right).

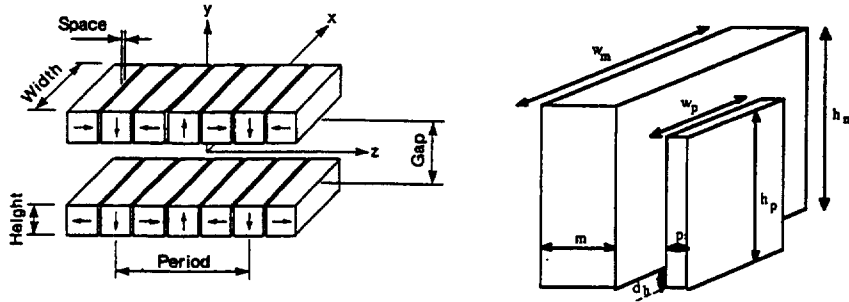


Fig. 29 Parameters to be optimized for pure-permanent magnet (left) and hybrid (right) structures

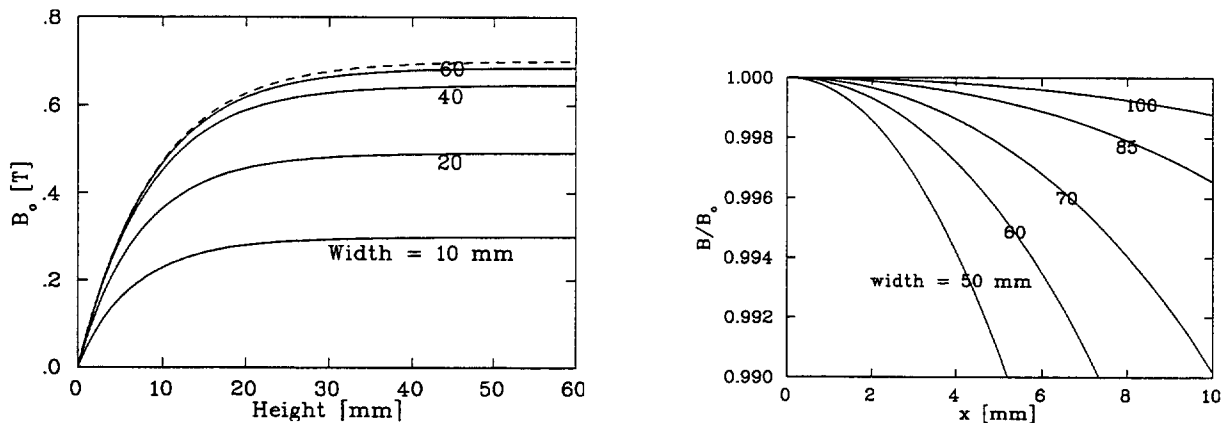


Fig. 30 Left - variation of field amplitude with block height for various block widths; dotted line - infinite width. Right - relative field variation with transverse position for various block widths (block height = 28 mm). Period = 56 mm, gap = 20 mm, space = 0, $B_r = 1.2$ T

In the case of the hybrid design the situation is more complicated since there are more parameters to optimize, as shown in Fig. 29, and more criteria to satisfy: achieving the required field and field homogeneity with a minimum volume of permanent magnet material, whilst avoiding excessive saturation in the iron and regions of reverse field in the permanent magnet material. Some account has to be taken of three dimensional effects. A possible solution is therefore to use a 3D code; several commercial ones are available such as AMPERES, FLUX3D, MAFA, MAXWELL and OPERA-3D. Such codes are however generally quite expensive and require a period of training before reliable results can be obtained. An alternative is to use a combination of a 2D code, which includes the effect of saturation in the iron, and 3D approximations valid for infinite μ [93,94].

By making use of symmetry only 1/8th of a complete period needs to be modelled; Figure 31 shows part of the field distribution in one case, together with the detailed region near the pole surface. A choice that has to be made is to use standard magnet steel, with a saturation magnetization of about 2.1 T, or iron-cobalt steel which has a significantly higher saturation of about 2.4 T. The latter is more costly but can lead to a smaller pole width and a reduction in the

volume of permanent magnet material and hence can be cost effective. In the hybrid design it is important that the permanent magnet is larger than the pole in order to reduce flux leakage [95].

Alternative geometries to the basic one discussed above that can lead to an enhanced field strength, but with an increase in complexity and cost, are the wedged-pole geometry [96], and the inclusion of side-magnets [97], as shown in Fig. 32.

A factor that has to be taken into account particularly in the hybrid design is the fact that the field can vary significantly from a pure sinusoid, containing a large 3rd harmonic component. The effect is to change the fundamental photon energy, and also the radiation opening angle.

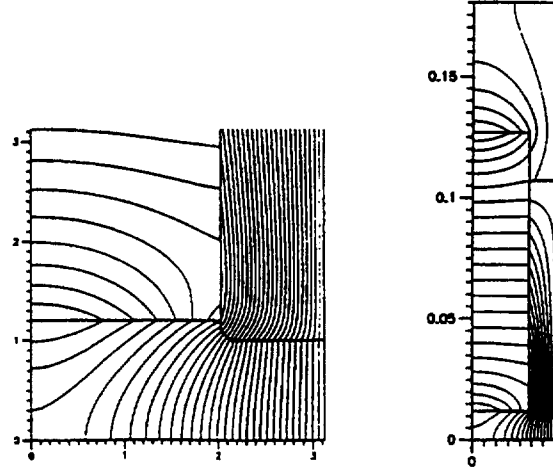


Fig. 31 Magnetic flux lines in a hybrid device; right - complete model of 1/8th period, left - detail of the pole-tip region

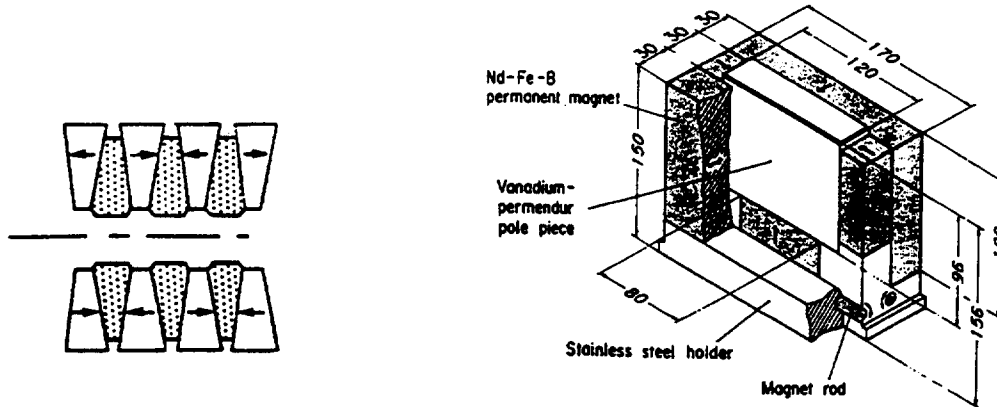


Fig. 32 Alternative hybrid designs; left - with wedged poles, right - with side magnets

6.2 Detailed magnetic design: ends

In order that the insertion device does not affect the storage ring closed orbit requires that there is no net change in transverse position or angle of the beam. In the horizontal plane for example the final angle and position at the end of the device ($z = L/2$) are given as follows:

$$x'(L/2) = \frac{e}{\gamma mc} \int_{-\infty}^{L/2} B_y(z) dz = \phi$$

$$x(L/2) = \frac{e}{\gamma mc} \int_{-\infty}^{L/2} dz \int_{-\infty}^z B_y(z') dz'$$

which are often referred to as the first and second field integrals respectively. It is convenient to refer to the position of the beam projected back to the centre of the device (δ), rather than at the end, as shown in Fig. 33. After integrating the second expression by parts and re-arranging one obtains the following result:

$$\delta = x(L/2) - x'(L/2) \frac{L}{2} = -\frac{e}{\gamma mc} \int_{-\infty}^{L/2} B_y(z) z dz$$

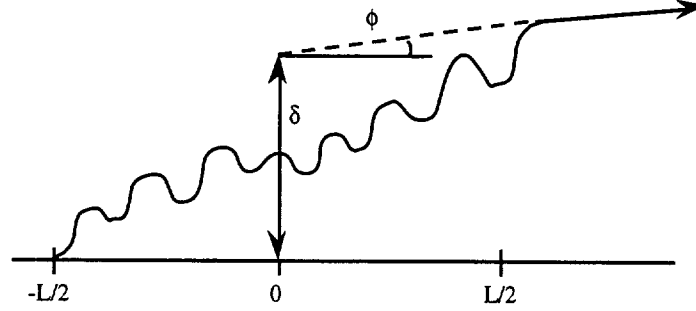


Fig. 33 Projection of electron beam exit conditions to the centre of the insertion device

We can now examine the requirement for no net change in angle (ϕ) or position (δ) of the beam. In the usual case the magnet design is symmetric with respect to the centre of the device. According to the above equation we have then by symmetry $\delta = 0$. The correct angular condition $\phi = 0$ is obtained by appropriate adjustment of the end-poles as illustrated in Fig. 34 (left). An alternative configuration is anti-symmetric with respect to the centre. In this case by symmetry $\phi = 0$, and the condition $\delta = 0$ is obtained by correct adjustment of the end-poles, as illustrated in Fig. 34 (right). In the case that the magnet consists of a series of equal strength central poles with a special end-pole at either end, then the required condition can be met in the case of the symmetric configuration by arranging that the end-poles have one half of the field integral of the centre poles. In the anti-symmetric case however no such simple result exists; the relative strength required for the end-poles depends on the number of poles in the magnet (M) as follows: $(M - \frac{1}{2})/(M - 1)$.

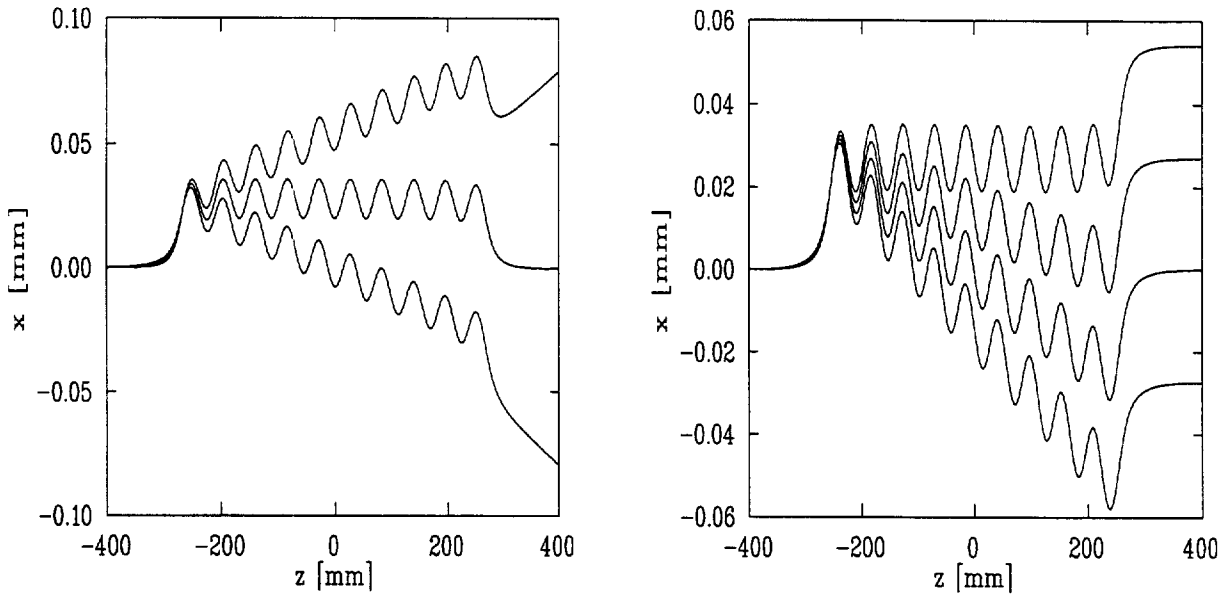


Fig. 34 Electron trajectories in insertion devices with symmetric (left) or anti-symmetric (right) field distributions

It can be seen from Fig. 34 that even when the magnet is perfectly compensated in angle and position the axis about which the electron beam oscillates is offset from the straight through direction either in position (symmetric case) or angle (anti-symmetric case). Such an offset is not generally a problem, but can be in certain applications. To overcome this requires a more sophisticated entrance and exit configuration involving more than one end-pole. Considering an arrangement of equally spaced poles, the simplest arrangement employs two reduced-strength end-poles with a sequence of $1/4, -3/4, 1 \dots$ as shown in Fig 35. It is clear that the same configuration can be applied to either the symmetric or anti-symmetric cases. For sequences involving three end-poles a variety of solutions are possible, which can be expressed as follows: $x, -(2x + 1/4), (x + 3/4), -1$.

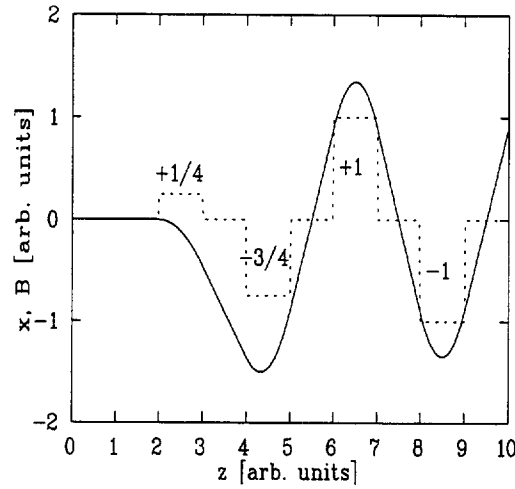


Fig. 35 Sequence of magnet poles (dotted line) resulting in no offset between the electron trajectory (solid line) and the magnet axis.

Implementation of the various end-pole sequences described above is straightforward in the case of the pure permanent magnet configuration because of the applicability of linear superposition. As a consequence of this, the field integral due to a single block depends linearly on the block length. A great advantage is the fact that apart from small effects due to non-unit permeability and magnetization errors the device remains compensated at all gaps. In the symmetric case overall compensation can be achieved simply by terminating with a half-block, as shown in the Fig. 36 (a). Some other schemes are shown (b-e) that give no offset of the oscillation axis using either half-sized blocks (b, c) or $1/4$ and $3/4$ sized blocks (d) [98].

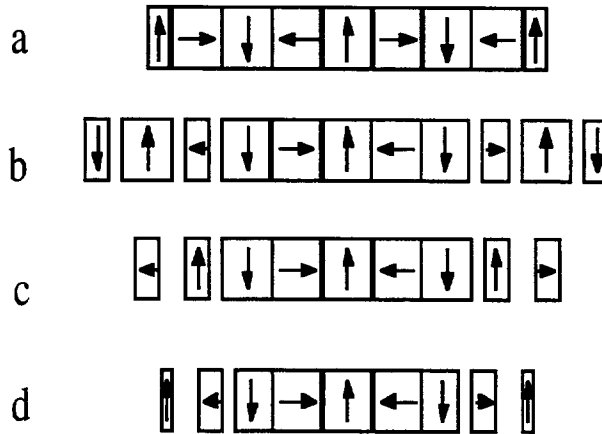


Fig. 36 Various end-sequences for the pure-permanent magnet structure

In the case of the hybrid construction no simple geometrical arrangement exists for the termination. Even after adjusting permanent magnet or pole dimensions to achieve compensation there still generally exists a significant variation of the field integrals with gap that needs to be overcome with an active system such as a coil [99] or rotating permanent magnet block [100], as shown in Fig. 37. Recently, there has been some success in developing a geometry that reduces the variation of field integral with gap, with the hope of eliminating completely the correction system. In one case empirical adjustments were made to the strength of the end magnets and the height of the next to last pole [101]. In another case 3D computer runs were made to optimize the end geometry [102]. Interestingly, in both cases the resulting solution produces close to no offset between the magnet and the oscillation axes.

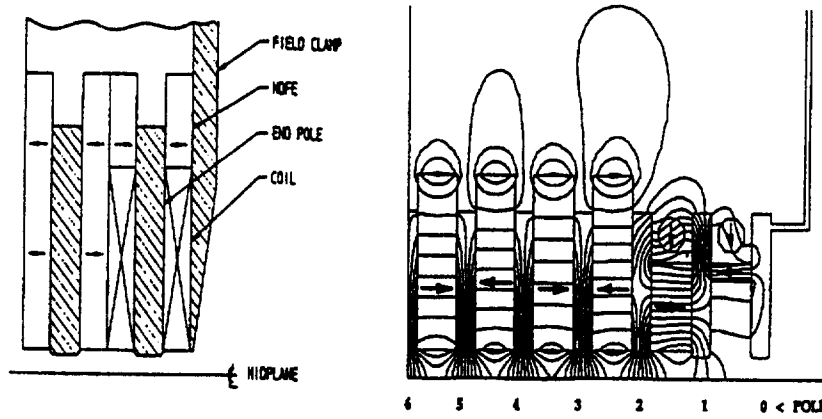


Fig. 37 Termination schemes for hybrid insertion devices using coils (left) and rotating magnets (right)

6.3 Magnetic field errors

Magnetic field errors arise due to variations in the dimensions and shape of the magnets and poles, variations in gap etc., as well as variations in magnetization strength and angle from block to block and inhomogeneity. The effects of such errors are two-fold: unwanted effects on the electron beam and deterioration of the quality of the emitted radiation. In the former case, the effects are due to the non-zero first and second field integrals along the beam axis, which produces a change in the closed orbit, as well as the transverse variation of these quantities. The transverse variation of the first field integral errors can be expressed in terms of multipole components which cause changes in focusing properties, i.e. changes in tune values and betatron functions (quadrupole errors), variations in coupling (skew-quadrupole errors) as well as non-linear beam dynamics effects (sextupole, octupole errors etc.). Effects on the radiation spectrum on the other hand depend on the details of the field distribution and hence electron trajectory. Effects arise due to both the deflection of the electron beam away from the nominal axis, as well as errors in the phase of the radiation. The latter cause a loss of constructive interference causing reduced angular flux density and brightness, particularly on the higher harmonics.

Initial work on the effects of field errors on the emitted radiation [103] attempted to characterize the effect in terms of the variation in field amplitude from pole to pole, which was assumed to be a random quantity. It was shown later however that the intensity is not well correlated with this quantity [104], but is well correlated to the radiation phase error [105]. The phase can be calculated simply from the magnetic field distribution as follows:

$$\phi = \frac{2\pi}{\lambda} \left(\frac{z}{2\gamma^2} + \int \frac{x'^2}{2} dz \right)$$

where the angle x' is given by Eq. (2). In practise the phase is calculated at each magnet pole and the linear variation subtracted, to remove the π increase in phase between each pole and also to take account of any small changes in the radiation wavelength from the nominal value [64,

106]. The rms value of the phase variation (σ_ϕ) is then calculated. Simulations show that the relative intensity of the n 'th harmonic (R_n) agrees well with the simple formula [64]:

$$R_n = \exp(-n^2 \sigma_\phi^2)$$

In order to combat field errors therefore requires a sufficiently precise mechanical construction as well as some method for overcoming the effect of magnetization errors. For the latter various techniques have been adopted. The first consists of measuring the individual permanent magnet blocks and then arranging them in the structure in such a way as to optimize the field quality. Sorting procedures of this kind are most widely applied to the pure permanent magnet case where linear superposition can be used. Various ways of characterizing the blocks have been tried based on measurement of either total magnetization strength, field integrals or field maps. The most common sorting procedure is based on the "simulated annealing" algorithm [107] although "genetic algorithms" have also been used [108]. Once the magnet has been constructed and measured, the remaining errors can be overcome by swapping blocks, or more commonly by "shimming" i.e. placing thin ferromagnetic sheets on the magnet surfaces (see Fig. 38). Shimming was first applied to hybrid devices to correct pole-to-pole field strength error [109] and subsequently as a means of correcting multipole errors [110,111] and then also radiation phase errors [112-116]. As a result of these developments, as well as better magnetic measurement techniques [117], recent insertion devices have been free from the multipole errors observed in some earlier devices and have also far exceeded expectations in terms of the radiation performance. When the present "third generation" of synchrotron radiation sources were planned it was thought that field errors would preclude use of harmonics higher than about the 5'th. Now, with the correction techniques described above, the undulator spectrum can be almost perfect up to much higher harmonic numbers so that the limiting factor is no longer the undulator quality but that of the electron beam itself (emittance and energy spread).

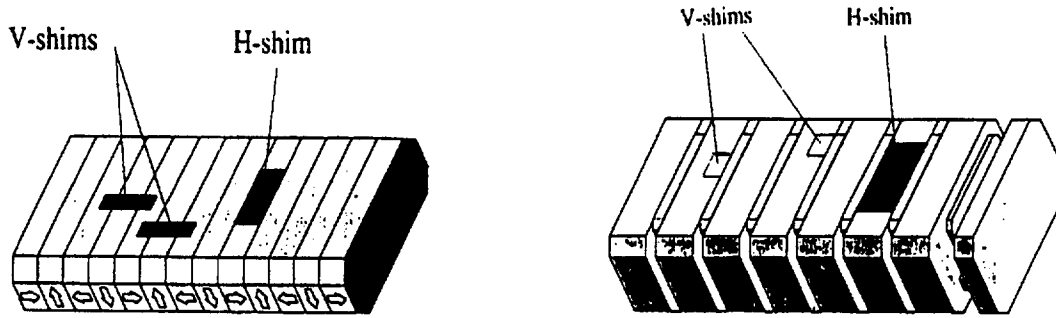


Fig. 38 Method of magnetic shimming to improve the magnetic field quality

7. INSERTION DEVICES FOR CIRCULARLY POLARIZED RADIATION

7.1 Introduction

There is an increasing interest in using circularly polarized radiation over a broad photon energy range for a wide range of experiments such as Circular Dichroism (CD) - which measures the difference in absorption between right- and left-handed polarized light, Magnetic Circular Dichroism (MCD) - the difference in absorption depending on the relative alignment of the magnetic field and the polarization directions in magnetic materials, and Circular Intensity Differential Scattering (CIDS) - the difference in scattering of right- and left-handed polarized radiation. Conventional insertion devices produce only linearly polarized radiation and so special types have been developed to generate circularly, or in general elliptically, polarized radiation. In this section we first introduce the concept and a basic description of the polarization, before considering the polarization properties of insertion devices.

The polarization of a radiation beam is determined by the phase relationship between the components of the Electric field [118]. For example given the horizontal component $E_x = E_{x0} \cos(\omega t)$ and the vertical component $E_y = E_{y0} \cos(\omega t + \phi)$, it can be seen that there are in general three independent parameters, E_{x0} , E_{y0} , and ϕ . When $\phi = 0$ the radiation is linearly polarized, in a direction defined by the amplitudes E_{x0} and E_{y0} ; when $\phi = 90^\circ$ and $E_x = E_y$ we have circularly polarized radiation. Such a description is not however of much practical use since none of these parameters are directly measurable, and in general we have to consider a superposition of different waves, not just a single monochromatic wave. A more useful description in terms of directly measurable quantities was introduced by G.G. Stokes in 1852, involving the following parameters:

$$\begin{aligned} S_0 &= I_x + I_y = I_{45^\circ} + I_{-45^\circ} = I_R + I_L \\ S_1 &= I_x - I_y \\ S_2 &= I_{45^\circ} - I_{-45^\circ} \\ S_3 &= I_R - I_L \end{aligned}$$

S_1 is the difference in intensity transmitted by a linearly polarized filter oriented in the horizontal (x) and vertical (y) directions, S_2 is the difference in intensity for the components polarized at $\pm 45^\circ$ with respect to the x-y axes and S_3 is the difference in intensity of right- and left-handed circularly polarized components. The total intensity is S_0 . Since only three independent parameters are needed it follows that there is some relationship between the above, and this is:

$$S_0^2 = S_1^2 + S_2^2 + S_3^2$$

The above description is true for a single monochromatic wave. In general there is a summation of waves from different source points in the insertion device as well as due to the range of wavelengths, the angular acceptance and the effect of electron beam emittance and energy spread. The effect of the summation is that the radiation can be divided into a polarized and an un-polarized part. For the polarized part the same decomposition into the three polarization states can be made, and so it follows that:

$$S_0 = \sqrt{S_1^2 + S_2^2 + S_3^2} + S_4$$

where S_4 is the intensity of the unpolarized radiation. Dividing through by the total intensity we obtain:

$$1 = \sqrt{P_1^2 + P_2^2 + P_3^2} + P_4$$

where $P_1 = S_1/S_0$ etc. The overall degree, or fraction, of polarization is therefore $\sqrt{P_1^2 + P_2^2 + P_3^2}$ and the fraction of unpolarized radiation is P_4 . The quantities P_1 , P_2 and P_3 are known as the polarization rates for the particular component.

The intensity of the radiation at a given frequency that is polarized in a given direction $\underline{u}$ is calculated by modifying Eq. (8) as follows:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{c}{4\pi^2} \left| \int_{-\infty}^{\infty} R(\underline{E} \cdot \underline{u}) e^{i\omega t} dt \right|^2 = |\underline{A} \cdot \underline{u}|^2 \quad (25)$$

The directions corresponding to the various polarization states are given below:

Polarization component	Direction vector, $\underline{u}$	Radiation amplitude
linear, x-direction	(1,0,0)	A_x
linear, y-direction	(-1,0,0)	A_y
linear, 45° to x-y	$(1,1,0)/\sqrt{2}$	$(A_x + A_y)/\sqrt{2}$
linear, -45° to x-y	$(1,-1,0)/\sqrt{2}$	$(A_x - A_y)/\sqrt{2}$
right circular	$(1,i,0)/\sqrt{2}$	$(A_x + iA_y)/\sqrt{2}$
left circular	$(1,-i,0)/\sqrt{2}$	$(A_x - iA_y)/\sqrt{2}$

In the above we also show the amplitude of the radiation polarized in the given direction in terms of the horizontal and vertical components, which are in general complex quantities. The intensities can then be expressed as follows:

$$\begin{aligned}
 I_x &= |A_x^2| & I_y &= |A_y^2| \\
 I_{45^\circ} &= [|A_x^2| + |A_y^2| + A_x A_y^* + A_x^* A_y] / 2 & I_{-45^\circ} &= [|A_x^2| + |A_y^2| - A_x A_y^* - A_x^* A_y] / 2 \\
 I_R &= [|A_x^2| + |A_y^2| - i(A_x A_y^* - A_x^* A_y)] / 2 & I_L &= [|A_x^2| + |A_y^2| + i(A_x A_y^* - A_x^* A_y)] / 2
 \end{aligned}$$

Finally, we can express the Stokes parameters in terms of the radiation amplitudes as follows:

$$\begin{aligned}
 S_0 &= A_x A_x^* + A_y A_y^* \\
 S_1 &= A_x A_x^* - A_y A_y^* \\
 S_2 &= A_x^* A_y + A_x A_y^* \\
 S_3 &= i(A_x^* A_y - A_x A_y^*)
 \end{aligned} \tag{26}$$

7.2 Bending magnet and conventional insertion devices

In order to understand the polarization properties of bending magnet and insertion device sources we will examine the electric field of the radiation, which according to Eq. (9) is proportional to the electron acceleration. To a first approximation we have the following relationships:

$$\begin{aligned}
 E_x &\sim \dot{\beta}_x - \theta_x \dot{\beta}_z \\
 E_y &\sim \dot{\beta}_y - \theta_y \dot{\beta}_z
 \end{aligned} \tag{27}$$

where θ_x, θ_y are the angles of emission with respect to the x, y axes (Fig. 6). In a bending magnet which bends in the horizontal plane $\dot{\beta}_y = 0$ and hence the vertical electric field component only exists for emission out of the orbit plane $\theta_y \neq 0$. It follows directly from the direction of acceleration that $E_x (\sim \dot{\beta}_x)$ is symmetric with respect to the tangent point whereas $E_y (\sim \theta_y \dot{\beta}_z)$ is anti-symmetric, as shown schematically in Fig. 39. Figure 40 shows the pattern traced out by the electric field for the particular case $\theta_y = 1/\gamma$. The circular motion of the electric field vector indicates the presence of circularly polarized radiation. In fact as the angle θ_y increases the fraction of circular polarization rate increases while the linear polarization decreases, as shown in Fig. 41. For angles below the orbit plane ($\theta_y < 0$) the vertical electric field component changes sign and hence the direction of circular polarization reverses.

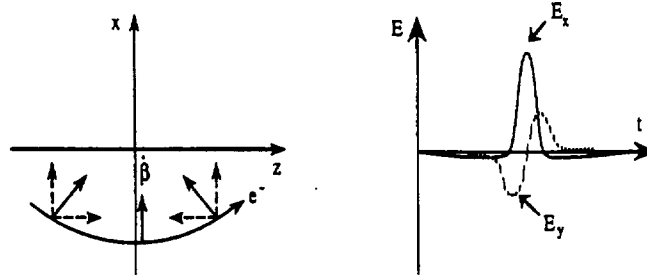


Fig. 39 Schematic diagram of radiation emission in a bending magnet: direction of electron acceleration (left), and electric field components as a function of time (right)

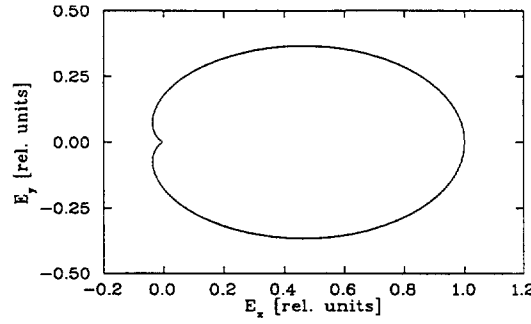


Fig. 40 Calculated electric field components, normalized to $(E_x)_{\max.}$, for a bending magnet source with $B = 1$ T at a vertical angle $\gamma\theta_y = 1$; arrows indicate the direction of rotation



Fig. 41 Stokes parameters (left) and relative polarization rates (right) as a function of vertical angle for bending magnet radiation, at a wavelength corresponding to the critical wavelength ($\lambda = \lambda_c$)

In terms of the radiation amplitudes, the symmetry of the electric field components implies that A_x is real whereas A_y is imaginary, which implies directly from Eqs. (26) that $S_2 = 0$. The Stokes parameters can therefore be written in this case as follows:

$$S_1 = I_x - I_y$$

$$S_2 = 0$$

$$S_3 = \pm 2\sqrt{I_x I_y}$$

We now consider what happens in the case of a conventional insertion device. Figure 42 shows one period of such a device and it can be seen that at the second pole $\hat{\beta}_x$ and hence E_x changes sign, whereas $\hat{\beta}_z$ and hence E_y remain the same, as indicated schematically. Figure 43 also shows the calculated electric field variation for a particular case and it can be seen that the direction of rotation reverses, indicating a cancellation of the circularly polarized component. In other words, S_3 has the opposite sign for the second pole compared to the first; summing the intensities of the two poles then results in $S_3 = 0$. All radiation that is not linearly polarized is therefore unpolarized (Fig. 44).

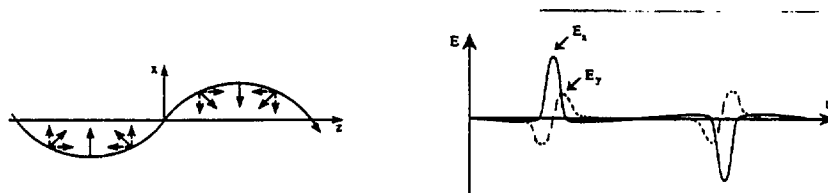


Fig. 42 Schematic diagram of radiation emission in an insertion device: direction of electron acceleration (left), and electric field components as a function of time (right)

radiation is viewed on axis. Another type is the asymmetric wiggler: in this case there is no horizontal field component, however the cancellation of the circularly polarized component that occurs off-axis in a conventional device is avoided by making the field strengths of the positive and negative poles unequal. A third type is the crossed-undulator: in this case circular polarization is generated by the interference of the radiation emitted in successive horizontal and vertically polarized undulators.

7.3 Helical/elliptical devices

7.3.1 Magnetic configurations

Many different schemes have been put forward that can generate helical or elliptical field distributions, using both electromagnets and permanent magnets in various configurations - either with a helical geometry, a combination of separate horizontal and vertical devices, or with an open-sided (planar) structure - and as result, with either fixed or variable polarization.

The classical method of producing a pure helical field is the "bifilar helix" (Fig. 45, left): a conductor is wound on a cylindrical former with diameter g and with a pitch λ_o forming a helical coil. A second winding is then added in between the windings of the first coil. When a current I is passed in opposite directions through the two windings the solenoidal field is cancelled, leaving a periodic helical field:

$$B_x = B_o \cos(kz) \quad B_y = B_o \sin(kz)$$

The field amplitude for the case of wires with infinitesimal cross-section is given by [120]:

$$B_o = \frac{2\mu_o I}{\lambda_o} [\xi K_0(\xi) + K_1(\xi)]$$

where $\xi = \pi g/\lambda_o$ and K_0, K_1 are modified Bessel functions. The above expression shows a rapid decrease in field amplitude as λ_o/g reduces, analogous to the behaviour of linearly polarized electromagnets and permanent magnets. Approximate expressions for a distributed winding were derived in Ref. [121]. Details of the field components generated on and off-axis were considered in [122] and end-effects in Ref. [123]. In this design the polarization direction (right- or left-handed) is determined by the sense of the winding. The possibility of superposing windings of opposite handedness to generate arbitrary polarization was suggested in Ref. [124]. The field generated by the bifilar helix is significantly less than for a conventional electromagnet [125] because of the lack of iron poles. Because of this fact, and the disadvantages associated with a closed structure for magnetic measurements and ease of installation etc., few magnets of this type have been built. The first FEL experiment at Stanford employed a superconducting helical magnet [126], while a pulsed magnet was used in the early FEL experiments at Frascati [127].

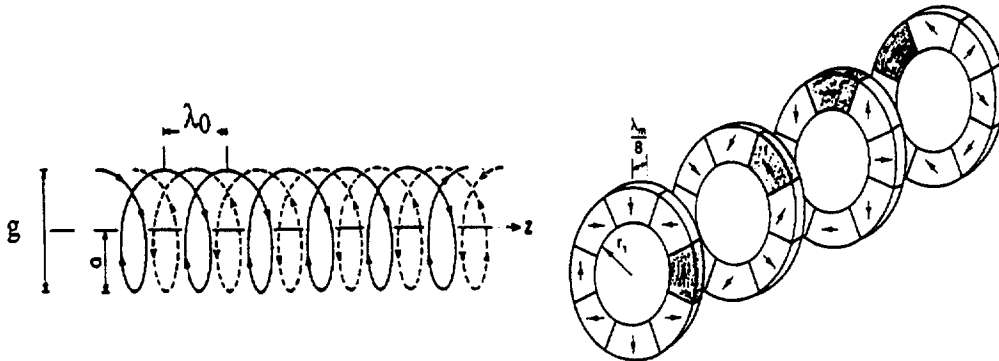


Fig. 45 Two methods for producing a pure helical field; left - the classical bifilar helix, right - a sequence of dipole magnets using permanent magnets

A higher field level can be obtained by incorporating iron poles. An example of a magnet of this type is the superconducting helical undulator installed in the VEPP-2M storage ring in Novosibirsk in 1984; prior to this a non-superconducting version was operated to which iron poles were added later to enhance the field level [86]. For the FEL experiment of the Bell Laboratories a novel scheme was devised in which the iron poles were fed by straight, rather than helical conductors [128]. Another scheme is presented in Ref. [129].

Helical arrangements of permanent magnets have also been considered. The design proposed in Ref. [130] consists of N "slices" per magnet period, where each slice is an annulus containing segments of permanent magnet material with a suitable direction of magnetization to create a dipole field (Fig. 45, right). The slices are arranged along the length of the undulator with a $2\pi/N$ rotation from one slice to the next, so creating a helical field. Similar field strengths can be generated in this way compared to the linear undulator geometry [125]. A method of producing blocks with the required magnetisation directions was put forward in Ref. [131] while an alternative consisting of a large number of small rectangular blocks was considered in [132].

A more flexible scheme consists of a superposition of separate horizontally and vertically polarized devices with a variable longitudinal displacement between them [133]. In general such a configuration allows the flexibility to change both the field amplitudes, B_{xo} , B_{yo} , and the phase between them, ψ :

$$B_x = B_{xo} \cos(kz + \psi) \quad B_y = B_{yo} \cos(kz) \quad (28)$$

and as a result, any polarization state can be created. We can see this in a simplified way by using Eqs. (25) and (27) above and equating the acceleration with the corresponding field component ($E_x \sim \dot{\beta}_x \sim B_y$ etc.). We obtain then that:

$$A_x \sim B_{yo} \quad A_y \sim B_{xo} (\cos \psi - i \sin \psi)$$

from which it follows from Eqs. (26) that:

$$\begin{aligned} S_0 &\sim B_{yo}^2 + B_{xo}^2 & S_1 &\sim B_{yo}^2 - B_{xo}^2 \\ S_2 &\sim 2 B_{xo} B_{yo} \cos \psi & S_3 &\sim 2 B_{xo} B_{yo} \sin \psi \end{aligned}$$

This analysis is valid for the first harmonic and for small K . The original proposal was an undulator consisting of identical permanent magnet arrays, as shown in Fig. 46, and therefore generating equal field strengths in both directions. A prototype device was subsequently constructed and installed in the TERAS ring in Japan and the polarization of the emitted radiation measured [134]. In such a structure changing the phase changes the polarization but does not change the field amplitudes, or the wavelength of the radiation (see 7.3.2). The polarization and the radiation wavelength can therefore be changed independently.

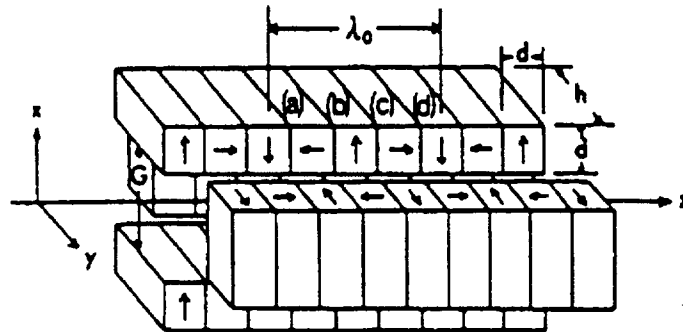


Fig. 46 Variably polarized undulator

With unequal field amplitudes in the two directions higher harmonics are generated with, in general, elliptical polarization [135]. Two permanent magnet devices of this kind were later constructed and installed in the TRISTAN Accumulator Ring and the Photon Factory [136]. These devices made use of the fact that the vertical magnet gap (30 mm) could be much smaller than the horizontal one (110 mm) so allowing a much higher field strength in the vertical direction, 1.0 T ($K_y = 15$) compared to the horizontal, 0.2 T ($K_x = 3$). Such a device became known as an elliptical multipole wiggler (EMPW). Later measurements confirmed that a high degree of circular polarization could be obtained at high photon energies [137].

The idea of replacing the permanent magnet arrays in the above scheme with an electromagnet in order to allow a rapid switching of the polarization state was first mentioned in [133]. Since an electromagnet requires a longer period length, and generates a smaller field strength than a permanent magnet, the idea is more applicable to the case of an elliptical wiggler. The first proposal for an electromagnetic elliptical wiggler (EEW) capable of reaching a switching speed of 100 Hz was made in [138]. A prototype device was later built by an APS/BINP/NSLS collaboration and installed in the NSLS X-ray ring [139]. A similar device was subsequently installed in the APS. A device consisting of two electromagnets, with the possibility also of shifting one magnet with respect to another has recently been proposed at LURE [140].

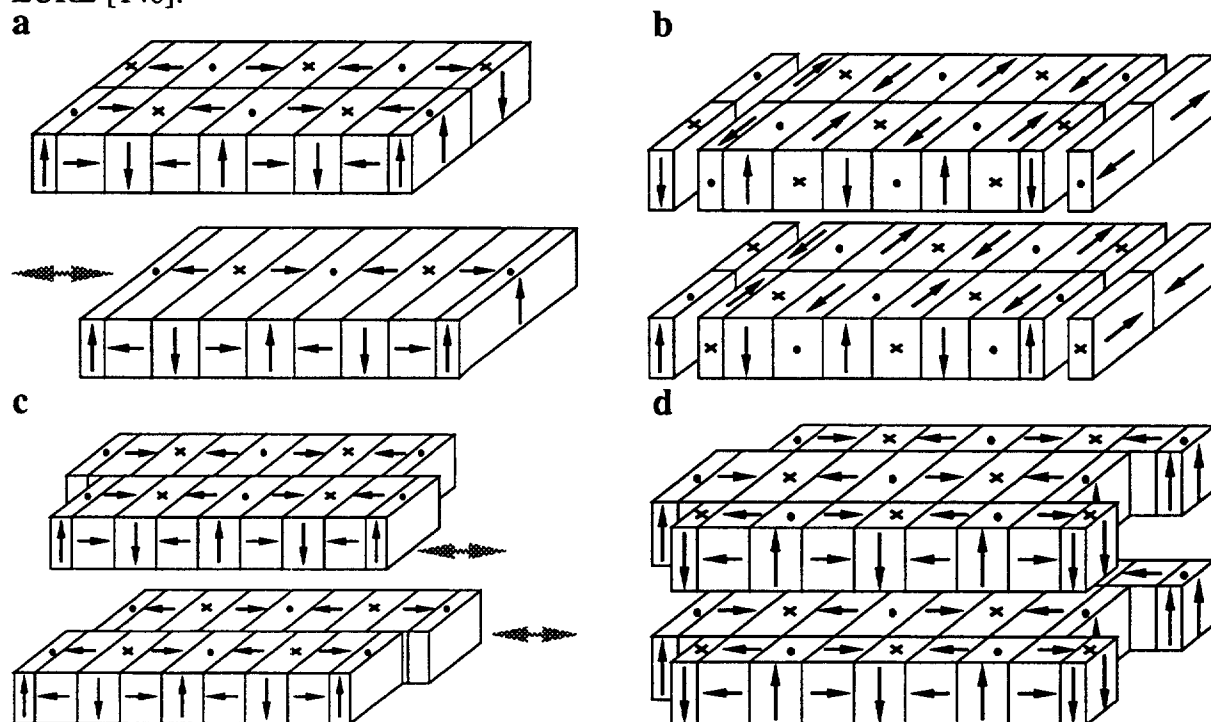


Fig. 47 Various planar permanent magnet arrangements.
a - HELIOS, b - planar helical undulator, c - APPLE-II, d - Spring-8

A disadvantage of all of the above schemes is the need for magnetic arrays on all sides of the electron beam tube. This not only complicates magnetic measurements and installation, but also means that the minimum gaps that can be achieved are restricted by the lateral size of the magnetic structures. Alternative "planar" structures have therefore been devised with magnets located only above and below the beam axis. The first proposal of this kind is shown in Fig. 47 (a) [141]. In this scheme the upper array produces a horizontal field, while the lower array produces a vertical field. The field components on-axis are therefore identical to those of the Onuki device given by Eq. (28), and hence by adjusting separately the upper and lower half-gaps and the longitudinal phase shift any polarization state can be generated. Owing to the lack of symmetry this device produces a linear variation of both field components in the vertical direction, thereby increasing the sensitivity of the radiation wavelength to beam position. A further potential problem is the fact that the fields give rise to a second-order deflection of the

trajectory in the horizontal direction when $\psi \neq 0$, and in the vertical direction when $B_{x0} \neq B_{y0}$ [142]. Since these effects are of second-order and therefore inversely proportional to Energy², the effects may become problematic in low energy rings. In the ESRF the horizontal deflection is overcome using two separate undulators with opposite helicity so that the two deflections cancel [143]. The two undulators are used in a "chicane" arrangement to produce radiation with opposite helicity displaced slightly to the left and right of the beam axis. The HELIOS device was installed in the ESRF in June 1993; no effect on beam lifetime was observed, and only small changes in closed orbit were produced, in agreement with expectations [67]. Two further single section structures have subsequently been installed [144].

A modification of the HELIOS device is the planar helical undulator [142], shown in Fig. 47 (b). In this case both arrays produce a pure helical field, with equal field amplitudes that exceed that of the HELIOS device in the helical mode. A further advantage of this structure is that the symmetrical arrangement eliminates any second-order deflections. A significant disadvantage however is that the helicity is fixed.

More recently a further device was developed called APPLE (Advanced Planar Polarized Light Emitter), which produces even higher field amplitudes. The first structure to be put forward (APPLE-I) employed magnet blocks with a 45° magnetization [145]. Subsequently a simpler structure (APPLE-II) was developed in which each of the four arrays has a conventional Halbach structure [146], shown in Fig. 47 (c). This version, as well as being easier to construct, produces a larger B_x field at the expense of smaller B_y and hence also a higher field in the helical mode. In both versions the polarization is altered by shifting the upper-back and lower-front arrays with respect to the upper-front and lower-back arrays. Considering for simplicity a shift of $\pm\psi$ for the pairs of arrays, the upper-back and lower-front arrays produce a linearly polarized field given by:

$$B_x = B_{x0}/2 \cos(kz - \psi) \quad B_y = B_{y0}/2 \cos(kz - \psi)$$

whereas the upper-front and lower-back arrays generate a field given by:

$$B_x = -B_{x0}/2 \cos(kz + \psi) \quad B_y = B_{y0}/2 \cos(kz + \psi)$$

Adding the two together and simplifying results in the following:

$$B_x = B_{x0} \sin(kz) \sin(\psi) \quad B_y = B_{y0} \cos(kz) \cos(\psi)$$

Thus, unlike the Onuki and HELIOS cases, the field amplitudes do depend on ψ , and so the polarization cannot be adjusted independently of the radiation wavelength. In addition, changing the gap changes the horizontal and vertical fields by a different amount, and so changes the polarization. An interesting property of this device is that the two field components remain 90° out of phase, and hence the polarization ellipse remains upright. The simple polarization analysis gives in this case (in the low K limit):

$$S_1 \sim B_{y0}^2 \cos^2 \psi - B_{x0}^2 \sin^2 \psi \quad S_2 \sim 0 \quad S_3 \sim B_{x0} B_{y0} \sin 2\psi$$

The APPLE configuration has a greater degree of symmetry than the HELIOS one, which eliminates problems due to second-order steering effects. The transverse field homogeneity remains considerably worse than in a conventional device, however experience with operating devices and beam dynamics calculations do not indicate any major problems.

The first test of the APPLE-I device was made with a seven-period prototype in the JAERI Storage Ring (JSR) [147]. Visible light was generated with variable polarization with a ring energy of 138 MeV. No beam deflections were observed while changing the gap or the phase.

An APPLE-II device was subsequently constructed for the SPEAR storage ring. This device includes a further feature, namely the ability to change the field strengths by phase rather than gap variation [148]. This is achieved by shifting the relative positions of the two upper arrays with respect to the two lower ones, or the two front arrays with respect to the two back

arrays. In this way no gap adjustment is required to set any desired output wavelength and ellipticity. The device was subsequently tested in SPEAR with no noticeable effect on the 3 GeV beam for any longitudinal motion of the arrays [149]. Further devices are under construction for ALS, BESSY-II and TLS, and are planned for the SRS.

Finally, a structure consisting of six magnet arrays has been proposed by the Spring-8 team [150], shown in Fig. 47 (d). In this case the upper and lower central arrays generate a vertical field while the outer four arrays generate a horizontal field. The longitudinal phasing of the outer arrays with respect to the central arrays allows the polarization to be adjusted. A disadvantage of the structure however is that it is not possible to achieve both pure horizontal and vertical polarization unless at least five of the arrays are moveable: if the central arrays are fixed, then there is always a vertical field present, while if the outer arrays are fixed there is always a horizontal field. Both versions are planned for Spring-8, while the former type has been constructed and installed recently in the UVSOR storage ring. It has been claimed that this structure results in a better transverse field homogeneity than the APPLE design, and therefore produces less effect on the electron beam dynamics.

7.3.2 Radiation properties

We firstly consider how the interference condition is modified in the presence of an additional horizontal field component, at an arbitrary phase with respect to the vertical field component:

$$B_x = B_{xo} \sin(kz + \psi) \quad B_y = B_{yo} \sin(kz)$$

The electron motion is then:

$$\beta_y = -\frac{K_x}{\gamma} \cos(kz + \psi) \quad \beta_x = \frac{K_y}{\gamma} \cos(kz)$$

where $K_x = \frac{eB_x \lambda_o}{2\pi mc}$, and $K_y = \frac{eB_y \lambda_o}{2\pi mc}$. Since the electron velocity is given by $\beta^2 = \beta_x^2 + \beta_y^2 + \beta_z^2$ we have therefore:

$$\beta_z \equiv \beta \left(1 - \frac{K_x^2}{4\gamma^2} - \frac{K_y^2}{4\gamma^2} - \frac{K_y^2}{4\gamma^2} \cos 2kz - \frac{K_x^2}{4\gamma^2} \cos 2(kz + \psi) \right) \quad (29)$$

For the interference condition only the average velocity along the z-axis is important, which is simply:

$$\bar{\beta} \equiv \beta \left(1 - \frac{K_x^2}{4\gamma^2} - \frac{K_y^2}{4\gamma^2} \right)$$

Applying the interference condition, Eq. (3), then gives:

$$\lambda = \frac{\lambda_o}{2\gamma^2} \frac{1}{n} \left(1 + \frac{K_x^2}{2} + \frac{K_y^2}{2} + \gamma^2 \theta^2 \right) \quad (30)$$

Thus the radiation wavelength depends only on the amplitudes of the two field components, independently of the phase ψ between them.

The analysis of the radiation emission in the general case is quite complicated (see Appendix). In the particular case of on-axis radiation for $\psi = \pi/2$ the angular flux density can be written, in practical units, as follows:

$$\left. \frac{d^2 \dot{n}}{d\omega/\omega \, d\Omega} \right|_{\theta=0} = 1.744 \cdot 10^{14} \, N^2 \, E_{[GeV]}^2 \, F_n(K_x, K_y) \, I_b$$

where

$$F_n(K_x, K_y) = \frac{n^2}{(1 + K_x^2/2 + K_y^2/2)^2} (A_x^2 + A_y^2)$$

and

$$A_x = K_y \left(J_{\frac{n+1}{2}}(Y) - J_{\frac{n-1}{2}}(Y) \right), \quad A_y = K_x \left(J_{\frac{n+1}{2}}(Y) + J_{\frac{n-1}{2}}(Y) \right)$$

with

$$Y = \frac{n (K_y^2 - K_x^2)}{4(1 + K_x^2/2 + K_y^2/2)}$$

The integrated flux is then given, see Eq. (18), by:

$$\frac{d\dot{n}}{d\omega/\omega} = 1.431 \cdot 10^{14} \, N \, Q_n(K_x, K_y) f(N \Delta\omega/\omega_1(0)) \, I_b$$

where $Q_n(K_x, K_y) = (1 + K_x^2/2 + K_y^2/2) F_n(K_x, K_y) / n$. The Stokes parameters are given by:

$$\begin{aligned} S_1/S_0 &= (A_x^2 - A_y^2) / (A_x^2 + A_y^2) \\ S_2/S_0 &= 0 \\ S_3/S_0 &= -2A_x A_y / (A_x^2 + A_y^2) \end{aligned}$$

In the case of purely circular motion, when $K_x = K_y = K$ (and $\psi = \pi/2$) the longitudinal velocity, Eq. (29), is constant. As a result the electric fields E_x and E_y are purely sinusoidal and so the radiation consists of a single harmonic on axis. This is evident also in the formula above since when $Y = 0$ the only non-zero term is $J_0(0)$ which occurs only for $n = 1$. We have therefore:

$$F_1(K) = \frac{2K^2}{(1 + K^2)^2} \quad \text{and} \quad Q_1(K) = \frac{2K^2}{(1 + K^2)}$$

and the radiation is purely circularly polarized:

$$S_1/S_0 = 0, \quad S_2/S_0 = 0, \quad S_3/S_0 = 1$$

Figure 48 shows the variation of angular flux density, circular polarization rate and their product (S_3) as a function of the ratio between the horizontal and vertical field strengths with a fixed radiation wavelength (i.e. fixed $K_x^2 + K_y^2$). For $K_x/K_y = 0$ we have the same result for a plane undulator, with pure linear polarization. As K_x/K_y increases both the flux and circular polarization rate increase for the fundamental. The maximum circularly polarized flux is therefore obtained with pure circular polarization with $K_x = K_y$. In the case of the higher harmonics however the circular polarization rate increases, but the flux eventually tends to zero at $K_x = K_y$. There is therefore an optimum value of K_x/K_y which maximizes S_3 , which depends on the harmonic number (and on $K_x^2 + K_y^2$).

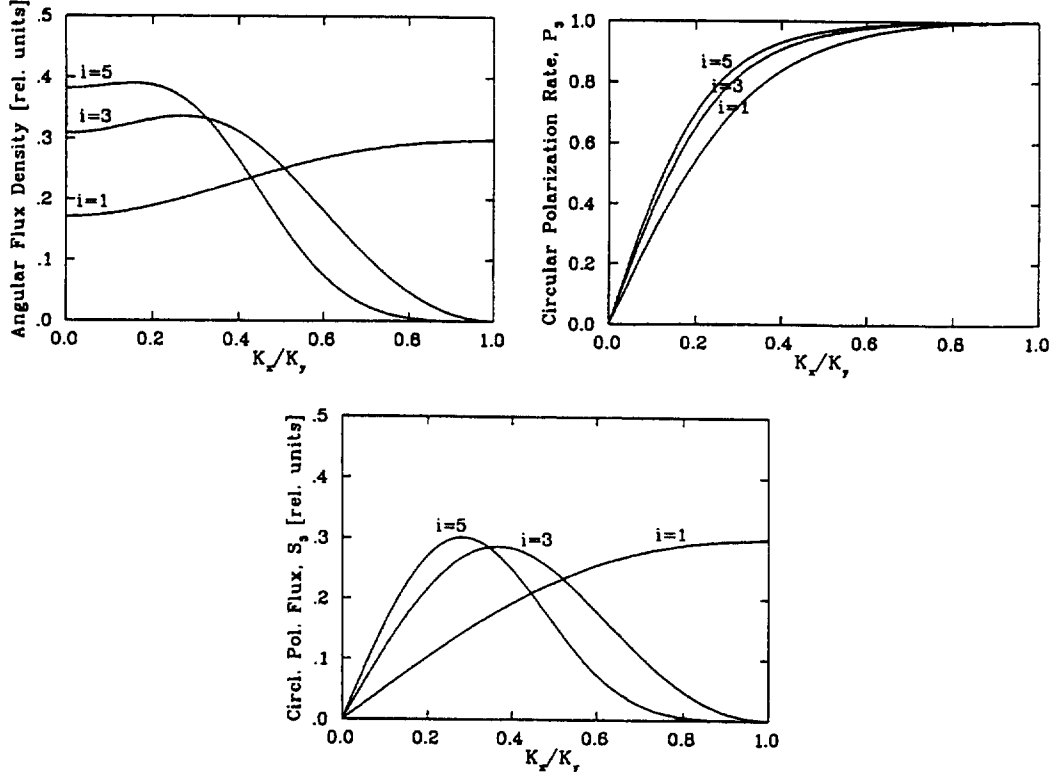


Fig. 48 Variation of (a) angular flux density, (b) circular polarization rate and (c) circularly polarized flux density with ratio K_x/K_y for $K_x^2 + K_y^2 = 3^2$

The angular distribution of the power density in the elliptical case ($\psi = \pi/2$) is obtained from Eq. (19) with $v_x = K_y \sin \alpha - \gamma \theta_x$ and $v_y = K_x \cos \alpha - \gamma \theta_y$ from which one obtains:

$$\frac{dP}{d\Omega} [W/\text{mrad}^2] = 13.44 \cdot 10^{-3} E_{[GeV]}^4 I_b \frac{2\pi}{\lambda_0} N \int_{-\pi}^{\pi} \left\{ \frac{(K_y^2 \cos^2 \alpha + K_x^2 \sin^2 \alpha)}{D^3} + \frac{[(K_y^2 - K_x^2) \sin 2\alpha - 2K_y \gamma \theta_x \cos \alpha + 2K_x \gamma \theta_y \sin \alpha]^2}{D^5} \right\} d\alpha$$

where $D = 1 + (K_y \sin \alpha - \gamma \theta_x)^2 + (K_x \cos \alpha - \gamma \theta_y)^2$. In the case of pure helical motion, $K_x = K_y$, the on-axis result becomes simply:

$$\frac{dP}{d\Omega} [W/\text{mrad}^2] = 13.44 \cdot 10^{-3} E_{[GeV]}^4 I_b \left(\frac{2\pi}{\lambda_0} \right)^2 L \frac{K^2}{(1 + K^2)^3}$$

which shows a maximum for $K = 1/\sqrt{2}$. For larger K the on-axis power density decreases rapidly as most of the power is emitted off-axis, with a maximum near the angle K/γ [15]. The total power emitted is given from Eq. (20) as:

$$P_{\text{tot}} [W] = 633 \cdot E_{[GeV]}^2 (B_{x0}^2 + B_{y0}^2) L I_b$$

7.4 Asymmetric wiggler

An asymmetric wiggler is a planar device with a single on-axis field component but differs from a conventional insertion device in that the positive and negative field strengths are of unequal magnitude [151]. The first asymmetric wiggler was constructed at HASYLAB in 1989 and employed a pure permanent magnet construction, illustrated in Fig. 49 (left) [152]. In a normal sinusoidal wiggler the radiation emitted at a given horizontal angle has two source points per period with equal positive and negative field amplitudes. In an asymmetric wiggler however this symmetry is broken and the field values are no longer equal, as illustrated in Fig. 49 (right). As a result, circularly polarized radiation is emitted off-axis in the vertical direction as in a single bending magnet. It can be seen that the difference between the field values is greatest at zero horizontal angle and decreases with increasing angle. The circular polarization rate therefore follows the same trend, as can be seen in Fig. 50 (left), calculated for the structure of Fig. 49. Another related feature of this device is that the total intensity peaks off-axis in the horizontal direction. It is clear that the highest circular polarization rate is obtained close to zero horizontal angle and also off axis vertically, and there is therefore a compromise between flux and degree of circular polarization. Figure 50 (right) shows the integrated flux, circularly polarized flux and circular polarization rate for a particular case, with horizontal acceptance of ± 0.25 mrad and vertical acceptance from 0.1 to 0.2 mrad. The characteristic feature can be seen that the circularly polarized flux S_3 changes sign at low photon energies when the lower field pole (with negative S_3) contributes more than the higher field pole (with positive S_3). On the other hand at the highest photon energies the contribution of the lower field pole becomes negligible, resulting in increasing polarization rate, but rapidly diminishing flux.

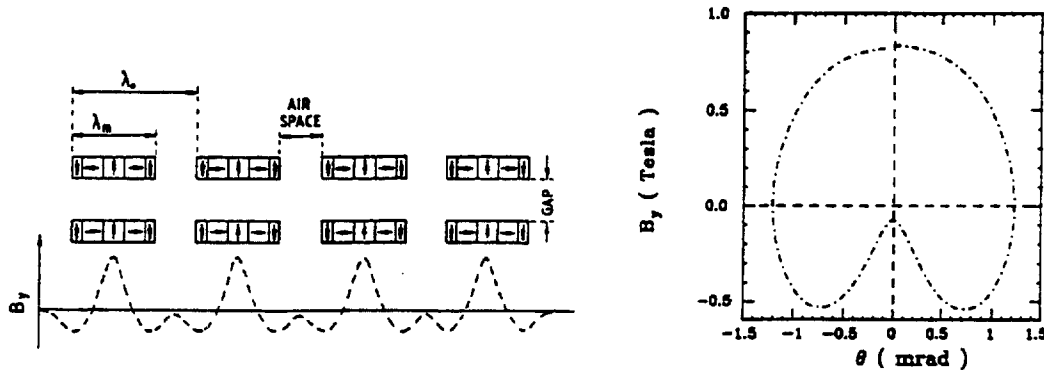


Fig. 49 Asymmetric wiggler; left - permanent magnet structure and field distribution, right - variation of magnetic field with emission angle [152].

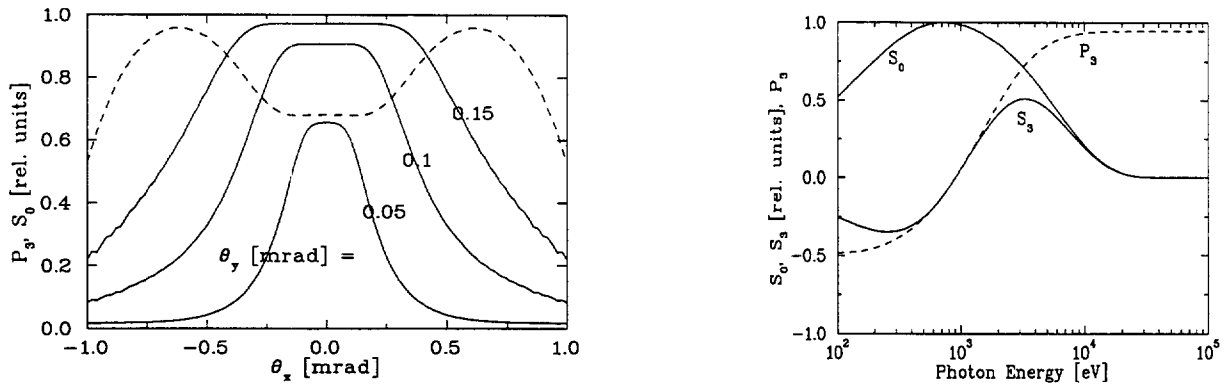


Fig. 50 Performance of the asymmetric wiggler shown in Fig. 49; left - circular polarization rate as a function of horizontal and vertical angle for 10 keV radiation, dotted line - total intensity at $\theta_y = 0.1$ mrad; right - integrated flux and circular polarization rate as a function of energy.

Further asymmetric wigglers have been built using the hybrid technology (i.e. with iron poles) at LURE [153], HASYLAB [154] and ESRF [144]. A superconducting asymmetric wiggler has also been constructed for DELTA [155]. The LURE and ESRF designs are similar and are obtained by splitting every alternate pole into two and separating by a drift space. The HASYLAB design on the other hand incorporates zero-potential field clamps rather than drift spaces. In the pure permanent magnet case, linear superposition guarantees that although the field amplitudes are different for each pole, the field integrals are equal. In the hybrid case this is no longer true, and the main problem in the design is therefore to achieve a compensation of the vertical field integral at all operating gaps. Three dimensional computations are therefore very useful in order to reduce the correction needed with coils etc.

7.5 The crossed undulator

The crossed undulator [156] consists of two separate linearly polarized undulators with polarization at right-angles to each other, separated by a "modulator" magnet (see Fig. 51). The purpose of the latter is to create a compensated bump in the electron trajectory and hence introduce a variable phase delay between the radiation emitted in the two undulators. The resulting interference produces elliptically polarized radiation on-axis.

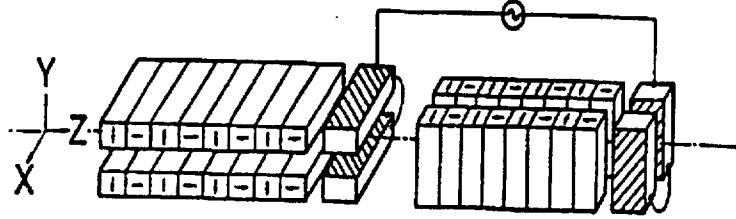


Fig. 51 The crossed undulator

A simple analysis of the polarization properties can be made by summing the amplitudes from the two undulators:

$$\underline{A} = \left[\underline{A}_1 + \underline{A}_2 e^{i(N\delta + \psi)} \right] \frac{\sin N\delta/2}{\sin \delta/2}$$

where $\delta = 2\pi\Delta\omega/\omega_1(\theta)$ is the phase advance per period in the first undulator and ψ is the extra phase change introduced by the modulator, which is assumed constant over the small range of frequencies that we will be considering. Assuming that the two undulators are identical (each with N periods) and that the first undulator is polarized in the horizontal direction and the second in the vertical direction we obtain:

$$A_x = A \frac{\sin N\delta/2}{\sin \delta/2} \quad A_y = A e^{i(N\delta + \psi)} \frac{\sin N\delta/2}{\sin \delta/2}$$

from which it follows that:

$$S_0 = 2A^2 \frac{\sin^2 N\delta/2}{\sin^2 \delta/2}$$

$$S_1 = 0$$

$$S_2/S_0 = \cos(N\delta + \psi)$$

$$S_3/S_0 = \sin(N\delta + \psi)$$

At zero detuning therefore ($\delta = 0$) the polarization changes with the modulator phase in the following way:

Phase (ψ)	S_2/S_0	S_3/S_0	Polarization
0°	1	0	linear at 45°
90°	0	1	right-handed circular
180°	-1	0	linear at -45°
270°	0	-1	left-handed circular
360°	1	0	linear at 45°

A rapid change of polarization can be made if the modulator is a suitably designed electromagnet. An interesting feature of the device is that, unlike helical magnets, even with pure circular polarization higher harmonics are generated on-axis, since each individual linear undulator generates higher harmonics.

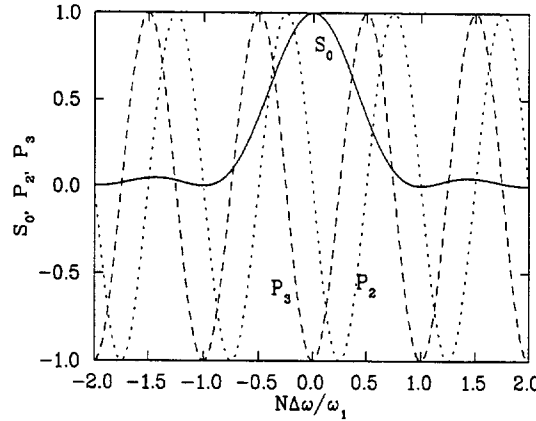


Fig. 52 Variation of intensity and circular polarization rates with frequency in a crossed undulator

From the equation above it is clear that the polarization changes rapidly with the detuning parameter, and this is illustrated in Fig. 52. For example, with $\psi = \pi/2$ we obtain pure circularly polarized radiation on axis. At $\Delta\omega/\omega_1 = 1/4N$ however this becomes zero, whereas the total flux becomes zero only at $\Delta\omega/\omega_1 = 1/N$. Certain conditions must therefore be met in order to obtain significant circular polarization:

- selection of a narrow range of photon energies, $\Delta\omega/\omega \leq 1/2nN$. Under normal circumstance this is not a problem since the monochromator provides a significantly smaller band-pass;
- selection of a narrow range of acceptance angles, $\theta \leq \sqrt{\lambda/2L}$. In other words, the acceptance angle must be restricted with a "pinhole" aperture so as to receive only a fraction of the available flux;
- it follows from the above that the electron beam divergence must also be sufficiently small, $\sigma_\theta \leq \sqrt{\lambda/2L}$;
- similarly, the energy spread of the electron beam must be sufficiently small, $\Delta E/E \leq 1/8nN$.

It should be noted that in all cases the sensitivity increases for the higher harmonics.

The first, and so far only, device of this type to be built is operational in BESSY [157]. The present modulator is not laminated and so does not permit a fast switching. A crossed undulator that was proposed for the ALADDIN storage ring [158] incorporated a modulator that was designed to reach a switching speed of 10 Hz [159].

Alternative geometries to the one presented above are also possible. With the two undulators oriented at 45° and -45° with respect to the x-y axes results in an exchange of the S_1 and S_2 parameters, i.e. allowing horizontal and vertical polarization to be obtained as well as circular polarization. A combination of right- and left-handed circularly polarized undulators creates linearly polarized radiation ($S_1 \neq 0$, $S_2 \neq 0$, $S_3 = 0$). In this case however the interesting situation arises that since only a single harmonic is generated on-axis in each device, the linearly polarized radiation consists only of a single harmonic [160].

8. UNDULATORS FOR FREE-ELECTRON LASERS

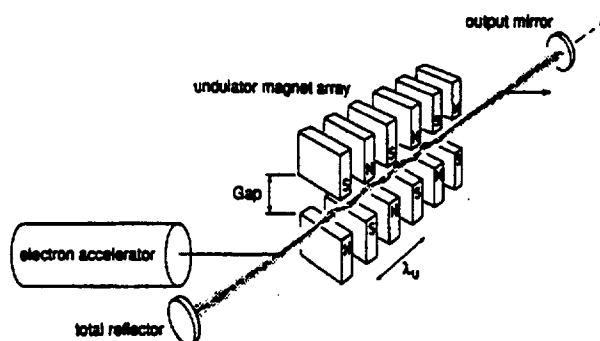


Fig. 53 Basic scheme of a free-electron laser

In a free-electron laser (FEL) the presence of a static periodic magnetic field produced by an undulator magnet allows an exchange of energy between an electron beam and a co-propagating radiation beam, resulting in the emission of coherent radiation [161]. The resonance condition governing this interaction and which determines the laser output wavelength is identical to the interference condition derived earlier, Eq. (3). An undulator is therefore one of the principle components of a FEL. The same type of undulator used as a synchrotron radiation source can be used, however there are a number of additional requirements and circumstances that apply in the case of the FEL and as a result many different kinds of magnet design have been developed for this specific application.

In the following we will firstly consider the various undulator configurations that have been developed in order to enhance the FEL interaction. Other important differences in the characteristics of FEL undulators compared to those used as synchrotron radiation sources arise in cases in which the electron beam passes only once through the undulator i.e. single-pass devices, driven by a linac beam for example. In this case a much smaller gap can be employed, since there are no restrictions due to the beam lifetime. A great deal of activity has therefore been devoted to the development of small-period devices of various types, in order to reach the shortest possible wavelength with a given electron beam energy. Some of these devices are pulsed electromagnets, since the electron beam itself is generally pulsed. Another important requirement in the case of FEL undulators is the need to incorporate additional focusing into the magnetic structure, in order to obtain a better overlap of the electron and photon beam sizes and so optimize the FEL interaction. Further details on FEL undulator construction and measurement can be found in the Proceedings of the annual International Free-Electron Laser Conferences [162].

8.1 Undulator configurations

The two main quantities that characterize the performance of a FEL oscillator are the small-signal gain, which determines how rapidly the signal builds up, and the efficiency, which determines what fraction of the electron beam energy can be converted into radiation intensity. The efficiency is limited by the fact that at saturation the electrons lose so much energy per pass that they fall out of resonance. To overcome this a *tapered undulator* is sometimes adopted in which the period and/or field strength are profiled along the length of the magnet in order to match the desired decrease in electron energy [163]. Several experiments were carried out to confirm the theoretical predictions in an amplifier configuration, using permanent magnet undulators with up to 9 % energy tapering [164]. An example of a tapered electromagnetic device is the 25 m long wiggler built for the PALADIN FEL experiments [165]. In this device each two-periods were excited with a separate power supply in order to allow a variable tapering. In order that the adjustment of the taper would cause no net displacement of the electron beam the excitation pattern applied was compensated as for a conventional entrance/exit sequence i.e. each five poles were excited with a sequence +1, -2, +2, -2, +1. More complex excitation patterns have also been studied [166].

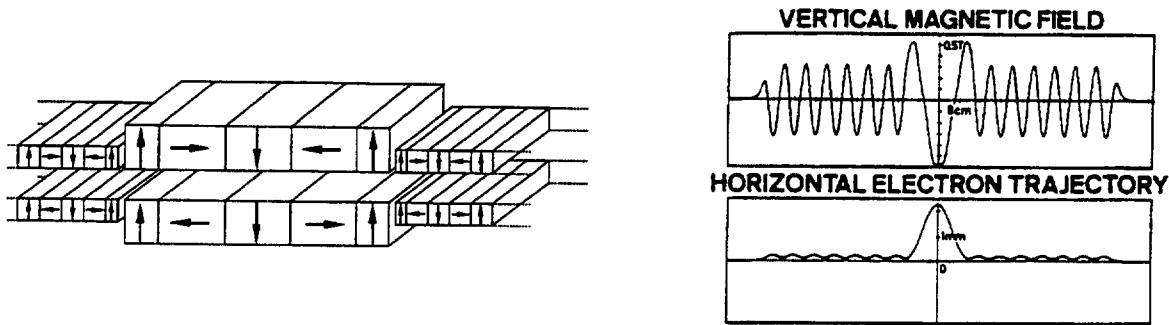


Fig. 54 Schematic of a pure permanent magnet optical klystron (left); field and electron trajectory in the Orsay device (right) [168].

Another variant of the basic FEL undulator is the so-called *optical klystron* [167], see Fig. 54. This consists of two identical undulators separated by a dispersive section, which introduces a compensated bump into the electron trajectory and hence a phase delay between the radiation emitted in the two magnets. The structure is therefore similar to that of the crossed undulator considered earlier (Section 7.5) except that in this case the undulators have the same polarization and the phase delay is much larger. The lineshape of the spontaneous or incoherent radiation spectrum becomes modulated, resulting in an enhanced gain. Alternatively one can consider that an energy modulation introduced in the first undulator by the interaction of the electron beam and radiation field is converted in the dispersive section into a density modulation, so that coherent emission takes place in the second undulator. Such a scheme is particularly useful in cases with small gain, but with sufficiently small energy spread, such as is often the case with a storage ring FEL. For this reason the optical klystron configuration has been chosen for all existing and previous SR-FEL experiments, on the ACO, DELTA, DUKE, NIJI-IV, SuperACO, UVSOR, and VEPP-3 storage rings. In almost all cases a linearly polarized undulator has been used, either permanent magnet or electromagnet. The exception is a helical optical klystron recently constructed for UVSOR [169].

A similar arrangement to the optical klystron is one in which the second undulator emits coherent radiation at a harmonic of the frequency of the first undulator [170] and is therefore called a *harmonic generation* scheme.

The disadvantage of a tapered undulator is that it results in a lower small-signal gain and is only optimized at a given power level. As a result it is not ideal in an oscillator configuration when a rapid increase of power is required. In order to overcome this problem and so provide a high output power oscillator a *multi-component* design was put forward, which includes a number of constant and tapered sections as well as a dispersive region as in an optical klystron [171]. Successful operation of an oscillator using this scheme was later reported [172].

8.2 Small-period devices

Various approaches have been taken to the construction of small-period FEL undulators. First of all, conventional permanent magnet devices can be scaled down quite successfully to smaller period and gap. For example, a 5 m tapered hybrid undulator with 21.8 mm period and 4.8 mm gap was constructed by Spectra Technology Inc. for a visible FEL oscillator experiment [173]. A 20 mm period pure permanent magnet in-vacuum device was built for the FELI facility (Japan), which generated 0.85 T at 5 mm gap [174]. Various other hybrid devices have been built, the most advanced is that developed at CREOL (USA) having 185 periods of 8 mm, and which achieves a field of 0.2 T at 6 mm gap [175]. A number of alternative hybrid schemes have been investigated at the Kurchatov Institute. In one scheme the field of a conventional hybrid structure is augmented by side permanent magnets which direct extra flux into the gap region via C-shaped pole pieces [176]. Alternative pure permanent magnet schemes have also been developed that are more easily adaptable to short period lengths, such as that developed UCSB which consist of grooves ground in large blocks of permanent magnet material [177]; prototypes of this concept were built with 4 mm period, 2 mm gap and achieved

fields of the order 0.1 T. The field was however considerably lower than could be achieved in a conventional pure permanent magnet, with the added difficulties of field offset, large end fields and significant field errors.

Various miniature electromagnetic schemes have also been developed, starting with a simple design in which a copper foil is wound between electric steel laminations in alternating directions [178]. A prototype with a period of 2.7 mm and gap of 1 mm produced a field amplitude of about 0.08 T in d.c. mode. Most later devices were designed to be pulsed in order to allow higher currents to be used without overheating. Another design based on a folded foil involved a double winding scheme, i.e. the foil is folded back at the end of the magnet and wound in the opposite sense, which has the advantage of producing a more symmetrical field distribution [179]. Periods as small as 5 mm were considered possible using this technique. Another approach which received a lot of attention was the slotted-tube device in which a high current pulse is passed along a cylindrical tube with appropriately positioned cut-outs to direct the current flow so as to generate the desired field configuration; several prototypes were built producing both helical and later planar fields and in one a field strength of 1 T was achieved with 5 mm period with a 25 kA pulse [180]. Another related idea was to use a bifilar helical sheet with ferromagnetic cores; a 10 mm prototype was constructed [181]. All of the above devices suffer from the disadvantage that there is no simple method for correcting field errors. An alternative approach which overcomes this problem uses individual small electromagnets, which allows independent tuning of the strength of each pole by means of a resistive current divider network feeding each pair of coils from a single power supply [182]. A 70 period device was constructed with a period of 8.8 mm and gap of 4.2 mm which generated a field amplitude of 0.42 T with excellent field quality [183].

Small-period devices have also been constructed using superconducting magnet technology. The design developed at BNL is based on a continuously wound conductor on a precisely machined iron yoke [184]. Two devices have been constructed for two different FEL experiments; the undulator with the shortest period (8.8 mm) produces a field of 0.5 T at 4.4 mm gap. Very good field uniformity was achieved within each section, although some errors appeared at the joints between sections [185].

Two other novel concepts that have been developed for producing sub-cm period undulators are an electromagnetic helical device based on a laminated construction with three poles per magnet period [186] and a device consisting of a staggered array of iron poles immersed in a solenoidal field produced by a superconducting magnet [187]. A 1 T sinusoidal field amplitude was generated for a 10 mm period at 2 mm gap with a 0.7 T solenoidal field.

8.3 Transverse focusing

A conventional linearly polarized undulator with a vertical field, causes a focusing of the beam in the vertical plane, whereas (ignoring the effects of any field errors) it acts like a drift space in the horizontal plane [29]. In the vertical plane therefore there exists a matching condition, such that if the electron beam ellipse is adjusted correctly at the entrance of the magnet, it will remain unaltered as it progresses along the magnet. It can be shown that under these conditions the longitudinal velocity (averaged over an undulator period) is constant for each electron during its motion in the vertical plane, and that this is the optimum setting in order to minimize the effect of the electron beam emittance on the FEL interaction. The same situation however does not exist in the horizontal plane due to the lack of horizontal focusing. To remedy this situation, certain types of helical undulator can be used. For example, the bifilar helix produces equal focusing in both planes [121]. The double-undulator scheme of Onuki in the helical mode, with identical magnet arrays and equal gaps, also gives equal focusing, even though the field distribution is quite different to that of the bifilar helix [142]. The same is not generally true however for the other helical magnet schemes capable of generating circularly polarized radiation (Section 7.3).

In many cases a plane polarized magnet is preferred to a helix and considerable attention has been given to different ways of incorporating focusing in the horizontal plane. In Ref. [188] it was shown that in general a small parabolic curvature of the pole profile (assuming an iron-dominated electromagnet or hybrid device) is sufficient to produce equal focusing in both planes and that this has the advantage over the alternative method of adding focusing by means of

linear field gradients that the longitudinal velocity, and hence the FEL resonance condition, remains constant. It was also shown that this difference can have a significant influence on FEL performance. One of the first magnets to be built incorporating this principle was the electromagnet for the PALADIN FEL experiment [165]. The idea is however quite an old one: focusing of this type was employed in the ubitron developed by Philips [5].

Various schemes have been investigated for incorporating both types of focusing, quadrupole and sextupole-like, into permanent magnet FEL undulators as shown in Fig. 55. In the case of the pure permanent magnet configuration, it has been shown that curved blocks (Fig. 55a) can introduce sextupole-like focusing [189], while trapezoidal (Fig. 55b) [189] and staggered blocks (Fig. 55c) [190] can introduce quadrupole-like focusing. In the case of hybrid designs, alternate pole canting (Fig. 55d) has been applied on several occasions to produce additional quadrupole focusing [173]; displacing the poles transversely in alternating directions (Fig. 55f) has a similar effect. It has been suggested that a simple method of introducing sextupole focusing is to cut a rectangular slot in the pole, rather than curving the pole surface [191]. Another possibility is the use of additional side magnets. If the side blocks are arranged so that their magnetization directions are opposite to the direction of the field in the gap on both sides of the magnet, the field amplitude is increased and a sextupole field is generated providing horizontal focusing. Such a scheme was adopted for the FOM-FEM project [192].

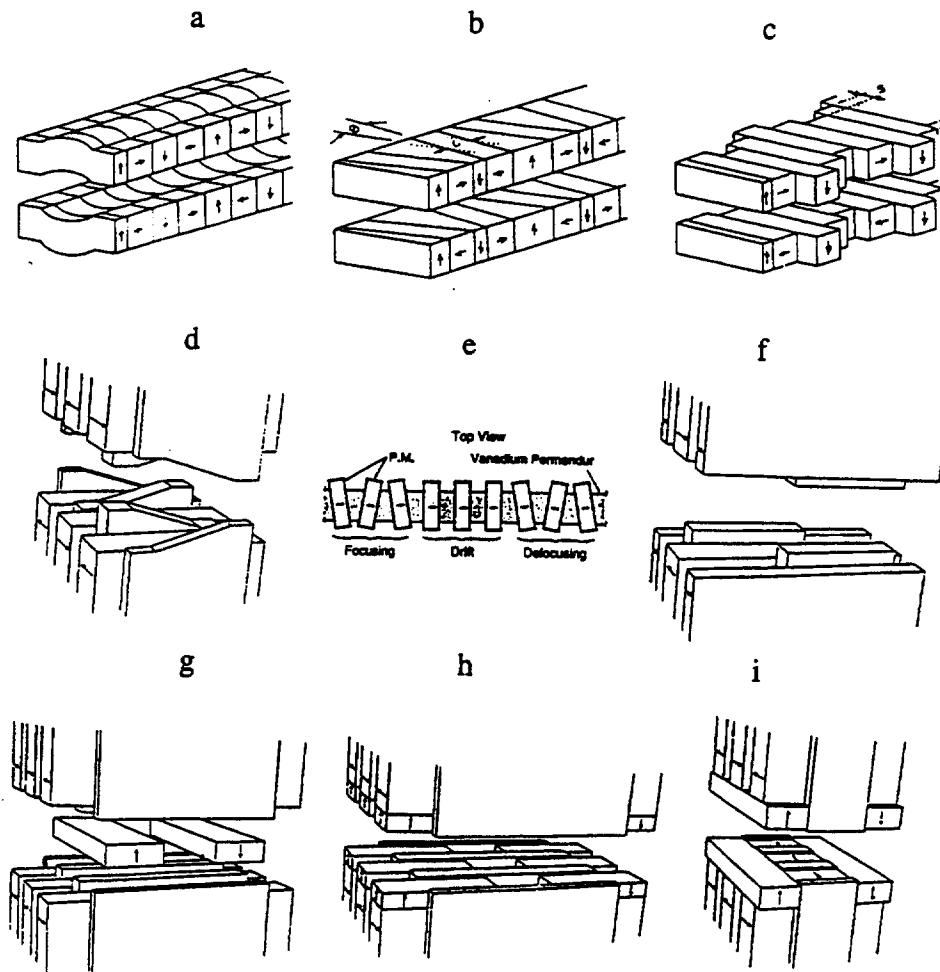


Fig. 55 Various schemes for introducing additional focusing in pure permanent magnet and hybrid undulators

In the case of short wavelength single-pass FEL experiments in particular there is a need for very strong focusing to maintain small beam dimensions over the required long undulator lengths of 15-50 m. In this case natural sextupole focusing is insufficient, and instead strong

superimposed quadrupole fields of 20-50 T/m must be used with alternating gradient. One of the possibilities being investigated for the LCLS project is a canted and wedged pole hybrid (Fig. 55e) [193]. Another possibility is the use of side magnets, but differing from the scheme outlined above for producing sextupole focusing. In this case the side magnets have opposite directions of magnetization, and are continuous along the length of the magnet, and so produce a superimposed quadrupole field (Fig. 55g) [194]. A similar concept, but using four magnets above and below the median plane, and therefore having the advantage of leaving the sides open, was introduced in Ref. [195]. Two further related schemes of incorporating the additional magnets into the hybrid structure (Figs. 55h and 55i) were reported in [196]. One of these, the so-called four magnet focusing undulator (Fig. 55h), was subsequently selected for the undulator for the TTF-FEL project and is presently under construction. Various other schemes are also being studied for the LCLS project, including superconducting and pulsed copper helical magnets [197].

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APPENDIX SPECTRAL AND ANGULAR DISTRIBUTION OF THE RADIATION EMITTED IN VARIOUS TYPES OF SINUSOIDAL UNDULATOR

Given the following general field distribution which includes both planar, circular and general elliptical undulator types:

$$B_x = B_{xo} \sin(kz + \psi) \quad B_y = B_{yo} \sin(kz)$$

we obtain, using the procedure outlined in section 2.1, for the transverse motion:

$$\begin{aligned} \beta_y &= -\frac{K_x}{\gamma} \cos(\Omega t + \psi) & \beta_x &= \frac{K_y}{\gamma} \cos(\Omega t) \\ y &= -\frac{K_x}{\gamma} \frac{c}{\Omega} \sin(\Omega t + \psi) & x &= \frac{K_y}{\gamma} \frac{c}{\Omega} \sin(\Omega t) \end{aligned}$$

and for the longitudinal motion:

$$\begin{aligned} \beta_z &\equiv \bar{\beta} - \frac{K_y^2}{4\gamma^2} \cos 2\Omega t - \frac{K_x^2}{4\gamma^2} \cos 2(\Omega t + \psi) \\ z &= \bar{\beta} ct - \frac{K_y^2}{4\gamma^2} \frac{c}{2\Omega} \sin 2\Omega t - \frac{K_x^2}{4\gamma^2} \frac{c}{2\Omega} \sin 2(\Omega t + \psi) \end{aligned}$$

where

$$\Omega = \frac{2\pi\bar{\beta}c}{\lambda_o}, \text{ and } \bar{\beta} \equiv 1 - \frac{1}{2\gamma^2} - \frac{K_x^2}{4\gamma^2} - \frac{K_y^2}{4\gamma^2}$$

The general expression for the radiated intensity in the case of a periodic magnet in the far-field approximation is given by the following (see section 3.1):

$$\frac{d^2 I}{d\omega d\Omega} = \frac{N^2 e^2 \omega^2}{(4\pi\epsilon_o)4\pi^2 c} \left| \int_{-\lambda_o/2\bar{\beta}c}^{\lambda_o/2\bar{\beta}c} \{\hat{n} \wedge (\hat{n} \wedge \beta)\} e^{i\omega(t-n \cdot r/c)} dt \right|^2 L(N\Delta\omega/\omega_1(\theta)) \quad (A1)$$

Expanding the term $\hat{n} \wedge (\hat{n} \wedge \beta)$ to order K/γ and using the small angle approximation $\hat{n} = (\theta \cos \phi, \theta \sin \phi, 0)$, one obtains $\hat{n} \wedge (\hat{n} \wedge \beta) \equiv (\theta \cos \phi - \beta_x, \theta \sin \phi - \beta_y, 0)$. Expanding also the phase terms we obtain:

$$\begin{aligned} \omega(t - n \cdot r/c) &= \omega t - \omega \theta \cos \phi x/c - \omega \theta \sin \phi y/c - \omega \cos \theta z/c \\ &= \omega t (1 - \bar{\beta} \cos \theta) - \frac{\omega \theta \cos \phi}{\Omega} \frac{K_y}{\gamma} \sin \Omega t + \frac{\omega \theta \sin \phi}{\Omega} \frac{K_x}{\gamma} \sin(\Omega t + \psi) + \\ &\quad \frac{\omega}{2\Omega} \left(\frac{K_y^2}{4\gamma^2} \sin 2\Omega t + \frac{K_x^2}{4\gamma^2} \sin 2(\Omega t + \psi) \right) \end{aligned}$$

With large N we only need evaluate the expression at the harmonic frequencies, where:

$$\omega = \frac{n\Omega}{(1 - \bar{\beta} \cos \theta)} = \frac{n\Omega 2\gamma^2}{1 + K_x^2/2 + K_y^2/2 + \gamma^2 \theta^2} \quad (A2)$$

Combining the trigonometric terms we then obtain:

$$\omega(t - n \cdot r/c) = n\Omega t - X \sin(\Omega t - \Phi) + Y \sin(2\Omega t + \Psi)$$

where,

$$X = \frac{2n\gamma\theta}{A} \sqrt{K_y^2 \cos^2 \phi + K_x^2 \sin^2 \phi - 2K_x K_y \sin \phi \cos \phi \cos \psi}$$

$$Y = \frac{n}{4A} \sqrt{K_y^4 + K_x^4 + 2K_x^2 K_y^2 \cos 2\psi}$$

and

$$A = 1 + K_x^2/2 + K_y^2/2 + \gamma^2 \theta^2, \quad \tan \Phi = \frac{-K_x \sin \phi \sin \psi}{K_x \sin \phi \cos \psi - K_y \cos \phi}, \quad \tan \Psi = \frac{K_x^2 \sin 2\psi}{K_x^2 \cos 2\psi + K_y^2}.$$

By means of the relation $\exp(ix \sin \phi) = \sum_{p=-\infty}^{\infty} J_p(x) \exp(ip\phi)$ the phase expression now becomes:

$$e^{i\{n\Omega t - X \sin(\Omega t - \Phi) + Y \sin(2\Omega t + \Psi)\}} = \sum_{p'=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} J_{p'}(X) J_p(Y) e^{i\{n\Omega t - p'(\Omega t - \Phi) + p(2\Omega t + \Psi)\}}$$

and hence Eq. (A1) becomes:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{N^2 e^2 \omega^2}{(4\pi\epsilon_0)4\pi^2 c} L(N\Delta\omega/\omega_1(\theta)) \times$$

$$\left| \sum_{p'=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} J_{p'}(X) J_p(Y) \int_{-\lambda_o/2\bar{\beta}c}^{\lambda_o/2\bar{\beta}c} (\theta \cos \phi - \beta_x, \theta \sin \phi - \beta_y) e^{i\{n\Omega t - p'(\Omega t - \Phi) + p(2\Omega t + \Psi)\}} dt \right|^2$$

where the two terms in the integrand represent the amplitudes polarized respectively in the x and y directions. It can be seen that the integral over one period is non-zero only when either $n - p' + 2p = 0$ for the constant terms, or when $n - p' + 2p = \pm 1$ for the terms in $\cos \Omega t$ or $\sin \Omega t$. Defining therefore $n - p' + 2p = -q$, where $q = -1, 0, 1$ we then have, using Eq. (A2):

$$\frac{d^2 I}{d\omega d\Omega} = \frac{N^2 e^2 \gamma^2 n^2}{(4\pi\epsilon_0)c A^2} L(N\Delta\omega/\omega_1(\theta)) |A_x, A_y|^2$$

with:

$$A_x = 2\gamma\theta \cos \phi S_o - K_y (S_1 + S_{-1})$$

$$A_y = 2\gamma\theta \sin \phi S_o + K_x (e^{i\Psi} S_1 + e^{-i\Psi} S_{-1})$$
(A3)

where:

$$S_q = \sum_{p=-\infty}^{\infty} J_p(Y) J_{n+2p+q}(X) e^{i(n+2p+q)\Phi} e^{ip\Psi}$$

On-axis $X = 0$ and hence $n + 2p + q = 0$ and $p = (-n - q)/2$. It follows that $S_o = 0$ and:

$$S_q = J_{\frac{-(n+q)}{2}}(Y) e^{-i(n+q)\Psi/2}$$

for $q = -1, 1$, with $n = \text{odd}$. After some simplification, and using the fact that $J_{-n}(x) = (-1)^n J_n(x)$ one obtains:

$$\begin{aligned} A_x &= K_y \left(J_{\frac{n+1}{2}}(Y) e^{i\frac{n+1}{2}\Psi} - J_{\frac{n-1}{2}}(Y) e^{i\frac{n-1}{2}\Psi} \right) \\ A_y &= K_x \left(-J_{\frac{n+1}{2}}(Y) e^{i\frac{n+1}{2}\Psi} e^{-i\psi} + J_{\frac{n-1}{2}}(Y) e^{i\frac{n-1}{2}\Psi} e^{i\psi} \right) \end{aligned} \quad (\text{A4})$$

Pure elliptical case

We have $\psi = \pi/2$ and hence $\tan \Phi = (K_x/K_y) \tan \phi$ and $\Psi = 0$. We obtain therefore from Eq. (A3):

$$\begin{aligned} A_x &= 2\gamma\theta \cos \phi S_0 - K_y(S_1 + S_{-1}) \\ A_y &= 2\gamma\theta \sin \phi S_0 + iK_x(S_1 - S_{-1}) \end{aligned}$$

where:

$$S_q = \sum_{p=-\infty}^{\infty} J_p(Y) J_{n+2p+q}(X) e^{i(n+2p+q)\Phi}$$

and:

$$X = \frac{2n\gamma\theta}{A} \sqrt{K_y^2 \cos^2 \phi + K_x^2 \sin^2 \phi}, \quad Y = \frac{n}{4A} (K_y^2 - K_x^2)$$

which is equivalent to the result of Yamamoto and Kitamura [135].

On-axis we obtain directly from Eq. (A4):

$$A_x = K_y \left(J_{\frac{n+1}{2}}(Y) - J_{\frac{n-1}{2}}(Y) \right) \quad A_y = iK_x \left(J_{\frac{n+1}{2}}(Y) + J_{\frac{n-1}{2}}(Y) \right)$$

Pure circular case

In this case we have in addition to the above, $K_y = K_x = K$ and hence $X = 2n\gamma\theta K/A$, $Y = 0$, $\Phi = \phi$. We have then:

$$S_q = J_{n+q}(X) e^{i(n+q)\phi}$$

and hence:

$$\begin{aligned} A_x &= e^{in\phi} \left\{ 2\gamma\theta \cos \phi J_n(X) - K \left(J_{n+1}(X) e^{i\phi} + J_{n-1}(X) e^{-i\phi} \right) \right\} \\ A_y &= e^{in\phi} \left\{ 2\gamma\theta \sin \phi J_n(X) + iK \left(J_{n+1}(X) e^{i\phi} - J_{n-1}(X) e^{-i\phi} \right) \right\} \end{aligned} \quad (\text{A5})$$

The general expressions for the amplitudes are therefore still quite complicated, however since the radiation pattern is circularly symmetric we can take any value of ϕ , for example $\phi = 0$:

$$\begin{aligned} A_x &= 2\gamma\theta J_n(X) - K(J_{n+1}(X) + J_{n-1}(X)) \\ A_y &= iK(J_{n+1}(X) - J_{n-1}(X)) \end{aligned}$$

By making use of the following relations:

$$J_{n+1}(x) + J_{n-1}(x) = \frac{2n}{x} J_n(x) \quad \text{and} \quad J_{n-1}(x) - J_{n+1}(x) = 2J'_n(x) \quad (\text{A6})$$

we can express the results in the following form:

$$A_x = \left(2\gamma\theta - \frac{2nK}{X} \right) J_n(X), \quad A_y = -2iK J'_n(X) \quad (\text{A7})$$

The expression for the total intensity then becomes:

$$|A_x|^2 + |A_y|^2 = 4K^2 \left[J_n'^2(X) + J_n^2(X) \left(\frac{\gamma\theta}{K} - \frac{n}{X} \right)^2 \right]$$

in agreement with Alferov et al. [14] and Kincaid [15].

On-axis $X = 0$ and hence in Eq. (A5) there is only one term with $n = 1$. We have then:

$$A_x = -K \quad \text{and} \quad A_y = -iK$$

The radiation has therefore complete circular polarization ($S_3 = 1$) with total intensity:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{N^2 e^2 \gamma^2}{(4\pi\epsilon_0)c} \frac{2K^2}{(1+K^2)^2} L(N\Delta\omega/\omega_1(\theta))$$

We note in passing that the circular polarization component reverses direction off-axis when $A_x = 0$ in Eq. (A7), i.e. when $\gamma\theta = \sqrt{1+K^2}$.

Linear case

In this case $K_x = 0$ and hence $\Phi = \Psi = 0$, $X = 2n\gamma\theta K_y \cos\phi/A$, $Y = nK_y^2/4A$ and $A = 1 + K_y^2/2 + \gamma^2\theta^2$. As a result we have from Eq. (A3):

$$\begin{aligned} A_x &= 2\gamma\theta \cos\phi S_0 - K_y(S_1 + S_{-1}) \\ A_y &= 2\gamma\theta \sin\phi S_0 \end{aligned} \quad (\text{A8})$$

with

$$S_q = \sum_{p=-\infty}^{\infty} J_p(Y) J_{n+2p+q}(X)$$

Making use of the first of the relations in Eq. (A4) we can write:

$$S_1 + S_{-1} = \frac{2n}{X} S_0 + \frac{4}{X} S_2 = \frac{A}{K_y \gamma\theta \cos\phi} \left(S_0 + \frac{2}{n} S_2 \right)$$

where

$$S_2 = \sum_{p=-\infty}^{\infty} p J_p(Y) J_{n+2p}(X)$$

Hence,

$$A_x = 2\gamma\theta\cos\phi S_o - \frac{A}{\gamma\theta\cos\phi}\left(S_o + \frac{2}{n}S_2\right) \quad A_y = 2\gamma\theta\sin\phi S_o$$

in agreement with the result of Aferov et al. [14] and Krinsky [63].

On-axis we obtain either from Eq. (A8) or (A4):

$$A_x = K_y\left(J_{\frac{n+1}{2}}(Y) - J_{\frac{n-1}{2}}(Y)\right) \text{ and } A_y = 0$$

The intensity is often written in the form:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{N^2 e^2 \gamma^2}{(4\pi\epsilon_o)c} L(N\Delta\omega/\omega_1(\theta)) F_n(K)$$

where:

$$F_n(K) = \frac{n^2 K^2}{(1 + K^2/2)^2} \left[J_{\frac{n+1}{2}}(Z) - J_{\frac{n-1}{2}}(Z) \right]^2 \quad \text{with } Z = \frac{n K^2}{4(1 + K^2/2)}$$

INSTABILITIES AND BEAM INTENSITY LIMITATIONS IN CIRCULAR ACCELERATORS

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Abstract

The main aim of these lectures will be to give an insight into the physics of the mechanisms of instabilities and beam intensity limitations in circular accelerators. Three different techniques will be used to evaluate the various instabilities. The first will be using ‘few-particle models’, the second will use matrix techniques involving eigenvalues, and the third technique will use Sacherer’s modal analysis technique. The threshold or the growth rate will be evaluated for each instability with particular attention to the parameter dependence.

1. CALCULATION TECHNIQUES

1.1 Using matrix techniques (Eigenvalues)

The position of a particle (U) in two-dimensional phase space can be defined by its instantaneous position u and angle u' (du/dt). Consequently n particles may be represented by a column matrix with N ($2n$) rows. A transition matrix may be derived which describes the transition of the n particles from one situation to another, i.e.

$$U_2 = \begin{bmatrix} u_1 \\ u'_1 \\ u_2 \\ u'_2 \\ u_n \\ u'_n \end{bmatrix}_2 = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} & t_{1N} \\ t_{21} & t_{22} & .. & .. & .. & .. \\ t_{31} & .. & t_{33} & .. & .. & .. \\ t_{41} & .. & .. & t_{44} & .. & .. \\ t_{51} & .. & .. & .. & t_{55} & .. \\ t_{N1} & .. & .. & .. & .. & t_{NN} \end{bmatrix}_{1 \rightarrow 2} \begin{bmatrix} u_1 \\ u'_1 \\ u_2 \\ u'_2 \\ u_n \\ u'_n \end{bmatrix}_1 = T_{1 \rightarrow 2} U_1.$$

This transition matrix is of necessity of dimension $N \times N$. Evaluation of the eigenvalues of the transition matrix allows determination of the growth rates, damping times, and frequency shifts experienced by the particles. For more than two particles it is usually necessary to evaluate the eigenvalues by computer.

1.2 Using frequency shifts

The well-known differential equation of a simple harmonic oscillator with a ‘normalised’ driving force is

$$\ddot{u} + \omega_0^2 u = Gu = (G_R + jG_I)u$$

where ω_0 is the natural frequency of the oscillation and G_R and G_I are the real and imaginary components of the normalised driving force. The equally well-known solution to this equation is

$$u = u_0 \exp(j\omega_n t)$$

where

$$\omega_n = \omega_0 + \Delta\omega, \text{ and } \Delta\omega = -\frac{G}{2\omega_0} = -\frac{G_R}{2\omega_0} - j\frac{G_I}{2\omega_0}$$

giving

$$u = u_0 \exp(j\omega_0 t) \cdot \exp\left(-\frac{jG_R}{2\omega_0} t\right) \cdot \exp\left(+\frac{G_I}{2\omega_0} t\right). \quad (1)$$

Consequently the exponential coefficient of the motion (α) is given by

$$\alpha = \frac{1}{\tau} = -\text{Im}(\Delta\omega) = +\frac{G_I}{2\omega_0}. \quad (2)$$

Hence the motion is unstable ($\alpha > 0$) if the imaginary component of the frequency shift is of negative sign or if the imaginary component of the driving force (G_I) is of positive sign.

1.3 Evaluation of the normalised driving force

In the case of beam instabilities the driving ‘force’ is normally an induced ‘force’ such as a voltage induced by the passage of the charged beam itself. Since the force is induced by the beam then it can only have components at frequencies corresponding to the modes of oscillation of the beam itself. Thus the force results from the spectrum of the beam oscillation ‘sampling’ the **impedance** Z seen by the beam. In certain cases the induced driving force is fairly obvious and can be derived almost by inspection, but in most cases the derivation is more complicated.

The induced voltage can be derived from the inverse Fourier transform of the product of the impedance and the bunch current (in frequency domain) i.e.

$$V(t) = \mathfrak{F}^{-1}\{V(\omega)\} = \mathfrak{F}^{-1}\{Z(\omega)I(\omega)\}.$$

The voltage induced by the beam gives the field which must be integrated over the bunch spectrum (ρ) in order to get the total force i.e.

$$G = jk \int V(t)\rho(t)dt = jk \int \mathfrak{F}^{-1}\{Z(\omega)I(\omega)\}\rho(t)dt.$$

Sacherer [1] has shown that, in general,

$$G \propto jI_0 \frac{\sum Z(\omega)h(\omega)}{\sum h(\omega)} \quad (3)$$

where I_0 is the average bunch current and $h(\omega)$ is the power spectrum of the bunch current distribution. Clearly, stability is determined by the sign of the **real part of the impedance** $\text{Re}\{Z(\omega)\}$. The stability situation is therefore investigated by evaluating the frequency dependence of the impedance seen by the beam and the power spectrum of the beam oscillations which ‘sample’ the impedance to produce an induced normalised force.

2. ‘ROBINSON’ INSTABILITY

The Robinson instability [2] has been analysed in many and complicated ways since it was pointed out more than 30 years ago. This instability is driven by the fundamental

accelerating modes of the RF cavities and is not a serious effect for modern accelerators since the cure is easy and well known. However, as will be seen later, the Robinson instability is a specific case of the more general coupled-bunch instability and therefore serves as a simplified introduction to instabilities and their evaluation in accelerators.

2.1 Robinson ‘physics’

The physics of the Robinson instability may be understood by simply examining the beam-induced voltage produced by the interaction of the beam with the fundamental accelerating mode of the RF cavities. It should be stated that, for the sake of simplicity, *only the induced voltage from the passage of the bunch on the previous turn is taken into account*. This is not an unreasonable approximation for room-temperature cavities but is certainly not applicable to the case of superconducting ones. The voltage induced by a bunch at time ‘zero’ is shown as a function of time in Fig. 1. After around four full oscillations of the induced voltage the bunch returns to sample the voltage which it induced in the cavity on the previous turn. In frequency domain this corresponds to operating at the peak of the resonance curve of the cavities. We now assume that the revolution frequency of the bunches (f_b) can be changed slightly and that we maintain the frequency of the induced voltage constant. If the bunch frequency is increased slightly then the bunch arrives sooner and experiences the induced sinusoidal voltage but with a negative gradient as shown in Fig. 1. From fundamental longitudinal phase space dynamics (phase stability) it is well known that this situation is stable, above transition energy. In the case where the bunch frequency is less than the mode frequency then the bunch arrives later and experiences a sinusoidal voltage with a positive slope. This situation is unstable and may be explained by considering a small energy oscillation with respect to the situation shown in Fig. 1. If the bunch at time zero has slightly more energy than the reference bunch then it will arrive somewhat later after one turn and (for $f_b < f_{\text{mode}}$) will experience a voltage gain which is higher than the reference bunch, clearly an unstable situation which is depicted in frequency domain by the lower trace on the left of Fig. 1.

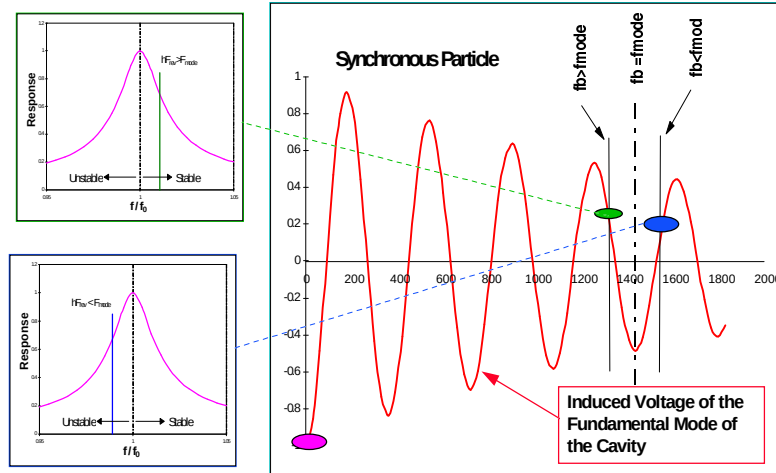


Fig. 1 Beam-induced voltage on right with the frequency spectrum of the resonance curves of the cavities on the left

3. CALCULATION OF ROBINSON-INDUCED VOLTAGE

This simplified analysis of the Robinson instability requires the analysis of the motion of a single particle driven by the induced voltage in the fundamental mode of the cavities.

Figure 2 is a schematic of a single turn of an accelerator with a single cavity. A single turn for convenience starts at the exit of the cavity, which has an induced voltage due to the passage of the particles, and is followed by a longitudinal ‘drift’ to the entrance of the cavity. The cavity is assumed to have no length and simply gives an energy boost to the particles.

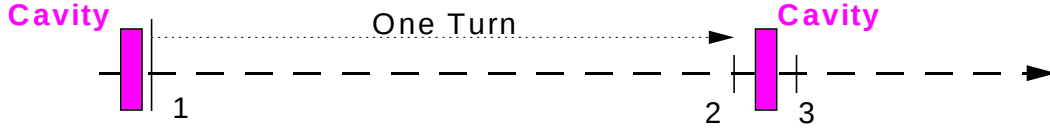


Fig. 2 Schematic of an accelerator with a single cavity

The induced (mode) voltage generated at location 1 and remaining at location 2 is given by

$$v_m = -V_m \exp \left\{ j\omega_m(t_2 - t_1) - \frac{(t_2 - t_1)}{T_f} \right\} \quad (4)$$

where ω_m is the radiancy of the excited mode, T_f is the filling time of the cavity, and the revolution time dependence on the relative energy offset ($\Delta = \delta E/E$) is

$$(t_2 - t_1) = t_{\text{rev}}(1 + \Delta\eta),$$

where t_{rev} is the average revolution time and

$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2},$$

where γ_t is the relative transition energy.

Substituting

$$\delta_0 = (\omega_m - h\omega_{\text{rev}})t_{\text{rev}},$$

where h is the RF harmonic number,

and

$$\tau_0 = \frac{t_{\text{rev}}}{T_f}$$

gives

$$v_m = -V_m \exp \{ j(\delta_0 + 2\pi h)(1 + \eta\Delta) - \tau_0(1 + \eta\Delta) \}$$

and neglecting small terms

$$v_m = -V_m \exp \{ j(\delta_0 + 2\pi h\eta\Delta) - \tau_0 \} \quad (5)$$

which by linearizing gives

$$v_m = -V_m e^{-\tau_0} [\cos \delta_0 - 2\pi h\eta\Delta \cdot \sin \delta_0 + j(\sin \delta_0 + 2\pi h\eta\Delta \cdot \cos \delta_0)]. \quad (6)$$

Taking only the oscillating part of Eq. (6)

$$v_m(\Delta) = V_m e^{-\tau_0} 2\pi h\eta\Delta \{ \sin \delta_0 - j \cos(\delta_0) \}. \quad (7)$$

The induced voltage may also be written in terms of ϕ by simply substituting

$$\varphi = \frac{j h \eta \Delta}{Q_s}$$

(Q_s is the synchrotron frequency, defined later, Section 5) which gives

$$v_m(\varphi) = V_m e^{-\tau_0} 2\pi Q_s \varphi \{\cos \delta_0 - j \sin \delta_0\}. \quad (8)$$

4. ROBINSON BY EIGENVALUES OF SINGLE-PARTICLE MOTION

Referring to Fig. 2, the energy at the exit of the cavity after one turn (location 3) is

$$\Delta_3 = \Delta_2 + \frac{eV_{RF}}{E} \{\sin(\Phi_s + \varphi_2) - \sin(\Phi_s)\} + \Re\left(\frac{ev_m(\Delta)}{E}\right)$$

$$\Delta_3 = \Delta_2 + \frac{eV_{RF}}{E} \{\sin(\Phi_s + \varphi_2) - \sin(\Phi_s)\} + \frac{eV_m}{E} e^{-\tau_0} 2\pi h \eta \Delta \cdot \sin \delta_0$$

where φ is measured with respect to the synchronous particle.

Consequently the linearized longitudinal phase space transition over a complete turn (single-turn matrix S) is given by

$$\begin{bmatrix} \varphi \\ \Delta \end{bmatrix}_3 = \begin{bmatrix} 1 & 0 \\ \frac{eV_{RF} \cos \Phi_s}{E} & 1 + \frac{eV_m}{E} e^{-\tau_0} 2\pi h \eta \cdot \sin \delta_0 \end{bmatrix}_{2 \rightarrow 3} \begin{bmatrix} 1 & 2\pi h \eta \\ 0 & 1 \end{bmatrix}_{1 \rightarrow 2} \begin{bmatrix} \varphi \\ \Delta \end{bmatrix}_1.$$

For a real 2×2 matrix the absolute value (amplitude) of the two eigenvalues is equal to the square root of the determinant, hence if the determinant is greater than unity the motion is unstable with exponential growth

$$\det(T) - 1 = \frac{2t_{rev}}{\tau}.$$

Hence the growth rate is
$$\frac{1}{\tau} = \frac{e\omega_{RF}\eta V_m e^{-\tau_0}}{2E} \cdot \sin \delta_0. \quad (9)$$

Clearly the motion is **unstable when the product $\eta\delta_0$ is positive**, i.e. above transition energy ($\eta > 0$) when δ_0 is positive (i.e. when $\omega_m > h\omega_{rev}$). This is of course the Robinson instability. However, the growth rate is not identical to that usually derived because of the approximation of only taking the induced voltage of the previous turn.

5. ROBINSON BY SOLUTION OF THE FORCED EQUATION OF MOTION

The general equation of longitudinal motion is

$$\ddot{\varphi} + \frac{\eta h \omega_{rev}^2 e}{2\pi E} [V_{RF}(\varphi) - V_{RF}(0) + v_m] = 0$$

which, for a normal sinusoidal RF voltage becomes

$$\ddot{\varphi} + \Omega^2 \varphi = -\left(\frac{eh\eta\omega_{rev}^2}{2\pi E}\right) \cdot v_m \quad (10)$$

where Ω is the synchrotron radiancy

$$\Omega = \omega_{\text{rev}} Q_s = \omega_{\text{rev}} \cdot \sqrt{\frac{eh\eta V_{\text{RF}} \cos \Phi_s}{2\pi E}}.$$

Substituting for v_m from Eq. (7) gives

$$\ddot{\phi} + \Omega^2 \phi = - \left(\frac{eh\eta \omega_{\text{rev}}^2}{E} \right) \cdot V_m e^{-\tau_0} Q_s (\cos \delta_0 - j \sin \delta_0) \phi.$$

As shown previously the growth rate is given by the imaginary part of the driving force, i.e.

$$\frac{1}{\tau} = \frac{1}{2\Omega} \frac{eh\eta \omega_{\text{rev}}^2 Q_s V_m e^{-\tau_0}}{E} \cdot \sin \delta_0 = \frac{e\omega_{\text{RF}} \eta V_m e^{-\tau_0}}{2E} \cdot \sin \delta_0.$$

Fortunately this gives the same result as Eq. (9).

6. SPECTRUM OF LONGITUDINAL OSCILLATIONS

In Section 1 it was shown that the stability situation can be investigated by evaluating the frequency dependence of the impedance seen by the beam and the power spectrum of the beam oscillations which ‘sample’ the impedance to produce an induced normalised force. In this section the power spectrum of oscillations in the longitudinal plane are investigated.

Figure 3 (from Ref. [3]) shows the time domain signal of a single bunch with Gaussian charge distribution circulating in an accelerator with a revolution time of T_0 and with vanishing synchrotron motion. This is the signal that would be detected by a current transformer with large frequency bandwidth. The current distribution is

$$I(t) = \frac{q}{\sqrt{2\pi\sigma_t}} \exp\left\{ -\frac{t^2}{2\sigma_t^2} \right\},$$

where q is the bunch charge and σ_t the bunch length in time.

The well-known Fourier transform of the sampled signal (also shown in Fig. 3) is given by

$$I_p(\omega) = \frac{q}{T_0} \exp\left\{ -\frac{p^2 \omega_0^2}{2\sigma_\omega^2} \right\}$$

where

$$\sigma_\omega = \frac{1}{\sigma_t} \quad \text{and} \quad \omega_0 = 2\pi/T_0.$$

Since in this case there is no synchrotron motion included in the spectrum then longitudinal instabilities are excluded; however, by using the derived spectrum to sample the real part of the spectrum of the longitudinal impedance (as described in Section 1), the voltage loss per turn can be derived. Similarly by using the imaginary part of the impedance the shift in the incoherent synchrotron frequency can be evaluated [3].

The next level in complexity is to introduce the synchrotron motion that causes a modulation of the bunch signal in the time domain. Transformed into the frequency domain this produces the familiar sidebands above and below the lines at revolution frequency and

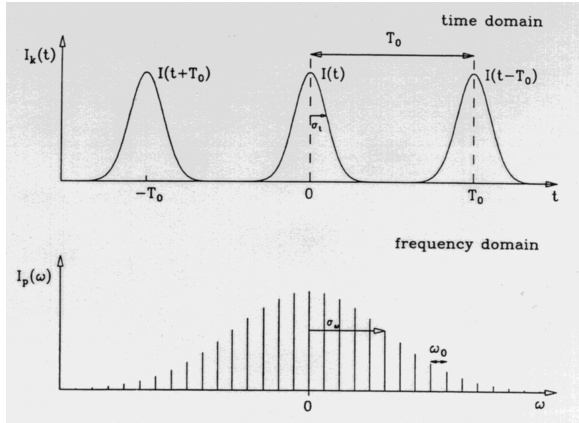


Fig. 3 Bunch signal of a stationary bunch in time and frequency domain

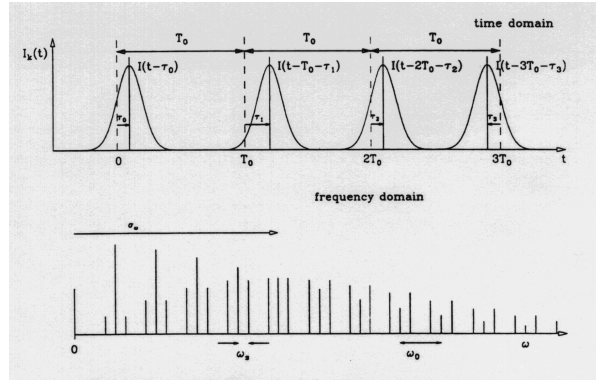


Fig. 4 Signals from a bunch with synchrotron motion, in time and frequency domain

separated by the synchrotron frequency. The lower plot in Fig. 4 shows the frequency spectrum as it would be seen on a real spectrum analyser where negative frequencies are ‘folded’ into positive frequencies. The real power spectrum of the bunch, with negative frequencies shown, is shown in Fig. 5 where the synchrotron sideband amplitudes (drawn as arrows) are all shown as equal for the sake of simplicity. These frequencies occur at

$$\omega_p = \sum_{-\infty}^{+\infty} (p + Q_s) \omega_{\text{rev}} .$$

Also superimposed on this spectrum is the real part of the impedance of a narrow-band resonator impedance which could represent the fundamental mode of an accelerating structure, i.e.

$$\frac{Z_R(\omega)}{\omega} = \frac{R_s / \omega}{1 + Q^2 \left\{ \frac{\omega_r}{\omega} - \frac{\omega}{\omega_r} \right\}^2}$$

where R_s is the shunt impedance, Q the quality factor, and ω_r the resonant frequency of the structure.

Note that for positive frequencies the impedance is positive in sign and the inverse for negative frequencies due to the Z/ω dependence of the impedance in the longitudinal plane.

In Fig. 5 the resonant frequency of the cavity is drawn slightly above the harmonic number (drawn as 3 for the sake of ease of illustration). In this case it may be seen that the amplitude of the impedance sampled at the synchrotron frequency at positive frequencies is greater than that at negative frequencies. This situation is depicted more simply in Fig. 6 where the negative frequencies are folded into the positive axis, and in order to represent the fact that the impedance is negative, the lines of negative frequencies are drawn with a downward arrow. It will be shown later that the downward arrows result in exponential growth of the synchrotron motion whereas positive arrows cause damping. Consequently in Fig. 6 (a) where the resonant frequency of the cavities is tuned slightly above the harmonic number, the summation of the lengths of the upward and downward arrows is upward, denoting overall damping. In Fig. 6 (b) the cavity is exactly tuned to the harmonic number and the summation is zero whereas Fig. 6 (c) with detuning below the harmonic number shows instability. This is

again the Robinson instability derived by using impedances and the mode spectrum of the bunch motion.

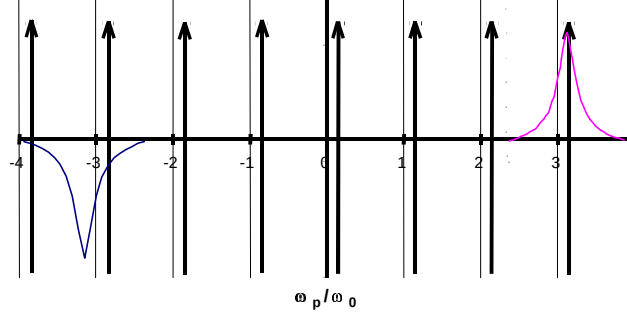


Fig. 5 Power spectrum with synchrotron sidebands drawn as arrows

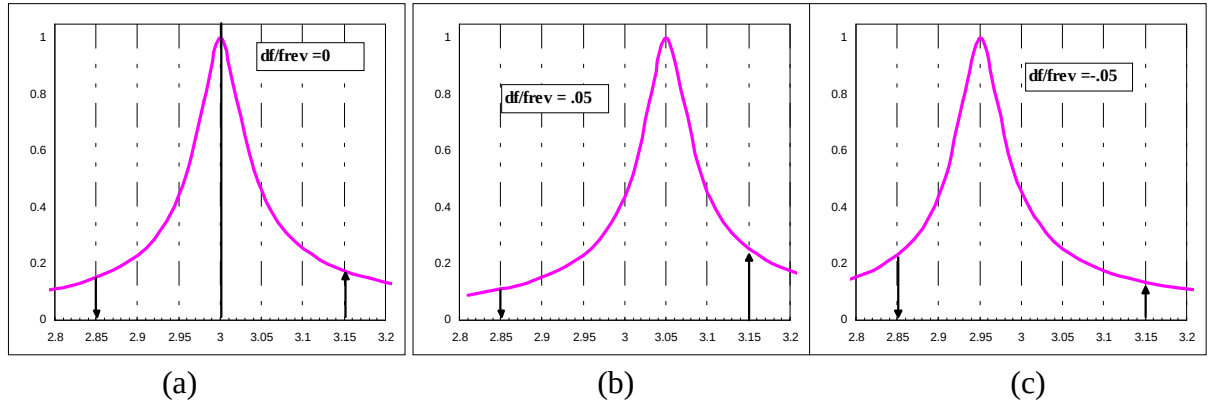


Fig. 6 Robinson instability depicted using impedance and mode spectra of the bunch motion

7. GROWTH RATES

Sacherer [1] has shown that the complex frequency shift of the synchrotron sidebands produced by a general impedance is given by

$$\Delta\omega_m = -j \frac{m\omega_s}{(m+1)} \frac{I_b \omega_{\text{rev}}}{3B_0^3 h V_{\text{RF}} \cos \Phi_s} \frac{\sum h_{mk}(\omega_p) Z(\omega_p)/\omega_p}{\sum h_{mm}(\omega_p)}$$

where m is the mode of oscillation (see later) and equals 1 for dipolar oscillations, I_b the bunch current, B_0 the bunching factor (bunch length τ_L /revolution time) and the summation is over the mode spectrum of the bunch oscillations.

The bunching factor can be written, for Gaussian bunches (with r.m.s. length σ_s) as

$$B_0 = \frac{16}{3} \sqrt{\frac{2}{\pi}} \frac{\sigma_s}{2\pi R} = 0.678 \frac{\sigma_s}{R}$$

with $2\pi R$ the circumference of the machine.

For $m = 1$, and for a narrow-band resonator covering two synchrotron sidebands (as in Fig. 6) this may be written

$$\Delta\omega_m = -j \frac{\omega_s}{2} \frac{I_{\text{total}}}{3B_0^2 h V_{\text{RF}} \cos \Phi_s} F_m \left(\frac{Z(p^+) - Z(p^-)}{p} \right)$$

where F_m is the form factor (amplitude of the envelope of the power spectrum of the bunch oscillations), p is $f_p/f_{\text{rev}} = h$ for the fundamental RF, and $Z(p\pm)$ refers to the impedance at the upper and lower synchrotron sideband frequencies.

Consequently

$$\text{Im}(\Delta\omega_m) = -j \frac{\omega_s}{6} \frac{I_{\text{total}}}{B_0^2 h V_{\text{RF}} \cos \Phi_s} F_m \left(\frac{Z_R(p^+) - Z_R(p^-)}{p} \right).$$

Here Z_R is the real part of the longitudinal impedance, i.e.

$$Z_R(\omega) = \frac{R_s}{1 + Q^2 \left(\frac{\omega_r}{\omega} - \frac{\omega}{\omega_r} \right)^2}.$$

For such a resonator the impedance difference between the synchrotron sidebands can be evaluated as

$$Z(p^+) - Z(p^-) \cong 16R_s Q^2 \frac{Q_s}{h} \frac{\Delta\omega}{\omega_r}$$

where $\Delta\omega$ is the frequency detuning of the accelerating structure ($\omega_r - h\omega_0$). This relationship only holds for

$$Q \ll \frac{\omega_r}{2(\Delta\omega \pm \omega_s)} \cong \frac{h}{2Q}$$

and will therefore not be valid for superconducting cavities where the loaded Q value is very high. When this inequality is satisfied, the growth rate can be reduced to

$$\frac{1}{\tau} = -\text{Im}(\Delta\omega_m) = \frac{4}{3\pi} \left(\frac{I_{\text{total}} F_m}{B_0^2} \right) \left(\frac{e\eta}{h^3 E} \right) (R_s Q^2 \Delta\omega)$$

where the first bracket refers to the beam conditions, the second refers to the machine parameters, and the third to the parameters of the cavities.

Since by our convention η is positive above transition, then stability is ensured by Z^+ being greater than Z^- which occurs when $\Delta\omega$ is positive, i.e. when the cavity frequency is tuned above $h\omega_{\text{rev}}$.

This relationship has a similar parameter dependence to that previously derived using the simpler model (Sections 4 and 5) but is not identical since in the simpler model only the influence of the cavity induced voltage on the previous turn is taken into account.

8. MODES AND SPECTRA OF MULTIPLE BUNCHES

The next level of complexity is to consider the spectrum of the dipolar motion of a beam with many bunches. Figure 7 shows the longitudinal phase space (energy deviation against RF phase) of the dipole modes of oscillation of four bunches. The different bunches are spaced horizontally left to right and the different modes of oscillation spaced vertically. The mode of oscillation (n) is defined by the phase advance between the motion of successive bunches, i.e.

$$\Delta\mu = \frac{2\pi}{k_b} n .$$

Consequently for the $n = 0$ mode of oscillation all four bunches move in an identical way, and there is zero phase difference between the motion of successive bunches. This is shown in Fig. 7 as are the other three possible modes of oscillation. It should be realised that in this phase space the motion is in the anticlockwise sense.

The bunches may also be subjected to ‘shape’ modes of oscillation. These are depicted for completeness in Fig. 8, where it can be seen that the $m = 1$ mode (dipolar mode which we have been considering) is a centre-of-gravity motion whereas the higher-mode numbers produce changes in the bunch shape.

In general the spectral lines of motion occur at frequencies

$$\omega_p = \omega_{\text{rev}}(k \cdot k_b + n + mQ_s) = \omega_{\text{rev}}(p + mQ_s)$$

where $p = n + k \cdot k_b$, $k = 0, \pm 1, \pm 2, \dots$ and $n = 0, 1, 2, \dots (k_b - 1)$.

An example of the spectrum is shown in Fig. 9. It is apparent that the spectrum simply repeats itself for different values of k . In addition, negative k values correspond to negative frequencies. Superimposed on this diagram is a narrow-band resonator impedance (plotted as Z_R/ω) which couples to mode $n = 1$ and $n = 3$. In this case since the real part of the impedance is negative at the negative frequency corresponding to mode $n = 1$, this mode will be unstable. The mode $n = 3$ is damped by the positive real part of the impedance. The previously described Robinson instability is a particular case of this coupled-bunch instability but affecting only mode $n = 0$.

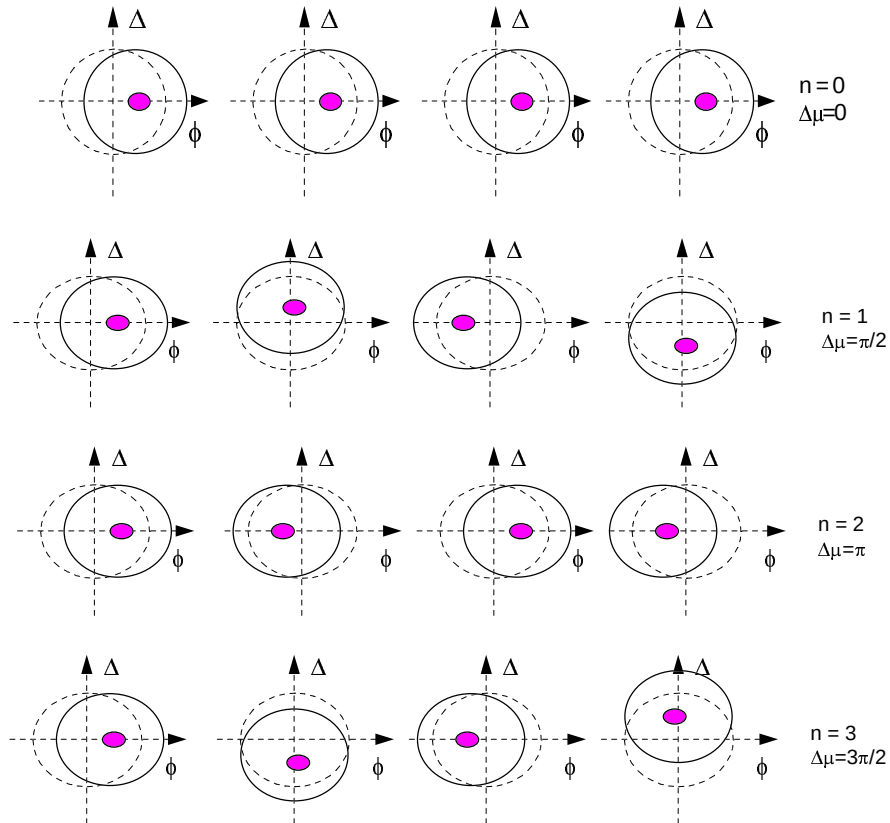


Fig. 7 Dipolar coupled-bunch modes

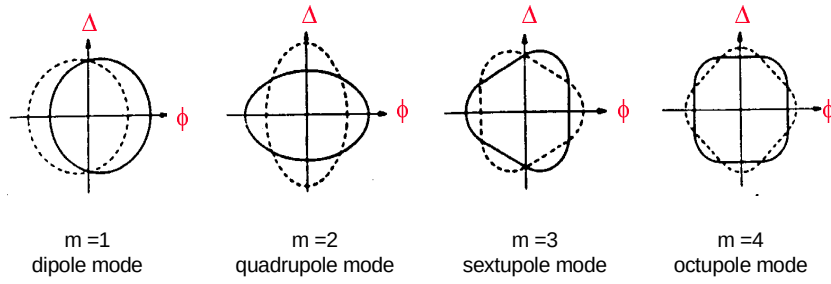


Fig. 8 Bunch shape modes of oscillation

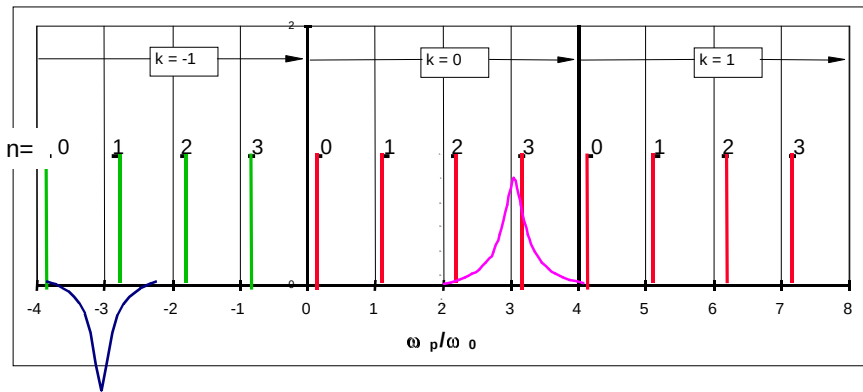


Fig. 9 Spectrum of longitudinal coupled-bunch motion for four bunches ($Q_s = 0.08$)

A more compact and useful way to draw spectra for coupled-bunch oscillations is shown in Fig. 10 where the negative frequencies are ‘folded’ into positive frequencies but drawn with negative amplitudes. In this way modes of oscillation with positive amplitudes are stable and those with negative amplitudes unstable. The growth rates can be evaluated in an identical way to the technique used previously for the Robinson instability.

In cases where the impedance is broadband then the summation of the impedance should be performed over frequency lines with the same mode number (n).

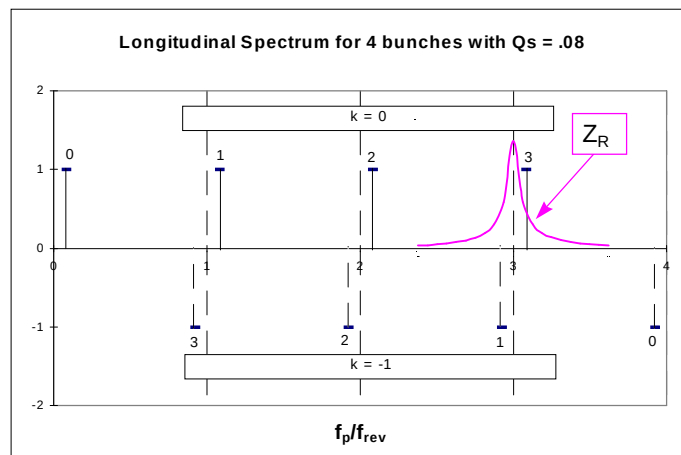


Fig. 10 Longitudinal spectrum of four bunches with the real part of the impedance superimposed

9. TRANSVERSE MOTION

9.1 Equation of motion and transverse impedance

The equation of motion of a single particle in a coasting beam in the transverse (horizontal or vertical) plane is

$$\ddot{x} + (Q_\beta \omega_{\text{rev}})^2 x = \frac{F_x}{m}$$

where Q_β is the betatron tune and F_x/m is the normalised driving force given by

$$\frac{F_x}{m} = \frac{e(E + B \times v)}{\gamma m_0} .$$

The definition of the transverse impedance is

$$Z^T = j \frac{\int_0^{2\pi} (E + B \times v) ds}{\beta I \Delta} .$$

Consequently the equation of motion for a single particle in a coasting beam is

$$\ddot{x} + (Q_\beta \omega_{\text{rev}})^2 x = -j \frac{e\beta}{\gamma m_0} \frac{Z^T I}{2\pi R} x .$$

Here the normalised driving force produces a frequency shift

$$\Delta\omega = \frac{j}{2Q_\beta \omega_{\text{rev}}} \frac{e\beta}{\gamma m_0} \frac{Z^T I}{2\pi R} .$$

This can be written

$$\Delta\omega = \frac{j}{4\pi Q_\beta} \frac{c}{(E/e)} Z^T I .$$

The transverse impedance is evaluated at the frequencies

$$\omega = (p + Q_\beta) \omega_{\text{rev}} + \Delta\omega .$$

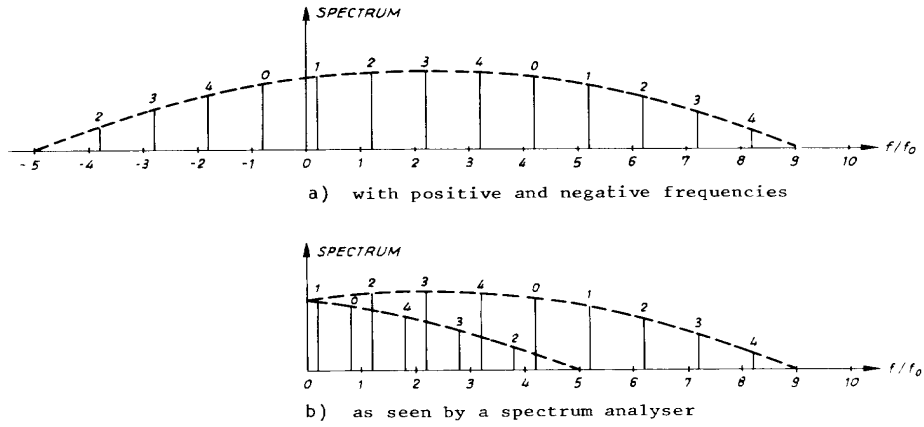
Clearly the imaginary part of the transverse impedance will produce a real frequency shift whereas the real part will cause instability for negative resistances and damping for positive resistances.

9.2 Modes of transverse oscillations

An example from Ref. [4] of the line spectrum of a bunched beam with transverse motion is shown in Fig. 11 which is drawn for a machine with five relatively long bunches (the PS Booster) and a $Q_\beta = 4.2$. This spectrum is similar to that shown previously for the longitudinal plane except that the sidebands are separated from the revolution frequency lines by the non-integer part of the transverse tune (q_β).

9.3 Chromaticity

In the case of transverse oscillations, the spectrum is significantly affected by the chromaticity (the tune dependence on the momentum offset). A particle that is performing



Spectrum for a machine with 5 long bunches (PS booster) and $Q = 4.2$. Envelope is drawn for head-tail mode = 0 and is shifted towards positive frequencies.

Fig. 11 Transverse spectrum of five bunches

synchrotron oscillations can be represented by the ellipse in Fig. 12: above transition a particle with higher energy than the synchronous particle drifts towards the 'tail' of the bunch. This means that in Fig. 12 a synchrotron oscillation moves on an approximate ellipse in the anticlockwise direction. Now consider what happens when there is a negative tune dependence on the particle momentum (or energy), i.e.

$$Q'_\beta = \frac{\Delta Q_\beta}{\Delta} \leq 0 .$$

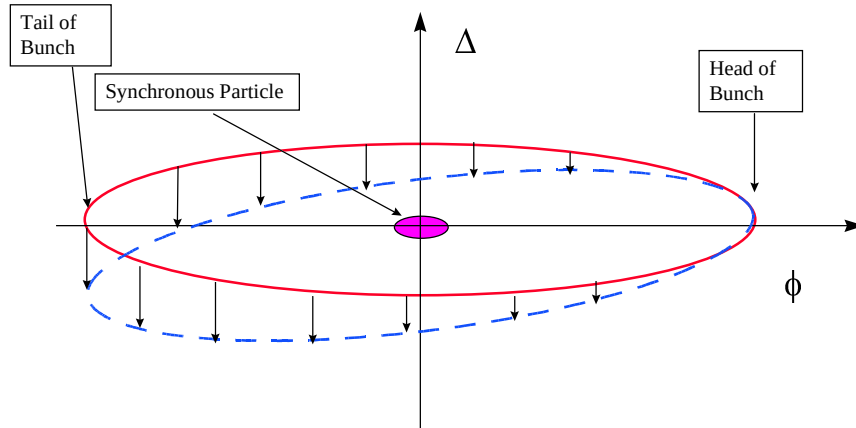


Fig. 12 Betatron phase along a synchrotron orbit

A particle at the head of the bunch has the same momentum deviation as the synchronous particle and therefore the same betatron frequency. As this particle moves along the synchrotron ellipse shown, the betatron frequency initially decreases and a betatron phase lag develops, shown as the downward arrows in the figure. When the particle reaches the tail of the bunch its phase lag is at a maximum. Continuing the motion towards the head of the bunch causes the particle to gradually regain betatron phase until it reaches the head of the bunch where the whole cycle restarts. It is also apparent that, if a large number of particles were distributed around the synchronous orbit with betatron phase advances as indicated in

Fig. 12, then the pattern would remain stationary. The net result of all this is that the head and the tail of the bunch oscillate at the same frequency but with a phase difference.

The total phase shift between the head and the tail is usually denoted by $\chi_{\max}$ and can be calculated as follows. Specify the longitudinal position of a particle within a bunch by its time delay (τ) from the head of the bunch. This time delay changes by an amount Δt_{rev} per turn

$$\frac{d\tau}{dk} = \Delta t_{\text{rev}} = \eta t_{\text{rev}} \Delta .$$

The betatron phase shift is

$$\chi = 2\pi \int \Delta Q_{\beta} dk = 2\pi Q'_{\beta} \int \Delta dk = \frac{2\pi Q'_{\beta}}{\eta t_{\text{rev}}} \int \frac{d\tau}{dk} dk = \frac{\omega_{\text{rev}} Q'_{\beta} \tau}{\eta} .$$

Thus the betatron phase varies linearly along the length of the bunch and reaches its maximum value at the tail where $\tau = \tau_L$. The same linear dependence of the phase shift on the distance along the bunch is true for higher modes and the oscillation amplitude is given by a standing wave pattern $[p_m(t)]$.

Figure 13 from Ref. [4] shows the difference signal which would be detected at a pick-up as a function of the number of turns (k) and the mode number (m) plotted for three different values of the head–tail phase shift (χ).

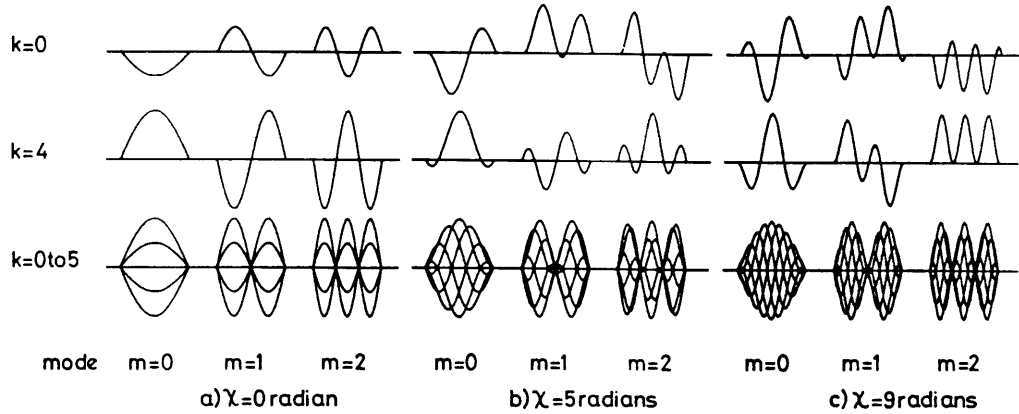


Fig. 13 Head–tail modes for the PS Booster for a tune of 4.833

This signal has the form

$$p_m(t) \exp(j\omega_{\xi} t + j2\pi k Q_{\beta})$$

where

$$\omega_{\xi} = \frac{\chi_{\max}}{\tau_L} = \frac{Q'_{\beta} \omega_{\text{rev}}}{\eta}$$

gives the frequency of the wiggles along the bunch. If the standing wave patterns $p_m(t)$ are taken to be sinusoidal i.e.

$$p_m(t) = \cos(m+1)\pi \frac{t}{\tau_L} \quad \text{for } m = 0, 2, 4, \dots$$

$$p_m(t) = \sin(m+1)\pi \frac{t}{\tau_L} \quad \text{for } m = 1, 3, 5, \dots$$

the envelope of the line spectrum of the power spectrum is given by [4]

$$h(\omega) = (m+1)^2 \frac{\tau_L^2}{2\pi^4} \frac{1 \pm \cos \pi \left(\frac{\omega \tau_L}{\pi} \right)}{\left[\left(\frac{\omega \tau_L}{\pi} \right)^2 - (m+1)^2 \right]^2}$$

which is drawn in Fig. 14 for the first four modes of oscillation.

When the chromaticity is finite, the travelling wave component, $\exp(j\omega\xi t)$, is present and shifts the power spectrum depending on the sign of $\omega\xi$ i.e.

$$h(\omega) = h(\omega - \omega_\xi) \quad .$$

The actual spectrum is a line spectrum as already shown in Fig. 11 where it may be seen that the envelope has been moved towards higher frequencies. This means that $\omega\xi$ is positive, implying positive chromaticity above transition energy or negative chromaticity below.

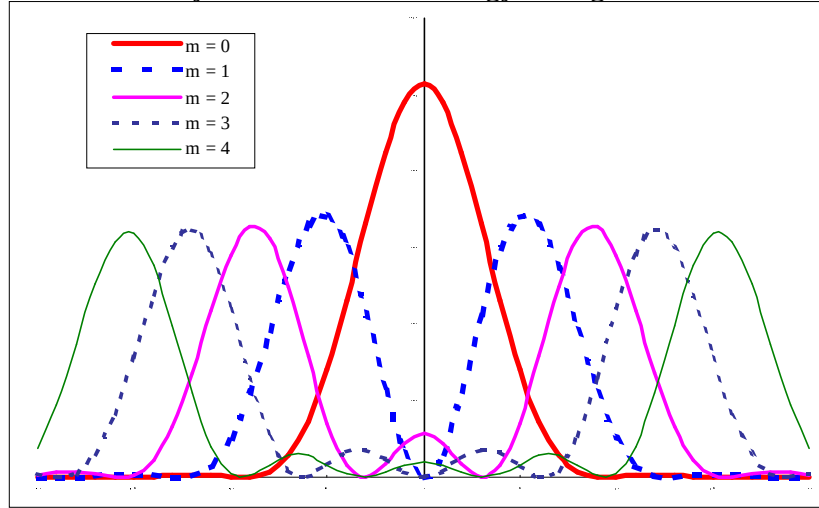


Fig. 14 Power spectrum of modes 0 to 4 with $\chi = 0$

9.4 Transverse growth rates

Sacherer [4] has derived the general result that the frequency shift is given by

$$\Delta\omega_m = \frac{1}{(1+m)} \frac{j}{2Q_\beta \omega_{\text{rev}}} \frac{e\beta}{\gamma m_0} \frac{I_b}{\tau_L} \frac{\sum Z^T(\omega) h(\omega - \omega_\xi)}{\sum h(\omega - \omega_\xi)}$$

where once again the summations are over the mode spectrum of the bunches. Hence the growth rate is

$$\frac{1}{\tau} = -\text{Im}(\Delta\omega_m) = -\frac{1}{(1+m)} \frac{j}{2Q_\beta \omega_{\text{rev}}} \frac{e\beta}{\gamma m_0} \frac{I_b}{\tau_L} \frac{\sum Z_R^T(\omega) h(\omega - \omega_\xi)}{\sum h(\omega - \omega_\xi)}$$

where Z_R^T is the real part of the transverse impedance. Consequently, in the transverse plane when the real part of the impedance is negative, an instability is provoked. In the case of a narrow-band resonator the real part of the transverse impedance is positive for positive frequencies and negative for negative frequencies. Hence transverse modes of oscillation at negative frequencies are unstable. It may therefore be appreciated that by moving the bunch

spectrum towards higher frequencies (positive $\omega\xi$, χ) the mode 0 is damped, whereas the higher order modes may become unstable. This means positive chromaticity above transition energy and negative below. This behaviour is shown graphically in Fig. 15.

Figures 16 to 21 show the bunch line-spectra coupling to a resonator impedance and a resistive wall impedance: also shown are the resultant growth rates as a function of the chromatic shift. It is interesting to study these plots to understand the influence of $\omega\xi$ and of the tune values.

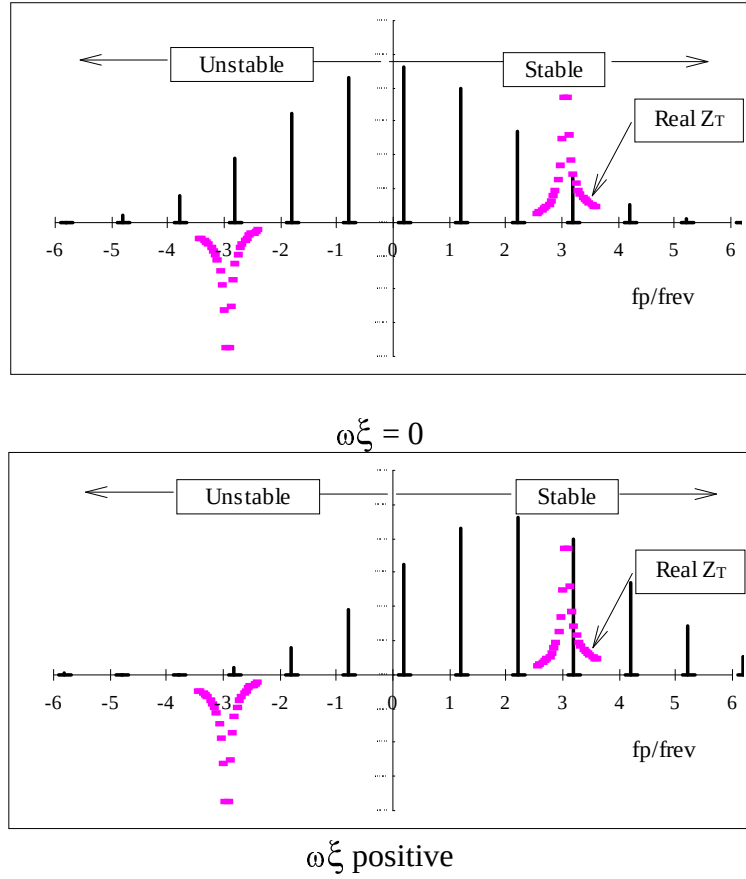


Fig. 15 Transverse spectra for two different values of $\omega\xi$

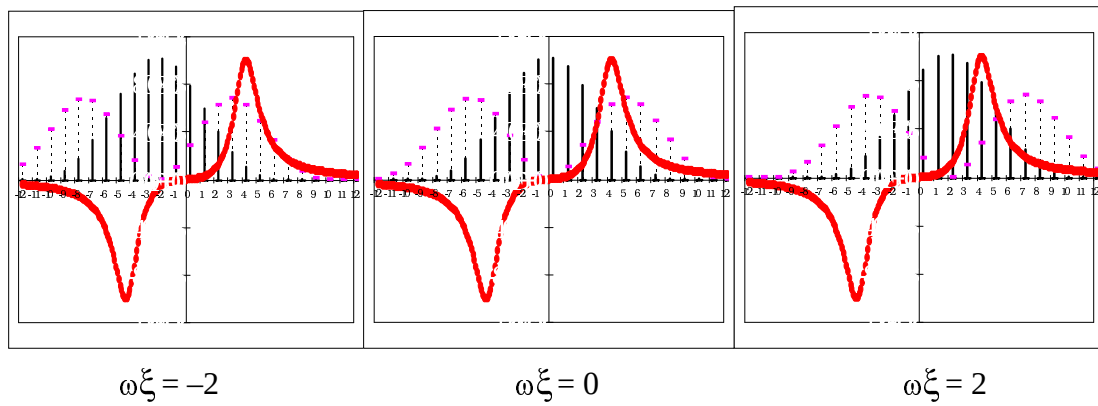


Fig. 16 Resonator impedance with varying $\omega\xi$ (for modes 0 and 1)

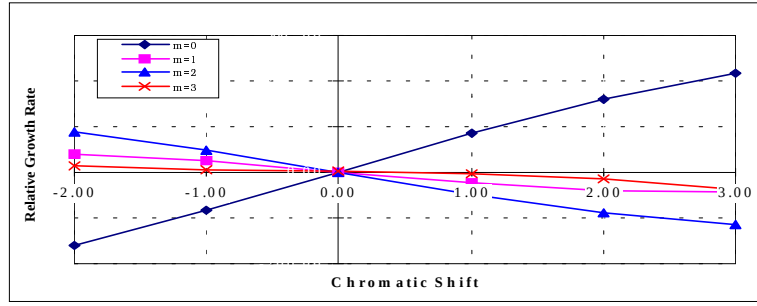


Fig. 17 Relative growth rates as a function of $\omega\xi$

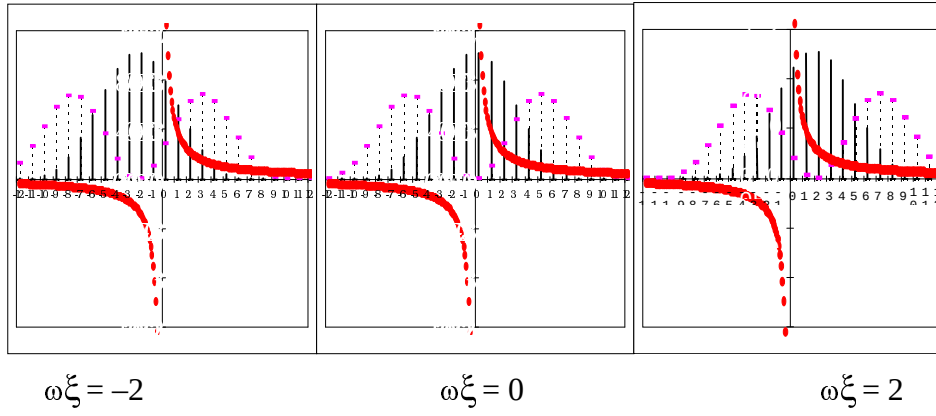


Fig. 18 Resistive-wall type impedance with varying $\omega\xi$ (for modes 0 and 1) ($Q\beta = 0.3$)

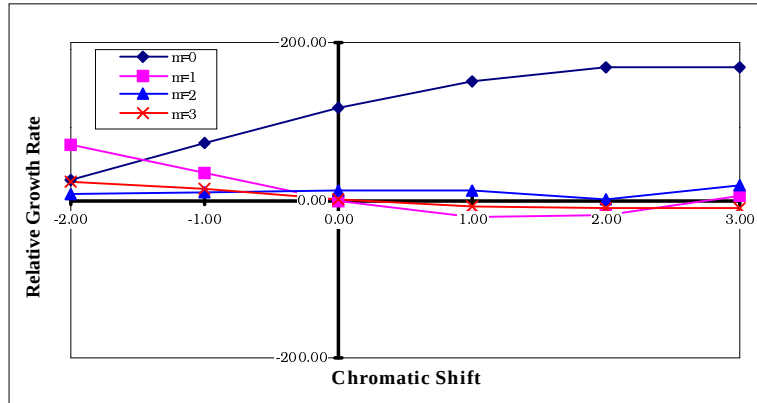


Fig. 19 Relative growth rates (resistive-wall impedance) as a function of $\omega\xi$ ($Q\beta = 0.3$)

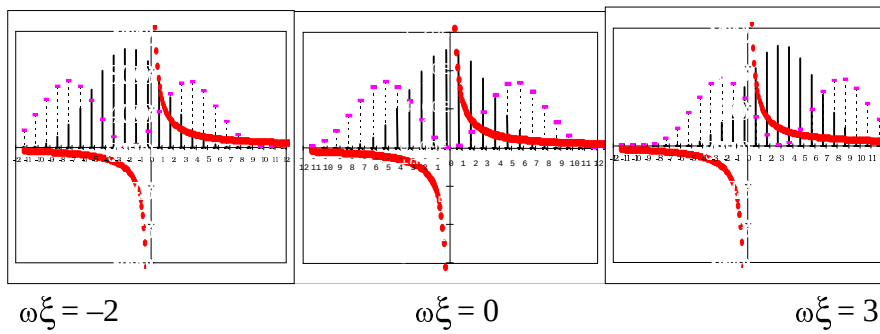


Fig. 20 Resistive-wall type impedance with varying $\omega\xi$ (for modes 0 and 1) ($Q\beta = 0.7$)

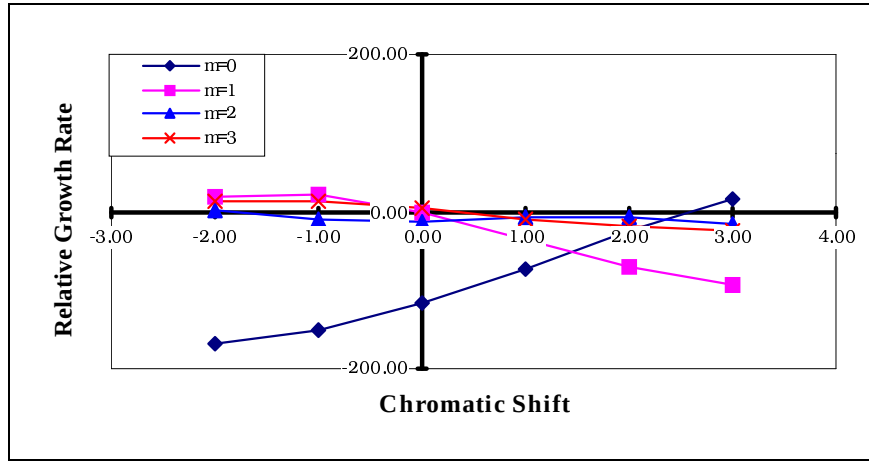


Fig. 21 Relative growth rates (resistive-wall impedance) as a function of $\omega\xi$ ($Q\beta = 0.7$)

10. THE TRANSVERSE MODE COUPLING INSTABILITY (TMCI)

In the previous sections the classical head–tail instability was shown to be a resonant effect driven by broad band impedances and controlled by the chromatic parameter $\omega\xi$. The ‘strong head–tail instability’ or the ‘transverse turbulent instability’ or more correctly the ‘transverse-mode coupling instability’ is a non-resonant instability which is unaffected by chromaticity and can be a severe and fundamental limitation to the intensity. This instability was first observed, but not understood in SPEAR, and later observed and explained in the PETRA machine [5] and in parallel by Talman [6]. The mechanism is identical to the classic head–tail in that the synchrotron motion is needed to interchange the head and tails of the bunch in order to drive the instability.

Probably the simplest way to gain a physical insight into the instability is to use a two-particle model, one at the head for the first half-synchrotron period and at the tail for the second half-synchrotron period. This situation is depicted in Fig. 22 along with the wakefield induced by the leading particle and experienced by the trailing one.

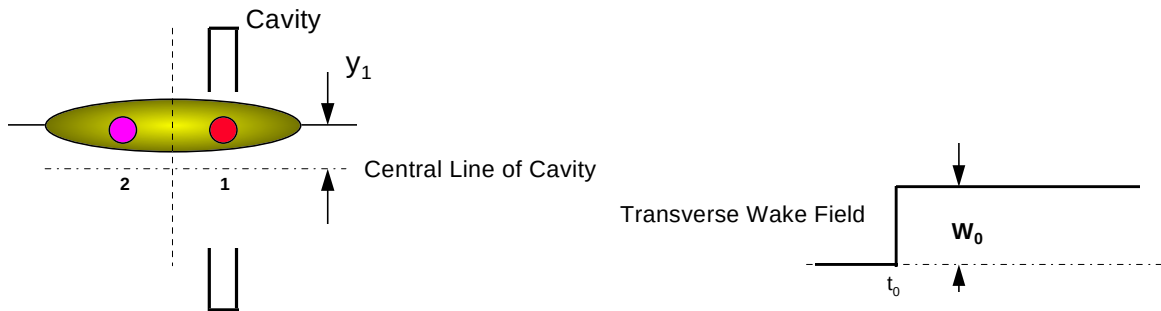


Fig. 22 Two-particle model and the transverse wakefield

For the first half-synchrotron period particle 2 is trailing and while traversing the cavity, experiences the wakefield (only one assumed per turn) induced by particle 1 and therefore receives a deflecting kick. Similarly, during the second half-synchrotron period, particle 1 is trailing and receives a deflecting kick.

Consequently the equations of motion for this situation are

$$\ddot{y}_1 + \omega_\beta^2 y_1 = 0 \quad \text{for } 0 < t < T_s / 2$$

$$\ddot{y}_2 + \omega_\beta^2 y_2 = \alpha y_1$$

$$\ddot{y}_2 + \omega_\beta^2 y_2 = 0 \quad \text{for } T_s / 2 < t < T_s$$

$$\ddot{y}_1 + \omega_\beta^2 y_1 = \alpha y_2$$

where

$$\alpha = \frac{Ne^2 W_0}{2m_0 \gamma}.$$

Using Laplace transforms it is easy to show that the solution to an equation like

$$\ddot{y}_2 + \omega_\beta^2 y_2 = \alpha y_1$$

is

$$y_2(t) = y_2(0) + y_2'(0) \frac{\sin \omega_\beta t}{\omega} + \alpha y_1'(0) \left\{ \frac{1}{2\omega_\beta^3} (\sin \omega_\beta t - \omega_\beta t \cos \omega_\beta t) \right\}.$$

Consequently the solution to the equations for the first half-synchrotron period is

$$\begin{bmatrix} y_1(t) \\ y_1'(t) \\ y_2(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} A(T_s / 2) & 0 \\ B(T_s / 2) & A(T_s / 2) \end{bmatrix} \begin{bmatrix} y_1(0) \\ y_1'(0) \\ y_2(0) \\ y_2'(0) \end{bmatrix} = T(0 \rightarrow T_s / 2) \begin{bmatrix} y_1(0) \\ y_1'(0) \\ y_2(0) \\ y_2'(0) \end{bmatrix}$$

where

$$A(t) = \begin{bmatrix} \cos \omega_\beta t & \frac{\sin \omega_\beta t}{\omega_\beta} \\ -\omega_\beta \sin \omega_\beta t & \cos \omega_\beta t \end{bmatrix}$$

and

$$B(t) = \begin{bmatrix} \frac{\alpha t \sin \omega_\beta t}{2\omega_\beta} & \frac{\alpha \sin \omega_\beta t}{2\omega_\beta^3} - \frac{\alpha t \cos \omega_\beta t}{2\omega_\beta^2} \\ \frac{\alpha t \sin \omega_\beta t}{2\omega_\beta} + \frac{\alpha t \cos \omega_\beta t}{2} & \frac{\alpha t \sin \omega_\beta t}{2\omega_\beta} \end{bmatrix}.$$

Similarly it can be shown that the transfer matrix for the second half-synchrotron period is

$$T(T_s / 2 \rightarrow T_s) = \begin{bmatrix} A(T_s / 2) & B(T_s / 2) \\ 0 & A(T_s / 2) \end{bmatrix}$$

and hence the total transfer matrix for a complete synchrotron period is

$$T(0 \rightarrow T_s) = \begin{bmatrix} A(T_s / 2) & 0 \\ B(T_s / 2) & A(T_s / 2) \end{bmatrix} \begin{bmatrix} A(T_s / 2) & B(T_s / 2) \\ 0 & A(T_s / 2) \end{bmatrix} = \begin{bmatrix} A^2 & AB \\ BA & A^2 + B^2 \end{bmatrix}.$$

All that remains to be done to determine stability is to evaluate the eigenvalues of this matrix (T). By eliminating resonant terms in the matrix B (it will be shown later that some interesting effects are dropped with this simplification), the eigenvalues (λ) may be evaluated [7] from the characteristic equation:

$$\lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_1\lambda + 1 = 0.$$

Here

$$c_1 = (2\eta^2 - 4)\cos\omega_\beta T_s$$

and

$$c_2 = (\eta^2 - 2)^2 + 4\cos^2\omega_\beta T_s - 2$$

where $\eta = \frac{\alpha T_s}{4\omega_\beta}$ (not to be confused with the frequency dispersion used previously).

Determination of the eigenvalues produces the rather simple result that there is instability if $|\eta| > 2$ and above this threshold the two modes of oscillation become degenerate. This corresponds to a threshold current of

$$I_{th} = \frac{16 f_{rev} E Q_s \omega_\beta}{ec(2\pi R) W_0}$$

where the factor $1/\omega_\beta$ is related to the β function. If the β function at the location of the wakefield is β_w then c/ω_β should be replaced by β_w giving

$$I_{th} = \frac{16 f_{rev} E Q_s}{e(2\pi R)\beta_w W_0}.$$

Figure 23 shows the frequency spectrum of the two modes of oscillation as a function of the parameter η .

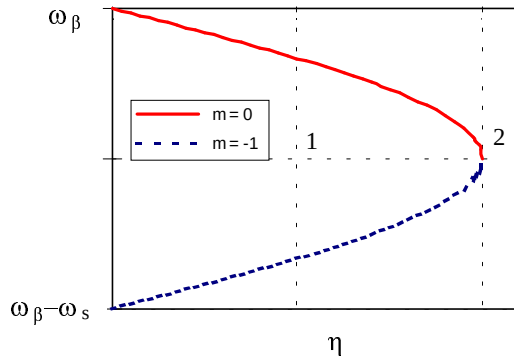


Fig. 23 Variation of frequencies of modes $m = 0$ and $m = -1$ as a function of η

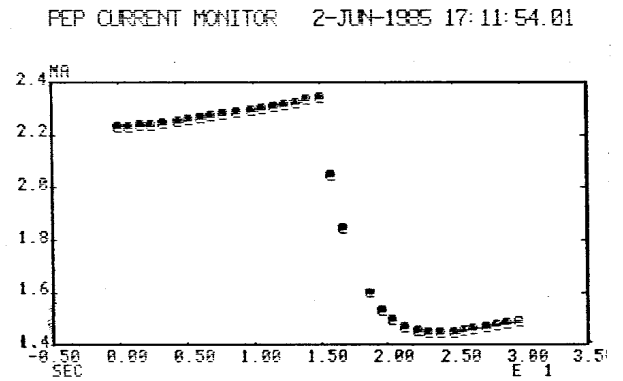


Fig. 24 Bunch current behaviour when the threshold of TMCI is exceeded

Closer examination of the equations of motion shows that, above threshold, the instability which occurs causes a very severe disruption to the bunches. For $\eta = 2$ the amplitude of the motion of the trailing particle grows by a factor of 2 during the half-synchrotron period. Above this value the growth of the particles 'bootstraps' into a severe instability with a very fast growth rate.

In a real machine where accumulation is going on, increasing the intensity beyond the threshold results in an ‘explosion’ of the transverse beam dimensions accompanied by a large loss of beam current. Figure 24 shows such behaviour of the bunch current (in a dedicated experiment on the PEP machine in 1985) as accumulation is continued beyond the threshold of the TMCI. In this plot accumulation is progressing normally until a single injection of around 1% of the total beam current causes the threshold to be surpassed and results in around 40% of the already accumulated beam being lost. It is also interesting to note that after the beam loss, accumulation begins again without problem.

The more precise equation for the threshold current (for more than two particles) is

$$I_{\text{th}} = \frac{2\pi f_{\text{rev}}}{e} \frac{Q_s E_b}{\sum \beta_i k_i (\sigma_s)}$$

which is very similar to that derived for two particles with the slight change that the wakefield is replaced by the loss parameter k_i at each source of impedance and the summation is performed to include all sources. It is also important to note that the loss parameter is bunch-length dependent. This has been shown to be important in LEP where the threshold current has been increased by increasing the bunch length at injection energy by the use of wiggler magnets. In addition, LEP is operated with a very high Q_s at injection, as well as an increased injection energy, in order to increase the threshold.

11. TMCI WITH FEEDBACK

Using the two-particle model it was shown that instability occurs when mode 0 is reduced by about half of the synchrotron frequency (see Fig. 23). Since mode $m = 0$ is simply the centre-of-gravity motion of the bunch, then the frequency of this mode may be controlled by a feedback system [8]. If for example in Fig. 23 the frequency of mode 0 is maintained constant as the bunch current (η) is increased, then coupling to mode 1 should be delayed to higher currents. Such a ‘reactive’ feedback system must measure the centre of gravity of the bunch and produce a ‘kick’ which is proportional to the measured displacement. In practice this means there must be a multiple of π radians betatron phase advance between the location of the ‘pick-up’ measuring the displacement and the location of the fast kicker magnet which produces the deflection.

The equations for the two-particle model with feedback are

$$\ddot{y}_1 + \omega_\beta^2 y_1 = \sigma(y_1 + y_2) \quad \text{for } 0 < t < T_s/2$$

$$\ddot{y}_2 + \omega_\beta^2 y_2 = \alpha y_1 + \sigma(y_1 + y_2)$$

$$\ddot{y}_2 + \omega_\beta^2 y_2 = \sigma(y_1 + y_2) \quad \text{for } T_s/2 < t < T_s$$

$$\ddot{y}_1 + \omega_\beta^2 y_1 = \alpha y_2 + \sigma(y_1 + y_2)$$

where σ is the feedback parameter and is proportional to the tune shift (ΔQ_{FB}) produced by the feedback on the motion of $(y_1 + y_2)$ at low intensities. Hence

$$\sigma = -\Delta Q_{\text{FB}} \omega_{\text{rev}} \omega_\beta.$$

It can be shown from the equations of motion that, in the absence of wakefields,

$$\Delta\omega_{\text{FB}} = -\frac{\sigma}{\omega_{\beta}}.$$

The solution to the coupled equations of motion with feedback can be performed in an identical manner to that used without feedback. In order to understand the stability limits with feedback [7] it is convenient to identify a feedback dimensionless quantity (ξ)

$$\xi = -\frac{\sigma T_s}{4\omega_{\beta}},$$

similar to the η parameter defined previously. Remember that η is simply proportional to the bunch current while ξ is proportional to the gain of the feedback system. The first idea for feedback called for compensation of the drop in frequency of the $m = 0$ mode as a function of current. This corresponds to making the feedback parameter ξ equal to $-\eta/2$. (ξ, η) plane. From this plot, it can be seen that, without feedback ($\xi = 0$) and by increasing the bunch current (η), the unstable region is reached when $\eta = 2$. This is depicted by the horizontal line starting at the origin. The line marked ($\xi = -\eta/2$) shows the situation Figure 25 shows the calculated regions of instability [7] (hatched areas) in the where the gain is controlled to compensate the linear part of the shift of mode 0. In this case the unstable region is reached at about four times the ‘current’ for zero feedback. Other stable regions are attainable with the feedback system e.g. the region indicated by $\sqrt{\xi(\xi + \eta)} = \pi$. This analysis clearly shows that the threshold for the transverse (microwave) instability can be increased by the use of a reactive feedback system.

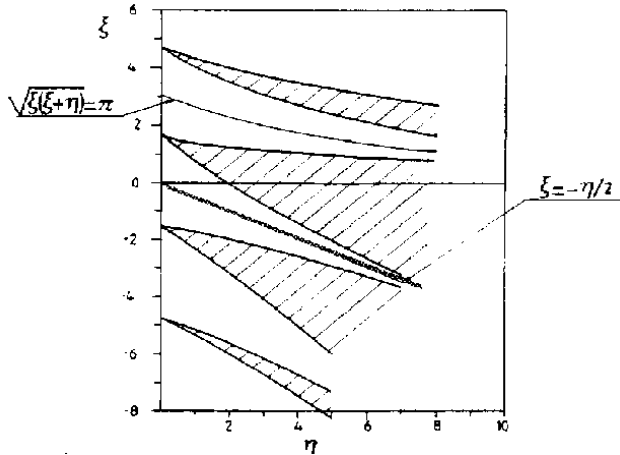


Fig. 25 Stability plots with feedback parameter ξ varied

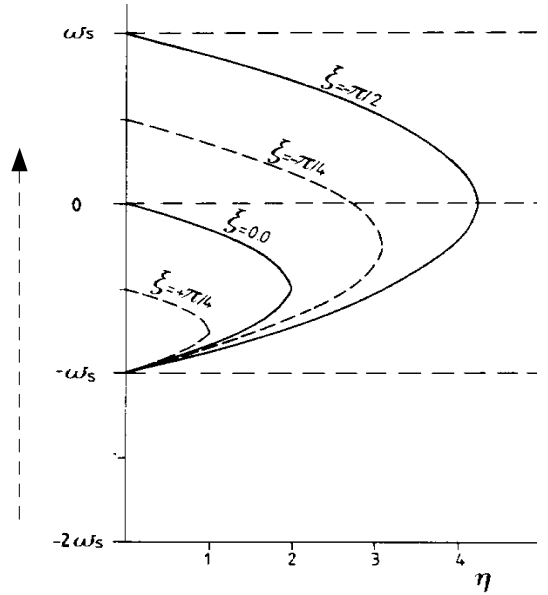


Fig. 26 Behaviour of modes as a function of η and ξ

Figure 26 shows the behaviour of the frequencies of modes 0 and 1 with increasing ‘current’ and in the presence of feedback. With the feedback parameter set to zero we obtain the same plot as previously shown in Fig. 23. However, if the feedback parameter is adjusted so as to push the modes closer at vanishing currents then the instability occurs at a lower threshold current ($\xi = +\pi/4$). In the cases where the feedback enhances the separation of the modes then the threshold is increased ($\xi = -\pi/4$ and $-\pi/2$).

12. IMPROVEMENTS IN THE MODEL

The analytical two-particle model discussed so far has two principal simplifying assumptions.

- The use of the continuous differential equations of motion implies that both the wakefields and the feedback are not localised elements but spread out continuously over the circumference of the machine. This of course implies that there is no information concerning the betatron phase advance between the wakefields, and between the feedback and the wakefields.
- In neglecting some of the ‘resonant’ terms in the analysis of Sections 10 and 11 some important resonances disappear.

Clearly, in order to improve the model it is necessary to introduce localised elements [9], [10].

12.1 Localised elements

Figure 27 shows one possible schematic representation of localised elements. For this situation it has been shown [10] that the single-turn transfer matrix is given by

$$S = L_{KP} \left[L_{WK} (W_{LBW})^{N_W - 1} W_{LPW} + K \right]$$

where L_{KP} is the betatron phase rotation between the **Kicker** and the **Pickup**, and the other phase rotations are as indicated in Fig. 27. The feedback (kicker) matrix is given by

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \sigma & 0 & \sigma & 0 \\ 0 & 0 & 0 & 0 \\ \sigma & 0 & \sigma & 0 \end{bmatrix}.$$

For two particles the wakefield matrix is given by

$$W_{1(2)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \alpha(0) & 0 \\ 0 & 0 & 1 & 0 \\ 0(\alpha) & 0 & 0 & 1 \end{bmatrix}$$

where the values in brackets refer to the second half-synchrotron period and α has already been defined as the wakefield strength parameter. For more than two particles there are $2n$ (n = number of particles) wakefield possibilities if the particles are equally spaced longitudinally.

For two particles it is clear that the matrix which must be inspected for stability is

$$S_T = S_1^{1/(2Q_s)} S_2^{1/(2Q_s)}$$

where the subscripts refer to the first and second half-synchrotron periods. In this procedure the synchrotron tune must be chosen such that $1/(2Q_s)$ is an integer. For more than two particles the matrix for a complete synchrotron period is

$$S_T = S_1^M S_2^M \dots S_p^M$$

where $Q_s = 1/pM$, S_p is the single-turn matrix which corresponds to the wakefield situation p , and p is the number of different wakefield situations ($= 2n$, where there are n particles equally spaced longitudinally).

The evaluation of the eigenvalues of S_T is too cumbersome to be done analytically but can be performed with ease on a computer.

A set of results for two particles is shown in Fig. 28 where the Q_s is 0.0833 (i.e. 1/12) and there is only a single wake per turn. With the feedback parameter set to zero one can clearly see the resonances at betatron tunes equal to multiples of Q_s .

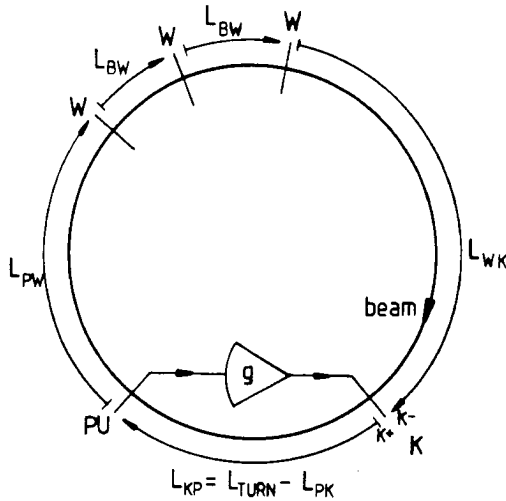


Fig. 27 Schematic representation of a circular accelerator with localised wakefields (W) and localised feedback elements

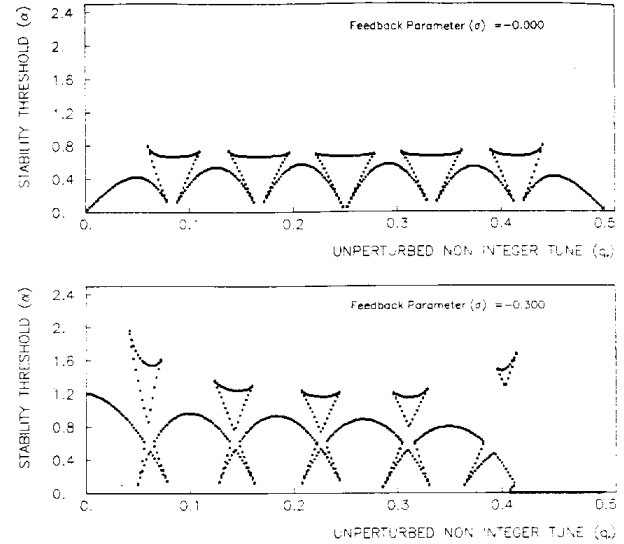


Fig. 28 Stability plots for two-particle model and with feedback (note a feedback parameter of $\sigma = -0.3$ makes a shift in mode 0 of $+Q_s/2$)

12.2 Summary of the results obtained with the ‘few-particle model’

The model described in the last section was used to study [11] the influence of the various parameters on the transverse mode coupling instability. The following summarises the main results obtained.

- *One wakefield per turn.* The threshold for instability is strongly dependent on the betatron tune value. The feedback enhances the threshold for a range of tune values. However, as the feedback gain increases, the tune range diminishes rapidly. The enhancement of the threshold is strongly influenced by the betatron phase shift between the wakefield and the kicker.
- *Many wakefields equally spaced over one betatron wavelength.* In this case for zero feedback gain there is absolutely no tune dependence (resonances). This indicates that the ‘coherent synchrotron–betatron resonances’ are more or less restricted to the case of a single wake per turn. In most circular accelerators the transverse impedance is indeed localised, but localised to many hundreds of positions around the circumference. Hence for most accelerators these resonances will not appear.

The maximum enhancement of the threshold current is greatly reduced when many wakefields (equally spaced over one betatron wavelength) are assumed. With many wakefields the kicker cannot be positioned in betatron phase for optimum enhancement for each wakefield.

13. COMPUTER SIMULATION OF TMCI

The few-particle models have been shown to predict the physical properties of the TMCI. However, as with all simplified models, they are limited in many ways. The previous section showed how some of the limitations can be removed to allow the introduction of localised elements in the model. The procedure uses a brute force technique for evaluation of the threshold of the instability. Although the model contains much of the physics there are, however, some simplifying assumptions which could be worrisome. Most of the remaining simplifications can be removed by a full-blown computer simulation that evaluates the thresholds in realistic conditions without the restriction of simplified wakefields and ‘few’ particles. In these simulations [12], [13] several hundreds or even thousands of superparticles are ‘tracked’ through a simplified machine lattice containing wakefields and feedback systems. The wakefields are evaluated offline using electromagnetic computer codes. Figure 29 shows the simulation of the PEP machine that was done in preparation for a dedicated experiment [14] designed to investigate the TMCI threshold in the presence of feedback. On the left is plotted the FFT of the centre-of-gravity motion for various values of the bunch current. It can be seen that as the current is increased the mode 0 decreases in frequency and approaches mode -1 . It is also apparent that as the current increases the amplitude of the mode -1 grows. Instability is reached when the amplitude of the two modes is approximately equal. (This behaviour is often seen in machines with short bunches. In machines with longer bunches where there is bunch lengthening with current, the situation becomes more complicated and the modes $m = 0$ and $m = -1$ can become parallel without ever crossing.) An almost identical behaviour was subsequently measured during the dedicated experiment. The motion of the modes is plotted on the right side of Fig. 29. Note the obvious similarity with the predicted motion for two particles in Fig. 23. These computer simulations predicted very well the PEP behaviour without feedback and less well the behaviour with feedback. Feedback in PEP allowed the threshold current to be increased by more than a factor of two.

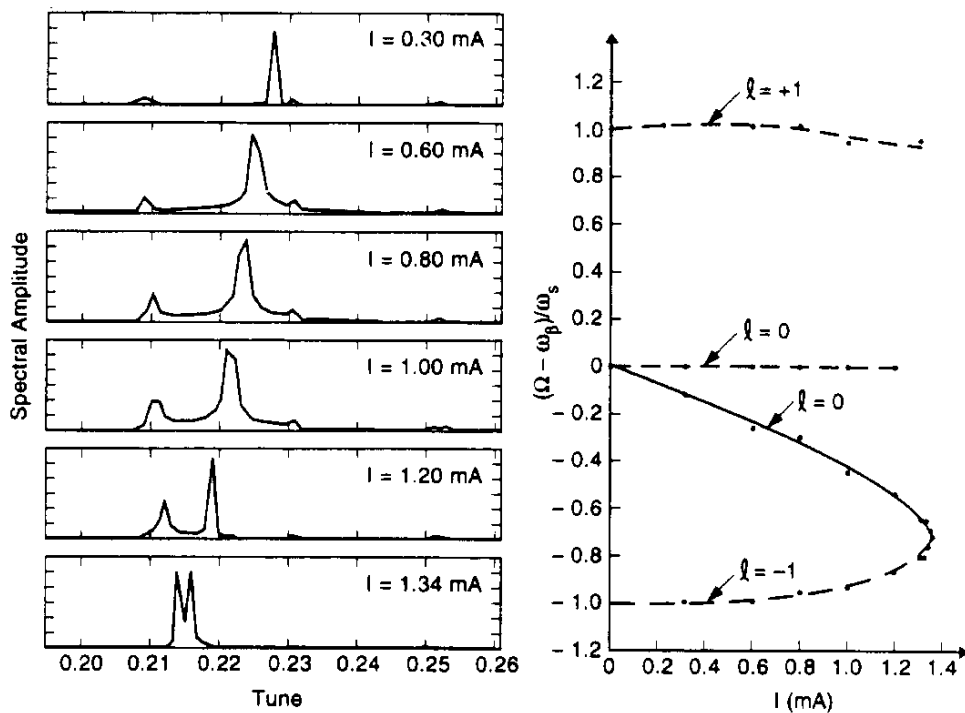


Fig. 29 Computer simulation of TMCI

14. SYNCHROBETATRON RESONANCES

The particle oscillation energy in the longitudinal plane can be coupled into the transverse plane by so-called ‘synchrobetatron’ coupling. The purpose in this section is to describe the physical processes which causes this coupling and finish by presenting some recent experimental results.

There are many well-known mechanisms for coupling the longitudinal and transverse motion of the particles. Of these, usually the most severe is momentum dispersion at the location of the RF cavities [15]. Consequently, most colliders are designed with zero horizontal momentum dispersion in all the RF straight sections and, of course, zero vertical dispersion everywhere. However, measurements have shown that the residual dispersion, produced by machine imperfections, is significant and particularly dangerous in the vertical plane.

The performance of LEP is fundamentally limited by the transverse-mode-coupling instability, (described in previous sections) which limits the current per bunch. In order to raise the threshold of this instability it has been proposed to increase significantly the synchrotron tune (Q_s) at injection energy. Since these high values of Q_s cannot be maintained during the energy ramp, it is inevitable that synchrobetatron resonances must be crossed. This experimental study [16] was initiated in order to understand the resonance behaviour and to allow a proposal of a scheme for crossing of synchrobetatron resonances during the energy ramp.

The mechanism for synchro-betatron coupling generated by dispersion at the RF cavities is depicted graphically in Fig. 30. Assume that a particle initially arrives at an RF cavity with the same energy as the synchronous particle and zero amplitude in betatron phase space as shown in Fig. 30. During the traversal of the cavity the particle gains energy and a new energy orbit is established about which the particle must oscillate. The particle which was initially at $x = 0$ in Fig. 30 has a new equilibrium orbit due to the energy increase. This is equivalent to the origin of the betatron axis being displaced by an amount equal to

$$\Delta x = D_x \frac{\Delta E}{E}$$

where D_x is the dispersion at the cavity and ΔE is the energy gained (relative to the synchronous particle) by the particle traversing the cavity. It may be seen that if the frequency of the oscillations in the transverse plane is an integer multiple of those in the longitudinal plane, then the amplitude of the betatron oscillations build up with each traversal of the cavity. More generally the resonant condition is

$$kQ_x + mQ_y + nQ_s = p$$

where k , m , n , and p are integers.

14.1 Experimental observation of synchrobetatron resonances

The measurement technique used was rather simple and automated by use of a ‘tune scan’. The horizontal and vertical tunes are incremented in a prescribed way by variation of the main quadrupole chains. At each incremental step the tunes are measured along with the beam sizes (as measured from the synchrotron light monitor), the bunch currents, and the current lifetime.

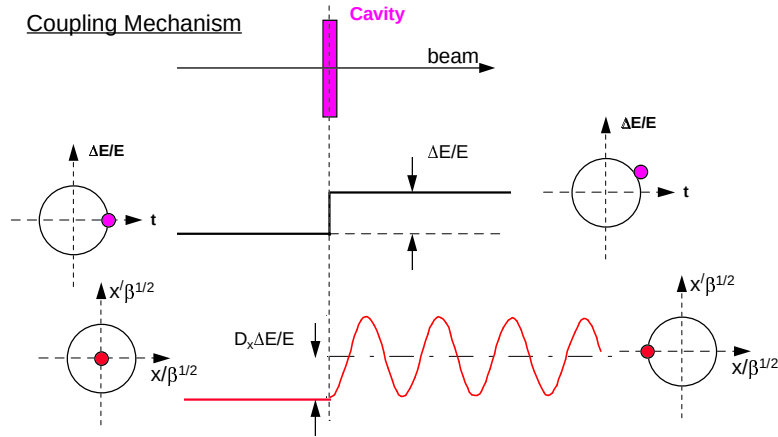


Fig. 30 Mechanism for synchrotron coupling driven by momentum dispersion at the RF cavities

Vertical tune scans were made at three intensity levels. The results are shown in Fig. 31 where the vertical beam size is plotted as a function of the **measured** vertical tune value. For ease of viewing a base line offset was added to each scan, creating a ‘mountain range’ where the increase in the varied parameter is depicted by an increase in the vertical offset.

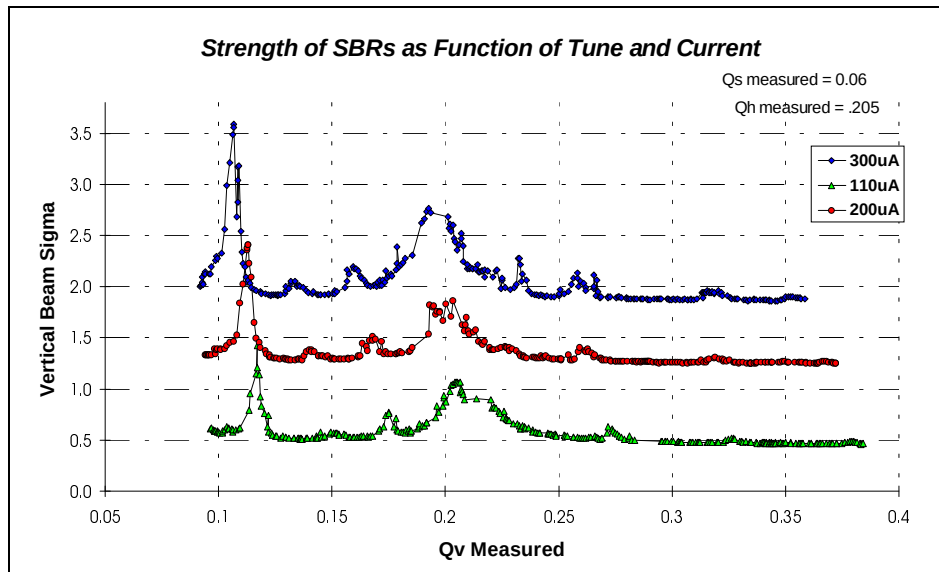


Fig. 31 Strength of SBRs as a function of tune and current

The vertical and horizontal tune shifts due to the LEP transverse impedance were subsequently measured as a function of intensity. These results were used to evaluate the incoherent or zero-current tune values (Q_{vinc}). In addition it is known that although the coherent (measured) synchrotron tune remains constant with intensity, the incoherent synchrotron tune increases with bunch current. If the hypothesis is made that, for example, the resonance $Q_y = 3Q_s$ is a single-particle resonance and therefore occurs at incoherent tune values, then by varying the value of Q_{sinc} , a perfect resonant condition can be found for various values of bunch current. This was done and the results are shown in Fig. 32.

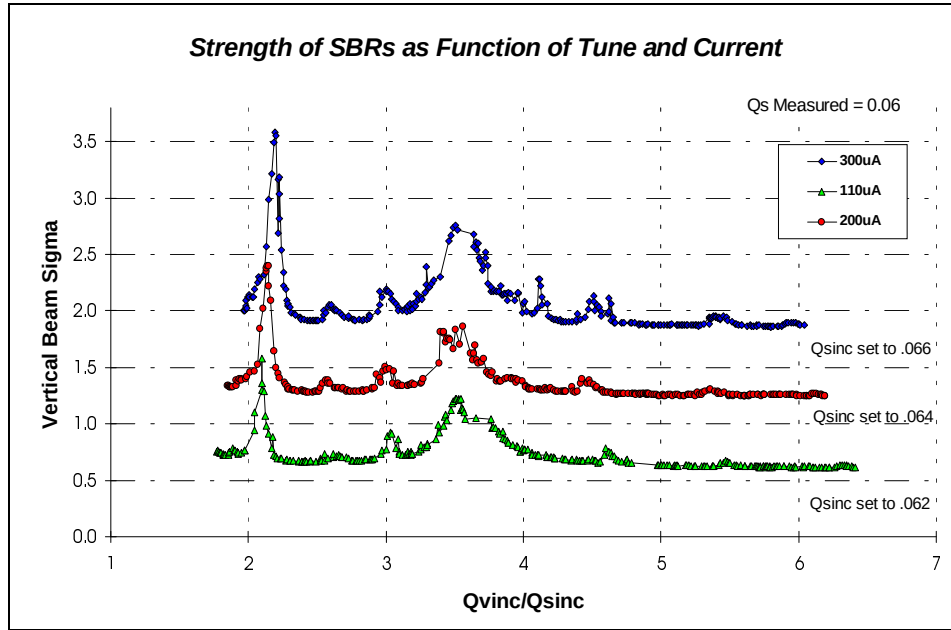


Fig. 32 Vertical beam size plotted against incoherent vertical tune shift normalised to a set value of the incoherent synchrotron tune which reproduces the resonant condition

The residual vertical dispersion in the RF straight sections is the main mechanism for coupling longitudinal motion into the vertical phase plane. In normal operation of LEP this dispersion is minimised by careful correction of the closed orbit. The dispersion can be measured by subtracting closed orbits measured with energy deviations. For the results presented thus far the measured r.m.s. dispersion was 8 cm. It is also well known that the application of ‘asymmetric’ orbit bumps through the interaction region creates a dispersion bump all around LEP. For this reason fairly large ‘asymmetric’ bumps were applied in all even interaction regions to increase the r.m.s. dispersion. By this technique the measured residual vertical r.m.s. dispersion was increased from 8 cm to 44 cm. The tune scan with this increased dispersion caused loss of nearly all the beam current as the tune was swept across the ‘second sideband’ ($Q_y = 2Q_s$). The tune scan in going from high tune values downward towards and eventually across the second sideband is shown in Fig. 33 along with the tune scan at lower dispersion.

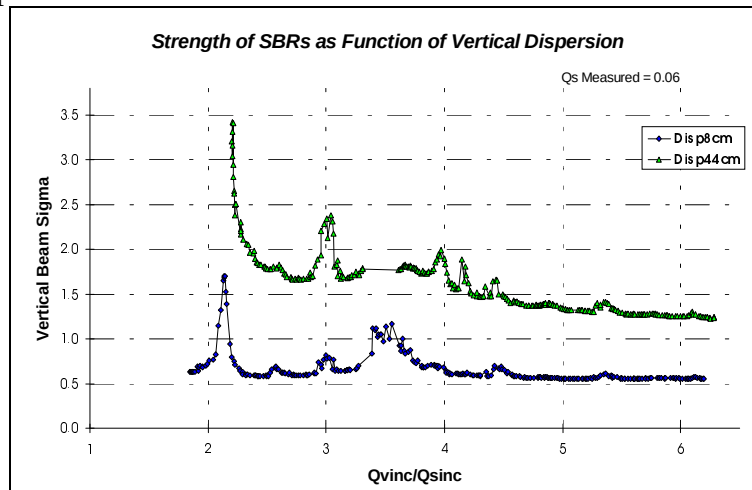


Fig. 33 Tune scans with two values of dispersion (the vertical tunes are corrected for the intensity and the Q_s is modified so as to meet the incoherent resonant condition)

It has been known for some time [17] that orbit displacement in RF cavities is a driving mechanism for synchrotron resonances. In another experiment the closed orbit through the RF cavities around interaction points 2 and 6 was well corrected so as to produce the minimum displacement through the room-temperature cavities. A vertical closed-orbit ‘bump’ (of amplitude 10 mm peak) was then applied to these straight sections. The tune scans for these two different situations are shown in Fig. 34. Examination of these plots shows that the second sideband ($Q_y = 2Q_s$) is very slightly less excited, whereas the excitations of the third, fourth, and fifth sidebands are significantly enhanced.

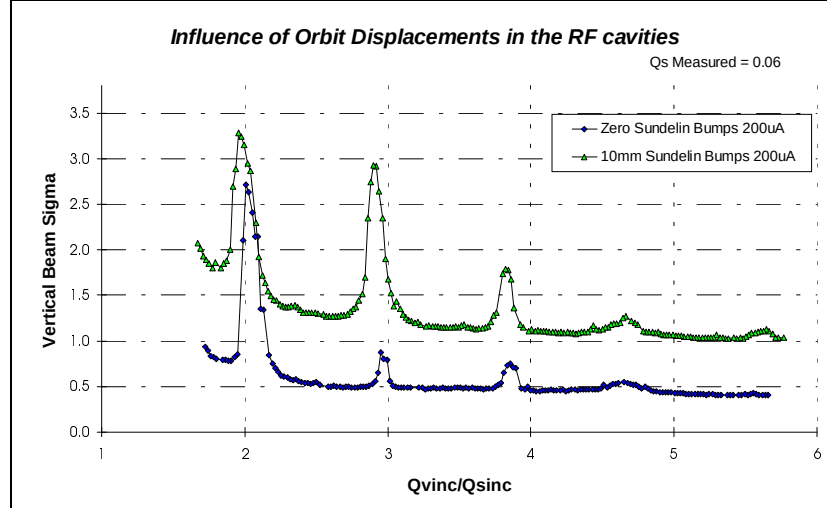


Fig. 34 Influence of orbit displacement in the RF cavities

In order to increase the threshold for the TMCI it is foreseen to operate LEP at very high values of Q_s at injection energy. For this reason an experiment was performed in which the value of Q_s was increased to nearly double its value used in the previous experiments. The results of these tests, shown in Fig. 35, indicate that the excitation of the synchrotron sidebands increases but only slightly as a function of the synchrotron tune.

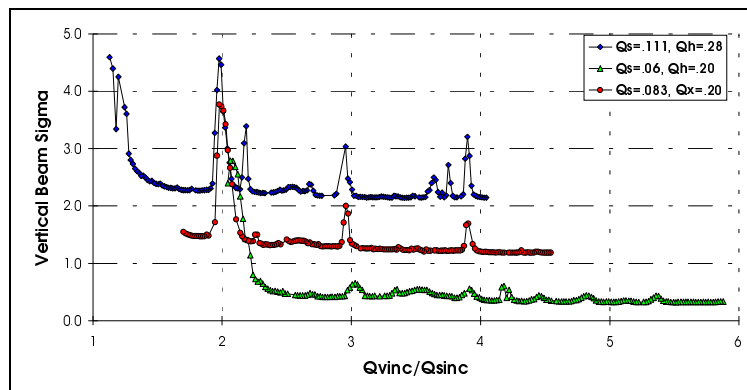


Fig. 35 Variation of the synchrotron tunes

ACKNOWLEDGEMENTS

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LIFETIME AND BEAM QUALITY

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Abstract

An overview is presented of lifetime and beam quality in a synchrotron radiation light source. Maintaining the brightness of a light source, which is determined by the beam emittance and stored current, is of fundamental importance. An introduction is given to the concept of brightness and the factors affecting it. These factors involve transverse beam stability, aperture limitations (physical and dynamic), beam gas scattering, quantum lifetime, intra-beam Coulomb scattering, beam instabilities and ion trapping. For each item an overview of the process/effect is given.

1. INTRODUCTION

The enormous scientific activity in the last few decades in the use of synchrotron radiation has produced several world-wide electron or positron storage ring facilities which provide increasingly more intense radiation [1]. Two photon regimes have emerged, which determine the energy of the stored beam, VUV to soft X-rays and hard X-rays. The latest facilities, the so called third generation light sources, have been specifically designed to cater for the production of highly intense photon beams. The characteristics of these machines are a small electron beam emittance and high currents. In addition the experimental user demands that the source be highly stable and long lived. These additional requirements are in conflict with the attainment of low emittance and the production of high beam currents [2]. The combination of emittance and current yields a figure of merit called the brightness.

Low emittance lattices come in various forms DBA, TBA, QBA [3]. All these lattices are characterised by the requirement of both small dispersion and horizontal betatron function in the bending magnets, which in turn requires strong focusing. The strong focusing in turn needs strong chromaticity correcting sextupoles. The strong quadrupoles also make the lattice sensitive to miss-alignment errors with consequent enhanced sensitivity to vibrations and other perturbations to the electron beam. The strong sextupoles make the electron motion deviate significantly from linear motion with a consequent reduction in the aperture available to the electron beam which has a negative impact on the beam lifetime.

High beam currents are limited by beam instabilities [4]. These instabilities arise from the interaction of the beam with its environment, the vacuum chamber and cavities. To lower the interaction special care has to be taken with the construction of the chamber to lower the impedance seen by the circulating beam. High beam currents also necessitate increased RF power with added difficulties in handling such power for a given cavity. To circumvent this more cavities are used which increases the susceptibility towards multibunch instabilities not to mention the added costs. Instabilities either limit the maximum stored current or give rise to a noisy beam either longitudinally or transversely.

Lifetime is an important parameter of a light source. High brightness means a small emittance and therefore high bunch densities. This increases the effect of intra-beam scattering, both small and large angle. The former leads to a new equilibrium emittance while the latter (the Touschek effect) reduces the lifetime. A higher beam energy alleviates the scattering effects, nevertheless, working with few bunches will still give Touschek limited operation even for the hard X-ray machines [5]. To overcome the lifetime effect with present-day third generation sources either the emittance (transverse or longitudinal) is compromised or the experimental User suffers frequent interruptions to his experiment due to the frequent machine refills. Future light sources are focusing effort to improve the lifetime

by enhancing the machine acceptance and/or examining the possibility of topping-off the lost electrons without interrupting the experiments [6].

A facet of modern light sources is the predominant use of insertion devices, undulators and wigglers, as the radiation source [7–9]. A significant fraction of the circumference of modern storage rings is covered by insertion devices. Undulators produce extremely bright tuneable light. The tuneability is provided by changing the gap of the device. Small gaps allow for a stronger device and permit the production of higher energy photons by going to smaller period lengths. Small gaps, however, will also have a negative impact on operation by reducing the aperture which leads to a reduction in lifetime, and by increasing chamber impedance effects and therefore instabilities. The devices also affect the beam dynamics: the linear focusing term may be compensated by additional quadrupoles in the machine, whereas the non-linear effect results in a reduction in the dynamic aperture with a consequent reduction in lifetime. The inevitable imperfections of the device also produce changes in the closed orbit, which are particularly damaging for the Users.

The stability of the beam is as important as the emittance or the lifetime [10, 11]. In general transverse orbit instability will dilute the emittance and/or shift the source point while in the longitudinal plane it will degrade the time structure of the beam and also leads to an increase in the effective energy spread which spoils the quality of the higher radiation harmonics from an insertion device. The time scales of the instabilities which range from weeks and hours to milliseconds determine the amount and type of beam degradation the experimental user experiences. Furthermore, high brightness machines put increasing demand on new technologies to achieve the desired stability on ever decreasing transverse beam dimensions.

In the following sections an overview will be given of what is meant by beam quality and lifetime and the things which affect them. Apart from the section dealing with ion trapping the considerations given below also apply to positrons as the circulating particles. The reader should note that the quoted references are by no means exhaustive and purely represent examples to the overview given for a particular topic. Extensive additional material may be found in the CERN "Yellow Reports" and in the latest proceedings of the American and European particle accelerator conferences.

2. BRIGHTNESS

2.1 A figure of merit [12, 13]

A quantity which defines the quality of a synchrotron radiation light source is the brightness, which is simply the phase space density of photon flux evaluated in the forward direction and at the centre of the source,

$$B = \frac{d^4 F}{d\theta d\phi dx dy} \bigg|_0 \quad (1)$$

θ ϕ and x y are the horizontal and vertical angles and co-ordinates and F the flux or the number of photons N per second. The importance of brightness arises from the fact that it is an invariant and is conserved while being propagated through linear optical elements.

Integration over either space or angle co-ordinates gives the angular and spatial flux densities,

$$\frac{d^2 F}{d\theta d\phi} = \int \frac{d^4 F}{d\theta d\phi dx dy} \cdot dx dy \quad (2)$$

$$\frac{d^2 F}{dx dy} = \int \frac{d^4 F}{d\theta d\phi dx dy} \cdot d\theta d\phi \quad (3)$$

The flux, also an invariant, is obtained by a further integration,

$$F = \frac{dN}{dt} \quad (4)$$

If we are interested in a small energy range about a given photon energy, then dividing by $d\omega/\omega$ we obtain the spectral brightness and spectral flux.

The units for the spectral brightness and spectral flux are,

$$B = \frac{\text{Number of photons}}{[\text{sec}] [\text{mm}^2] [\text{mrad}^2] [0.1\% \text{ bandwidth}]} \quad (5)$$

$$F = \frac{\text{Number of photons}}{[\text{sec}] [0.1\% \text{ bandwidth}]} \quad (6)$$

A systematic approach to calculate the brightness is to compute the intensity of the electric field of a single electron and convolute it with the probability distribution function of the electrons [12],

$$B = \int B_e \cdot \eta dV \quad (7)$$

where B_e is the brightness of a single electron and η the probability distribution function. The brightness can be considered as an emittance. It obeys Liouville's theorem and is conserved by linear transformations. Optical elements in a beam line propagate the brightness from one point to another,

$$B_1 = \mathbf{M} B_0, \quad (8)$$

$$\mathbf{M} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}. \quad (9)$$

The linear transformations are drift spaces or focusing elements. From Liouville's theorem of conservation of phase space area we find that the requirement to have a small photon spot size somewhere in the beamline will by necessity also give a large divergence there. This in turn implies large optical elements in the beamline which are more difficult to make and control with associated loss of flux and increased costs. To have an intense beam of photons on a sample it is therefore desirable to start with a source which has a small emittance [14].

If we consider the photons to be particles rather than waves and that they have a Gaussian distribution in spatial and angular co-ordinates, then the brightness can be written as,

$$B = \frac{F}{4\pi^2 \Sigma_x \Sigma_{x'} \Sigma_y \Sigma_{y'}} \quad (10)$$

where F is the flux in the forward direction and the denominator is the phase space volume of the photons. Σ_k ($k = x, x', y, y'$) are the spatial and angular standard deviations of the photons

in the two planes. The Σ_k are constructed from the convolution of the electron beam distribution with that of the photon distribution from a single electron,

$$\Sigma_{x,y} = (\sigma_{x,y}^2 + \sigma_r^2)^{1/2} \quad (11)$$

$$\Sigma_{x',y'} = (\sigma_{x',y'}^2 + \sigma_{r'}^2)^{1/2} \quad (12)$$

$\sigma_{x,y,x',y'}$ and $\sigma_{r,r'}$ are standard deviations of Gaussian distributions of position and angle with respect to the longitudinal co-ordinate.

Expression (10) allows an examination of effects which will degrade the brightness. We see immediately that any increase in the transverse dimensions or angles of the electron beam will result in a loss of quality. A reduction in the number of photons per second from a low lifetime also lowers machine performance.

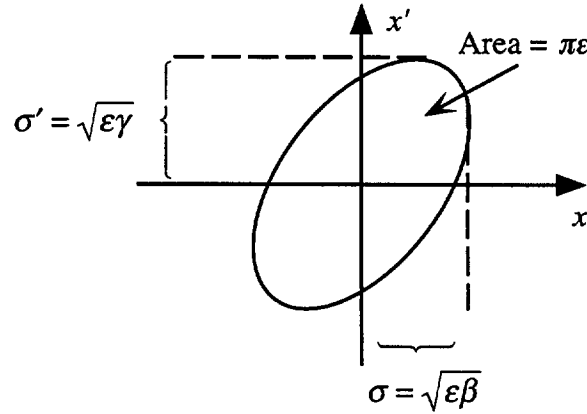
2.2 Source size

2.2.1 Electron beam [15]

The electron beam sizes and divergences are given in terms of the beam emittance and the Courant-Snyder parameters,

$$\sigma_x = \sqrt{\beta_x \epsilon_x} \quad \sigma_{x'} = \sqrt{\gamma_x \epsilon_x} \quad \gamma_x = \frac{1 + \alpha_x^2}{\beta_x} \quad (13)$$

The emittance ϵ and Courant-Snyder parameters α , β and γ are properties of the accelerator lattice. The emittance times π is the area in phase space which is bounded by one standard deviation.



The natural beam emittance is determined by radiation damping and quantum emission. The resulting beam distribution is Gaussian in the six dimensions (we do not consider here beam gas scattering which effects the emittance nor intra-beam scattering which alters the tails of the distribution). At points in the accelerator with dispersion the beam size is given by,

$$\sigma_x = \sqrt{\beta_x \epsilon_x + D^2 \left(\frac{\Delta p}{p} \right)^2} \quad (14)$$

$$\sigma_{x'} = \sqrt{\gamma_x \epsilon_x + D'^2 \left(\frac{\Delta p}{p} \right)^2} \quad (15)$$

D and D' is the dispersion function and its derivative.

The ideal vertical emittance for a lattice with no errors and no coupling of horizontal and vertical motion is of the order 10^{-4} nmrad (again ignoring gas and intra-beam scattering) and arises from the finite vertical emission angle of the photons. A real lattice, however, has coupling of both betatron motion and dispersion due to misalignments and closed orbit errors. Finite spurious vertical dispersion at the bending magnets will also contribute to the vertical emittance. Considering here only the betatron coupling the natural horizontal emittance ϵ_0 will be partitioned between the two planes. We can define an empirical coupling coefficient κ as,

$$\kappa = \frac{\epsilon_y}{\epsilon_x} \quad \text{with} \quad \epsilon_x + \epsilon_y = \epsilon_0 \quad (16)$$

$$\Rightarrow \epsilon_y = \frac{\kappa}{1 + \kappa} \epsilon_0 \quad (17)$$

Typical values for third generation machines are around 0.1 to 3 %. With no dispersion in the vertical plane we have,

$$\sigma_y = \sqrt{\kappa \beta_y \epsilon_x} \quad (18)$$

$$\sigma_{y'} = \sqrt{\kappa \gamma_y \epsilon_x} \quad (19)$$

Since the vertical beam size is small (small coupling) beamlines use that plane as the dispersive plane, i.e. monochromators then have maximum resolution and it is of greatest importance to conserve vertical emittance. Usually the source point at the centre of an insertion device is a symmetry point for the straight section and at that point $\alpha = 0$. Furthermore the dispersion is also made to vanish at positions with insertion devices leading to the simple relation for the emittance.

$$\epsilon_{x(y)} = \sigma_{x(y)} \sigma_{x'(y')} \quad (20)$$

2.2.2 Photon beam [16, 17]

To discuss the size and divergence of the photons emitted from a single electron we consider the diffraction effects of an individual source. A point-like source emitting radiation of wavelength λ viewed far away is described by a Fraunhofer diffraction pattern. The Fraunhofer diffraction pattern of an object is the Fourier transform of that object. For a source of light with a Gaussian distribution,

$$g(r) = \frac{1}{\sqrt{2\pi} \sigma_r} e^{-\frac{x^2}{2\sigma_r^2}} \quad (21)$$

where σ_r is one sigma of the distribution, the Fourier transform gives another Gaussian, from which we find the relation between size, divergence and the wavelength of the radiation to be,

$$\sigma_r \sigma_{r'} = \frac{\lambda}{4\pi}. \quad (22)$$

The relation can be viewed as a kind of uncertainty condition [17]. We call the product $\sigma_r \sigma_{r'}$ the diffraction limited emittance $\lambda/4\pi$. For wavelengths in the range $0.2 \lambda_c < \lambda < 100 \lambda_c$ the natural vertical opening angle of the radiation from a single electron in a bending magnet can be approximated by [18],

$$\sigma_{r'} = \frac{570}{\gamma} \left(\frac{\lambda}{\lambda_c} \right)^{0.43} \quad [\text{mrad}] \quad (23)$$

From which the diffraction limited source size σ_r associated with $\sigma_{r'}$ can be computed from Eq. (21).

2.2.3 Coupling, dispersion and emittance

We now briefly consider a few consequences of the convolution of the electron and photon distributions.

Firstly we see immediately that if $\sigma_r > \sigma_{x,y}$ then reducing the electron beam emittance gives no further increase in brightness. An effect which is more pronounced for low energy photons. The maximum brightness occurs when,

$$\sigma_x, \sigma_y < \sigma_r \quad \text{and} \quad \sigma_{x'}, \sigma_{y'} < \sigma_{r'} \quad (24)$$

in which case we have the diffraction dominated brightness [8]

$$B = \frac{F}{(\lambda/2)^2}. \quad (25)$$

Secondly the brightness is a function of the electron beam coupling and decreases with increased coupling,

$$B \propto \frac{1}{\sigma_x \sigma_{x'} \sigma_y \sigma_{y'}} \bigg|_{\substack{\sigma_{xy} > \sigma_r \\ \sigma_{x'y'} > \sigma_{r'}}} = \frac{1}{\kappa \epsilon_x^2}. \quad (26)$$

Thirdly, non-zero dispersion in either plane at the source point, especially along insertion devices, will also adversely affect the brightness (excluding here considerations of quantum excitation in insertion devices [19]) since both the beam size and its divergence will increase proportionally to the relative energy spread and the dispersion functions. There are situations, however, where a lower emittance may be obtained for a modified optics with a finite radial dispersion in the straight sections [20]. This is only of benefit at the undulator position if

$$\underbrace{\beta_0 \epsilon_0}_{\text{old}} > \underbrace{\beta_n \epsilon_n + \left(D_n \frac{\Delta p}{p} \right)^2}_{\text{new}}. \quad (27)$$

Tight control is then needed in maintaining a small relative energy spread, e.g., controlling longitudinal multi-bunch instabilities. At positions in the bending magnets the situation is nearly always more favourable, since the already existing dispersion is reduced.

2.3 Undulators [7, 8, 12, 21]

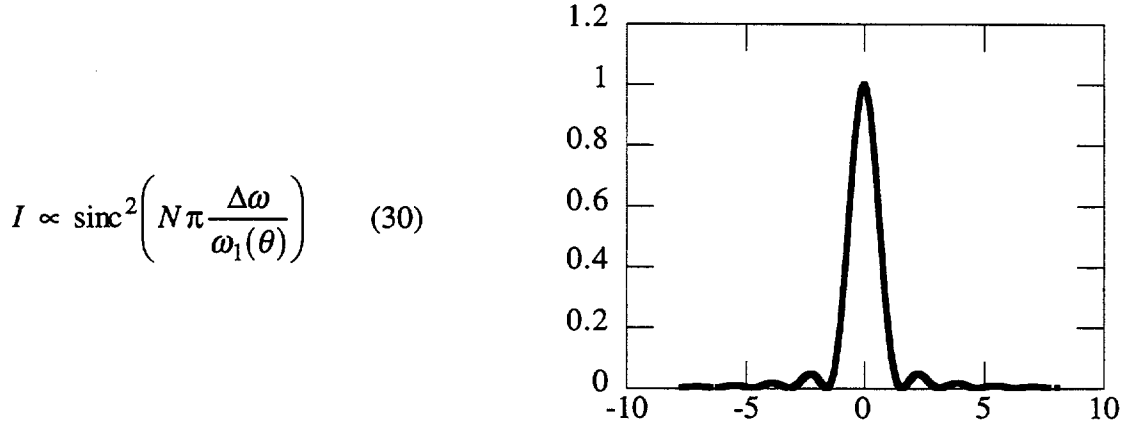
Undulators force the electron beam to perform a serpentine trajectory as it passes through the device. At each deflection photons are emitted. The intensity of the light from a single electron is given by the interference of the beamlets from each deflection point. The characteristic of undulator radiation is its sharply peaked harmonic emission. For a plane undulator the frequency of radiation emitted in an odd harmonic k is given by:

$$\omega_k = \frac{\omega_0 2k \gamma^2}{(1 + K^2/2 + \gamma^2 \theta^2)} \quad (28)$$

$$\omega_0 = \frac{2\pi c}{\lambda_0} \quad (29)$$

λ_0 length of undulator period
 γ relativistic factor $E/m_0 c^2$
 K magnetic field strength parameter $0.934 B[T] \lambda_0[\text{cm}]$
 θ observation angle with respect to the closed orbit

The equation is valid for the central coherent cone of the radiation. The intensity of the emitted radiation has a sinc function dependence, shown in the next figure



$$I \propto \text{sinc}^2 \left(N \pi \frac{\Delta \omega}{\omega_1(\theta)} \right) \quad (30)$$

where N is the number of periods and ω_1 is the frequency of the first harmonic, $\Delta \omega = (\omega - k \omega_1)$. The sinc function is sharply peaked when its argument is zero, i.e., at $\omega = k \omega_1(\theta)$, and has a first zero when the argument is equal to π . Writing

$$\frac{\omega}{\omega_1(\theta)} = \frac{\omega}{\omega_1(0)} \left(1 + \frac{\gamma^2 \theta^2}{1 + K^2/2} \right) \quad (31)$$

and looking about the exact location of the k 'th harmonic, i.e., at $\omega/\omega_1(0) = k$, gives

$$\text{sinc}^2 \left(\frac{N \pi \omega}{\omega_1(\theta)} \right) \approx \text{sinc}^2 \left(\frac{N \pi k \gamma^2 \theta^2}{1 + K^2/2} \right). \quad (32)$$

To determine the angular opening of the radiation of the k 'th harmonic we equate θ^2 to $2\sigma_\theta^2$, where θ is the angle at which $\text{sinc}(x)$ has its first zero or equivalently fit the function to a Gaussian and require that after integration the intensity be the same, i.e.,

$$\text{sinc}^2\left(\frac{N\pi k \gamma^2 \theta^2}{1 + K^2/2}\right) \sim \exp\left(\frac{-\theta^2}{2\sigma_{r'}^2}\right) \quad (33)$$

yielding,

$$\sigma_{r'} \sim \frac{1}{2\gamma} \sqrt{\frac{1 + K^2/2}{kN}} = \sqrt{\frac{\lambda}{2L}} \bigg|_{\substack{\text{forward direction} \\ \theta=0}} \quad (34)$$

where $L = N\lambda_o$ is the length of the undulator and λ the wavelength of the radiation. We find a vertical opening angle that is about $\sqrt{kN}$ smaller than that from a bending magnet.

The photon source is found from the requirement of constant phase space area,

$$\sigma_r \sigma_{r'} = \frac{\lambda}{4\pi} \Rightarrow \sigma_r = \frac{\sqrt{\lambda L}}{2\sqrt{2}\pi}. \quad (35)$$

We now examine the dependence of the line width as a function of the beam parameters [22]. Taking the limit of $N \rightarrow \infty$,

$$\text{sinc}^2\left(\frac{N\pi\Delta\omega}{\omega_1(\theta)}\right) \rightarrow \frac{1}{N} \delta\left(\frac{\Delta\omega}{\omega_1(\theta)}\right) = \frac{1}{kN} \delta\left(\frac{\Delta\omega}{\omega - \Delta\omega}\right) \sim \frac{1}{kN} \delta\left(\frac{\Delta\omega}{\omega}\right) \quad (36)$$

The natural line width for the k 'th harmonic is then,

$$\frac{\Delta\omega}{\omega} \bigg|_{\theta=0} \sim \frac{1}{kN} \quad (37)$$

The radiation also depends on the viewing angle θ and electron energy γ . From Eq. (28),

$$\omega = \frac{\omega_0 2k \gamma^2}{(1 + K^2/2 + \gamma^2 \theta^2)} \quad (38)$$

the spread in frequencies when the radiation is viewed through an aperture $\Delta\theta$ is,

$$\frac{\Delta\omega}{\omega} \bigg|_{\theta \sim 0} \sim \frac{\gamma^2 \Delta\theta^2}{1 + K^2/2} \quad (39)$$

Because of the θ^2 dependence of ω_k the peaks are shifted to lower energies when the radiation is viewed off axis. The energy γ of the radiating electron also affects the spread in frequencies and is given by,

$$\frac{\Delta\omega}{\omega} \bigg|_{\theta \sim 0} = 2 \frac{\Delta\gamma}{\gamma} \quad (40)$$

The full fractional bandwidth of the k 'th harmonic is then given by the convolution of these three effects,

$$\left(\frac{\Delta\omega}{\omega}\right)_{\Delta\theta, \Delta\gamma} \sim \left[\left(\frac{1}{kN}\right)^2 + \left(\frac{\gamma^2 \Delta\theta^2}{1 + K^2/2}\right)^2 + 4\left(\frac{\Delta\gamma}{\gamma}\right)^2 \right]^{1/2}. \quad (41)$$

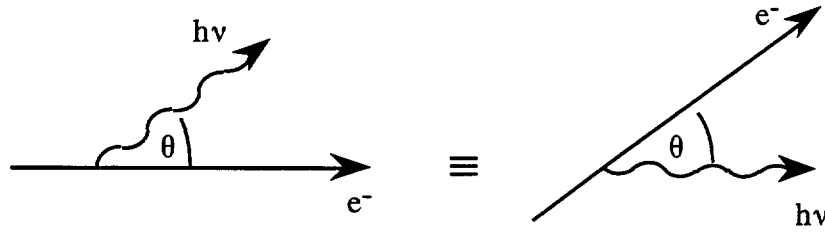
Since the brightness is defined per bandwidth $\Delta\omega/\omega$, we find a reduction in the brightness for fixed bandwidth if $\Delta\theta$ and $\Delta\gamma \neq 0$. The peak brightness is reduced by,

$$B = B_0 \frac{(1/kN)}{(\Delta\omega/\omega)_{\Delta\theta, \Delta\gamma}} \quad (42)$$

If we ignore the effect of the energy spread $\Delta\gamma/\gamma$, we note that to keep the line width comparable to the natural line width, we require the condition that the acceptance angle be comparable to the natural photon divergence,

$$\Delta\theta \sim \sigma_{\gamma'} \quad (43)$$

However since it is not possible to distinguish the source of the photons accepted in an angle $\Delta\theta$, the electron beam emittance is equivalent to the acceptance angle, see the figure below.



We can therefore write,

$$\left(\frac{\Delta\omega}{\omega}\right)_{\Delta\sigma_{x,y}, \Delta\gamma} \sim \left[\left(\frac{1}{kN}\right)^2 + \left(\gamma^2(\sigma_{x'}^2 + \sigma_{y'}^2)\right)^2 + 4\left(\frac{\Delta\gamma}{\gamma}\right)^2 \right]^{1/2}. \quad (44)$$

This last equation indicates a loss of energy resolution if either the beam emittance deteriorates or an increase in energy spread occurs. An increase in the emittance can arise from orbit instabilities whilst beam instabilities, notably longitudinal multibunch, will affect the energy spread. The expression also shows that the angular divergence should be kept small, however, since $\sigma_{y'} = \sqrt{\epsilon_y/\beta_y}$, $\sigma_{y'}$ should not be too small since a large β may adversely affect the gas-scattering lifetime (see section 5).

Typical values of the energy variation given by the natural relative energy spread of the ring are a few 10^{-4} to 10^{-3} rms. However, instabilities, especially multibunch, can increase this to 10^{-2} with loss of bandwidth and therefore brightness. Furthermore if spurious dispersion exists in the straight section the increased energy spread leads to a larger transverse beam size. The control of multibunch instabilities is one of most important aspects of present-day synchrotron radiation storage ring operation (see section 9).

A finite emittance causes broadening of the undulator lines and a reduction in the flux at fixed frequency. For machines of comparable emittance, the effect is greater on the higher energy machine since,

$$\sigma_{r'} \propto \frac{1}{\gamma}. \quad (45)$$

The following sections will overview aspects which affect the brightness via either a flux reduction or an increase in beam size and divergence. Flux is reduced if the stored current is reduced, in other words a long lifetime is desirable. It is also reduced if the photon beam passing through an entrance pinhole is miss-steered. To maintain the brightness the stability of the electron beam has to be guaranteed since beam movement is translated into an effectively increased beam emittance.

3. STABILITY [10, 11, 23–25]

3.1 Beam motion with respect to small apertures

A beam line is made up of many components that restrict positional and angular motion. These components are masks, collimators, mirrors, slits, monochromators and the experiment and detector. Not all experiments need high brightness some may be interested in high flux, however, beam motion is still detrimental to both types since beam line functionality is compromised. Intensity changes as small as 0.1% degrade measurement resolution if they occur over the time scale of a measurement. Electron beam orbit miss-steering can miss samples if these are very small and micro-radian angle changes affect the transmission energy of diffraction gratings or crystal monochromators thereby reducing the spectral resolution. If the time scale for the variation is faster than the measurement time the beamline user sees a resolution error, otherwise they suffer a calibration error. We can consider two cases: A focused beam configuration where the source is imaged on a sample and affected mostly by position stability and an unfocused configuration determined by angle stability.

For the first case the photon beam is focused on slits or a pinhole which is matched to the beam size or smaller and defines a fixed source position for downstream optics. We will consider the intensity of the photon beam that is described by a two dimensional Gaussian distribution with equal dimensions in both planes. The fraction of beam contained in radius r is, $1 - \exp(-r^2/2\sigma^2)$. A circular aperture of radius 3σ will then contain 98.9% of the incident flux (a radius of $\sqrt{6}\sigma$ contains 95%). If this beam ($\pm 3\sigma$) is miss-steered by 10% of its rms value at a horizontal slit we find a reduction of the integrated intensity of 0.01%. The reduction is more pronounced if the slit is smaller, say $\pm 2\sigma$, then a 0.1σ miss-steered beam suffers a reduction in intensity of 0.1%. If the photon beam size is dominated by the electron emittance, a fixed 10% increase in beam size ($\sim 20\%$ ϵ) corresponds to a reduction of 1/75 (1.3%) in intensity for a circular aperture.

For an unfocused beam the divergence will dominate. At a slit positioned L meters away from the source, the beam size given by transformation of the photon ellipse, is [25]:

$$\Sigma_y^2(L) = \Sigma_y^2(0) + L^2 \Sigma_{y'}^2(0) \quad (46)$$

The expression shows that at large distances intensity fluctuations due to positional changes will be smaller than those caused by angle changes. A 10% change in angle translates to a 10% change in beam size. Changes in angle also result in changes in the energy of photons transmitted by a monochromator. In general keeping the beam angle constant to within a few percent of the beam divergence will be tolerable.

The simultaneous constraints of keeping the angle and position stable to 10% of the beam size can be combined by requiring that the effective beam emittance averaged over all movements be no more than 10% of the ideal photon emittance. This constraint can be relaxed if the time to make the averaging is long compared to the time to perform a

measurement. Closed orbit errors displace the beam's centre of mass. If the time scale of the disturbance is greater than the time it takes to make an experiment the result is a simple miss-steering. This will manifest itself as energy calibration errors and loss of brightness. If the time scale of the disturbance is less than the time to make an experiment the beam is blurred in phase space [10].

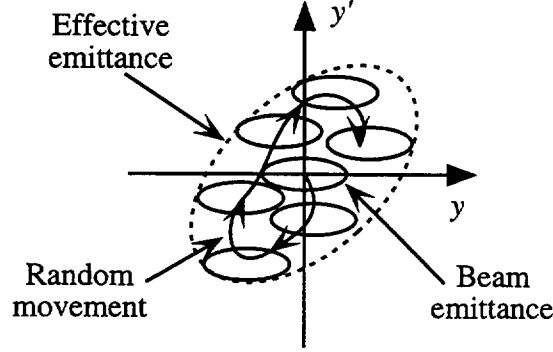


Fig. 1 Random closed orbit movements in angle and position results in an effective increase of the beam emittance.

Time varying field errors and magnet misalignments (section 3.3) which distort the closed orbit will generate at a source point a time varying closed orbit position having randomly distributed orbit amplitudes and angles with respect to the ideal machine. These amplitudes and angles we take to have a Gaussian distribution. The beam dimensions and the amplitude of closed orbits are uncorrelated and can be added in quadrature to get an effective beam dimension,

$$\sigma_{eff}^2 = \sigma_{\beta}^2 + \sigma_{c.o.}^2. \quad (47)$$

The closed orbit distribution is also determined by the betatron function, in which case we can write,

$$\varepsilon_{eff.} = \varepsilon_{\beta} \left(1 + \frac{\Delta\varepsilon}{\varepsilon} \right) \quad (48)$$

where

$$\frac{\Delta\varepsilon}{\varepsilon} = \frac{\sigma_{c.o.}^2}{\sigma_{\beta}^2} = \frac{\beta \varepsilon_{c.o.}}{\beta \varepsilon_{\beta}} \quad (49)$$

We can utilise this result for fast orbit fluctuations compared to the observation time and find if we fix the maximum emittance increase to be 10% that the corresponding acceptable closed orbit distortion is ~32% of the beam size. Take for example the situation at ELETTRA which has a vertical beam dimension of 13 μm at the insertion straight (β_y at ID is 2.6 m with 0.2 nm-rad emittance for 1% coupling at 2.0 GeV). To have 10% emittance stability the closed orbit variation has to be kept within 4 μm which is comparable to the resolution of storage ring BPM's ~ 2 μm .

3.2 Sources of instability

Stability requirements in a third generation light source are particularly stringent simply from the fact that the source sizes are small. The use of strong focusing also exacerbates the situation since the strong magnetic fields lead to enhanced amplification of errors (see below). The sources of beam instability are many and occur at varying frequencies. In this section we consider frequencies up to a roughly a few hundred Hz that are driven by

mechanical or electrical stability of the accelerator components. We distinguish three rough time scales for the movements.

- Long term – months to years [24, 26]: Mostly caused by seasonal temperature changes (building/ground expansion/shrinking), seismic motion (tectonic motion, earthquakes, tidal motion) and ground settlement (addition of shielding, ground water levels).
- Medium term – hours to days [27-29]: The main sources of disturbance are temperature effects (on component support systems of magnets, mirrors, chambers, etc., air conditioning, magnet cooling water fluctuations and diurnal changes), cultural noise (nearby traffic) and cranes and heavy machinery. We include here also reproducibility effects which are determined by magnet hysteresis after cycling and storage ring energy ramping and orbit changes which may arise from chromaticity or tune variations to combat beam instabilities.
- Short term – fractions of seconds to minutes [9, 23, 28, 30-32]: These disturbances are driven by power supply ripple and stability, booster synchrotron cycling, mechanical vibration (cooling water turbulence, vacuum pumps, compressors, traffic), ground waves, beam instabilities driven by coupled bunch oscillations and ion trapping and, most importantly, insertion device gap movements. Undulators are tuned by changing the gap of the device thereby changing the energy of the emitted photons. However the devices introduce distortions to beam optics and the closed orbit. Orbit distortions are principally due to non-zero first and second field integrals through the device. Well constructed insertion devices are possible with small field integrals (the measured fields at the limit of measurement techniques), however, these integrals still affect the electron beam. The situation is further aggravated by the fact that many insertion devices will be used and gap changes performed at random times and manners. Wigglers in general operate at a fixed gap and will predominantly affect the optics.

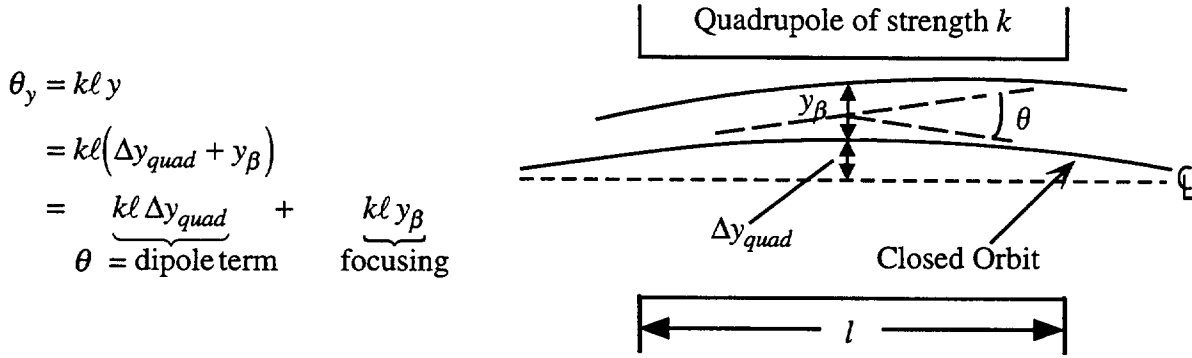
The above mentioned instabilities can be cured or alleviated in many ways. We distinguish between passive and active correction or compensation.

- Passive: This can be done firstly by the proper choice of site and distance from cultural noise. A rock ground base is good against settling but may couple vibrations. Micro fractures in the rock structure will help damp ground vibrations. A clay base will damp vibrations but also allows settling (the ESRF and ALS have $\sim 100\mu\text{m}/10\text{m}/\text{year}$). Experimental and accelerator areas can be separated from each other to limit interference. We can also separate experimental and accelerator areas from structures that can move (building and ground movement from seasonal/diurnal temperature changes). Mechanical pumps/compressors etc. are isolated from experiments and accelerators using vibration attenuation materials. Temperature effects are regulated and controlled (most materials have expansion coefficients in the range 10 to $30 \times 10^{-6}/^\circ\text{C}$). Homogeneous and regulated air-conditioning (no turbulence) with stabilisation to $\pm 0.1^\circ\text{C}$ is used in accelerator tunnels. Careful control of temperature is also essential in the experimental floor and beamline hutches. Vibration attenuation material is used in magnet and beam line support systems. The correct choice of power supply specifications ensures acceptable levels of stability and ripple. With regard to insertion device effects, these can be minimised by proper construction of the device with the adoption of appropriate techniques when measuring the field of the magnetic blocks, by careful sorting and by the application of shimming algorithms to minimise the field integrals and to boost the emission of the higher harmonics via a reduction of phase errors.
- Active: Regular measurement and alignment of magnets is essential but is a time consuming task especially if the accelerator has many magnets. The ESRF has adopted a dynamic alignment system whereby ground settling can be compensated by using hydrostatic levelling and motorised jacks on magnet girders and front ends. The alignment of the machine is done with a stored beam and takes two hours to perform. On a shorter time scale the most powerful technique is closed orbit feedback using either electron beam and/or

photon beam information. Feedforward techniques are used when behaviour of the beam distortion is known beforehand, for example when gap changes of insertion devices are performed. In this case storage ring steerer magnets or insertion device correction coils or rotating magnet blocks are utilised. Feedforward may also be applied to compensate the main vibration modes of quadrupole support systems.

3.3 Closed orbit distortions

External noise from whatever source ultimately couples to the beam via the magnetic fields of the magnets. In an ideal machine the magnetic centres of the magnets coincide with the closed orbit of the stored beam. If a magnet is displaced from this ideal orbit the beam centroid will experience a force. For example, a quadrupole displaced Δy_{quad} (by a vibration) will deflect the closed orbit in proportion to the displacement, see Fig. 2. The closed orbit trajectories follow the betatron motion. The kick produced by the quadrupole is termed a feed down effect. Magnets with higher multipoles will also produce these effects, although weaker (e.g., a sextupole plus an equivalent beam displacement gives dipole, quadrupole and sextupole terms).



$$\begin{aligned}\theta_y &= k\ell y \\ &= k\ell(\Delta y_{quad} + y_\beta) \\ &= \underbrace{k\ell \Delta y_{quad}}_{\theta = \text{dipole term}} + \underbrace{k\ell y_\beta}_{\text{focusing}}\end{aligned}$$

Fig. 2 A transverse displacement of the quadrupole from the ideal position generates a dipole kick which in turn causes a closed orbit distortion.

Magnets are positioned to a finite alignment accuracy. The position accuracy is typically little better than 100 μm and for angles a few tenths of a milli-radian. Misplacement of magnets leads to tilts, rotations and displacements which will distort the closed orbit, couple the beam oscillation planes, distort the optics and generate spurious dispersion. The spurious dispersion is particularly damaging as it modifies the natural emittance giving an enhanced contribution to the vertical beam dimension and additionally increases the beam size via the beam energy spread. The sensitivity of a machine to spurious dispersion depends on the lattice [3]. Given the low geometric coupling of third generation light sources ($\sim 1\%$) the vertical uncorrected spurious dispersion accounts for most of the vertical beam size. Correction of the spurious vertical dispersion significantly improves the photon source [33, 34]

To compute the effect of magnet displacements we consider in a perfectly aligned machine one quadrupole which is misaligned giving a deflection θ . Using the one turn transfer matrix $\mathbf{M}$ [15],

$$\mathbf{M} = \begin{pmatrix} \cos \psi + \alpha \sin \psi & \beta \sin \psi \\ -\gamma \sin \psi & \cos \psi - \alpha \sin \psi \end{pmatrix} \quad (50)$$

where α , β , γ are the Courant-Snyder parameters at the quadrupole and ψ the total phase advance for a complete turn ($\psi = 2\pi Q$ with Q the tune), we propagate the orbit (y, y') by one turn just downstream of the kick. The orbit is then closed by adding the kick to get back to (y, y') ,

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \mathbf{M} \begin{pmatrix} y \\ y' \end{pmatrix} + \begin{pmatrix} 0 \\ \theta \end{pmatrix} \quad (51)$$

Solving for the orbit position and angle at the quadrupole we obtain,

$$\begin{pmatrix} y \\ y' \end{pmatrix} = (1 - \mathbf{M})^{-1} \begin{pmatrix} 0 \\ \theta \end{pmatrix} = \frac{\theta}{2 \sin \pi Q} \begin{pmatrix} \beta \cos \pi Q \\ \sin \pi Q - \alpha \cos \pi Q \end{pmatrix} \quad (52)$$

The displacement y at this point can then be propagated elsewhere in the ring,

$$y(s) = \frac{\theta \sqrt{\beta(s) \beta}}{2 \sin \pi Q} \cos(|\psi(s) - \psi| + \pi Q) \quad (53)$$

For a sum of dipole kicks the closed orbit, for a linear lattice, is a linear superposition of the above equation. For a lattice containing sextupoles or other non-linear elements the solution depends on the orbit itself and requires iteration to be found. The orbit for N displaced quadrupoles is then,

$$y(s) = \sum_{i=1}^N \theta_i \frac{\sqrt{\beta(s) \beta_i}}{2 \sin \pi Q} \cos(|\psi(s) - \psi_i| + \pi Q) \quad (54)$$

The expectation value for a sum of statistically independent kicks is given by the rms $\sqrt{\langle y^2 \rangle}$,

$$\langle y^2(s) \rangle^{1/2} = \frac{\sqrt{\beta(s)}}{2\sqrt{2} \sin \pi Q} \sqrt{\sum_i^N \langle \theta_i^2 \beta_i \rangle} \quad (55)$$

For statistically distributed kicks, the central limit theorem states that the distribution of orbit distortions will be Gaussian with a standard deviation given by the rms. ($\sigma^2(s) = \langle y^2(s) \rangle$). For a storage ring with N uncorrelated quadrupole displacements with a distribution σ_{quad} , having a betatron function β at the quadrupoles with strength k the expected orbit distortion is:

$$\sigma_y(s) = \langle y^2(s) \rangle^{1/2} = \sigma_{quad} A(s) \quad (56)$$

where

$$A(s) = \sqrt{\frac{N \beta(s) \langle (k\ell)^2 \beta \rangle}{8 \sin^2 \pi Q}} \quad (57)$$

The quantity A is termed the error amplification factor. Typical A_x values found are, ELETTRA ~ 40 , ESRF ~ 100 , Spring8 ~ 110 , therefore quadrupoles aligned to a reasonable value of ~ 0.1 mm will lead to closed orbit distortions of up to 10 mm. This is not the maximum which could be up to twice this. The amplification factor can be used to find the maximum tolerable amplitude of vibration, for example in ELETTRA, at the ID straight with a coupling of 1% the beam size is 13 μm (2.0 GeV and 7 nm-rad emittance). The vertical amplification factor at that point is 12 and to keep the beam stable to 1/10th of the beam size requires quadrupole vibration amplitudes to be less than 0.1 μm .

3.4 Correction of closed orbit distortions [35, 36]

The correction of the closed orbit due magnet misalignments can be done in a straight forward manner using corrector magnets. The closed orbit is detected by beam position monitors (BPM's) placed about the ring. Carefully constructed ring BPM's have a relative resolution of a few microns. The correction of the closed orbit is performed by powering the correctors to flatten the beam at the BPM's. The most naive correction method is as follows,

- Measurement of the closed orbit at the N BPM's gives a vector of displacements,

$$\mathbf{y} = (y_1, y_2, y_3, \dots, y_N) \quad (58)$$

- The M correctors are powered to give deflections θ_k which change the orbit at the monitors,

$$\Delta y_i = \frac{\sqrt{\beta_i}}{2 \sin \pi Q} \sum_{k=1}^M \theta_k \sqrt{\beta_k} \cos(|\psi_i - \psi_k| + \pi Q) \quad (59)$$

- The kicks to be applied at each corrector is then found from,

$$\theta = -\mathbf{M}^{-1} \Delta \mathbf{y} \quad (60)$$

The elements of $\mathbf{M}$ are given by Eq. (59). The solution is exact for $N = M$, otherwise it is under or over determined. This method can be generalised to correct to an arbitrary orbit, usually known as the "golden orbit". Such an orbit will be one that provides the best conditions for the beamline users. The effectiveness of the correction is related to the performance of the monitors, the positioning of the correctors, and the number of both of these which are available. To perform a correction several numerical methods are used to find the optimum corrector strengths that will minimise the kicks to be applied whilst at the same time guaranteeing the best correction. Some of these methods are: Singular value decomposition (SVD), harmonic analysis and least squares techniques [35].

It is extremely useful to have the means to locally alter the position and/or slope of the closed orbit without disturbing the orbit elsewhere in the ring. For example the correction of the source point in an insertion device straight section to fix the photon beam on an experiment or on the entrance slits of a beam line. The local adjustment of angle and position requires a minimum of four correctors, two to define the position and slope and two to close the bump [37]. The monitors may either be in the ring or in the beamline front end. In the former case the monitors measure the electron beam position whilst in the latter case the monitors measure the position of the photon beam. Photon BPM's are made to intercept the outer edge of the synchrotron light profile by two or more pairs of tungsten blades [38, 39]. The beam position in the monitor is calculated from the difference in photo-emission currents between the blades. Photon BPM's have very good resolution of the order of a few microns or less. This means that two monitors separated by 10m will detect sub micro-radian orbit changes. They do, however, suffer from the drawback when used to measure insertion device radiation that they also detect dipole radiation which makes the signals gap dependent [40].

The most powerful methods to guarantee stability of the photon beam on the experiment is electron beam feedback either global or local and generally a combination of both [10, 11, 41–43]. A local orbit feedback system may be built up using storage ring correctors and photon BPM's or storage ring BPM's or a combination of both monitors. Given that data acquisition at the experimental stations can be taken at several tens of Hz for kinetic experiments [44] and envisaging faster data acquisition in the near future the frequency range to correct is typically up to 100 Hz and higher, (experiments with even faster acquisition rates suffer less since the amplitude of the disturbance decreases with frequency

[45]). For the system to be effective the following conditions have to be satisfied or taken into consideration:

- The system must not couple to adjacent systems [46].
- Magnet system sensitivity should be better than $1 \mu\text{rad}$. Magnets are specially constructed for fast linear response. Air core magnets are faster and more linear but weaker than iron core magnets which, however, suffer from hysteresis effects.
- The positions of the monitors with respect to the beam determine the sensitivity and have to be appropriately positioned to give the necessary resolution [10]. With regard to photon BPM's their location should not be too close to the source which would otherwise lower the resolution.
- Due to the quickly varying magnetic fields eddy currents are set up in the chamber walls which produce phase lags. For stainless steel the attenuation is small up to 100 Hz whereas for chambers made of aluminium it is strong at 10 Hz. The attenuation can be compensated but the system requires increased power.
- The system must have interlocks incorporated to protect the vacuum chamber against beam miss-steering.

4. THE APERTURE [47]

4.1 Introduction

We now address the important topic of machine aperture which defines the space available for the storage of the circulating electrons. It is clear that to have a long lifetime, i.e., to maintain the brightness, adequate aperture has to be given to compensate effects which lead to particle loss. The aperture of the accelerator is not only defined physically by the vacuum chamber but also by the electromagnet fields which guide and accelerate the beam in addition to the unwanted fields from magnet imperfections. High brightness low-emittance light sources are highly non-linear machines [2, 3, 48, 49]. The non-linearity stems from the need to compensate the strong chromatic effects which are generated by the strong focusing used to obtain the low emittance. In addition, insertion devices are additional sources of non-linearities especially devices which are used to generate elliptically polarised light. Insertion devices also break the symmetry of a ring leading to the excitation of structure resonances. The non-linearity of the transverse guide fields limits the amplitude of stable motion and defines a dynamic aperture for both on and off-momentum particles. In the longitudinal plane the rf bucket size given by the accelerating voltage usually defines the aperture for off-momentum particles.

The question may be asked as to why we need a large aperture when the beam size is fractions of millimetres in cross-section? The answer is the need to efficiently accumulate a stored beam and to account for particle-particle interactions within the vacuum chamber. During injection particles execute large betatron oscillations which sample strong non-linear fields. The aperture must be sufficiently large to accommodate these particles and others which undergo large betatron oscillations caused by elastic collisions with residual gas molecules. The dynamic aperture for off-momentum particles has to contain those electrons which suffer energy changes because of interaction with other electrons in the beam or with residual gas molecules — the Touschek effect and inelastic gas collisions. Off-momentum particles execute synchrotron oscillations which lead to transverse displacements at points of non-zero dispersion in addition to the betatron oscillation amplitude. In general the aperture for off-momentum particles is less than that for on-energy particles.

4.2 Dynamic aperture and transverse acceptance [50, 51]

The Dynamic aperture is defined to be the maximum stable initial transverse amplitude in the presence of non-linearities. Non-linear fields will blow-up the betatron oscillation compared to a linear machine. The concept is schematically shown in Fig. 3 [47]. The

starting amplitude A_{initial} at which a particle goes to infinity, i.e., the particle has unbounded motion, is usually referred to the dynamic aperture, which can either be inside or outside the physical aperture. The physical aperture is represented in the figure by a horizontal line parallel to the initial co-ordinate axis. In the presence of non-linearities the betatron amplitude is increased compared to a linear machine. Particle loss always occurs at the physical aperture, however, this now occurs for smaller initial co-ordinates.

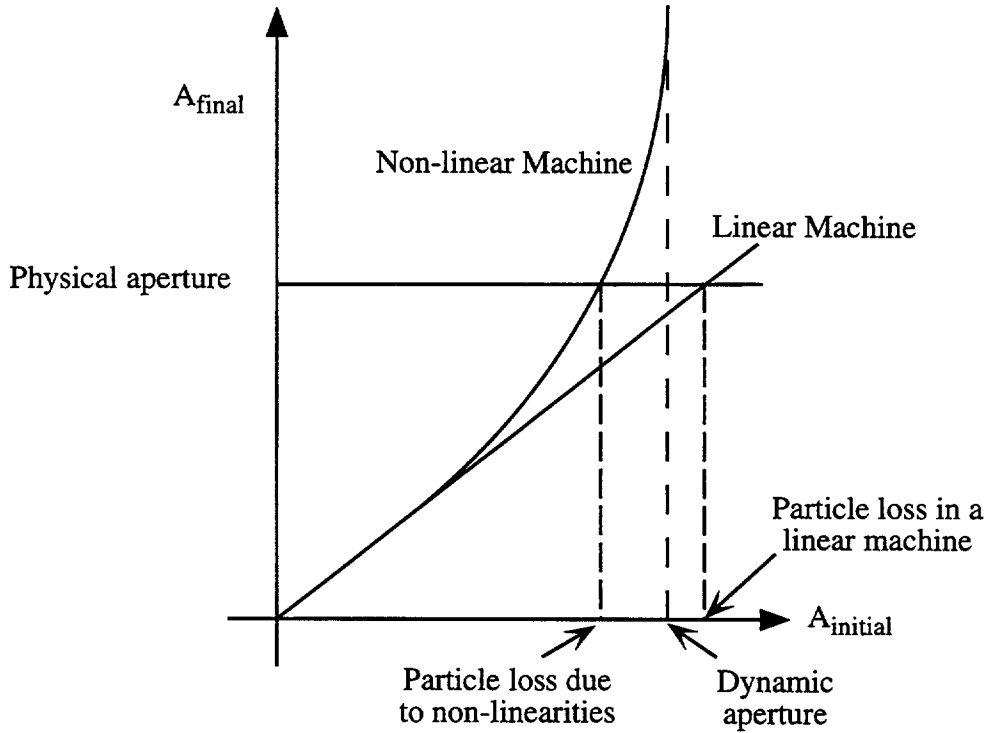


Fig. 3 Schematic representation of the dynamic aperture which is given by the initial amplitude that approaches infinity, shown here to be less than the physical aperture [47].

The strong focusing quadrupoles lead to large chromatic effects which necessitates the use of sextupoles to compensate the chromaticity ξ (tune shift with off-momentum $\Delta Q = \xi \Delta p/p$). This is done to avoid the head-tail instability and to prevent the crossing of resonances for particles which suffer energy losses due to collisions (Bremsstrahlung or Touschek). The compensation is performed by sextupole magnets positioned at points with a non-zero dispersion function. The sextupoles, however, also introduce geometric aberrations and to alleviate the effect of the resonances driven by the chromatic sextupoles, additional sextupoles are also used to cancel some of these resonances, i.e., the harmonic sextupoles. The equations of motion in the presence of sextupoles define a phase space area which has quasi-linear motion close to the origin characterised by elliptical trajectories for the particles. For particles further away from the origin the motion becomes increasingly non-linear characterised by a distortion of the ellipses. At this point the oscillation frequency also shifts from a constant value to a quadratic function of the amplitude. At larger amplitudes islands of stability form which eventually become chaotic and beyond this point the particle is rapidly lost. In addition to the non-linearity of particle motion from sextupoles the unavoidable imperfections in the construction of storage ring magnets give residual magnetic multipole fields which also contribute non-linear terms to the equations of motion. These in turn further reduce the available dynamic aperture.

The equations of motion in the presence of insertion devices contain both linear and non-linear terms [52]. For a planar insertion device the main effects are in the vertical plane. The linear contribution gives additional focusing which distorts the periodicity of the lattice and leads to the excitation of structure resonances with consequent reduction in dynamic aperture. The non-linear term affects the large amplitude motion in a way similar to

sextupole magnets and is again associated with a reduction of the dynamic aperture. The main non-linear contribution is an octupole term. The effects are inversely proportional to the square of the beam energy and are more severe for low energy machines, however, larger machines typically have more devices installed. The finite pole width introduces horizontal field components (quadratic and higher fields) and also contributes to reducing the dynamic aperture. Elliptical wigglers have a strong effect on the dynamic aperture and those which are crossed impose restrictions on the horizontal physical aperture.

To determine the dynamic aperture numerical tracking of a number of particles is used. Several programs exist for this purpose [53]. Particles are typically tracked for several hundred turns to see their survival. Ideally the dynamic aperture should be outside the physical aperture. Figure 4 shows an example of the dynamic aperture in the presence of sextupole magnets and when in addition insertion devices are symmetrically and asymmetrically placed in the storage ring.

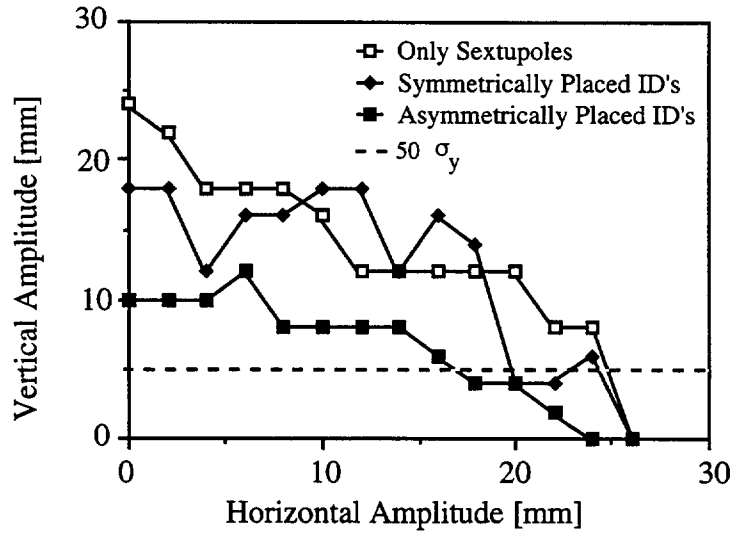


Fig. 4 Reduction in dynamic aperture when insertion devices are placed asymmetrically in the ELETTRA storage ring [54].

Recalling that a betatron trajectory has the following form[15],

$$x(s) = \sqrt{\beta_x \varepsilon_x} \cos(\psi(s) + \phi) \quad (61)$$

we see that if A is the aperture at any point in the ring, then A^2/β defines an emittance for that point. The minimum value of A^2/β anywhere in the ring is the acceptance which is also the largest emittance the storage ring can sustain. A particle whose trajectory is generated (e.g., by collisions) to lie outside this emittance will be lost. The acceptance can be limited either by the physical or dynamic aperture. The physical aperture is determined by the magnets and the vacuum chamber that has to stay inside them. In the vertical plane the physical aperture is generally determined by the smallest gap required for an insertion device. Typical values for the full internal gap at an insertion device is 10 to 20 mm. Future developments will push this down to 10 to 6 mm and perhaps lower. Injection into the storage ring normally proceeds in the horizontal plane and, as discussed below, sufficient aperture should be allowed for this procedure to be efficient.

4.3 The longitudinal acceptance [55]

In the longitudinal plane the acceptance is defined by the rf system and is a function of the rf voltage and the magnet lattice. A large aperture in this plane is needed to contain

particles which suffer energy losses, inelastic gas collisions and the Touschek effect, and to guarantee sufficient quantum lifetime. The rf voltage and the lattice delimit a potential energy well in which the electrons are bound.

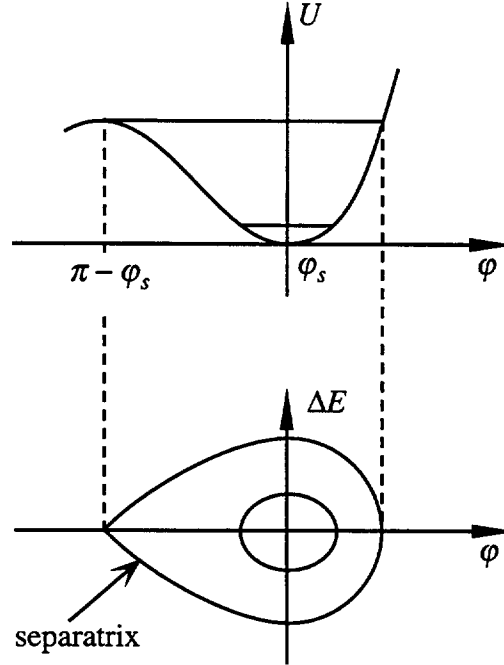


Fig. 5 The rf potential well and the bucket of stable longitudinal motion defined by it.

The Separatrix is the limit of stable motion. From the equation describing this curve the maximum energy deviation can be found by which we derive the maximum relative momentum acceptance,

$$\left(\frac{\Delta p}{p}\right)_{rf\,acc}^2 = \frac{eV \sin \psi_s}{\pi h \alpha_c E} 2 \left(\sqrt{q^2 - 1} - \arccos\left(\frac{1}{q}\right) \right) \quad (62)$$

where ψ_s is the synchronous phase, h the harmonic number, E the beam energy, V the accelerating voltage and q is the overvoltage factor ($q = eV/U = 1/\sin \psi_s$, with U the energy lost per turn). The momentum acceptance scales in proportion to the square root of the accelerating voltage. Typical values for the relative momentum acceptance are two to three percent, although machines are being proposed where this is being increased up to six percent [56].

We note that the energy acceptance has the momentum compaction [15] in the denominator which depends purely on the lattice parameters. The momentum compaction relates the path length of an electron in the ring to its energy deviation.

$$\frac{\Delta p}{p} \alpha_c = \frac{\Delta L}{L} \quad (63)$$

For strong focusing lattices, as used in third generation sources, the momentum compaction factor is a small number and thereby increases the momentum acceptance. A small momentum compaction factor, however, also leads to lower thresholds for beam instabilities (section 9) [4].

4.4 The injection process [57, 58]

To reach high beam intensities a beam is accumulated by repeated injection of particles from an injection system, which may be a booster synchrotron or a linac. Ideally the injection system should be full energy, thereby minimising the time it takes for a refill and increasing the reproducibility of the machine. Full energy injection, furthermore, has the advantage that top-off procedures may be applied whereby particle losses are exactly compensated for by many frequent injections performed during normal user operation. This is equivalent to having an infinite lifetime. At present no operating storage ring works in this mode, but future developments may make this option a reality.

The injection process which transfers particles from an injector is performed by a system of fast kicker magnets and septum magnets. The kickers move the already stored beam close to the septum magnet the moment a newly injected pulse arrives. The septum magnet then deflects the incoming bunch parallel to the stored beam without affecting the latter. The full process has the following three basic steps,

- (i) During the injection the stored beam is brought close to the septum sheet by using a closed orbit bump. The distance from the beam centre to the sheet should not be greater than half the beam dimension including the tail of the bunch distribution. Four to five standard deviations σ_{stored} of the beam size is considered acceptable.
- (ii) The injected beam follows the septum sheet at a distance of about six standard deviations σ_{inj} . The distance to the sheet is not critical since the incoming beam passes the sheet only once.
- (iii) The injected beam enters the storage ring off-axis and executes betatron oscillations, the maximum value of these oscillations occurs where the betatron function is largest. After a time to complete $1/\text{fractional}(Q_x)$ turns, the amplitude of the closed bump should be reduced to a value that does not allow the injected beam to be lost on the septum sheet. After about 3 damping times the injected pulse will have merged with the stored beam.

The minimum horizontal half aperture A_x defined by the injection process must satisfy the following condition,

$$\frac{A_x}{\sqrt{\beta_x}} = \frac{1}{\sqrt{\beta_{inj}}} \{5\sigma_{stored} + S_{eff} + 6\sigma_{inj}\} + \sigma_{c.o.} \quad (64)$$

where β_x is the betatron function at the position of the aperture, S_{eff} is the effective septum thickness including injection jitter and $\sigma_{c.o.}$ is the peak closed orbit error. Values of A_x are around 15 to 30 mm, if the dynamic aperture is less than this the injection efficiency is compromised. By not closing the bump completely, i.e., powering the downstream kickers more than the upstream ones a more efficient injection can be obtained at the expense of disturbing the closed orbit, since the injected beam will then have a smaller oscillation amplitude [57]. Before leaving this section we note that the horizontal aperture must also take into account Touschek scattered particles which have suffered energy changes due to intra-beam collisions (see section 8).

5. GAS SCATTERING [47, 59]

5.1 Introduction

Particles from the circulating beam are lost by scattering off the residual gas molecules in the vacuum chamber. The effect is controlled by providing sufficient pumping to reach

low vacuum pressures and by careful construction and preparation of the chamber to minimise photo desorption of gas molecules. There are two main single-particle effects:

- Elastic collisions:

Elastic collisions deflect the trajectories of the circulating electrons causing an increase of the betatron amplitudes. If the amplitude is large enough the scattered electron is lost either at the physical or dynamic aperture.

- Inelastic collisions:

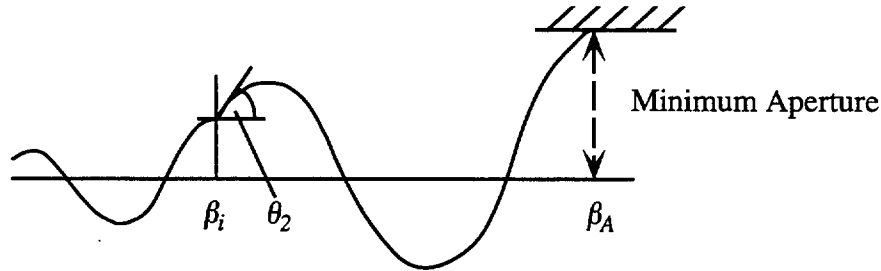
In this case an electron loses energy either by transfer to the residual gas molecule or by emitting a photon as it is deflected by the electric fields within the gas atom. This latter effect is called Bremsstrahlung. In both cases the electron is lost either at the rf acceptance limit or at the off-momentum physical or dynamical aperture limit.

5.2 Elastic collisions

5.2.1 The loss mechanism

A circulating electron suffering a deflection of θ_i radians will perform betatron oscillations given by [37],

$$u(s) = \theta_i \sqrt{\beta(s)\beta_i} \sin(\varphi(s) - \varphi_i) \quad (65)$$



where $\beta_i, \beta(s)$ are the betatron functions at the deflection point and at an arbitrary longitudinal position in the ring and $\varphi_i, \varphi(s)$ are the phases at these points.

The maximum amplitude is,

$$\text{Max } |u(s)| = A = \theta_i \sqrt{\beta_A \beta_i} \quad (66)$$

If the amplitude A exceeds the physical or dynamic aperture H , where H is the minimum value of A^2/β_A anywhere in the ring, the particle is lost. This occurs for deflections that satisfy,

$$\theta_i \geq \sqrt{\frac{1}{\beta_i} \left(\frac{A^2}{\beta_A} \right)_{\min}} = \sqrt{\frac{H}{\beta_i}} \quad (67)$$

5.2.1 The cross-section

The probability of a collision resulting in a deflection between θ and $\theta + d\theta$ per unit time is proportional to the differential cross-section $d\sigma/d\Omega$ for the encounter and the number of scattering centres per unit volume. We consider here only atomic nuclei as the scattering centres since electron-electron elastic collisions are much weaker [59]. The differential

cross-section for a collision off a residual gas nucleus is given by the classical Rutherford scattering formula,

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z r_0}{2\gamma} \right)^2 \frac{1}{\sin^4 \frac{\theta}{2}} \quad (68)$$

where r_0 is the classical electron radius,

$$r_0 = \frac{1}{4\pi\epsilon_0} \cdot \frac{e^2}{m_0 c^2} = 2.82 \cdot 10^{-15} [\text{m}] \quad (69)$$

Z the atomic number of the nucleus and γ the usual relativistic factor for the electron beam. Ω is the solid angle with $d\Omega = \sin\theta d\theta d\phi$, and θ the scattering angle with respect to the incident direction. The cross-section has to be corrected for two extremes, at small angles the electrons around the gas atom screen the nucleus, and at large angles the finite size of the nucleus has to be taken into account [60]. These effects do not significantly alter the results and are ignored here. To determine the loss rate of electrons we calculate the rate of collisions which lead to a deflection angle greater than a maximum $\theta_{\max}$ which defines the acceptance of the ring. Integrating $d\sigma$ from $\theta_{\max}$ to π gives,

$$\sigma_{\text{loss}} = 2\pi \int_{\theta_{\max}}^{\pi} \frac{d\sigma}{d\Omega} \cdot d\Omega = \frac{\pi}{2} \left(\frac{Z r_0}{\gamma} \right)^2 \cot^2 \left(\frac{\theta_{\max}}{2} \right) \quad (70)$$

since $\theta_{\max} = \sqrt{H/\beta_i}$ is generally a small angle and $\tan\theta_{\max} \sim \theta_{\max}$,

$$\sigma_{\text{loss}} = \frac{2\pi Z^2 r_0^2}{\gamma^2} \cdot \frac{1}{\theta_{\max}^2} = \frac{2\pi Z^2 r_0^2}{\gamma^2} \frac{\beta_i}{H} \quad (71)$$

Averaged over all positions in the ring, the cross-section is

$$\sigma = \frac{2\pi Z^2 r_0^2}{\gamma^2} \cdot \frac{\langle \beta \rangle}{H} = 2\pi r_0^2 \cdot Z^2 \cdot \frac{1}{\gamma^2} \frac{\langle \beta \rangle \beta_A}{A^2}. \quad (72)$$

The cross-section, or probability of loss, becomes, for a given aperture, smaller at higher energies and with smaller values of the betatron functions. It has also a strong dependence on the atomic number varying as Z^2 . The derivation for non-isotropic aperture restrictions can be found in Volume I of Ref. [21].

5.3 Inelastic scattering

5.3.1 The loss mechanism

There are two effects:

- 1) Bremsstrahlung scattering, where an electron emits a photon when it is deflected by the nucleus or the electrons within a residual gas atom and leaves the atom in an unexcited state.
- 2) Direct energy transfer from the colliding electron to the atom of the residual gas.

An electron which suffers an energy loss $\Delta E/E$ will be lost either at the transverse aperture limit (physical or dynamic) or at the rf limit. Transversely, if the energy loss occurs at a position with non-zero dispersion, the electron will perform betatron oscillations about a new off-momentum orbit (see section 7). In the longitudinal plane it will perform synchrotron oscillations.

5.3.2 The cross-section [61]

Only radiative losses (Bremsstrahlung) will be treated. For highly relativistic particles energy loss through atomic excitation and ionization of the residual gas have much smaller cross-sections compared to the radiative contributions [62].

The differential cross-section for energy loss from photon emission at the nucleus is,

$$\left(\frac{d\sigma}{d\varepsilon}\right)_N = \alpha \frac{4Z^2 r_0^2}{\varepsilon} \left\{ \left[\frac{4}{3} \left(1 - \frac{\varepsilon}{E}\right) + \frac{\varepsilon^2}{E^2} \right] \left[183 - \frac{1}{3} \ln Z \right] + \left[\frac{1}{9} \left(1 - \frac{\varepsilon}{E}\right) \right] \right\} \quad (73)$$

where α is the fine structure constant = $1/137$. The numerical constants within the brackets are related to screening of the nucleus by the atomic electrons [47]. The formula is valid for highly relativistic energies.

The differential cross-section for photon emission at a bound electron on a residual gas atom is (relativistic energies),

$$\left(\frac{d\sigma}{d\varepsilon}\right)_e = \alpha \frac{4Zr_0^2}{\varepsilon} \left\{ \left[\frac{4}{3} \left(1 - \frac{\varepsilon}{E}\right) + \frac{\varepsilon^2}{E^2} \right] \left[1194 - \frac{2}{3} \ln Z \right] + \left[\frac{1}{9} \left(1 - \frac{\varepsilon}{E}\right) \right] \right\} \quad (74)$$

We note that the Z dependence on the former goes as Z^2 , whilst for electron-electron interactions it goes as Z .

The total differential cross-section is given by the sum of the two (independent events). The total cross-section is obtained by integrating from ε_m , the lowest energy loss which results in particle loss, to E the highest energy loss.

$$\sigma(\varepsilon, E) = \int_{\varepsilon_m}^E \left(\frac{d\sigma}{d\varepsilon}\right) d\varepsilon \quad (75)$$

$$= 4\alpha r_0^2 \left\{ F(Z) \frac{4}{3} \left[\ln \left(\frac{E}{\varepsilon_m} \right) - \frac{5}{8} \right] + \frac{Z(Z+1)}{9} \left[\ln \left(\frac{E}{\varepsilon_m} \right) - 1 \right] \right\} \quad (76)$$

for $\varepsilon_m \ll E$, and

$$F(Z) = Z^2 \ln \left(\frac{183}{Z^{1/3}} \right) + Z \ln \left(\frac{1194}{Z^{2/3}} \right) \quad (77)$$

The cross-section shows a strong dependence of the atomic number of the residual gas species, but a weak dependence on the maximum energy acceptance which goes as $-\ln(\varepsilon_m/E)$.

5.4 Beam gas scattering lifetime

For independent events the total cross section is the sum of the cross-sections for the individual events, then,

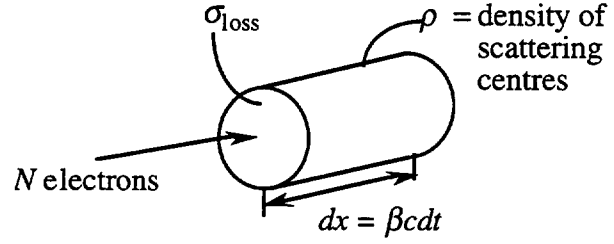
$$\sigma_{loss} = \sigma_{elastic} + \sigma_{inelastic} \quad (78)$$

The number of particles lost dN per unit time is proportional to the cross-section, the number of scattering centres and the number of incident particles.

$$dN = -N\rho\sigma_{loss}dx$$

(79)

$$\frac{1}{N} \frac{dN}{dt} = -\rho\sigma_{loss}\beta c$$



which leads to an exponential decay of the stored beam ($N = N_0 e^{-t/\tau}$), with a lifetime given by,

$$\tau = \frac{1}{\beta c \sigma_{loss} \rho} \quad (80)$$

The gas density is determined by the composition of molecules present. For a given molecular species i the gas equation gives,

$$\rho_i = \frac{p_i}{kT} \quad (81)$$

where p_i is the partial pressure, k the Boltzmann constant and T the absolute temperature. If each molecule is made up of different numbers of atoms then $\sigma\rho$ becomes:

$$\sum_{ij} \sigma_j \alpha_{ij} \rho_i \quad (82)$$

where α_{ij} is the number of atoms of type j in molecule i .

Desorption of gas molecules by synchrotron radiation is the main source of residual gas in light sources. The desorption depends on the circulating current (number of electrons N), therefore we write for ρ ,

$$\rho = \rho_0 + GN \quad (83)$$

G is a desorption coefficient [68]. We now have,

$$\frac{dN}{dt} = -\beta c \sigma_{loss} (\rho_0 + GN) N \quad (84)$$

giving

$$\frac{N(\rho_0 + GN_0)}{N_0(\rho_0 + GN)} = e^{-\rho_0 \beta c \sigma_{loss} t} \quad (85)$$

with N_0 the starting number of electrons. We define the lifetime to be the time it takes for the starting number of electrons to be reduced by $1/e$, and get,

$$\tau_e = \frac{1}{(\beta c \sigma_{loss} \rho_0)} \ln \left\{ \frac{e \rho_0 + G N_0}{\rho_0 + G N_0} \right\} \quad (86)$$

For $G = 0$ we are led to the previous result. For situations where desorption effects dominate $\rho_0 \rightarrow 0$, and,

$$\tau_e = \frac{e - 1}{\beta c \sigma_{loss} G N_0}. \quad (87)$$

The lifetime is inversely proportional to the initial current, which accounts for the increasing lifetime as the current decays.

The total lifetime is given by the sum of individual reciprocal lifetimes,

$$\frac{1}{\tau_{total}} = \frac{1}{\tau_{elastic}} + \frac{1}{\tau_{inelastic}} \quad (88)$$

Figure 6 shows as an example the individual effects predicted for the SOLEIL project [63].

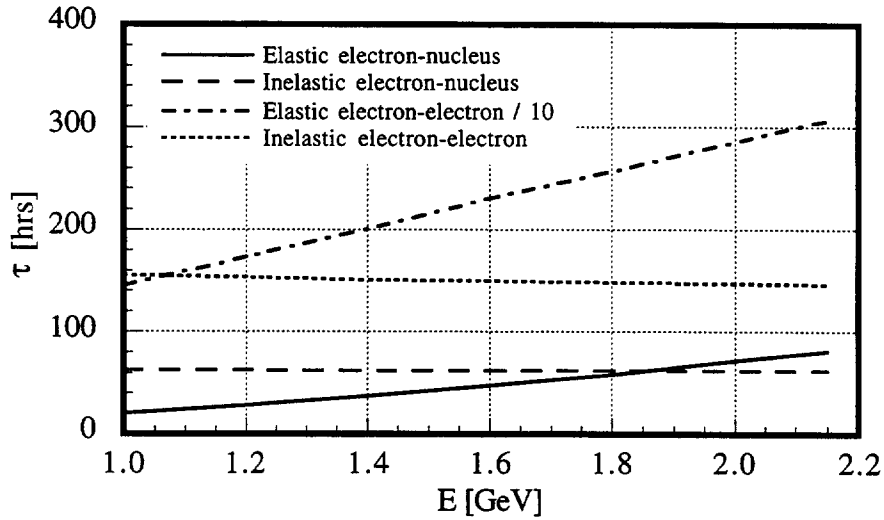


Fig. 6 Example of contributions to the lifetime from different beam-gas interactions. (Data taken from [63], 300 mA, 1 nTorr, $Z=7$, 10 and 16 mm vertical and horizontal half apertures.)

The aperture dependence of the elastic scattering contribution can be used to provide, via a scraper, useful information on machine operating conditions. If A is the distance from the beam centre to the scraper blade, then we can write the lifetime dependence as,

$$\frac{1}{\tau} = C_1 + \frac{C_2}{A^2} \quad (89)$$

where C_1 depends on effects which are independent of aperture (scraper position), and C_2 depends only on the elastic scattering loss on the scraper. The coefficient C_1 will contain contributions to the lifetime arising from inelastic and Touschek scattering events (see section 7). By measuring the lifetime as a function of scraper position dynamic aperture effects can be measured [2, 64, 65].

The use of low gap insertion devices to produce high brightness photon sources gives a limiting physical aperture at the insertion device. Of importance to all light sources is the minimum insertion device gap that can be tolerated without significant beam lifetime degradation [66]. Smaller gaps allow higher energy photons from the use of mini-undulators with smaller period lengths. Since the elastic gas scattering effect scales as the square of the beam energy a higher energy machine is less sensitive to the effect. As a comparison (ignoring betatron function differences) measurements with a scraper at the ESRF operating at 6.0 GeV have shown 24 hours lifetime (100 mA) with 4.6 mm full aperture (3×10^{-9} mbar)[67] whereas at ELETTRA at 2.0 GeV the lifetime starts to deteriorate at around 10 mm full aperture.

5.5 Gas desorption

To reduce the effects of gas scattering and ion trapping (see section 10) we require ultra high vacuum (UHV), which corresponds to pressures of less than 10^{-7} Torr. For long gas scattering lifetimes pressures of 10^{-9} Torr are required. The synchrotron radiation from the electron beam is responsible for the major part of the gas load in the vacuum chamber. Typical static pressures (without beam) are a few hundredth's of a nTorr, whereas dynamic pressures (with beam) are a few nTorr.

The main mechanisms for an increase in pressure with beam are:

- Electron stimulated desorption.
- Thermal heating.
- Beam induced rf fields — heating from Higher Order Modes (HOM).

Even after cleaning and careful preparation of the chamber surface, gas molecules are still found bound to the surface. The binding energy of adsorbed molecules is of order of a few eV. Synchrotron radiation impinging on the surface of the chamber releases these molecules and is the major contributor to the vacuum pressure in the chamber. A simple example taken from reference [68] reveals that a monolayer of N_2 released from the surface of a 10 cm^3 box would result in a pressure increase of $1.7 \cdot 10^{-2}$ Torr.

5.5.1 Electron stimulated desorption [68, 69]

The mechanism for gas desorption by synchrotron radiation is as follows:

- Molecular excitation levels are excited by photons in the range 2–100 eV. The synchrotron radiation X-rays produced by the electron beam are too energetic for this.
- Instead the X-rays cause emission of photo-electrons from the chamber surface (atomic core electrons) and the molecules are desorbed during emission.
- The photo electrons then couple to adsorbed molecules causing further electron-stimulated desorption.

Experiments show that the desorption is proportional to the photo-electron current which depends on the chamber material. Of importance is the gas desorption yield η defined as the number of molecules produced per incident photon,

$$N_{mol} = \eta N_{ph}. \quad (90)$$

Typical values for η are in the range 10^{-5} to 10^{-7} depending on the material of the chamber and its treatment. The desorption yield is a function of dose, i.e., the length of time the chamber has been exposed to radiation. The desorption decreases with the accumulated photon dose and new machines are scrubbed by synchrotron radiation before they reach the design vacuum pressures. Typical values for this conditioning time are around 50 to 100 Ahrs.

5.5.2 Thermal desorption

A second source of gas desorption arises from heating of the chamber material. Many molecules are weakly bound to the surface of the chamber. One of the most troublesome molecules is water which can form many surface molecular layers the last few of which are only weakly bound. Thermal excitation will release copious amounts of molecules and this fact can be used to treat the chamber, i.e., baking-out. An in-situ bake-out of the chamber is usually performed on new chambers to improve the base pressure before scrubbing with synchrotron radiation.

The desorption rate is given by,

$$\frac{dN_{mol}}{dt} = \nu N_{mol} e^{-E_b/kT} \quad (91)$$

with E_b binding energy (0.1 to 10 eV), ν a rate constant and N_{mol} the number of molecules/cm². The rate constant depends on the previous conditioning of the chamber. For baked stainless steel (250 °C),

$$\nu \sim 2 \cdot 10^{-12} \text{ Torr } \ell \text{ s}^{-1} \text{ cm}^{-2}$$

whereas for an unbaked chamber,

$$\nu \sim 10^{-10} \text{ to } 10^{-11} \text{ Torr } \ell \text{ s}^{-1} \text{ cm}^{-2}$$

Even with careful treatment of the chamber, the outgassing due to synchrotron radiation is very high. To overcome this problem the construction of the chamber geometry to collect synchrotron radiation in well defined points, where cooling and additional pumping is provided, is of fundamental importance both in terms of the photo-electron induced desorption and that due to thermal heating.

5.5.3 Beam induced radio-frequency fields [69]

An additional source of thermal desorption arises from the heating of the chamber by the beam-induced radio-frequency fields, i.e., Higher Order Mode (HOM) heating. The circulating beam leaves wake fields (see section 9) at discontinuities in the chamber, i.e., at bellows, valves, pumping slots, cavities, tapers and the like. The average power loss depends on the resistive part of the chamber impedance seen by the beam,

$$P_{rf,\parallel} = (e N_b)^2 n_b f_0 k(\sigma) \quad (92)$$

where eN_b is the charge per bunch, n_b the number of circulating bunches, f_0 the revolution frequency and k the loss parameter in Volts/m defined by the impedance [4]. The loss parameter depends strongly on the bunch length and is greater for shorter bunches which leads to greater heating of chamber components [70].

6. QUANTUM LIFETIME [3, 71–73]

6.1 Introduction

The distribution of the electrons in a storage ring bunch is principally determined by the containment forces, both transverse and longitudinal, and by events originating within the bunch itself. The main events originating within the bunch are the emission of synchrotron radiation and intra-beam scattering. Synchrotron radiation leads to two effects: a damping of

betatron and synchrotron oscillations and a growth of the oscillation amplitudes due to quantum fluctuations. The former depends on the average energy loss $\langle u \rangle$ while the latter depends on deviations of the loss about the average $\langle u^2 \rangle$. The number of photons emitted per electron per turn is a rather small number. For a single electron the number of emissions per meter is about 6.2 B[T] (with B the dipole field). This small number will give rise to noticeable statistical effects and leads to a broadening of the energy distribution due to quantum fluctuations proportional to $\langle u^2 \rangle$. In the transverse plane the emission of a quantum of radiation changes only the magnitude of the particle's momentum but leaves the position and slope the same. The physics, however, is different if there is dispersion or not at the emission point. If the dispersion is zero, the emission of a photon does not modify the amplitude of the orbit (ignoring the small angle changes $\pm 1/\gamma$). With non-zero dispersion, the position and slope still remain the same but the electron now has a new reference energy orbit $E = E_0 - u$ which results in an excitation of a new betatron amplitude. The amount of energy lost during a quantum event is small compared to the energy spread of the beam (i.e., in ELETTRA the critical energy is 3.2 keV whereas the natural energy spread is 1.6 MeV) implying small changes to the amplitude compared to the beam width. The change can happen at any phase of the oscillation and can lead to an increase or a decrease in amplitude depending on the distance between the original and final orbits which is given by the dispersion. The process is similar to a random walk.

The appearance of damping arises from the emission of synchrotron radiation which is proportional to $\langle u \rangle$ and the subsequent energy gain in the rf cavity. Damping in the longitudinal plane occurs because of the dependence of the energy loss on the particle's energy, a high energy electron loses more energy than a low energy electron. Consider two electrons with energies above E_a and below E_b an electron with an ideal energy E_0 . The energy lost by E_a will be greater than that of E_0 and that lost by E_b will be less than E_0 . The result is that the energy difference has been reduced. The energy deviation and the longitudinal co-ordinate of an electron with respect to the synchronous particle are conjugate variables implying that the longitudinal bunch dimension is also damped. To consider damping in the transverse plane we note that the energy lost from the emission of a photon results in a loss of both transverse and longitudinal momentum. The lost energy, however, is restored in the cavities as an increase in the longitudinal momentum whilst the transverse momentum remains the same. This results in a reduction of the slope with respect to the closed orbit and is equivalent to a damping of the transverse motion.

Intrabeam scattering, which will be dealt with in section 8, is an effect arising from multiple small angle electron-electron Coulomb collisions within the bunch. The collisions increase the "temperature" of the bunch by transferring energy from the longitudinal plane to the transverse plane. The sum of a multitude of small random variations in an electron's co-ordinates will by virtue of the central limit theorem will lead to a distribution which will asymptotically approach that given by Gaussian distribution independent of the initial electron's distribution. The phase space distribution of the electron bunch results from an equilibrium between radiation damping, quantum excitation and intra-beam scattering. The distribution is altered by the presence of the vacuum chamber or by the potential well of the rf system, the finite boundaries of which will lead to particle loss.

6.2 Quantum lifetime

The equilibrium established between quantum fluctuations and radiation damping results in a Gaussian phase space distribution for the electrons,

$$\frac{1}{2\pi\epsilon_x} \exp\left\{-\frac{1}{2\epsilon_x}(\gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2)\right\} dx dx' \quad (93)$$

where ε_x is the beam emittance. Particles in the distribution have varying phases and to see the effect of finite apertures on this distribution we need to examine the Courant Snyder invariant for a particular trajectory x, x' ,

$$W_x = \gamma_x x^2 + 2\alpha_x x x'^2 + \beta_x x'^2 = A^2, \quad (94)$$

The invariant is related to the maximum excursion of an electron by,

$$x_{max} = \sqrt{W_x \beta_x} \quad (95)$$

The position and slope with respect to the closed orbit expressed in terms of the invariant is,

$$\begin{aligned} x &= (W_x \beta_x)^{1/2} \cos \phi \\ x' &= -(W_x / \beta_x)^{1/2} (\sin \phi + \alpha_x \cos \phi) \end{aligned} \quad (96)$$

where ϕ is an arbitrary phase, and the phase space distribution can be now be written as,

$$h(W_x) dW_x d\phi = \frac{1}{2\pi \langle W_x \rangle} \exp\left(\frac{-W_x}{\langle W_x \rangle}\right) dW_x d\phi \quad (97)$$

satisfying

$$\int h(W_x) dW_x = 1. \quad (98)$$

The average value of W_x is $\langle W_x \rangle = 2\varepsilon_x$. In a stored beam of N electrons the number having values of W between W_x and $W_x + dW_x$ is,

$$dN = N h(W_x) dW_x = \frac{N}{\langle W_x \rangle} \exp\left(\frac{-W_x}{\langle W_x \rangle}\right) dW_x. \quad (99)$$

The distribution being Gaussian, in principle extends to infinity. Any restriction, however, be it physical or dynamic will truncate the distribution and lead to a constant particle loss at the tails. With reference to Fig. 7 without the restriction a stationary situation exists where the number of particles crossing an arbitrary point W_0 due to quantum excitation (process A) equals the number entering due to radiation damping (process B). With a restriction of the amplitudes at W_0 far away from the core, so as not to significantly alter the distribution, the number of electrons crossing W_0 and being lost (process C) will be very nearly the same as if there were no aperture i.e., $C \equiv A = B$.

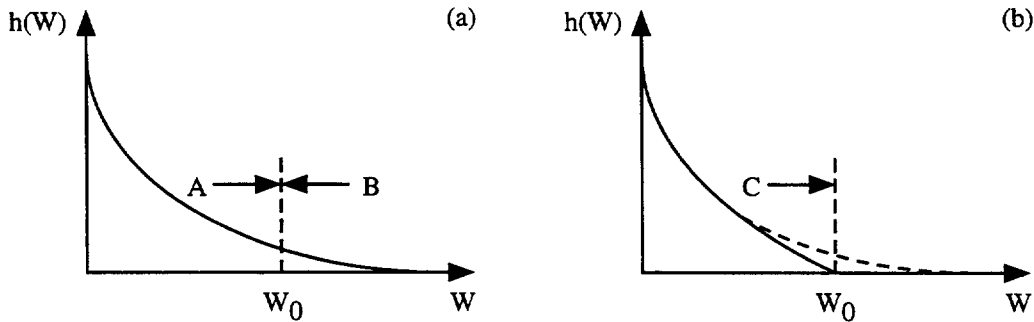


Fig. 7 Distribution of invariants with no aperture (a) and in the presence of a limiting aperture (b), taken from [72].

The loss rate can therefore be estimated as that due to process B, due to radiation damping. Evaluated at the restriction we have,

$$\left(\frac{dN}{dt}\right)_{W_0} = \left(\frac{dN}{dW} \cdot \frac{dW}{dt}\right)_{W_0} \quad (100)$$

and

$$dN = Nh(W)dW \Rightarrow \frac{dN}{dW} = Nh(W). \quad (101)$$

For a given electron the rate of change of W_x due to radiation damping is [3],

$$\frac{dW_x}{dt} = -\frac{2W_x}{\tau_x} \quad (102)$$

where τ_x is the radial damping time and

$$\frac{dN}{dt} = -N \frac{2}{\tau_x} \frac{W_0}{\langle W \rangle} \exp\left(\frac{-W_0}{\langle W \rangle}\right) \quad (103)$$

$$\Rightarrow N = N_0 \exp\left(\frac{-t}{\tau_q}\right). \quad (104)$$

the quantum lifetime τ_q is then found to be,

$$\begin{aligned} \tau_q &= \frac{\tau_x}{2} \frac{\langle W \rangle}{W_0} \exp\left(\frac{W_0}{\langle W \rangle}\right) \\ &= \frac{\tau_x}{2} \frac{\exp(\xi^2)}{\xi^2} \end{aligned} \quad (105)$$

where

$$\xi^2 = \frac{W_0}{\langle W \rangle}. \quad (106)$$

Assuming the limiting aperture occurs at a point with maximum betatron function β_{max} then,

$$\xi^2 = \frac{x_{max}^2/\beta_{max}}{2\sigma_x^2/\beta_{max}} = \left(\frac{x_{max}}{\sqrt{2}\sigma_x}\right)^2 \quad (107)$$

Because of the exponential factor the quantum lifetime increases rapidly with x_{max}/σ_x . The above relation for the quantum lifetime is valid for $x_{max} \gg \sigma_x$. As x_{max} approaches σ_x the assumption that the restriction does not alter the distribution is no longer valid. In fact the expression gives a minimum lifetime of $\tau_x/2$ at $x_{max} = \sqrt{2}\sigma_x$ which is incorrect since smaller apertures must give shorter lifetimes. A more complete derivation of the distribution using the Fokker-Planck equation yields the correct solution [74] from which we also find for $x_{max} < \sigma$ the approximate solution,

$$\tau_q = \tau_x \left(\frac{x_{\max}}{2.405 \sigma_x} \right). \quad (108)$$

For values of $x_{\max} > \sim 5\sigma_x$ accurate values of τ_q can be computed using Eq. (107) and some numbers are given in Table 1. For a damping time of 10 ms we find the "golden rule" of choosing $x_{\max}$ to be greater than $6.5\sigma_x$ in order to have a quantum lifetime of around 100 hours.

Table 1
Quantum lifetime as a function of limiting aperture

$x_{\max}/\sigma_x$	5	5.5	6.0	6.5	7.0
τ_q	1.8 min	20.4 min	5.1 hrs	98.3 hrs	103 days

In the horizontal plane the situation may be additionally more complicated. Finite dispersion can exist which will also give quantum lifetime contributions from synchrotron oscillations, i.e.,

$$x = x_\beta + D \frac{\Delta p_x}{p} \quad (109)$$

If the beam size given by betatron motion is comparable to the width of the beam due to the rms energy spread (σ_e/E), i.e.,

$$\sigma_\beta \sim D \left(\frac{\sigma_e}{E} \right) \ll W_x, \quad (110)$$

then, the lifetime is [74],

$$\tau_q = \tau_x \frac{\exp(\xi^2)}{\sqrt{2\pi} \xi^3} \cdot \frac{2\sqrt{2}}{(1+r)\sqrt{r(\beta-r)}} \quad (111)$$

where

$$r = \left(\frac{(D \sigma_e/E)^2}{\sigma_x^2 + D^2 \left(\frac{\sigma_e}{E} \right)^2} \right) \quad \text{and} \quad \xi = \frac{1}{\sqrt{2}} \frac{x_{\max}}{\sqrt{\sigma_x^2 + D^2 \left(\frac{\sigma_e}{E} \right)^2}} \quad (112)$$

The minimum value of τ_q occurs when $r = 0.640$ and,

$$\tau_{q,\min} \equiv \frac{\tau_x \sqrt{2}}{\xi^3} \exp\left(\frac{\xi^2}{2}\right) \quad (113)$$

The lifetime is lower than in the case with no dispersion, however, since the horizontal aperture must already be large enough to accommodate the injection process and to give adequate Touschek lifetime the quantum lifetime effect can be ignored in the horizontal plane.

In the longitudinal plane a Gaussian distribution also exists for the energy and phase deviations. Here the aperture restriction is the finite potential well given by the radio-frequency system. The oscillation "amplitude" is now defined by the invariant of the energy oscillations,

$$W = \varepsilon^2 + \left(\frac{E_0 \Omega}{\alpha_c} \right) \tau^2 = A^2 \quad (114)$$

With Ω the synchrotron oscillation frequency. We obtain a similar result for the quantum lifetime as for the transverse case,

$$\tau_q = \frac{\tau_\varepsilon}{2} \frac{\exp \xi^2}{\xi^2} \quad (115)$$

where τ_ε is the longitudinal damping time and

$$\xi^2 = \frac{W_0}{\langle W \rangle} = \frac{\varepsilon_{\max}^2}{2 \sigma_\varepsilon^2} \quad (116)$$

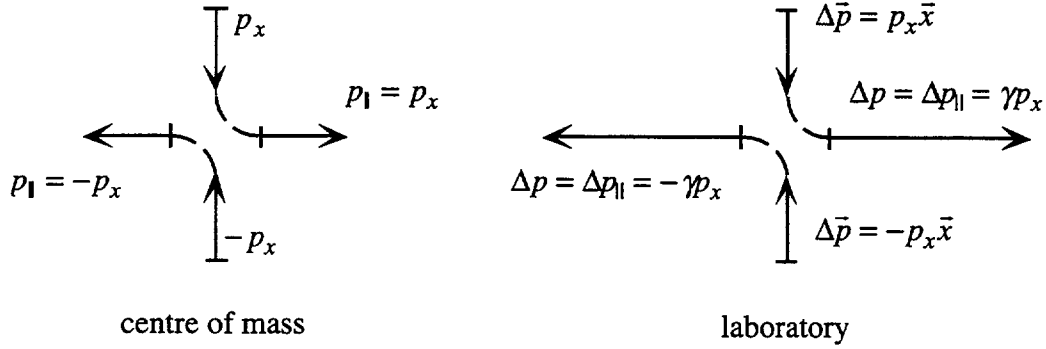
with σ_ε the rms energy spread and the maximum energy deviation defined by the rf system is given by Eq. (62). As for the radial case the rf acceptance is determined by the need to have an acceptable Touschek lifetime. Typical values for the acceptance are 2 to 3%. With relative energy spreads of the order a few 10^{-4} and damping times of the order of 10 ms the longitudinal quantum lifetime is practically infinite.

7. THE TOUSCHEK EFFECT [75–77]

7.1 Introduction

The high bunch densities present in low emittance high brightness machines leads to an enhanced rate of elastic collisions between electrons within the bunch. Collisions occur in all planes, however, the energy transfers involved from the longitudinal plane to the transverse are insufficient to generate a betatron oscillation capable of leading to particle loss. In this section we will only consider the processes that transfer sufficient energy from the transverse plane to the longitudinal for loss of both colliding particles. The change in the longitudinal momentum leads to particle loss if the momentum exceeds the rf acceptance or the transverse (physical or dynamic) acceptance. The Touschek effect is one of the limiting mechanisms for present day synchrotron radiation sources [78–79]. It was first explained by Bruno Touschek [75] after observations of the lifetime on the small storage ring ADA. Only recently has the effect become a critical issue in the construction and running of present day light sources having low emittance. The effect of longitudinal to transverse momentum transfer will be treated in section 8 along with the remaining collisions that generate many small momentum transfers that lead to an increase in the beam emittance.

To understand the mechanism we must consider the motion of the electrons in a frame which moves with them. The betatron motion in this frame is purely transverse and a collision will transfer momentum into the longitudinal plane. Transforming back to the laboratory frame the transferred momentum is boosted by a factor γ . The process is shown in the figure below [80].

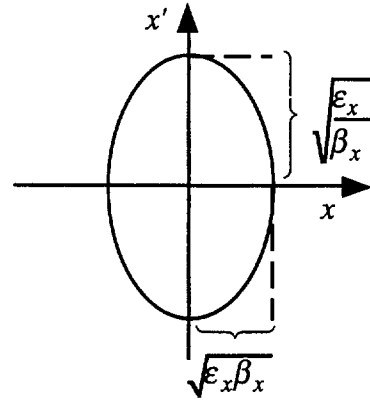


We can calculate in a qualitative manner the magnitude of the transferred momentum during such a collision [77]. At a position where the electron's amplitude is σ_x having a maximum betatron value of β_x , ($\alpha_x = 0$) the maximum divergence is,

$$\sigma_{x'} = \frac{\sigma_x}{\beta_x} = \frac{p_x}{p}$$

since

$$\sigma_{x'} = \sqrt{\frac{\epsilon_x}{\beta_x}}, \quad \sigma_x = \sqrt{\beta_x \epsilon_x}$$



If the transverse momentum p_x is all transferred to the longitudinal plane it is boosted by a factor γ ,

$$\Delta p = \gamma p_x = \gamma \frac{p \sigma_x}{\beta_x} \quad (117)$$

Therefore a displacement of $100 \mu\text{m}$ at a point with $\beta_x = 10 \text{ m}$ and beam energy 2.0 GeV ($\gamma = 3914$), gives $\Delta p = 78 \text{ MeV}/c$, which is 3.9% energy deviation. This is the same order of magnitude as the energy acceptance. In the vertical plane the beam is usually very small because of the small coupling resulting in roughly an order of magnitude less effect for a 1% coupled beam. It has, however, been recently reported that vertical Touschek losses can be dominant if the vertical chromaticity is large [56]. The loss mechanism in this case is through resonant crossing of the off-momentum particles.

7.2 The cross-section

To determine the cross-section we consider the collision in the beam centre of mass system, in which the transverse momenta of the two colliding particles are equal and opposite. We make the assumption that the motion is non-relativistic in this frame. The collision geometry is shown in Fig. 8:

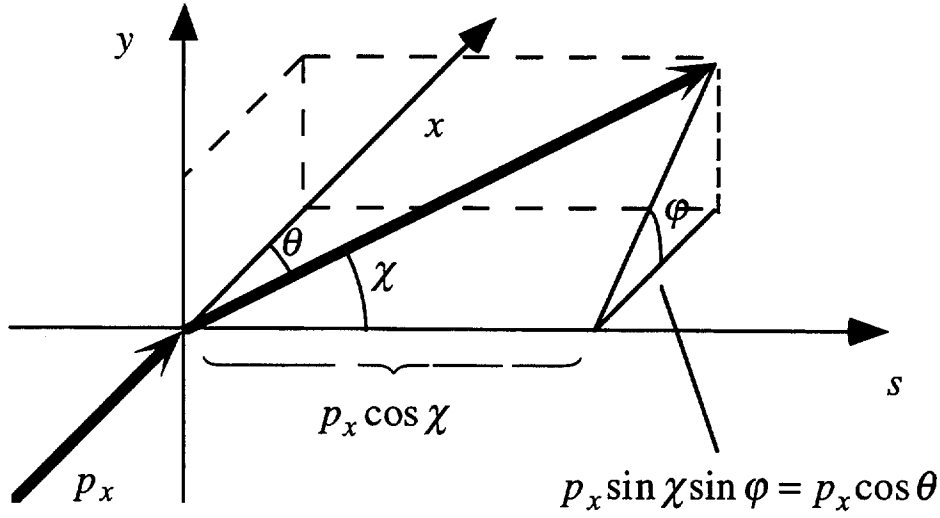


Fig. 8 Co-ordinates used for the Touschek collision.

The momentum transferred into the longitudinal laboratory direction is $\gamma p_x \cos \chi$. The particle is lost if this exceeds the longitudinal acceptance Δp_{rf} , i.e., if,

$$|\cos \chi| \geq \frac{\Delta p_{rf}}{\gamma p_x} = \mu \quad (118)$$

The non-relativistic differential cross-section in the centre of mass frame is given by the Möller cross-section,

$$\frac{d\sigma}{d\Omega} = \frac{4r_0^2}{(v/c)^4} \left[\frac{4}{\sin^4 \theta} - \frac{3}{\sin^2 \theta} \right] \quad (119)$$

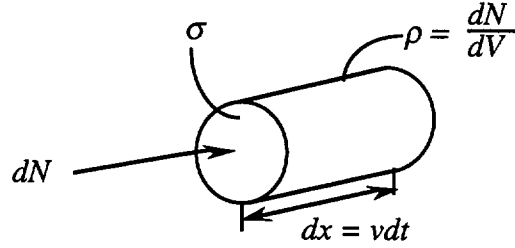
where v is the relative velocity of the two colliding particles (cm frame), r_0 the classical electron radius and θ the deflection angle. For relativistic considerations see references [81, 82], which show that the results may vary by a few percent for large momentum deviations. The total cross-section leading to particle loss is now given by,

$$\sigma_T = \int_{|\cos \chi| > \mu} \left(\frac{d\sigma}{d\Omega} \right) d\Omega \quad (120)$$

Using the geometrical relationship $\cos \theta = \sin \chi \cos \varphi$ and $d\Omega = \sin \chi d\chi d\varphi$, we have in the centre-of-mass system,

$$\begin{aligned} \sigma_T &= \frac{4r_0^2}{(v/c)^4} \int_0^{\cos^{-1} \mu} \sin \chi d\chi \int_{-\pi}^{\pi} d\varphi \left[\frac{4}{(1 - \sin^2 \chi \cos^2 \varphi)^2} - \frac{3}{(1 - \sin^2 \chi \cos^2 \varphi)} \right] \\ &= \frac{8\pi r_0^2}{(v/c)^4} \left[\frac{1}{\mu^2} - 1 + \ln \mu \right]. \end{aligned} \quad (121)$$

7.3 The lifetime



The fraction dN_i/N_i per unit time of incoming particles with momentum p_i which suffer a collision is equal to the cross-section, the density ρ_j of the scattering centres with momentum p_j and the distance traversed,

$$\begin{aligned} dN_i &= N_i \sigma \rho_j v dt \\ \Rightarrow \frac{dN_i}{dt} &= N_i \sigma v \rho_j = (\sigma v) \rho_i \rho_j dV \end{aligned} \quad (122)$$

since,

$$N_i = \rho_i dV. \quad (123)$$

The total loss rate is now determined by fully integrating the above equation over all space and momentum co-ordinates which lead to particle loss,

$$\left(\frac{dN}{dt} \right)_{\text{loss}} = 2 \left(\frac{dN}{dt} \right)_{\text{collisions}} = -\frac{2}{\gamma^2} \int (\sigma v) \rho_i \rho_j dV \quad (124)$$

The factor γ^2 comes from transforming the integral from the cm system to the laboratory frame. For a laminar beam, flat in the vertical plane, the phase space distribution in the laboratory is,

$$\rho_i = N \rho_x(x_i, x'_i) \rho_y(y_i) \rho_z(z_i) \quad (125)$$

with

$$\begin{aligned} \rho_x(x_i, x'_i) &= \frac{\beta_x}{2\pi\sigma_x^2} \exp \left\{ -\frac{x_i^2 + (\beta_x x'_i + \alpha_x x_i)^2}{2\sigma_x^2} \right\} \\ \rho_y(y_i) &= \frac{1}{\sqrt{2\pi}\sigma_y} \exp \left\{ -\frac{y_i^2}{2\sigma_y^2} \right\} \\ \rho_z(z_i) &= \frac{1}{\sqrt{2\pi}\sigma_z} \exp \left\{ -\frac{z_i^2}{2\sigma_z^2} \right\} \end{aligned} \quad (126)$$

with

$$\sigma_{xy} = \sqrt{\epsilon_{xy} \beta_{xy}}$$

and

$$\sigma_z = \frac{\alpha R}{Q_s} \left(\frac{\sigma_e}{E} \right) \quad (127)$$

In the last expression the bunch rms length is related to the rms relative momentum spread (σ_e/E) via the momentum compaction factor and the synchrotron tune, R is the average ring radius. The integration is performed assuming the collision occurs at $(x_1, y_1, z_1) = (x_2, y_2, z_2)$. The integrals over y and z are performed immediately yielding,

$$\begin{aligned} \frac{dN}{dt} &= -\frac{N^2}{2\pi\gamma^2\sigma_y\sigma_z} \int \sigma(v)v \rho_x(x_1, x'_1) \rho_x(x_1, x'_2) dx_1 dx'_1 dx'_2 \\ &= -\frac{N^2\beta_x^2}{8\pi^3\gamma^2\sigma_x^4\sigma_y\sigma_z} \int \sigma(v)v \exp\left\{-\frac{A_0x_1^2 + A_1x_1 + A_2}{2\sigma_x^2}\right\} dx_1 dx'_1 dx'_2 \end{aligned} \quad (128)$$

with

$$\begin{aligned} A_0 &= 2(1 + \alpha_x^2) \\ A_1 &= 2\alpha_x\beta_x(x'_1 + x'_2). \\ A_2 &= \beta_x^2(x_1'^2 + x_2'^2) \end{aligned} \quad (129)$$

Making use of

$$\sigma'_x = \frac{\sigma_x}{\beta_x} = \sqrt{\frac{\epsilon_x}{\beta_x}} \text{ and } x'_2 - x'_1 = \frac{1}{\gamma} \left(\frac{v}{c} \right) = \frac{2p_x}{\gamma m_0 c} \quad (130)$$

where v is the relative velocity in the cms, we integrate over dx_1 express the resulting integral in terms of $p_x dx'_1$ and finally integrate over dx'_1 , giving,

$$\frac{dN}{dt} = -\frac{N^2}{2\pi^2\gamma^3 m_0^2 c \sigma_{x'} \sigma_x \sigma_y \sigma_z} \int_{\frac{\Delta p_{rf}}{\gamma}}^{\infty} \sigma(p_x) p_x \exp\left\{-\left(\frac{p_x}{\gamma \sigma'_x m_0 c}\right)^2\right\} dp_x \quad (131)$$

The lower limit reflects particle loss if the momentum exceeds the momentum acceptance in the centre-of-mass system. Defining,

$$\begin{aligned} \epsilon &= \left(\frac{\Delta p_{rf}}{\gamma^2 \sigma'_x m_0 c} \right)^2 \\ &= \frac{1}{(\gamma \sigma'_x)^2} \left(\frac{\Delta p}{p} \right)_{rf}^2, \quad \gamma \gg 1 \end{aligned} \quad (132)$$

with $(\Delta p/p)_{rf}$ the rms relative momentum acceptance, the loss rate can be expressed as,

$$\frac{dN}{dt} = -\frac{\sqrt{\pi} c r_o^2 N^2}{\gamma^3 V_b \sigma'_x \left(\frac{\Delta p}{p}\right)_{rf}^2} \varepsilon \int_{\varepsilon}^{\infty} \frac{1}{u^2} \left[\left(\frac{u}{\varepsilon}\right) - \ln\left(\frac{u}{\varepsilon}\right)^{1/2} - 1 \right] e^{-u} du = -\frac{\sqrt{\pi} c r_o^2 N_b^2}{\gamma^3 V_b \sigma'_x \left(\frac{\Delta p}{p}\right)_{rf}^2} C(\varepsilon) \quad (133)$$

$V_b = 8\pi^{3/2} \sigma_x \sigma_y \sigma_z$ is the bunch volume. The function $C(\varepsilon)$ for $\varepsilon < 1$ can be approximated by,

$$C(\varepsilon) \sim -[\ell n(1.732 \varepsilon) + 3/2]. \quad (134)$$

This is a weak function of ε , and in the range of validity ($10^{-4} < \varepsilon < 10^{-1}$) varies from 7.1 to 0.2. Solving to find the time it takes for the number of particles to be reduced by one half,

$$\frac{dN}{dt} = \dot{N} = -a N^2 \Rightarrow N(t) = \frac{N_0}{1 + N_0 a t} \quad (135)$$

gives the time dependent half-life as,

$$\tau_{1/2} = \frac{1}{aN_0} = \frac{N_0}{\dot{N}_0} = \frac{\gamma^3 V_b \sigma'_x \left(\frac{\Delta p}{p}\right)_{rf}^2}{\sqrt{\pi} c r_o^2 N_b} \cdot \frac{1}{C(\varepsilon)} \quad (136)$$

which is related to the $1/e$ lifetime by,

$$\tau_{1/2} = \ln 2 \cdot \tau_{1/e}. \quad (137)$$

Expression (136) applies to one point in the ring and to find the overall lifetime an average is taken over the ring circumference: $1/\tau = \langle 1/\tau(s) \rangle$. The assumptions of non-relativistic motion in the beam rest frame, a flat beam and the neglect of non-zero dispersion all affect the outcome of the computation. Relativistic effects become noticeable for large momentum deviations and will lead to a lower lifetime because the cross-section is larger. With regard to beam thickness, low emittance light sources will generally operate with small horizontal-to-vertical coupling and the approximation holds. For those situations where increased coupling is used to increase the lifetime the expression for the Touschek effect has been derived in reference [83],

$$\tau = \frac{\gamma^4}{2\pi r_o^2 c} \cdot \frac{V_b \sigma'_x \sigma'_y}{N_b} \cdot \left(\frac{\Delta p}{p}\right)_{rf} \cdot \frac{1}{D(\varepsilon)} \quad (138)$$

where,

$$D(\varepsilon) = \sqrt{\varepsilon} \int_{\varepsilon}^{\infty} \frac{1}{u^{3/2}} \left[\left(\frac{u}{\varepsilon}\right) - \frac{1}{2} \ell n\left(\frac{u}{\varepsilon}\right) - 1 \right] e^{-u} du. \quad (139)$$

The situation with dispersion is treated by LeDuff in reference [77]. The effect of finite horizontal dispersion is to modify the phase space co-ordinates and the horizontal beam size,

$$x \rightarrow x + D \frac{\Delta p_x}{p} \quad (140)$$

$$x' \rightarrow x' + D' \frac{\Delta p_x}{p} \quad (141)$$

$$\sigma_x \rightarrow \left(\epsilon_x \beta_x + D^2 \left(\frac{\sigma_e}{E} \right)^2 \right)^{1/2} \quad (142)$$

where $\Delta p_x/p$ is the relative momentum deviation of an electron in the laboratory frame prior to the collision and (σ_e/E) is the rms relative momentum spread of the bunch. In general the inclusion of dispersion increases the bunch volume and leads to a reduction in the collision rate thereby increasing the lifetime.

7.4 Transverse momentum acceptance

Even though a scattered electron may remain inside the longitudinal acceptance, it may still be lost in the transverse plane either at the physical or dynamic limit. An energy loss at a point with zero dispersion will to first order result in a betatron trajectory having the same Courant-Snyder invariant having an off-set in the machine determined by the dispersion function. A particle which suffers an energy change Δp_x at a position with dispersion will in addition to the energy off-set find itself performing betatron oscillations with respect to a new off-momentum closed orbit (see Fig. 9) and may eventually have a maximum amplitude at that point of,

$$x(s) = \left| \frac{\Delta p_x}{p} \cdot D_x(s) \right| + \sqrt{\beta_x(s) \bar{\epsilon}_x} \quad (143)$$

where $\bar{\epsilon}_x$ is the Courant-Snyder invariant for that particle,

$$\bar{\epsilon}_x = \gamma_x dx^2 + 2\alpha_x dx dx' + \beta_x dx'^2 \quad (144)$$

which is not to be confused with the emittance and dx and dx' are related to the original co-ordinates prior to energy loss by,

$$dx = D_x \frac{\Delta p_x}{p} - x \quad (145)$$

$$dx' = D'_x \frac{\Delta p_x}{p} - x' \quad (146)$$

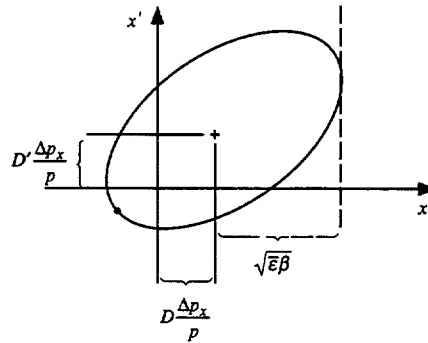


Fig. 9 An electron represented by the dot on losing energy will perform betatron oscillation about a new off-momentum reference orbit that is centred on the cross.

Consider a particle on the on-momentum closed orbit suffering an energy loss at a position with dispersion D_T , β_T and $\alpha_s = D'_T = 0$. Equating the Courant-Snyder invariant of that particle elsewhere in the ring (at position s with betatron value $\beta(s)$) we can write the maximum deviation after a Touschek event from the reference orbit as,

$$\Delta x(s) = \left\{ D_T \sqrt{\frac{\beta(s)}{\beta_T}} + D(s) \right\} \left(\frac{\Delta p}{p} \right)_T \quad (147)$$

The amplitude is greatly influenced by the optical functions. If the betatron function is small at the point where the dispersion function is large the resulting betatron oscillation elsewhere can be very large [49].

7.5 Improving the Touschek lifetime

The energy dependence of the Touschek lifetime is very strong and several factors affect the lifetime: the bunch volume, the accelerating voltage, energy spread, beam divergence, Lorentz contraction and intra-beam scattering (see later). The lifetime scales somewhere between the third and fifth power of the energy. The Touschek effect is more severe for low energy storage rings and is one of the limiting factors for the performance of these machines. Even high energy light sources are affected by the Touschek effect, for example, the ESRF when operating in 16-bunch mode observe a strong decrease in lifetime as the coupling is altered: 31 to 17 hours lifetime change at 90 mA when the coupling is varied from 10 to 1.5% [67]

Methods to improve the Touschek lifetime at a fixed energy could work on improving the limiting momentum acceptance, longitudinal or transverse, or to decrease the probability of collision by decreasing the bunch density.

The simplest method to decrease the bunch density is to operate with many bunches. This is done by choosing a high radio-frequency. Present day third generation light sources operate either at 350 or 500 MHz. The many bunches, though, will increase the difficulty in combating multi-bunch instabilities (see section 9) which are easier to control if few bunches are present.

To increase the longitudinal acceptance one would need to increase the rf voltage,

$$\left(\frac{\Delta p}{p} \right)_{rf}^2 = \frac{e V \sin \psi_s}{\pi h \alpha_c E} 2 \left(\sqrt{q^2 - 1} - \arccos \left(\frac{1}{q} \right) \right) \propto V$$

$$q = \frac{e V}{U_0} = \frac{1}{\sin \psi_s}.$$

With increasing rf voltage, however, the bunch length is also decreased,

$$\sigma_z = \frac{c \sqrt{2\pi}}{\omega_{rev}} \sqrt{\frac{\alpha_c E}{h e V \cos \psi_s}} \left(\frac{\sigma_e}{E} \right) \propto V^{-1/2}. \quad (148)$$

The lifetime, nevertheless, increases faster with rf acceptance than it decreases with bunch length. The effect is weak and the costs involved in providing additional rf power soon makes this method unattractive. Recently proposals have been made to provide additional rf acceptance only for Touschek scattered particles with the use of a super-conducting cavity. The cavity operates at zero phase since there is no need to restore the energy lost due to the

emission of synchrotron radiation. The cavity is passive and is powered by the circulating electron beam [56].

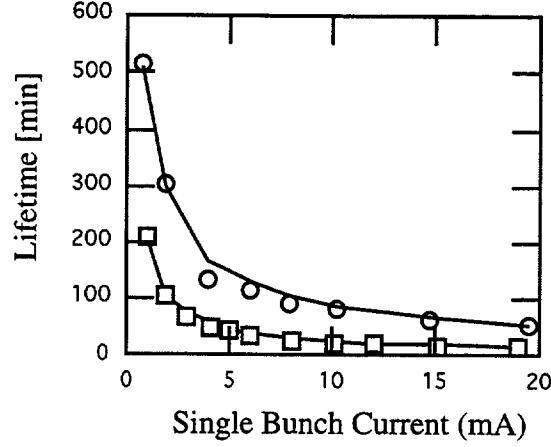


Fig. 10 Beam lifetime at 2 GeV (0.35% coupling) in ELETTRA as a function of single-bunch current for two different rf. voltages (circles 1.76 MV and squares 0.84 MV). The solid line is the prediction taking into account bunch lengthening effects [79].

The lifetime is proportional to the bunch volume and the goal of high brightness light sources is to minimise this quantity, which unfortunately will lower the Touschek lifetime. A compromise may be found by adjusting the coupling. This, however, will affect the vertical beam emittance with consequent reduction in the energy resolution of the emitted light. Alternatively the bunch density can be decreased by increasing the bunch length. This is done by operating with a higher harmonic cavity that changes the slope of the accelerating field [84]. The cavity can be powered either by the circulating beam or by an external source. At ELETTRA another technique is adopted, a controlled residual longitudinal multi-bunch excitation is allowed which dilutes the longitudinal bunch volume. The mechanism relies on the synchrotron tune shift with amplitude to smear the phase space volume [85]. This method to increase the Touschek lifetime compared to changing the emittance coupling results in more intense and sharper line spectra from undulators, especially for the lower harmonics.

8. INTRA-BEAM SCATTERING [47, 77, 86–88]

Intra-beam scattering deals with all those collisions which were ignored in deriving the Touschek effect. The many small angle collisions lead to a re-distribution of the six-dimension phase space distribution. In the centre-of-mass frame the transverse momentum spreads are larger than the longitudinal.

$$\left. \begin{array}{l} \sigma'_x p \\ \sigma'_y p \\ \sigma'_y p / \gamma \end{array} \right\} cms$$

The multiple collisions between particles will predominantly transfer momentum from the transverse plane to the longitudinal. We must, however, consider synchrotron oscillations. The Touschek effect showed that the transverse collision in the bunch frame resulted in a re-distribution of momenta. The transferred transverse momentum p_x is boosted to γp_x . The longitudinal blow-up is compensated by the transverse collapse. The Touschek event neglected transfer of momentum from the longitudinal plane to the transverse because of the larger transverse acceptance. Following Sørensen [80] we examine the emittance (at a symmetry point) prior to the collision of two electrons which have co-ordinates,

$$\begin{aligned}x_\beta &= x \\x'_\beta &= \pm p_x/p\end{aligned}\tag{149}$$

where the positive sign corresponds to one particle and the negative sign to the other. The excitation of betatron motion will result after a collision with full transfer of momentum to the longitudinal plane at a position with dispersion. The new co-ordinates are,

$$\begin{aligned}x_\beta &= x \pm D \frac{p_x}{p} \\x'_\beta &= 0\end{aligned}\tag{150}$$

If the original emittance of the particles was

$$\varepsilon_x = \frac{1}{\beta_x} (x_\beta^2 + \beta_x^2 x_\beta'^2)\tag{151}$$

the change in the sum of the emittances for the two particles after the collision is

$$\Delta(\varepsilon_{x1} + \varepsilon_{x2}) = \frac{2}{\beta_x} \left(\frac{p_x}{p} \right)^2 [(D\gamma)^2 - \beta_x^2].\tag{152}$$

Therefore if

$$\left(\frac{D\gamma}{\beta_x} \right)^2 > 1,\tag{153}$$

we have simultaneous longitudinal and transverse growth since the beam can absorb any amount of energy from the rf system. The full treatment of intrabeam scattering is mathematically rather complicated since it involves integration over 12 dimensions for two particles. The recipe for performing the computations will just be given and results quoted (see also [47]). Computation of the growth rate of the intra-beam effect given by the recipe of Piwinski [87], is as follows

- 1) Transformation of the momenta of the two colliding particles to the centre-of-mass frame.
- 2) Calculation of the change in momenta due to an elastic collision.
- 3) Transformation of the momenta back to the laboratory frame.
- 4) Relate the changes in momenta to changes in emittances.
- 5) Average over the distribution of scattering angles using the small angle Möller cross-section
- 6) Average over the distribution of the particles within a bunch to get the growth time.

The derivation for a weak focusing machine yields the following condition

$$\langle H \rangle \left(\frac{1}{\gamma^2} - \alpha_c \right) + \frac{\langle \varepsilon_x \rangle}{\beta_x} + \frac{\langle \varepsilon_y \rangle}{\beta_y} = 0\tag{154}$$

$\langle \rangle$ denotes the average around the ring circumference ε_x and ε_y the transverse emittances and H is the longitudinal emittance given by,

$H = \eta^2 + \frac{\dot{\eta}^2}{\Omega_s}$ with $\eta = \frac{\Delta p}{p}$ the relative momentum and Ω_s the synchrotron frequency.

Below transition $1/\gamma^2 < \alpha_c$ the sum of the three positive invariants is limited, they only exchange energy. Above transition there is no equilibrium distribution and the invariants can grow without bounds until radiation damping counteracts the effect. The growth times for the longitudinal phase space momentum and bunch distributions are [97],

$$\begin{aligned}\frac{1}{\tau_p} &= \frac{1}{2\sigma_p^2} \frac{d\sigma_p^2}{dt} = A \frac{\sigma_h^2}{\sigma_p^2} f(a, b, c) \\ \frac{1}{\tau_x} &= \frac{1}{2\sigma_{x\beta}^2} \frac{d\sigma_{x\beta}^2}{dt} = A \frac{\sigma_h^2}{\sigma_p^2} \left[f\left(\frac{1}{a}, \frac{b}{a}, \frac{c}{a}\right) + \frac{\eta^2 \sigma_p^2}{\sigma_{x\beta}^2} f(a, b, c) \right] \\ \frac{1}{\tau_y} &= \frac{1}{2\sigma_{y\beta}^2} \frac{d\sigma_{y\beta}^2}{dt} = A \frac{\sigma_h^2}{\sigma_p^2} f\left(\frac{1}{b}, \frac{a}{b}, \frac{c}{b}\right)\end{aligned}\quad (155)$$

with

$$A = \frac{r_e^2 c N_b}{64\pi^2 \sigma_s \sigma_p \sigma_{x\beta} \sigma_{y\beta} \sigma_{x'} \sigma_{y'} \beta^3 \gamma^4} \quad (156)$$

the particle bunch density. The function f is

$$f(a, b, c) = 8\pi \int_0^1 \left\{ \ln \left[\frac{c^2}{2} \left(\frac{1}{\sqrt{p}} + \frac{1}{\sqrt{q}} \right) \right] - 0.577.. \right\} \frac{1-3x^2}{\sqrt{pq}} dx \quad (157)$$

and

$$p = \left(\frac{\sigma_h}{\gamma \sigma_{x'}} \right)^2 + x^2 \left(1 - \left(\frac{\sigma_h}{\gamma \sigma_{x'}} \right)^2 \right) \quad (158)$$

$$\sigma_h^2 = \frac{\sigma_p^2 \sigma_{x\beta}^2}{\sigma_{x\beta}^2 + \alpha_c^2 \sigma_p^2} \quad (159)$$

$$q = \left(\frac{\sigma_h}{\gamma \sigma_{y'}} \right)^2 + x^2 \left(1 - \left(\frac{\sigma_h}{\gamma \sigma_{y'}} \right)^2 \right) \quad (160)$$

$$c^2 = \beta^2 \sigma_h^2 \frac{\sqrt{2\pi} \sigma_{y\beta}}{r_e} \quad (161)$$

From the dependence on energy of factor A we see that the growth time diminishes rapidly with energy. Intra-beam scattering is only of importance at low energies where the emittance is low and high particle densities can arise. For light sources above 1.0 GeV its effect rapidly diminishes with increasing energy. Finally in the presence of radiation damping an equilibrium will exist since,

- radiation damping becomes stronger with increasing amplitudes
- intra-beam scattering becomes weaker with increasing amplitudes

- and quantum fluctuations are independent of amplitudes.

Programs such as ZAP [89] are available for the computation of the intra-beam effect and provide as output the equilibrium emittances. Piwinski [87] treated the problem for a weak focusing machine, Bjorken and Mtingwa [88] used quantum electrodynamics for a strong focusing machine. The results, however, are essentially the same. Figure 11 [54] shows the emittance blow-up due to intra-beam scattering for ELETTRA, at high energies the emittance increases with the square of the energy and intra-beam effects are negligible. At 1.0 GeV the emittance increase is ~33% larger than the theoretical value. The effect is even more noticeable at lower energies. The ANKA [90] storage ring operates by energy ramping from 500 MeV to its final energy of 2.5 GeV. At the injection energy the equilibrium emittance is a factor of 8 times larger than the natural emittance. At the operating energies of present day light sources intra-beam effects can be ignored. We note, however, that the scaling of the emittance goes as $E^2\theta^3$, therefore large rings operating at low energies will particularly suffer from intra-beam scattering [91].

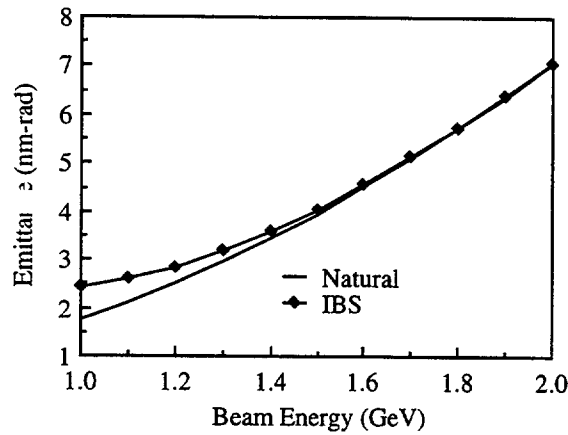


Fig. 11 Emittance increase due to intra-beam scattering (IBS) in the ELETTRA [54] storage ring computed by ZAP [89]. The calculation takes into account bunch lengthening effects as a function of energy.

9. INSTABILITIES [92]

9.1 Introduction

In order to have high brightness we desire to have as high a stored current as possible (we ignore here power requirements and heating of the vacuum chamber by synchrotron radiation). In addition, to lessen the effects of Touschek scattering, the beam current is distributed in many bunches. The bunches are intense and short of the order of a few millimetres. The electric fields of these bunches act back on themselves via the chamber, giving rise to collective phenomena. Instabilities arise, growths of betatron or synchrotron oscillations, which if not damped by radiation damping or Landau damping will lead to deterioration or loss of the beam. The onset of the instabilities depends on the current. Some instabilities increase gradually with increasing current whilst others have sharp thresholds. Instabilities are of two types, transverse and longitudinal, for both single and multibunch filling of the ring. In this section only some highlights will be given of this topic. Several texts found in references [4, 93–97] cover the area in depth.

9.2 Wakefields [4, 97, 98]

The inclusion of forces which originate from the circulating bunch in the equations of motion will lead to current dependent effects. The electric field of a relativistic electron is

contracted longitudinally into a disk perpendicular to the direction of motion. For a perfectly conducting smooth chamber the field is perpendicular to the direction of motion and does not affect trailing electrons. With finite resistivity the field lags the electron and affects other electrons, the leading electron loses energy. The trailing electric field is termed a wake field. Discontinuities along the vacuum chamber are also "good" sources for the production of wake fields and act by "scraping" away electric field lines. Bunch lengths are of the order of some millimetres and small discontinuities along the chamber act like cavities, where the electron bunch can deposit energy. The trapped fields will have a decay time over which the energy is dissipated by heating the vacuum chamber. We can distinguish two regimes, broad- and narrow-band parasitic losses. In the first case the wake field decays quickly, with a time scale of less than one turn. These fields are produced by small discontinuities:

- bellows
- vacuum chamber transitions (insertion devices)
- vacuum ports
- beam position monitors
- strip lines
- fluorescent screens
- flanges
- injection elements

The fields affect electrons in the same bunch leading to single-bunch instabilities, i.e. electrons at the head of the bunch act on the tail.

In the second regime, the narrow-band parasitic losses, wake fields are generated which remain for a long time in the chamber that act back on the same bunch or another for many turns. The fields are produced in cavity-like objects and the most notable object of this sort in a storage ring is a radio frequency cavity. The cavity can be excited by the beam which deposits energy in the higher order modes (HOM's). The interaction of the beam with these modes results then in the excitation of multibunch instabilities.

The wake function of a single electron is defined to be the distance-integrated force felt by a trailing electron. The integration allows an averaging to be performed and absorbs the details of the environment which is usually the situation for most cases of interest. The forces depend on the distribution of the electromagnetic fields generated by the leading electron which in turn depend on the geometry of the vacuum chamber. Writing the fields in terms of an expansion in multipole components allows the integrated force or wake function to be expressed in terms of these components. Similarly an arbitrary charge distribution will have multipole components. In the longitudinal plane the lowest order multipole $m = 0$ or monopole is usually the most important, whilst in the transverse plane the dipole, $m = 1$, component is of greatest interest. We note that a monopole transverse component cannot exist because of symmetry reasons. These lowest-order multipoles express effects related to the displacements of the centre of gravity of the charge distribution. The higher-order multipoles reflect the effect of various charge distributions within the bunch. The wake field can be used to define an impedance-induced voltage. For an electron distribution in the centre of the chamber we can write the longitudinal voltage as,

$$V_{\parallel}(z') = - \int \rho(z) W_{\parallel}(z' - z) dz \quad (162)$$

with ρ the charge distribution. Using a Fourier transform the integral is deconvoluted to get,

$$V_{\parallel}(\omega) = -I(\omega)Z_{\parallel}(\omega) \quad (163)$$

where $I(\omega)$ is the Fourier transform of the current ($c\rho(z)$) and $Z(\omega)$ is the impedance in frequency space representing the Fourier transform of the Lorentz force caused by the wake

field. Strong coupling between the beam and the vacuum chamber occurs if both the beam signal and chamber impedance have components at the same frequency. The induced voltage leads to an energy loss via the resistive part of the impedance which is resorted by the rf cavities.

A transverse impedance also exists arising from wakes generated by electrons which have transverse displacements in asymmetric or discontinuous chambers. The resistive transverse impedance can be related to the longitudinal impedance for the lowest modes for a circular pipe of radius b by

$$Z_{\perp}(\omega) = \frac{2c}{b} \frac{Z_{\parallel}(\omega)}{\omega}. \quad (164)$$

A cavity-like structure can be described by an equivalent parallel resonant circuit [95] having a quality factor Q . A high Q value implies a chamber object where a wake field will ring for a long time and the resonant frequencies have a narrow range. Chamber components such as bellows and small discontinuities can also be described in an equivalent way but with a small Q value, typically with a value of one. The impedance is a function of frequency and the spectrum depends on the accelerator. The entire ring impedance may be defined by a broadband-impedance model which contains contributions from the low-frequency resistive-wall impedance, a contribution from all low Q chamber components and contributions from narrowband cavity modes which show up as sharp peaks [4]. At high frequencies the impedance rolls off at the cut-off frequency of the vacuum chamber at which point the wakefield propagates freely down the chamber. The cut-off frequency is customarily taken to be $\omega = c/b$, with b the chamber radius and c the speed of light. Instabilities are directly related to the impedance and it is of importance to reduce the impedance as much as possible. Typical longitudinal broadband impedance values for third generation source are around 1Ω , while for second generation sources these can be as high as 10Ω . The longitudinal and transverse impedances account for the beam instabilities in a storage ring which limit the maximum attainable current or degrade the beam characteristics.

How wake fields (impedances) affect particle motion can be seen by taking a simplistic picture of describing either transverse or longitudinal motion by harmonic motion and adding an extra force due to the wake field. For the longitudinal plane we can write the displacement z_n of an electron on the n th turn with respect to the reference particle as [92],

$$\frac{dz_n}{dn} + \Omega_s^2 z_n = \text{const} \cdot \sum_{k=-\infty}^n W_{\parallel}(k, n) \quad (165)$$

where $W_{\parallel}$ is the wake field accumulated on one turn and the summation over k accounts for the wake fields left on all turns previous to the n th. To solve the equation we assume a solution of the form $z_n = A \exp(-i2\pi n\Omega/\omega_0)$, express the wake function in terms of a longitudinal impedance and solve for Ω in terms of it. The real part of Ω gives a frequency shift. The imaginary part can give rise to an instability if $\text{Im}[\Omega] > 0$, with a growth time $1/\tau = \text{Im}[\Omega]$. If $\text{Im}[\Omega] < 0$ the disturbance is damped. The real part of the impedance gives the energy loss for an electron bunch interacting with the chamber, which is compensated by the rf system. The energy lost for a bunch of total charge q is

$$\Delta U = q^2 k(\sigma) \quad (166)$$

with k the loss factor which depends on the bunch length,

$$k = \frac{2}{q^2} \int_0^{\infty} \text{Re}[Z(\omega)] I^2(\omega) d\omega. \quad (167)$$

9.3 Single-bunch instabilities

Single-bunch instabilities arise from the interaction of the bunch with wake fields which are short, comparable to the bunch length. Broadband impedances will dominate in this case where the effect of a passing bunch is lost before the arrival of a trailing bunch. The wake fields generated by the low Q storage ring structures will affect particles within the same bunch, that is, the leading electrons will interact with the trailing electrons. In certain cases synchrotron oscillations will then interchange the roles of the electrons giving rise to a feedback system.

The most notable single-bunch instability occurs in the longitudinal plane namely the microwave instability or turbulent bunch lengthening. The wake field perturbs the longitudinal bunch density leading to an increase in the bunch length and energy spread. The instability is fast and occurs before the head and tail electrons can interchange via synchrotron oscillations. The bunch current threshold for the instability is,

$$I_{\text{threshold}} = \frac{\sqrt{2\pi} \alpha_c \sigma_z (E/e)}{R |Z_{\parallel}/n|_{bb}} \left(\frac{\sigma_e}{E} \right)^2 \quad (168)$$

$|Z_{\parallel}/n|_{bb}$ is the normalised longitudinal broadband impedance where n is the harmonic number, α_c is the momentum compaction factor, σ_z the bunch length, (σ_e/E) is the relative energy spread and R the average ring radius. The instability is self stabilising and growth stops when the current falls below the threshold and a new equilibrium is reached.

In the case of the transverse fast head-tail instability the leading electrons cause transverse wake fields which perturb the tail electrons. Synchrotron oscillations exchange the head and tail electrons and weakens the effect, however, if the wake interaction is intense enough an instability starts to grow before the completion of a synchrotron period. The bunch current threshold is,

$$I_{\text{threshold}} = \frac{\sqrt{2\pi} \alpha_c (E/e)}{\langle \beta |Z_{\perp}|_{bb} \rangle} \left(\frac{\sigma_e}{E} \right) \quad (169)$$

the denominator is the average transverse broadband impedance weighted by the betatron function. The value for the threshold is generally higher than the longitudinal turbulent bunch lengthening threshold. The fast head tail is the most limiting threshold and requires special care in keeping the broadband impedance low.

Another instability which is practically current independent is the head-tail instability. The mechanism is similar to the fast head-tail but involves in addition the betatron tune dependence with energy. This instability is avoided by operating with a slightly positive chromaticity (above transition), and is the main reason for having chromaticity correcting sextupoles in the storage ring. For present day machines single bunch currents in excess of 20 mA can be stored albeit with increased bunch length and energy spread [1, 78, 99]. Because of the single bunch instabilities and Touschek lifetime limitations, to achieve high currents and brightness (increased flux) many bunches are placed around the circumference. These multiple bunches, however, give rise to multibunch instabilities.

9.5 Multibunch instabilities

The beam generated wake fields that persist for a long time in the chamber will couple with the circulating bunches and define a feedback system which may become unstable. Cavity like objects, especially the rf system, are receptacles for these wake fields via their higher order modes (HOM's). Interaction of the bunches with the wakefields induces centre-

of-mass or bunch-density modulations in both longitudinal and transverse phase space. Instability occurs if the rise time of a coupled-bunch mode exceeds the radiation or Landau damping time. The longitudinal rise time is generally much faster than the transverse. Experimental observations show that longitudinal coupled-bunch instabilities rarely cause beam loss. The longitudinal oscillations grow in amplitude until some other damping mechanism such as Landau damping limits further growth. The non-linear accelerating field given by the sinusoidal shape of the rf voltage introduces significant tune variation with amplitude. This can give rise to de-coherence of a mode with subsequent radiation damping. For a few strong modes a pick-up would see this as low frequency oscillations characterised by the radiation damping time. Transverse instabilities will increase the betatron motion leading to partial or total beam loss, however, it has been observed that in the presence of longitudinal instabilities transverse excitation is greatly reduced or even absent [85, 100]. With the appearance of multi-bunch instabilities the following detrimental situations can exist,

- Difficulty in accumulating and storing beam.
- Sudden partial or total loss of the beam.
- Decreased beam lifetime.
- Increased bunch length and energy spread and consequent degradation of ID spectra.
- Jitter in arrival times of bunches.
- Increased transverse jitter.
- Increase in effective emittance.
- Increased source size from bending magnet radiation.
- Fluctuations in the intensity of the light.
- Low frequency beam and energy oscillations of the order of damping times.

9.5.1 Longitudinal motion

A single synchronous circulating Gaussian bunch will be seen by a pickup at a fixed longitudinal position as a series of line signals at the revolution harmonics ω_0 with an envelope centred about zero decaying with a width given by the inverse of the rms bunch length [95]. For a non-synchronous bunch which performs rigid synchrotron oscillations, side bands appear on either side of the revolution harmonics at a distance ω_s . The envelope of these side bands vanishes at low frequencies and peaks at approximately $1/\sigma_\tau$ where σ_τ is the bunch length. The lower sidebands about a given revolution harmonic correspond to negative frequencies which have been folded onto the positive frequency side as seen by a spectrum analyser. If the bunch has internal density oscillations then additional side bands will be seen around a revolution harmonic at $m\omega_s$ where m denotes the bunch modulation, (i.e., $m = 2$, corresponds to a quadrupolar density modulation).

To compute the effect of narrow-band impedances on the beam, we have to fold the beam spectrum with that of impedance to compute the induced voltage that eventually is responsible for energy gain or loss. The impedance sampled by the beam is restricted to a narrow range in frequency and will overlap with one revolution harmonic and its associated sidebands. The growth rate of an instability is associated with the difference in the real part of the impedance sampled at the upper and lower sidebands [95]. Considering a single bunch interacting with the fundamental cavity mode, leads to the interpretation of the Robinson instability, which can be cured by detuning the cavity such that the lower side-band sees a larger impedance (above transition). We can understand this by noting that a bunch with higher energy will have a lower revolution frequency which interacts with a larger impedance to lose more energy than a bunch which has less energy and a higher revolution frequency. This leads to a damping of the oscillation. The beam signal at the revolution frequency also couples to the impedance, the resistive part leads to an energy loss while the reactive part to a change in the incoherent synchrotron frequency.

A beam of M bunches will also have signals at the revolution harmonics. For two bunches performing rigid dipole oscillations about the synchronous phase, they can oscillate

either in phase or π out of phase, corresponding to two modes of oscillation. For four bunches we have four oscillation modes corresponding to phase differences of $0, \pi/2, \pi, 3\pi/2$. M bunches will have M modes with phase differences of $\Delta\phi = 2\pi n/M$, $n = 0, 1, 2, \dots, M-1$. Observation at a single point in the ring will see frequencies at

$$f_{nmp} = (pM + n)f_0 + mf_s \quad (170)$$

where n labels the multibunch mode, f_0 is the revolution frequency, f_s the synchrotron frequency, m the bunch oscillation frequency ($m = 1$, dipole; $m = 2$, quadrupole) and p an integer ($-\infty$ to ∞). With more than two bunches a given revolution harmonic (other than the rf harmonics) will have positive and negative side bands corresponding to different coupled-bunch modes [96].

As mentioned above the imaginary part of the complex frequency shift accounts for the growth of an instability. The general formula for the shift for mode n is,

$$\Delta\Omega_{\parallel}^n = i \frac{\alpha_c h \omega_o I}{4\pi Q_s (E/e)} \sum_{p=-\infty}^{\infty} \frac{\omega_p}{\omega_{rf}} Z_{\parallel}(\omega_p) e^{-(\omega_p \sigma_{\tau})^2} \quad (171)$$

with

$$\omega_p = (pM + n + Q_s)\omega_o \quad (172)$$

where I is the average circulating current. The summation gives the effective impedance weighted by the beam spectrum. The growth rate for an instability is given by $-\text{Im}(\Delta\Omega_{\parallel})$ and is related to the real part of the impedance. For a single narrow-band impedance tuned at $pM\omega_o$, for example a higher-order mode of an rf cavity, and a short bunch we may write [96],

$$-\text{Im}(\Delta\Omega_{\parallel}^n) = \frac{\alpha_c \omega_o I}{4\pi \omega_s (E/e)} \left\{ \omega_{p+} \text{Re } Z_{\parallel}(\omega_{p+}) - \omega_{p-} \text{Re } Z_{\parallel}(\omega_{p-}) \right\} \quad (173)$$

where $\omega_{p\pm} = (pM \pm n \pm Q_s)\omega_o$. The expression shows that the upper (or positive) sidebands are destabilising and the lower (negative) sidebands are stabilising. Figure 12 illustrates the excitation of the possible 432 dipole modes in ELETTRA when the beam interacts with the cavity higher-order modes without compensation.

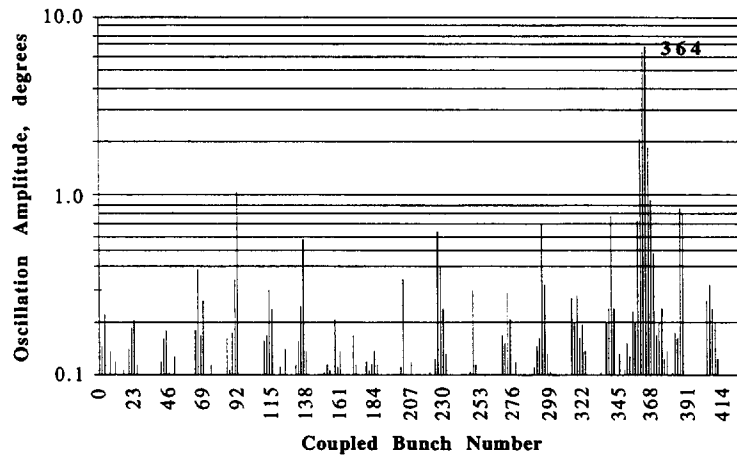


Fig. 12 Longitudinal dipole mode excitation by the cavity HOM's in ELETTRA before compensation. The figure shows the synchrotron oscillation amplitudes of the interaction from which an estimate of the effective energy spread may be obtained [101].

9.5.2 Transverse motion

A situation similar to the longitudinal plane also exists for transverse multibunch instabilities. The important differences are:

- The spectrum of the modes now depends also on the transverse tune.

$$f_{nm,p} = (pM + n + Q) f_0 + m f_s \quad (174)$$

where Q is the transverse tune and the other quantities are as before.

- The betatron phase ϕ at a fixed point varies as a function of particle energy and will have a contribution from the chromaticity which relates the tune shift with momentum.
- Mode 0 exists and corresponds to bunch dipole oscillations about the closed orbit. The line spectrum of mode 0 is centred about the chromatic frequency ($\omega_\xi = Q_o \omega_o \xi / \alpha_c$) and for a chromaticity of zero will sample low frequencies. The resistive wall impedance which dominates at low frequencies is now important and will give rise to noticeable effects. Since present day synchrotron radiation sources are built to accommodate increasingly smaller insertion device gaps and the transverse impedance varies as b^{-3} (b is the distance of the closed orbit to the chamber surface), this effect has to be carefully evaluated.
- Positive side bands are now stabilising and negative side bands de-stabilising.

9.6 Cures of multibunch instabilities [102–104]

The cure of multibunch instabilities is one of the more important aspects when operating a light source. The effects of uncontrolled instabilities will result in unacceptable performance degradation especially with regard to the quality of light from the higher harmonics of an insertion device, see Fig. 13.

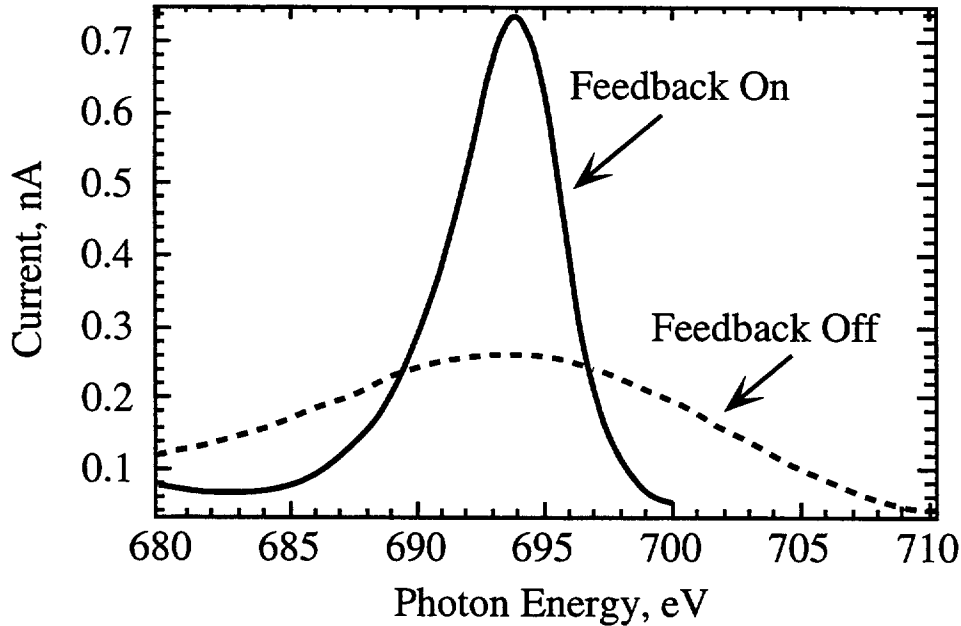


Fig. 13 Spectrum of the 5th harmonic of undulator U5.0 on beamline 7 at the ALS when the beam suffers from multibunch instabilities and when these are corrected by a longitudinal multibunch feedback system. Figure adapted from ref [70].

If a coupled-bunch mode is not damped by radiation damping or natural Landau damping other methods have to be found. Several techniques exist:

- The most straight forward method would be the elimination or reduction of the strength of those cavity higher-order modes which couple to the beam. This can be done by adopting [103]:
 - 1) Mono-mode cavities. This cavity has a wide beam-pipe opening allowing the free propagation of higher-order modes out of the cavity which are damped by ferrite loads. The geometry, however, leads to a reduction in the strength of the fundamental mode which therefore necessitates the use of super-conducting technology.
 - 2) Normal conducting cavities with HOM dampers, which can take on the form of antennas or waveguides. Antennas are used to couple out the power in a HOM and are useful if there are a few modes to damp. Waveguides orthogonally positioned on the surface of a cavity allow the propagation of HOM's to loads, however, as in the case for super-conducting cavities the fundamental is also affected and requires additional power to provide the desired voltage to the beam.
- If multiple cavities are used they should be arranged so that their modes do not all coincide.
- If a mode cannot be reduced then its resonant frequency can be shifted away from beam resonant frequencies either by temperature tuning or through the use of a cavity plunger. This is feasible for narrow modes compared to the spacing between beam signals. The method which is adopted at ELETTRA to control multi-bunch instabilities uses temperature tuning of the cavities [105]. The ELETTRA cavities are surrounded by a dense number of cooling tubes which guarantee a temperature stabilisation of better than $\pm 0.05^\circ\text{C}$. The distribution in frequency space of the higher-order modes of the cavity is a function of the cavity volume. To avoid a HOM interaction with the beam spectrum the mode is shifted by heating or cooling the cavity which changes the cavity volume. This also affects the fundamental mode which is re-installed at the correct frequency by a mechanical tuning system. The mechanical tuner acts by compressing the cavity in the longitudinal direction. For each $100\text{ }\mu\text{m}$ of compression the fundamental is shifted by 100 kHz . The shifting of the modes can be analytically computed with an accuracy of better than 1°C . Figure 14 shows the distribution of the growth rates (above damping) driven by the HOM's in an ELETTRA cavity as a function of the cavity temperature. We see that there are temperature windows which are free from HOM excitation at 49 to 53°C . Additional control may be given by installing in the cavity a frequency shifter, i.e., a plunger. The placement of a plunger on the longitudinal mirror plane of the cavity acts on the fundamental frequency which as before is re-installed via the mechanical tuner. This also changes the cavity volume and therefore the position of the HOM's.
- For HOM's which cannot be damped or shifted the growth of a coupled-bunch mode can be counteracted by increasing the Landau damping. For longitudinal modes a higher harmonic cavity can be adopted to increase the bunch length (this will also increase the Touschek lifetime) or run the rf cavities at lower voltage (which also lengthens the bunch, however, in this case the Touschek lifetime would worsen). For transverse modes the use of octupole magnets will give tune shifts with amplitude, although it may be difficult to operate in a low emittance lattice as the beam size is small necessitating strong magnets which would in turn reduce the dynamic aperture.
- A technique which is adopted at the ESRF [24] is to introduce a bunch to bunch synchrotron tune spread. This is done by operating the storage ring with a partial filling ($1/3$ filled) and operating with a cavity voltage regulation loop which has a long time constant. In this way the leading bunches in the bunch train successively remove power from the cavity and the trailing bunches then see differing voltages resulting in a Landau damping of the centre of mass of the bunches. The technique becomes more

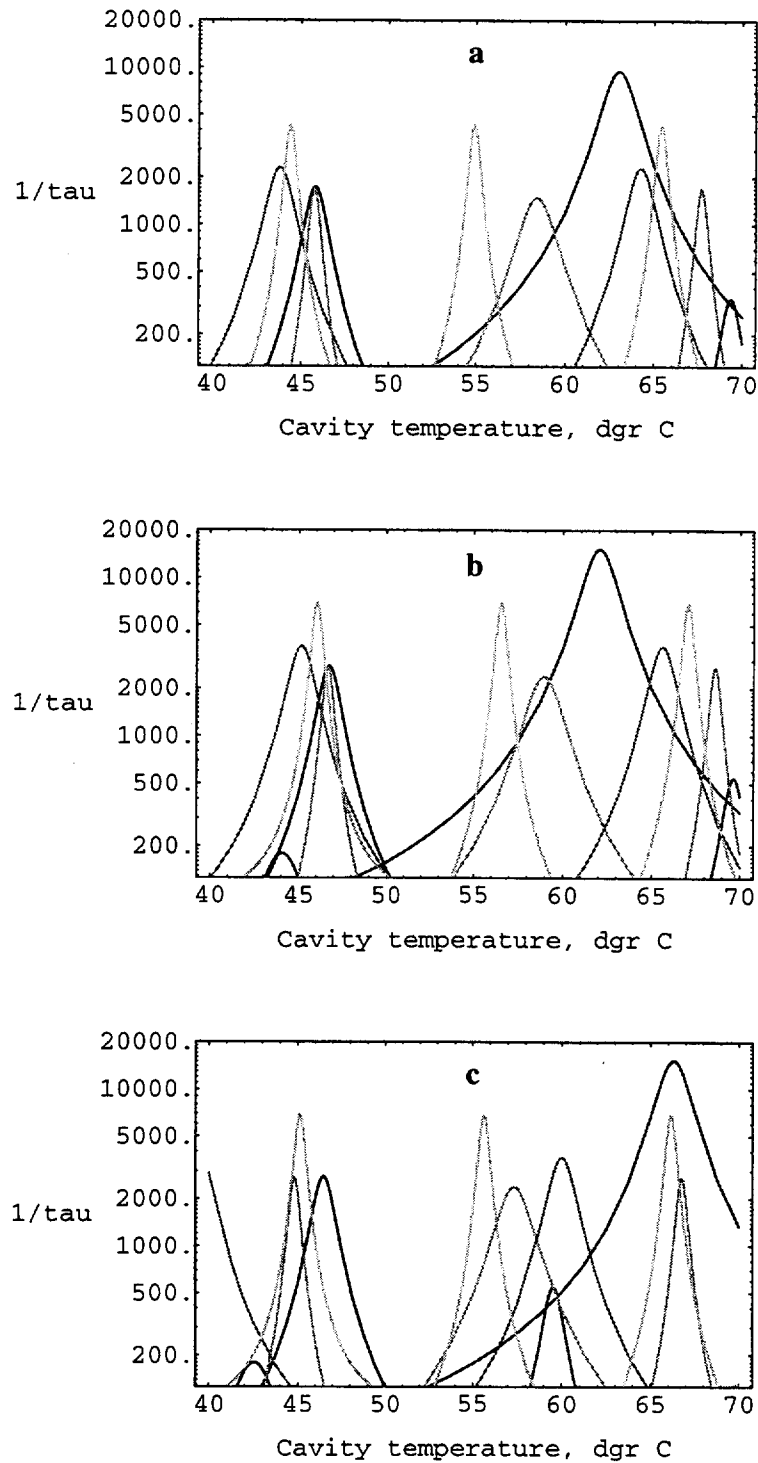


Fig. 14 Growth rates for longitudinal and transverse coupled bunch modes driven by cavity higher-order modes as a function of cavity temperature in the ELETTRA storage ring. (a) Stable operating conditions are obtained by operating in the window 49-53°C (250 mA, 2.0 GeV). (b) At higher current, 400 mA, a broad transverse mode covers this window, which is shifted away from the operating point by the installation of a frequency shifter (c).

effective at higher currents, however, it works best on machines with a large circumference. In a similar manner modulation of the rf voltage can be used.

- Longitudinal and transverse feedback [106–108]: With increasing current it becomes more difficult to find stable operating windows via mechanical or temperature tuning since the growth time for an instability is proportional to the current whereas damping due to radiation is a single particle effect. Storage rings with large circumferences are also more problematic since the spacing of coupled bunch modes corresponds to the revolution frequency which is lower for the larger machines, making it more difficult to find a stable temperature window. The most powerful technique is then to control multi-bunch instabilities by an active feedback system which damps all modes, both in the longitudinal and transverse planes. The technique can be realised both in the time domain (bunch by bunch) or in the frequency domain (mode feedback). In the time domain each bunch is treated as a single oscillator. The system is made up of a detector capable of continuously measuring a single bunch, for each bunch a dedicated Digital Signal Processor (DSP) is used for filtering the signal and computing the kick to be applied which is phase shifted by 180° and finally a broadband power amplifier and kicker is required. The set-up requires state-of-art electronics and power amplifiers and care in construction of the kicker. The ALS [107] has adopted this method to combat both longitudinal and transverse multi-bunch instabilities successfully and can maintain a stable beam with 400 mA of stored current. A mode feedback system which works on selected modes could be envisaged for radio frequency systems utilising temperature tuning since the number of modes to be damped will be small, however, with diminishing costs for DSP's and associated electronics the broadband (bunch by bunch) approach may well be the most promising for future developments.

The full control of both longitudinal and transverse multi-bunch instabilities will result in a bunch volume which is close to the theoretical value, i.e., dense bunches with no dilution in either plane. The Touschek lifetime will consequently decrease as has been observed in various laboratories. This necessitates frequent refills or the adoption of advanced techniques to increase the Touschek lifetime by increasing both the transverse and longitudinal momentum acceptance.

10. ION TRAPPING [109, 110]

10.1 Introduction

The ionisation of the residual gas molecules in the vacuum chamber, either by direct electron-molecule interaction or via photo-electron emission will result in the production of positive ions. These ions may then become trapped in the potential well of the circulating beam. When trapping occurs two main points are observed: an increase of local gas pressure and the appearance of an electric field due to the ions. The increase in gas pressure leads to an increase in elastic scattering off the gas nuclei and beam-gas Bremsstrahlung (inelastic scattering), which cause beam blow-up, and loss of particles. The presence of the ion space charge leads to incoherent electron-beam tune shifts, electron-beam tune spreads and coherent effects given by electron-ion cloud oscillations, which may lead to resonances and particle loss. The increase in tune spread, however, may be beneficial in combating instabilities via Landau damping. Ion accumulation will depend on several factors: the number of circulating electron bunches, the bunch charge, the transverse size of the beam, the circumference of the storage ring, the mass of the ion and external fields. Trapped ions can be cleared by leaving a gap in the circulating bunch structure, by adopting clearing techniques such as clearing electrodes or by avoiding trapping altogether by using positrons rather than electrons. All second generation synchrotron radiation light sources have reported ion trapping whereas few third generation sources have clearly seen the effect [111, 112]. The reason for the lack of trapping is not clear and is most probably associated with the low emittance and the low coupling of these new machines in addition to strong non-linear ion clearing (see below).

This has had a beneficial impact on the design of new light sources since a costly positron option can be avoided.

In addition to the trapping of molecular/atomic ions, there is the possibility of trapping macro-molecules [113]. Dust particles in the chamber can be slowly ionised by synchrotron radiation and are then attracted by the beam and destroyed. The passage of a macro molecule can lead to sudden partial or total beam loss.

10.2 The trapping mechanism

No laboratory vacuum is perfect and gaseous molecules are present even after the most careful conditioning and treatment of the storage ring vacuum chamber. The molecular species found will depend on the past history of the chamber and the materials forming it. Exotic species (hydrocarbons etc..) can be excluded if the chamber has been treated properly. A chamber made of stainless steel, for example, may have the following gas species, molecular hydrogen, carbon monoxide and carbon dioxide which are initially contained in the steel and desorbed into the vacuum, methane which may come from heating filaments of certain vacuum pumps and molecular nitrogen, oxygen and argon if leaks are present. The circulating electron bunches will ionise the residual gas species present in the chamber either by collision or photo-ionisation. The ions formed may then be trapped by the electron beam. The limit of trapping occurs when the entire beam is effectively neutralised by the ion-cloud which is formed. The time for this to occur is of the order of seconds. We define the average neutralisation factor η [109] to be,

$$\eta = \frac{\text{local ion density}}{\text{average electron density}} = \frac{d_i}{d_e} \quad (175)$$

with

$$d_e = \frac{d_{ne}}{B} = \frac{d_{ne} n \sigma_z}{C} \quad (176)$$

where B is the bunching factor, B^{-1} gives the fraction of the ring circumference covered by electrons. For a uniformly distributed electron beam with elliptical cross-section the local ion density can be written as,

$$d_i = \eta \frac{I}{ec\beta} \frac{1}{2\pi\sigma_x\sigma_y}. \quad (177)$$

With respect to the electron bunches an ion can be considered to be stationary. At each passage of a bunch an ion will feel a transverse impulsive kick towards the bunch centre and then will drift freely between bunch traversals. The ions in this way may execute stable oscillations about the bunch centres. The electron bunches have three dimensional Gaussian distributions travelling close to the speed of light. As they pass an ion they impart a transverse kick via the electric field. The ion's position is left unchanged and the given impulse is,

$$M_i \Delta \dot{X} = \int_0^{\Delta T} e E_{\perp} dt. \quad (178)$$

where ΔT is the bunch passing time and M_i the mass of the ion. To evaluate this kick we borrow results from the beam-beam problem [114, 115]. The instantaneous change in the ion's transverse velocity is found to be

$$\begin{pmatrix} \Delta \dot{x} \\ \Delta \dot{y} \end{pmatrix} = \frac{Nr_p c}{An} \sqrt{\frac{2\pi}{(\sigma_x^2 - \sigma_y^2)}} \begin{pmatrix} \text{Im} \\ \text{Re} \end{pmatrix} \left(W \left[\frac{x + iy}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}} \right] - \exp \left[-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} \right] W \left[\frac{x \frac{\sigma_y}{\sigma_x} + iy \frac{\sigma_x}{\sigma_y}}{\sqrt{2(\sigma_x^2 - \sigma_y^2)}} \right] \right) \quad (179)$$

where N is the total number of stored electrons, n the number of bunches, r_p the classical proton radius, c the speed of light, A the atomic mass of the ion in units of the proton mass, and $\sigma_{x,y}$ the transverse electron beam dimensions. The function W is the complex error function. The impulse is a non-linear function of the ion's position with respect to the bunch centre for distances greater than the bunch rms dimensions.

10.2.1 Linear theory

Most of the theory for ion trapping can be taken directly from the theory of betatron motion in circular accelerators. Expecting the motion to be linear close to the bunch centre we expand the complex error function and keep only linear terms,

$$W[x + iy] \sim 1 + \frac{2}{\sqrt{\pi}}(x - iy). \quad (180)$$

The change in ion velocity is now expressed as,

$$\begin{pmatrix} \Delta \dot{x} \\ \Delta \dot{y} \end{pmatrix} = \frac{2Nr_p c}{An(\sigma_x + \sigma_y)} \begin{pmatrix} x/\sigma_x \\ y/\sigma_y \end{pmatrix} \quad (181)$$

The force is linear in the transverse co-ordinates and equivalent to a kick given by an electron beam with uniform charge density and elliptical cross section. The motion is decoupled in the two planes and we now consider only one plane. The effect of the bunch on the ion is simply

$$\begin{aligned} x_j &= x_i \\ \dot{x}_j &= \dot{x}_i + \Delta \dot{x} = \dot{x}_i + ax_i \end{aligned} \quad (182)$$

with the indices i, j indicating before and after the passage of the bunch and the kick parameter a is given as,

$$a = \frac{2Nr_p c}{An\sigma_x(\sigma_x + \sigma_y)}. \quad (183)$$

The result of the combination of kicks and drifts, with times t_b , between the passage of successive electron bunches can be written in matrix form,

$$\begin{pmatrix} x \\ \dot{x} \end{pmatrix}_j = \begin{pmatrix} 1 & t_b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix}_i \quad (184)$$

i.e.,

$$\mathbf{X}_j = \mathbf{M}\mathbf{X}_i \quad (185)$$

The electron bunches have been considered as thin focusing lenses. The drift time of ions is simply, $t_b = C/nc$, with C the ring circumference. The matrix $\mathbf{M}$ defines the transport matrix from one bunch to the next. Floquet's theorem states that the motion is stable if [15],

$$|\text{Trace}(\mathbf{M}_x)| < 2 \quad (186)$$

Evaluating the above in terms of the ion mass allows the definition of a critical mass $A_{x,c}$, for which motion is stable if an ion has a mass greater than this, i.e., if $A > A_{x,c}$ then the ion is trapped. The critical mass is given as

$$A_{x,c} = \frac{NCr_p}{2n^2\sigma_x(\sigma_x + \sigma_y)} \quad (187)$$

A similar expression exists for the vertical plane. Since we generally have $\sigma_x > \sigma_y$ then $A_{x,c} < A_{y,c}$ and ions that perform stable oscillations in the horizontal plane may not be stable in the vertical plane. Flat beams are therefore preferable against the possibility of ion trapping, in other words weak coupling between horizontal and vertical electron motion. $A_{y,c}$ is small and conditions are therefore good for trapping when the transverse beam dimensions are large, when the beam current is low, when the ion charge state is small and when there is a large number of circulating electron bunches.

10.2.2 Effects of the ion charge cloud

- The ion ladder [47]

As trapping proceeds the drift space between bunches becomes more like a thick defocusing lens due to the trapped ions. This alters the condition for trapping which now occurs for ion masses that satisfy,

$$A > (1 - \eta)A_{y,c}. \quad (188)$$

Initially η is zero and the condition for trapping is as before, however, as trapping proceeds the effective critical mass is reduced. At the same time there is an increase in the transverse dimensions of the electron bunch due to gas scattering which also decreases $A_{y,c}$. These effects lead to smaller ion masses being trapped, and this continues until the electron beam is completely neutralised. This process is called the ion ladder.

- The tune shift [109]

The ion cloud which is formed is an additional force on the electrons that provides focusing in both planes. Assuming the ion cloud has the same dimensions as the electron bunch, the tune shift induced by an additional “quadrupole” of strength $k(s)$ is

$$\Delta Q = \frac{1}{4\pi} \int \beta(s)k(s)ds \quad (189)$$

where β is the betatron function and ΔQ is given as,

$$\begin{aligned}
\Delta Q_{x,y} &= \frac{r_e}{\gamma} \int \frac{\beta_{x,y} d_i}{(1 + \sigma_{x,y}/\sigma_{y,x})} ds \\
&= \eta \frac{IC^2 r_e}{4\pi^2 ec \gamma Q_{x,y}} \frac{1}{\langle \sigma_{x,y} (\sigma_x + \sigma_y) \rangle} \\
&= \eta IG
\end{aligned} \tag{190}$$

The average β function has been taken to be $C/(2\pi Q)$. Values for G can range from 10^2 to 10^4 depending on the lattice parameters and can lead to large tune shifts for small neutralisations. The model, however, overestimates the ion-cloud dimensions and taking into account non-linearities alleviates the situation.

- Electron bunch – ion cloud oscillations [109, 116]

The two beam instabilities arise from the coherent interaction of the accumulated ion cloud and the electron bunch. In the limit of many symmetrically placed bunches the transport matrix for a bunch-kick drift sequence is equivalent to an attractive force acting over the same time-span,

$$\begin{pmatrix} 1 - t_b a_i & t_b \\ -a_i & 1 \end{pmatrix} = \begin{pmatrix} \cos(\omega_i t_b) & (1/\omega_i) \sin(\omega_i t_b) \\ -\omega_i \sin(\omega_i t_b) & \cos(\omega_i t_b) \end{pmatrix} \tag{191}$$

and the motion is

$$\ddot{u}_i = -\omega_i^2 u_i, \tag{192}$$

with u_i the ion co-ordinate and,

$$\omega_i^2 = \frac{2\lambda_e r_p c^2}{A\sigma_{i=x,y}(\sigma_x + \sigma_y)} \tag{193}$$

where $\lambda_e (=N/C)$ is the number of electrons per unit length. The coupled equations of motion describing the interaction of the ion cloud (having the same transverse dimensions as the electron bunch) and the electron bunch is,

$$\begin{aligned}
\ddot{y}_e + Q_y^2 \omega_o^2 y_e &= -\omega_e^2 (y_e - y_i) \\
\ddot{y}_i &= -\omega_i^2 (y_i - y_e),
\end{aligned} \tag{194}$$

the subscripts e and i in the last two differential equations refer to electron and ion bunch centre-of-mass co-ordinates and,

$$\omega_e^2 = \frac{2\lambda_i r_e c^2}{\gamma \sigma_y (\sigma_x + \sigma_y)} = \frac{Am_p}{\gamma m_e} \eta \omega_i^2, \tag{195}$$

where $\omega_o = 2\pi f_o$. Taking the solutions to be

$$y_e = Ae^{i(n\theta - \omega t)} \tag{196}$$

and

$$y_i = B e^{i\omega t}, \quad (197)$$

inserting in the differential equations and eliminating A and B gives the following dispersion relation for ω ,

$$(x^2 - v_i^2) [(x - n)^2 - v_e^2 - Q_y^2] = v_e^2 v_i^2 \quad (198)$$

with $v_{i,e} = \omega_{i,e}/\omega_0$ and $x = \omega/\omega_0$. The roots of this equation determine the stability of motion. If the roots are complex the motion of the system grows exponentially, until the ions are lost and the electron beam damps down to its equilibrium value whereupon the process starts again. Given the non-uniform distribution of ions in the ring, the phenomena may give rise to beating and low frequency beam oscillations.

10.2.3 Asymmetric bunch filling [117]

We now consider ion motion when there is asymmetric bunch filling of the ring, i.e., a gap is left in the bunch train. The transport matrix for one complete revolution is required because of symmetry breaking. The one turn matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \left[\begin{pmatrix} 1 & t_b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} \right]^n \begin{pmatrix} 1 & t_b \\ 0 & 1 \end{pmatrix}^{(h-n)} \quad (199)$$

where h is the harmonic number and n the number of bunches. Analytical evaluation of the trace is difficult, but easily done numerically. The critical mass is no longer applicable but stable mass bands are now found. The ion ladder is disrupted, since ions that become trapped in a mass band perturb the beam dimensions, decrease the beam current and introduce space charge, all of which changes $\mathbf{M}$ and hence shifts the mass bands to new patterns. We note that the ions are not lost because there is more drift space, but because the focusing structure has imperfections, which in turn introduce resonance behaviour. The ions are overfocused. Figure 15 shows an example of stable mass bands as a function of gap.

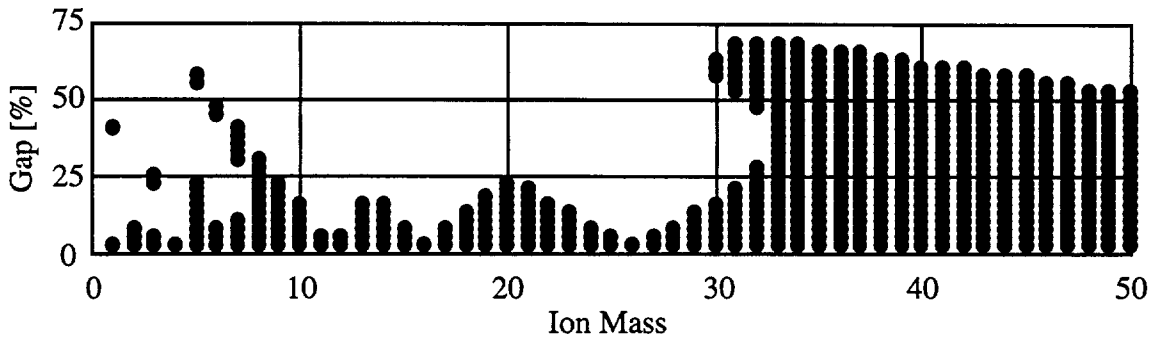


Fig. 15 Prediction of linear theory for the trapping of ions in the ANKA [90] storage ring (400 mA, 2.5 GeV, 3% coupling at the position for maximum trapping).

10.2.4 Non-linear electron bunch deflections [118] and ion clearing

The above considerations are valid only for motion close to the bunch centre, but says nothing about large ion amplitudes about the bunch centre. To do this we need to examine the motion using the non-linear kick equation (179), which upon numerical integration shows, as expected that the amplitude and oscillation frequency of motion depend on the coordinates the ion had when it was created. Simulations show that ions which are predicted to be stable using linear theory may be unstable or stable only close to the bunch centre when

non-linearities are included. The converse is rarely found. In general, non-linearities lead to a reduction in the trapping of ions, see Fig. 16.

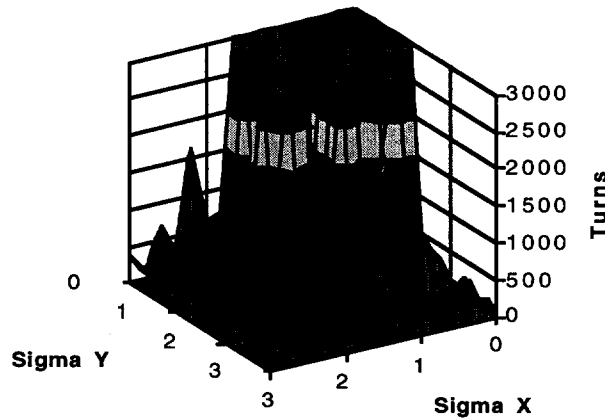


Fig. 16 Survival times in terms of electron bunch train traversals for ion mass 44 created at different transverse locations in the electron bunch (only one quadrant shown), under the action of non-linear kicks (400 mA, 3% coupling and a gap of 32 bunches) [90].

There are several methods to clear ions [119–123], the simplest is to leave a gap in the circulating bunch train. If, however, trapping persists then static or oscillatory clearing electric fields may be used which "pull" the ions into the non-linear region whereby the non-linear electron bunch fields will clear the ion. The oscillation frequency depends on the ion mass and on the electron beam dimensions at a given point in the ring. The electron beam itself may be shaken at the ion frequency or frequency modulation is applied to the beam from high to low frequency, such that the ion locks onto an oscillation driven by the non-linear force of the electron beam ion interaction. For ions trapped in a bending magnet the beam may be shaken at the ion cyclotron frequency inside the magnet.

Present day synchrotron radiation facilities use insertion devices as the prime source of radiation. These devices have magnetic fields which act like magnetic bottles and mirrors in the vertical direction which may enhance the trapping of ions. Simulations show that an enhancement of trapping is predicted for ions that are created at positions with longitudinal magnetic fields [119]. Experimental evidence, however, has so far not shown any clear indication that ion trapping is enhanced by insertion devices.

In addition to the trapping of ions, there is the possibility that macromolecules or dust particles may be ionised by stray photons in the chamber [113, 124, 125]. The ionisation proceeds until the induced charge to mass ratio is sufficient for the macro-particle to overcome gravity and pass through the beam. This will cause partial or complete loss of the stored beam. Since the mass of the macro-particle is large the oscillation frequency will be slow so that the time structure of the beam is not felt, i.e., a gap in the bunch structure is ineffective. The only solution to this is high cleanliness of the chamber.

Recently a fast ion instability has been proposed [126] whereby ions are produced on a single turn of the electron bunch train. The theory outlined above is not applicable to this situation. The instability arises from the continuous production of ions and involves the interaction of ions created at the head of the bunch train perturbing the trailing electron bunches before the ion cloud is over focused by a gap in the bunch train. The instability is fast, growing exponentially with an exponent proportional to the square root of time. The effect has not been clearly seen but may be important for future machines having high stored currents in many bunches.

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BEAM STABILITY

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Abstract

Small emittances are one of the main target performances of a Synchrotron Radiation Source. As a consequence the beam stability must be considered from the design phase as a critical point: effort has to be put in various domains to avoid any undesired beam motion. Ways of quantifying the beam stability, sources of instabilities and correction procedures will be described.

1. DEFINITIONS

1.1 Emittance growth

A natural way to measure the beam stability is to refer to the equilibrium beam sizes. This applies of course both in position and angle. In phase space, this immediately suggests the definition of a macroscopic “emittance growth”.

When the centre-of-mass position varies with time, an integral measurement of the disturbance due to the motion is the emittance value ε containing all the instantaneous displaced emittances. The ratio $\Delta\varepsilon/\varepsilon = (\varepsilon - \varepsilon_0)/\varepsilon_0$ where ε_0 is the unperturbed emittance is a measurement of the degradation of performance.

Usual values for beam stability tolerances are for instance of 10% of the beam size. The equivalent figure of 20% of emittance growth corresponds to either 10% of the size, or 10% of the divergence, or any intermediate combination of both.

The reference emittance and the macroscopic emittance should be photon emittances. However these values are usually computed for the electron emittances, which gives pessimistic results (and very close values for machines far from the diffraction limit).

This definition in terms of emittance growth is independent of the local values of the β function. It ensures all along the machine a fair balance between position and angle. This stability figure will also conserve along a beam line for any drift space or focusing.

The displacement of the centre-of-mass will be measured by its invariant ε_{cm} . The relationship between ε_{cm} and $\Delta\varepsilon/\varepsilon$ will be discussed in two cases, depending on the frequency of the motion compared to the observation time.

1.1.1 Fast motion

If the motion is fast compared to the observation time, the macroscopic emittance is taken as the statistical combination of the distributions of the beam density and of the centre-of-mass position. The “centre-of-mass emittance” ε_{cm} can be expressed in standard Twiss parameters:

$$\begin{aligned}
\varepsilon_{cm} &= \sqrt{\langle \Delta x^2 \rangle \langle \Delta x'^2 \rangle - \langle \Delta x \Delta x' \rangle^2} \\
\beta_{cm} &= \langle \Delta x^2 \rangle / \varepsilon_{cm} \\
\gamma_{cm} &= \langle \Delta x'^2 \rangle / \varepsilon_{cm} \\
\alpha_{cm} &= -\langle \Delta x \Delta x' \rangle / \varepsilon_{cm}
\end{aligned}$$

The macroscopic emittance is then the combination of the two distributions:

$$\varepsilon = \sqrt{(\beta_0 \varepsilon_0 + \langle \Delta x^2 \rangle)(\gamma_0 \varepsilon_0 + \langle \Delta x'^2 \rangle) - (\alpha_0 \varepsilon_0 - \langle \Delta x \Delta x' \rangle)^2}$$

For “natural” motion (induced by random displacements of magnetic elements) this centre-of-mass emittance will be matched with the electron emittance. In such a case, the macroscopic emittance becomes $\varepsilon = \varepsilon_0 + \varepsilon_{cm}$ and the emittance growth is:

$$\Delta \varepsilon / \varepsilon = \varepsilon_{cm} / \varepsilon_0$$

For residual motion after closed orbit correction, the emittance ε depends on the location of sensors, on sensor errors, and on the orbit correction algorithm. The expression of the emittance growth is less simple but the combination of the two emittances is possible.

This definition of emittance growth corresponds to a quadratic combination of beam sizes and centre-of-mass distribution. In the simple case of matched emittances the usual tolerance of 20% of emittance growth allows:

$$\sqrt{\langle \Delta x^2 \rangle} = \Delta x_{rms} = 0.458 \sigma_x \quad \text{and} \quad \sqrt{\langle \Delta x'^2 \rangle} = \Delta x'_{rms} = 0.458 \sigma'_x$$

This is a favourable case where the beam motion is integrated. In most cases the observation time is longer than 1s, so we shall use this definition for beam motion at frequencies above 1 Hz. Figure 1 illustrates this definition, the standard deviation of the displacement is taken as (0.4 x beam size) and the apparent enlargement of the beam size is 8%.

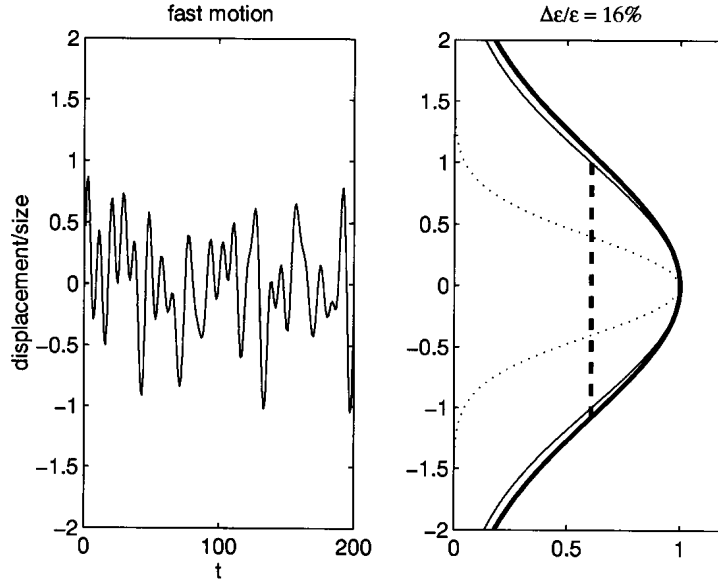


Fig. 1 Emittance growth for “fast motion”

1.1.2 Slow motion

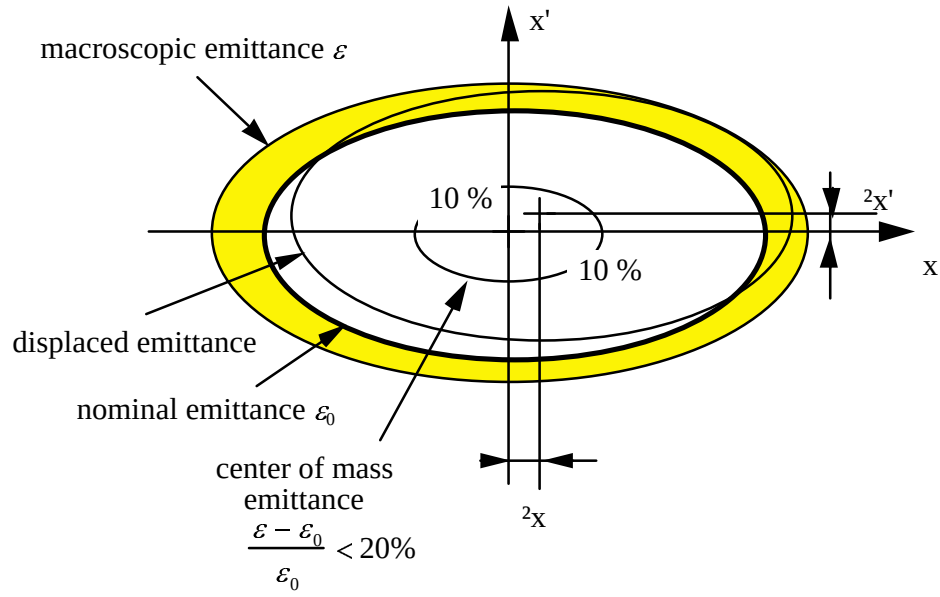
For slow motion, the macroscopic emittance will rather be defined in terms of envelope: we look for the macroscopic emittance which encloses all displaced unperturbed emittances. If the beam centre-of-mass is displaced by $\bullet x$, $\bullet x'$ from a reference position, its invariant can be defined at any point along the machine by:

$$\varepsilon_{cm} = \gamma \Delta x^2 + 2\alpha \Delta x \Delta x' + \beta \Delta x'^2$$

Alternatively, at a given location, one can define a statistical “emittance” for the centre-of-mass position. For matched beams, and with the assumption that $\varepsilon_{cm} \ll \varepsilon_0$, the macroscopic emittance growth is given by:

$$\Delta \varepsilon / \varepsilon = 2 \sqrt{\frac{\varepsilon_{cm}}{\varepsilon_0}}$$

For non-matched emittances, a pessimistic assumption consists in considering the matched centre-of-mass invariant surrounding the real one.



With this definition, the tolerance of 20% of emittance growth gives:

$$\Delta x \leq 0.1\sigma_x \text{ and } \Delta x' \leq 0.1\sigma'_x$$

In this definition the tolerance is much more severe in terms of centre-of-mass displacements. This definition will be used for motion at frequencies below 1 Hz. Figure 2 illustrates this definition, with the same standard deviation for the beam centre-of-mass displacement as in the previous section (0.4 x beam size). The beam size is now increased by 40%:

1.2 Amplification factors

When characterising various lattices, a common practice is to refer to amplification factors. These are ratios between the excitation (quadrupole motion...) and the consequence (beam motion). $\bullet q$ being the quadrupole displacement applied on all elements and $\bullet x$ the beam displacement at a given observation point, the amplification factor A is:

$$A = \sqrt{\frac{\langle \Delta x^2 \rangle}{\langle \Delta q^2 \rangle}}$$

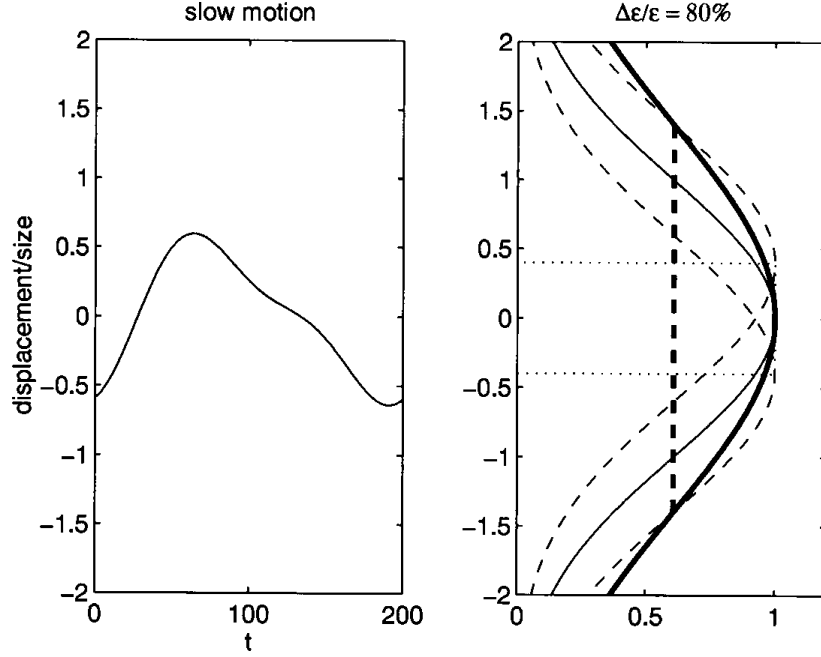


Fig. 2 Emittance growth for “slow motion”

This is easily expressed for each plane in terms of optical functions at the quadrupole locations:

$$A = \frac{\sqrt{\beta_{obs}}}{2\sqrt{2} \sin \pi\gamma} \sqrt{\sum_i \beta_i (Kl_i)^2}$$

The influence of:

- the size of the machine (proportional to the square root of the number of elements)
- the focusing of the machine (proportional to Kl and to $\sqrt{\beta}$)
- the tune

immediately appears. However the result depends on the choice of the observation point, and the performance in terms of beam size and emittance growth does not appear. This criteria is therefore useful for comparisons between different designs for similar machines.

1.3 Reference point

Beam centre-of-mass displacements have to be measured with respect to a well known reference. An absolute reference is useless since a rigid motion of the whole site, including the Storage Ring, beam lines and experiments would have no consequence at all. This implies that any motion with a wavelength larger than the site diameter will have a negligible impact on the beam stability. This will be mentioned when dealing with ground settlement or very low frequency vibrations. But even over limited distances ($\bullet 1$ km), it is unrealistic to have a stable referential in the desired range ($1 \mu\text{m}$) and for long periods of time.

A possibility to make things easier is to define one reference per beam line: the position reference is attached to the source point, the angle reference can be defined as the horizontal

plane at the source point, for vertical motion, and as an arbitrary direction, for horizontal motion. Stability measurements then rely only on differential position measurements over distances of about the length of a beam line. Experience shows that even this is at the limit of feasibility.

Another possibility is to make use of a large number of beam position sensors. The reference is taken as the beam position at the initial time T_0 . Later on the deviations of the beam position reading from the initial position will be filtered (through harmonic analysis or similar...) to separate the real beam motion from the sensor motion. This reference system (the “average beam position”) is the most stable, however it has no physical representation and cannot be easily related to the position of individual users.

2. SOURCES OF INSTABILITY

The major part of the beam instability is induced by the displacement of quadrupoles. Other effects could come from:

- displacement of other magnetic elements (dipoles or sextupoles): much weaker effect.
- magnetic field variations in the magnetic elements (fluctuations of power supplies or geometry modification).
- external magnetic fields.

As soon as some position feedback is operated (slow or fast), the stability of sensors will also become important. In any case it will affect any possibility of monitoring the beam position stability.

2.1 Long term

2.1.1 Ground settlement

Ground settlement is responsible for deformations with long wavelengths, and therefore not extremely detrimental to the beam position stability. On the other hand, large amplitudes may be reached ($\pm 500 \mu\text{m}$ / year for instance). The ESRF example is shown on Fig. 3.

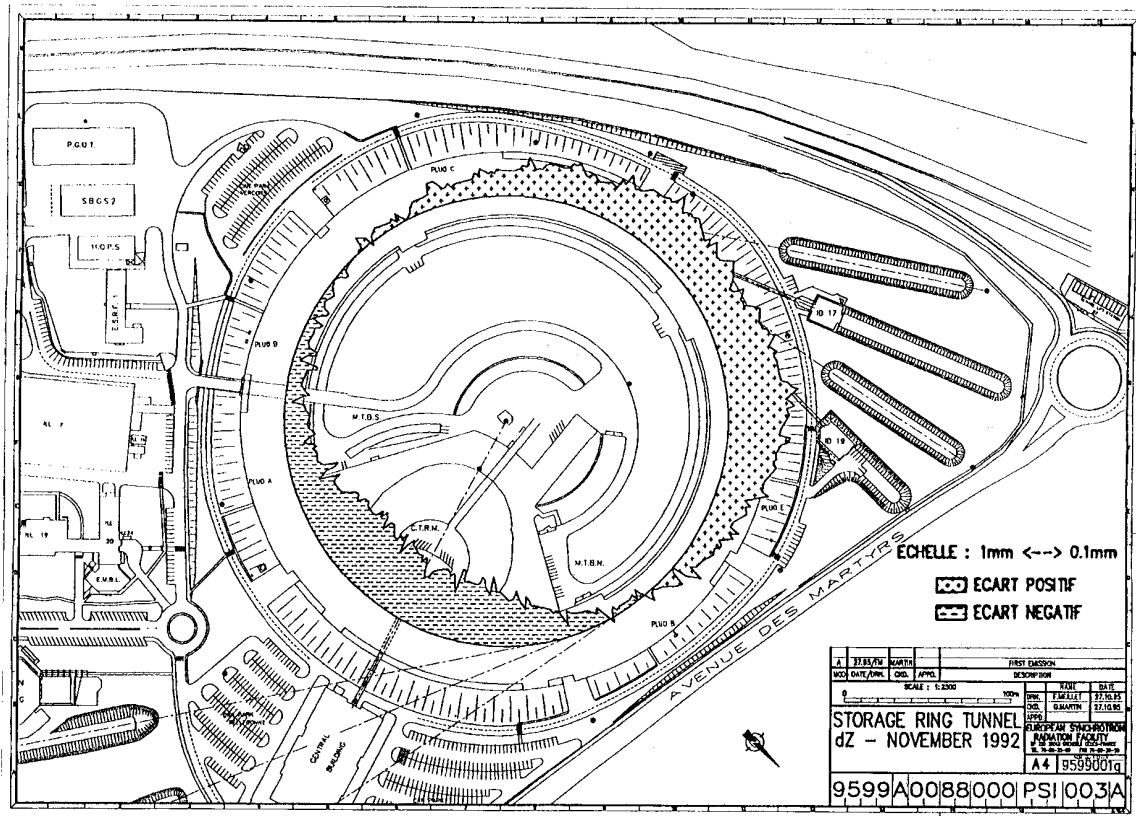


Fig. 3 Ground settlement

2.1.2 Seasonal variations

In addition to the non-reversible ground settlement, one can also observe alternating ground motion with one period per year. The maximum variation happens in Spring and Autumn, this is specially visible at the location of slab joints where the amplitude reaches 100 μm .

2.2 Medium term

2.2.1 Tides

This effect is now well known, but as for ground settlement it concerns mainly very long wavelengths. It appears on vertical position, and also on the machine perimeter (measured with a extremely good accuracy by the RF frequency). For both effects, the consequences on beam stability are usually negligible.

2.2.2 Surrounding activities

Any activity around the Storage Ring modifying the load applied on the ground may produce a local deformation: construction activities in the Experimental Hall and crane usage are good examples. The ground reaction to such a change is fast (~ 30 s) and the motion is not fully reversible. Amplitudes are about 5 μm .

2.2.3 Girder deformation

This turns out to be the major component involved in medium-term stability. The ground motion is transmitted to the magnetic elements and to the sensors through the supporting elements. In addition, thermal effects will modify the geometry of these supports.

Depending on the material, design, height,... of these supporting elements, the results may be very different (in terms of amplitudes and time constants). One can distinguish between:

- Beam intensity related effects: The heat load induced by the Synchrotron Radiation modifies the thermal equilibrium. These effects may be easily measured and possibly compensated.
- Other effects: These effects are usually slower, but may reach large amplitudes. Unless the mechanism is understood (optimistic view...), these are difficult to compensate.

2.2.4 Sensor motion

Strictly speaking, the motion of a position sensor does not induce any beam instability. However because of other effects, a beam position correction is permanently necessary, and consequently the sensor stability becomes crucial. The motion of sensors has two origins:

- Mechanical motion: this is mainly reflecting thermal effects, and it is linked with the beam intensity, and possibly the filling pattern of the Storage Ring.
- Electrical motion: this includes any kind of drift of the beam position detection, either correlated with the beam intensity through saturation effects, or not (sensitivity to the filling pattern, drift of electrical components, ageing, maintenance...).

When looking at the limit of stability which can be obtained with a permanent orbit correction, the resolution of the beam position measurements will also play a role. However the resolution can usually be made much better than the stability itself.

2.2.5 Insertion devices

The field integral of an insertion device may be compensated by different methods. But a variation of this field integral as a function of the gap will induce a beam displacement on each gap variation. For synchrotron radiation sources with many simultaneous and independent users, this becomes a severe problem. Sometimes, the variations of the second field integral (position instead of angular displacement) will also be noticeable. This can be compensated by:

- Field adjustment by shimming.
- For “exotic” devices, and if the magnetic environment modifies the performance resulting from the shimming, local compensation with electro-magnet correctors, experimentally calibrated.

2.3 Short term

2.3.1 Ground vibrations

In the lower part of the frequency range ($0.01 \text{ Hz} < f < 1 \text{ Hz}$), the vibration is mainly caused by ocean waves and micro-seismic motion. The corresponding wavelength is then above 1 km. The coherence of this motion is very good over the surface of the site of the machine, so that these effects can be neglected: as an example, the ground vibration in the bandwidth 0.05 to 1 Hz, measured at two points distant by 500 m on the ESRF site is represented on Fig. 4:

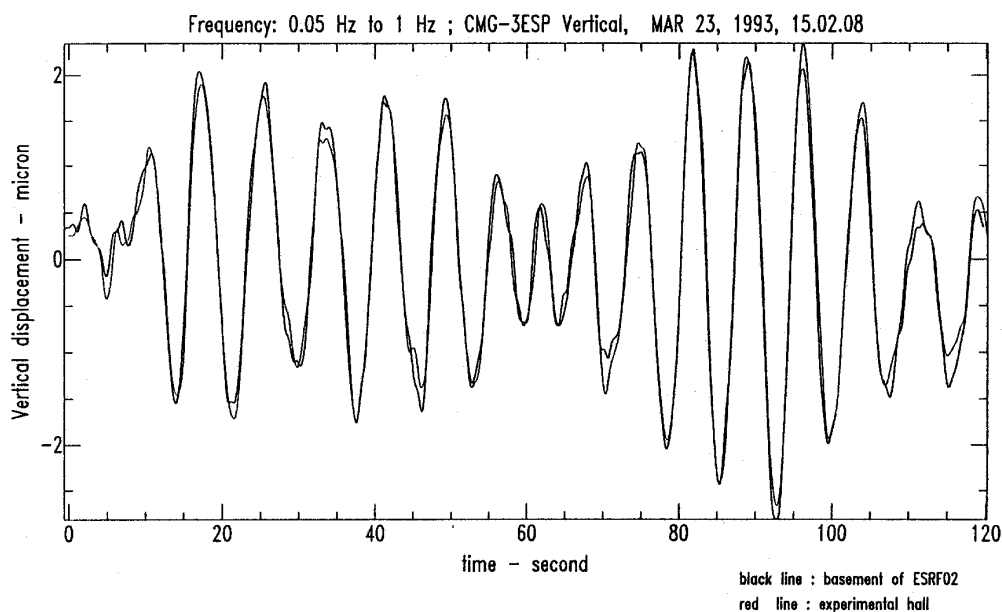


Fig. 4 Coherence of low frequency vibrations

In the upper part of the range ($1 \text{ Hz} < f < 50 \text{ Hz}$) the vibration is generated by the circulation of vehicles or trains, operation of heavy machines, wind, rivers,... Vibration in the range is approximately isotropic. A typical power density spectra is shown on Fig. 5:

The design of mechanical components such as magnet supports has to be made such that sharp resonances are avoided and the first Eigen frequency is pushed as high as possible.

2.3.2 Fluid induced vibrations

The cooling of the magnets, vacuum chamber and absorbers is a source of vibration in the upper frequency range. Interaction with the Eigen modes of the magnet supports has to be avoided by pushing the frequency to high values, using short and flexible connections.

2.3.3 Ripple of power supplies

Ripple coming through the main magnets is limited by the usually strong inductance of the magnets. However noise may appear through RF cavities, and interaction may occur if the noise (harmonic of 50 Hz or of the power supply switching frequency) interacts with the betatron frequency. Though this has been observed, it is a negligible effect in normal conditions.

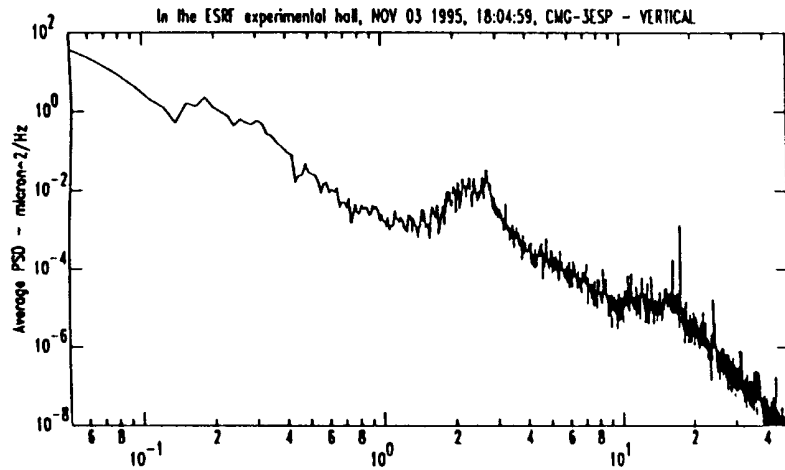


Fig. 5 Power spectral density measured at the ESRF

3. REMEDIES

3.1 Girder design

The design of girders may be optimised to minimise the vibrations, as already mentioned, but also to reduce the effect of thermal drifts: as shown in Table 1, the amplification factor for quadrupole displacements depends strongly on the correlation between the magnet displacements. Uncorrelated displacements give the worse factor, while coupled motion of opposite quadrupoles (foc. + defoc.) will partially compensate. It is then better to find the best couplings between the magnet families (obviously doublets or triplets on the same girder).

Table 1
Closed orbit amplification factors

	H	V
Uncorrelated motion	69	37
Girder motion	30	14

(Values computed for the ESRF lattice when applying random motion to individual quadrupoles or to each end of 2.5m girders.)

3.2 Machine realignment

Machine realignment is performed to compensate for the ground settlement. Though the effects of ground settlement are small, this may be necessary for mechanical reasons.

3.3 Orbit correction

3.3.1 Steerer resolution

As long as orbit corrections are continuously performed, the steerer currents will change and the effect of the resolution of the steerer power supplies will spoil the stability. Assuming a large number of randomly distributed steerers, the centre-of-mass emittance due to the resolution errors, at any point along the machine, is:

$$\varepsilon_{cm} = \frac{n \bar{\beta} \Delta x'_{rms}{}^2}{2 (2 \sin \pi \nu)^2}$$

where:

- $\Delta x'_{rms}$ is the angular kick corresponding to the rms steerer resolution error,
- $\bar{\beta}$ is the average β value at the location of steerers,
- n is the number of steerers being changed.

The emittance growth is computed (“slow motion” case) as:

$$\Delta \varepsilon / \varepsilon = \frac{\Delta x'_{rms}}{\sin \pi \nu} \sqrt{\frac{n \bar{\beta}}{2 \varepsilon_0}}$$

3.3.2 Local correction

The orbit stability when using local correction is limited by the stability of the two sensors defining the local position. The centre-of-mass emittance can be computed easily in the middle of the two monitors. x_1, x_2 are the readings on the monitors, Δx is the error on the monitor position. The distance between the monitors is $2l$. If we assume a perfect local correction following exactly the centre of the BPMs, the beam position in the middle of the sensors is:

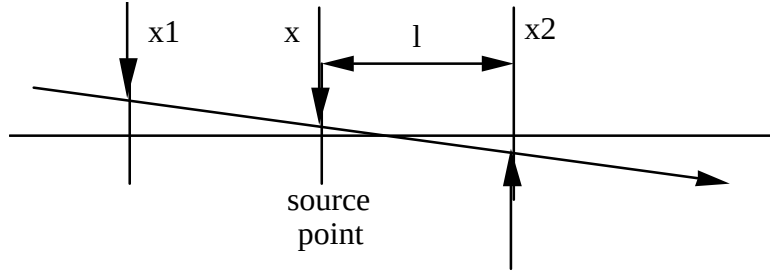
$$\begin{aligned} x &= \frac{x_2 + x_1}{2} & x' &= \frac{x_2 - x_1}{2l} \\ \langle x^2 \rangle &= \frac{\Delta x^2}{2} & \langle x'^2 \rangle &= \frac{\Delta x^2}{2l^2} \end{aligned}$$

and the centre-of-mass emittance is defined by:

$$\begin{aligned} \varepsilon_{cm} &= \frac{\Delta x^2}{2l} \\ \beta_{cm} &= l \\ \alpha_{cm} &= 0 \end{aligned}$$

This centre-of-mass emittance has to be transported to the reference point (source point in the middle of the Insertion Device straight section) and combined with the electron beam emittance at that point. Two different possibilities are studied: e-BPMs and X-BPMs:

e-BPM: These monitors may be located on each side of the observation point, but their distance is limited by the length of the machine straight section:



The middle of the two BPMs is exactly the point where we want to evaluate the stability. The best matching is reached when the BPMs are at a distance equal to the β value. In the general case (no matching), we will take an upper limit of the emittance growth by using the matched centre-of-mass emittance surrounding emittance:

$$\varepsilon_{cm} = \frac{\Delta x^2}{2l} \max\left(\frac{\beta}{l}, \frac{l}{\beta}\right)$$

This is a pessimistic assumption, and depending on the focusing at the observation point, the emittance growth may never reach that value. However the emittance growth for the “slow motion” convention is:

$$\begin{aligned} l > \beta \quad \Delta\varepsilon/\varepsilon &= 2 \frac{\Delta x}{\sqrt{2} \sigma_x} && \text{position limitation} \\ l < \beta \quad \Delta\varepsilon/\varepsilon &= 2 \frac{\Delta x}{\sqrt{2} l \sigma'_x} && \text{angle limitation} \end{aligned}$$

This shows immediately the limitations of the “local correction” option. The degradation due to the wrong matching is shown for example on Fig. 6 (phase spaces in μm and μrad). The centre-of-mass emittance, in solid line, has to be maximised to match the electron beam emittance and reaches the dashed line value:

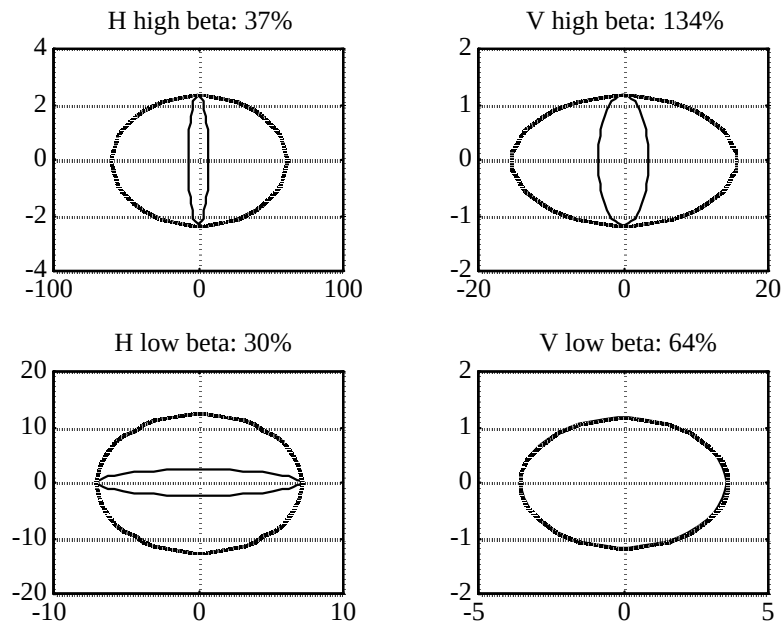
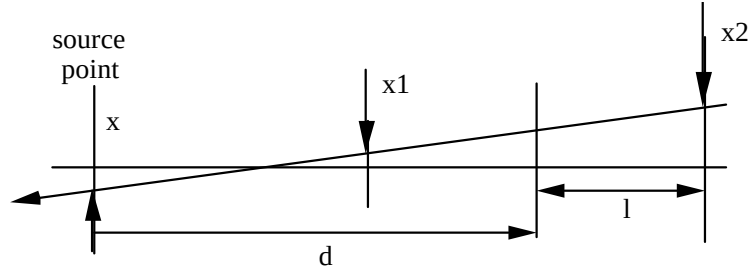


Fig. 6 Local correction matching

X-BPM: these sensors are located downstream the beam line.



In this case the distance between the sensors can be more favourable, but when transporting the emittance backwards to the source point, the matching may be worse. The centre-of-mass emittance at the source point is:

$$\varepsilon_{cm} = \frac{\Delta x^2}{2l}$$

$$\beta_{cm} = l \left(1 + \frac{d^2}{l^2} \right)$$

$$\alpha_{cm} = -\frac{d}{l}$$

The surrounding matched emittance can be written as:

$$\varepsilon_{cm} = \frac{\Delta x^2}{2l} \left(C + \sqrt{C^2 - 1} \right)$$

and the emittance growth becomes:

$$\Delta \varepsilon / \varepsilon = 2 \Delta x \sqrt{\frac{C + \sqrt{C^2 - 1}}{2l \varepsilon_0}}$$

with

$$C = \frac{\beta^2 + l^2 + d^2}{2\beta l}$$

In spite of the better lever arm, the results may be worse than for e-BPMs, because of the worse emittance matching. It is in particular extremely bad in low beta sections, where a position error due to a X-BPM error can be extremely large compared to the very small beam size at that point. This is demonstrated on

Fig. 7, showing the phase space (in μm and μrad) corresponding to four cases of local correction.

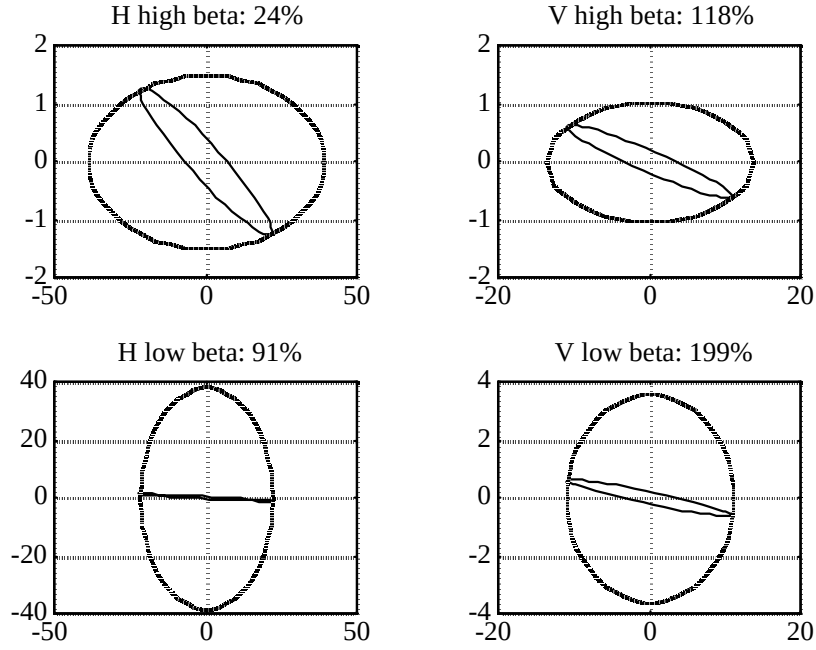


Fig. 7 Local correction matching

3.3.3 Global correction

The stability when using only global orbit correction is limited by the steerer resolution and the average stability of all BPMs. The sensitivity to the sensor motions depends on the correction method. The best flexibility is obtained with the SVD correction algorithm where one can tune the number of Eigen correction vectors between 0 and the total number of available steerers. A large number gives better correction possibilities but also a larger sensitivity to sensor position fluctuations. The best compromise depends of course on the amplitude of the distortions one has to correct.

This sensitivity to sensor errors can be studied by simulation. A machine with a realistic set of random errors and random sensor displacements is corrected with n , number of Eigen vectors, varying from 0 to the maximum number. All special conditions like sensor attachments to magnets, rigid girder motion... may be taken into account. For each correction, the emittance growth is averaged over similar points of the lattice. Average values for the high- β straight section are plotted on Fig. 8. For small n , the stability rapidly improves with increasing n , since the first vectors are the most efficient. For large n , additional vectors only allow the orbit to follow more accurately the mispositioned monitors, and the emittance growth increases. Repeating this on a few error sets gives a good indication of the best compromise, as shown on Fig. 8.

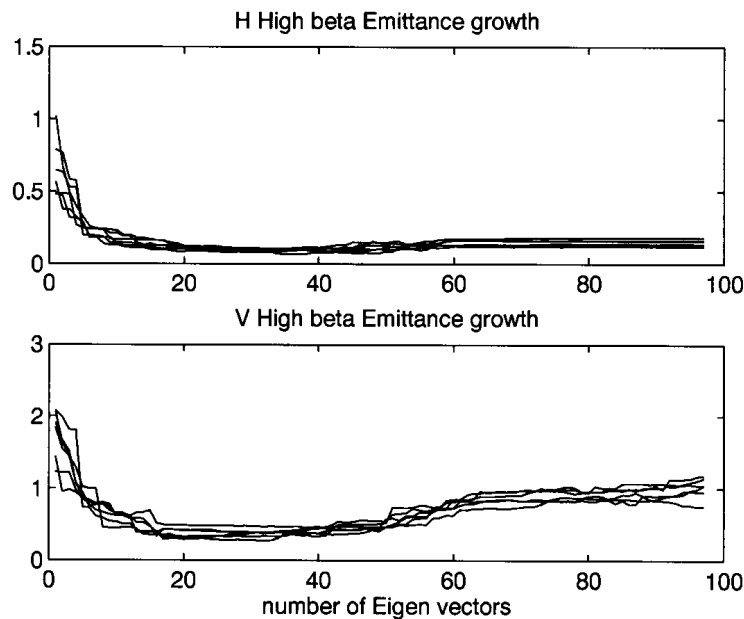


Fig. 8 Optimisation of a global correction

3.4 Vibration dampers

Vibration dampers using sandwich structure with visco-elastic material proved very efficient to reduce the vibration of magnets (APS example): the vibration amplitude may be reduced by a factor 5 in wide band (4 to 50 Hz) or even 10 on the first Eigen mode. However,

- The damping solution is very specific to each structure.
- It has to be located in a region with large shear strain.

3.5 Fast feedback

The same arguments concerning local and global correction apply for a fast feedback. However the question of the sensor drift can be avoided to a certain extent by limiting the bandwidth of the system: as sensor drifts due to saturation, thermal effects,... are slow, the feedback system must reject the low frequencies ($T < \text{a few minutes}$).

3.5.1 Local feedback

- is technically easier,
- must reject the low frequencies,
- the matching of the beam motion with the correction error ellipse must be good.

3.5.2 Global feedback

- may be more difficult, depending on the upper frequency limit,
- Same advantages as the global orbit correction.

4. EXAMPLES

Experimental values for some of these effects are given, in the case of the ESRF Storage Ring. Some relevant parameters are listed in Table 2:

Table 2
ESRF parameters

	H	V
Perimeter	844.39 m	
Number of BPMs	224	
Number of quadrupoles	320	
Number of steerers	96	
Emittances	$4 \cdot 10^{-9}$ m	$4 \cdot 10^{-11}$ m
Tunes	36.44	11.39

4.1 Long term (4 weeks)

The main contributions are summarised as follows:

Table 3
Contributions to long-term motion

Effect	Conditions	Horizontal (μm)	Vertical (μm)
Ground settlement		$\bullet 0$	$\bullet 0$
Girder motion ($\bullet$ x adjacent girders)	peak/4 weeks	15	20 μm 15 μrad
Quadrupole/girder	peak/4 weeks	2	1
BPM/quadrupole	peak/4 weeks	1–10	1–10
BPM replacement	2*rms	20	20

In the long term the dominant effect is the slow motion of girders. An example of the differential motion between two adjacent girders versus time is shown on Fig. 9:

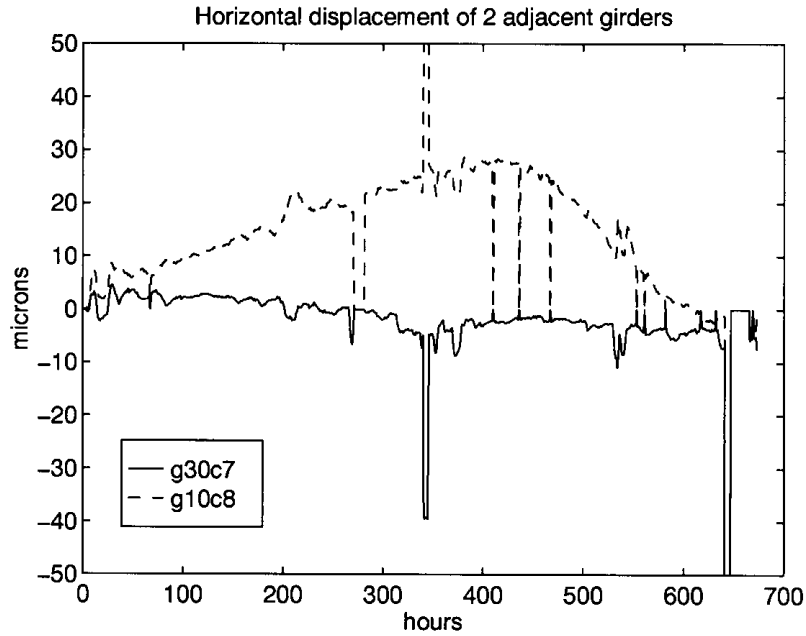


Fig. 9 Differential motion between two girders

Errors plus global or local corrections may be simulated, which gives the following estimations:

Table 4
Simulation of long term emittance growth

Emittance growth	Horizontal	Vertical
No correction	520%	2800%
Local correction	75%	480%
40 vectors	25%	250%

A control can be performed on X-BPMs: with only one sensor, one cannot derive both position and angle, but the ratio position/beam size gives an order of magnitude of half the emittance growth. The results measured on X-BPMs show:

- horizontally a peak value of 10% in size to be compared with 25% of emittance growth
- vertically 150% in size to be compared with 250% in emittance

The orders of magnitude for the errors and corrections look consistent with the simulations.

4.2 Medium term (a day or one beam decay)

The main contributions are summarised in Table 5. In the medium term, the instability is mainly generated by

- individual quadrupole displacements.

— girder motion linked with intensity.

Table 5
Contributions to medium term motion

Effect	Conditions	Horizontal (μm)	Vertical (μm)
Girder motion (•x adjacent girders)	peak/1 day	1.5	1
Quadrupole/girder		2	1
BPM/quadrupole	peak/1 day	1–10	1–10
BPM drift	2*rms	20	20
Crane	peak/1 day		1.5

The correction is limited by the sensitivity of sensor to beam intensity (mechanically and electrically). In the medium term, simulations give:

Table 6
Simulation of medium term emittance growth

Emittance growth	Horizontal	Vertical
No correction	60%	180%
Local correction (high β)	30%	220%
Best correction (high β)	10%	40%

The control with X-BPM readings is given in Fig. 10 which shows the evolution of the vertical position over 250 hours, with local correction ($150 < t < 280$) and with only global SVD correction ($280 < t$). With local correction, the drift with beam intensity is clearly visible and the motion is compatible with the predicted value of 220%. With only global correction, the fluctuations look slightly smaller than the expected value of 20%.

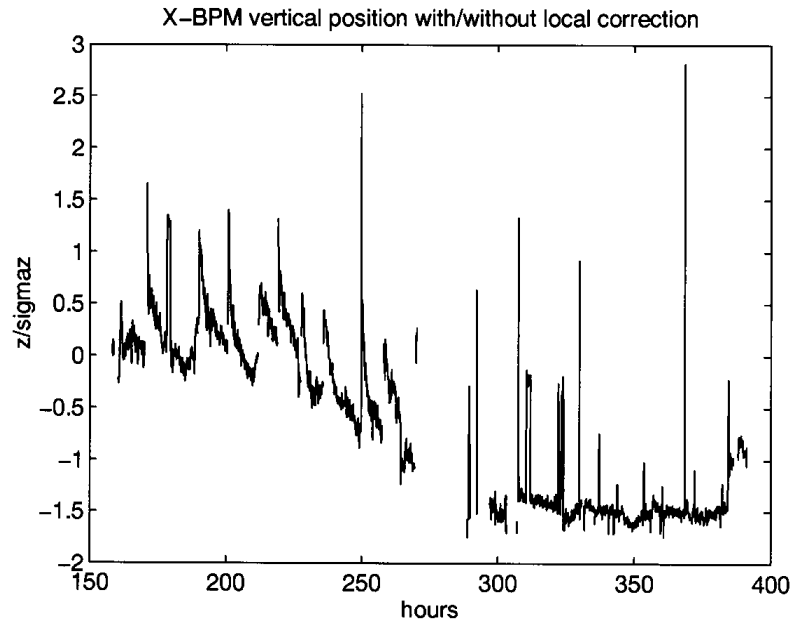


Fig. 10 X-BPM vertical position

4.3 Short term

Without correction:

Table 7
Contributions to short term motion

Effect	Conditions	Horizontal (μm)	Vertical (μm)
Ground vibration noise		1	0.5
Road traffic	peak	2*girder amplification	2*girder amplification
Absorber movement	peak	35	10

Emittance growths measured on feedback electron BPMs give, using fast motion quadratic combination:

Table 8
Simulation of short term emittance growth

Emittance growth	Horizontal	Vertical
No correction	$5 \cdot 10^{-4}$	1.2%
Local correction	$7 \cdot 10^{-5}$	0.4%

Even in the pessimistic case of the “slow motion” combination, the values would be small:

Table 9
Simulation of short term emittance growth

Emittance growth	Horizontal	Vertical
No correction	4.6%	22%
Local correction	1.7%	13%

5. CONCLUSIONS

- Vibrations should not introduce significant disturbances to the beam stability. In addition a correction is proven to be efficient. A significant part of vibrations comes from the facility itself, and the influence of the site is of secondary importance.
- In the medium term, thermal or electronic effects linked with the beam intensity are dominating. They can be corrected to a satisfactory level by using a global correction scheme filtering out most of the sensor drifts. However the stability of sensors is the limiting parameter.
- Slow motion due to girder displacements is by far the more difficult problem. The reference for measuring such a motion is even doubtful.

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Abstract

Synchrotron radiation is used mainly to measure the dimensions of an electron beam. The transverse size is obtained by forming an image of the beam cross section by means of the emitted synchrotron radiation. The obtained resolution is limited by diffraction. The angular spread of the particles in the beam can be obtained by direct observation of the radiation. Here, the natural opening angle of the emitted light sets a limit to the resolution. Measuring both beam cross section and angular spread gives the emittance of the beam. In most cases only one of the two parameters is observed and the other obtained from the known properties of the particle optics. The longitudinal particle distribution is directly obtained from the observed time structure of the emitted radiation. In most cases the observed radiation is emitted in long bending magnets. However, short magnets and undulators are also useful sources for some measurements. For technical reasons the beam diagnostics is carried out using visible or ultraviolet light. This part of the spectrum is far below the critical frequency and corresponding approximations can be applied for the radiation properties. Synchrotron radiation is an extremely useful tool for diagnostics in electron (or positron) rings. In some cases it has also served in proton rings using special magnets.

1 INTRODUCTION

There are mainly three types of measurement made with synchrotron radiation: imaging to measure the beam cross section, direct observation to measure the angular spread of the particle and observation of the longitudinal structure of the radiation to obtain the bunch length.

For the most common measurement the radiation emitted tangentially in the bending magnet is extracted from the vacuum chamber through a window. A lens is then used to form an image of the source point on a screen. This is illustrated in Fig. 1.

It is also possible to observe the synchrotron radiation directly without using focus-

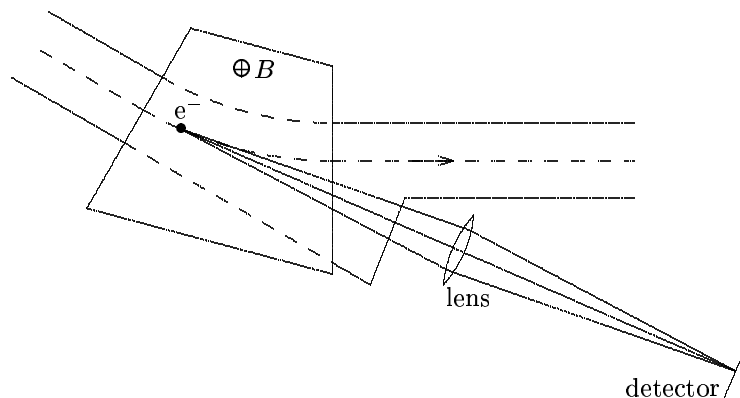


Figure 1: Imaging of the beam cross section with synchrotron radiation

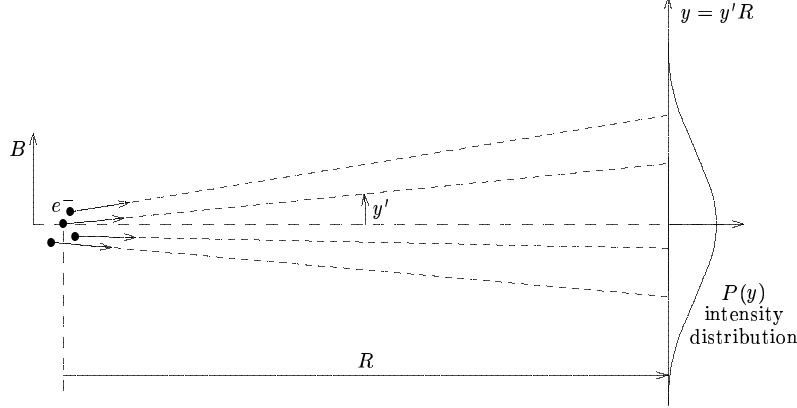


Figure 2: Direct observation of synchrotron radiation

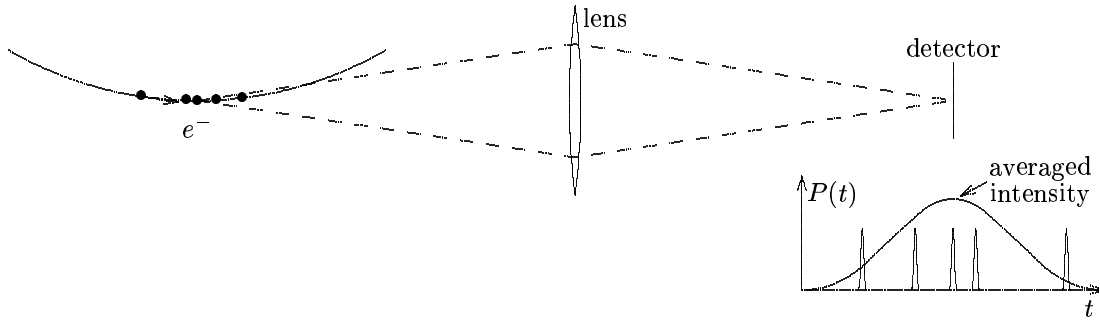


Figure 3: Observation of the synchrotron radiation time structure to measure bunch length

ing elements as shown in Fig. 2. In this case one measures the angular distribution of the particles in the beam. If a horizontal bending magnet serves as source of the radiation only the vertical angles can be measured. The resolution is limited by the natural opening angle of the radiation itself.

The bunch length can be measured by observing the time structure of the emitted radiation. Since the light pulse observed from each individual particle is very short the time distribution of the radiation reflects directly the longitudinal bunch shape as indicated in Fig. 3. A fast photon detector is needed to measure this distribution.

2 PROPERTIES OF SYNCHROTRON AND UNDULATOR RADIATION

The properties of ordinary synchrotron radiation emitted in long bending magnets has been treated in an earlier lecture of this school [1] and we will in the following refer to the derivation carried out there. However, for convenience we summarize here the most important results.

2.1 Qualitative treatment of the radiation

We start with a qualitative treatment of the synchrotron radiation emitted in long magnets, in short magnets and in undulators.

We consider an electron moving in the laboratory frame F on a circular orbit and

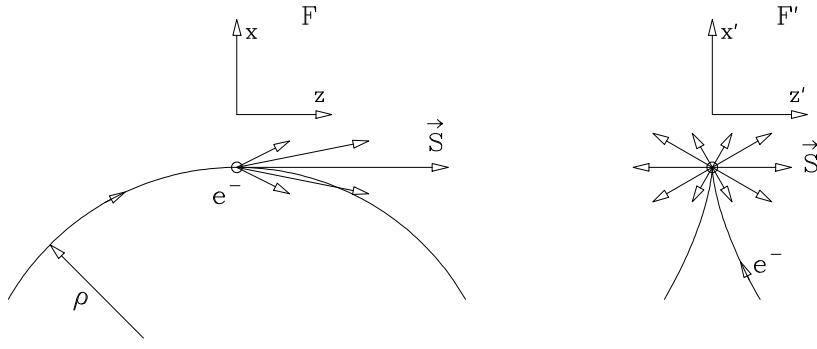


Figure 4: Opening angle of synchrotron radiation caused by the moving source

emitting synchrotron radiation, Fig. 4. In a frame F' which moves at one instant with the same velocity, $\mathbf{v} = \beta c$, as the electron the particle trajectory has the form of a cycloid with a cusp where the electron undergoes an acceleration in the $-x'$ direction. Like any accelerated charge it will emit radiation which in this frame F' is approximately uniformly distributed. Going now back to the laboratory frame F , by applying a Lorentz transformation, this radiation will be peaked forward. A photon emitted along the x' -axis in the moving frame F' will appear at an angle $1/\gamma$ in the laboratory frame F . The typical opening angle of synchrotron radiation is therefore expected to be of the order of $1/\gamma$. For ultra-relativistic particles $\gamma \gg 1$ the radiation is confined to very small angles around the direction of the electron motion.

Next we try to estimate the typical frequency of the emitted synchrotron radiation spectrum and consider an electron going through a long magnet where it emits radiation which reaches an observer P , Fig. 5. We ask ourselves how long the pulse of radiation will last. Due to the small opening angle this observer will see the light for a rather short time only. The radiation seen first is emitted at the point A , where the electron trajectory has an angle of $1/\gamma$ with respect to the direction towards the observer, and the last time at point A' where this angle is $-1/\gamma$. The length of the radiation pulse seen by the observer is just the difference in travel time between the electron and the photon in going from point A to point A'

$$\Delta t = t_e - t_\gamma = \frac{2\rho}{\beta\gamma c} - \frac{2\rho \sin(1/\gamma)}{c}.$$

For the ultra-relativistic case we consider here, $\gamma \gg 1$, we can expand the trigonometric function

$$\Delta t \sim \frac{2\rho}{\beta\gamma c} \left(1 - \beta + \frac{\beta}{6\gamma^2}\right) \sim \frac{2\rho}{\gamma c} \left(\frac{1}{2\gamma^2} + \frac{1}{6\gamma^2}\right) = \frac{4}{3} \frac{\rho}{c\gamma^3},$$

where we used the approximation

$$1 - \beta = \frac{1 - \beta^2}{1 + \beta} \sim \frac{1}{2\gamma^2}.$$

The typical frequency is approximately

$$\omega_{typ} \sim \frac{2\pi}{\Delta t} \sim \frac{c\gamma^3}{\rho}.$$

This frequency is proportional to γ^3 . A factor γ^2 comes from the difference in velocity between electron and photon and another factor of γ is due to the difference in trajectory length of the two particles in the magnet.

We consider now the radiation emitted in a short magnet having a length $L < 2\rho/\gamma$. An observer will receive the radiation emitted during the whole passage of the electron through this magnet, Fig. 6. The duration of the received pulse is now determined by the length L of the deflecting magnet. Again, the length of the radiated pulse is given by the difference in traveling times of the electron and photon through the magnet

$$\Delta t_{\text{sm}} = \frac{L}{\beta c} - \frac{L}{c} = \frac{L(1 - \beta)}{\beta c} \approx \frac{L}{2c\gamma^2},$$

and the typical frequency is

$$\omega_{\text{sm}} \approx \frac{4\pi c\gamma^2}{L}. \quad (1)$$

This frequency contains only a factor γ^2 since the difference in trajectory length is small if the magnet is sufficiently short.

An interesting source of synchrotron radiation is an undulator. It consists of specially periodic magnetic fields with period length λ_u in which the particles move on a sinusoidal orbit, Fig. 7. Each of the periods represents a source of radiation. These contributions emitted towards an observer at an angle θ will interfere with each other. We get maximum intensity at a wavelength λ for which the contributions from different undulator periods are in phase. The time difference ΔT between the arrival of adjacent contributions is

$$\Delta T = \frac{\lambda_u}{\beta c} - \frac{\lambda_u \cos \theta}{c} = \frac{\lambda_u(1 - \beta \cos \theta)}{\beta c}.$$

For a relativistic particle the angle θ where radiation of reasonable intensity can be observed is small; $\theta \approx 1/\gamma$. We can approximate $\cos \theta \approx 1 - \theta^2/2$.

$$\Delta T \approx \frac{\lambda_u}{\beta c} \left(1 - \beta + \frac{\theta^2}{2} \right) \approx \frac{\lambda_u}{2c\gamma^2} (1 + \gamma^2 \theta^2). \quad (2)$$

The frequency for which we get constructive interference is just $\omega = 2\pi/\Delta t$

$$\omega = \frac{4\pi c\gamma^2}{\lambda_u(1 + \gamma^2 \theta^2)}.$$

Harmonics of this frequency might also be emitted.

3 Radiation emitted by a relativistic charge

3.1 The time scales for emission and observation of the radiation

Synchrotron radiation is emitted by a moving charge and received by a stationary observer. To describe these processes one uses two time scales: the time t' of emission and the time t of observation. This is shown in Fig. 8 where the charge q moves on a trajectory $\mathbf{R}(t')$ emitting radiation which is received by an observer P , being at a distance $r(t') = |\mathbf{r}(t')|$, at the later time t

$$t = t' + \frac{r(t')}{c}.$$

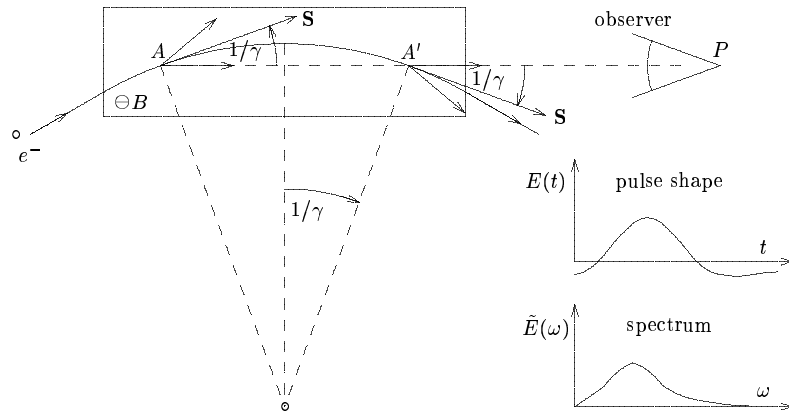


Figure 5: Spectrum of synchrotron radiation emitted in a long magnet

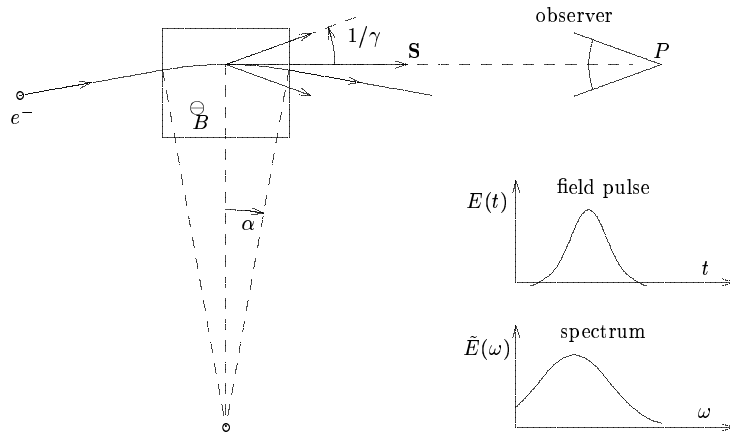


Figure 6: Spectrum of synchrotron radiation emitted in a short magnet

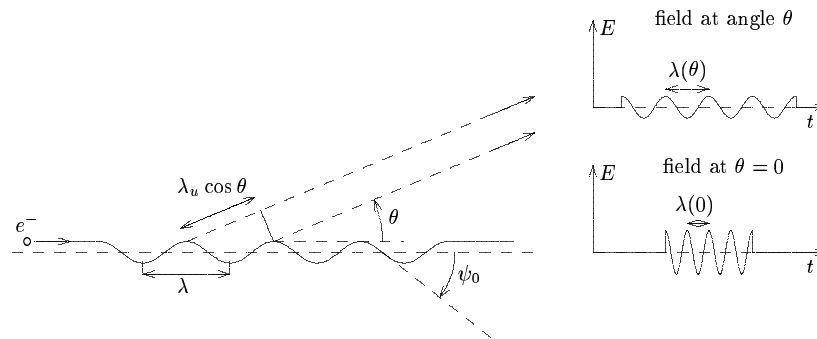


Figure 7: Spectrum of synchrotron radiation emitted in an undulator

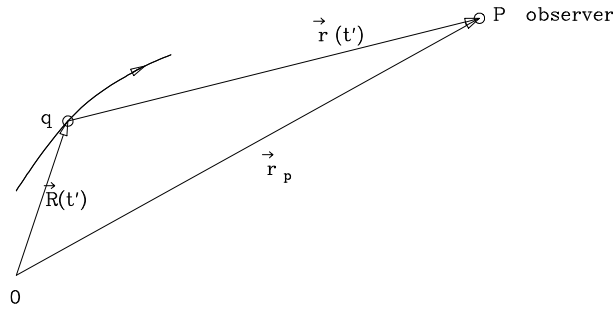


Figure 8: Particle trajectory and radiation geometry

The vectors $\mathbf{R}(t')$ and $\mathbf{r}_p$, pointing from the origin to the charge and the observer respectively, and the vector $\mathbf{r}(t')$, pointing from the charge to the observer, are related by

$$\mathbf{R}(t') + \mathbf{r}(t') = \mathbf{r}_p.$$

We differentiate this with respect to the time t' and use the expression for the particle velocity and the unit vector pointing from the charge to the observer

$$\frac{d\mathbf{R}(t')}{dt'} = \mathbf{v} = \boldsymbol{\beta}c, \quad \mathbf{n} = \frac{\mathbf{r}(t')}{r(t')}$$

to get the time derivative of $r(t')$

$$\mathbf{r} \frac{d\mathbf{r}}{dt'} = \frac{1}{2} \frac{d(\mathbf{r}^2)}{dt'} = r \frac{dr}{dt'} = -\mathbf{r} \cdot \mathbf{v}, \quad \frac{dr}{dt'} = -\mathbf{n} \cdot \mathbf{v}, \quad \frac{d\mathbf{r}}{dt'} = -\mathbf{v} = -\boldsymbol{\beta}c.$$

From this we obtain the relation between increments of the observation time t and the emission time t'

$$dt = (1 - \mathbf{n} \cdot \boldsymbol{\beta}) dt'.$$

This relation is important for many aspects of synchrotron radiation and appears in various expressions. In the forward direction, where $\mathbf{n}$ and $\boldsymbol{\beta}$ are nearly parallel, the observation time interval is in the relativistic case much shorter than the emission time interval. This compaction in time leads to the very high frequencies observed in synchrotron radiation.

3.2 The fields of a moving, accelerated charge

One starts from the potentials of a moving charge and gets the field from Maxwell's equation taking the relation between the two time scales into consideration. As a result we obtain the Liénard-Wiechert equation for the electric and magnetic field of a moving and accelerated charge [1],

$$\begin{aligned} \mathbf{E}(t) &= \frac{q}{4\pi\epsilon_0} \left(\frac{(1 - \beta^2)(\mathbf{n} - \boldsymbol{\beta})}{r^2(1 - \mathbf{n} \cdot \boldsymbol{\beta})^3} + \frac{[\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]]}{cr(1 - \mathbf{n} \cdot \boldsymbol{\beta})^3} \right)_{ret.} \\ \mathbf{B}(t) &= \frac{[\mathbf{n} \times \mathbf{E}]}{c}. \end{aligned} \quad (3)$$

These equations are the basis for calculating the radiation emitted in long and short magnets as well as in undulators. The index ‘ret.’ indicates that the expression inside the bracket has to be evaluated at the time t' of emission in order to get the field received at the time t by the observer. For a general motion of a charge the relation between the time scales can be very complicated.

The expression for the electric field has two terms. The first one, proportional to $1/r^2$, does not contain the acceleration and can be reduced to a Coulomb field by a Lorentz transformation. Therefore, it does not contribute to the radiated power. The second term is proportional to the acceleration and to $1/r$. It will dominate at large distances and is for this reason often referred to as ‘far-field’. We will from now on concentrate on this far-field and use

$$\mathbf{E}(t) \approx \mathbf{E}_{far-field} = \frac{q}{4\pi\epsilon_0 c} \left(\frac{[\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]]}{r(1 - \mathbf{n} \cdot \boldsymbol{\beta})^3} \right)_{ret.} \quad (4)$$

This is fine as long as we use the field to discuss the polarization properties and to calculate the radiation power. It should however be pointed out that the ‘far-field’ alone does not satisfy Maxwell’s equations.

3.3 The power radiated by the particle

The power flux of this field is determined by the Poynting vector $\mathbf{S}$

$$\mathbf{S} = \frac{1}{\mu_0} [\mathbf{E} \times \mathbf{B}] = \frac{1}{\mu_0 c} [\mathbf{E} \times [\mathbf{n} \times \mathbf{E}]] = \frac{1}{\mu_0 c} (E^2 \mathbf{n} - (\mathbf{n} \cdot \mathbf{E}) \mathbf{E}).$$

For the far field we have $(\mathbf{n} \cdot \mathbf{E}) = 0$ and therefore

$$\mathbf{S} = \frac{E^2 \mathbf{n}}{\mu_0 c}.$$

It represents the energy passing through a unit area per unit time t of observation. To get the power P radiated by the particle into a unit solid angle we have to consider the energy W emitted by the charge per unit time t' of emission

$$\frac{dP}{d\Omega} = \frac{d^2 W}{d\Omega dt'} = (\mathbf{n} \cdot \mathbf{S}) \frac{dt}{dt'} r^2 = S(1 - \mathbf{n} \cdot \boldsymbol{\beta}) r^2 = \frac{r^2 E^2}{\mu_0 c} (1 - \mathbf{n} \cdot \boldsymbol{\beta}), \quad (5)$$

where $S = |\mathbf{S}|$ is the absolute value of the Poynting vector and Ω is the solid angle. To calculate the power distribution we use a coordinate system (x, y, z) in which the particle is momentarily at the origin moving in the z -direction and express the three components with the angles θ and ϕ of the corresponding spherical coordinate system. The unit vector $\mathbf{n}$ pointing from the particle to the observer and the normalized velocity vector $\boldsymbol{\beta}$ are

$$\mathbf{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \quad \text{and} \quad \boldsymbol{\beta} = \beta (0, 0, 1).$$

We take now the case of an acceleration being perpendicular to the velocity and pointing in the $-x$ direction. This corresponds to synchrotron radiation emitted by an elementary charge $q = e$ going through a magnetic field B_y with a curvature

$$\frac{1}{\rho} = \frac{eB}{\beta \gamma m_0 c}.$$

The normalized acceleration is

$$\dot{\boldsymbol{\beta}} = \frac{\beta^2 c}{\rho} (-1, 0, 0).$$

The distribution of the power radiated by the particle is

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{e^2}{(4\pi)^2 \epsilon_0 c} \frac{[\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]]^2}{(1 - \mathbf{n} \cdot \boldsymbol{\beta})^5} \\ &= \frac{cr_0 m_0 c^2 \beta^4}{4\pi \gamma^2 \rho^2} \left(\frac{\gamma^2 (1 - \beta \cos \theta)^2 - \sin^2 \theta \cos^2 \phi}{(1 - \beta \cos \theta)^5} \right), \end{aligned} \quad (6)$$

where we introduced the classical particle radius.

$$r_0 = \frac{e^2}{4\pi \epsilon_0 m_0 c^2} = \begin{array}{l} 2.818 \cdot 10^{-15} \text{ m for electrons} \\ 1.535 \cdot 10^{-18} \text{ m for protons} \end{array}.$$

Integrating (6) over the solid angle gives the total power radiated by the particle

$$P_0 = \frac{2r_0 c m_0 c^2 \beta^4 \gamma^4}{3\rho^2} \approx \frac{2r_0 c m_0 c^2 \gamma^4}{3\rho^2} = \frac{2r_0 e^2 c^3 (m_0 c^2 \gamma)^2 B^2}{3 (m_0 c^2)^3} \quad (7)$$

where we approximated for an ultra-relativistic particle. If the orbit is a closed circle we get the energy U_s lost per turn by multiplying the power by the revolution time $T = 2\pi\rho/c$

$$U_s = \frac{4\pi r_0 m_0 c^2 \gamma^4}{3\rho} = \frac{4\pi r_0 e c (m_0 c^2 \gamma^2)^3 B}{3 (m_0 c^2)^3}.$$

This expression is also valid for a ring having all magnets of the same strength and field-free straight sections in between.

For the instantaneous angular distribution (6) we assume the ultra-relativistic case $\beta \approx 1$, $\gamma \gg 1$ for which the radiation is peaked forward confined to a cone of opening angle $\theta \sim 1/\gamma$. We use the corresponding approximations and the expression for the total power P_0 to obtain

$$\frac{dP}{d\Omega} = P_0 \frac{3\gamma^2}{\pi} \left(\frac{1 + 2\gamma^2 \theta^2 (1 - 2 \cos^2 \phi) + \gamma^4 \theta^4}{(1 + \gamma^2 \theta^2)^5} \right).$$

3.4 Fourier transformed radiation field and angular spectral power density

We derived the electric and magnetic radiation fields emitted by a moving charge $q = e$ as a function of time (4). As we said before, the difficulty to calculate these fields lies in the fact that the expressions involving the particle motion have to be evaluated at the earlier time t' which has, in general, a rather complicated relation to the time t of observation. For this reason it is often easier to calculate directly the Fourier transform $\tilde{\mathbf{E}}(\omega)$ of the electric field

$$\tilde{\mathbf{E}}(\omega) = \frac{e}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{E}(t) e^{-i\omega t} dt.$$

This integration involves the time t since we are interested in the spectrum of the radiation as seen by the observer. We can however make a formal substitution of the integration variable t by $t = t' + r(t')/c$, $dt = (1 - \mathbf{n} \cdot \boldsymbol{\beta})dt'$ and get

$$\tilde{\mathbf{E}}(\omega) = \frac{q}{4\pi c\epsilon_0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{[\mathbf{n} \times [(\mathbf{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]]}{r(1 - \mathbf{n} \cdot \boldsymbol{\beta})^2} \right) e^{-i\omega(t' + r(t')/c)} dt'. \quad (8)$$

We omitted in the above equation the index ‘ret’ since the integration variable is anyway the time t' at which the expressions are evaluated. By partial integration we obtained in [1] the Fourier transformed electric field

$$\tilde{\mathbf{E}}(\omega) = \frac{i\omega e}{4\pi\sqrt{2\pi}\epsilon_0 c r} \int_{-\infty}^{\infty} [\mathbf{n} \times [\mathbf{n} \times \boldsymbol{\beta}]] e^{-i\omega(t' - r(t')/c)} dt' \quad (9)$$

which is an easier equation to deal with than (4) giving the field in time domain. However, we should remember the expression (9) is less accurate than (4).

Based on this Fourier transformed field the angular spectral energy density, i.e. the energy radiated per unit solid angle and frequency band, has been calculated [1]

$$\frac{d^2 W}{d\Omega d\omega} = \frac{2r^2 |\tilde{E}(\omega)|^2}{\mu_0 c}. \quad (10)$$

The factor 2 on the right hand side indicates that the spectral energy density is taken at positive frequencies only, contrary to the field which is taken at positive and negative frequencies. This is common practice since power can be measured directly but the sign of the frequency cannot be observed during such measurements. The field, however, is rarely accessible to direct measurements.

Sometimes one likes to give the power radiated per unit solid angle and frequency band, called the angular spectral power density. This only makes sense if this power can be averaged over some interval. In the case of synchrotron radiation emitted on a closed circular orbit of radius ρ and revolution frequency $\omega_0 = \beta c/\rho \approx c/\rho$ the observer receives $c/2\pi\rho$ such flashes per second from the particle and an *average* spectral angular power density

$$\frac{d^2 P}{d\Omega d\omega} = \frac{2r^2 |\tilde{E}(\omega)|^2}{2\pi\mu_0\rho}. \quad (11)$$

This expression (11) gives the average received power which is also the power radiated by the particle. For a magnet of finite length L_u , like an undulator, radiation is emitted during the whole traversal time $\Delta t' = L_u/c$ which reaches the observer. The average power emitted per solid angle during this traversal is

$$\frac{d^2 P}{d\Omega d\omega} = \frac{2r^2 |\tilde{E}(\omega)|^2}{\mu_0 L_u}. \quad (12)$$

4 Synchrotron radiation

4.1 The synchrotron radiation field

We consider now the radiation emitted by a charge which moves momentarily with a constant ultra-relativistic speed on a circular trajectory of bending radius ρ as shown

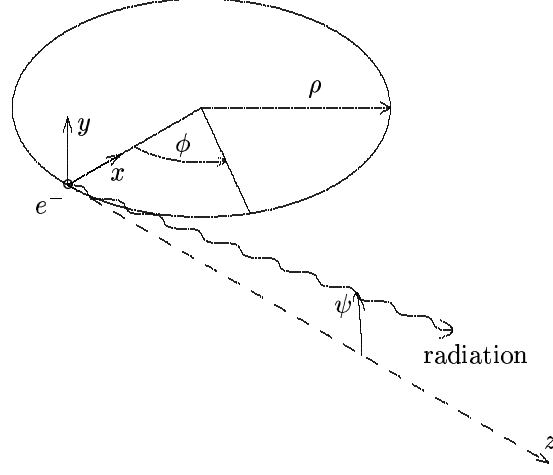


Figure 9: Geometry used to describe the synchrotron radiation

in Fig. 9. This is the case of ordinary synchrotron radiation emitted by a charge $q = e$ in a bending magnet where the curvature is

$$\frac{1}{\rho} = \frac{eB}{m_0 c \beta \gamma} \approx \frac{eBc}{m_0 c^2 \gamma}.$$

Using the expression (9) one obtains the horizontal and vertical field components

$$\begin{aligned} \tilde{E}_x(\omega) &= \frac{e\gamma}{\sqrt{2\pi}\epsilon_0 c r} \left(\frac{3\omega}{4\omega_c}\right)^{1/3} \text{Ai}'\left(\left(\frac{3\omega}{4\omega_c}\right)^{2/3} (1 + \gamma^2 \psi^2)\right) \\ \tilde{E}_y(\omega) &= \frac{ie\gamma}{\sqrt{2\pi}\epsilon_0 c r} \left(\frac{3\omega}{4\omega_c}\right)^{2/3} \gamma \psi \text{Ai}\left(\left(\frac{3\omega}{4\omega_c}\right)^{2/3} (1 + \gamma^2 \psi^2)\right) \end{aligned} \quad (13)$$

where we used the Airy function $\text{Ai}(z)$ and its derivative $\text{Ai}'(z)$ and introduce the critical frequency ω_c

$$\omega_c = \frac{3c\gamma^3}{2\rho}. \quad (14)$$

The fact that the expression for the vertical field $\tilde{E}_y(\omega)$ has an imaginary factor in front while this factor is real for the horizontal component indicates that the two fields are 90° out of phase for a given frequency ω . There is therefore some circular polarization present which vanishes in the median where the polarization is purely horizontal.

The angular spectral angular power distribution is obtained from this field and the expression (11)

$$\frac{d^2 P}{d\Omega d\omega} = \frac{2r^2 |\tilde{E}(\omega)|^2}{2\pi\mu_0 \rho} = \frac{P_0 \gamma}{\omega_c} [F_\sigma(\omega, \psi) + F_\pi(\omega, \psi)]. \quad (15)$$

Here, P_0 is the total radiated power given by (7). The solid angle element can be approximated by $d\Omega \approx d\phi d\psi$ due to the small vertical opening angle of the radiation $\psi \ll 1$.

The form of the distribution is determined by the two expressions $F_\sigma(\omega, \psi)$ and $F_\pi(\omega, \psi)$ which give the contributions of the two linear polarization components. The first

one, called σ -mode, has the electrical field in the plane of the particle orbit (usually the horizontal plane) and the second one, called π -mode, has the electric field perpendicular to this plane. The two functions are given by

$$\begin{aligned} F_\sigma(\omega, \psi) &= \frac{9}{2\pi} \left(\frac{3\omega}{4\omega_c} \right)^{2/3} \text{Ai}'^2 \left(\left(\frac{3\omega}{4\omega_c} \right)^{2/3} (1 + \gamma^2 \psi^2) \right) \\ F_\pi(\omega, \psi) &= \frac{9}{2\pi} \left(\frac{3\omega}{4\omega_c} \right)^{4/3} \gamma^2 \psi^2 \text{Ai}^2 \left(\left(\frac{3\omega}{4\omega_c} \right)^{2/3} (1 + \gamma^2 \psi^2) \right). \end{aligned} \quad (16)$$

As mentioned earlier, diagnostics with synchrotron radiation is carried out mostly with visible (or close to visible) light. In electron rings this light is at the lower part of the spectrum $\omega \ll \omega_c$ and we can make some approximation. In this case the argument of the Airy function or its derivative in (13) is small except if $\gamma^2 \psi^2$ becomes very large. Therefore, we make a small error by replacing $(1 + \gamma^2 \psi^2)$ by $\gamma^2 \psi^2$ inside the argument of the Airy functions. Using the expression (14) for the critical frequency we get then for the argument of the Airy functions

$$\left(\frac{3\omega}{4\omega_c} \right)^{2/3} (1 + \gamma^2 \psi^2) \approx \left(\frac{3\omega}{4\omega_c} \right)^{2/3} \gamma^2 \psi^2 = \left(\frac{\omega \rho}{2c} \right)^{2/3} \psi^2.$$

With this we get for the electric field (13) and spectral angular distribution (15) at small frequencies $\omega \ll \omega_c$

$$\tilde{\mathbf{E}}(\omega) = \frac{e}{\sqrt{2\pi\epsilon_0 cr}} \left(\frac{\omega \rho}{2c} \right)^{1/3} \left[\text{Ai}' \left(\left(\frac{\omega \rho}{2c} \right)^{2/3} \psi^2 \right), i \left(\frac{\omega \rho}{2c} \right)^{1/3} \psi \text{Ai} \left(\left(\frac{\omega \rho}{2c} \right)^{2/3} \psi^2 \right) \right] \quad (17)$$

and

$$\frac{d^2 P}{d\Omega d\omega} = \frac{2r_0 m_0 c^2}{\pi \rho} \left[\left(\frac{\omega \rho}{2c} \right)^{2/3} \text{Ai}'^2 \left(\left(\frac{\omega \rho}{2c} \right)^{2/3} \psi^2 \right) + \psi^2 \left(\frac{\omega \rho}{2c} \right)^{4/3} \text{Ai}^2 \left(\left(\frac{\omega \rho}{2c} \right)^{2/3} \psi^2 \right) \right]. \quad (18)$$

It is interesting to note that these expressions do not depend on γ . At low frequencies the properties of synchrotron radiation are independent of the particle energy and depend only on the radius ρ of curvature. This fact is important for diagnostics applications which usually uses the lower part of the spectrum. The angular power distribution for this case is plotted in Fig. 10.

The rms opening angle of synchrotron radiation can be calculated [1] which gives for the low frequency part of the spectrum $\omega \ll \omega_c$

$$\begin{aligned} \psi_{\sigma\text{-rms}} &\sim \sqrt{\frac{5}{12} \frac{\text{Ai}(0)}{-\text{Ai}'(0)}} \left(\frac{c}{\rho \omega} \right)^{1/3} = 0.4097 \left(\frac{\lambda}{\rho} \right)^{1/3} \\ \psi_{\pi\text{-rms}} &\sim \sqrt{\frac{9}{12} \frac{\text{Ai}(0)}{-\text{Ai}'(0)}} \left(\frac{c}{\rho \omega} \right)^{1/3} = 0.5497 \left(\frac{\lambda}{\rho} \right)^{1/3} \\ \psi_{\text{rms}} &\sim \sqrt{\frac{6}{12} \frac{\text{Ai}(0)}{-\text{Ai}'(0)}} \left(\frac{c}{\rho \omega} \right)^{1/3} = 0.4488 \left(\frac{\lambda}{\rho} \right)^{1/3}. \end{aligned} \quad (19)$$

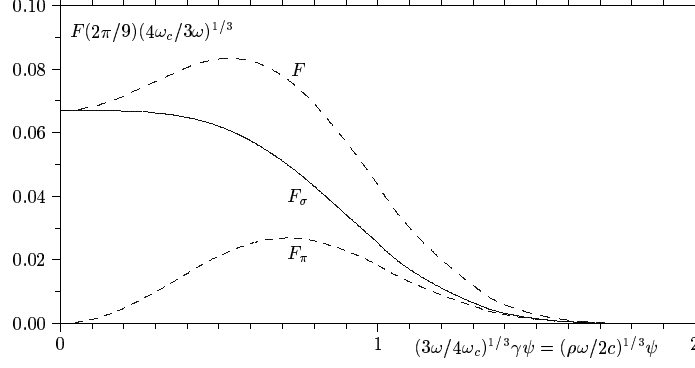


Figure 10: Vertical distribution of synchrotron radiation at low frequencies $\omega \ll \omega_c$

4.2 Undulator radiation

An undulator is a spatially periodic magnetic structure designed to produce quasi-monochromatic radiation from relativistic particles. We consider here a plane harmonic undulator with period length λ_u , Fig. 11. It has in the median plane ($y = 0$) a magnetic field of the form

$$B(z) = B_y(z) = B_0 \cos(k_u z)$$

with $k_u = 2\pi/\lambda_u$. If the field is not too strong the trajectory of a particle going along the axis is of the form

$$x(z) = a \cos(k_u z), \quad a = \frac{eB_0}{m_0 c \gamma k_u^2}$$

and

$$\frac{dx}{dz} = -\psi_0 \sin(k_u z), \quad \psi_0 = \frac{eB_0}{m_0 c \gamma k_u} = \frac{K}{\gamma}.$$

Here, we introduced the undulator parameter

$$K = \frac{eB_0}{m_0 c k_u} = \gamma \psi_0$$

which gives the ratio between the maximum deflecting angle ψ_0 and the natural opening angle of the radiation $1/\gamma$. For the case $K < 1$ the emitted light is deflected by angle ψ_0 smaller than the natural opening angle. An observer will receive a weakly modulated field which is quasi-monochromatic. However, for $K > 1$ the deflection is larger than the natural opening angle and the observer will receive strongly modulated light containing harmonics of the basic modulation frequency.

The Fourier transformed electric field of the undulator radiation has been derived [1] to be

$$\tilde{\mathbf{E}}(\omega) = \frac{4r_0 c B_0 \gamma^3}{\sqrt{2\pi} r_p} \frac{[1 - \gamma^2 \theta^2 \cos(2\phi), \gamma^2 \theta^2 \sin(2\phi)]}{(1 + \gamma^2 \theta^2)^3} \frac{\pi N_u}{\omega_1} \frac{\sin\left(\frac{(\omega - \omega_1)\pi N_u}{\omega_1}\right)}{\frac{(\omega - \omega_1)\pi N_u}{\omega_1}} \quad (20)$$

where we assume that the undulator has many periods $N_u \gg 1$ and introduced the frequency

$$\omega_1 = k_u c \frac{2\gamma^2}{1 + \gamma^2 \theta^2} = \frac{\omega_{10}}{1 + \gamma^2 \theta^2}.$$

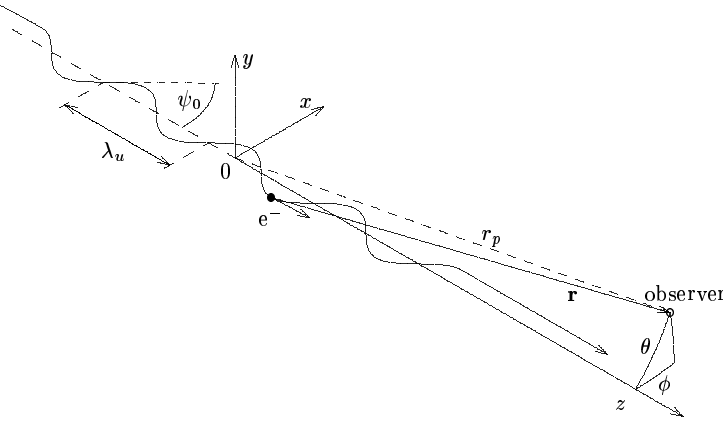


Figure 11: Geometry of undulator radiation

The angular spectral power distribution of a weak undulator is obtained from the above field

$$\frac{d^2 P_u}{d\Omega d\omega} = P_u \gamma^2 (F_{u\sigma}(\theta, \phi) + F_{u\pi}(\theta, \phi)) f_N(\Delta\omega),$$

where P_u is the total radiated power and the two functions $F_{u\sigma}$ and $F_{u\pi}$ determine the contributions of the two polarization modes

$$F_{u\sigma}(\theta, \phi) = \frac{3}{\pi} \frac{(1 - \gamma^2 \theta^2 \cos(2\phi))^2}{(1 + \gamma^2 \theta^2)^5}, \quad F_{u\pi}(\theta, \phi) = \frac{3}{\pi} \frac{(\gamma^2 \theta^2 \sin(2\phi))^2}{(1 + \gamma^2 \theta^2)^5}.$$

The function $f_N(\Delta\omega)$ gives the spectral distribution at a given angle θ which depends on the number N_u of undulator periods

$$f_N(\Delta\omega) = \frac{N_u}{\omega_1} \left(\frac{\sin\left(\frac{\Delta\omega \pi N_u}{\omega_1}\right)}{\frac{\Delta\omega \pi N_u}{\omega_1}} \right)^2 \quad \text{with } \Delta\omega = \omega - \omega_1.$$

This function is normalized and approaches the Dirac delta function for a large number N_u of periods

$$\int_{-\infty}^{\infty} f_N(\Delta\omega) d\omega = 1, \quad f_N(\Delta\omega) \rightarrow \delta(\Delta\omega) \text{ for } N_u \rightarrow \infty.$$

In this latter case the radiation is monochromatic at each observation angle θ with the frequency ω_1 .

We consider now a magnet which is sufficiently short and weak that the deflection it produces for the beam stays within an angle smaller than $1/\gamma$. We assume that the particle trajectory lies in the x, z -plane and follows closely the z -axis and give the magnetic field in the form

$$B_y = B_y(z).$$

Within this approximation and using the geometry shown in Fig. 11 the particle trajectory is determined by

$$\frac{1}{\rho} = \frac{eBc}{m_0 c^2 \gamma} \approx -\frac{1}{c^2} \frac{d^2 x}{dt^2} = -\frac{\dot{\beta}}{c}.$$

It can be regarded as a weak undulator with a generalized field $B_y(z)$ having a Fourier transform

$$\tilde{B}(k_{sm}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} B_y(z) e^{-ik_{sm}z} dz.$$

The radiation observed at frequency ω and angle θ is determined by the Fourier component $\tilde{B}(k_{sm})$ of the magnetic field. We get for the radiation in frequency domain [1]

$$\begin{aligned} \tilde{\mathbf{E}}(\omega) &= \frac{2r_0\gamma}{r_p} \frac{[1 - \gamma^2\theta^2 \cos(2\phi), \gamma^2\theta^2 \sin(2\phi)]}{(1 + \gamma^2\theta^2)^2} \tilde{B}_y(k_{sm}) \\ &= \frac{2r_0\gamma}{r_p} \frac{[1 - \gamma^2\theta^2 \cos(2\phi), \gamma^2\theta^2 \sin(2\phi)]}{(1 + \gamma^2\theta^2)^2} \tilde{B} \left(\frac{1 + \gamma^2\theta^2}{2c\gamma^2} \omega \right) \end{aligned} \quad (21)$$

and the angular spectral energy distribution is obtained from the relation

$$\frac{dW^2}{d\Omega d\omega} = \frac{2r_p^2 |\tilde{E}(\omega)|^2}{\mu_0 c}$$

giving

$$\begin{aligned} \frac{d^2W}{d\Omega d\omega} &= \frac{2r_0 c e^2 (m_0 c^2 \gamma)^2 [(1 - \gamma^2\theta^2 \cos(2\phi))^2 + (\gamma^2\theta^2 \sin(2\phi))^2]}{\pi (m_0 c^2)^3 (1 + \gamma^2\theta^2)^4} \\ &\quad \left| \tilde{B} \left(\frac{1 + \gamma^2\theta^2}{2c\gamma^2} \omega \right) \right|^2. \end{aligned} \quad (22)$$

This expression gives the radiation from a general magnet provided that the deflection is weak and nowhere exceeds an angle of $1/\gamma$.

A special case of a short magnet is an undulator having a harmonic field which is modulated by a Lorentz function

$$B(z) = B_0 \frac{\cos(k_u z)}{1 + (z/z_0)^2} = \frac{K_0 m_0 c k_u}{e} \frac{\cos(k_u z)}{1 + (z/z_0)^2}.$$

For the case of many periods N_u within its characteristic length $2z_0$ the total energy radiated by an electron is

$$W_u = \frac{\pi r_0 (m_0 c^2 \gamma)^2 K_0^2 k_u^2 z_0}{6 m_0 c^2}.$$

We treat this undulator as a short magnet and filter out the frequency component ω_{10} from the radiation

$$\omega = \omega_{10} = k_u c 2\gamma^2$$

which is given by the spacial Fourier component at

$$k_{sm} = \omega_{10} \frac{1 + \gamma^2\theta^2}{2\gamma^2} = k_u (1 + \gamma^2\theta^2).$$

Using (21) we get for the radiation field

$$\tilde{\mathbf{E}}(\omega) = \frac{\sqrt{2\pi} r_0 m_0 c^2 \gamma K_0 k_u z_0}{r_p e c} [1, 0] e^{-\pi N_u \gamma^2 \theta^2} \quad (23)$$

where we are left with the horizontal polarization mode only. From (10) we obtain emitted angular spectral energy distribution

$$\frac{d^2 W_u}{d\Omega d\omega} = W_u \frac{6z_0}{\pi} e^{-2\pi N_u \gamma^2 \theta^2}. \quad (24)$$

The radiation from this Lorentz modulated undulator, filtered at ω_{10} , has a Gaussian angular energy distribution with rms values for the polar angle θ and the two Cartesian angles x' and y'

$$\theta_{rms} = \frac{1}{\gamma\sqrt{2\pi N_u}}, \quad x'_{rms} = y'_{rms} = \frac{1}{2\gamma\sqrt{\pi N_u}}.$$

5 Imaging with SR — Qualitative treatment

5.1 Diffraction of radiation emitted in long magnets

We use synchrotron radiation from a long magnet to form an image of the beam cross section with an arrangement shown in Fig. 12. For simplicity a single lens is considered to form a 1:1 image. This is no restriction since we could also use a magnification and project the image back to the beam to get the resolution in terms of beam size. Since the vertical opening angle of the radiation σ'_γ is small, only the central part of the lens is illuminated. The situation is therefore similar to optical imaging with a limited lens aperture. This case is known to lead to a limitation of the resolution by diffraction [2]

$$d \approx \frac{\lambda}{2D/R}, \quad (25)$$

where d is the half size of an image from a point source and D is the full lens aperture. For synchrotron radiation observed at the low frequency part of the spectrum we found for the horizontal polarization an rms opening angle (19) which is

$$\sigma'_\gamma = \psi_{\sigma-rms} \approx 0.41(\lambda/\rho)^{1/3}$$

for the σ -mode. Relating this to (25) we take $D \approx 4\sigma_\gamma R$ since the lens represents the full aperture which we approximate with $\pm 2\psi_{\sigma-rms}$ to get

$$d \approx 0.3(\lambda^2 \rho)^{1/3}.$$

We find that the resolution improves with shorter wavelength and with smaller radius of curvature. Due to the latter dependence synchrotron radiation monitors have a poor resolution in large machines as indicated in the examples shown in Table 1. A wavelength as short as possible can help to improve the resolution. Special magnets with strong curvature could help to improve the resolution but the weak dependence on ρ makes this approach not very attractive.

Diffraction represents often a serious limitation of the resolution for large machines. It is caused by the small opening angle of synchrotron radiation. Sometimes the question arises if a large angular spread of the particles in the beam helps the resolution since a larger part of the lens is now illuminated. This is not the case since the diffraction results in a finite size image of each particle. The radiation originating from different particles has no phase relation and does not produce the corresponding diffraction pattern.

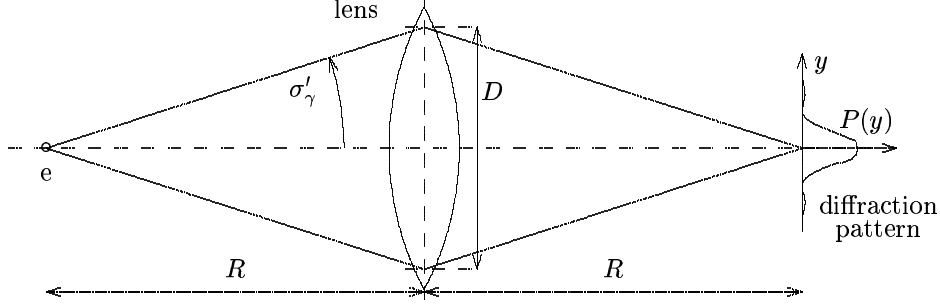


Figure 12: Imaging the beam cross section with synchrotron radiation

machine	ρ m	λ nm	σ'_γ m rad	d mm
EPA (CERN)	1.43	400	2.7	0.018
LEP (CERN)	3096	400	0.21	0.24

Table 1: Resolution for imaging with synchrotron radiation in different machines

5.2 Depth of field effect for the radiation emitted in long magnets

We consider now the effect of the depth of field on the resolution of an image of the beam cross section. The situation is illustrated in Fig. 13. We discussed at the beginning the length of orbit from which radiation can be received by an observer, Fig. 5. There, we assumed an opening angle of $\pm 1/\gamma$ for the radiation. In forming the image of the beam we use σ -mode of the low frequency part of the spectrum for which the opening angle is $\sigma'_\gamma \approx 0.41(\lambda/\rho)^{1/3}$. The part of the orbit from which radiation of wavelength λ can reach the observer has therefore a length of about $\pm 2\sigma'_\gamma \rho$. We consider now three points A , B and C along this orbit where B is located at the nominal distance $R = 2f$ from the lens having a focal length f and the other points deviate by $\pm 2\sigma'_\gamma \rho$ from it. For a 1:1 image and assuming $\sigma_\gamma \rho \ll R$ we find that the images A' and C' have also about the spacing $\pm 2\sigma'_\gamma$ from the central image B' . At this central point the radiation from A and B has an extension

$$d \approx 2\sigma'^2_\gamma \rho = 0.34(\lambda^2 \rho)^{1/3}.$$

It is interesting that the resolution limitation due to the depth of field effect has the same parameter dependence as the one caused by diffraction.

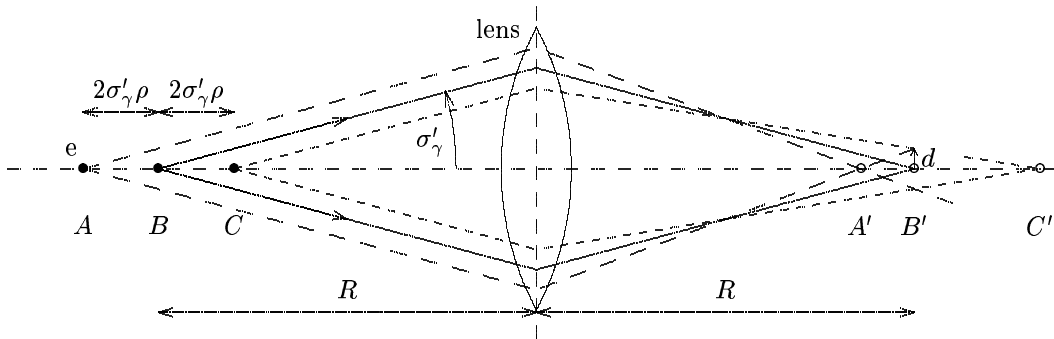


Figure 13: Depth of field effect

5.3 Diffraction and depth of field effect for undulator radiation

We derived the radiation from a weak undulator in frequency domain 20

$$\tilde{\mathbf{E}}(\omega) = \frac{4r_0cB_0\gamma^3}{\sqrt{2\pi}r_p} \frac{(1 - \gamma^2\theta^2 \cos(2\phi), \gamma^2\theta^2 \sin(2\phi))}{(1 + \gamma^2\theta^2)^3} \frac{\pi N_u}{\omega_1} \frac{\sin\left(\frac{(\omega - \omega_1)\pi N_u}{\omega_1}\right)}{\frac{(\omega - \omega_1)\pi N_u}{\omega_1}}.$$

with

$$\omega_1 = \frac{\omega_{10}}{1 + \gamma^2\theta^2} = \frac{2\gamma^2 k_u c}{1 + \gamma^2\theta^2}.$$

We filter now the frequency $\omega = \omega_{10}$ to use it for imaging the beam and get for the spectral function

$$\frac{\pi N_u}{\omega_1} \frac{\sin\left(\frac{(\omega - \omega_1)\pi N_u}{\omega_1}\right)}{\frac{(\omega - \omega_1)\pi N_u}{\omega_1}} = \frac{\pi N_u (1 + \gamma^2\theta^2)}{\omega_{10}} \frac{\sin(\gamma^2\theta^2 \pi N_u)}{\gamma^2\theta^2 \pi N_u}.$$

This equation contains the function $\sin x/x$ which has a maximum value of unity at $x = 0$ and is small for large values of x . It has a first zero at $x = \pi$ which corresponds to the angle

$$\theta_{zero} = \frac{1}{\gamma\sqrt{N_u}}. \quad (26)$$

We assume now an undulator with many periods $N_u \gg 1$ for which the radiation at $\omega = \omega_{10}$ is concentrated within an angle much smaller than $1/\gamma$. We can then approximate in (20 for $1 + \gamma^2\theta^2 \approx 1$ and get a radiation field containing only an x -component

$$\tilde{E}_x(\omega_{10}) \approx \frac{4r_0cB_0\gamma^3}{\sqrt{2\pi}r_p} \frac{\pi N_u}{\omega_{10}} \frac{\sin(\gamma^2\theta^2 \pi N_u)}{\gamma^2\theta^2 \pi N_u} \quad (27)$$

and a angular spectral power distribution

$$\frac{d^2P(\omega_{10})}{d\Omega d\omega} \approx P_u \gamma^2 \frac{3}{\pi} \frac{N_u}{\omega_{10}} \left(\frac{\sin(\gamma^2\theta^2 \pi N_u)}{\gamma^2\theta^2 \pi N_u} \right)^2. \quad (28)$$

This function is plotted in Fig. 14 together with an exponential approximation.

For the evaluation of the diffraction we would like to use the rms width of this power distribution. However, the variance of function $(\sin x/x)^2$ diverges. This is due to the unphysical undulator field which is harmonic within $\pm L_u/2$ but vanishes abruptly outside this range. For the qualitative treatment of the diffraction we relate again to the case of a lens with diameter D shown in Fig. 12 and approximate $D \approx 2\theta_{zero}R$ with the opening angle (26) of the first zero. Using also the relation $\lambda_{10} = \lambda_u/2\gamma^2$

$$d \approx \frac{\lambda_{10}}{2D/R} = \frac{\lambda_{10}\gamma\sqrt{N_u}}{4} = \frac{\sqrt{\lambda_{10}L_u}}{4\sqrt{2}}.$$

Since the resolution is proportional to $\sqrt{L_u}$ one would like to work with a short undulator. However, we obtained the above expression with the assumption $N_u \gg 1$. A more detailed calculation is necessary to treat the more general case of an undulator having few periods or of a short magnet.

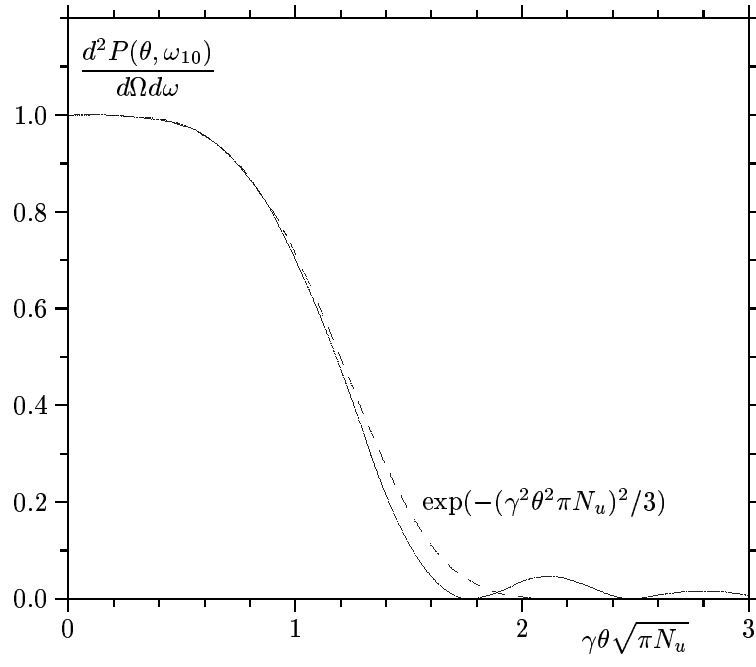


Figure 14: Angular power distribution of undulator radiation at ω_{10}

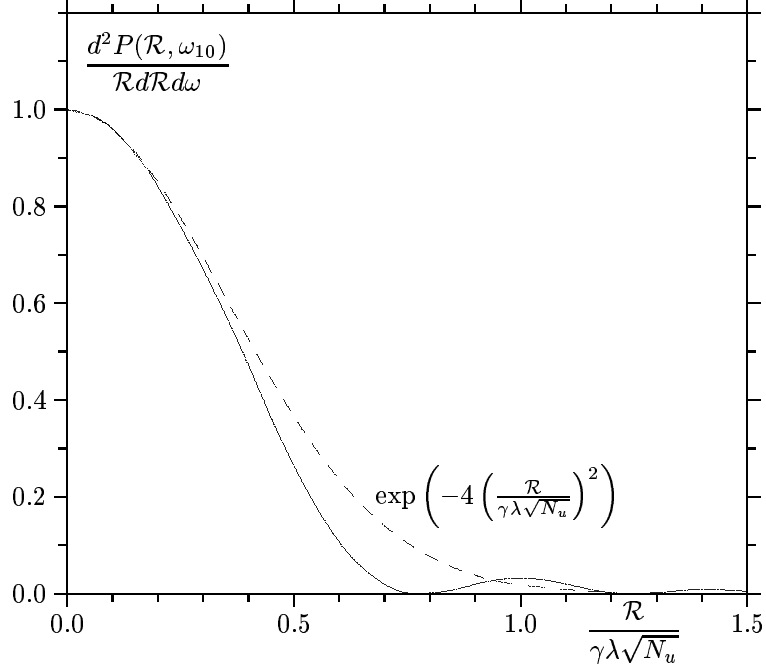


Figure 15: Fraunhofer diffraction for imaging with undulator radiation at ω_{10}

We can also investigate the depth of field effect by replacing the source length $2\sigma'_\gamma\rho$ by the undulator length L_u and σ'_γ by θ_{zero} in Fig. 13

$$d \approx \frac{\theta_{zero}L_u}{2} = \frac{\sqrt{\lambda_{10}L_u}}{\sqrt{2}}.$$

Again, diffraction and depth of field effect have the same parameter dependence and are of similar magnitude.

5.4 Resolution for imaging with short magnet radiation

We treated the radiation from short magnets before and obtained the angular spectral power distribution (22) for a general, but weak field $B(z)$. Without knowing the details of this field we can estimate the opening angle of the radiation from the length L_{sm} of the source. Considering two points, one at the beginning and one at the end of the short magnet, emitting radiation with the wavelength λ for which we get positive interference in the forward direction

$$\frac{\lambda}{c} = \frac{L_{sm}}{\beta c} - \frac{L_{sm}}{c} = \frac{L_{sm}}{\beta c}(1 - \beta) \approx \frac{L_{sm}}{2c\gamma^2},$$

or

$$\lambda = \frac{L_{sm}}{2\gamma^2}.$$

We look now for the angle θ for which we get destructive interference

$$\frac{3\lambda}{2c} = \frac{L_{sm}}{\beta c} - \frac{L_{sm}\cos\theta}{c} = \frac{L_{sm}}{\beta c}(1 - \beta\cos\theta) \approx \frac{L_{sm}(1 + \gamma^2\theta^2)}{2c\gamma^2},$$

or

$$1 + \gamma^2\theta^2 = \frac{3}{2} \rightarrow \theta = \frac{1}{\sqrt{2}\gamma}.$$

The radiation emitted at wavelength λ has a first minimum at the above angle. Of course this is a rough estimate since we should also take the radiation emitted in between the end point of the magnet into account. Using this angle $\sigma'_\gamma = 1/(\sqrt{2}\gamma)$ we get for the resolution

$$d \approx \frac{\lambda}{2D/R} \approx \frac{\sqrt{2}\lambda\gamma}{2} = \frac{1}{2}\sqrt{\lambda L_{sm}}.$$

Using this very general argument we find a similar resolution to the one obtained for an undulator.

6 Imaging with SR — Fraunhofer approximation

6.1 Fraunhofer diffraction

We treat now the diffraction in a more quantitative way and consider an image formed with synchrotron radiation as illustrated in Fig. 16. We form a 1:1 image with a single lens at a distance R from the source. The emitted Fourier transformed field components have a horizontal and vertical angular distribution of the form

$$\tilde{E}_x = \tilde{E}_x(x'_\gamma, y'_\gamma) \quad , \quad \tilde{E}_y = \tilde{E}_y(x'_\gamma, y'_\gamma).$$

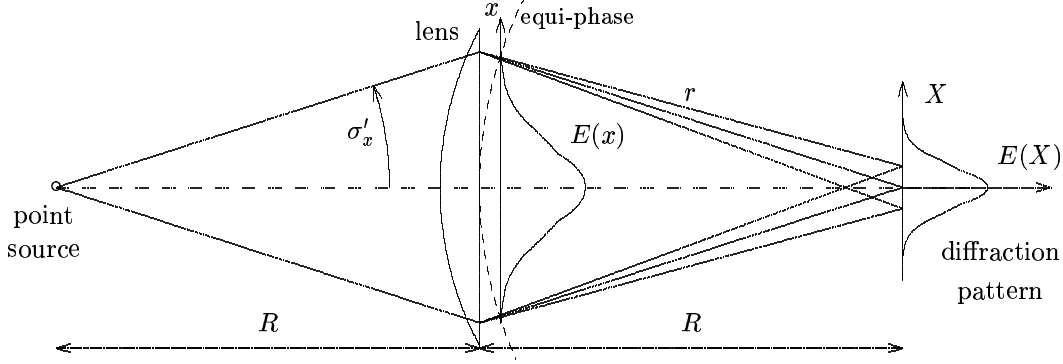


Figure 16: Diffraction in imaging with synchrotron radiation

At the lens this is converted into a spacial distribution

$$\tilde{E}_x(x, y) = \tilde{E}_x(Rx'_\gamma, Ry'_\gamma) \quad , \quad \tilde{E}_y(x, y) = \tilde{E}_y(Rx'_\gamma, Ry'_\gamma).$$

The point source in the figure is imaged on to the image point at a distance R from the lens. In this case all rays between source and image points have the same optical length. The dashed circular arc with radius R around the image point, shown in the figure, represents therefore a cut through an equi-phase surface. Using Huygens's principle we consider each point on this surface as a source of a radiation field of strength proportional to $(\tilde{E}_x(x, y), \tilde{E}_y(x, y))$. We restrict ourselves to a scalar field E which can stand for either the horizontal or vertical two field components. From this secondary source point (x, y, z) this field propagates towards the image plane (X, Y) in the form of a wave

$$\delta \tilde{E}(X, Y) \propto \tilde{E}(x, y) e^{(kr - \omega t)}, \quad (29)$$

where $k = 2\pi/\lambda$ is the wave number of the radiation and r the distance between the secondary source and the observation point in the image plane

$$r^2 = (R - z)^2 + (x - X)^2 + (y - Y)^2.$$

On the equi-phase surface we have $(R - z)^2 = R^2 - (x^2 + y^2)$ which gives

$$r = \sqrt{R^2 - 2xX - 2yY + X^2 + Y^2} \approx R \left(1 - \frac{xX}{R^2} - \frac{yY}{R^2} + \frac{X^2}{R^2} + \frac{Y^2}{R^2} \dots \right).$$

The small opening angle of synchrotron radiation and the limited extension of the image permit to use an approximation and we neglect from now on higher order terms in (X/R) and (Y/R) .

We get the field in the image plane by integrating the contribution (29) from each surface element of the secondary source

$$\tilde{E}(X, Y) \propto e^{-i\omega t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{E}(x, y) e^{i(kR - \frac{kxX}{R} - \frac{kyY}{R})} dx dy.$$

We omit the oscillatory term as well as fixed phase term $\exp(ikR)$ which are both of no interest and replace the coordinates (x, y) at the lens by the angles (x', y') at which the radiation is emitted

$$\tilde{E}(X, Y) \propto \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{E}(x, y) e^{-i(x'kX + y'kY)} dx' dy'. \quad (30)$$

This integral represents a Fourier transform. In other words, the field distribution $\tilde{E}(X, Y)$ in the image plane is just proportional to the two dimensional Fourier transform of the field distribution at the lens, or of the angular distribution of the emitted radiation

$$\tilde{E}(kX, kY) \propto \mathcal{F}\tilde{E}(x', y').$$

Instead of the source angles x' and y' we can use a spherical coordinate system (R, ϕ, θ) having the origin at the source and the relations

$$x = R \sin \theta \cos \phi \approx R \theta \cos \phi; \quad y = R \sin \theta \sin \phi \approx R \theta \sin \phi.$$

We also express the image coordinates X, Y by polar coordinates $(\mathcal{R}, \Phi)$ with the relations

$$X = \mathcal{R} \cos \Phi, \quad Y = \mathcal{R} \sin \Phi$$

and get

$$\tilde{E}(\mathcal{R}, \Phi) \propto \int_{-\infty}^{\infty} \int_0^{2\pi} \tilde{E}(\theta, \phi) e^{-ik\theta\mathcal{R}\cos(\phi-\Phi)} \theta d\theta d\phi.$$

In some cases the emitted radiation is independent of the azimuthal angle ϕ which leads also to the same symmetry for the diffraction pattern at the image plane. The integration over ϕ becomes

$$\tilde{E}(\mathcal{R}) \propto \int_{-\infty}^{\infty} \int_0^{2\pi} \tilde{E}(\theta) e^{-ik\theta\mathcal{R}\cos(\phi-\Phi)} \theta d\theta d\phi = 2\pi \int_0^{\infty} \tilde{E}(\theta) J_0(k\mathcal{R}\theta) \theta d\theta \quad (31)$$

where we used the integral representation of the Bessel function

$$J_0(z) = \frac{1}{\pi} \int_0^{\pi} e^{iz \cos \xi} d\xi.$$

In all these calculations of the diffraction we assumed a point source located at the distance R from the lens. This is an approximation for the case of synchrotron radiation. In the treatment of the depth of field we said that the length of the source is $\pm 2\sigma'_\gamma \rho$ and used for the rms opening angle $\sigma'_\gamma \approx 0.41(\lambda/\rho)$. If we take the finite longitudinal extension of the source into account the sphere of radius R around the image center is no longer an equi-phase surface. The calculation becomes more complicated and the exponent in the integral (30) will contain quadratic terms of coordinates x and y at the lens. This leads to the case of Fresnel diffraction which will not be treated here but it can be found in more profound treatments [3, 4]. Since we found before that the depth of field effect is of the same order as the diffraction we expect the improved treatment to make a sizable correction. However, we will in the following still use the Fraunhofer diffraction to illustrate some of the underlying physics.

6.2 Diffraction of synchrotron radiation emitted in long magnets

We use here synchrotron radiation from a long magnet to image the cross section of the beam. The radiation depends only on the vertical emission angle y' which we called $y' = \psi$ before. The Fourier transformed electric field in the approximation of small frequencies $\omega \ll \omega_c$ is

$$\tilde{\mathbf{E}}(\psi, \lambda) = \frac{e}{\sqrt{2}\epsilon_0 c r} \left(\frac{\pi \rho}{\lambda} \right)^{\frac{1}{3}} \left(Ai' \left(\psi^2 \left(\frac{\pi \rho}{\lambda} \right)^{\frac{2}{3}} \right), \psi \left(\frac{\pi \rho}{\lambda} \right)^{\frac{1}{3}} Ai \left(\psi^2 \left(\frac{\pi \rho}{\lambda} \right)^{\frac{2}{3}} \right) \right).$$

The corresponding power distribution is proportional to the square of this field and is plotted in Fig. 17. To get the image given by Fraunhofer diffraction we have to use the expression (30). The field distribution in the horizontal direction is uniform by nature and will be terminated by some aperture limitation due to a slit or lens size. Since this aperture will determine the horizontal diffraction one does not want to make it too small. On the other hand, by making it too large we will increase the depth of field effect. As a compromise one makes the horizontal angular acceptance comparable to the natural vertical distribution. Considering that the horizontal limitation has a sharp edge a value of the order $|x'| \leq 2\sigma'_\gamma$ is a reasonable compromise. We restrict ourselves here to the vertical resolution and integrate (30) only over the vertical coordinate and get for the corresponding distribution of the image field

$$\tilde{\mathbf{E}}(Y, \lambda) \propto \int_{-\infty}^{\infty} \tilde{\mathbf{E}}(\psi, \lambda) e^{-i(\psi k Y)} d\psi.$$

This integration has to be done numerically. The corresponding image power is proportional to $|\tilde{\mathbf{E}}(Y, \lambda)|^2$ and is plotted in Fig. 18 for the horizontal and vertical polarization as well as for the total radiation. The rms values of the image power are

$$\sigma_{Y,\sigma} = 0.206(\lambda^2 \rho)^{1/3}, \quad \sigma_{Y,\pi}(\lambda^2 \rho)^{1/3}, \quad \sigma_{Y,total} = 0.279(\lambda^2 \rho)^{1/3}. \quad (32)$$

From the figure it is evident that the image is narrowest for the σ -mode of the radiation. Using a horizontal polarizing filter will therefore improve the resolution of the image by about 25%.

6.3 Diffraction for the undulator having a Gaussian angular distribution

Since a normal undulator has a divergent rms value for the opening angle it is difficult to define the imaging resolution. However, the angular distribution and its Fourier transform are shown in Figs. 14 and 15 to illustrate the general behavior and are both fitted by an exponential.

We discussed a better behaved undulator with a Lorenz modulation having a magnetic field of the form

$$B(z) = B_0 \frac{\cos(k_u z)}{1 + (z/z_0)^2}$$

giving a radiation field at the frequency ω_{10} which has for a large number of periods only an x -component (23).

$$\tilde{E}_x(\omega_{10}) = \frac{\sqrt{2\pi} r_0 m_0 c^2 \gamma K_0 k_u z_0}{r_p e c} e^{-\pi N_u \gamma^2 \theta^2}$$

and the emitted angular spectral energy distribution (24)

$$\frac{d^2 W}{d\Omega d\omega} = W_0 \frac{6z_0}{\pi} e^{-2\pi N_u \gamma^2 \theta^2}$$

with the rms opening angle

$$\theta_{rms} = \frac{1}{2\gamma\sqrt{\pi N_u}}.$$

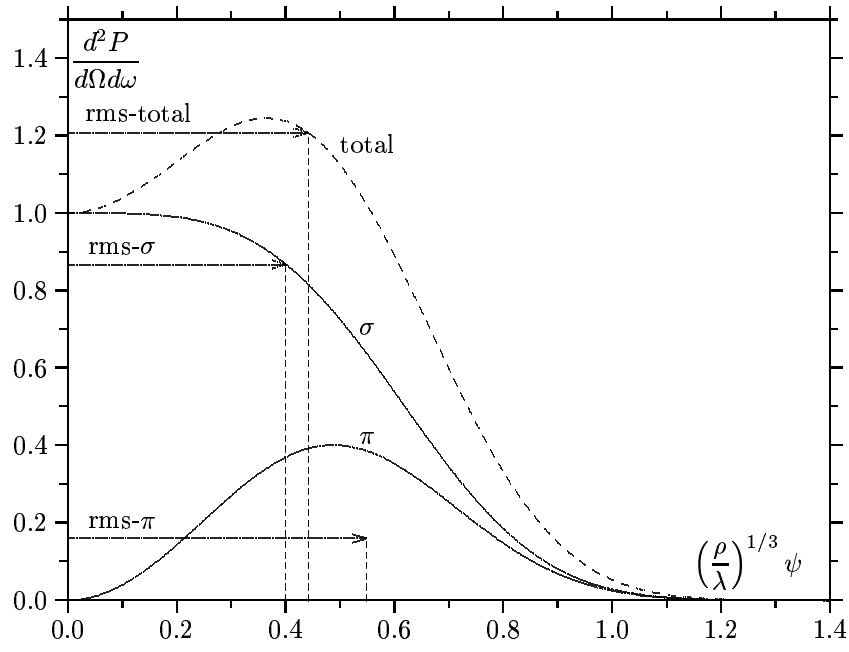


Figure 17: Vertical distribution of synchrotron radiation from long magnets

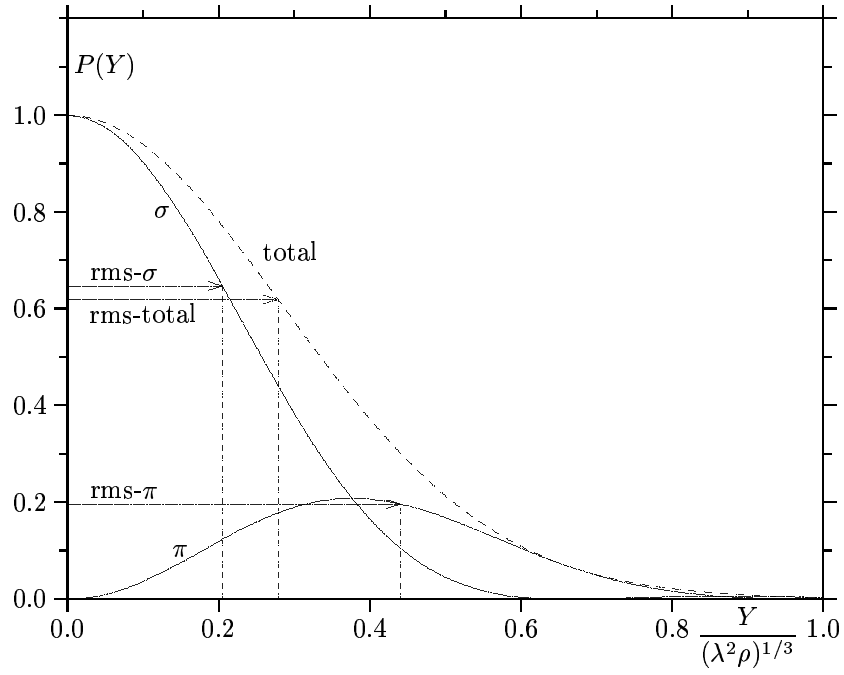


Figure 18: Fraunhofer diffraction for synchrotron radiation from long magnets

To calculate the diffraction for the field distribution we make use of its azimuthal symmetry and use the relation (31)

$$\tilde{E}(\mathcal{R}) \propto \int_0^\infty e^{-\pi N_u \gamma^2 \theta^2} J_0(k\mathcal{R}\theta) \theta d\theta = \frac{1}{\pi N_u \gamma^2} e^{-\frac{k^2 \mathcal{R}^2}{4\gamma^2 \pi N_u}}.$$

The integral appearing above can be found in [5]. The energy distribution of the diffraction pattern has the form

$$\frac{dW}{\mathcal{R} d\mathcal{R} d\omega}(\omega_{10}) \propto e^{-\frac{k^2 \mathcal{R}^2}{2\gamma^2 \pi N_u}}.$$

It is also Gaussian with an rms width

$$\sigma_{\mathcal{R}} = \frac{\sqrt{2\pi N_u} \gamma \lambda}{2\pi}, \quad \sigma_X = \sigma_Y = \frac{\sqrt{\pi N_u} \gamma \lambda}{2\pi}.$$

7 Direct observation of SR

Instead of forming an image of the beam cross section we can observe the synchrotron radiation directly and measure its angular distribution. This experiment can determine the angular spread of the particles in the beam. The resolution is clearly limited by the natural opening angles of the synchrotron light, as shown in Fig. 19. In order to correct for this it is advantageous to use quasi-monochromatic light for which the distribution and the sensitivity of the detector is well known. The angular spread of the particles can then be obtained by a deconvolution.

The angular spread and the dimension of the particle beam in a ring are related by an invariant emittance ϵ . It is defined as the rms phase space area divided by π . At locations where the beam size has a maximum or minimum the emittance is simply given by the product of rms beam size and rms angular spread $\epsilon = \sigma \sigma'$. At other locations the phase space area forms a tilted ellipse and the relation is more complicated and determined by the local particle beam optics described by the lattice function $\beta(s)$, $\alpha(s)$, $\gamma(s)$ in the two planes. At the location of maximal or minimal beam size the ratio between size and angular spread is simply given by $\sigma/\sigma' = \beta$. A convenient method has been developed [6] to use the lattice functions at the source to treat the photon beam distribution measured on a screen at a distance s from the source. If the lattice functions are known at the source $s = 0$ we can calculate their propagation in a drift space being free of focusing elements to get the values at a distance s

$$\beta(s) = \beta(0) - 2\alpha(0)s + \gamma(0)s^2.$$

Apart from the finite opening angle we can treat the synchrotron radiation emitted in the forward direction like particles and describe their propagation by the same lattice functions. We can therefore define a beta function for the photon beam at the screen $\beta_\gamma(R) = \beta(R)$. Neglecting the finite opening angle of the radiation the photon beam size σ_R on the screen is determined by the emittance of the electron beam ϵ and the beta function $\beta(R)$

$$\sigma_\epsilon = \sqrt{\epsilon \beta_\gamma(R)}.$$

To take into account the effect of the rms opening angle σ_γ of the emitted synchrotron radiation we just have to deconvolute the picture on the screen with this photon distribution. In many cases we can approximate all distributions by a Gaussian and obtain the

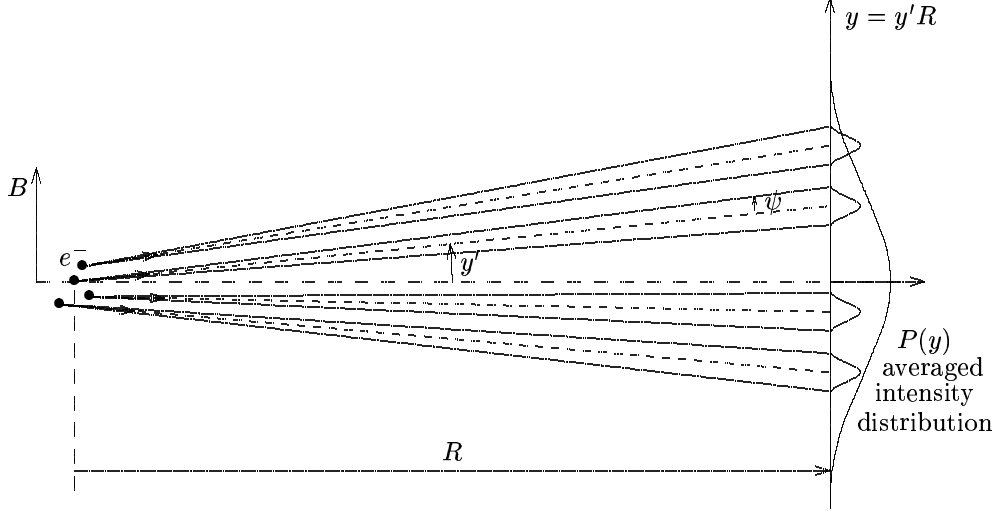


Figure 19: Limitation of the angular spread measurement by the natural opening angle of synchrotron radiation

relation

$$\sigma^2 = \sigma_\gamma^2 + \sigma_\epsilon^2.$$

Of course in an actual measurement there are other contributions to the beam size like the limited resolution of the radiation detector or the energy spread of the particle beam in case of a finite dispersion at the source, etc.

The finite opening angle of the radiation limiting the resolution of this direct observation is smaller for a shorter wavelength. For radiation from long magnets the vertical rms opening angle of the σ -mode at the lower part of spectrum $\omega \ll \omega_c$ is

$$\psi_{\sigma-rms} = 0.41 \left(\frac{\lambda}{\rho} \right)^{1/3}.$$

For a Lorentz modulated undulator with the distribution Gaussian with the rms values

$$\theta_{rms} = \frac{1}{\gamma\sqrt{2\pi N_u}} \quad , \quad \sigma'_x = \sigma'_y = \frac{1}{2\gamma\sqrt{\pi N_u}}.$$

Because the direct observation does not involve any focusing elements we are free to chose short wavelength radiation, like x -rays.

8 Emittance measurement

We investigated the radiation emitted by a relativistic charge in long magnets and in undulators which can be used to form an image of the beam cross section and to get its dimension σ or to obtain the angular spread σ' of the particles. Concentrating for simplicity on locations at which the beam size has a maximum or minimum, the product of these two quantiles is the emittance which is an invariant around the ring, and their ratio is given by the beta function which depends on the local focusing properties

$$\epsilon = \sigma\sigma' = \frac{\sigma^2}{\beta} = \sigma'^2\beta \quad , \quad \frac{\sigma}{\sigma'} = \beta. \quad (33)$$

Source	σ_γ	σ'_γ	ϵ_γ
Long magnet (y)	$0.206\sqrt[3]{\lambda^2\rho}$	$0.41\sqrt[3]{\lambda/\rho}$	$1.06\lambda/4\pi$
Lorentz undulator (x, y)	$\lambda\gamma\sqrt{\pi N_u}/(2\pi)$	$1/(2\gamma\sqrt{\pi N_u})$	$\lambda/4\pi$

Table 2: Fraunhofer diffraction, natural opening angle and emittance of different radiation sources

We can determine the emittance using synchrotron radiation by measuring both, σ and σ' . In the first case diffraction σ_γ limits the resolution, in the second case it is the natural opening angle σ'_γ of the radiation. We can define the product of these two limitations as the effective emittance of the photon beam $\epsilon_\gamma = \sigma_\gamma\sigma'_\gamma$ which represents some measure of limitation with which the particle beam can be measured. We compare now in Table 2 two different sources of radiation with respect to their resolution. For undulators the angular spread can be measured in both planes while the radiation from long magnets can only give the vertical angles. In all cases we consider only the horizontal polarization component (σ -mode) which gives a better resolution.

The photon beam from the Lorentz modulated undulator has a Gaussian angular distribution and gives the smallest emittance. It is shown in optics that there is no other distribution which gives a smaller emittance than the Gaussian. Of course these photon beam emittances do not represent a hard limit for measuring the particle beam. The measured data can be corrected for the known size and angular spread of the radiation leading to a considerably better resolution.

In most cases one does not measure both the beam dimension and the angular spread of a beam with synchrotron radiation. It is much better to measure either the beam size or the angular spread and use the relations (33) to find the emittance. To measure the beam size it is best to choose as source a location of high beta function where the beam size is large for a given emittance. For direct observation a small beta function is advantageous since it gives large spread

$$\sigma = \sqrt{\epsilon\beta} \quad , \quad \sigma' = \sqrt{\frac{\epsilon}{\beta}}.$$

For such measurements involving only one of two parameters the beta function has to be known to deduce the emittance.

9 Measurement examples

9.1 Introduction

In the following we discuss some beam observations carried out with synchrotron radiation. Such measurements are done at all electron storage rings and the ones selected here represent just some typical examples which are also discussed in [7]. In all cases either the beam size or the angular spread was measured and the emittance was calculated from the lattice function at the source. In most storage rings these functions are not too well known locally. Focusing errors produce beta beating around the machine. It is recommended to provide some measurements of the beta function at the source point. This could be realized by making a small change of a neighboring quadrupole and measure the resulting variation of the betatron tune.

9.2 Imaging with synchrotron radiation in LEP

The electron positron storage ring LEP has operated at an energy of 91.5 GeV per beam and will be able to reach about 100 GeV once all the RF cavities are installed. To minimize the power radiated by synchrotron radiation this ring has a large bending radius in the dipole magnets of $\rho = 3096$ m. There are monitors which image the beam cross section using the synchrotron radiation from these magnets [8]. They consist of telescopes of the kind shown in Fig. 20 looking at the beam at locations having different values for the dispersion function. To get a good resolution they can operate with ultra-violet light of a wavelength $\lambda = 200$ nm. We expect a vertical resolution limitation by Fraunhofer diffraction (32) if only the horizontal polarization is used

$$\sigma_Y(\sigma) = 0.206(\lambda^2 \rho)^{1/3} = 0.1 \text{ mm.}$$

A more detailed calculation of the effect of diffraction and depth of field has been carried out giving a resolution which is about twice as large.

We have to compare this resolution with the actual size of the beam. For physics experiments LEP is operating either at 46 GeV per beam to produce Z^0 -particles or at about 90 GeV to create W^+, W^- -pairs. At the lower energy the horizontal beam emittance determined by synchrotron radiation in a 90° FODO lattice is $E_x \approx 12$ nm rad. At the higher energy it is larger. The vertical emittance is not well determined since it is due to coupling and residual dispersion. After careful correction of these effects an emittance ratio of about 0.5 % can be obtained giving $E_y \approx 0.06$ nm rad. The vertical beta functions at the synchrotron radiation monitors are 78.6 m and 137.1 m giving rms beam sizes of $\sigma_y = 0.07$ mm and $\sigma_y = 0.09$ mm respectively. These values are somewhat smaller than the resolution. However, the instrument can make a correction for the diffraction and is able to measure beams down to an emittance of about 0.1 nm rad. The horizontal resolution of the instrument is optimized by limiting the horizontal acceptance. It is a little larger than the vertical resolution but does not represent a limitation since the horizontal beam size is large.

The image of the beam cross section is measured with a CCD camera and memorized. It can now be presented as a horizontal and vertical profile or as a three dimensional plot. It is also possible to get this information for successive revolutions and thereby observe fast bunch shape oscillation as shown in Fig. 21.

9.3 Imaging an electron beam with an x-ray pin-hole camera

Since the diffraction limited resolution of the image is proportional to $\lambda^{2/3}$ it can be improved by going to a shorter wavelength. However, lenses and other optical elements are not readily available for ultra-violet light or x-rays. It is possible to obtain an image using x-rays and a simple pin-hole camera. Such a measurement was carried out for the electron-positron storage ring CEA [9]. The lay-out of this experiment is shown in Fig. 22. The radiation originates in a bending magnet having $\rho = 26.2$ m, reaches at a distance of 8 m a pin-hole of 0.07 mm diameter and is detected 16 m further down stream on a film. The x-rays had to pass through 1.3 mm of Al and about 6 m of air which cut most of the radiation with $\lambda > 0.1$ nm. Taking also the spectral sensitivity of the film into account it is estimated that radiation with $\lambda \approx 0.05$ nm contributed mostly to the picture. For this wavelength and the pin-hole size used the diffraction contributes less than 0.01 mm to the resolution while the direct effect of the finite pin-hole size gives about 0.07 mm. Taking

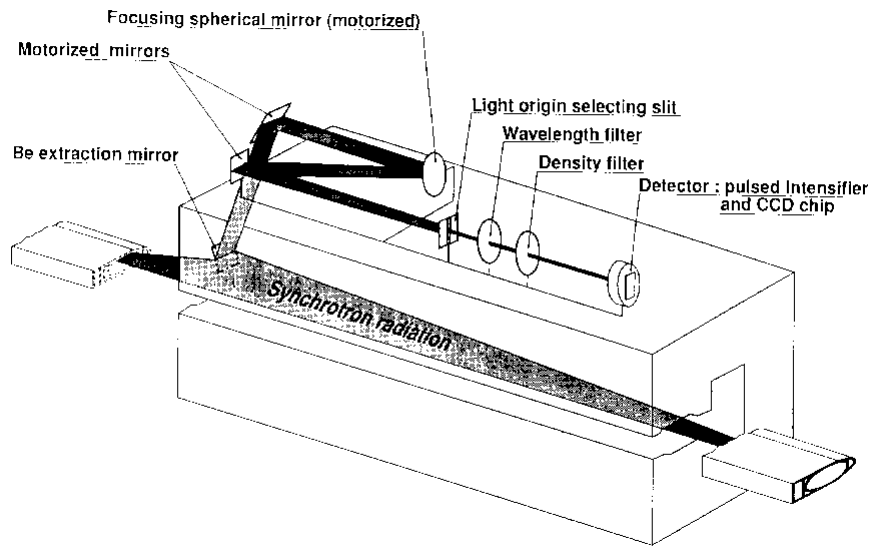


Figure 20: Telescope for imaging the beam cross section in LEP

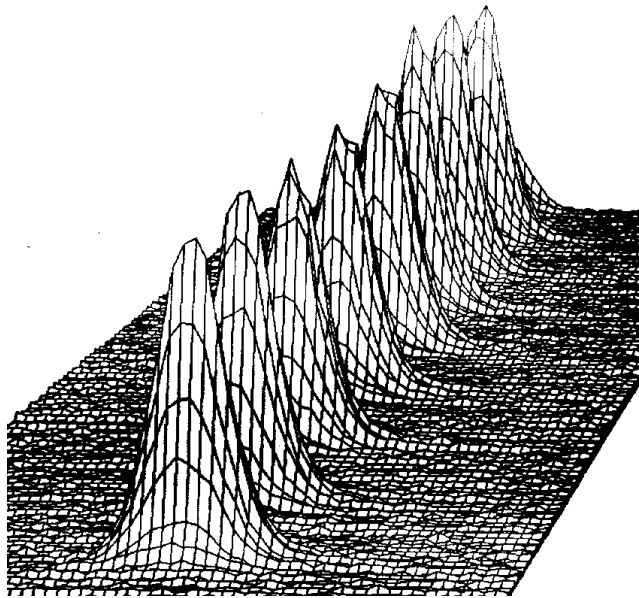


Figure 21: Three dimensional representation of the beam cross section for eight successive revolutions

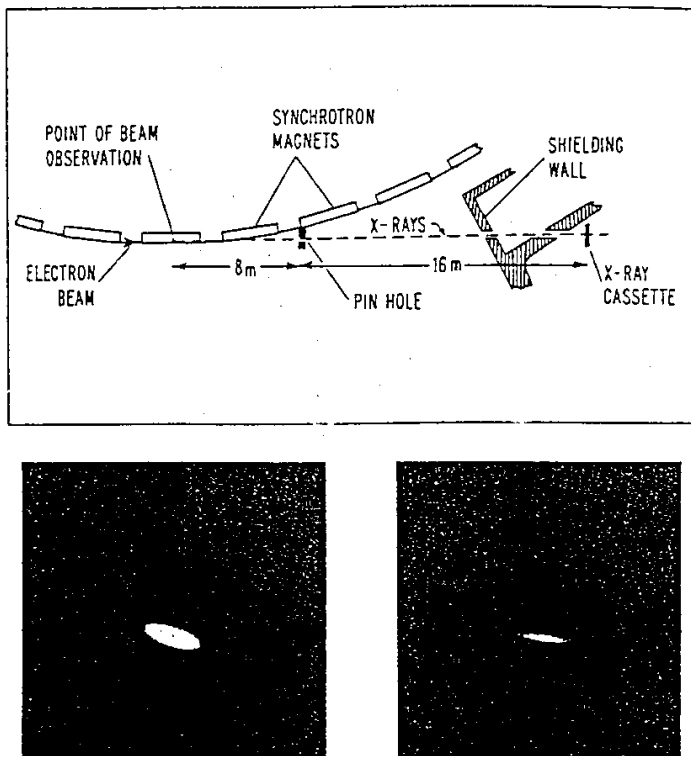


Figure 22: Measuring the beam cross section with an x-ray pin-hole camera to evaluate and compensate coupling

also the resolution of the film into account we estimate an overall resolution of about 0.1 mm. The lower part of the figure shows two measured cross sections of the beam. The left picture indicates some strong coupling which is reduced by powering a quadrupole which separates the horizontal and vertical tunes as shown by the right side.

9.4 Imaging proton beams in the SPS

The CERN-SPS (Super Proton Synchrotron) is a synchrotron accelerating protons from a momentum of about 26 to 450 GeV/c. The radius of curvature in the dipole magnets is $\rho = 741.3$ m. This machine has been operated as a storage ring at 315 GeV/c to make proton-antiproton collisions. For luminosity optimization it is desirable to measure the beam cross section. This could be done with synchrotron radiation. However, for the bending magnets the critical wavelength at 315 GeV/c ($\gamma = 336$) is

$$\lambda_c = \frac{2\pi c}{\omega_c} = \frac{4\pi\rho}{3\gamma^3} = 82\mu\text{m}.$$

The spectrum lies therefore in the far infra-red where optical elements are not easily available and which will result in a poor resolution.

It is possible to obtain radiation of higher frequencies from a short magnet. In section 2 we estimated a typical wavelength emitted by a short magnet of length L (1)

$$\lambda_{sm} = \frac{2\pi c}{\omega_{sm}} \approx \frac{L}{2\gamma^2}.$$

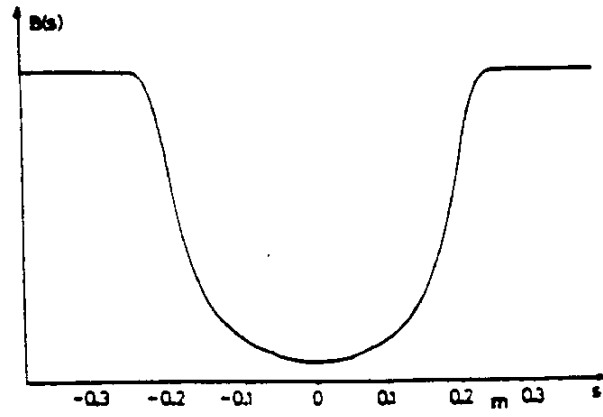


Figure 23: Magnetic field between two SPS dipole magnets

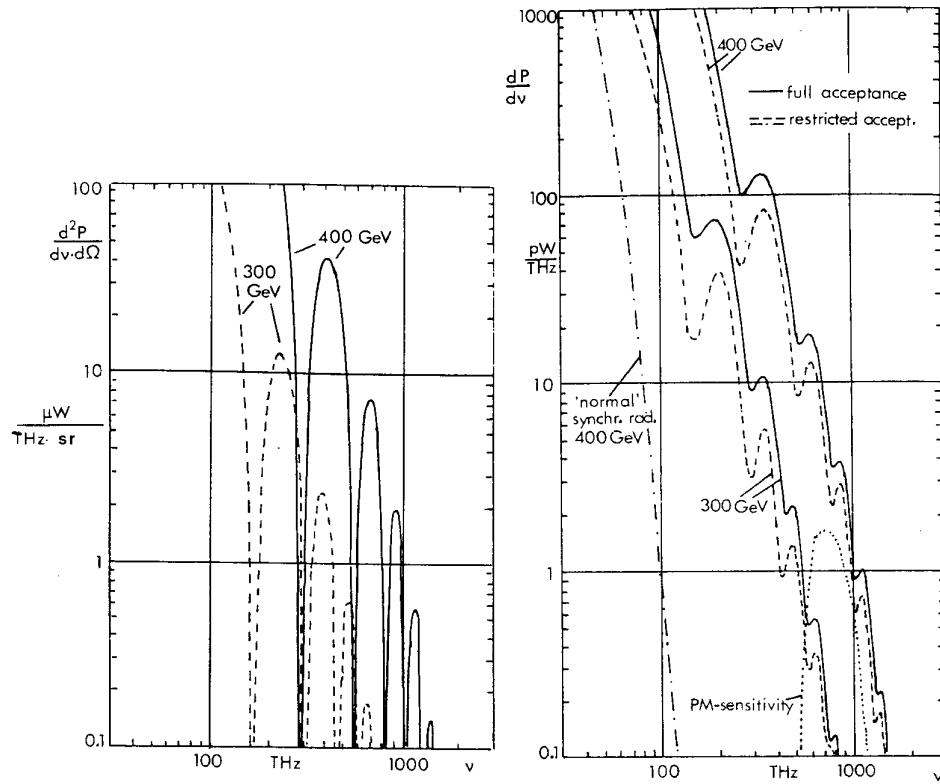


Figure 24: Spectrum of the 'short magnet' radiation. Left: observed on axis ($\theta = 0$). Right: solid line, integrated over the solid angle; dashed line, integrated over the detector acceptance; dot-dash line, radiation from the adjacent long magnet; dotted line, spectral sensitivity of the photo-multiplier used for detection.

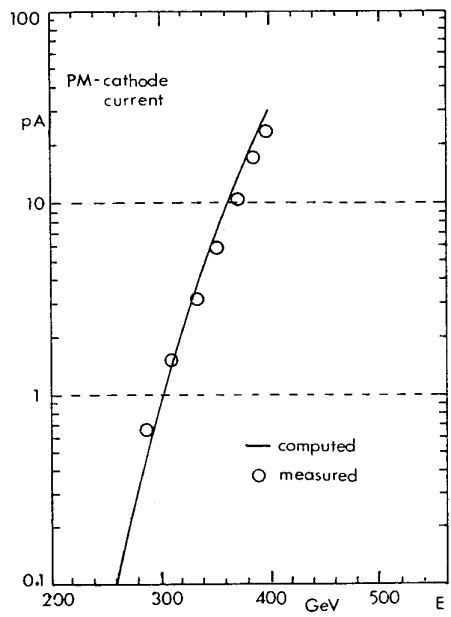


Figure 25: Expected and measured radiation as a function of proton energy expressed in photo-multiplier current

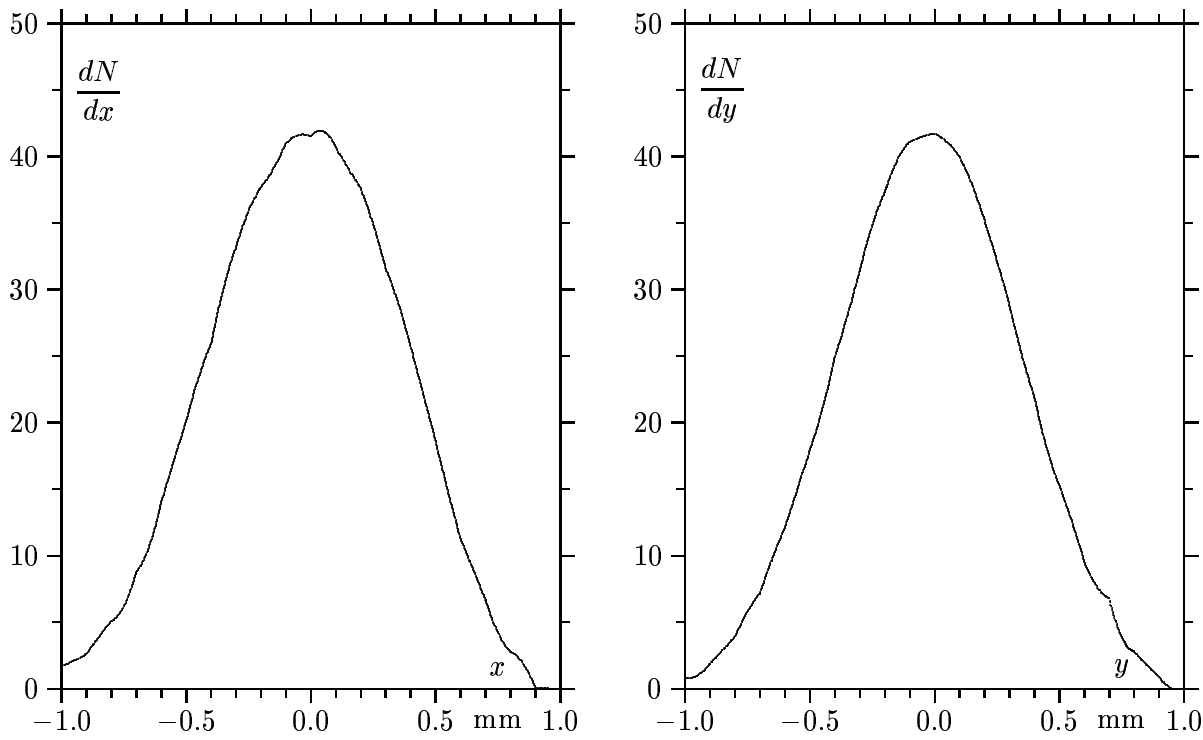


Figure 26: Horizontal and vertical proton beam profile measured with 'short magnet' radiation

In order to get visible light of $\lambda = 600$ nm from protons at 315 GeV/c we need a magnet of length $L \approx 0.135$ m which looks reasonable.

The spectrum of the radiation from a short magnet observed at a given angle θ is given by the Fourier transform of the magnetic field (22). It is the same for the radiation coming from a short magnet with field $B_{sm}(s)$ or from a depression of a dipole field B_0 between two magnets having the form $B_0 - B_{sm}(s)$. This is only true for the short magnet radiation and obviously not for the one emitted in the long magnets. For the protons in the SPS a short magnet of length ≈ 0.15 m produces a spectrum which is far above the one of the radiation coming from the long magnets and can be treated separately.

In the case of the SPS the field variation between two dipole magnets shown in Fig. 23 has been used as source of radiation to image the proton beam [10]. The spectrum has been calculated with the equation (22) and is plotted in Fig. 24 for the radiation emitted at two proton energies. The figure on the left shows the spectrum observed on axis $\theta = 0$. It shows sharp maxima and minima created by constructive and destructive interference between the radiation originating from the two magnet edges. On the right side of Fig. 24 this radiation is integrated over all angles (solid line) which results in a smoother curve because the interference frequencies depend on the observation angle θ . The dashed curves give the radiation integrated over the angular acceptance of the detector. The radiation coming from the long magnets is also shown in the figure as a ‘dot-dash’ line. At low frequencies this radiation will of course dominate. Finally the spectral response of the photo-multiplier used for detection is shown as a dotted curve. A convolution of this spectral power within the acceptance and the spectral sensitivity of the photo-multiplier gives the expected output current of the latter. This is plotted in Fig. 25 together with the measured data which shows very good agreement. A measurement of the horizontal and vertical beam profile with this system is shown in Fig. 26. Since the frequency emitted by the short magnet is proportional to $1/\gamma^2$, the emitted spectrum lies in the visible range only for relatively large proton energies above about 270 GeV. Later an undulator has been installed [11] which produced more light than the magnet gap and extended the energy range over which radiation could be observed. Furthermore, a wire scanner was installed later which can measure the beam size also at injection.

9.5 Measurement of the angular spread with undulator radiation

A direct measurement of the synchrotron radiation distribution gives the angular spread of the particles in the beam. The resolution is limited by the natural opening angle of the emitted radiation. Such a measurement has been carried out to measure the beam in the electron-positron storage ring PEP at SLAC [12]. An undulator has been used as a radiation source which permits to measure the vertical and horizontal angular spread in the beam. A wavelength of the emitted radiation is selected by a monochromator and observed on a screen at a distance L from the source as shown in Fig. 27. The measured undulator spectrum and photon beam pictures taken at two different selected wavelengths are shown in Fig. 28. The first picture is within the fundamental peak of the undulator spectrum (but unfortunately not quite at its maximum). It shows an elliptic distribution of the radiation around the axis. This distribution was scanned with a pinhole to get a horizontal and a vertical cut through the distribution with the rms widths σ_x and σ_y at the screen. The second measurement was only taken for comparison. It shows that the second undulator harmonics has a distribution with vanishing intensity on axis $\theta = 0$.

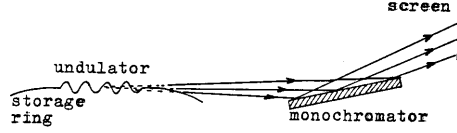


Figure 27: Direct angular distribution measurement of the monochromatized undulator radiation

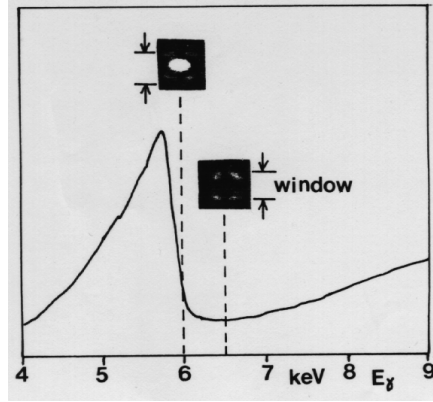


Figure 28: Measured undulator spectrum and photon beam picture

To analyze the measurement we have to consider the other effects which influence the measured photon beam size at the screen. The pinhole used for scanning had the form of a square of side $a = 0.5$ mm which corresponds to $\sigma_{pinhole} = a/\sqrt{12} = 0.144$ mm. The energy spread together with the dispersion and its derivative at the source give a contribution $\sigma_D = (D(0) + D'(0)L)\sigma_E/E$ at the screen. For the natural angular distribution of the photon beam at the fundamental frequency ω_{10} we take the one given by (28) and plotted in Fig. 14 valid for a weak field undulator as an approximation for our experiment which uses a somewhat stronger field. We said before that this distribution has a diverging variance for the opening angle θ . However, in this experiment cuts in the x and y -direction were made for which we find

$$\sigma_\gamma = \frac{L}{\gamma} \sqrt{\frac{3}{4\pi N_u}}.$$

Finally the emittance of the beam gives the contribution we want to measure. We showed in section 7 that we can define a beta function $\beta(L)$ for a photon beam on the screen and get the relation between its size and the emittance ϵ of the electron beam

$$\sigma_\epsilon = \sqrt{\epsilon\beta_\gamma(L)}.$$

The measured size σ of the photon beam on the screen is therefore composed of the contributions

$$\sigma^2 = \sigma_D^2 + \sigma_{pinhole}^2 + \sigma_\gamma^2 + \sigma_\epsilon^2$$

from which the emittance can be obtained. Its value in this measurement was about 35% larger than expected, probably due to the limited accuracy with which the beta function is known.

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THE STORAGE RING FREE-ELECTRON LASER

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Abstract

In a storage-ring free-electron laser (SRFEL) the circulating current of a ring forms the driver for the free-electron laser. The storage ring provides excellent beam parameters and it seems an ideal candidate for a tuneable laser for the ultra-violet and vacuum ultraviolet spectral range. At present experiments demonstrate stable operation with good spectral properties in the UV spectral range, i.e. down to a wavelength of 240 nm. Construction of such a device is non-trivial, however, and optimisation of both the laser and the storage ring is necessary. The SRFEL also has a more complex dynamic behaviour as compared to linac-driven FEL's since the dynamics of both the laser and the storage ring are involved. Here follows a discussion of both the design and the operational aspects of this device.

1. INTRODUCTION

The storage ring free-electron laser (SRFEL) employs the circulating current in a storage ring to drive the gain of the laser. Figure 1 shows the fundamental layout of such a device. An undulator is located in a dispersion-free section of the ring. Next, the synchrotron radiation is captured in a resonator to enable the interaction between the electrons and the optical field. As a FEL driver, the storage-ring has excellent beam parameters, e.g., a low energy spread and a low emittance. Because of the typical beam-energy of rings, the laser is specially suited for operation at short wavelengths (< 700 nm). Difficulties with optics will, most probably, limit the use to the vacuum ultra-violet (VUV) spectral range.

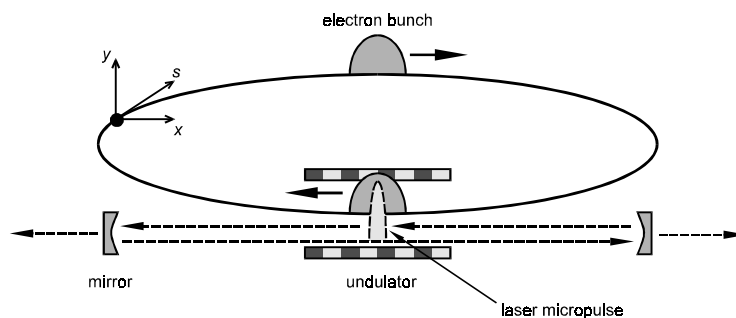


Fig. 1 Schematic layout of the Storage Ring Free Electron Laser (SRFEL). The radiation that is produced by the circulating electrons in the storage-ring passing through an undulator is captured between two mirrors. Due to the interaction between the circulating electron bunches and the stored optical pulse, the characteristic FEL process can take place. The example sketched is typical for a two-bunch mode of operation, i.e. two bunches in the storage ring that both interact with one optical pulse in the resonator.

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The first operation of a storage-ring based FEL was demonstrated at a wavelength of 650 nm on ACO in Orsay, France in 1983 [1]. Since then the field has become maturer and on several rings such as VEPP-3 (Novosibirsk, Russia) [2], UVSOR (Okazaki, Japan) [3] and Super ACO (Orsay, France) [4] the FEL process has been studied. New challenging projects are starting, or close to, operation, e.g. at Duke University (Durham, USA) [5] and DELTA (Dortmund, Germany) [6]. A more complete overview is given in Table 1. Recent developments also demonstrated the feasibility of the SRFEL as a light source for user experiments. For example at Super ACO where up to 50% of the FEL beam-time is used to facilitate user experiments [11]. The SRFEL has the advantage that it is naturally synchronised with the additional beam lines along the ring. One thus has two synchronised high-brilliance polarised light sources with tuneable wavelength in the UV or VUV. The SRFEL forms an ideal tool for pump-probe experiments with a high repetition rate. For example at Super ACO such experiments are performed with a repetition rate of 8.3 MHz. Here the FEL either serves as pump or as probe [11].

Table 1

Storage-ring free-electron laser facilities

Project	Location	Ref.	Status ¹⁾	E [GeV]	λ [nm]	Gain [%]	Remarks
ACO	France, Orsay	[1]	a	0.16-0.25	460-650	0.4	First SRFEL
DELTA	Germany, Dortmund	[6]	c	0.5-1.5	100-400		Dedicated
Duke Univ.	USA, Durham	[5]	o	1	25-400		Dedicated
KEK	Japan, Tsukuba	[7]	c	0.75	170-220		
NIJI-IV	Japan, Tsukuba	[8]	o	0.24	350	2.3	Dedicated
Soleil	France	[9]	p	1.5	100-350		Synchrotron
Super ACO	France, Orsay	[4]	o	0.6-0.8	350-600	2.5	Synchrotron
TERAS	Japan, Tsukuba	[10]	a	0.24	598	> 0.1	
UVSOR	Japan, Okazaki	[3]	o	0.5	290-500		
VEPP-3	Russia, Novosibirsk	[2]	a	0.35-0.5	240-690	10.0	Nucl. phys.

¹⁾ a - abandoned, c - construction, o - operational, p - proposed

In the SRFEL the beam in the storage ring serves as a source for the FEL interaction. Apart from this the SRFEL is not fundamentally different from other types of free-electron lasers. Hence, all aspects that are of importance for a (linac driven) FEL are similarly important in the SRFEL. The storage ring has several properties that make it a good choice as a driver. In a modern ring it is possible to obtain a high current in combination with an excellent beam quality, for example, a stable beam with a low energy-spread, a small emittance, and a high peak current. It is important to note that in a storage ring the beam is recirculated. This gives the storage ring the advantages of 100% duty cycle (the beam runs continuously for many hours). Because of limitations to the rf power needed, conventional linacs generally operate with macro pulses with a duty cycle less than one percent. The recirculation also causes a coupling between the dynamics of the laser and ring. This makes the storage ring a different and a more complex beam source. Consequently the performance, characteristics and dynamic behaviour of the SRFEL are different from FEL's using other types of accelerators. The achievable output-power of the SRFEL is, for example, different from one driven by a linear accelerator. In the storage ring the recirculation limits the achievable peak-power. The peak-power of an SRFEL is therefore lower than the peak-power of a linac driven FEL. Because of its 100 % duty cycle the average power level is much higher, however. Typical values are depicted in Figs. 2 and 3.

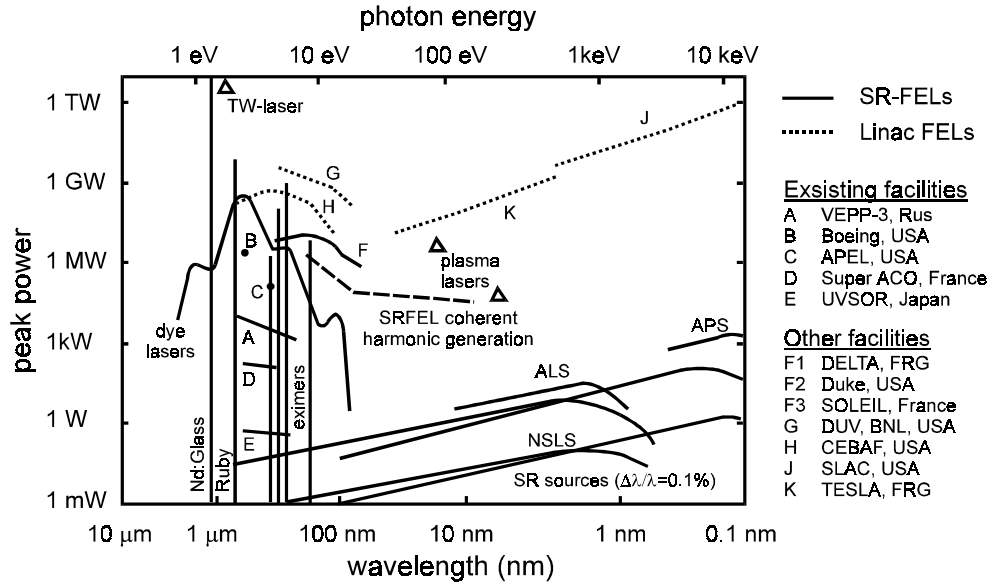


Fig. 2 Overview of achieved and predicted peak powers for several synchrotron and FEL light-sources [12]. See Table 1 for references to the SRFEL light-sources.

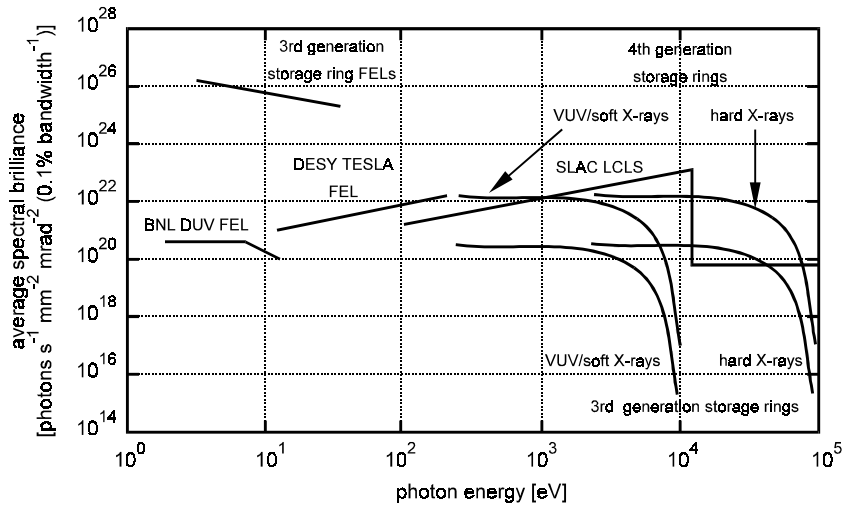


Fig. 3 Achieved and predicted average spectral brightness of several synchrotron and FEL light-sources [13]. See Table 1 for references to the SRFEL light-sources.

The performance of the linac-based and the ring-based FEL's are different by other factors as well. The laser micropulse length in the linac is typically shorter than that of the ring system, i.e. down to a sub-pico-second level in a linac versus several tens of pico-seconds for a ring. It is expected that the achievable peak current and, hence, the laser gain, of a linac-based system can be much higher. As a result the linac based FEL is a better-suited candidate for a single-pass mode of operation where spontaneous emission will be amplified to saturation in a single-pass (SASE - Self-Amplified Spontaneous Emission). This is especially important for short-wavelength operation where presently no mirror material is available for resonators, i.e. wavelengths less than 50 nm. In terms of stability and the quality (e.g. spectral width) of the emitted radiation, it is more advantageous to use a resonator. Due to this and the fact that the beam can be more stable in a ring, the SRFEL is a better candidate for stable user

source at wavelengths where optics are available. Furthermore, unlike the alternative, the SRFEL already has a record of obtained results. An overview is given in Table 2. It is expected that several of these parameters will be improved soon since two new lasers at Duke University and at Dortmund University are close to being operational. Both machines have the novel feature that the storage ring has been specially optimised for FEL operation. Hence, their performance, in terms of the small-signal gain and the expected output power, are expected to overshadow the performance of the presently operational SRFEL's.

Table 2

Experimental results obtained

Shortest wavelength:	λ_{min}	240	nm	VEPP-3
		239	nm	UVSOR
Rms minimum line-width:	$\Delta\lambda/\lambda$	$3 \cdot 10^{-7}$		VEPP-3
Maximum small-signal gain:	G	10	%	VEPP-3
Maximum peak power:	P_{max}	60	kW	VEPP-3
Maximum average power:	P_a	100	mW	Super ACO
Maximum duration of CW operation:	T	10	hours	Super ACO

The contents of this chapter can be divided into two parts. Sections 2 and 3 describe the process of gain and saturation in the SRFEL while sections 4 and 5 discuss the essential parts for the design of a storage-ring FEL: the resonator and the optimisation of the ring. Additionally there are two appendices. The major parts of the experimental examples of laser operation are obtained from the Super ACO FEL experiment. For reference Appendix A gives an overview of this machine and some of its diagnostic equipment. Appendix B presents the details of a gain model that is used in the calculations presented in Sections 2, 4 and 5.

2. THE FEL GAIN

The SRFEL is generally a low-gain device. That is, the laser saturates after the light has passed many times through the undulator. For such a laser to operate, the gain must initially exceed the threshold losses. Hence, the single-pass small-signal gain is an important parameter. Quantitative analyses of the FEL gain are given in Appendix B. It is often useful to start with a more qualitative description. The Madey theorem [14] states that the single-pass small-signal FEL gain is proportional to the derivative of the spontaneous emission spectrum. It thus follows that any modification of the spontaneous emission spectrum of the undulator can be translated into a change in the small-signal gain. Experimentally this is an important aspect since it is often easy to measure the spontaneous emission of an undulator. The spontaneous emission can, therefore, serve as a tool to optimise the FEL gain.

A quantitative prediction of the small-signal gain for a planar undulator shows that in a storage ring the regular planar undulator is not an optimum choice. More gain can be obtained through a modification of the undulator into a so-called Optical Klystron (OK). A description and quantitative analyses of this device is given in Section 2.3.

2.1 The Madey theorem

The physics behind the Madey theorem is best explained in the so-called rest frame of the electrons, i.e. the coordinate system in which the net forward velocity of the electrons is zero. This frame is illustrated in Fig. 4. Photons originating from the optical field move from

the left to the right. Virtual undulator photons move in the opposite direction. In the rest frame both emission and absorption can be described as Compton scattering. In the case of emission a virtual undulator photon, moving in backward direction with energy $\hbar\omega'_u$ transforms into a light photon $\hbar\omega'_e$ with forward velocity due to scattering. Here $\hbar$ denotes Planck's constant divided by 2π . The prime indicates that the frequencies are related to the rest frame. In case of absorption the opposite process results in the transformation of a light photon $\hbar\omega'_a$ into a virtual undulator photon $\hbar\omega'_u$. In the laboratory frame, i.e. the frame of the observer, the corresponding wavelength transforms back into the well-known relation of the fundamental wavelength of the undulator radiation λ :

$$\lambda = \frac{\lambda_u}{2\gamma^2} \left(1 + \frac{K^2}{2} + \gamma^2 \theta^2 \right) \quad (1)$$

$$K = \frac{eB_u \lambda_u}{2\pi mc}$$

where λ_u , γ , θ denote the undulator period, the Lorentz factor corresponding to the beam energy ($\gamma \approx E [\text{MeV}]/0.511$), and the angle between the forward propagation direction of the electron beam with respect to the magnetic axis of the undulator, respectively. The magnetic field-strength of the undulator is transformed into the dimensionless parameter K in which B_u denotes the peak magnetic field strength on axis.

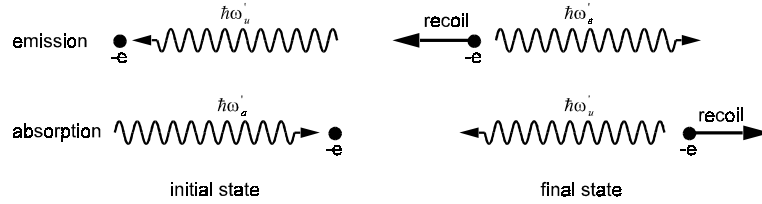


Fig. 4 Scattering interaction for a single electron in the rest frame. In the figure $\hbar\omega'_u$, $\hbar\omega'_e$, and $\hbar\omega'_a$ denote the virtual undulator photon, the emitted photon and the absorbed photon, respectively. The prime indicates that all frequencies are in the rest frame.

By the definition of the rest frame the net electron velocity is zero. Hence, the scattering process always increases the kinetic energy of an electron due to recoil, both in the case of absorption and emission. Because of energy conservation, the energy contained by a scattered photon is always less than the energy of the original photon. Hence, $\hbar\omega'_u > \hbar\omega'_e$ in the case of emission and $\hbar\omega'_a > \hbar\omega'_u$ for absorption. Since the frequency of the virtual undulator photon is related to the definition of the rest frame, i.e. see Eq. (1), it follows that the energy of the electron beam must increase in order to emit a photon with the same frequency. Similarly the electron beam energy must decrease in order to absorb a photon at the central frequency ω'_s . Hence, emission and absorption of a photon with energy $\hbar\omega'_s$ require different electron beam energies. This is illustrated in Fig. 5a. A quantitative value for the shift, $\Delta\gamma$, depends on the magnitude of the electron recoil. The derivative of the spontaneous emission spectrum, therefore, maps to the gain spectrum, see Fig. 5b. Electrons injected into the FEL slightly above the resonant energy contribute to net stimulated emission. Electrons injected at a slightly too low energy give rise to net absorption. In the former case the working of the FEL is reversed and it operates as an electron accelerator.

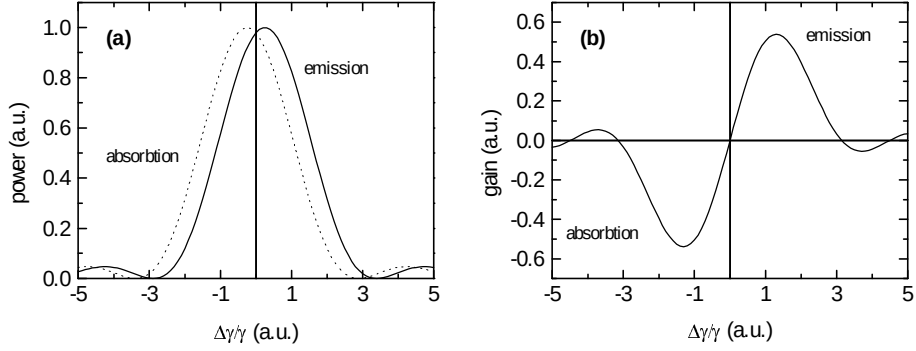


Fig. 5 Emission and absorption spectrum of an electron pass through a two-wave FEL (a) and the resulting gain spectrum (b).

2.2 The FEL gain for a planar undulator

This text follows the gain calculations made by Dattoli et al. [15]. The peak single-pass gain for a perfect one-dimensional electron beam, i.e. no energy spread and infinitely small transverse beam-size, can be found to be

$$G_0 = 0.27 \pi g_0 \quad (2)$$

where g_0 is the so-called gain parameter which for a planar undulator and low gain ($g_0 < 1$) is given by

$$g_0 = 2\pi \frac{J}{I_A} K^2 N^3 \frac{\lambda_u^2}{\gamma^3} JJ \quad (3)$$

where J , I_A , and N denote the electron current density, the Alfvén current (≈ 17 kA), and the number of undulator periods, respectively. The parameter JJ

$$JJ = (J_0(\xi) - J_1(\xi))^2$$

$$\xi = \frac{K^2}{4 + 2K^2} \quad (4)$$

accounts for the gain reduction due to the periodic detuning of the electrons in a planar undulator (see Ref. [16] p. 73), where J_0 and J_1 are Bessel functions of the first kind.

The transverse overlap of the electron beam and the EM wave is usually taken into account by multiplication of the gain parameter with a heuristic filling factor, for a narrow electron beam defined as the ratio of the transverse cross section of the electron beam and the EM wave. Under the assumption of a diffraction-limited Gaussian beam, the average transverse cross section of the EM wave along the undulator is proportional to the undulator length $N\lambda_u$, and the wavelength λ , see Section 4.1. For an optimum overlap the filling factor, F , is found to be [17]

$$F = \frac{3}{2} \frac{\Sigma_e}{N\lambda_u \lambda} \quad (5)$$

where the transverse cross section of the electron beam, averaged along the undulator, is denoted as Σ_e . Substitution of the Eqs.(5), (3) and (1) into Eq. (2) leads to

$$G_0 = 3.76 \cdot 10^{-3} N^2 I(A) \frac{\varepsilon}{\gamma} JJ \quad (6)$$

where I also substituted the beam current, $I = J\Sigma_e$. It should be realised that this gain formula is only valid for ‘ideal’, continuous electron beams, i.e. for continuous-wave (cw) beams with negligible energy spread and emittance. This assumption is certainly not correct for a realistic beam and G_0 needs to be corrected.

First, the influence of the finite beam emittance is discussed, i.e. the influence of the finite beam radius and divergence. Due to the fact that the magnetic field has a sinusoidal z -dependence on the axis of the undulator, off-axis electrons experience a slightly different field (because $\nabla \cdot \mathbf{B} = 0$), and thus also a slightly different K that leads to a different wavelength than is given by Eq. (1). Also, electrons moving under a small angle θ give rise to a different wavelength. It can be calculated that this leads to the following emittance-induced inhomogeneous broadening of the line-width [15]:

$$\left(\frac{\Delta\omega}{\omega} \right)_{x,y} = \frac{2\gamma^2}{1 + \frac{K^2}{2}} \sqrt{\frac{1}{2} \sigma_{x',y'}^4 + \frac{1}{2} \left(\frac{\pi K}{\gamma \lambda_u} \right)^4 h_{x,y}^2 \sigma_{x,y}^4} \quad (7)$$

where σ_x and σ_y denote the standard deviation of the particle distribution in the x - and y -direction respectively; the wiggle motion takes place in the x - z plane. Further, $\sigma_{x'}$ and $\sigma_{y'}$ denote the standard deviation of the corresponding transverse velocity distributions. The parameter h denotes the sextupole term of the undulator, i.e. $h_x = h_y = 0$ for a helical undulator and $h_x \approx 0$, $h_y \approx 2$ for a planar undulator. The parameters $\sigma_{x,y}$ and $\sigma_{x',y'}$ are coupled through the unnormalized beam emittance $\varepsilon_{x,y}$. In the case of a 100 % coupled cylindrical beam, i.e. $\varepsilon = \varepsilon_x = \varepsilon_y$ and $\sigma_x = \sigma_y$, it is found that the gain-reduction due to the finite emittance can be expressed as [15]:

$$\begin{aligned} C_{x,y} &= \frac{1}{1 + \mu_{x,y}^2} \\ \mu_x &= \pi N \sqrt{2} \frac{\gamma \varepsilon}{\lambda_u} \frac{K^2}{4 + 2K^2} \\ \mu_y &= 2\pi N \frac{\gamma \varepsilon}{\lambda_u} \frac{K^2}{4 + 2K^2} \end{aligned} \quad (8)$$

where $\mu_{x,y}$ expresses the relative broadening of the spontaneous emission spectrum, i.e. $\mu_{x,y} = (\Delta\omega / \omega)_{x,y} / (\Delta\omega / \omega)_0$. The corrections are valid up to (at least) $\mu_{x,y} = 1$.

Also important is the influence of a finite energy spread. It follows from Eq. (1) that the spectrum of spontaneous emission broadens in the case of non-zero electron-beam energy spread and, the gain is reduced (Madey theorem). A quantitative estimate of the gain-reduction is found to be [15]:

$$\begin{aligned} C_e &= \frac{1}{1 + 1.7\mu_e^2} \\ \mu_e &= \frac{1}{4N(\sigma_\gamma / \gamma)} \end{aligned} \quad (9)$$

where σ_γ denotes the standard deviation associated with the energy distribution. Eq. (9) is valid for $\mu_e \leq 2$. Substitution of Eq. (8) and Eq. (9) into Eq. (2) leads to:

$$G_0 = 3.76 \cdot 10^{-3} N^2 I C_e C_x C_y \frac{\xi}{\gamma} JJ. \quad (10)$$

From Eq. (9) it follows that the influence of energy spread, and hence the gain reduction due to the energy-spread, increases as the number of undulator periods increases. However, the maximum gain also increases, see Eq. (10). It is thus possible to calculate an optimum undulator length:

$$L_{opt} = \frac{\lambda_u}{2(\sigma_\gamma / \gamma)}. \quad (11)$$

In storage rings the energy spread is generally low and, hence, the optimum undulator length can be considerable. As a result the optimum undulator length is normally much more than the length of the straight sections. For example at Super ACO ($\lambda_u = 12.9$ cm, $\sigma_\gamma / \gamma = 0.65 \times 10^{-3}$) it follows that $L_{opt} = 99$ m. The available space in the straight section is less than 4 m, however. This dilemma can be solved through a modification of the undulator.

2.3 The Optical Klystron

The Optical Klystron is a modified version of the undulator that was originally proposed to improve the gain of an FEL with a limited straight section length and small energy spread [18], for example, the SRFEL. The OK consists of a set of two undulators separated by a dispersive section, see Fig. 6. The first undulator serves as a modulator, i.e. due to the interaction with the optical field, the electron beam energy is modulated *on the scale of the fundamental wavelength*. In the dispersive section, low-energy electrons are retarded with respect to the high-energy electrons. Hence, the energy-modulation is transformed into a longitudinal modulation (micro bunching). As a result the coherent radiation and gain in the second undulator is enhanced. The main advantage of an OK is the gain enhancement for a given interaction length. The saturated power level is reduced, however.

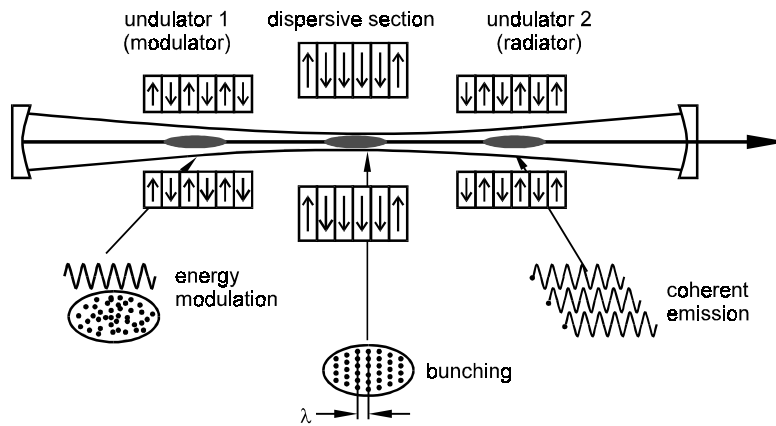


Fig. 6 Schematic diagram of a transverse optical klystron

Insight into the magnitude of the gain enhancement can be deduced from the spontaneous-emission power-spectrum of the OK which serves as a delay between the electrons and the optical wavefront. Consequently the spontaneous emission of both undulators interfere similarly to the interference between two slits. The total radiated intensity I_{OK} can be written as [19]:

$$I_{OK} = 2I_0(1 + \cos \delta) \quad (12)$$

where I_0 denotes the intensity of the spontaneous radiation of a single undulator section. The phase delay δ between the entrance of the two undulators has to be separated into two components: the delay in the first undulator section (slippage = $N\lambda$) and that in the dispersive section.

$$\delta = 2\pi(N + N_d) \frac{\Delta\lambda}{\lambda_R} \quad (13)$$

Here λ_R is the resonant wavelength defined in Eq. (1). The new parameter N_d defines the number of wavelengths over which the electrons are delayed in the dispersive section, relative to the optical wavefront. For a standard magnetic dispersive section this can be written as [20]:

$$N_d = \frac{L_d}{2\gamma^2\lambda} \left[1 + \frac{1}{L_d} \left(\frac{e}{mc} \right)^2 \int_0^{L_d} \left(\int_0^u B(z) dz \right)^2 du \right] \quad (14)$$

where $B(z)$ denotes the transverse magnetic field and L_d the length of the dispersive section respectively. An example of the spontaneous emission spectrum of the optical klystron is given in Fig. 7. The ideal case is depicted in Fig. 7a, i.e. the spectrum that one can expect from an ideal electron beam traversing an ideal optical klystron. From the steeper slopes of the spectrum it follows that the gain is significantly enhanced. Generally the spectrum is deteriorated, however, and the modulation depth f defined as

$$f = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} \quad (15)$$

is reduced, see Fig. 7b. Here the reduction of the modulation depth due to a finite energy spread is the most important. For a Gaussian distribution it can be derived that [20]:

$$f = e^{-2 \left(2\pi(N + N_d) \frac{\sigma_\gamma}{\gamma} \frac{\lambda}{\lambda_R} \right)^2}. \quad (16)$$

The gain for an OK is found to be [21]:

$$G_{ok} = 0.93 \frac{G_0(2N)}{N} (N + N_d) f C_f \quad (17)$$

where G_0 is the homogeneous gain as defined in Eq. (2). The parameter C_f is used to compensate for the transverse overlap between the electron beam and the optical beam. In the case of a Gaussian transverse electron distribution and a TEM_{00} resonator mode a good estimate can be obtained by [20]:

$$C_f = \frac{2}{\sqrt{4\sigma_x^2 + w_0^2}} \frac{2}{\sqrt{4\sigma_y^2 + w_0^2}} \quad (18)$$

where additional to the previous definition, w_0 is the waist size of the optical field in the resonator (see Sec. 3). From Eq. (16) it can be found that the optimum value, i.e. the maximum gain is obtained when [21]:

$$N + N_d = \frac{1}{4\left(\frac{\sigma_r}{\gamma}\right)}. \quad (19)$$

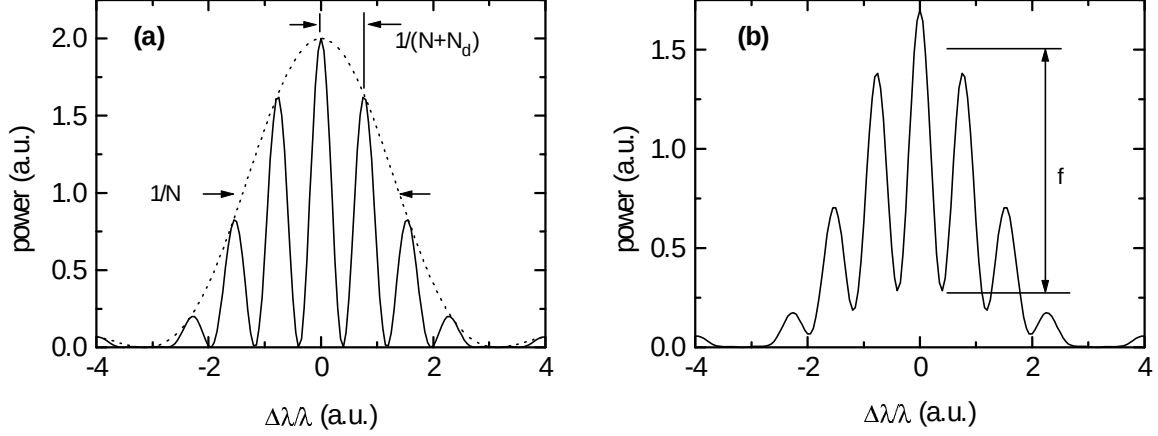


Fig. 7 Spontaneous emission spectrum of an optical klystron: (a) homogeneous spectrum and (b) deteriorated spectrum due to inhomogeneous effects.

An experimental example of the gain enhancement of an OK is shown in Fig. 8. Figure 8a is the measured spectrum of the OK used in the Super ACO FEL (solid line). The corresponding spectrum of this device used as a regular undulator, i.e. $N_d = 0$, is depicted as a dashed line. The gain curve shown in Fig. 8b is obtained using the derivative of the spectrum shown in Fig. 8a (Madey's theorem). The gain enhancement obtained is clearly visible.

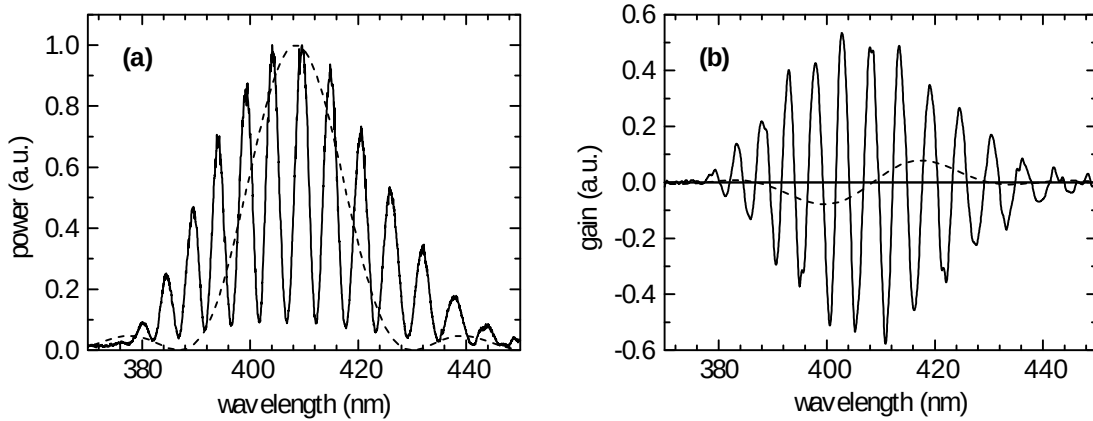


Fig. 8 The spontaneous emission spectrum (a) and the derived small signal gain (b) measured with the Super ACO optical klystron SU7 (solid line). The dashed line corresponds to the spontaneous emission spectrum and gain of an undulator with similar length as the OK, i.e. $2N$ periods.

3. SATURATION

The saturation mechanism of an SRFEL is fundamentally different from a linac driven FEL. In a linac-driven FEL the electron beam is dumped after it has passed through the undulator. In a storage-ring free-electron laser the electron beam is recirculated. Hence, any

perturbation of the electron beam due to the FEL interaction is coupled back to the laser on the next pass through the undulator. Initially this effect is small since, in the small signal case, there is virtually no perturbation of the electron beam. As the optical power grows more energy is extracted from the electron beam and the energy spread increases. This causes a decrease in FEL gain for a multiple of reasons. Firstly because of inhomogeneous broadening of the small-signal gain profile: see Eqs. (9) and (19). Secondly the increase in energy spread can cause bunch lengthening. In a storage ring the bunch length and the energy spread are coupled through the so-called momentum-compaction factor [22]:

$$\alpha \approx \frac{\Delta L / L}{\Delta E / E} \quad (20)$$

which expresses the path lengthening ΔL of a particle with an energy difference ΔE , relative to the reference energy E and the circumference of the storage ring L . The change in bunch duration σ_t depends on both α and the angular synchrotron oscillation frequency ω_s [23]

$$\sigma_t = \frac{\alpha}{\omega_s} \frac{\sigma_\gamma}{\gamma}. \quad (21)$$

Bunch lengthening automatically translates into a reduction of the peak current and, hence, the gain. Finally this leads to an equilibrium state where the small-signal gain is reduced to the cavity losses. The saturation process is schematically depicted in Fig. 9. The left-hand side (a) shows that the gain drops as the optical power increases. On the right-hand side (b) the evolution of the gain and the energy spread is plotted.

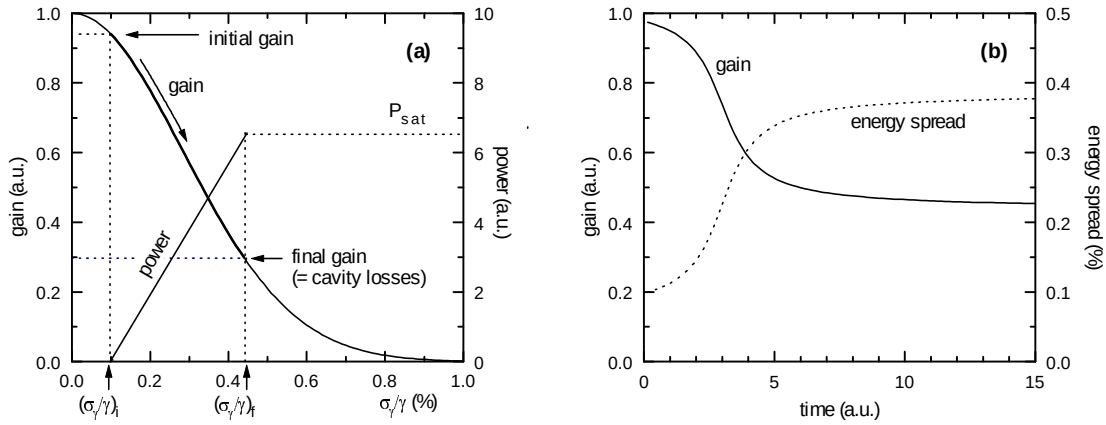


Fig. 9 The saturation process of the SRFEL: as the optical power increases the FEL gain drops (a) due to a decrease in gain induced by an increased energy spread (b).

The magnitude of the extracted power and the induced energy spread depend on the design of both the FEL and the storage ring. For example, an optical klystron with a high value for $N+N_d$ has a higher initial gain but is more sensitive to an increase in energy spread, see Eq. (19). Hence, with a high value for $N+N_d$ the laser power grows more quickly but saturates at a lower level. Simultaneously the induced energy spread at saturation is reduced.

An experimental example taken from the UVSOR FEL is depicted in Fig. 10 [24]. It shows the evolution of the laser power measured simultaneously with the electron bunch length in the ring. The origin of the horizontal axis corresponds with the moment at which the laser is switched “on”. That is, the electron beam parameters are tuned such that the gain

exceeds the cavity losses. During the increase of power the electron bunch becomes longer and the peak current decreases and, hence, the gain decreases. As the gain drops below threshold the optical power decreases.

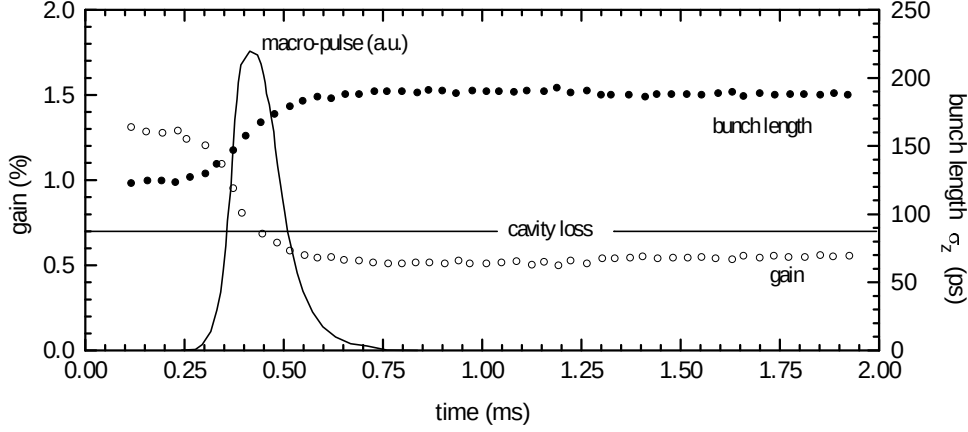


Fig.10 Evolution of the bunch length and the gain in a Q-switched mode. The bunch length has been measured with a dual-sweep streak camera. The gain is derived from the bunch length. The solid line shows the macropulse measured with a photo-diode [24]. The streak camera is described in Appendix A.

From Fig. 10 it follows that the actual saturation process is much more complicated than the situation depicted in Fig. 9. For a better understanding both the dynamics of the FEL and the storage ring need to be considered. Most important here is the damping of induced energy spread in the storage ring. Generally the power, gain and energy spread do not evolve as smoothly as depicted in Fig. 9. The following sub-sections discuss a quantitative description of the steady-state parameters (Sec. 3.1 and 3.2) and the dynamic properties (Sec. 3.3 to 3.5).

3.1 The Average Output Power

The average output power of the SRFEL was first studied by Renieri [25] who solved the Fokker-Plank equation for the electron-distribution in the presence of the FEL. He found that the average output power is limited to a small fraction of the synchrotron power emitted all around the ring (P_{SR}), i.e. the “Renieri limit”:

$$P_{FEL} = 2e^{-\frac{1}{2}} \frac{\sigma_{\gamma,f}^2 - \sigma_{\gamma,i}^2}{\sigma_{\gamma,f}} P_{SR} \quad (22)$$

where $\sigma_{\gamma,f}$ and $\sigma_{\gamma,i}$ denote the initial and final energy spread of the beam, respectively. The total emitted synchrotron power is given by [26]:

$$\begin{aligned} P_{SR} &= \frac{1}{3} \frac{e}{\epsilon_0} \frac{\gamma^4}{\rho} I_a \\ &= 88.46 \cdot 10^3 \frac{E^4 I_a}{\rho} \frac{[\text{GeV}][\text{A}]}{[\text{m}]} \end{aligned} \quad (23)$$

where I_a is the average beam current and ρ the effective beam radius of the synchrotron. A more sophisticated formula, adapted for an optical klystron reads [27]:

$$\begin{aligned}
P_{OK} &= \eta_c \frac{f}{\pi(N + N_d)} \frac{\log(G_0 / \Gamma)}{G_0 / \Gamma} P_{SR} \\
\eta_c &= \frac{\text{mirror transmission losses}}{\text{total transmission losses}} \\
f &= e^{-2 \left(2\pi(N + N_d) \frac{\sigma_\gamma}{\gamma} \frac{\lambda}{\lambda_R} \right)^2} \\
\Gamma &= \text{total cavity losses}
\end{aligned} \tag{24}$$

Typically the emitted synchrotron radiation is a few kW for a storage ring with a beam-energy up to 1 GeV, e.g. 10 kW for Super ACO operating at 800 MeV. The energy acceptance of a ring, and hence, the maximum possible energy spread, can go up to a few percent. It thus follows that the total emitted FEL power is much lower than the emitted synchrotron power. For example, at Super ACO the total emitted laser power is of the order of 100 mW. Note, however, that this radiation is fully coherent. Hence, the brilliance of this radiation is orders of magnitude higher than synchrotron radiation from an undulator beam-line, see Fig. 3.

It follows from Eqs. (23) and (24) that more power can be obtained at higher beam energy and by allowing a larger increase in energy spread. Note however, that there is a practical limit to both parameters. The energy of the beam can not increase too far without changing the lasing wavelength, see Eq. (1). For this, either the undulator period, or the magnetic field strength, must be increased. The former has the disadvantage that for a given length of the straight section, the number of periods, and hence the gain, is reduced. The latter cause problems with the mirrors, see Sec. 4.2.

An increase of energy-spread requires a significant increase in the energy acceptance of a ring:

$$\sigma_p \approx \sqrt{\frac{2}{\pi} \frac{e}{mc^2} \frac{V_{rf}}{\alpha h \gamma}} \tag{25}$$

where h is the harmonic number ($L_c \omega_r / 2\pi c$) and V_{rf} is the total voltage over the rf cavities in the ring. With a too high energy spread, as compared to the energy acceptance of the ring, the lifetime of the beam will be significantly reduced. An increase in energy acceptance is costly since $\sigma_p \propto (P_{rf})^4$.

3.2 The FEL bandwidth

Both the gain bandwidth of the FEL and the eigenmodes of the optical resonator determine the final bandwidth of the FEL where, normally, the gain bandwidth of the laser covers many eigenmodes of the cavity. In this, the SRFEL works similarly to any other type of free-electron laser where the spectrum narrows on successive passes through the undulator. That is, on each pass through the undulator the spectrum of the light pulse is multiplied with the gain spectrum of the laser. In contrast to linac driven FEL's there is no macro-pulse and the process of line-width narrowing can continue over many more round-trips. The line width of an SRFEL is therefore much smaller. Assuming a stable electron beam the line width can narrow down to the transform limit. A study by Dattoli and Renieri [25] predict a laser pulse-width $\sigma_{z,l}$ of

$$\sigma_{z,l} = \sqrt{\sigma_z N \lambda} \tag{26}$$

for an undulator with N periods, a wavelength λ , and an electron bunch length σ_z . Note that the laser pulse-duration is much smaller than the electron bunch duration. This is due to the fact that the optical pulse is also multiplied with the longitudinal gain profile of the laser, i.e. the longitudinal electron bunch shape. The corresponding relative bandwidth is

$$\frac{\Delta\lambda}{\lambda} = \frac{1}{\pi} \frac{\lambda}{\sigma_{z,l}} = \frac{1}{\pi} \sqrt{\frac{\lambda}{N\sigma_z}}. \quad (27)$$

In practice the bandwidth is determined by the stability of the beam in the ring, however. Typical relative bandwidths for an SRFEL are in the order of 10^{-4} to 10^{-5} depending on the way the laser is operated. An example is given in the next section. A smaller line width can be obtained by adding additional filters inside the resonator, thus artificially narrowing the gain spectrum. An experiment performed at VEPP3 has demonstrated a relative laser bandwidth of $3 \cdot 10^{-7}$ using an intra-cavity etalon [28].

3.3 Influencing the FEL output 1: Cavity resynchronisation

From Eq. (23) it follows that the saturated laser power depends on the initial small-signal gain. An optimum gain requires perfect longitudinal and transverse overlap between the optical pulse and the electron bunches. The transverse overlap is mainly determined by the design of the resonator and the machine functions inside the undulator. This will be discussed further in Sections 4 and 5. The longitudinal overlap can only be maintained when the round-trip time of the optical pulses in the laser resonator matches the electron-bunch distance. Hence, the optical cavity length must be matched with the electron bunch distance. Figure 11 depicts this for a storage ring in a two-bunch mode of operation. In that case the resonator length must be one quarter of the circumference of the storage ring. This is, for example, the case for the Super ACO FEL. Other FEL's (e.g. the Duke-FEL and the DELTA FEL) operate in single-bunch mode with a cavity length equal to half the circumference of the ring.

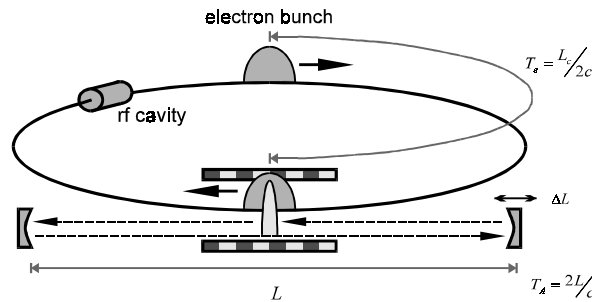


Fig. 11 Longitudinal synchronisation of the FEL: To maintain longitudinal overlap between the optical pulse and the electron bunches, the length of the optical cavity must be tuned to match half the electron-bunch distance. Tuning of the synchronisation can either be done by changing the cavity length or changing the rf frequency of the accelerating field (i.e. changing the distance between successive electron bunches).

As the cavity length is detuned with respect to the spacing between bunches, the longitudinal overlap is reduced and, hence, the small-signal gain is reduced as well. While scanning the synchronisation different (dynamic) behaviour can be observed. An example of the modified output of the Super ACO FEL is given in Fig. 12 [29]. The figure shows the major properties that can be observed in the case of saturation. The main curve displays the average power of the laser. Maximum power is obtained when the cavity length is

synchronised with the bunch distance. When the system is desynchronised, the saturated power level drops. This process is similar to other FEL's. Note that the curve shown in Fig 12 is virtually symmetric around its origin. This is caused by the fact that the electron-bunch length of the Super ACO FEL is long compared to the slippage length. In the case of (much) shorter electron bunches the curve would become more asymmetric, similar to linac driven FEL's.

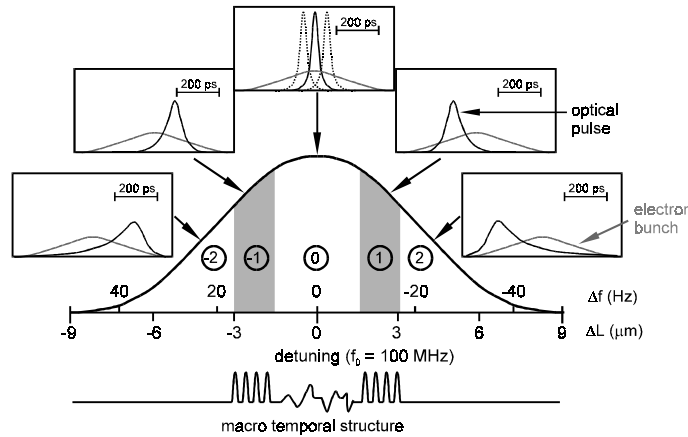


Fig. 12 Longitudinal detuning curve of the Super ACO FEL [29]. The main curve shows the average output power as a function of the detuning. The top graphs give a snapshot of both the electron bunch and the optical pulse at saturation. The bottom curve gives an indication of the macro-temporal structure of the saturated laser output. See text for details.

The graphs above depict the shape of the optical pulse and the electron bunch. The shortest optical pulses are obtained for perfect synchronisation. As the synchronisation changes the optical pulse moves forward or backwards with respect to the electron bunch, depending on the sign of the resynchronisation. Simultaneously the length of the optical pulse increases. This is due to the fact that on one side the optical pulse, on successive round-trips, moves ahead or behind the bunch, will cease to experience gain, and dies out with the ring-down time of the cavity. On the other side of the optical pulse new light is created in the amplification process. Note that the pictures shown are not measured but serve as an example only.

The bottom curve in Fig. 12 shows the macro-temporal structure of the laser. At Super ACO five distinctive zones can be observed. As the laser operates far from synchronisation fairly low power, stable laser output is observed (zones ± 2). As the system is tuned closer to synchronisation a pulsed time-structure is present. In this zone (± 1) a semi-stable competition takes place between the rise time of the optical power, i.e. the increase in energy spread, and its damping given by the damping time of the storage ring. In the next sub-section this process is discussed in more detail. At perfect synchronisation (zone 0) the output power becomes unstable and large irregular fluctuations ($\sim 80\%$) of the output power are observed. Simultaneously the optical pulse starts to jitter with respect to the electron bunch position. The cause of this jitter is not clear yet but can be counteracted with a feedback system. Note further that the size and number of zones depend on the maximum achievable gain. The zones that are indicated in Fig. 12 are typical for the Super ACO FEL running under optimum conditions. As the available gain drops the central zone becomes smaller or can even

disappear completely. Also at UVSOR, where the gain is lower, the central zone can not be observed [24].

The pulsed time structure that was mentioned in the previous sub-section is an intrinsic effect of the SRFEL and has been explained by Elleaume [31] using a very simple model. This model is based on Eq. (24), which indicates that small changes in the energy spread can have a large effect on the saturated power level. This influence has been modelled as

$$\frac{di}{dt} = \frac{(1-\Sigma)i}{\tau_0}, \quad \frac{d\Sigma}{dt} = 2\frac{1-\Sigma}{\tau_s}, \quad \Sigma = \sigma_{\gamma^2} - \sigma_{\gamma}^2 \quad (28)$$

where i is the dimensionless laser power normalised to unity at saturation. The laser rise time τ_0 is found to be

$$\tau_0 = T / (G_0 - \Gamma) \quad (29)$$

where, additional to the previous definitions, T is the roundtrip time of the optical resonator. τ_0 can be understood as the rise time of the first laser pulse train once the gain has been set higher than the losses. τ_s is the synchrotron damping time. Generally $\tau_0 \ll \tau_s$ (for example at ACO: $\tau_0 \approx 50 \mu\text{s}$ and $\tau_s \approx 200 \text{ ms}$). Within this restriction the general solution of Eq. (28) is:

$$\Sigma = 1 + \Sigma_0 e^{-t/\tau_s} \cos \left[2\pi \frac{t-t_0}{T_R} \right], \quad T_R = \pi \sqrt{2\tau_s \tau_0} \quad (30)$$

where Σ_0 is the value of Σ at the beginning of the pulse.

The equations imply that the FEL tends to reach a stable equilibrium state after a series of damped oscillations of period T_R damped with the synchrotron damping time τ_s . An example of a solution of Eq. (30) is given in Fig. 13. It shows that starting from the laser-off condition the laser intensity has a pulsed behaviour before reaching a stable level. The shape of each pulse is approximated by

$$\frac{i}{i_{\max}} = \cosh^{-2} \left[t \sqrt{\frac{i_{\max}}{\tau_s \tau_0}} \right], \quad i_{\max} = \frac{(1-\Sigma_0)\tau_s}{2\tau_0} \quad (31)$$

Moreover, it can be shown that the equilibrium state is very unstable and resonant with any perturbation of period close to T_R , so that the laser has in practice a pulsed structure. For example, a noise of 3% affecting the energy spread can account for the randomly pulsed behaviour.

More accurate modelling can be done with the aid of numerical simulation codes, e.g. for the Super ACO case depicted in Fig. 12. A numerical model developed by Hara et al. [32] can make a full prediction of the curves shown in this figure.

3.4 Stabilising the laser: feedback

For use as a light source the central zone is the most interesting since it provides the highest power levels as well as the narrowest line width. The main drawback is the instability of the laser power and the temporal jitter. For this reason Billardon et al. [30] have developed a feedback system capable of stabilising temporal jitter. In the feedback system, a dissector measures the position of the optical pulse, relative to the positron bunch. When the optical pulse moves, the position of the bunch is adjusted accordingly through a change in the rf frequency. With the feedback the laser becomes much more stable. Power level fluctuations,

for example, drop below 5 %. Figures 14 and 15 show the change in temporal jitter, measured with a streak-camera and a dissector, respectively. In Appendix A, the diagnostic equipment is described. Table 3 gives a summary of the output parameters of the laser. Note the reduction in optical pulse length with the feedback in zone 0.

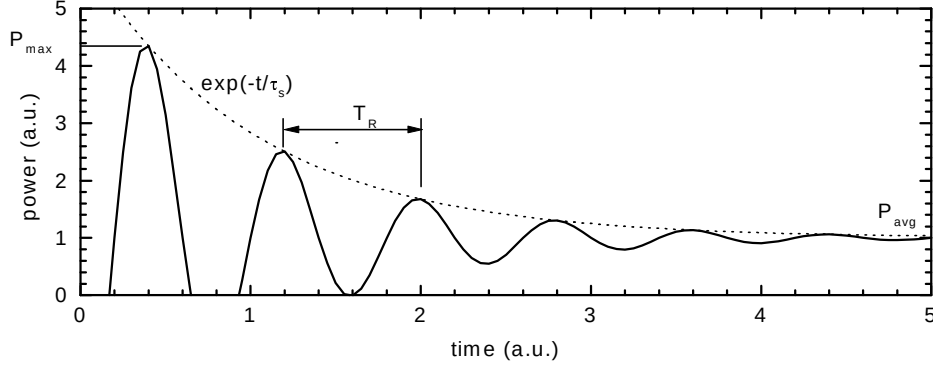


Fig. 13 Evolution of the optical power of the laser: the rise time of the laser and the damping of the synchrotron lead to a damped oscillation of the laser power as predicted by Eqs. (28) to (31).

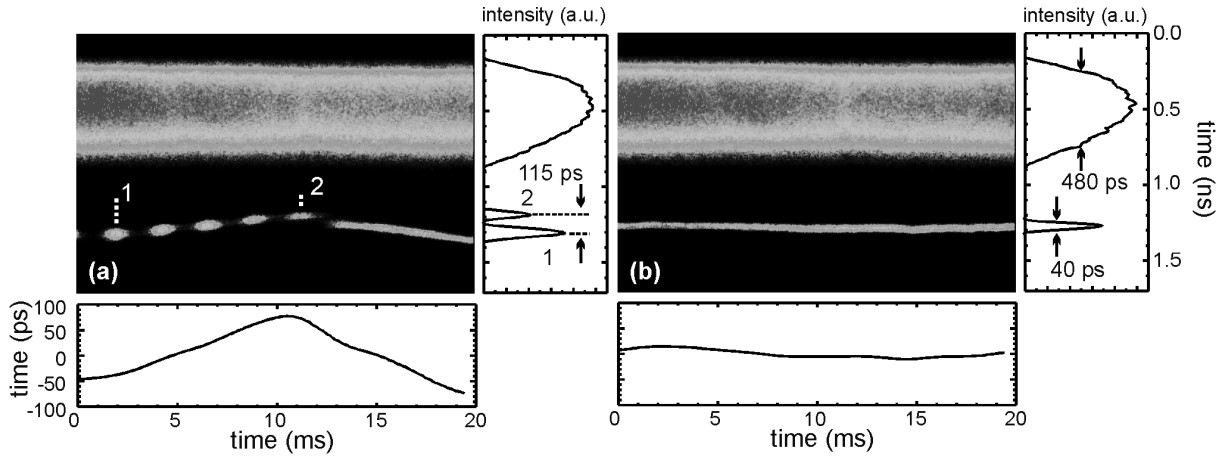


Fig. 14 Results of the longitudinal feedback system installed at Super ACO measured with a double sweep streak-camera: (a) without feedback and (b) with feedback. The fast sweep is along the vertical axis and the slow sweep along the horizontal axis. The photo's show both the positron bunch (top) and the laser pulse (bottom). Note the difference in duration between the positron bunch and the laser pulse.

Table 3

Temporal and spectral features (bandwidth and drift) of the Super ACO FEL operated at 350 nm [33]

Zone	σ_t (ps)	σ_λ/λ ($\times 10^{-4}$)	Δ_λ/λ ($\times 10^{-4}$)	Macro structure	Remarks
0	17			Cw	Feedback
0	20	0.3-1.2	< 3	Cw	No feedback: Jitter up to 200 ps
1, -1	25-35	0.6-1.5	6-9	Pulsed	Rapid wavelength shifts
2, -2	35-45	0.9-1.5	< 3	Cw	Low power

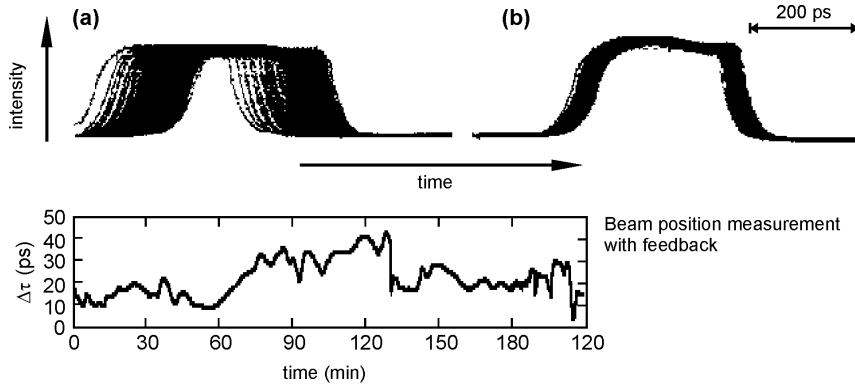


Fig. 15 Results of the longitudinal feedback system installed at Super ACO [30]. The top graph shows the jitter of the base of the optical pulse measured with a disector (in the centre of the pulse the detector is saturated) without feedback (a) and with feedback (b). The bottom graph shows the jitter of the optical pulse over two hours with feedback. Without feedback the jitter exceeds the positron bunch duration of 200 ps.

3.5 Influencing the FEL output 2: Q-switching (gain-switching)

One interesting consequence of the laser instability is the possibility of driving these oscillations externally, i.e. Q- or gain-switching. By modulating the optical gain with a period of the order of magnitude of T_R it is possible to obtain very regularly spaced and reproducible pulses, see Fig. 16. The Q-switching of SRFEL's has two great advantages over working with the natural time structure: (i) the laser is stabilised and (ii) the laser peak power is enhanced by several orders of magnitude. If T_0 is the Q-switching period a maximum is reached at $T_0 \sim \tau_0$ and the peak power enhancement is about $\tau_s/4\tau_0$. This factor is $\sim 10^3$ for the ACO experiment (and will be about the same on future experiments) although only 10^2 could be reached in practice. Remember that the higher the small signal gain g_0 , the smaller the rise time τ_0 and the bigger the peak power enhancement. However, the estimates given above do not take into account saturation by over bunching in the case of a high-power SRFEL. When the dimensionless laser field $|a|$ [35] ($|a| = 4KN_u eKL_u |E|/\gamma^2 mc^2 \approx 0.1$ on ACO, Q-switched regime, where $|E|$ is the optical field strength) becomes of the order of unity, over-bunching will occur which will modify the above results. In most cases, use of the Q-switching technique will drive the peak power up to the over-bunching limit.

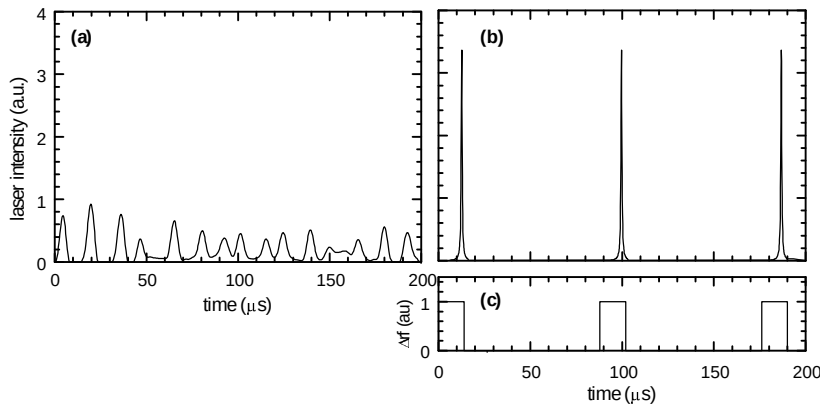


Fig. 16 Time record of the ACO laser intensity [34] for (a) “natural” operation and (b) low-frequency gain switched operation. The trace in (c) shows the switching of the gain through a

change of the longitudinal overlap between the optical pulses and the bunches, i.e. changes of the rf frequency.

4. THE RESONATOR

The most commonly used resonator in an FEL is the stable two-mirror normal-incidence resonator, see Fig. 17, which will be discussed in more detail in Sec. 4.1. Its main advantages are its straightforward design and the possibility to obtain low loss (less than 1%) resonators with the aid of multi-coated dielectric mirrors. Furthermore, two-mirror normal-incidence resonators are also common practice in other laser systems. There are also some drawbacks to the design, however. For example, in Sec. 4.1 it will be shown that the tolerances to resonator dimensions become more critical as the resonator length increases relative to the undulator length. This is specifically an important factor in the SRFEL since more space is required to include elements for the electron beam optics (e.g., quadrupoles and dipoles). Furthermore, the cavity length must be adjusted to match half of the bunch spacing in the storage ring in order to obtain overlap between the optical pulses and the electron bunches on each passage through the cavity. Other, and more fundamental, drawbacks are related to the optics required for the resonator. In Sec. 4.2 the two most prominent problems are discussed, i.e. the occurrence of mirror degradation due to the harmonics present in the spontaneous undulator radiation and the lack of available high-reflecting optics at shorter wavelengths. Because of these problems, other (resonator) schemes have been considered of which a brief overview is given in Sec. 4.3.

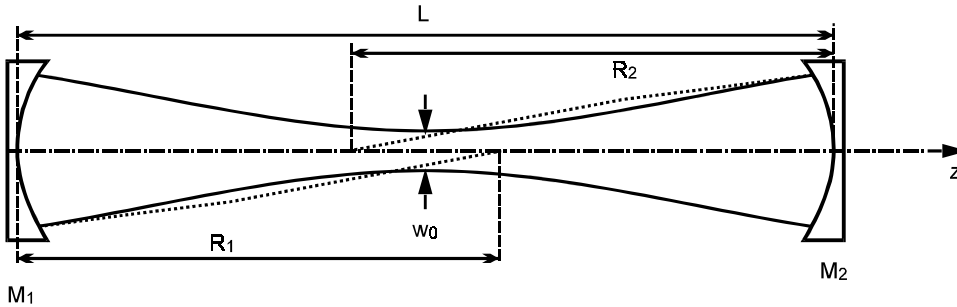


Fig. 17 Schematic diagram of a cavity defined by the mirrors M_1 , M_2 and the cavity length L . The mirrors have a radius of curvature R_1 and R_2 , respectively.

4.1 Standard resonator geometry

In the two-mirror normal-incidence resonator two curved mirrors are used to obtain a stable periodic focusing system, see Fig. 17. If the transverse dimensions of these mirrors are large enough and edge diffraction effects can be neglected it can trap the lowest or higher order Gaussian modes which bounce back and forth between the two mirrors. An overview of the properties of such systems can be found in the Refs. [36, 37, 17]. Here the most important aspects for the FEL are summarised.

The resonator of Fig 17, can be characterised by its length L and the radius of curvature of both mirrors R_1 and R_2 . As discussed in Sec. 2.1 it is important to minimise the average transverse optical mode area to optimise the small-signal gain of the FEL. For this the so-called Rayleigh length β_0 plays an important role, i.e. the distance over which the transverse mode-area of a TEM_{00} mode increases with a factor 2 relative to the minimum mode area in the waist.

$$\beta_o = \sqrt{\frac{L(R_1 - L)(R_2 - L)(R_1 + R_2 - L)}{(R_1 + R_2 - 2L)^2}} \quad (32)$$

Note that β_o plays the same role as the machine functions $\beta_{x,y}$ used to describe the electron beam optics of the ring. The minimum spot size (waist) of the E-field of a TEM₀₀ mode can be found to be:

$$\sigma_0 = \sqrt{\frac{\lambda \beta_o}{2\pi}} \quad (33)$$

with the waist position, relative to the centre of the cavity, given by

$$z_0 = \frac{1}{2} \frac{R_1 - R_2}{R_1 + R_2 - 2L} L. \quad (34)$$

Note that for a symmetric cavity ($R_1 = R_2$) the waist position is in the centre of the cavity ($z_0 = 0$). The transverse mode area at an arbitrary position is

$$\Sigma_A(z) = \pi \sigma_0^2 \left[1 + \left(\frac{z - z_0}{\beta_0} \right)^2 \right] \quad (35)$$

To optimise the gain one has to optimise the filling factor, i.e. maximise $\langle \Sigma_e / \Sigma_A \rangle_{L_u}$ where the brackets denote the average over the whole undulator. It can be deduced that the optimum Raleigh length can be approximated by $\beta_0 \approx L_u/3$ [17] leading to the result presented in Sec. 2.2.

Next it is also important that the resonator is stable, i.e. that it can store the optical field on multiple passages through the resonator. It is customary to describe the stability of a two-mirror resonator with the so-called stability parameter g [37]:

$$g_1 = \left(1 - \frac{L}{R_1} \right), \quad g_2 = \left(1 - \frac{L}{R_2} \right) \quad (36)$$

where, for a stable resonator, the following condition needs to be fulfilled:

$$0 \leq g_1 g_2 < 1. \quad (37)$$

In an FEL the cavity length is generally much longer than the undulator length since space is required to install additional components for the electron beam such as quadrupolar lenses and dipoles. This is specifically true for SRFEL's since the cavity length must also equal half of the spacing between successive electron bunches. The difference between the undulator length and the cavity length can, therefore, be considerable. For example at Super ACO the undulator has a total length of 3 m with a resonator length equal to 18 m. For the Duke FEL project an undulator of 15 m with a cavity length of 50 m is anticipated.

The requirement of $\beta_0 \approx L_u/3$ and $L \gg L_u$ generally leads to a situation where $g_1 g_2 \rightarrow 1$ and, hence, a nearly unstable cavity. The effect of this is illustrated in Fig. 18. As the resonator length increases with respect to the undulator length it becomes more difficult to obtain the optimum filling factor, firstly because the tolerances on the radius of curvature become more restricted (Fig. 18b) and secondly because the resonator comes closer to the stability limit (Fig. 18c).

When a cavity approaches the stability limit, it also becomes more sensitive to misalignment errors. The effect of misalignment is illustrated in Fig. 19. The optical axis in a two-mirror resonator is, by definition, the line passing through the centres of curvature C_1 and C_2 of the two end mirrors. The quadratic base curvatures of the two mirrors are centred on or are normal to this axis. If the cavity also contains any kind of aperture (including the apertures defined by the mirrors themselves), rotation of an end mirror will translate the optical axis relative to this aperture or, alternatively, will cause the aperture to be effectively off centre with respect to the resonator axis. The presence of an off-centre aperture will tend to produce resonator eigenmodes that are mixtures of the even and odd eigenmodes of the aligned resonator. From simple geometry considerations it can be deduced how far the optical axis will be translated and rotated by a small angular rotation of either end mirror. Let θ_1 and θ_2 be the small angular rotations of the two end mirrors and Δx_1 and Δx_2 be the small sideways translations of the optical axis at the point where it intercepts the end mirrors, as shown in Fig. 19. Alternatively, Δx_1 and Δx_2 can represent the off-centre translations of the apertures at those two mirrors. From Fig. 19 and some simple geometry we can then evaluate these displacements as [36]:

$$\begin{aligned}\Delta x_1 &= \frac{g_2}{1-g_1g_2} L \theta_1 + \frac{1}{1-g_1g_2} L \theta_2 \\ \Delta x_2 &= \frac{1}{1-g_1g_2} L \theta_1 + \frac{g_1}{1-g_1g_2} L \theta_2\end{aligned}\tag{38}$$

One criterion for judging the seriousness of misalignment effects is to compare the transverse displacement of the centre of the cavity with the optical waist size and the transverse electron beam dimensions.

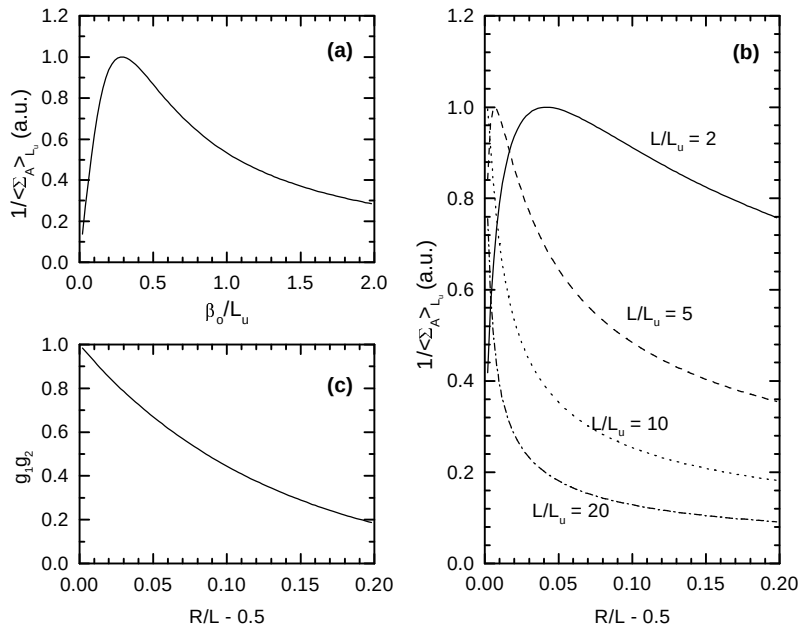


Fig. 18 Influence of the resonator geometry on the FEL gain and the resonator stability: (a) The filling factor as a function of the Raleigh length, see Eq. (32) normalised with respect to the undulator length L_u . (b) Filling factor as a function of the radius of curvature of the mirrors

for a symmetric cavity, i.e. $R = R_1 = R_2$, for different undulator lengths with respect to the cavity length L . (c) The resonator stability parameter a function of the radius of curvature. Note that $(R/L - 0.5) = 2(\beta_0/L)^2$.

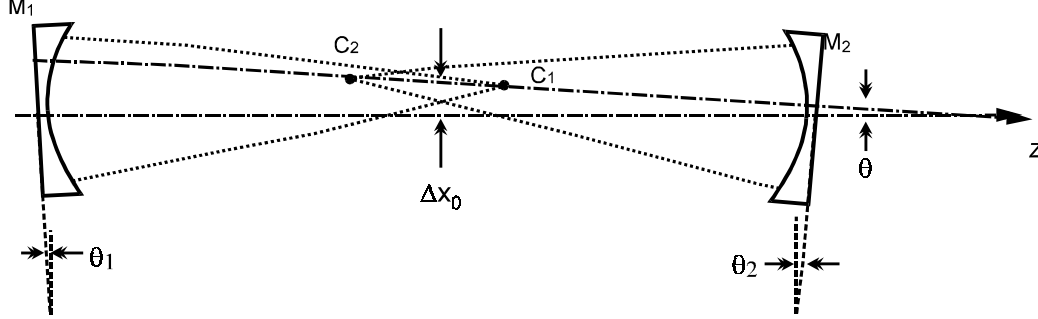


Fig. 19 Geometry of a misaligned optical cavity

The angular displacement of the resonator axis that can be important in evaluating far-field pointing accuracy, for example, can be evaluated from [36]:

$$\theta = \frac{\Delta x_2 - \Delta x_1}{L} = \frac{(1 - g_1)\theta_1 - (1 - g_2)\theta_2}{1 - g_1 g_2}. \quad (39)$$

This parameter can be specifically important for user light sources where the FEL radiation has to be transported over large distances to a user experiment. Note that the sensitivity of these angular misalignment blows up as $g_1 g_2 \rightarrow 1$, i.e. as the resonator design approaches the stability boundary. Note further that with $g_1 g_2 \rightarrow 1$ and $\sigma_0 \ll$, see Eq. (33), it is conceivable that the resonator will be more sensitive to higher-order transverse mode which impedes both the laser gain and/or the stability at saturation. At Super ACO such modes are common. The order of the mode and its stability depends on the beam current: small changes at the mirror due to temperature induce mechanical stress, see Sec. 4.2.

Finally it is important to consider the mirror dimensions since too small mirrors will cause radiation leakage along the edges of the mirrors. These losses can be evaluated with the aid of the resonator Fresnell number N_f :

$$\frac{\text{resonator mirror surface area}}{\text{confocal TEM}_{00} \text{ mode area}} = \pi N_f \quad (40)$$

To reduce cavity losses a large Fresnell number, i.e. $N_f > 10$ is preferable, see Fig. 20.

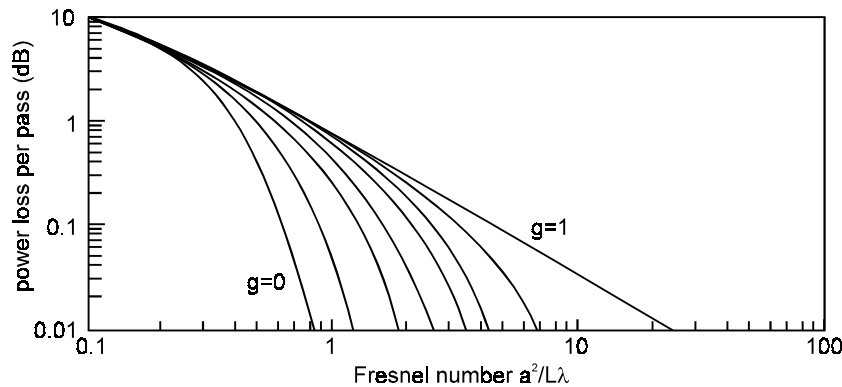


Fig. 20 Power loss per round trip (measured in dB) versus Fresnell number for a TEM_{00} mode in a two-mirror symmetric resonator, i.e. $R_1 = R_2$, with g-values ranging from $g = 0$ (confocal resonator) to $g = 1$ (planar resonator). The intermediate g values are $g = 0.5, 0.8, 0.9, 0.95, 0.97$ and 0.99 respectively [36].

4.2 Optics

SRFEL's are low gain devices, i.e. a laser where a resonator is required to reach saturation on multiple passes. The problem of making good mirrors for the visible to the VUV spectral region is a difficult one and deserves at least a separate chapter. For general information the reader is referred to textbooks and review articles [3840]. Specially for the UV and VUV spectral region it becomes more difficult to find suitable materials for mirrors. As a result the operational wavelength range of the SRFEL is reduced. An overview of the state of the art mirror technology is given in Fig. 21.

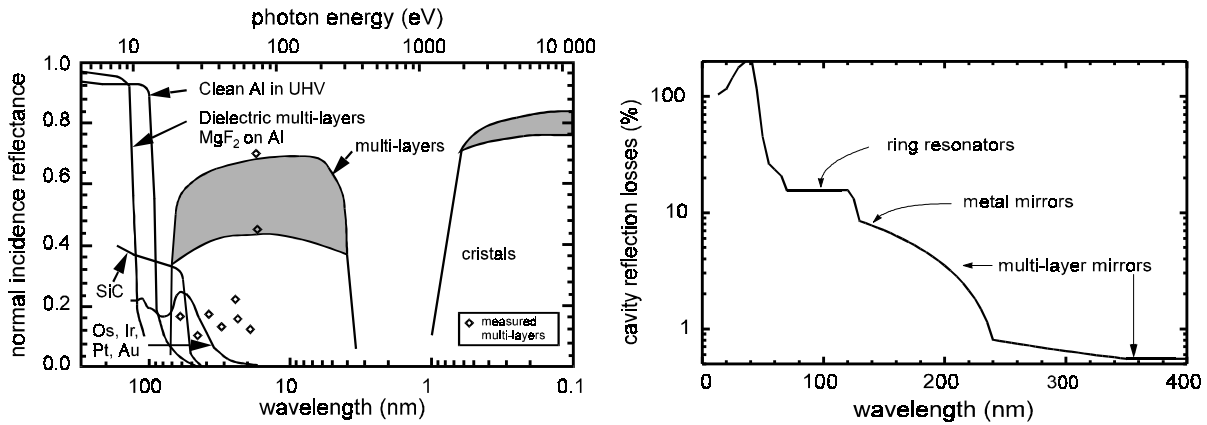


Fig. 21 Reflectivity of normal-incidence optics [40] and the estimated cavity loss due to absorption at the mirror surface. For details on ring-resonators, see Sec. 4.3.

Apart from these problems there are additional ones which arise due to the nature of the SRFEL:

- The gain of a SRFEL is, at present, up to a few percent. To obtain the lasing threshold condition, i.e. cavity losses $<$ gain, highly-reflecting mirrors are required, i.e. multi-coated mirrors. Highly-reflecting multi-coated mirrors have a narrow spectral bandwidth and, as a consequence, impede the wavelength tunability of the FEL.
- As will be explained below, mirrors of an SRFEL are exposed to a harsh environment of high power radiation ranging from the visible to the X-ray region. As a result thermal effects and mirror damage or mirror degradation forms a serious problem in the operation of the SRFEL.

To overcome the first restriction metal mirrors, e.g. coated Al mirrors [40], could be used. However, the losses associated with these types of mirrors are of the order of a few percent and all SRFEL's that are presently operational do not have sufficient gain to use such mirrors.

A multi-coated mirror consists of a stack of Fabry-Perot interferometers where each layer has an optical thickness of $\lambda/4$. Two different layers are deposited alternately, one with a low- and one with a high index of refraction. The residual diffusion and absorption in the layers limit the maximum reflectivity. For a mathematical treatment of such mirrors refer to

Ref. [41]. In the visible region reflectivity's as high as 99.95 % can be obtained. However, in this case the ratio of the transmission over the losses is very low and does not permit good energy extraction [42]. Multi-layers with a reflectivity over 99.5 % are currently manufactured down to a wavelength of 240 nm. An overview of some materials suitable for mirrors in the UV is given in Table 4.

Table 4

Optical constants of oxide layers deposited by reactive sputtering [43]

λ (nm)	SiO ₂		Al ₂ O ₃		[Hf _x Si _{1-x}] ₂ O ₇		HfO ₂		ZrO ₂		Ta ₂ O ₅		TiO ₂	
	n	k $\times 10^{-4}$	n	k $\times 10^{-4}$	n	k $\times 10^{-4}$	n	k $\times 10^{-4}$	N	k^d $\times 10^{-4}$	n	k $\times 10^{-4}$	n	k $\times 10^{-4}$
200	1.50	0.00	1.83	29.1	2.13	78.0	2.51	152	—	—	—	—	—	—
300	1.47	0.00	1.70	11.9	1.99	9.8	2.12	33.7	2.32	37.0	2.38	45.6	—	—
400	1.46	0.00	1.65	3.7	1.90	4.4	2.05	12.3	2.21	23.6	2.26	5.3	2.7	21.6

Apart from a good choice of non-absorbing materials suited for use as layers for the multi-coatings, the intrinsic losses of the mirrors are also determined by the surface roughness of the mirror, which determines the diffusion at the mirror surface. Especially at shorter wavelengths this aspect becomes more important [45]. An example of a multi-coated mirror with a central wavelength of $\lambda = 300$ nm is given in Fig. 22. From the plot it follows that the mirrors have an intrinsic loss of 0.4%. However, as the surface roughness goes up the losses increase. Above a rms average roughness of 0.2 nm the scattering losses dominate the absorption losses of the mirror.

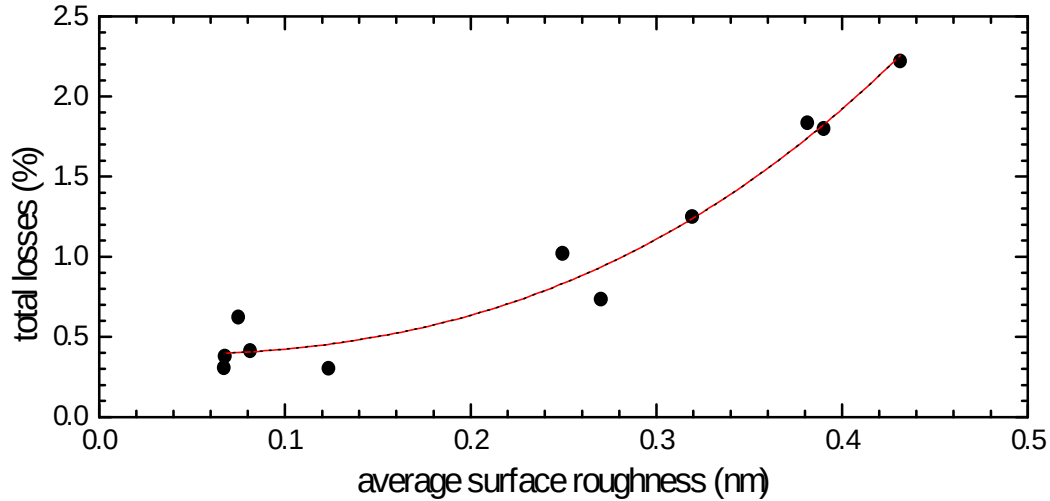


Fig. 22 Total losses as a function of the average rms surface roughness of a mirror at $\lambda = 300$ nm. The example given involves a multi-layer mirror with HfO₂/SiO₂ layers [44].

In an SRFEL it is not sufficient to produce mirrors which have an initially high reflectivity coefficient. While operating the laser the resonator mirrors can be exposed to a very intense flux of spontaneous radiation emitted by the undulator. For this it is important to remember that the typical beam energy of a storage ring will be of the order of (at least) a few hundred MeV. Hence, the total emitted power from an undulator is given by [26]:

$$P_u (\text{W}) = 7.28 E^2 (\text{GeV}) NI (\text{A}) \frac{K^2}{\lambda_u (\text{cm})} \quad (41)$$

where the same notation as in Sec. 2 has been used.

From Eq. (41) it follows that the spontaneous radiation can easily go up to a level of several (tens of) Watts. All this radiation will be deposited on the resonator mirror at the down-stream side of the undulator and thermal effects can occur. Note that due to the low gain of the SRFEL it is not possible to install the resonator mirrors outside the vacuum system. Hence, mirrors can only be cooled by either the mirror support or by thermal radiation. Since the radiation stored in the cavity is extracted through transmission of the mirrors it is important that both the mirror support and the mirror substrate have a good thermal conductivity. At Super ACO, for example, 18 W of spontaneous radiation hits the downstream mirror. As a result isolated mirrors on a SiO_2 substrate were heated with $\Delta T = 80^\circ\text{C}$ relative to room temperature. Under the same conditions mirrors on an Al_2O_3 substrate were heated by $\Delta T = 35^\circ\text{C}$. With a mirror support with a good thermal contact the rise in temperature could be reduced to $\Delta T = 58^\circ\text{C}$ and $\Delta T = 5^\circ\text{C}$, respectively [46].

Next, the spectrum of the radiation plays an important role as well. The spontaneous radiation of a planar undulator does not only contain the fundamental harmonic such as given in Eq. (1) but also a large number of harmonics. For on-axis radiation ($\theta = 0$) the harmonics per opening angle Ω are given by [26]:

$$\left. \frac{d^2 I}{d\omega d\Omega} \right|_{\theta=0} = \frac{1}{4\pi\epsilon_0} \frac{4(e\gamma N)^2}{c} \sum_{n=1,3,\dots}^{\infty} F_n(K) \left(\frac{\sin \nu_n}{\nu_n} \right)^2 \quad (42)$$

$$F_n(K) = 4\xi_n \left(J_{\frac{n+1}{2}}(\xi_n) - J_{\frac{n-1}{2}}(\xi_n) \right)$$

where n denotes the harmonic number. The parameters ξ_n and ν_n are given by:

$$\xi_n = \frac{1}{21 + K^2} n K^2 \quad (43)$$

$$\nu_n = n\pi N \frac{\lambda_n - \lambda}{\lambda_n}$$

A graph of Eq. (42) is presented in Fig. 23. Especially for SRFEL's the undulator strength K can be high because of the high energy of the beam stored. As a consequence the power of the harmonics radiated by the undulator can be high as well. For this it is convenient to recall the critical wavelength λ_c or critical harmonic n_c , i.e. the wavelength for which half of the power is radiated at wavelengths shorter than this value.

$$\lambda_c (\text{nm}) = \frac{1.86}{B_u (\text{T}) E^2 (\text{GeV})} \quad (44)$$

$$n_c = \frac{3}{2\sqrt{2}} K(1 + K^2)$$

In Table 5 it can be seen that for typical SRFEL experiments the critical wavelength is much shorter than the fundamental wavelength. Generally the mirror absorbs these harmonics and mirror damage can occur.

An example of such mirror degradation is shown in Fig. 24. At present mirror degradation can not be avoided. However, it is possible to produce multi-coated mirrors that are more resistant. This topic is still very much under development. One of the aspects that play an important role is the fabrication process involved. Mirrors fabricated with an ion-beam sputtering method are more resistant to synchrotron radiation, see Fig. 25. The vacuum level, i.e. the number of residual gas molecules near the mirror surfaces also influences mirror degradation. Ultra-high vacuum is required in order to reduce the absorption of residual molecules such as carbon which reduces the overall reflectivity of the mirrors. However, loss of reflectivity due to carbon contamination can be recovered by exposing the mirror surface to RF-discharged oxygen plasma [46-48].

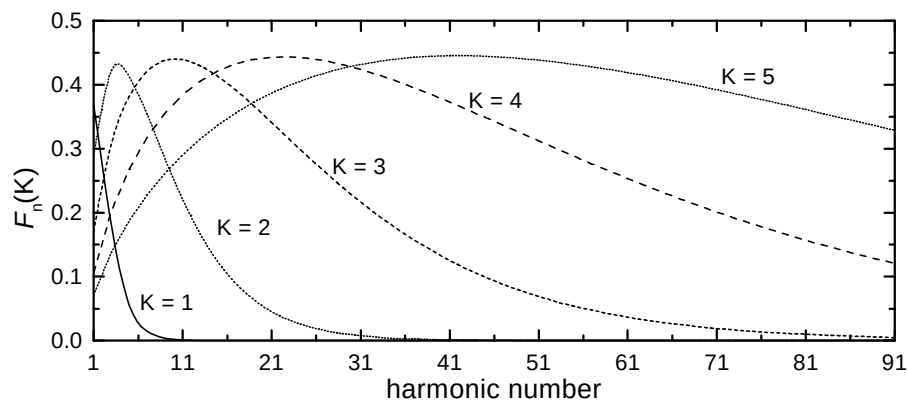


Fig. 23 Contents of the on-axis harmonics of a planar undulator for several values of the undulator strength, see Eq. (42)

Table 5

Undulator radiation for various SRFEL experiments. Estimated values are given in italic [46]

Experiment	Beam char.		Undulator characteristics			Radiation characteristics			
	E (MeV)	I (mA)	λ_u (cm)	N	K	P_u (W)	λ_c (nm)	λ_{FEL} (nm)	P_{FEL} (mW)
Super ACO	800	100	12.9	20	5.0	18	7.1	350	100
UVSOR	500	20	11.1	16	2.9	0.4	26.6	300	0.2
VEPP3	350	40	10.0	67	1.6	0.6	89.3	250	2.5
DELTA	<i>700</i>	<i>100</i>	<i>25.0</i>	<i>17</i>	<i>2.1</i>	<i>1</i>	<i>54.2</i>	<i>213</i>	—
Duke	<i>1000</i>	<i>100</i>	<i>10.0</i>	<i>67</i>	<i>6.6</i>	<i>210</i>	<i>2.6</i>	<i>300</i>	—

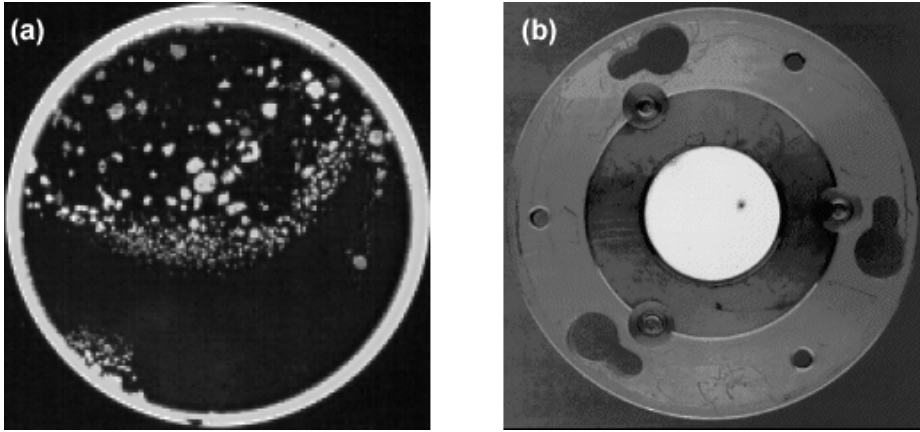


Fig. 24 Examples of mirror degradation due to the spontaneous undulator radiation. (a) Severe degradation of a test mirror deposited on a SiO_2 substrate. (b) Photo of a mirror in its support after 60 hours of FEL operation showing the presence of a large carbon spot [26].

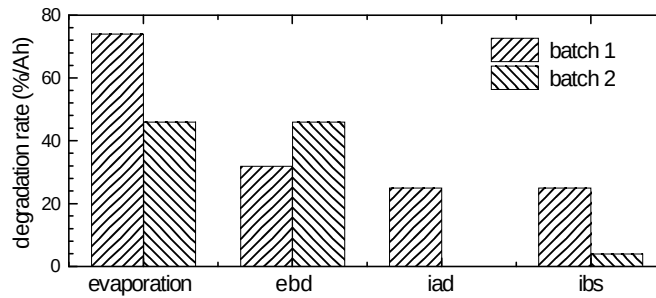


Fig. 25 Summary of the degradation rates applying to different deposition techniques: classical evaporation, electron beam deposition (ebd), ion-assisted deposition (iad) and ion beam sputtering (ibs) [46].

4.3 Alternative solutions

The problems related to the optics remain a difficult one. Nevertheless several solutions have been considered and/or implemented in FEL experiments. Some of these ideas are listed below.

Helical undulators

Helical undulators have the advantage of having a higher gain since the periodic dephasing of the electrons with respect to the optical wave is not present. Next, the use of a helical undulator or optical klystron can significantly reduce the amount of harmonics emitted. The main disadvantages of the helical undulator or optical klystron are a more complex design and a reduced flexibility to change the undulator strength K . At present only the UVSOR SRFEL experiment employs such a device as an OK [49] and lasing has been obtained down to a wavelength of 239 nm. A calculation on the expected transverse spectral distribution of this device is shown in Fig. 26. Note the different distribution of the high-energy photons in the form of a ring. It is thus possible to filter out this radiation with the aid of an iris, thus reducing the mirror degradation on the mirror surface due to the interaction of these photons with residual gas molecules near the mirror surface.

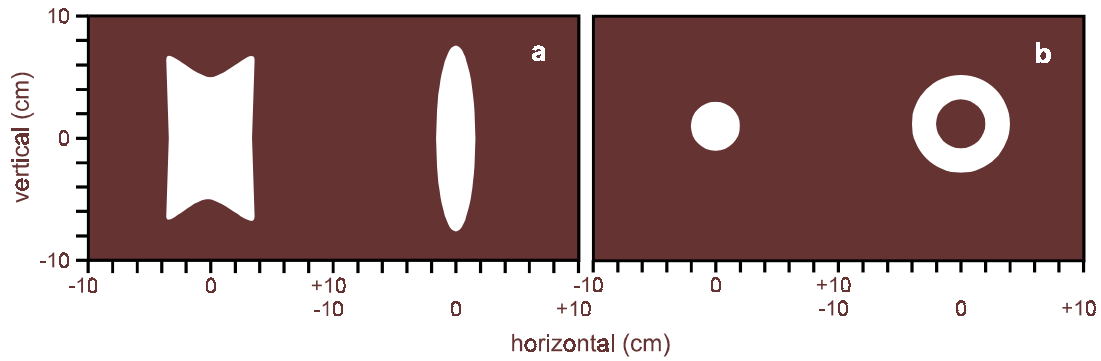


Fig. 26 Calculated spatial distribution of radiation from a planar OK (a) and a helical OK (b) such as in use at the UVSOR SRFEL experiment [49]. The left-hand sides of both figures show the integrated intensity of the low energy photons (< 10 eV). The right-hand side depicts the high-energy photons (10-200 eV).

Alternative resonator design

An alternative resonator design can also enhance the performance of the SRFEL. Two options are depicted in Figs. 27 and Fig. 28. In Fig. 27 two corner-cube resonators replace the normal-incidence end mirrors of the resonator. This design has been proposed for the CEBAF linac driven FEL [50]. The main advantage of this design is the increased stability. In the case where the cavity is filled with multiple optical pulses it also has the advantage that the return path of the light is separated from the electron beam. Hence, Compton back scattering with electrons passing through the undulator while the optical pulse returns can not take place. Obvious disadvantage is the increased resonator losses since the number of reflections are increased from two to six. At present no FEL system in the visible or UV was capable of using such a design due to lack of gain.

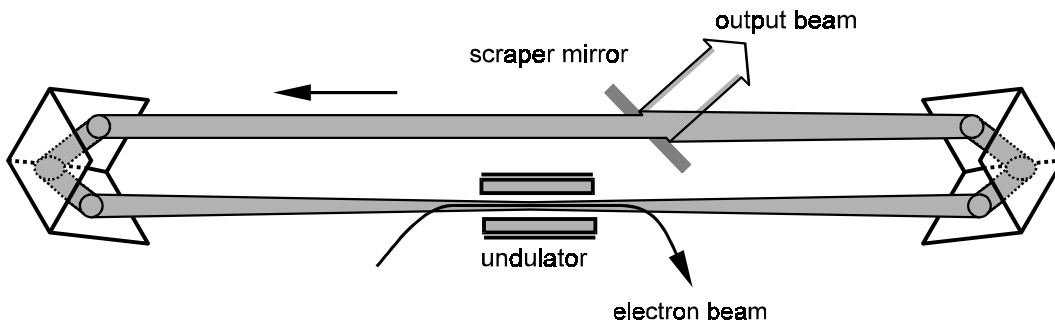


Fig. 27 Self-imaging confocal unstable ring FEL resonator with corner-cube reflectors [50]

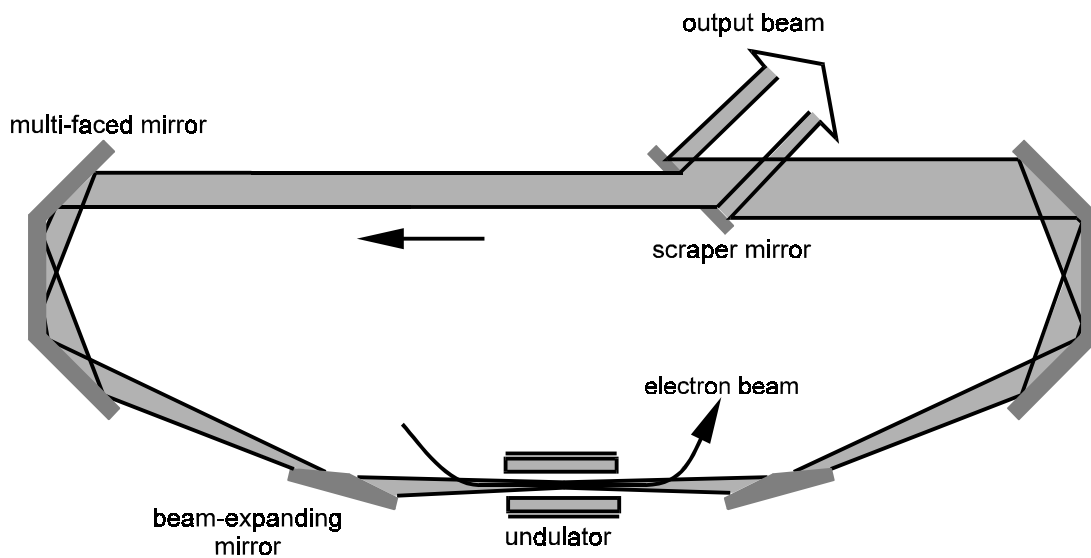


Fig. 28 Grazing incidence or ring resonator [51]

The most commonly considered alternative is the so-called whispering-mode or ring resonator, Fig. 28. In laser technology ring resonators were understood and demonstrated very early on, and have since been extensively developed for applications in ring-laser gyroscopes [36]. For free-electron lasers this type of resonator is not much used, however. Until now only one FEL experiment has made use of such a resonator at a wavelength of 640 nm [51]. Nevertheless, ring resonators have several advantages [36, 52]:

- Increased cavity design flexibility, alignment insensitivity, and insensitivity for higher order transverse modes.
- Higher level of power extraction.
- Possible operations with grazing incidence reflectance at wavelengths where no optical material for normal-incidence mirrors are available.
- Separation of the return path of the light-pulse. This can be specifically advantageous for cavities with a multiple of optical pulses where Compton back-scattering of the light pulses with the electron beam needs to be avoided.

The main disadvantage is the complexity of the design, especially the initial alignment that is required to stack the optical pulse in the resonator.

Harmonics generation

In Fig. 23 it was seen that the (planar) undulator of the FEL is an excellent harmonics generator. Furthermore, as the electrons pass through the undulator they are bunched on the scale of the fundamental wavelength. This also causes an increase in the component of higher order modes in the longitudinal density distribution of the electron beam. As explained above the optics cause a serious problem for lasing at shorter wavelengths. As an alternative it is, therefore, worthwhile considering the options for extracting harmonics out of the FEL, either through seeding with another laser or by FEL operation at a longer fundamental wavelength.

One of the options would be to use an external laser to produce a substantial up-conversion in the FEL interaction region. In the first demonstration [53, 54] of the technique, a YAG laser bunched the electrons either on the infrared or the green line, and emission was observed on the harmonics of the bunching frequency. Although the efficiency was low (10^5 to 10^7 photons per pulse were obtained), better results are expected in future experiments.

As an alternative to the seed laser it would be better to employ the FEL itself. If the FEL intensity builds up to saturation, the laser generates its own harmonics. Reasonable conversion

efficiencies have now been reported from the infrared FEL oscillator experiments. This approach has the advantage of tunability since an external laser is not required. It will also be possible to reach the highest frequencies this way since the FEL itself is anticipated to be the shortest-wavelength laser available. No such coherent harmonic radiation was observed at ACO since the laser saturated due to bunch lengthening long before saturation due to over-bunching could be reached. However, if sufficient gain is available, SRFEL's can reach high peak power in the Q-switched mode and will therefore produce strong harmonic emission. Possible solutions for extracting the harmonics are sketched in Fig. 29 where in (a) an intra-cavity ethalon is used to extract the harmonics out of the cavity. Note that this option only works if there is sufficient gain as the ethalon induces extra cavity losses. For example at Super ACO lasing in this mode of operation has not yet been demonstrated due to too high cavity losses as compared with the maximum obtainable small-signal gain. In Fig. 29 an additional undulator with a 3- or 5-times shorter period has been drawn. In this situation the harmonic compound of the electron beam, generated in the FEL interaction, is amplified in the second undulator. Note that for this option careful tuning of the saturated power level of the fundamental wavelength is required in order to avoid over-bunching before the electrons enter the second undulator: $|a| = 4KN_u eKL_u |E|/\gamma^2 mc^2 < 1$ [35].

5. DESIGN OF THE STORAGE RING

From Eq. (10) and Eq. (17) it follows that the gain of an FEL scales with N^2 in the case of an undulator, and with N in the case of an optical klystron. Hence, a high gain can only be obtained in long straight sections where sufficient space is available for the undulator. To avoid energy dependent behaviour (the FEL extracts energy from the beam and increases the energy spread) it is also convenient if this section is dispersion free. From Sec. 2 it follows that an optimum gain can only be obtained with a high peak current in combination with a low energy spread and emittance, i.e. with an as high as possible density in phase space. For a detailed discussion on the optimisation of a storage ring with respect to such parameters the reader is referred to the other chapters of these proceedings. Here the parameters specifically important for the FEL are discussed. In Sec. 5.1 and 5.2 an overview of some aspects to obtain a high peak current and a low energy spread and emittance is given. The optimum performance of the SRFEL also requires a maximum transverse overlap between the optical pulse and the electron beam. This optimisation process is presented in Sec. 5.3.

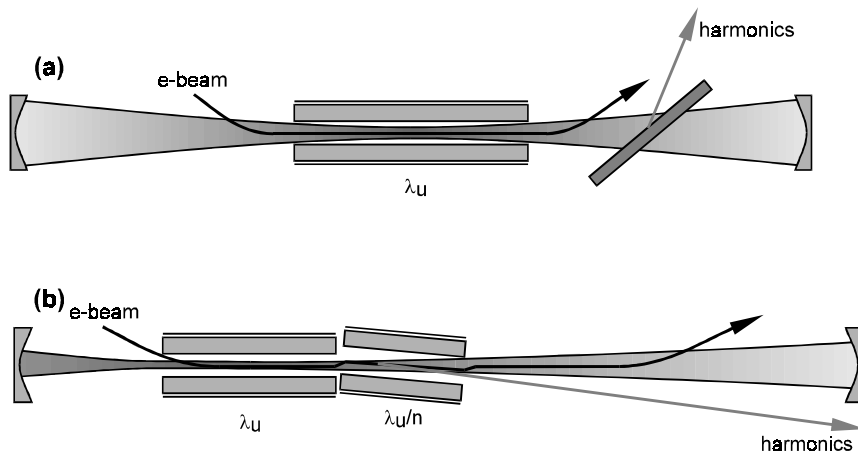


Fig. 29 Possible options for harmonics generation using the a free-electron laser that lases on the fundamental wavelength as a seed: (a) reflecting the present harmonics through an intra-cavity ethalon, (b) enhancing the output of an undulator tuned to a wavelength of λ/n by using

the electron beam that has passed through the laser as a seed. Note that the undulator has been set at a small angle so that the harmonics are not intercepted by the resonator mirrors.

5.1 Ring geometry

One of the easiest ways to increase the peak current, while keeping the average current low, is to reduce the number of bunches in the ring. Maximum current can be obtained in a single-bunch mode of operation. For this it is important to keep in mind that the resonator length must match half the distance between successive bunches. For large storage rings this implies very long cavities. Assuming a standard two-mirror resonator, see Sec. 4.1, it is difficult to keep such a cavity stable. For this reason the optimum number of bunches depends on the shape of the ring. In Fig. 30 two examples are given. As the ring is more circular, the straight sections, and hence the undulator length, will be relatively short. A multi-bunch mode is then advantageous since it reduces the intra-bunch distance to reduce the cavity length. The combination of the short cavity with a single-bunch mode is not advantageous since the optical pulse then has to pass the cavity twice before encountering interaction with the electron bunch. Hence, the effective cavity loss is doubled.

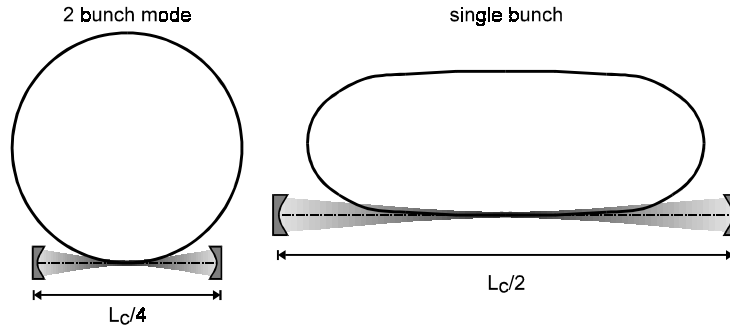


Fig. 30 Ring geometries

For the SRFEL it is advantageous to shorten the circumference of the ring while maximising the undulator length. The racetrack design, depicted in the left-hand side of Fig. 30 is, therefore, advantageous. Dedicated rings such as the one at Duke in Durham, North Carolina and DELTA in Dortmund both have this design and are designed to run in a single-bunch mode.

5.2 Beam current and electron beam quality

The most straightforward method to increase the peak current is to simply increase the average current in the ring. However, as the average beam current goes up some undesired effects need to be considered. Two aspects are of special importance: the lifetime of the beam and current-induced instabilities.

The Touschek effect

For electron beams with a high volume density the Touschek effect plays a significant role. This is the process whereby, due to Coulomb scattering within a single beam bunch the transverse-to-longitudinal momentum exchange can lead to loss of longitudinal stability, if the (energy) variation exceeds the magnitude of the energy acceptance of the longitudinal phase-stable domain [55]. The Touschek lifetime is proportional to:

$$\tau_{\text{tou}} \propto \gamma^3 \frac{I_p}{\sigma_x \sigma_y} \quad (45)$$

A long beam lifetime is important because of the long-time (thermal) effects that take place. For example thermal effects on the mirrors of the FEL, see Sec. 4. To maximise lifetime several options are available:

- An optimum choice of the emittance: Modern storage rings can have a very small emittance, down to a few nm in the horizontal plane. Depending on the spectral domain of interest this can be lower than that required for proper FEL operation, see Eq. (8) and Eq. (18).
- A full coupling of the horizontal and vertical emittance of the ring, e.g. by installing a skewed quadrupole. The coupling results in a larger phase-space and, hence, a longer Touschek lifetime. The FEL gain will, most probably, not change too much. Note that this is not a general rule, however. For specific cases it is necessary to consider the impact on the FEL gain as seen in Eq. (8) and Eq. (18).
- An increase in the longitudinal phase-stable domain of the ring, i.e. an increase of the rf-power employed. This is also advantageous since the saturation process of the FEL can increase the energy spread in the ring.
- An increase of the beam energy. The Touschek lifetime increases with increasing beam energy. An additional advantage is the increases saturated power level of the laser, see Eqs. (23) and (24). Due to this the radiation flux on the mirrors increases as well, however, firstly because of the increased synchrotron power, Eq. (24) and secondly because of the increase in harmonics: as the beam energy increases the undulator strength must increase as well to keep the radiated wavelength constant. The higher magnetic field strength causes a significant increase in harmonics, Fig. 38.

Current-induced effects

As the beam circulates in the ring the electromagnetic field of a bunch is coupled with the impedance of the environment (discontinuity of the vacuum chamber, rf cavity, etc.) to the same bunch [55, 56]. This effect becomes more important as the intensity, i.e. the average beam current in the ring, increases. The stationary solution gives the so-called “potential-well” distortion at rather low current. At higher currents one reaches the threshold of anomalous bunch lengthening. At Super ACO, for example, this threshold is found to be at approximately 8 mA/bunch [57]. Above this limit the so-called “microwave instability regime” is entered. In addition, the bunch or bunches can also start to oscillate coherently close to their natural oscillation synchrotron frequency and its harmonics. The presence of modes of oscillations (dipolar, quadrupolar, hexapolar and octupolar) depends on the beam current. In the upper limit of the current the beam becomes completely unstable. At Super ACO this limit is reached at roughly 100 mA/bunch [33].

For the FEL it is clear that the laser would work best in the absence of any type of instability. However, as a general rule the FEL operates in a regime where the instabilities are present. For example at Super ACO the under-threshold for lasing is reached at roughly 15 mA/bunch, far above the threshold for anomalous bunch lengthening. Here it is important to note that not all instabilities are of the same importance.

- (a) *Coherent bunch-oscillations.* In Fig. 31 the coherent synchrotron modes of oscillation, as observed at Super ACO, are shown. As can be deduced from the left-hand side of the figure these types of modes are devastating for the FEL: from the plot of the line-intensity it follows that the position of the bunch starts to oscillate. Hence, the effective small-signal gain is reduced since, over multiple roundtrips, the effective longitudinal overlap of the bunch with the optical pulse is reduced. Next it can be seen that also the effective energy-spread increases, causing a further decrease

of the small-signal gain. At Super ACO the dipolar modes are damped with a Pedersen type of feedback system [58, 59]. With this system it is possible to obtain stable operation up to a current of 35 mA/bunch. Above this threshold quadrupolar modes occur that make the start-up of the laser more difficult. Moreover, with quadrupolar modes present the laser becomes less stable. However, once lasing is achieved, the FEL interaction tends to damp these modes [33]. Recently, feedback on the quadrupolar modes has been installed providing the means to have stable FEL operation at up to 50 mA/bunch [60].

- (b) *Microwave instability.* At higher circulating beam current in a storage ring significant dilution of longitudinal phase-space density may occur. Here the interaction of the bunch with its own short-range wake field causes high-order coupled oscillatory modes to be induced. This results in short-wavelength (compared with the bunch length) particle density modulation of the bunch, effective lengthening of the bunch, and enhancement of the energy spread. The threshold peak current for the longitudinal single-bunch instability is given by [23]:

$$I_p = 2\pi\alpha \left(\frac{\sigma_\gamma}{\gamma} \right)^2 \frac{(E_0/e)}{(Z_{\parallel}/n)} \quad (46)$$

where E_0 and $Z_{\parallel}$ are the beam energy and the longitudinal impedance of the ring, respectively. Above the threshold the peak current may still increase but at the expense of an increase in energy spread [23]:

$$\begin{aligned} \frac{\sigma_\gamma}{\gamma} &= \sqrt[3]{\frac{1}{2\pi} \frac{\nu_s I_0 (Z_{\parallel}/n)}{\alpha^2 (E_0/e)^2 h}} \\ \nu_s &\approx \sqrt{\frac{V_{rf} h \alpha}{2\pi E}} \end{aligned} \quad (47)$$

where I_0 is the average beam current, V_{rf} is the rf voltage, and h is the number of bunches stored in the ring.

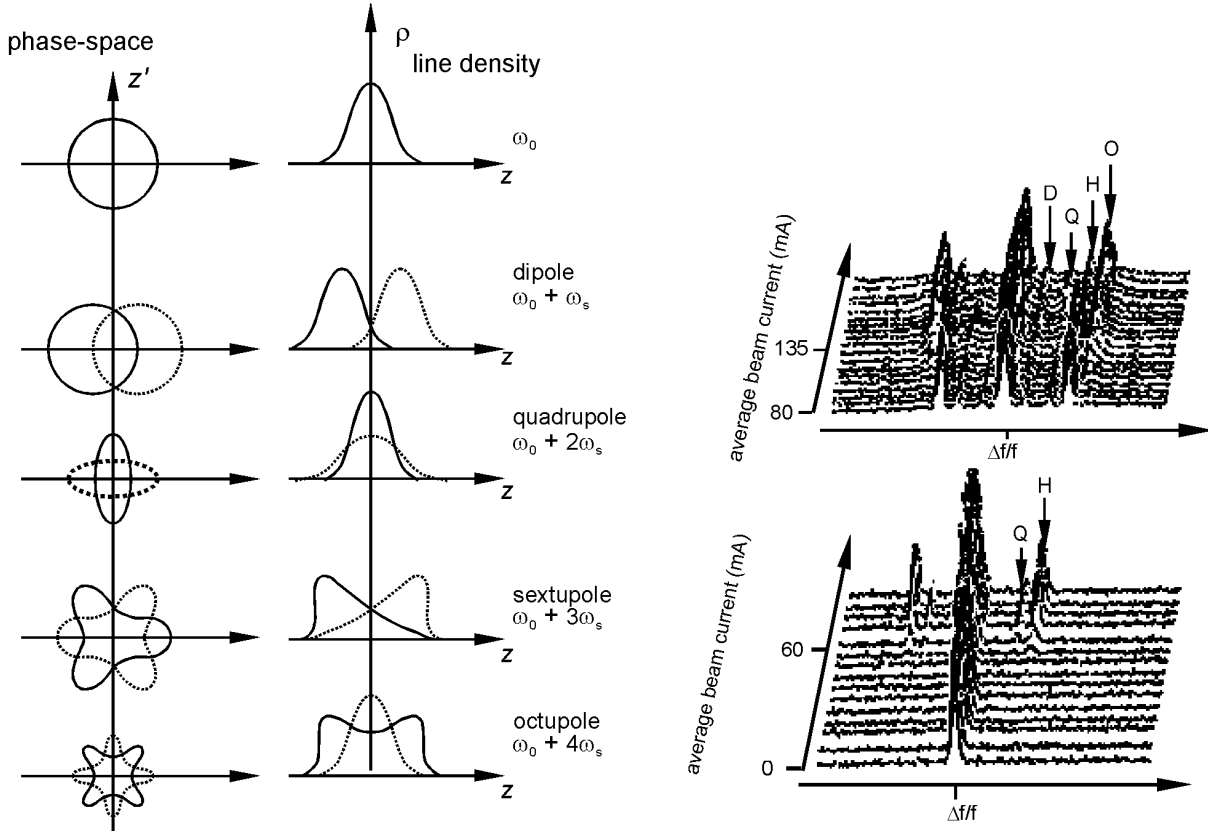


Fig. 31 Synchrotron modes of oscillations observed at Super ACO in a two-bunch mode of operation [33]. Left-hand side: Representation of the coherent synchrotron modes of oscillation in phase space and in time space. Right-hand side: Spectrum of the modes observed on the sidebands of high harmonics of the revolution frequency versus the stored beam current. With increasing current one can observe the presence of dipolar modes (D), quadrupolar modes (Q), hexapolar modes (H), and octupolar modes (O).

To first order the gain of the laser scales with the square of the energy spread. Hence, even at the occurrence of microwave instability, it remains worthwhile to increase the beam current. Nevertheless, it is also important to design the storage ring such that instabilities are avoided to an as high as possible current. To do so it is important to reduce the number of bunches in the ring, see Sec. 5.1. Increasing the rf voltage also helps but is not very effective, see Eq. (47). A high rf voltage remains important, however, since it also determines the energy-spread acceptance of the ring, see Sec. 3.

In the optimisation process the momentum compaction factor α plays an important role: as the compaction factor becomes smaller the storage ring becomes more sensitive to instabilities. Hence, a too small value for α is to be avoided. However, big compaction factors are non-ideal either. From Eq. (3.2) it follows that a small value for α leads to short bunches and, hence, to high peak currents and gain. Furthermore, there is a (indirect) relation between the compaction factor and the emittance since both are related to the so-called “synchrotron radiation integrals” [23]:

$$\alpha = \frac{I_1}{R}$$

$$\varepsilon_x = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \gamma^2 \frac{I_5}{I_2 - I_4} \quad (48)$$

where the synchrotron integrals are given by:

$$\begin{aligned} I_1 &= \int_{\text{dipoles}} \frac{\eta}{\rho} ds, & I_2 &= \int_{\text{dipoles}} \frac{1}{\rho^2} ds, & I_3 &= \int_{\text{dipoles}} \frac{1}{|\rho^3|} ds \\ I_4 &= \int_{\text{dipoles}} \frac{(1-2n_d)\eta}{\rho^3} ds, & I_5 &= \int_{\text{dipoles}} \frac{H}{|\rho^3|} ds \end{aligned} \quad (49)$$

with:

$$\begin{aligned} H &= \gamma_x \eta_x^2 + 2\alpha_x \eta_x \eta'_x + \beta_x \eta'^2_x && \text{dispersive invariant} \\ n_d &= \frac{r}{B} \frac{\partial B}{\partial r} && \text{dipole magnetic field index} \end{aligned} \quad (50)$$

and $\eta = \eta(s)$ the dispersion function. The parameters $\alpha_x(s)$, $\beta_x(s)$, $\gamma_x(s)$ are the machine Twiss coefficients. Note that the differentiation is with respect to s , the lattice azimuthal coordinate, and integration is over the extent of the dipole regions only.

From Eqs. (48) and (49) it follows that a careful optimisation of the dispersion is necessary: small dispersions lead to a small emittance and, through the compaction factor, to short bunches and a high peak-current. However, a too small compaction factor causes more instability. In modern machines it is feasible to reduce the emittance to only a few nm. Normally this goes in parallel with a decrease in α . For the FEL it might be advantageous to tune the compaction factor independent of the emittance, e.g. by adding additional dispersive elements in a straight section of the ring. It might thus be possible to operate the ring at any (positive or negative) value of the compaction factor. At present only A few storage rings have the possibility to change α and more studies are required to find an optimum. A study at Super ACO [57] indicates that a negative compaction factor might be advantageous: running with negative α resulted in shorter bunches but higher energy spread. With more tuning it might thus be possible to find optimum conditions for FEL operation in a storage ring.

The most promising option would be to run a ring at $\alpha = 0$. This situation is known as the isochronous SRFEL [61]. In normal SRFEL operation the micro bunching on the scale of the wavelength, a process that occurs naturally as the electrons pass through the undulator, is lost as the electrons make their turn through the ring. If the compaction factor would be sufficiently small, however, this bunching could be maintained on successive turns through the ring. The advantages are obvious since an already bunched beam would start radiating coherently directly when entering the undulator. Hence, the gain would be increased by orders of magnitude. At present no storage ring has demonstrated this, however. The main reason is the small compaction factors required. For example, lasing at 500 nm in a small ring with a circumference of 50 m would already require $|\alpha| < 10^{-6}$. It might be possible to demonstrate the effect in the (far) infrared in a small, low-energy ring, however.

5.3 Machine functions

In Sec. 4.1 the average transverse optical mode area was minimised for the case of a standard two-mirror resonator. Here this process is repeated for the electron beam. Since the transverse mode area of the electron beam is generally less than the optical mode area (from Eq. (8) it follows that $\varepsilon_x \varepsilon_y \ll \lambda^2$) this process is less important. Nevertheless this optimisation is a useful exercise. An example of such an optimisation is shown in Fig. 32. In the figure the

gain of the Super ACO FEL is plotted as a function of the horizontal Twiss parameter β_x . The results are based on the model presented in Appendix B.

From the figure it follows that an optimum gain is obtained for a β_x in the range from 4 to 7 m, depending on the resonator geometry, i.e. the Raleigh length β_0 . Most important for a proper optimisation is the matching of the curvature of the phase-front of both the optical wave and of the electron beam. Hence, in the case of a shorter Raleigh-length the Twiss parameters β_x and β_y need to be reduced as well. The figure also shows that the gain is fairly insensitive for moderate increases of the transverse beam size. For Super ACO ($\beta_0 \approx 3$ m) an optimum gain of 2 % is reached for $\beta_x \approx 5$ m. For reasons concerning the stability of the optical resonator, see Sec. 4.1, this is somewhat lower than the maximally achievable gain when $\beta_0 = 1.5$ m and $\beta_x = \beta_y = 4$ m.

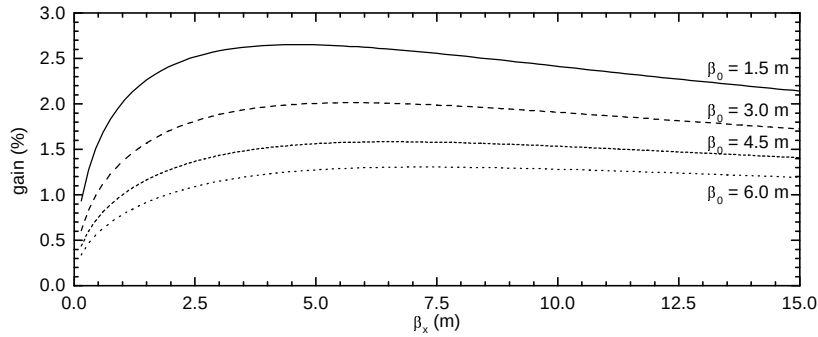


Fig. 32 The gain of Super ACO as a function of the machine Twiss parameter β_x in the centre of the undulator for several values of the Raleigh-length of the resonator. For Super ACO ($\beta_0 \approx 3$ m) the optimum gain is reached for $\beta_x \approx 5$ m.

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APPENDIX A

THE SUPER ACO FEL

In this chapter the free-electron laser installed in the synchrotron light source Super ACO [7, 62] takes a special place. Together with the ACO FEL this facility has the longest history of operation. Moreover, the author of this chapter is most familiar with this SRFEL. Consequently a major part of the operational examples presented here originate from these facilities.

An overview of the machine is given in Fig. A.1 and its parameters are given in Table A.1. A full energy positron linac injects the particles into an octagon-shaped ring with a circumference of 72 m. For FEL operation the ring is filled with two identical, equidistant bunches with a maximum current of 45 mA/bunch, i.e. up to 20 A of peak current. A 2-times-10-period optical klystron in a dispersive free section of the ring forms the centre of the laser. The OK is enclosed in an 18-m long optical resonator. Light is coupled out through the multi-coated mirrors at both sides of the undulator. The south-mirror can move along the axis of the resonator to synchronize the length of the cavity to half the bunch distance. Tuning the frequency of the 100-MHz rf cavity achieves fine tuning of the bunch distance, see Sec. 3.3.

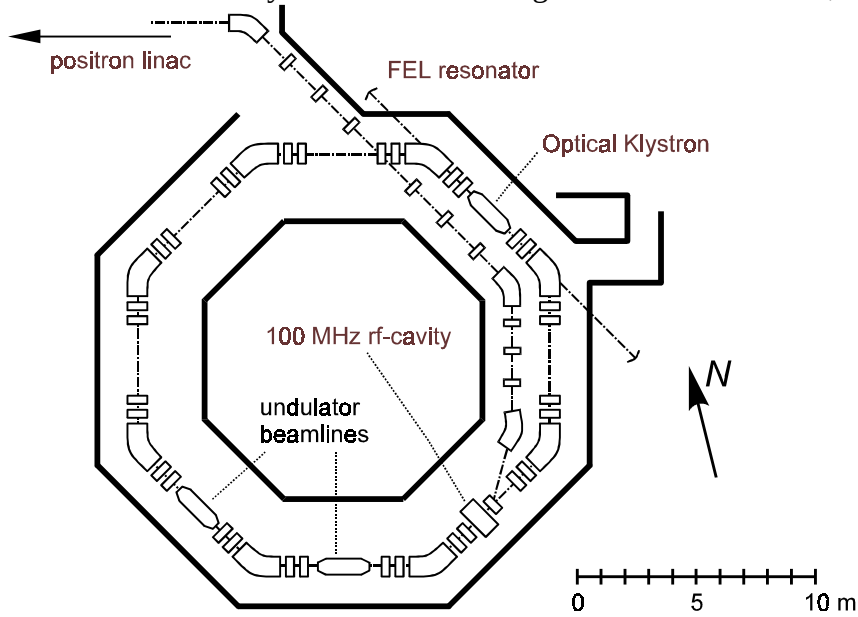


Fig. A.1 Layout of Super ACO

During a typical week the synchrotron provides 113 hours of operation (including injection time) of which an average of 21.5 hours are dedicated to FEL operation. The parameters of the laser are chosen to be compatible with a standard two-bunch mode of operation at the default beam energy of Super ACO of 800 MeV. The main differences are the reduced beam current and the slightly altered machine functions in the straight section of the OK. The former is necessary to avoid beam-instability, see Sec. 5.2. The latter enables an optimisation of the gain, see Sec. 5.3. Since this is a minor adjustment, laser operation can take place in parallel with the use of synchrotron beam lines. As such the FEL can also be considered as just an advanced beam line. Moreover, it is possible to perform experiments where, in a single experiment, the output of the FEL is used in combination with the output of a second (undulator) beam line. This way it is possible to do two-color pump-probe

experiments with a repetition rate of 8.3 MHz. At Super ACO roughly 50 % of the FEL beam time is used for this and other types of user experiments [11].

Table A.1
Super ACO parameters

Beam Energy	E	600-800	MeV
Circumference	L_c	72	m
Momentum compaction factor	α	0.0148	
rf-frequency	f_{rf}	100	MHz
Number of bunches		2	
Max. average beam current	I_a	200	mA
Damping time	τ_s	10-20	ms
Synchrotron frequency	f_s	14-17	kHz
Beam life time	τ	6	hours
Energy spread	σ_γ/γ	0.065	%
Emittance	$\epsilon_{x,y}$	20	π nm rad
Transverse beam size	$\sigma_{x,y}$	0.3	μ m
Transverse machine functions in OK	$\beta_{x,y}$	5	m
Energy dispersion in OK	$\eta_{x,y}$	0	
Cavity length	L	18	m
Maximum undulator strength	K	3.9	
Undulator period	λ_u	12.9	cm
Number of periods	N	2x10	
Strength dispersive section	$N+N_d$	<120	
Wavelength range	λ	600-350	nm
Laser gain	G_0	2	%
Average output power	P	100	mW
Micro-pulse duration	σ_t	12	ps
Spectral width	$\Delta\lambda/\lambda$	10^{-4}	

Two types of temporal diagnostic are presented in Fig. A.2 and Fig. A.3: a double-sweep streak-camera and a dissector. With a dissector (synchrotron or laser) light is projected onto a pinhole with a photo-cathode. The intensity evolution on the pinhole is then transformed into a spatial distribution with a fast transverse sweep on two deflecting electrons that deflect the passing electron beam. Behind the anode a fluorescent screen makes it possible to display the distribution on a photo or CCD camera. With a double-sweep camera an additional synchronised slow sweep is applied perpendicularly to the fast sweep. This way it is possible to record the evolution of a repeating signal, e.g., an electron bunch providing synchrotron radiation at a dipole beam line on each turn through the ring. The image in Fig. A.2 is a recording of the Super ACO FEL pulse in a pulsed mode of operation, i.e. zone ± 1 in Fig. 23. The streak-camera at Super ACO (Hamamatsu C5680) has a resolution of a few ps. The slow sweep can be set in a range from 100 ns to 1 s.

A dissector, developed at Novosibirsk [63], is in fact identical to a streak camera equipped with a single-pixel CDD camera. The temporal information can now be obtained in a stroboscopic way, i.e. by applying a slowly varying bias voltage to the fast sweep of the deflecting electrodes. The temporal resolution of the dissector at Super ACO is of the order of 10 ps. The main advantage of a dissector is its capability to process its signal directly by means of an oscilloscope or other electronic equipment, e.g. the feedback system described in Sec. 3.3. Due to its stroboscopic nature it is not possible to follow fast changing phenomena

such as the synchrotron modes of oscillation described in Sec. 5.2. Furthermore, it can only be used to record periodic events.

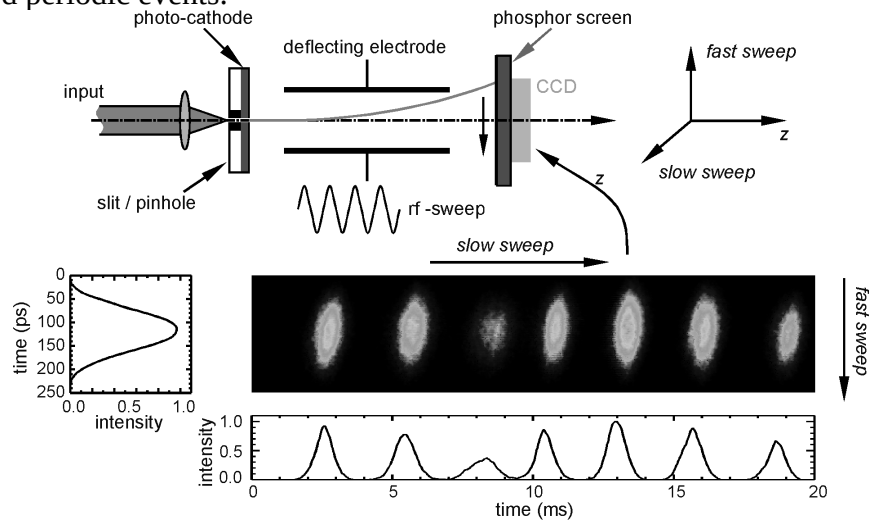


Fig. A.2 Schematic view of a double-sweep streak-camera. The lower image shows the pulsed output of the Super ACO FEL.

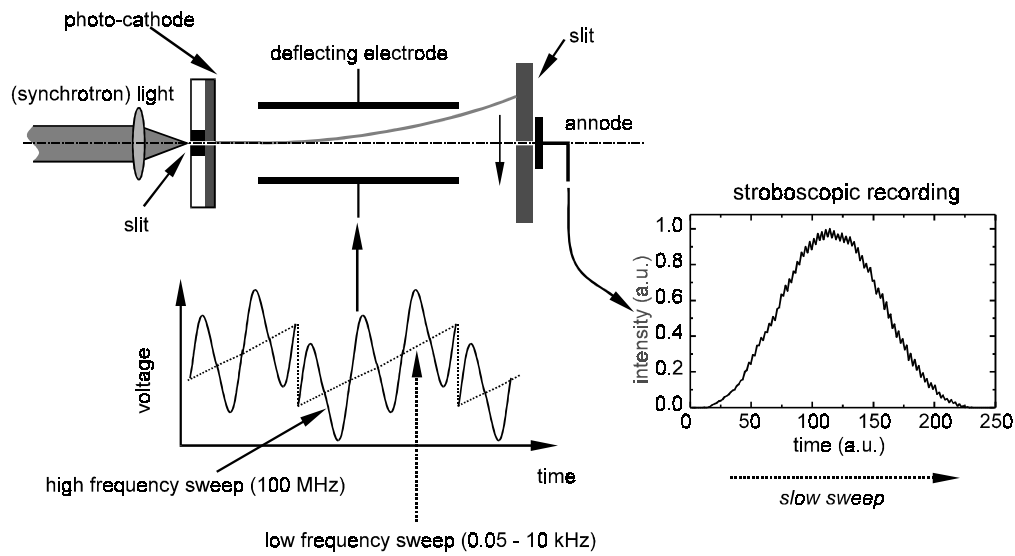


Fig. A.3 Schematic view of a dissector. The image shows a stroboscopic recording of the laser output at low intensity, i.e. zone 2 in Fig. 12.

APPENDIX B

3D SEMI-ANALYTICAL GAIN CALCULATIONS

In the following a formula for the gain for an FEL and optical klystron is derived*, including 3D effects, i.e. transverse interaction with the optical mode, emittance, and energy spread. Within this context the following assumptions are used:

- A continuous electron beam, i.e. the slippage effect is neglected.
- Operation in the small-signal, low-gain regime, i.e. the slowly varying amplitude and phase approximation.
- Interaction with a Gaussian TEM₀₀ transverse optical mode (not a fundamental restriction).
- Interaction with an electron beam with an arbitrary Gaussian distribution in phase space.

To derive the equation this Appendix has been divided into four sections. Section B1 serves as an introduction. In this section the well-known 1D gain formula for a regular FEL where an ideal electron beam interacts with a plane wave has been derived. In Sec. B2 the gain formula is modified to describe the effect of energy spread on the electron beam. This section serves as a simple example for Sec. B3 where all the above-mentioned 3D effects are included in the gain formula. In Sec. B4 the modifications required to describe an OK are discussed.

B.1 Gain calculation for an ideal beam

In this section a plane wave with angular frequency ω and a wave vector k , propagating in z direction is assumed. For this wave the electric field looks like:

$$\begin{aligned} E_x &= E_0 \Re(e^{i(\omega t - kz + \phi)}) \\ &= E_0 \cos(\omega t - kz + \phi) \end{aligned} \quad (B1)$$

where ϕ denotes an arbitrary phase of the field. An electron, with a velocity component parallel to the electric field will experience a force $-eE$ and, hence, will gain or lose kinetic energy. The energy change for an individual electron with energy γmc^2 can be expressed as

$$d\gamma = -\frac{e}{mc^2} E_x(z, t) dx \quad (B2)$$

where γ is the Lorentz factor, m the electron rest mass, c the vacuum velocity of light, and e the electron charge. Integration of Eq. (B2) leads to

$$\begin{aligned} \Delta\gamma(z) &= -\frac{e}{mc^2} \int_{z_i}^z E_x(z_1, t) \frac{dx}{dz} dz_1 \\ &= -\frac{e}{mc^2} \int_{z_i}^z E_x(z_1, t) x' dz_1 \end{aligned} \quad (B3)$$

When bunching of the electrons is expressed as a deviation in arrival time (t_0) at a given position z , the energy deviation can be expressed as a Taylor expansion around t_0

*•Calculations have been taken from the thesis of V. Litvinenko: "VEPP-3 Storage Ring Optical Klystron: Lasing in the Visible and the UV" Dissertation of the Institute of Nuclear Physics, Novosibirsk, 1989

$$\begin{aligned}\Delta\gamma(z) &= \Delta\gamma_1(z) + \Delta\gamma_2(z) + \dots \\ &= -\frac{e}{mc^2} \int_{z_i}^z E_x(z_1, t_0) x'(z_1) dz_1 - \frac{e}{mc^2} \int_{z_i}^z E_x(z_1, t_0 + \Delta t) x'(z_1) dz_1 + \dots\end{aligned}\quad (B4)$$

The first term $\Delta\gamma_1$ describes the FEL interaction without bunching. In this case there is no net energy loss for a distribution of electrons within one ponderomotive well, hence

$$\langle \Delta\gamma_1(z) \rangle_\lambda = 0 \quad (B5)$$

The second term, $\Delta\gamma_2$, describes the effects of bunching. For this the time deviation can be expressed as a phase-fluctuation with $\Delta\phi = -\omega\Delta t$. Hence, $\Delta\gamma_2$ can be expressed as

$$\Delta\gamma_2 = \frac{e}{mc^2} \int_{z_i}^z \frac{dE_x(z_1, t_0 + \Delta t)}{d\phi} \Delta\phi(z_1) x'(z_1) dz_1 \quad (B6)$$

where $\Delta\phi(z_1)$ is a function of the dispersion D in the undulator

$$\begin{aligned}\Delta\phi(z) &= \int_{z_i}^z \Delta\gamma(z_1) \int_{z_i}^{z_1} E_x(z_2) x'(z_2) dz_2 dz_1 \\ &= \frac{e}{mc^2} \int_{z_i}^z E_x(z_2) x'(z_2) \int_{z_i}^{z_1} D_z(z_1) dz_2 dz_1 \\ &= \frac{e}{mc^2} \int_{z_i}^z E_x(z_2) x'(z_2) D_z(z_2, z) dz_2 dz_1\end{aligned}\quad (B7)$$

For a planar undulator $D(z_2, z_1)$ reads

$$D(z_1, z_2) = \frac{2k_u}{\gamma} (z_1 - z_2) \quad (B8)$$

A combination of Eqs. (B6) and (B7) leads to

$$\Delta\gamma_2 = -\left(\frac{e}{mc^2}\right)^2 \int_{z_i}^{z_f} \int_{z_i}^{z_1} \frac{dE_x(z_1, t_0 + \Delta t)}{d\phi} x'(z_1) E_x(z_2) x'(z_2) D(z_1, z_2) dz_2 dz_1 \quad (B9)$$

The average energy lost per electron over one ponderomotive well thus becomes

$$\langle \Delta\gamma_2 \rangle_\lambda = -\left(\frac{e}{mc^2}\right)^2 \int_{z_i}^{z_f} \int_{z_i}^{z_1} \left\langle \frac{dE_x(z_1, t_0 + \Delta t)}{d\phi} x'(z_1) \right\rangle_\lambda \langle E_x(z_2) x'(z_2) \rangle_\lambda D(z_1, z_2) dz_2 dz_1 \quad (B10)$$

For an FEL with a planar undulator the terms in Eq. (B10) can be expressed as

$$\begin{aligned}\langle E_x(z_2) x'(z_2) \rangle_\lambda &= E_0 \frac{K}{\gamma} \langle \cos(\omega t - kz + \phi) \sin(k_u z) \rangle_\lambda \\ &= E_0 \frac{K}{2\gamma} (J_0(\xi) - J_1(\xi)) \cos(\delta k z + \phi)\end{aligned}\quad (B11)$$

where $J_{0,1}$ represent the Bessel functions of the first kind. K and δk are the undulator strength and the detuning parameter, respectively.

$$\begin{aligned}
K &= \frac{eB_u \lambda_u}{2\pi mc} \\
\xi &= \frac{K^2}{4 + 2K^2} \\
\delta k &= k - 2k_u \gamma^2
\end{aligned} \tag{B12}$$

Combination of Eqs. (B10) and (B11) leads to an average energy loss (averaged over one wavelength)

$$\begin{aligned}
\Delta\gamma_2 &= -\frac{1}{4} \left(\frac{e}{mc^2} \right)^2 \left(\frac{K}{\gamma} \right)^2 (J_0(\xi) - J_1(\xi))^2 E_0^2 \\
&\quad \int_{z_i}^{z_f} \int_{z_i}^{z_1} \cos(\delta k z_1 + \phi) \cdot \cos(\delta k z_2 + \phi) D(z_1, z_2) dz_2 dz_1 \\
&= -\frac{1}{8} \left(\frac{e}{mc^2} \right)^2 \left(\frac{K}{\gamma} \right)^2 (J_0(\xi) - J_1(\xi))^2 E_0^2 \int_{z_i}^{z_f} \int_{z_i}^{z_1} \sin(q(z_1 - z_2)) D(z_1, z_2) dz_2 dz_1
\end{aligned} \tag{B13}$$

The integral can be made dimensionless by rescaling the whole integral over the total interaction length $L = z_f - z_i$. The expression above then becomes

$$\Delta\gamma_2 = -\frac{1}{8} \left(\frac{e}{mc^2} \right)^2 \left(\frac{KL}{\gamma} \right)^2 (J_0(\xi) - J_1(\xi))^2 E_0^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{z_1} \sin(q(z_1 - z_2)) D(z_1, z_2) dz_2 dz_1 \tag{B14}$$

where δk has been substituted with $q = L \delta k$. Also the dispersion D has been normalised with respect to the interaction length. Eq. (B8) thus becomes

$$D(z_1, z_2) = \frac{4\pi L}{\lambda_u \gamma} (z_1 - z_2) \tag{B15}$$

which leads to

$$\begin{aligned}
\Delta\gamma_2 &= -\frac{\pi}{2} \left(\frac{e}{mc^2} \right)^2 \frac{K^2}{\lambda_u} \left(\frac{L}{\gamma} \right)^3 (J_0(\xi) - J_1(\xi))^2 E_0^2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{z_1} \sin(q(z_1 - z_2)) (z_1 - z_2) dz_2 dz_1 \\
&= -\frac{\pi}{2} \left(\frac{e}{mc^2} \right)^2 \frac{K^2}{\lambda_u} \left(\frac{L}{\gamma} \right)^3 (J_0(\xi) - J_1(\xi))^2 E_0^2 f(q)
\end{aligned} \tag{B16}$$

Here $f(q)$ describes the detuning curve, i.e. the gain as a function of detuning. The actual average energy loss per electron is $\Delta\gamma_2 mc^2$. Since the electron density per time-interval equals J/e , the total power density equals $P = \Delta\gamma_2 mc^2 J/e$. The power density, gained by the optical field thus becomes

$$P = \frac{\pi}{2} \frac{e}{mc^2} J \frac{K^2}{\lambda_u} \left(\frac{L}{\gamma} \right)^3 (J_0(\xi) - J_1(\xi))^2 E_0^2 f(q) \tag{B17}$$

The original average power density of the optical field equals $P_i = 4\pi\epsilon_0 c E_0^2/8\pi$. Hence the gain, $G = P/P_i$, equals

$$\begin{aligned}
G &= 4\pi^2 \frac{1}{4\pi\epsilon_0} \frac{e}{mc^3} J \frac{K^2}{\lambda_u} \left(\frac{L}{\gamma} \right)^3 (J_0(\xi) - J_1(\xi))^2 f(q) \\
&= 4\pi^2 \frac{J}{I_A} \frac{K^2}{\gamma^3} \frac{L^3}{\lambda_u} (J_0(\xi) - J_1(\xi))^2 f(q)
\end{aligned} \tag{B18}$$

with

$$\begin{aligned}
f(q) &= \int_{-\frac{1}{2}-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{z_1} \sin(q(z_1 - z_2)) (z_1 - z_2) dz_2 dz_1 \\
&= \frac{2 - 2\cos(q) - q\sin(q)}{q^3}
\end{aligned} \tag{B19}$$

B.2 Influence of finite energy spread

Due to a finite energy spread the phase of the electrons in the ponderomotive well will be disturbed. This phase disturbance can be described via the dispersion. Hence, the additional phase advance $\mathcal{G}$ between z_1 and z_2 due to an energy deviation $\delta\gamma = (\gamma - \gamma_0)/\gamma_0$ can be written as

$$\begin{aligned}
\mathcal{G}(z_1, z_2, \delta\gamma) &= -\omega\Delta t(z_1, z_2, \delta\gamma) \\
&= -D(z_1, z_2) \delta\gamma
\end{aligned} \tag{B20}$$

When assuming a Gaussian energy distribution with variance σ_e for the electron beam:

$$f_e = \frac{1}{\sigma_e \sqrt{2\pi}} \exp\left[-\frac{1}{2} \left(\frac{\delta\gamma}{\sigma_e} \right)^2\right] \tag{B21}$$

the average influence of a finite energy spread can be calculated through a modification of $f(q)$ to

$$\begin{aligned}
f(q) &= \int_{-\infty-\frac{1}{2}-\frac{1}{2}}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{z_1} f_e(\delta\gamma) \Im\{\expi[q(z_1 - z_2) + \mathcal{G}(z_1, z_2, \delta\gamma)]\} D(z_1, z_2) dz_1 dz_2 d\gamma \\
&= \frac{1}{\sigma_e \sqrt{2\pi}} \int_{-\infty-\frac{1}{2}-\frac{1}{2}}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{z_1} \Im\{\expi[q(z_1 - z_2)]\} \exp\left\{-\frac{1}{2} \left(\frac{\delta\gamma}{\sigma_e} \right)^2\right\} D(z_1, z_2) dz_1 dz_2 d\gamma \\
&= \frac{1}{\sigma_e \sqrt{2\pi}} \int_{-\infty-\frac{1}{2}-\frac{1}{2}}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{z_1} \Im\{\expi[q(z_1 - z_2)]\} \\
&\quad \exp\left\{-\frac{1}{2} \left[\frac{\delta\gamma}{\sigma_e} - \sigma_e D(z_1, z_2) \right]^2\right\} \exp\left\{-\frac{1}{2} [\sigma_e D(z_1, z_2)]^2\right\} D(z_1, z_2) dz_1 dz_2 d\gamma \\
f(q) &= \int_{-\frac{1}{2}-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{z_1} \Im\{\expi[q(z_1 - z_2)]\} F_e(z_1, z_2) D(z_1, z_2) dz_1 dz_2
\end{aligned} \tag{B22}$$

with

$$F_e(z_1, z_2) = \exp\left\{-\frac{1}{2}[\sigma_e D(z_1, z_2)]^2\right\} \quad (\text{B23})$$

B.3 Gain calculation for a real beam

In the following a Gaussian is assumed for both the distribution of the energy spread and the distribution in all directions in phase-space, i.e. $\delta\gamma$, x , x' , y , and y' . Properties of the electron beam are defined through the variance of the energy-spread distribution, the β -functions, and the emittance, i.e. σ_e , ε_x , $\beta_x(z)$, ε_y , and $\beta_y(z)$. Hence, the distribution of the electron beam can be described as a 3D Gaussian function

$$\begin{aligned} f(\mathbf{X}) &= \sqrt{\frac{\det|\mathbf{A}|}{(2\pi)^5}} \exp\left(-\frac{1}{2}\mathbf{X}^T \mathbf{A} \mathbf{X}\right) \\ &= \frac{1}{\varepsilon_x \varepsilon_y \sigma_e} \frac{1}{(2\pi)^{5/2}} \exp\left(-\frac{1}{2}\mathbf{X}^T \mathbf{A} \mathbf{X}\right) \end{aligned} \quad (\text{B24})$$

In a waist, i.e. at the centre of the undulator $\mathbf{A}$ can be expressed as

$$\mathbf{A}_{z=0} = \begin{pmatrix} \frac{1}{\beta_x \varepsilon_x} & 0 & 0 & 0 & 0 \\ 0 & \frac{\beta_x}{\varepsilon_x} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\beta_y \varepsilon_y} & 0 & 0 \\ 0 & 0 & 0 & \frac{\beta_y}{\varepsilon_y} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sigma_e^2} \end{pmatrix} \quad \text{and} \quad \mathbf{X} = \begin{pmatrix} x \\ x' \\ y \\ y' \\ \delta\gamma \end{pmatrix} \quad (\text{B25})$$

Assuming a linear transverse motion, both the evolution of the phase-space distribution can be described via the transport matrix $\mathbf{M}$ of the undulator. For a planar undulator this reads

$$\mathbf{M}(z) = \begin{pmatrix} 1 & z & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cos(z/\beta_{sc}) & \beta_{sc} \sin(z/\beta_{sc}) & 0 \\ 0 & 0 & -\frac{1}{\beta_{sc}} \sin(z/\beta_{sc}) & \cos(z/\beta_{sc}) & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{B26})$$

with $\beta_{sc} = \gamma/K k_u = \gamma mc^2/eB$. Note that the focussing properties of the undulator are neglected. Gain reduction, due to a finite emittance and energy spread, can be calculated by averaging the electron motion with the phase-advance and intensity of the optical mode. For the latter a fundamental Gaussian (TEM₀₀) with Rayleigh-length β_o will be assumed with a waist in the centre of the interaction region, i.e. $z = 0$. For this case Eq. (B1) expands to

$$E_x = \Re(E_0(z) e^{i(\omega t - kz - \zeta(z, y, z) + \phi)}) \quad (\text{B27})$$

$$\begin{aligned} E_0(z) &= \frac{E_0}{1 - i(z/\beta_o)} \\ \zeta(z, y, z) &= \frac{k(x^2 + y^2)}{2(\beta_o - iz)} \end{aligned} \quad (\text{B28})$$

Note that both x and y are functions of z , i.e. $x(z)$ and $y(z)$.

Similar as in Sec. B.2, $f(q)$ will be modified to

$$f_x(q) = \int_{-\infty-\frac{1}{2}}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathfrak{I} \left\{ \frac{1}{1-i(z_1/\beta_o)} \frac{1}{1+i(z_2/\beta_o)} \exp[\zeta(\mathbf{X}, z_1) + \zeta^*(\mathbf{X}, z_2)] \right. \\ \left. \exp i[q(z_1 - z_2) + \mathcal{G}(z_1, z_2, \mathbf{X})] \right\} f(\mathbf{X}) D(z_1, z_2) dz_1 dz_2 d\mathbf{X} \quad (\text{B29})$$

$$f_x(q) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathfrak{I} \left\{ \frac{1}{1-i(z_1/\beta_o)} \frac{1}{1+i(z_2/\beta_o)} e^{iq(z_1-z_2)} ff(z_1, z_2) \right\} D(z_1, z_2) dz_1 dz_2$$

where $\mathcal{G}(z_1, z_2, \mathbf{X})$ accounts for the additional phase advance of the electrons, either due to energy spread (see Sec. B.2), or due to transverse motion of the electrons. The latter can be expressed as

$$\mathcal{G}(x', y, y', z_1, z_2) = -k(z_1 - z_2) \left(x'^2 + y'^2 + \frac{y^2}{\beta_{sc}^2} \right) \quad (\text{B30})$$

The terms with x' and y' arise from the fact that an electron with transverse motion must have a reduced longitudinal motion. The dependence on y arises from the increased magnetic field strength as an electron moves closer to the poles of the undulator. Note that because of energy conservation the term $y'^2 + (y/\beta_{sc})^2 = \text{const.}$

To simplify calculations averaging over the electron beam will be done in the centre of the undulator, i.e. $z = 0$. The beam parameters will be expanded according to the transport matrix $\mathbf{M}$. Equation (B28) therefore expands to

$$\zeta(\mathbf{X}, z) = \frac{1}{2} \frac{k}{\beta_o - iz} \left[(x + zx')^2 + \left(y \cos(z/\beta_{sc}) + \frac{y'}{\beta_{sc}} \sin(z/\beta_{sc}) \right)^2 \right] \quad (\text{B31})$$

and ff becomes

$$ff(z_1, z_2) = \sqrt{\frac{\det|\mathbf{A}|}{(2\pi)^5}} \int d\mathbf{X} \frac{1}{1-i(z_1/\beta_o)} \frac{1}{1+i(z_2/\beta_o)} \\ \exp - \frac{1}{2} \left[\frac{x^2}{\beta_x \varepsilon_x} + \frac{\beta_x x'^2}{\varepsilon_x} + \frac{k(x + z_1 x')^2}{\beta_o - iz_1} + \frac{k(x + z_2 x')^2}{\beta_o + iz_2} - ik(z_1 - z_2)x'^2 \right] \\ \exp - \frac{1}{2} \left[\frac{y^2}{\beta_y \varepsilon_y} + \frac{\beta_y y'^2}{\varepsilon_y} + \frac{k(y \cos(z_1/\beta_{sc}) + y' \beta_{sc} \sin(z_1/\beta_{sc}))^2}{\beta_o - iz_1} \right. \\ \left. + \left[\frac{k(y \cos(z_2/\beta_{sc}) + y' \beta_{sc} \sin(z_2/\beta_{sc}))^2}{\beta_o + iz_2} - ik(z_1 - z_2) \left(y'^2 + \frac{y^2}{\beta_{sc}^2} \right) \right] \right] \\ \exp - \frac{1}{2} \left[\left(\frac{\delta\gamma}{\sigma_e} \right)^2 + 2D(z_1, z_2) \delta\gamma \right] \quad (\text{B32})$$

In a more general form the integral of $\mathbf{X}$ can be expressed as

$$ff(z_1, z_2) = \sqrt{\frac{\det|\mathbf{A}|}{(2\pi)^5}} \int \exp\left[-\frac{1}{2} \mathbf{X}^T \mathbf{W}(z_1, z_2) \mathbf{X} - \mathbf{V}^T(z_1, z_2) \mathbf{X}\right] d\mathbf{X} \quad (\text{B33})$$

In the equation the matrix $\mathbf{W}$ represents all terms with a squared dependence on $\mathbf{X}$, e.g., $f(\mathbf{X})$ and $\mathfrak{I}(\mathbf{X}, z)$. The vector $\mathbf{V}$ represents all terms that are linear with $\mathbf{X}$, e.g. the influence of dispersion.

Similar to the approach used in Eq. (B22) a general solution for Eq. (B33) can be found through a linear transformation

$$\mathbf{Y} = \mathbf{X} + \mathbf{W}^{-1}(z_1, z_2) \mathbf{V}(z_1, z_2) \quad (\text{B34})$$

which transforms Eq. (B33) into

$$ff(z_1, z_2) = \sqrt{\frac{\det|\mathbf{A}|}{(2\pi)^5}} \int \exp\left[-\frac{1}{2} \mathbf{Y}^T \mathbf{W}(z_1, z_2) \mathbf{Y} + \mathbf{V}^T(z_1, z_2) \mathbf{W}(z_1, z_2) \mathbf{V}(z_1, z_2)\right] d\mathbf{X} \quad (\text{B35})$$

Because $\mathbf{W}$ is a symmetric matrix with its real part positive definite (all density distributions must be real and positive), Eq. (B35) can be integrated, leading to

$$ff(z_1, z_2) = \sqrt{\frac{\det|\mathbf{A}|}{(2\pi)^5}} \exp\left[-\frac{1}{2} \left[\mathbf{V}^T(z_1, z_2) \mathbf{W}(z_1, z_2) \mathbf{V}(z_1, z_2)\right]\right] \quad (\text{B36})$$

From Eq. (B32) it follows

$$\mathbf{W}(z_1, z_2) = \begin{pmatrix} w_{11} & w_{12} & 0 & 0 & 0 \\ w_{12} & w_{22} & 0 & 0 & 0 \\ 0 & 0 & w_{33} & w_{34} & 0 \\ 0 & 0 & w_{34} & w_{44} & 0 \\ 0 & 0 & 0 & 0 & w_{55} \end{pmatrix} \quad \text{and} \quad \mathbf{V}(z_1, z_2) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ D(z_1, z_2) \end{pmatrix} \quad (\text{B37})$$

with

$$\begin{aligned} w_{11} &= \frac{1}{\beta_x \varepsilon_x} + \frac{k}{\beta_o - iz_1} + \frac{k}{\beta_o + iz_2} \\ w_{12} &= \frac{kz_1}{\beta_o - iz_1} + \frac{kz_2}{\beta_o + iz_2} \\ w_{22} &= \frac{\beta_x}{\varepsilon_x} + \frac{kz_1^2}{\beta_o - iz_1} + \frac{kz_2^2}{\beta_o + iz_2} - ik(z_1 - z_2) \\ w_{33} &= \frac{1}{\beta_y \varepsilon_y} + \frac{k \cos^2(z_1 / \beta_{sc})}{\beta_o - iz_1} + \frac{k \cos^2(z_2 / \beta_{sc})}{\beta_o + iz_2} - ik \frac{z_1 - z_2}{\beta_{sc}^2} \\ w_{34} &= \frac{1}{2} \frac{k \sin(2z_1 / \beta_{sc})}{\beta_o - iz_1} + \frac{1}{2} \frac{k \sin(2z_2 / \beta_{sc})}{\beta_o + iz_2} \\ w_{44} &= \frac{\beta_y}{\varepsilon_y} + \frac{k \beta_{sc}^2 \sin^2(z_1 / \beta_{sc})}{\beta_o - iz_1} + \frac{k \beta_{sc}^2 \sin^2(z_2 / \beta_{sc})}{\beta_o + iz_2} - ik(z_1 - z_2) \\ w_{55} &= \frac{1}{\sigma_e^2} \end{aligned} \quad (\text{B38})$$

leading to $\det|\mathbf{W}| = \det|\mathbf{W}_x| \det|\mathbf{W}_y| \det|\mathbf{W}_e|$ with

$$\begin{aligned}
\det|\mathbf{W}_x| &= \frac{1}{\varepsilon_x^2} \left\{ \left[1 + k\varepsilon_x \beta_x \left(\frac{1}{\beta_o - iz_1} + \frac{1}{\beta_o + iz_2} \right) \right] \right. \\
&\quad \left[1 + k \frac{\varepsilon_x}{\beta_x} \left(\frac{z_1^2}{\beta_o - iz_1} + \frac{z_2^2}{\beta_o + iz_2} - i(z_1 - z_2) \right) \right] \\
&\quad \left. - \left[(k\varepsilon_x)^2 \left(\frac{z_1}{\beta_o - iz_1} + \frac{z_2}{\beta_o + iz_2} \right)^2 \right] \right\} \\
\det|\mathbf{W}_y| &= \frac{1}{\varepsilon_y^2} \left\{ \left[1 + k\varepsilon_y \beta_y \left(\frac{\cos^2(z_1 / \beta_{sc})}{\beta_o - iz_1} + \frac{\cos^2(z_2 / \beta_{sc})}{\beta_o + iz_2} - i \frac{z_1 - z_2}{\beta_{sc}^2} \right) \right] \right. \\
&\quad \left[1 + k \frac{\varepsilon_y}{\beta_y} \left(\frac{\beta_{sc}^2 \sin^2(z_1 / \beta_{sc})}{\beta_o - iz_1} + \frac{\beta_{sc}^2 \sin^2(z_2 / \beta_{sc})}{\beta_o + iz_2} - i(z_1 - z_2) \right) \right] \\
&\quad \left. - \left[\left(\frac{k\varepsilon_y}{2} \right)^2 \left(\frac{\sin(2z_1 / \beta_{sc})}{\beta_o - iz_1} + \frac{\sin(2z_2 / \beta_{sc})}{\beta_o + iz_2} \right)^2 \right] \right\} \\
\det|\mathbf{W}_e| &= \frac{1}{\sigma_e^2}
\end{aligned} \tag{B39}$$

and

$$\exp - \frac{1}{2} [\mathbf{V}^T \mathbf{W} \mathbf{V}] = \exp - \frac{1}{2} (\sigma_e D)^2 \tag{B40}$$

identical to F_e in Eq. (B23). Hence $f(q)$ can be written as (see Eq. (B29))

$$\begin{aligned}
f &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathfrak{I} \left\{ \frac{1}{1 - i(z_1 / \beta_o)} \frac{1}{1 + i(z_2 / \beta_o)} \right. \\
&\quad \left. F_x(z_1, z_2) F_y(z_1, z_2) F_e(z_1, z_2) e^{iq(z_1 - z_2)} \right\} (z_1 - z_2) dz_1 dz_2
\end{aligned} \tag{B41}$$

which leads to a gain of

$$\begin{aligned}
G &= 8\pi^2 \frac{I}{I_A} \frac{K^2}{\gamma^3} \frac{L^3}{\lambda_u \lambda \beta_o} (J_0(\xi) - J_1(\xi))^2 \\
&\quad \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathfrak{I} \left\{ \frac{1}{1 - i(z_1 / \beta_o)} \frac{1}{1 + i(z_2 / \beta_o)} \right. \\
&\quad \left. F_x(z_1, z_2) F_y(z_1, z_2) F_e(z_1, z_2) e^{iq(z_1 - z_2)} \right\} (z_1 - z_2) dz_1 dz_2
\end{aligned} \tag{B42}$$

Here J has been substituted for the peak current I divided by the mode area of the electric field:

$$\Sigma_A = \frac{\lambda\beta_o}{2} \quad (\text{B43})$$

and F is given by

$$\begin{aligned} F_x &= \left\{ \left[1 + k\varepsilon_x \beta_x \left(\frac{1}{\beta_o - iz_1} + \frac{1}{\beta_o + iz_2} \right) \right] \cdot \left[1 + k \frac{\varepsilon_x}{\beta_x} \left(\frac{z_1^2}{\beta_o - iz_1} + \frac{z_2^2}{\beta_o + iz_2} - i(z_1 - z_2) \right) \right] \right. \\ &\quad \left. - \left[(k\varepsilon_x)^2 \left(\frac{z_1}{\beta_o - iz_1} + \frac{z_2}{\beta_o + iz_2} \right)^2 \right] \right\}^{-\frac{1}{2}} \\ F_y &= \left\{ \left[1 + k\varepsilon_y \beta_y \left(\frac{\cos^2(z_1 / \beta_{sc})}{\beta_o - iz_1} + \frac{\cos^2(z_2 / \beta_{sc})}{\beta_o + iz_2} - i \frac{z_1 - z_2}{\beta_{sc}^2} \right) \right] \right. \\ &\quad \left[1 + k \frac{\varepsilon_y}{\beta_y} \left(\frac{\beta_{sc}^2 \sin^2(z_1 / \beta_{sc})}{\beta_o - iz_1} + \frac{\beta_{sc}^2 \sin^2(z_2 / \beta_{sc})}{\beta_o + iz_2} - i(z_1 - z_2) \right) \right] \\ &\quad \left. - \left[\left(\frac{k\varepsilon_y}{2} \right)^2 \left(\frac{\sin(2z_1 / \beta_{sc})}{\beta_o - iz_1} + \frac{\sin(2z_2 / \beta_{sc})}{\beta_o + iz_2} \right)^2 \right] \right\}^{-\frac{1}{2}} \\ F_e &= \exp - \frac{1}{2} (\sigma_e D)^2 \end{aligned} \quad (\text{B44})$$

B.4 The optical klystron

For the OK the assumption is made that the interaction between the electrons and the optical wave can be neglected within the drift section of the OK which thus becomes a linear combination of two undulators, separated by a drift section. Now the symmetric OK will be investigated, i.e. the drift section occupies a system with a total length L where the centre part L_d . In the drift section the optical wave-front advances N_d wavelengths, relative to the electron beam. A drift length $(z_1 - z_2)$ thus modifies to

$$\Delta z = z_1 - z_2 \mapsto \begin{cases} z_1 + N_d / N - z_2, & z_1 \geq \frac{1}{2} L_d \wedge z_2 \leq -\frac{1}{2} L_d \\ z_1 - z_2 & \text{other} \end{cases} \quad (\text{B45})$$

and the gain becomes

$$\begin{aligned} G &= 8\pi^2 \frac{I}{I_A} \frac{K^2}{\gamma^3} \frac{N^3 \lambda_u^2}{\lambda \beta_o} (J_0(\xi) - J_1(\xi))^2 \\ &\quad \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \Im \left\{ \frac{1}{1 - i(z_1 / \beta_o)} \frac{1}{1 + i(z_2 / \beta_o)} \right. \\ &\quad \left. g(z_1) g(z_2) F_x(z_1, z_2) F_y(z_1, z_2) F_e(z_1, z_2) e^{iq\Delta z} \right\} \Delta z dz_1 dz_2 \end{aligned} \quad (\text{B46})$$

$$g(z) = \begin{cases} 0, & |z| < L_d / 2L \\ 1, & |z| \geq L_d / 2L \end{cases}$$

where $g(z)$ arises from the fact that the interaction in the drift space of the OK is neglected. Note that N is the total number of undulator periods (divided over two undulators).

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