Geometry of Strings and Branes

Foar Heit en Mem

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Introduction

Elementary particle physics aims to describe the fundamental constituents of Nature and their interactions. Experiments indicate that elementary particles fall into two classes: leptons, containing among others the electron and the neutrino; and quarks, which form the building blocks of protons and neutrons. The four known forces between these building blocks of matter are the gravitational, the electromagnetic, the weak, and the strong interaction.

At small length scales, the gravitational interaction is many orders of magnitude weaker than all the other forces[1], and it can therefore safely be neglected. The remaining three interactions of elementary particles can be described by an elegant theory called the Standard Model. This theory is a gauge theory: it has an internal local symmetry group in which each interaction is described by the exchange of gauge fields. These gauge fields are called the photon, the W-bosons and Z-boson, and the gluons for the electromagnetic, the weak, and the strong interaction, respectively. Gauge fields are different from matter particles in several aspects: the former fall into the class of bosons, particles with integer spin and commuting statistics; the latter are called fermions, particles with half-integer spin and anti-commuting statistics. It can be shown that internal symmetry groups, such as those of the Standard Model, cannot mix bosons with fermions [1].

Microscopic physics is described by quantum mechanics, which can be seen as a deformation of classical dynamics. It has several non-intuitive properties: one cannot simultaneously measure all observables with infinite accuracy, and many quantities can only be expressed in terms of probabilities. The Standard Model is quantum mechanically completely consistent, and the theory is in excellent agreement with experiments.

At macroscopic scales, the interactions of the Standard Model are virtually absent: the strong interaction is confined to small distances; the weak interaction has an exponential decay with distance; and although the electromagnetic force has an infinite range, all large configurations of matter are approximately electrically neutral. Hence, the gravitational interaction becomes the dominant force at large length scales.

Gravity is described by the theory of General Relativity. The basic ingredients of General Relativity are that space and time merge into a spacetime, that matter induces a curved geom-

[1] The ratio of the gravitational and the electric force between a proton and an electron is 10^{-40}.

etry on spacetime, and that this geometry in turn determines the dynamics of matter. One can also try to cast General Relativity in the form of a gauge theory: in this case a gauge theory of spacetime symmetries, known as general coordinate transformations, rather than internal symmetries. The corresponding gauge field in this case is called the graviton. General Relativity is a purely classical theory. It successfully explains physics in the range of terrestrial to cosmological length scales.

However, this split of physics into the macroscopic theory of General Relativity and the microscopic Standard Model is not without caveats, because General Relativity has some peculiar properties. First of all, it turns out that certain solutions to the classical field equations, known as black holes, have as a generic feature the occurrence of spacetime singularities [2] around which the gravitational field becomes infinitely large. This undermines the reason for ignoring gravitational interactions in elementary particle physics, and it becomes necessary to treat the gravitational field quantum mechanically.

Most of these spacetime singularities are predicted not to be directly observable. Instead, they are conjectured always to be hidden behind event horizons – surfaces from which not even light can return. Singularities are therefore thought not to be directly observable. However, the quantum mechanical behavior of elementary particles around such event horizons is problematic, since the one-way nature of event-horizons interferes with the probabilistic interpretation of quantum mechanics. This gives rise to information paradoxes [3].

Although the energy scales necessary to probe microscopic gravitational effects are not easily obtained in laboratory experiments, they did occur in the early universe. In order to develop good cosmological models, it is therefore necessary to have a description of gravity at small length scales. As a final remark, there is the related problem of the cosmological constant, a parameter in General Relativity for which the Standard Model predicts a value many orders of magnitude larger than the value inferred from astronomical observations [4].

To solve the problems sketched above, it is necessary to construct a theory of quantum gravity. To see what problems can arise in quantizing gravity, it is instructive to compare electromagnetism and gravity since at the classical and semi-classical level there are many parallels between the two interactions, as we have summarized in table 1. They both share a characteristic long range force, although gravity can never be repulsive. Both interactions also fit into a relativistic framework, and covariant field equations for both theories were found by Maxwell, and by Einstein, respectively. Both actions are invariant under local symmetries. For electromagnetism, these symmetries form the group of phase transformations, known as $\mathrm{U}(1)$; for General Relativity, they form the group of general coordinate transformations. There is one particular classical effect of the gravitational interaction that has not yet been observed directly: namely the radiation of gravitational waves[2], the gravitational counterpart of optics.

The quantum mechanical motion of particles in the background of classical force fields is sometimes called first quantization. For the electromagnetic force, this was studied in the first few decades of the twentieth century during which in particular the nature of black

[2]Indirect evidence for gravitational waves comes from the rotation time decay of binary star systems [5].

Process	Electromagnetism	Gravity
Force	$F_{\rm el} = \frac{q_1 q_2}{r^2}$	$F_{\rm gr} = -\frac{G m_1 m_2}{r^2}$
Relativistic	$\partial^\nu F_{\nu\mu} = J_\mu$	$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}$
Classical action	$\mathcal{L}_{\rm em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$	$\mathcal{L}_{\rm GR} = \frac{1}{16\pi G}\sqrt{\lvert g\rvert} R$
Symmetry	$\mathrm{U}(1)$	General coordinate transf.
Radiation	Optics	*Gravitational waves*
Spectrum	Black body	Hawking radiation
Phenomenology	H-atom spectral lines	Black hole entropy
Microscopic origin	Energy levels	Density of states

Table 1: (Semi-)classical electromagnetism versus gravity.

body radiation and the origin of the energy levels of the hydrogen atom were clarified. In the last few decades of the last century, the quantum mechanical behavior of particles in gravitational fields has been clarified: in particular, the process of Hawking radiation [6] and the microscopic origin [7] of entropy [8,9] for certain classes of black holes were discovered.

To continue the discussion of quantum gravity, it is more useful to compare the gravitational with the weak or the strong interaction, as we have summarized in table 2, since electromagnetism has no self-interactions at the quantum level, in contrast to the other three interactions. For the electromagnetic interaction, one can apply quantization methods to the classical action $\mathcal{L}_{\rm em}$ given in table 1, but this procedure fails for the action of General Relativity since it has an energy-dependent coupling constant G – this makes the theory non-renormalizable.

Some progress towards solving this non-renormalizability problem was obtained by the discovery of supergravity in 1976 [10]. Supergravity is a modified version of General Relativity having spacetime symmetries as well as internal symmetries. A characteristic property of this so-called supersymmetry is that it mixes bosons with fermions [11]. In chapter 5, we will be more precise about the structure of supersymmetry and its cousin conformal supersymmetry. Although supergravity is better behaved at high energies than General Relativity, it is still non-renormalizable. The best one can hope for is that supergravity is a low-energy effective description of a theory of quantum gravity. This is rather similar to the situation concerning the weak interaction where Fermi's theory of beta-decay is also a non-renormalizable theory, but it can be seen to arise from the Standard Model.

In order to go beyond the low-energy effective description of a theory, a prescription for calculating scattering amplitudes at higher energies is necessary. For the strong interaction, this so-called S-matrix theory was developed during the nineteen sixties, and it uses

Process	Weak or strong interaction	Gravity
Low-energy theory	Fermi's theory of beta-decay	Supergravity
Scattering amplitudes	S-matrix theory:	Perturbative string theory:
	Feynman graphs	Riemann surfaces
Classical action	Standard Model:	*String field theory*:
	$\mathcal{L}_{\mathrm{SM}} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \ldots$	$\mathcal{L}_{\mathrm{SFT}} = \frac{1}{2}\Psi \star Q\Psi + \ldots$
Symmetry	$\mathrm{SU}(2)$ or $\mathrm{SU}(3)$	*Unknown*
Solitonic solutions	Monopoles	Branes
Duality	Electric/magnetic charges	Strong/weak coupling
Quantization method	BRS-method	BV-formalism

Table 2: Quantizing the weak or strong interaction versus gravity.

a perturbative expansion over Feynman graphs in order to calculate amplitudes. The precise prescription is fixed by a Lagrangian formulation. In the case of the strong interaction, as well as the electroweak interactions, all the Feynman rules can be derived from the Lagrangian of the Standard Model.

A corresponding formalism yielding scattering amplitudes for gravity involves the concept of strings: i.e. at small length scales, particles are postulated to be tiny vibrating strings. The motivation is that the spectrum of a closed string contains the graviton, the gauge field for gravity. Since strings sweep out worldsheets rather than worldlines, as particles do, the idea of Feynman graphs has to be extended to surfaces. It was shown in the nineteen eighties that a perturbative expansion over Riemann surfaces gives quantum mechanically consistent scattering amplitudes.

The string theory analog of the Standard Model was developed in the nineteen eighties, this goes under the name of string field theory. In this theory, one single string field describes all string vibrations simultaneously. For the simplest models of perturbative string theory, it can be shown that the corresponding string field theory yields the same answers for scattering amplitudes, but for more complicated perturbative string theories, there are technical complications in constructing the corresponding string field theories.

The fields in the Lagrangian of the Standard Model can be rotated by two- or three-dimensional unitary matrices, in which case the gauge group is called $\mathrm{SU}(2)$ or $\mathrm{SU}(3)$, respectively. Since matrices do not commute, such theories are called non-Abelian gauge theories. The quantization of the classical action of an interaction is often called second quantization, and for the weak and the strong interactions this can be consistently done using the methods of BRS-quantization [12, 13]. The symmetry groups of string field theories are much larger and much more complicated than the gauge groups of the Standard Model, and in many cases not known explicitly. This means that traditional methods of quantization fail, and one needs to use more sophisticated methods such as the BV anti-field formalism [14]. Just as the quantization of the weak interaction required more sophisticated tools than the quantization of electromagnetism, it seems also likely that the quantization of gravity will require new methods in this area.

Gauge theories often have solitons – solutions of the classical field equations with finite energy. In modified theories of the weak interaction there are for example magnetic monopoles. The presence of such magnetic monopoles can imply that there is a duality between electric and magnetic charges. Such dualities are powerful symmetries, since they often relate separate regimes of a given theory. String theory has higher-dimensional solitonic solutions called branes[3]. In string theory, there is also a number of dualities, such as dualities between strongly and weakly coupled regimes of different versions of string theory. In all of these dualities, branes play an essential role. The overall framework of string theory and branes is called M-theory, where the M can mean anything ranging from Mystery to Membrane, according to taste. It is not clear yet whether strings are the fundamental degrees of freedom of quantum gravity, or if there is perhaps a formulation in terms of branes.

The organization of this thesis is as follows. We will start in chapter 1 with a more elaborate treatment of the string theory framework, including the basic features of string theory and supergravity, as well as the various dualities and brane solutions of these theories. In chapter 2, we will describe the AdS/CFT correspondence – a recently discovered duality between theories of gravity in Anti-de-Sitter spacetimes and conformal quantum field theories. This is a remarkable duality, because several quantities within quantum gravity can be expressed in terms of concepts known from quantum field theory. A central theme in the AdS/CFT correspondence is a special brane solution of string theory: the D3-brane.

In chapter 3, we will present our results [15] that show how this duality can be extended to a duality between gravity in more general curved spacetimes called domain-walls and more general quantum field theories – the DW/QFT correspondence. In particular, we will discuss a large class of brane solutions that includes the D3-brane. After choosing a suitable coordinate frame, the so-called dual frame, we will study the near-horizon geometry of these brane solutions of supergravity, and we will analyze what kinematical information can be extracted from the dual field theories.

The domain-walls that appear in the analysis mentioned above describe spaces that are separated into several domains by a boundary surface – the domain-wall. Across such domain-walls, physical quantities can change their values in a discontinuous fashion. Domain-walls that have such discontinuities are sometimes called "thin" domain-walls. On the other hand, domain-walls that can be interpreted as smooth interpolations between different supergravity vacua go under the name of "thick" domain-walls. At the end of chapter 3, we will explain how these thick domain-walls have the interpretation of renormalization group flows in their

[3] Compare 0,1,2, ... *many* with particle, string, membrane, ... *brane*.

dual quantum field theories.

Domain-wall spacetimes have attracted renewed attention recently: they are a member of the class of brane world scenarios. In chapter 4, we will describe such brane world scenarios in more detail: the basic idea is that our four-dimensional universe is actually a hypersurface within a five-dimensional supergravity theory. The size of the extra fifth dimension transverse to the so-called brane world can be used to gain insight in the origin of some unnatural properties of four-dimensional physics. For instance, the so-called Randall-Sundrum scenarios were used to obtain a better understanding of the cosmological constant problem, as well as the unnatural ratio of the strength of the gravitational force and the remaining three interactions, the so-called hierarchy problem.

Supersymmetric versions of such theories have proven to be hard to find. The main obstacle is realizing supersymmetry on the four-dimensional brane world solution: it is related to finding the vacuum structure of the corresponding five-dimensional supergravity theory. This, in turn, requires a detailed knowledge of all possible couplings of five-dimensional matter models to supergravity. The scalar fields of these matter models can be interpreted as coordinates on an abstract space. Many properties of the matter-coupled supergravity theory can then be expressed in terms of the geometrical properties of the corresponding space of scalar fields.

In particular, the scalar fields generate a potential that determines the vacuum structure of the supergravity theory. For supersymmetric brane worlds to exist, this scalar potential needs to possess two different, stable minima that need to satisfy some additional constraints. Moreover, one needs to find a suitable solution that smoothly interpolates between two such minima. Such an analysis, which had been started in the nineteen eighties (albeit for different reasons), has recently been renewed, but still does not encompass the most general five-dimensional matter-coupled supergravity theory.

We will take a systematic approach to construct these five-dimensional matter-couplings. This so-called superconformal program starts from the most general spacetime symmetry group, the group of superconformal transformations, which considerably simplifies the analysis of matter-couplings to supergravity. The different models possessing superconformal symmetry are called multiplets. First of all, there is the so-called Weyl multiplet: this is the smallest multiplet of the superconformal group that possess the graviton. On the other hand, there are the matter multiplets: they interact with the Weyl multiplet that forms a fixed background of conformal supergravity. Matter-couplings to non-conformal supergravity can then be obtained by breaking the conformal symmetries.

In chapter 5, we will present our results [16] on the five-dimensional Weyl multiplets. We will see that there are two versions of this multiplet: the Standard Weyl multiplet and the Dilaton Weyl multiplet. Multiplets similar to the Standard Weyl multiplet also exist in four and six dimensions, but the Dilaton Weyl multiplet had so far only been found in six dimensions. We will use a well-known method to deduce the transformation rules for the different fields: the so-called Noether method. In particular, we will construct the multiplet of conserved Noether currents for the various conformal symmetries. A remarkable detail is that the current multiplet that couples to the Standard Weyl multiplet contains currents that satisfy differential equations, a mechanism that so far had only been known from ten-dimensional conformal supergravity.

Our results [17] on five-dimensional superconformal matter multiplets will be presented in chapter 6. We will discuss so-called vector multiplets: these are multiplets that contain the gauge field of the gauge group under which the multiplet transforms. We will analyze vector multiplets that transform under arbitrary transformations of the gauge group: the so-called vector-tensor multiplets. In particular, we will consider representations of the gauge group that are reducible but not completely reducible. This gives rise to previously unknown interactions between vector fields and tensor fields. The conformal symmetries can only be realized on the tensor fields if these satisfy their equations of motion. By dropping the usual restriction that the equations of motion have to follow from an action principle, we can also formulate vector-tensor multiplets with an odd number of tensor fields.

Apart from vector-tensor multiplets, we will also consider hypermultiplets in chapter 6. These multiplets also possess scalar fields but not gauge fields. The scalar fields span a vector space over the quaternions. Realizing the conformal algebra on the scalar fields will induce a non-trivial geometry called hyper-complex geometry on the space of scalars. Similarly as for tensor fields, the superconformal algebra can only be realized on the fields of the hypermultiplet with the use of equations of motion. Also in this case, we will consider equations of motion that do not follow from an action principle. The special cases for which there is an action correspond to hyper-complex manifolds possessing a metric: the so-called hyper-Kähler manifolds. Furthermore, we will analyze the interaction of hypermultiplets with vector multiplets, and we will also make use of the scalar field geometry in this case. At the end of chapter 6, we will give an overview of all the geometrical concepts that we will make use of.

The matter-couplings to conformal supergravity that we will construct in this way can be used as a starting point to construct matter-couplings to non-conformal supergravity. At the end of chapter 6, we will sketch some ingredients of this procedure. Whether the five-dimensional matter-couplings of supergravity that can be obtained in this way will actually modify the vacuum structure in such a way that supersymmetric brane world scenarios can be realized, remains an open question that will have to be answered by future research.

Chapter 1

The string theory framework

In this chapter, we will give an overview of the string theory framework. We will start with describing several basic features of string theory, after which we we will discuss some aspects of supergravity, the low-energy effective description of string theory. In the last two sections of this chapter, we will review some recent developments in string theory: in particular, we will discuss string theory dualities and brane solutions of supergravity.

1.1 String theory

String theory was born out of attempts to explain the hadron resonance spectrum of the strong interaction. Soon after the discovery of the Veneziano scattering amplitude [18], which expressed a duality between resonances coming from the so-called s-channel and t-channel, it was realized that this amplitude described the dynamics of an open relativistic string.

Open strings have in their spectrum a massless spin-1 particle, which is reminiscent of a gauge field. However, after new experimental results were shown to be in conflict with the Veneziano amplitude, string theory as a model for the strong interaction was replaced by the gauge theory QCD. Relativistic closed strings, however, have in their spectrum a massless spin-2 particle, which corresponds precisely to the characteristic properties of a graviton. It was then argued that closed string theory could be a theory of gravity [19].

We will first explain in some detail the geometrical and dynamical setup of classical bosonic string theory. After that, we will be less detailed as we discuss the quantized and the supersymmetric versions of string theory, since most of the research described in this thesis has been performed at the level of supergravity. For more details and proper references, see the classic textbooks of [20, 21], and the more modern approach of [22, 23]. We will finish this section with a discussion of the Kaluza-Klein mechanism, and a description of interacting strings in non-trivial backgrounds.

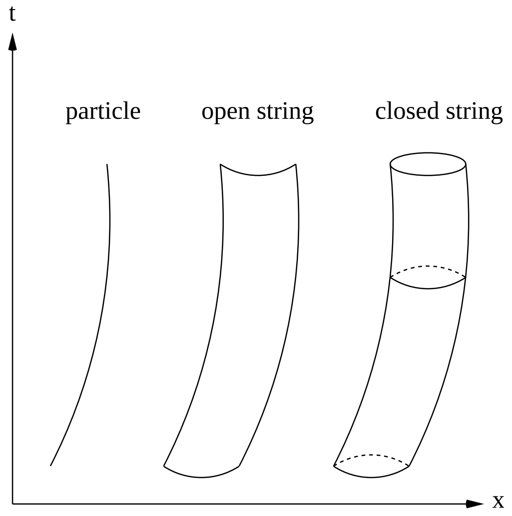

Figure 1.1: A particle worldline and string worldsheets.

1.1.1 Free bosonic string theory

The mathematical formulation of string theory proceeds along similar ways as the relativistic motion of particles, as we have indicated in figure 1.1. Consider a particle with mass m, moving through a flat D-dimensional spacetime with coordinates X^μ. Here, one assigns a parameter τ to the worldline Λ that the particle sweeps out in spacetime, and the action is simply the length of the worldline

$$S_{\text{particle}} = -m \int_\Lambda \mathrm{d}\tau \, \sqrt{|\partial_\tau X^\mu \partial_\tau X_\mu|}\,. \tag{1.1}$$

The dynamics of a relativistic string with tension T can likewise be formulated by assigning coordinates $\sigma^a = (\tau, \sigma)$ to the two-dimensional world-sheet Σ. The action is given by the surface that the string worldsheet sweeps out in spacetime

$$S_{\text{string}} = -T \int_\Sigma \mathrm{d}^2\sigma \, \sqrt{|\det(\partial_a X^\mu \partial_b X_\mu)|}\,. \tag{1.2}$$

For historical reasons, the tension of the string is often expressed in terms of the Regge-slope parameter α', which is related to the length of the string ℓ_s by

$$T = \frac{1}{2\pi\alpha'}\,, \quad \alpha' = \frac{\ell_s^2}{\hbar}\,. \tag{1.3}$$

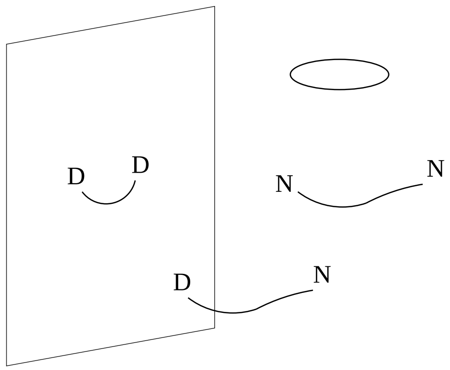

Figure 1.2: The periodic, Neumann, and Dirichlet boundary conditions for strings.

The equation of motion following from the action (1.2) is nothing else than the two-dimensional wave equation for the embedding coordinates

$$\frac{\partial}{\partial\sigma^-}\frac{\partial}{\partial\sigma^+}X^\mu(\tau,\sigma) = 0\,, \quad \sigma^\pm \equiv \tau \pm \sigma\,. \tag{1.4}$$

As can be seen from the particular form in which we have written the wave equation, there are two independent directions along which vibrations of the string can propagate, usually called left and right.

To be able to solve the equations of motion, one has to supplement them by suitable boundary conditions, as we have indicated in figure 1.2. For the closed string of length ℓ_s, one has to impose periodic boundary conditions

$$X^\mu(\tau,0) = X^\mu(\tau,\ell_s)\,, \tag{1.5}$$

but for open strings there are two different possibilities, depending on how the right-moving vibrations turn into left-moving modes at the endpoints

$$\frac{\partial}{\partial\sigma^-}X^\mu(\tau,\sigma) = \pm\frac{\partial}{\partial\sigma^+}X^\mu(\tau,\sigma)\,, \quad \sigma = 0,\ell_s\,. \tag{1.6}$$

The choice of the plus sign goes under the name of Neumann boundary conditions and corresponds to freely moving open strings. The case of the minus sign is known as Dirichlet boundary conditions, where the endpoints of the strings are actually fixed at some hyperplanes in spacetime. We will later see that these hyperplanes correspond to solitonic objects called Dirichlet-branes, or D-branes [24] for short.

The final result is that the coordinates X^μ are given as linear superpositions of all possible solutions of (1.4), subject to the boundary conditions (1.5), or (1.6). Each of the elementary vibration modes of the string worldsheet corresponds to a particle in spacetime. In particular, the energies and the polarizations of the vibration modes are related to the masses and spins of the corresponding elementary particles.

1.1.2 Quantization and superstrings

The string action (1.2) has many symmetries, including reparametrizations and rescalings of the two-dimensional worldsheet, both of which are essential in order to solve the equations of motion in full generality. The symmetry group of the worldsheet is actually infinite-dimensional [25] and goes under the name of the conformal or the Virasoro algebra.

If one tries to quantize the oscillators on the string worldsheet while retaining the conformal structure, then the string can no longer move in spacetimes of arbitrary dimension, but instead the spacetime in which the string propagates is restricted to be 26-dimensional. At first sight, this seems to rule out string theory as a realistic description of four-dimensional Quantum Gravity, but we will see in section 1.1.3 how the Kaluza-Klein mechanism solves this apparent contradiction.

There is a more severe problem with the quantized bosonic string: namely the oscillator with the lowest energy actually has an imaginary mass, meaning that it is a tachyon. The appearance of tachyons in field theory usually means that one is expanding around the wrong vacuum and that by redefining the vacuum the tachyon will disappear. Recent developments [26] indicate that this may also be the case in string theory, but a complete understanding of this will require sophisticated string field theory methods [27].

Another problem of bosonic string theory is the absence of fermions in its spectrum: if string theory is to provide a unification scheme of elementary particles and all their interactions, then one would like to have matter included as well. A modification of bosonic string theory, called superstring theory, addresses both the fermion and the tachyon problem.

There are two different approaches to superstring theory. In the Neveu-Schwarz-Ramond formulation [28, 29], one adds worldsheet fermions to the action (1.2). These world-sheet fermions have to satisfy appropriate boundary conditions. This divides the oscillators into two classes: a Neveu-Schwarz or NS-sector and a Ramond or R-sector. On the other hand, the Green-Schwarz formulation [30] starts from a spacetime supersymmetric action. These two, a priori different, formulations turn out to be equivalent in the sense that they give the same answers for scattering amplitudes.

Supersymmetry already restricts the dimension in which classical superstrings can live to $3, 4, 6$ or 10 dimensions, but in order to obtain quantum mechanical consistency, the spacetime in which superstrings move has to be ten-dimensional. The quantization of the NSR-formulation can be done in a manifestly covariant manner, but spacetime supersymmetry can only be obtained by performing the so-called GSO-projection [31] that eliminates the tachyon from the spectrum. In the GS-formulation, spacetime supersymmetry is manifest from the outset but covariance is lost and one has to resort to light-cone gauge quantization.

There are five consistent superstring theories. The first two are called Type IIA and Type IIB superstring theory. They are both theories of closed strings only, and they posses what is technically known as $\mathcal{N} = 2$ supersymmetry: the difference being that Type IIA has two spinors of opposite chirality, called $(1,1)$ supersymmetry; whereas Type IIB is a chiral theory with $(2,0)$ supersymmetry.

Then there are three theories with $\mathcal{N} = 1$ supersymmetry. First there is the Type I superstring theory of open strings, which also has a closed string sector. This theory has massless gauge fields in its spectrum that transform under the gauge group $\mathrm{SO}(32)$. Finally, there are two Heterotic string theories [32]: these are rather exotic theories of closed strings in which the right-moving and left-moving modes on the world-sheet are taken to be different. These Heterotic theories also have a gauge symmetry, and the gauge group can be $\mathrm{E}_8 \times \mathrm{E}_8$ or $\mathrm{SO}(32)$ in this case.

All these five superstring theories were shown to be free of anomalies and to give consistent quantum mechanical scattering amplitudes [33].

1.1.3 Dimensional reduction

We saw how bosonic strings and superstrings had to move in spacetimes of dimensions 26 or 10. This problem does not have to be fatal per se, since the extra dimensions of spacetime can be taken care of by a well-defined mathematical procedure called Kaluza-Klein compactification. We will illustrate this mechanism with the toy model example of a massless two-dimensional scalar field satisfying the wave equation

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}\right)\phi(t,x) = 0\,. \tag{1.7}$$

We now take the x-direction to be a circle of radius R, and since the scalar field has to periodic in the compact direction, we can Fourier expand the scalar field in this compact direction

$$\phi(t,x) = \phi(t, x + 2\pi n R) \rightarrow \phi(t,x) = \sum_n \phi_n(t) e^{\pi i n x/R}\,. \tag{1.8}$$

If we substitute the expansion (1.8) into the equation of motion (1.7), then we find

$$-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi_n(t) = m^2\phi_n(t)\,, \quad m^2 = \frac{(\pi n)^2}{R^2}\,, \tag{1.9}$$

from which one observes that each Fourier-mode describes a massive particle in the remaining non-compact spacetime with a mass that is inversely proportional to the radius.

Taking the limit $R \rightarrow 0$, we see that the zero-mode decouples from all the other modes, since these become infinitely massive. The result is that, after dimensional reduction over a circle of infinitesimal radius, a massless two-dimensional scalar field is effectively described by a massless scalar field in one dimension. On the other hand, taking the limit $R \rightarrow \infty$ makes the spectrum in (1.9) continuous, and we will regain the uncompactified two-dimensional theory. In string theory, the limits $R \rightarrow 0$ and $R \rightarrow \infty$ are equivalent to each other, as we will see in section 1.3.1 when we discuss T-duality.

The analog of (1.7) in supergravity is a set of ten-dimensional tensor fields satisfying non-linear differential equations in a spacetime forming a product of four-dimensional Minkowski spacetime times a compact six-dimensional manifold. After a Fourier-expansion of the tensor fields in eigenfunctions of the differential operator on the compact manifold, the higher Fourier-modes will decouple in the limit $R \to 0$, and the higher-dimensional fields are described by a set of lower-dimensional tensor fields[1].

A closely related mechanism to Kaluza-Klein reduction is called spontaneous compactification: this occurs when the zero-modes on the compact manifold do not appear as sources in the equations of motion for the higher Fourier-modes. In this rather special case, one can consistently truncate these higher modes to zero. What this means is that the solutions of the compactified lower-dimensional theory formed by the zero-modes are also solutions of the original, uncompactified, higher-dimensional theory.

For the two-dimensional scalar field example, it is consistent to truncate to the zero-mode $\phi_0(t)$, since it satisfies not only the reduced equation of motion (1.9) but also the original equation of motion (1.7). At the level of the linearized equations of motion or for compactifications on manifolds as simple as tori or group manifolds, the consistency of such truncations is always guaranteed. But for compactification on more complicated manifolds such as spheres, the zero-modes generically appear in the equations of motion of the higher Fourier-modes, and a consistent truncation is generically no longer possible: the higher Fourier-modes only decouple in the limit $R \to 0$.

1.1.4 Backgrounds and interactions

So far, we have discussed superstrings propagating in flat ten-dimensional spacetimes, and we argued that since closed strings had massless spin-2 particles in their spectrum that string theory could be a theory of gravity. General Relativity tells us that the geometry of spacetime should actually be a dynamical variable, fixed by the equations of motion. We will now discuss how to generalize the action (1.2) to strings moving in more complicated backgrounds.

In addition to a massless symmetric traceless tensor $G_{\mu\nu}$, closed superstrings have two more massless modes[2]: an anti-symmetric tensor $B_{\mu\nu}$ and a massive scalar Φ called the dilaton. The tensor $G_{\mu\nu}$ will be identified with the spacetime metric, which in (1.2) was given by the flat Minkowski spacetime metric $\eta_{\mu\nu}$. We will denote the metric on the string worldsheet Σ by γ_{ab}.

The tensor $B_{\mu\nu}$ can be interpreted as a generalized gauge field: analogously to how particles can be charged under vector fields, higher-dimensional objects such as strings can be charged under higher-rank tensor fields. Finally, the scalar Φ will couple to the string worldsheet through its curvature $R(\gamma)$. The generalization of the action (1.2) is given by a two-

[1] Tensor-components in compact directions behave as tensors of lower rank in the remaining dimensions.

[2] We will not discuss the massless fermions or Ramond-Ramond gauge fields in this section.

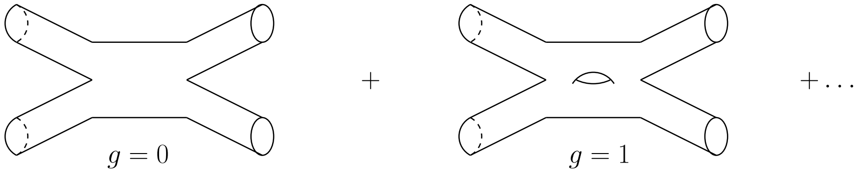

Figure 1.3: The genus expansion of string theory interactions.

dimensional non-linear sigma-model with a ten-dimensional target space

$$S = -\frac{T}{2}\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{\gamma}\,\left[\left(\gamma^{ab}G_{\mu\nu}(X) + \varepsilon^{ab}B_{\mu\nu}(X)\right)\partial_a X^\mu \partial_b X^\nu - \alpha'\Phi(X)R(\gamma)\right]\,. \quad (1.10)$$

The discussion so far only involved freely propagating strings. Scattering amplitudes for interacting particles can conveniently be calculated by the technique of Feynman diagrams in which there is a one-to-one map from a graph to a contribution to an amplitude. In string theory, the analog of this is given in terms of Riemann surfaces as we have indicated in figure 1.3.

The most convenient way to obtain scattering amplitudes is through the same path-integral methods that are used to quantize the free strings [34, 35]. In particular, the partition function corresponding to the action (1.10) is given by a series expansion over Riemann surfaces of genus g

$$\mathcal{Z}_{\text{string}} = \sum_{g=0}^{\infty}\int \mathcal{D}\gamma_{(g)}\mathcal{D}X\, e^{-S[\gamma_{(g)},X]}\,. \quad (1.11)$$

Even though the specific contribution of a Riemann surface to a string theory scattering amplitude is harder to calculate [36] than a corresponding Feynman diagram in field theory, the number of diagrams at any given genus is exactly one, whereas in field theory the number of diagrams per loop grows rapidly. The high-energy behavior of string theory scattering amplitudes is also a lot better: this is intuitively clear from the observation that vertices in Feynman diagrams are singular whereas Riemann surfaces are smooth everywhere.

In quantum electrodynamics, there is a dimensionless constant α that can be formed out of the dimensionful parameters $e^2, \hbar$, and c. The partition function can be calculated as a series expansion in Feynman diagrams with L loops: after assigning every vertex a factor α, this becomes a series expansion in α

$$\alpha = \frac{e^2}{\hbar c}\,, \quad \mathcal{Z}_{\text{QED}} = \sum_{L=0}^{\infty}\alpha^{2(L-1)}\mathcal{Z}_L\,. \quad (1.12)$$

In the expression (1.11) for the string theory partition function, we did not write any dimensionless parameter. However, string theory has as dimensionful parameters the gravitational coupling κ and the string length ℓ_s from which it is possible to form a dimensionless parameter g_s. One should therefore expect that the genus-expansion will become a series expansion in g_s

$$g_s^2 = \frac{4\pi\kappa^2}{(2\pi\ell_s)^{D-2}}\,, \quad \mathcal{Z}_{\text{string}} = \sum_{g=0}^{\infty} g_s^{2(g-1)} \mathcal{Z}_g\,. \tag{1.13}$$

It would be disappointing if reconciling gravity with quantum mechanics involved the introduction of a new fundamental dimensionless parameter. Happily, this is not the case. What comes to rescue is that the power of g_s in (1.13) is a topological quantity: it is minus the Euler number χ of the corresponding Riemann surface. But the Euler number of a two-dimensional surface Σ is also related to an integral over its curvature through the Gauss-Bonnet theorem

$$\chi = \frac{1}{4\pi}\int_\Sigma \mathrm{d}^2\sigma\,\sqrt{\gamma}\,R(\gamma)\,, \tag{1.14}$$

which is precisely the coupling to Φ in the action (1.10). This implies that we can define the string coupling to be the expectation value of the dilaton exponential

$$g_s = \left\langle e^{\Phi}\right\rangle\,. \tag{1.15}$$

Instead of being a one-parameter family of theories labeled by a fundamental dimensionless parameter g_s, string theory is a single theory with a one-parameter family of vacuum states labeled by the expectation value of the dilaton exponential.

1.2 Supergravity

Historically, four-dimensional supergravity [10] was discovered as a gauge theory of supersymmetry, a procedure that we shall mimic for conformal supergravity and conformal supersymmetry in chapter 5. In this section, we will emphasize a different viewpoint: namely we will show that supergravity is the low-energy effective description of string theory. For each superstring theory mentioned in the previous section, we will give its supergravity action. We will also make some remarks about eleven-dimensional supergravity. For more details and an accurate historical account, we refer to [37, 38].

1.2.1 Low-energy effective actions

As we mentioned before, the worldsheet for a string propagating in a flat spacetime has an infinite-dimensional symmetry group including a two-dimensional scaling symmetry which allowed for the complete solution to the equations of motion as well as a consistent quantization.

For strings propagating in non-trivial backgrounds, this is no longer guaranteed: the action (1.10) describes a string as a non-linear sigma-model in which the spacetime fields appear as dimensionful coupling-constants on the string worldsheet. This means that the essential scale-symmetry will be broken in general.

In order to obtain a consistent description, the coupling constants should not transform under the scale-symmetry: in other words, the beta-functions of the corresponding coupling constants should vanish. These beta-functions can be calculated as a perturbation series in α' for which the lowest-order approximation yields

$$\begin{aligned} \beta(G_{\mu\nu}) &= R_{\mu\nu} + 4\partial_\mu\Phi\partial_\nu\Phi - \tfrac{1}{4}H_\mu{}^{\lambda\rho}H_{\nu\lambda\rho} + \mathcal{O}(\alpha')\,, \\ \beta(B_{\mu\nu}) &= \nabla^\lambda\left(e^{-2\Phi}H_{\mu\nu\lambda}\right) + \mathcal{O}(\alpha')\,, \\ \beta(\Phi) &= 4\nabla^\mu\partial_\mu\Phi - 4\partial^\mu\Phi\partial_\mu\Phi + R - \tfrac{1}{12}H^{\mu\nu\lambda}H_{\mu\nu\lambda} + \mathcal{O}(\alpha')\,, \end{aligned} \tag{1.16}$$

where we have defined the field-strength of the gauge field by

$$H_{\mu\nu\lambda} = 3\partial_{[\mu}B_{\nu\lambda]}\,. \tag{1.17}$$

So, we see that demanding quantum mechanical consistency through the vanishing of the beta-functions (1.16) gives constraints on the massless modes. These constraints can be interpreted as equations of motion, since they are equivalent to the Euler-Lagrange equations for the familiar Einstein action of General Relativity in the presence of a generalized gauge field and a scalar field

$$S = \frac{1}{2\kappa_0^2}\int \mathrm{d}^D x\,\sqrt{|G|}\,e^{-2\Phi}\left(R + 4\,(\partial\Phi)^2 - \frac{1}{12}H^2\right)\,. \tag{1.18}$$

The constant κ_0 is not fixed by the equations of motion, and in order to relate it to the gravitational coupling κ, we redefine the dilaton in such a way that is has a vanishing expectation value

$$e^\phi \equiv \frac{e^\Phi}{g_s}\,. \tag{1.19}$$

The action then takes on the form

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^D x\,\sqrt{|G|}\,e^{-2\phi}\left(R + 4\,(\partial\phi)^2 - \frac{1}{12}H^2\right)\,. \tag{1.20}$$

where the gravitational coupling is now *defined* using (1.13)

$$\frac{1}{2\kappa^2} \equiv \frac{2\pi}{g_s^2(2\pi\ell_s)^{D-2}}\,. \tag{1.21}$$

The force coming from the dilaton exchange breaks the equivalence principle of General Relativity: free-falling frames are no longer equivalent to the absence of gravity. In particular, the beta-functions (1.16) are derived in a frame called the string frame:

$$g^{\mathrm{S}}_{\mu\nu} \equiv G_{\mu\nu}\,. \tag{1.22}$$

It is often convenient to scale the metric in such a way that the curvature term in the action has no dilatonic pre-factor. This metric is called the Einstein frame: it is related to the string frame by the transformation

$$g^{\mathrm{E}}_{\mu\nu} = e^{-\frac{4}{D-2}\phi} g^{\mathrm{S}}_{\mu\nu} \,. \tag{1.23}$$

In this frame the action (1.20) takes on the form

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^D x \sqrt{|g^{\mathrm{E}}|} \left(R - \frac{4}{D-2} (\partial\phi)^2 - \frac{1}{12} e^{-\phi} H^2 \right) . \tag{1.24}$$

To avoid cluttering actions with a lot of constants, we will often put factors of α' and g_s equal to unity: the former can always be restored by dimensional analysis; and for each term, we can find the correct factor of g_s in any metric frame from the relative scaling between that frame and the string frame, the factor of e^{Φ} for the corresponding term in the string frame, and the use of (1.19).

Any closed string theory with massless modes $G_{\mu\nu}$, $B_{\mu\nu}$, and ϕ as described by the action (1.10) has the spacetime action (1.20) as its low-energy description. We will now review how the various massless modes of the different versions of superstring theory give rise to different modifications of the action (1.20).

1.2.2 $\mathcal{N} = 2$ supergravities

The Type II superstrings each have their own version of supergravity describing their massless modes. Since closed strings have both left and right-moving modes, there are in total four different sectors for the massless modes, depending on the boundary conditions. We will only look at the bosonic sectors of the supergravity actions, since we will need their structures in chapters 2 and 3. This corresponds to keeping the massless modes of the NSNS and RR-sectors of the superstrings.

The massless modes of the NSNS-sector of all the superstrings are given by the familiar metric $g_{\mu\nu}$, the anti-symmetric tensor field $B_{\mu\nu}$, and the dilaton ϕ. To simplify the structure of the supergravity actions, we will use differential form notation in this section. In this language the tensor $B_{\mu\nu}$ is written as the two-form $B_{(2)}$, and the volume element $\sqrt{|g|}$ as $\star\mathbb{1}$. For more details on our notation, see appendix A. The RR-sector of Type IIA string theory consists of a set of two gauge potentials $\{C_{(1)}, C_{(3)}\}$. The bosonic part of the Type IIA supergravity action is given by

$$\begin{aligned}\mathcal{L}_{\mathrm{IIA}} \;=\; & e^{-2\phi}\left(R \star \mathbb{1} + 4 \star \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{2} \star H_{(3)} \wedge H_{(3)} \right) - \frac{1}{2} \star G_{(2)} \wedge G_{(2)} \\ & - \frac{1}{2} \star G_{(4)} \wedge G_{(4)} + \frac{1}{2} B_{(2)} \wedge \mathrm{d}C_{(3)} \wedge \mathrm{d}C_{(3)} \,,\end{aligned} \tag{1.25}$$

where the field-strengths of the various gauge potentials are defined as

$$H_{(3)} = \mathrm{d}B_{(2)} \,, \quad G_{(2)} = \mathrm{d}C_{(1)} \,, \quad G_{(4)} = \mathrm{d}C_{(3)} - H_{(3)} \wedge C_{(1)} \,. \tag{1.26}$$

For the Type IIB superstring one finds in the RR-sector a set of three gauge potentials $\{C_{(0)}, C_{(2)}, C_{(4)}\}$ which appear in the Type IIB supergravity action according to

$$\begin{aligned} \mathcal{L}_{\mathrm{IIB}} &= e^{-2\phi}\left(R\star\mathbb{1}+4\star \mathrm{d}\phi\wedge\mathrm{d}\phi-\frac{1}{2}\star H_{(3)}\wedge H_{(3)}\right)-\frac{1}{2}\star G_{(1)}\wedge G_{(1)} \\ &\quad -\frac{1}{2}\star G_{(3)}\wedge G_{(3)}-\frac{1}{4}\star G_{(5)}\wedge G_{(5)}-\frac{1}{2}C_{(4)}\wedge \mathrm{d}C_{(2)}\wedge\mathrm{d}B_{(2)}\,, \end{aligned} \tag{1.27}$$

where we have

$$H_{(3)}=\mathrm{d}B_{(2)}\,,\quad G_{(1)}=\mathrm{d}C_{(0)}\,,\quad G_{(3)}=\mathrm{d}C_{(2)}-H_{(3)}\wedge C_{(0)}\,. \tag{1.28}$$

The five-form field-strength $G_{(5)}$ satisfies a self-duality condition

$$G_{(5)}=\mathrm{d}C_{(4)}-\frac{1}{2}C_{(2)}\wedge\mathrm{d}B_{(2)}+\frac{1}{2}B_{(2)}\wedge\mathrm{d}C_{(2)}\,,\quad G_{(5)}\equiv\star G_{(5)}\,, \tag{1.29}$$

which does not follow from the equation of motion [39] but which has to be imposed as an extra constraint [40].

1.2.3 $\mathcal{N}=1$ supergravities

The $\mathcal{N}=1$ superstrings have $\mathcal{N}=1$ supergravities as their low-energy effective description. They share the NSNS-sector of the Type II strings, but none of the $\mathcal{N}=1$ superstrings have RR-gauge potentials, although they do have ordinary gauge fields $A^I_{(1)}$ for their respective $\mathrm{E}_8\times\mathrm{E}_8$ and $\mathrm{SO}(32)$ symmetry groups. For the Heterotic string theories, one has the following kinetic terms of the bosonic part of the supergravity actions

$$\mathcal{L}_{\mathrm{Het}}=e^{-2\phi}\left(R\star\mathbb{1}+4\star\mathrm{d}\phi\wedge\mathrm{d}\phi-\frac{1}{2}\star H_{(3)}\wedge H_{(3)}-\frac{1}{2}\mathrm{Tr}\,\star F_{(2)}\wedge F_{(2)}\right)\,, \tag{1.30}$$

where the trace is taken over all gauge group generators and where the field-strengths are given by

$$H_{(3)}=\mathrm{d}B_{(2)}+\frac{1}{2}\mathrm{Tr}\,A_{(1)}\wedge\mathrm{dA}_{(1)}\,,\quad F_{(2)}=\mathrm{d}A_{(1)}+A_{(1)}\wedge A_{(1)}\,. \tag{1.31}$$

The Type I superstring has the same field content as the Heterotic strings but a slightly different supergravity action

$$\mathcal{L}_{\mathrm{I}}=e^{-2\phi}\left(R\star\mathbb{1}+4\star\mathrm{d}\phi\wedge\mathrm{d}\phi\right)-\frac{1}{2}\star H_{(3)}\wedge H_{(3)}-\frac{1}{2}e^{-\phi}\mathrm{Tr}\,\star F_{(2)}\wedge F_{(2)}\,. \tag{1.32}$$

We have left out some terms in the actions (1.30) and (1.32): they are necessary to cancel the gauge and gravitational anomalies [33]; we refer to the literature for the complete expressions [21].

1.2.4 $D = 11$ supergravity

Quantum versions of superstring theory can only live in ten dimensions, and we have shown above that they all have a ten-dimensional low-energy limit. However, there also exists an eleven-dimensional supergravity theory [41]. The field content consists of a metric $g_{\mu\nu}$ and a three-form gauge potential $C_{(3)}$ described by the following bosonic Lagrangian

$$\mathcal{L}_{11} = R \star \mathbb{1} - \frac{1}{2} \star G_{(4)} \wedge G_{(4)} + \frac{1}{6} C_{(3)} \wedge G_{(4)} \wedge G_{(4)} \,, \tag{1.33}$$

where as usual

$$G_{(4)} = \mathrm{d}C_{(3)} \,. \tag{1.34}$$

For a long time, it was not clear what the meaning of this eleven-dimensional theory was, until it was discovered that if one generalizes the concept of superstrings to supermembranes, then the spacetimes in which such supermembranes can consistently move are precisely those satisfying the equations of motion of eleven-dimensional supergravity [42].

Attempts to quantize the supermembrane and to obtain its spectrum failed however, and it was shown that the supermembrane has a continuous spectrum with no discrete non-zero energy vibration modes [43]. However, the supermembrane and eleven-dimensional supergravity were a turning point in the development of string theory, since they provided many insights in the relationships between the different versions of string theory, as we will now discuss in the remainder of this chapter.

1.3 Dualities

The possibility of no less than five consistent superstring theories is an embarrassment of riches. In this section, we will sketch how all string theories are related to each other by a web of dualities.

1.3.1 T-duality

The first duality that we will discuss is a duality in which string theories compactified on circles of different radii are mapped into each other. This is possible since the embedding coordinates X^μ of a string are not ordinary scalar fields, satisfying periodic boundary conditions as in (1.8), but instead they can wrap around the compact dimension according to

$$X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + \ell_s) + 2\pi n R \,. \tag{1.35}$$

This means that the solution to the wave equation for the coordinates X^μ has momentum modes proportional to the inverse radius, but also winding terms proportional to the radius. This is also reflected in the mass formula

$$X^\mu_\pm(\tau, \sigma) \sim \left(\frac{m}{R} \pm wR \right) \sigma_\pm \,, \quad M^2 \sim \left(\frac{m^2}{R^2} + w^2 R^2 \right) \,. \tag{1.36}$$

The mass spectrum is symmetric under inversion of the radius with a simultaneous interchange of the momentum modes with the winding modes

$$R \leftrightarrow \frac{1}{R}\,, \quad m \leftrightarrow w\,: \quad M^2 \to M^2\,, \quad X^\mu_\pm(\tau,\sigma) \to \pm X^\mu_\pm(\tau,\sigma)\,. \tag{1.37}$$

For the coordinates, the effect is equivalent to a parity transformation on the right-moving modes. For the Type II superstrings this parity transformation changes the chirality of the spinors and the overall result is that the (1,1) supersymmetric Type IIA superstring theory is mapped into the (2,0) supersymmetric Type IIB superstring theory. This relation holds for any value of the radius: in particular it relates the limits $R \to 0$ and $R \to \infty$.

This has no counterpart in field theories, since particles cannot wind around a compact dimension. The effect of T-duality in supergravity is not that e.g. Type IIA supergravity and Type IIB supergravity are T-dual to each other in the sense of the Type II string theories, but rather that there is a discrete symmetry relating the two supergravity theories when both are reduced to nine dimensions over a circle of zero radius [44].

The effect of T-duality for the Heterotic superstrings is more difficult to explain, but it is related to the fact that one can see the right-moving modes as bosonic string theories compactified on sixteen-dimensional lattices, which are precisely the root lattices of the corresponding gauge groups. Under the map (1.37), the lattices of the $\mathrm{E}_8 \times \mathrm{E}_8$ and $\mathrm{SO}(32)$ are interchanged[3] making the Heterotic superstrings T-dual to each other [45].

T-duality in Type I string theory is even more astonishing, since the effect of a parity transformation on the right-moving modes interchanges the Neumann and Dirichlet boundary conditions, as can be seen from (1.6). As we argued before, and as we will show in the next section, the hyperplanes on which endpoints of open strings with Dirichlet boundary conditions end are actually solitonic solutions of string theory. The effect of T-duality on these D-branes is that it maps branes of different dimensions to each other.

The examples of T-duality which we have discussed here are only the tip of a mathematical iceberg: there are also dualities known as mirror-symmetries in which ten-dimensional string theories compactified on different six-dimensional spacetimes, known as Calabi-Yau manifolds, are related to each other. The perturbative expansions using sigma-model actions with mirror-related target spaces give the same quantum mechanical scattering amplitudes [46].

1.3.2 S-duality

String theory also possesses non-perturbative dualities in which the strong coupling regime of a string theory is related to the weak coupling regime of another theory. This class of dualities is called S-duality. These dualities are non-trivial to prove, but substantial evidence

[3] This involves breaking the gauge group in both theories to an $\mathrm{SO}(16) \times \mathrm{SO}(16)$ subgroup by turning on appropriate Wilson lines.

from string theory compactifications has been obtained. For a good review see [47]. We will now indicate how these dualities come about using the supergravity approximation.

If we transform the Heterotic $\mathrm{SO}(32)$ action (1.30) to the Einstein frame (1.23), then we find

$$\mathcal{L}^{\mathrm{E}}_{\mathrm{Het}} = R \star \mathbb{1} - \frac{1}{2} \star \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{2} e^{-\phi} \star H_{(3)} \wedge H_{(3)} - \frac{1}{2} e^{-\frac{1}{2}\phi}\, \mathrm{Tr}\ \star F_{(2)} \wedge F_{(2)}\,, \quad (1.38)$$

and for the rescaled Type I action (1.32) we obtain

$$\mathcal{L}^{\mathrm{E}}_{\mathrm{I}} = R \star \mathbb{1} - \frac{1}{2} \star \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{2} e^{\phi} \star H_{(3)} \wedge H_{(3)} - \frac{1}{2} e^{\frac{1}{2}\phi}\, \mathrm{Tr}\ \star F_{(2)} \wedge F_{(2)}\,. \quad (1.39)$$

It is clear upon inspection that the two actions (1.38) and (1.39) are transformed into each other under the discrete mapping

$$\phi \to -\phi\,. \quad (1.40)$$

Since the exponential of the dilaton corresponds to the string coupling constant, this suggests that the strong and weak coupling regimes of the Heterotic $\mathrm{SO}(32)$ and Type I superstring are mapped into each other [48]. This is a surprising result: it relates a theory of both closed and open strings to a theory of only closed strings.

Transforming the IIB supergravity action to the Einstein frame yields

$$\begin{aligned} \mathcal{L}^{\mathrm{E}}_{\mathrm{IIB}} &= R \star \mathbb{1} - \frac{1}{2} \star \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{2} e^{-\phi} \star H_{(3)} \wedge H_{(3)} - \frac{1}{2} e^{2\phi} \star G_{(1)} \wedge G_{(1)} \\ &\quad - \frac{1}{2} e^{\phi} \star G_{(3)} \wedge G_{(3)} - \frac{1}{4} \star G_{(5)} \wedge G_{(5)} - \frac{1}{2} C_{(4)} \wedge \mathrm{d}C_{(2)} \wedge \mathrm{d}B_{(2)}\,. \end{aligned} \quad (1.41)$$

This action has a symmetry mixing the two scalars and two two-form potentials. In particular, it can be shown [40] that the action (1.41) remains invariant under so-called $\mathrm{S}\,\ell(2,\mathbb{R})$ transformations of the form

$$\tau \to \frac{a\tau + b}{c\tau + d}\,, \quad \begin{pmatrix} C_{(2)} \\ B_{(2)} \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_{(2)} \\ B_{(2)} \end{pmatrix}\,, \quad ad - bc = 1\,, \quad (1.42)$$

where we have grouped the two real scalars together into one complex scalar τ

$$\tau = C_{(0)} + \mathrm{i}\, e^{-\phi}\,. \quad (1.43)$$

For the special case $a = d = C_{(0)} = 0$ and $b = -c = 1$, the transformation (1.42) is equivalent to (1.40). This makes it plausible that the strong coupling regime of Type IIB superstring theory[4] is actually dual to its own weak coupling regime [49].

[4]The duality symmetry is restricted to $\mathrm{S}\,\ell(2,\mathbb{Z})$ in Type IIB string theory.

1.3.3 M-theory

The strong coupling limit of Type IIA string theory is even more surprising. First we transform (1.25) to the Einstein frame to obtain

$$\begin{aligned}\mathcal{L}^{\mathrm{E}}_{\mathrm{IIA}} &= R\star\mathbb{1}-\frac{1}{2}\star\mathrm{d}\phi\wedge\mathrm{d}\phi-\frac{1}{2}e^{-\phi}\star H_{(3)}\wedge H_{(3)}-\frac{1}{2}e^{\frac{3}{2}\phi}\star G_{(2)}\wedge G_{(2)}\\ &\quad -\frac{1}{2}e^{\frac{1}{2}\phi}\star G_{(4)}\wedge G_{(4)}+\frac{1}{2}B_{(2)}\wedge\mathrm{d}C_{(3)}\wedge\mathrm{d}C_{(3)}\,. \end{aligned}\tag{1.44}$$

Then we group the various fields of this action together in the following way

$$\begin{aligned}\widehat{g}_{\mu\nu} &= e^{-\frac{1}{6}\phi}g_{\mu\nu}+e^{\frac{4}{3}\phi}C_\mu C_\nu\,,\\ \widehat{g}_{\mu z} &= e^{\frac{4}{3}\phi}C_\mu\,,\quad \widehat{g}_{zz}=e^{\frac{4}{3}\phi}\,,\\ \widehat{C}_{(3)} &= C_{(3)}+B_{(2)}\wedge\left(\mathrm{d}z+C_{(1)}\right)\,.\end{aligned}\tag{1.45}$$

Note that this is precisely the field content of $D=11$ supergravity. In fact, substituting (1.45) into the $D=11$ supergravity action (1.33), we obtain

$$\mathcal{L}_{11}=\mathcal{L}^{\mathrm{E}}_{\mathrm{IIA}}\wedge\mathrm{d}z\,,\tag{1.46}$$

meaning that Type IIA supergravity is a Kaluza-Klein reduction of $D=11$ supergravity over a circle.

From (1.45) and (1.15), we see that the exponential of the dilaton relates the string coupling constant to the radius of the eleventh dimension in units of the eleven-dimensional Planck length

$$R_{11}=g_s^{\frac{2}{3}}\,\ell_p\,.\tag{1.47}$$

The eleven-dimensional gravitational couplings constant has dimensions of ℓ_p^9: using (1.46) and (1.47), we can determine the ten-dimensional Newton's constant

$$\kappa_{11}^2\equiv\kappa_{10}^2R_{11}\rightarrow\kappa_{10}^2=\frac{\ell_p^8}{g_s^{\frac{2}{3}}}\,.\tag{1.48}$$

If we compare this with the previous expressions (1.3) and (1.13), then we obtain a relation between the ten-dimensional string length, string coupling, and Planck length. We can then express the eleven-dimensional radius in ten-dimensional quantities

$$\ell_p=g_s^{\frac{1}{3}}\,\ell_s\rightarrow R_{11}=g_s\ell_s\,.\tag{1.49}$$

This means that the strong coupling limit of ten-dimensional Type IIA string theory is an eleven-dimensional theory [50]. This theory goes under the name of M-theory [51]: it is defined to be the theory that has $D=11$ supergravity as its low-energy limit. In a similar way, there are reasons to believe that the strong coupling limit of the Heterotic $\mathrm{E}_8\times\mathrm{E}_8$ string theory is related to the same M-theory [52], but this time the extra dimension is not a circle but an interval [53].

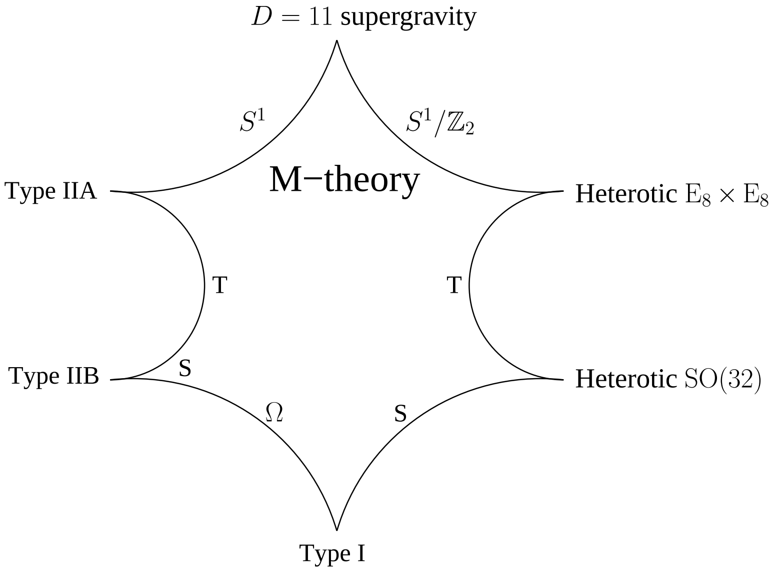

Figure 1.4: The M-theory web of string theories and their dualities.

1.3.4 The duality web

We have summarized this web of dualities in figure 1.4, where S^1 and $S^1/\mathbb{Z}_2$ indicate a circle and a line interval, respectively. There is also a duality between the Type IIB and Type I string theories which we have not discussed, but a parity operator Ω can be applied to spectrum of Type IIB string theory to obtain Type I string theory.

After this web of dualities emerged, the term M-theory was no longer used for the strong coupling limit of Type IIA string theory but for the whole framework of string theories and supergravities in ten and eleven dimensions. The overall picture is that all these theories are different vacua of a single underlying theory around each of which one can perform perturbation theory.

The various dualities interpolate between the different vacua and relate the various perturbative results. A detailed microscopic description of the full M-theory is still lacking, although there have been some attempts in this direction [54].

1.4 Branes

In this section, we will consider some aspects of branes. We will start with looking at how branes appear as solutions of the supergravity equations of motion. Then we will describe the worldvolume actions describing the fluctuations around these solutions, and we will discuss the different metric frames in which one can work. We will finish with a discussion about the tensions and charges related to the brane solutions. The geometrical aspects of these branes will be discussed in chapter 3.

1.4.1 Two-block solutions

Our approach will be to first consider a general class of solutions called two-block solutions to a generic D-dimensional supergravity action in the Einstein frame consisting of the kinetic terms for the metric, the dilaton[5] and a $p+1$-form gauge field

$$\mathcal{L}^{\mathrm{E}}_{(D,\,p)} = R \star \mathbb{1} - \frac{4}{D-2} \star \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{2} e^{a\phi} g_s^{2k} \star F_{(p+2)} \wedge F_{(p+2)}\,, \tag{1.50}$$

where the exponent k of g_s is the remnant of the dilaton coupling $e^{2k\Phi}$ to the field-strength in the string frame. Using (1.19) and (1.23), we find

$$k = \frac{a}{2} + \frac{2d}{D-2}\,. \tag{1.51}$$

Anticipating that this action will describe both an electric p-brane and a magnetic $\tilde{p}$-brane, we will introduce

$$\begin{cases} d = p+1\ : & \text{worldvolume dimension of the electric } p\text{-brane} \\ \tilde{d} = \tilde{p}+1\ : & \text{worldvolume dimension of the magnetic } \tilde{p}\text{-brane} \end{cases} \tag{1.52}$$

The equations of motion for the action (1.50) in the electric formulation have as solution an

$$\text{electric } p\text{-brane} = \begin{cases} \mathrm{d}s^2_{\mathrm{E}} & = & H^{\frac{-4\tilde{d}}{(D-2)\Delta}} \mathrm{d}x^2_{(d)} + H^{\frac{4d}{(D-2)\Delta}} \mathrm{d}y^2_{(\tilde{d}+2)}\,, \\ e^{\Phi} & = & g_s H^{\frac{(D-2)a}{4\Delta}}\,, \\ F_{(p+2)} & = & g_s^{-k} \sqrt{\frac{4}{\Delta}}\, \mathrm{d}^d x \wedge \mathrm{d}H^{-1}\,, \\ H(y) & = & 1 + \left(\frac{R}{y}\right)^{\tilde{d}}\,. \end{cases} \tag{1.53}$$

The parallel coordinates x^a ($a = 0, \ldots, p$) span the worldvolume of the brane, and the coordinates y^m ($m = p+1, \ldots, D-1$) are transverse to the brane. The parameter Δ of the solution is given by

$$\Delta = \frac{(D-2)a^2}{8} + \frac{2d\tilde{d}}{(D-2)}\,. \tag{1.54}$$

The function H is a harmonic function in the transverse dimensions, depending on the transverse coordinates y^i only

$$\Delta_{(\tilde{d}+2)} H = 0 \rightarrow \quad H(y) = 1 + \left(\frac{R}{y}\right)^{\tilde{d}}\,, \quad y^2 \equiv \sum_m (y^m)^2\,, \tag{1.55}$$

[5]In our conventions the scalar kinetic term has a nonstandard normalization in $D < 10$.

where R is an integration parameter depending on the charge of the brane, as we will see later.

Since the metric splits up into two diagonal pieces, and since the function H depends only on the transverse coordinates, such solutions are called brane solutions. Furthermore, the field strength is proportional to the worldvolume $\mathrm{d}^d x$ of the brane which corresponds to the fact that the brane couples to a gauge potential $A_{(p+1)}$.

Note that we do not consider solutions for which $\Delta = 0$: they correspond to the cases $a = d = 0$, $a = \tilde{d} = 0$, or $a \neq 0$ with $\tilde{d} < 0$. These cases correspond to a -1-brane or instanton, which is only a solution of the Wick-rotated action; the $D-3$-brane, for which the harmonic function is logarithmic; the $D-2$-brane, which falls under the class of domain-walls; and the $D-1$-brane, which is a spacetime-filling brane. We will not discuss these exotic branes, since they do not have a regular near-horizon limit, except for the domain-walls: they will be the subject of chapter 3.

We can transform the field strength to its magnetic dual

$$g_s^{2-k} F_{(\tilde{p}+2)} \equiv e^{a\phi} g_s^k \star F_{(p+2)} \,, \quad \tilde{p} \equiv D - p - 4. \tag{1.56}$$

This gives for the action

$$\mathcal{L}^{\mathrm{E}}_{(D,\,\tilde{p})} = R \star \mathbb{1} - \frac{4}{D-2} \star \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{2} e^{-a\phi} g_s^{2-4k} \star F_{(\tilde{p}+2)} \wedge F_{(\tilde{p}+2)} \,. \tag{1.57}$$

The magnetic dual formulation (1.57) supports a

$$\text{magnetic } \tilde{p}\text{-brane} = \begin{cases} \mathrm{d}s^2_{\mathrm{E}} & = & H^{\frac{-4d}{(D-2)\Delta}} \mathrm{d}x^2_{(\tilde{d})} + H^{\frac{4\tilde{d}}{(D-2)\Delta}} \mathrm{d}y^2_{(d+2)} \,, \\ e^{\Phi} & = & g_s H^{\frac{-(D-2)a}{4\Delta}} \,, \\ F_{(\tilde{p}+2)} & = & g_s^{k-2} \sqrt{\frac{4}{\Delta}}\, \mathrm{d}^{\tilde{d}} x \wedge \mathrm{d}H^{-1} \,, \\ H(y) & = & 1 + \left(\frac{R}{y}\right)^d \,, \end{cases} \tag{1.58}$$

where the function H is now harmonic on the $d+2$ transverse directions.

We will now give some explicit brane solutions of supergravities in ten and eleven dimensions. We will start with giving solutions of the Type IIA and Type IIB supergravity equations of motion following from the actions (1.44) and (1.41), after which we will give the eleven-dimensional two-block brane solutions.

Strings and five-branes

Since all $D = 10$ supergravities are low-energy limits of superstring theories, one expects that they should have string-like solutions. We can obtain such solutions if we truncate the actions (1.41) and (1.44) to only their first three terms. This gives the action (1.50) with

$D = 10$, $p = 1$ and $a = -1$. If we substitute this into the general two-block solutions (1.53), we obtain the fundamental string solution [55]

$$\mathrm{F1} = \begin{cases} \mathrm{d}s^2 &= H^{-\frac{3}{4}}\mathrm{d}x^2_{(2)} + H^{\frac{1}{4}}\mathrm{d}y^2_{(8)}\,, \\ e^{\Phi} &= g_s H^{-\frac{1}{2}}\,, \\ H_{(3)} &= \mathrm{d}^2x \wedge \mathrm{d}H^{-1}\,, \\ H(y) &= 1 + \left(\frac{R}{y}\right)^6 . \end{cases} \tag{1.59}$$

The magnetic dual of this is a magnetic five-brane, also known as the Neveu-Schwarz five-brane [56, 57].

$$\mathrm{NS5} = \begin{cases} \mathrm{d}s^2 &= H^{-\frac{1}{4}}\mathrm{d}x^2_{(6)} + H^{\frac{3}{4}}\mathrm{d}y^2_{(4)}\,, \\ e^{\Phi} &= g_s H^{\frac{1}{2}}\,, \\ \widetilde{H}_{(7)} &= \mathrm{d}^6x \wedge \mathrm{d}H^{-1}\,, \\ H(y) &= 1 + \left(\frac{R}{y}\right)^2 . \end{cases} \tag{1.60}$$

Dp-branes

The Type II string theories have RR-potentials $C_{(p)}$ in their massless spectrum, where $p = 1, 3$ for Type IIA and $p = 0, 2, 4$ for Type IIB. We notice that the kinetic terms of the gauge potentials have a factor of $a = \frac{3-p}{2}$ in their dilaton exponential. The branes coupling to these gauge potentials are called Dp-branes [58], with solutions given by

$$\mathrm{D}p = \begin{cases} \mathrm{d}s^2 &= H^{\frac{p-7}{8}}\mathrm{d}x^2_{(p+1)} + H^{\frac{p+1}{8}}\mathrm{d}y^2_{(D-p-1)}\,, \\ e^{\Phi} &= g_s H^{\frac{3-p}{2}}\,, \\ G_{(p+2)} &= g_s^{-1}\mathrm{d}^{p+1}x \wedge \mathrm{d}H^{-1}\,, \\ H(y) &= 1 + \left(\frac{R}{y}\right)^{D-p-3}\,, \end{cases} \tag{1.61}$$

where for $p > 3$ the magnetic field strength has been given[6]. In Type IIA one finds D0 and D2-branes as well as their magnetic duals, the D4 and D6-branes. Type IIB contains D1 and self-dual D3-branes as well as the magnetic D5-branes. The gauge field $C_{(0)}$ supports a D(-1)-brane called the D-instanton and a D7-brane.

It was a major breakthrough in string theory when it was realized that these Dp-branes could be identified as the hyperplanes on which open strings can end [24]. In chapter 2, we will look in more detail at the implications of the different aspects of D-branes.

Membranes and five-branes

The action of $D = 11$ supergravity given in (1.33) can be truncated to only its first two terms, giving the form (1.50) with $D = 11$ and $p = 2$. This should come as no surprise since we

[6] The D3-brane solution will be given in (2.1).

mentioned before that there exists an eleven-dimensional supermembrane. The solution of this M2-brane is given by [59]

$$\text{M2} = \begin{cases} \mathrm{d}s^2 &= H^{-\frac{2}{3}}\mathrm{d}x_{(3)}^2 + H^{\frac{1}{3}}\mathrm{d}y_{(8)}^2\,, \\ G_{(4)} &= \mathrm{d}^3x \wedge \mathrm{d}H^{-1}\,, \\ H(y) &= 1 + \left(\frac{R}{y}\right)^6\,. \end{cases} \tag{1.62}$$

The magnetic dual of the M2-brane is the M5-brane which has as solution [60]

$$\text{M5} = \begin{cases} \mathrm{d}s^2 &= H^{-\frac{1}{3}}\mathrm{d}x_{(6)}^2 + H^{\frac{2}{3}}\mathrm{d}y_{(5)}^2\,, \\ \widetilde{G}_{(7)} &= \mathrm{d}^6x \wedge \mathrm{d}H^{-1}\,, \\ H(y) &= 1 + \left(\frac{R}{y}\right)^3\,. \end{cases} \tag{1.63}$$

Dualities between branes

In the previous section, we have discussed dualities in string theory. In particular, we saw how S-duality mixes the different scalar fields and gauge fields of Type IIB supergravity and how T-duality changes the boundary condition of open strings. We also saw how $D = 11$ supergravity could be dimensionally reduced to Type IIA supergravity in ten dimensions. Since branes couple to the various gauge fields, and since Dp-branes are also the hyperplanes on which open strings can end, dualities will relate the various branes to one another. In addition to that, the eleven-dimensional branes reduce to solutions of Type IIA supergravity.

In figure 1.5, we have indicated the various branes in ten and eleven dimensions and the dualities between them. Dimensional reduction is given by arrows, T-duality by solid lines, and S-duality by dashed lines. Electric/magnetic duality and some T-dualities are not indicated. Also depicted are some branes which do not fall in our two-block solutions, such as the waves (W) and Kaluza-Klein monopoles (KK), but which are related to ordinary branes by duality or reduction. Another exotic brane, the D-instanton, is left out altogether. The 8-branes and 9-branes on the right of figure 1.5 are special branes: they correspond to domain-walls and spacetime-filling branes. For more details, see [61]. In chapters 3 and 4, we will discuss domain-walls in more detail.

1.4.2 Worldvolume actions

Branes are not just static solutions but dynamical objects, since they couple to gravity and to gauge potentials. The fluctuations around the static solutions are described by worldvolume actions, which are generalizations of the actions for the particle and the string given in (1.1) and (1.2). An additional remark is that some of the brane solutions we gave in (1.53) are singular, which means that the target space action (1.50) needs to be supplemented with a source term in these cases

$$S_{\text{total}} = \frac{1}{2\kappa^2}\int_{\mathcal{M}} \mathrm{d}^D x\, \mathcal{L}_{(D,\,p)} + \int_{\Sigma} \mathrm{d}^d\sigma\, \mathcal{L}_{p-\text{brane}}\,. \tag{1.64}$$

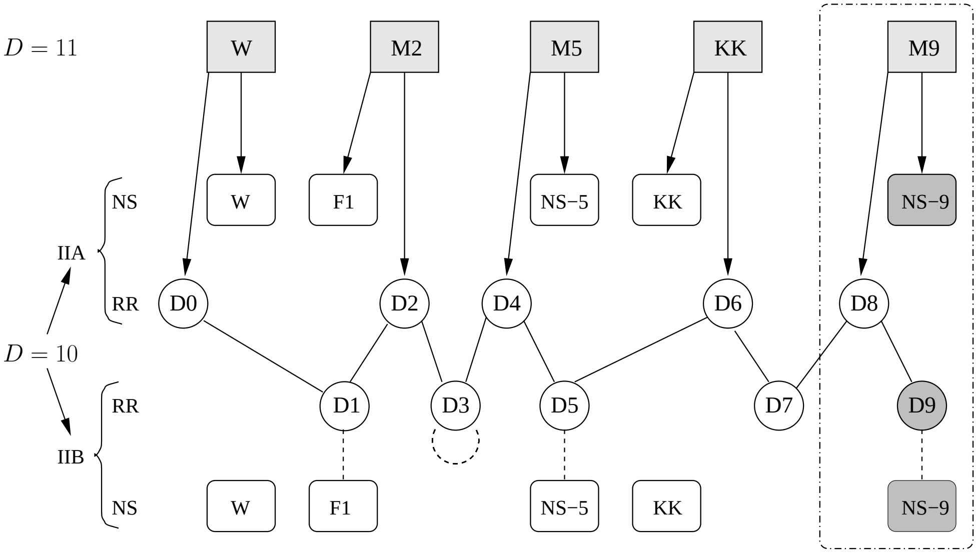

Figure 1.5: The various branes in $D = 10$ and $D = 11$ and their dualities.

In order to obtain the worldvolume action, we first assign coordinates $\sigma^a\,, a = 0 \ldots p$ to the brane. This gives as induced metric g_{ab} on the brane worldvolume Σ the pull-back of the metric $G_{\mu\nu}$ on the target space $\mathcal{M}$

$$g_{ab} = G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \,. \tag{1.65}$$

A general Ansatz for the action is then given by

$$\begin{aligned} \mathcal{L}_{p-\text{brane}} &= \mathcal{L}_{\text{kinetic}} + \mathcal{L}_{\text{WZ}} \\ &= -\tau_p \sqrt{|g|} + \mu_p C_{(p+1)} + \ldots \,. \end{aligned} \tag{1.66}$$

Here, τ_p is the energy-density, or tension, and μ_p is the charge-density. The last term is the generalization of the coupling of a particle to a gauge field called the Wess-Zumino action. The kinetic term modifies the condition on the harmonic function

$$\Delta_{(\tilde{d}+2)} H(y^m) = \kappa^2 \tau_p \, \delta(y^m) \,. \tag{1.67}$$

Generically, the embedding coordinates X^μ can be identified with the Goldstone modes corresponding to the translational symmetries that are broken by the brane. This means that the modes form a scalar multiplet; this is the case for the branes such as the M2-brane. However, other branes, such as the D3-brane or the M5-brane, can have a vector multiplet [62] or a tensor multiplet [63] as their massless modes. In chapter 2, we will look in more detail at the worldvolume action of the D3-brane.

1.4.3 Tensions and charges

For the general two-block solutions we gave before, we can actually calculate the tension and charge-density. We first define the deviation of the flat metric $h_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}\,, \tag{1.68}$$

and then we substitute the two-block solution (1.53) into the ADM-formula which expresses the tension as a spatial surface integral over a sphere surrounding the brane

$$\begin{aligned}\tau_p &= \frac{1}{2\kappa^2}\int_{S^{\tilde{d}+1}} \mathrm{d}^{\tilde{d}+1}\Sigma^m \left(\partial^n h_{mn} - \partial_m h^a{}_a\right) \\ &= \frac{4}{\Delta}\frac{\tilde{d}R^{\tilde{d}}\Omega_{(\tilde{d}+1)}}{2\kappa^2}\,,\end{aligned} \tag{1.69}$$

where the indices m, n run over the transverse coordinates, and where the index a runs over all spatial coordinates $1 \ldots D-1$. The surface area of the sphere is given by

$$\Omega_{(\tilde{d}+1)} = \frac{2\pi^{\frac{\tilde{d}+2}{2}}}{\Gamma\left(\frac{\tilde{d}+2}{2}\right)}\,. \tag{1.70}$$

The charge of the brane is given by a generalized Gauss-law

$$\begin{aligned}\mu_p &= \frac{1}{2\kappa^2}\int_{S^{\tilde{d}+1}} e^{a\phi} \star F_{(p+2)} \\ &= \sqrt{\frac{4}{\Delta}}\frac{\tilde{d}R^{\tilde{d}}\Omega_{(\tilde{d}+1)}}{2\kappa^2}\,.\end{aligned} \tag{1.71}$$

Comparing (1.69) with (1.71), we obtain what is known as the BPS-condition

$$\tau_p = \sqrt{\frac{4}{\Delta}}|\mu_p|\,. \tag{1.72}$$

The BPS-condition (1.72) is the limiting case of a more general condition called the Bogomol'nyi bound

$$\tau_p \geq \sqrt{\frac{4}{\Delta}}|\mu_p|\,. \tag{1.73}$$

This bound is valid for solutions of supersymmetric theories such as the branes of supergravities we discussed. In eleven dimensions, the superalgebra is generated by 32-component supercharges Q_α

$$\{Q_\alpha, Q_\beta\} = (\gamma^\mu \mathcal{C})_{\alpha\beta}\, P_\mu + \frac{1}{2!}\,(\gamma^{\mu\nu}\mathcal{C})_{\alpha\beta}\, Z_{\mu\nu} + \frac{1}{5!}\left(\gamma^{\mu\nu\lambda\rho\sigma}\mathcal{C}\right)_{\alpha\beta} Z_{\mu\nu\lambda\rho\sigma}\,, \tag{1.74}$$

where $\gamma^{\mu\cdots}$ are Dirac-matrices and where $\mathcal{C}$ is the charge conjugation matrix. For our conventions on gamma-matrices see appendix A. The operator P_μ represents the generator of translations, and the tensors $Z_{(2)}$ and $Z_{(5)}$ are called central charges, since they commute with the supersymmetry charges.

All the operators in (1.74) can be thought of as charges: they can be expressed as integrals over conserved currents which, using Noether's theorem, correspond to the symmetries of $D = 11$ supergravity such as supersymmetry, general coordinate invariance, and gauge invariance. Since the left-hand side of (1.74) is a positive operator, the sum of the momentum operator and the central charges are bounded from below. This bound is equivalent to (1.73), and it has to be satisfied by every solution of a supersymmetric theory.

A generic solution of a supersymmetric theory will itself not be completely supersymmetric, but when the BPS-bound (1.73) is exactly saturated, as it is in (1.72), then the corresponding solution preserves some of the supersymmetries of the underlying theory. In particular, for the branes in $D = 10$ and $D = 11$, we have $\Delta = 4$ which means that all these branes preserve half of the 32 spacetime supersymmetries.

So far, we have neglected the fermions in the supergravity action, since their structure is rather complicated in general. For the brane solutions that we have presented, only the bosonic fields have a non-vanishing value. In order for this to be consistent with supersymmetry, the fermions that are set to zero also need to have a vanishing supersymmetry variation.

For instance for the Dp-branes, the supersymmetry variations for the supersymmetric partners of the graviton and the dilaton, the gravitino ψ_μ and the dilatino λ, are given by [64]

$$\begin{aligned} \delta\psi_\mu &= \partial_\mu\epsilon - \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\epsilon + \frac{(-)^p}{8(p+2)!}e^\phi\,\gamma^{(p+2)}\cdot F_{(p+2)}\gamma_\mu\epsilon'_p\,, \\ \delta\lambda &= \partial\!\!\!/\phi\epsilon + \frac{3-p}{4(p+2)!}e^\phi\,\gamma^{(p+2)}\cdot F_{(p+2)}\epsilon'_p\,, \end{aligned} \tag{1.75}$$

where $\omega_\mu{}^{ab}$ denotes the spin-connection corresponding to the metric of the Dp-brane solution (1.61), and where we have used the notation of appendix A. The spinor ϵ'_p is equal to ϵ up to a chirality projection matrix, depending on the value of p [64]. Substituting the solutions for the metric, dilaton, and field-strength of the Dp-brane solution (1.61), we find the following condition on the spinor ϵ

$$\epsilon + \gamma_{01\ldots p}\epsilon'_p = 0\,. \tag{1.76}$$

Spinors for which the supersymmetry variation of the gravitino vanishes are called Killing spinors. The condition (1.76) implies that the Killing spinor ϵ is projected to only half of its original degrees of freedom, in other words, only half of the supersymmetries of the Type IIA and Type IIB supergravity actions are realized on the Dp-brane solutions.

There are also more general brane solutions, which can be thought of as intersections of the elementary branes and which have different values for Δ. If $\Delta = 4/n$ the corresponding brane preserves $32/2^n$ supersymmetries. The importance of the BPS-bound lies in the fact that it is independent of any coupling constant, meaning that the brane tension is stable for quantum corrections. However, recent developments show that there are also stable objects in string theory which are non-BPS [65].

The supersymmetry algebra (1.74) contains much more information than we have room to discuss here. As an example, we remark that the complete spectrum of branes can be deduced from it [66]. The spatial components of every central charge $Z_{(p)}$ correspond to a p-brane, giving the wave, the M2-brane and M5-brane in $D = 11$. One can also dualize the timelike component to a $Z_{(D-p)}$ charge[7] which should correspond to a $D - p$-brane. In $D = 11$ this gives a Kaluza-Klein monopole and a M9-brane. This explains the extra branes appearing in figure 1.5.

1.4.4 Metric frames

We have seen that the dilaton can be used to rescale the metric: this enabled us to go from the string frame, in which supergravities are derived, to the Einstein frame, where the curvature term has no dilatonic pre-factor. In this section, we will discuss two other metric frames that we will need later.

The dilaton dependence of the effective tension τ_p will in general depend on the frame being used. We define the sigma-model frame $g^\sigma_{\mu\nu}$ as the frame in which a particular brane tension is independent of the dilaton. We denote this intrinsic tension with T_p. The world-volume action of a BPS p-brane in the sigma-model frame couples to the induced metric g^σ_{ab} and is given by

$$\mathcal{L}^\sigma_{p-\mathrm{brane}} = -T_p \left(\sqrt{|g^\sigma|} + C_{(p+1)} \right) . \tag{1.77}$$

It scales homogeneously under the scale transformations

$$g^\sigma_{ab} \to \lambda^2 g^\sigma_{ab} \, , \quad C_{(p+1)} \to \lambda^{p+1} C_{(p+1)} \, . \tag{1.78}$$

This so-called trombone symmetry is a symmetry of the p-brane equations of motion [67], which implies that the combined system (1.64) of the target space action and the worldvolume action has to scale homogeneously as λ^{p+1}. For this to happen, we have to let the dilaton scale as

$$e^\phi \to \lambda^\alpha e^\phi \, , \quad \alpha = -\frac{2d\tilde{d}}{(D-2)a} \, . \tag{1.79}$$

Using the expressions (1.52) for the worldvolume dimensions of the electric and magnetic brane, we can relate the sigma-model frame to the Einstein frame by

$$g^\sigma_{\mu\nu} = e^{\omega_\sigma \phi} g^{\mathrm{E}}_{\mu\nu} \, , \quad \omega_\sigma = -\frac{a}{d} \, . \tag{1.80}$$

[7]Except for $p = 0$ which has no timelike components and for P_μ which has the Hamiltonian as its timelike component. Self-dual charges $Z_{(D/2)}$ also correspond to only a single brane.

The target space action (1.50) in the sigma-model frame (1.80) has an overall dilaton factor and a modified kinetic term for the dilaton

$$\mathcal{L}^{\sigma}_{(D,\,p)} = e^{\delta_\sigma \phi}\left(R \star \mathbb{1} + \gamma_\sigma \star \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{2} \star F_{(p+2)} \wedge F_{(p+2)}\right) . \tag{1.81}$$

which are given by

$$\delta_\sigma = \frac{(D-2)a}{2d}\,, \quad \gamma_\sigma = \frac{D-1}{D-2}\delta_\sigma^2 - \frac{4}{D-2}\,. \tag{1.82}$$

We have already seen an example of such a sigma-model frame: the string frame defined in (1.23) which is in agreement with (1.80) if we substitute the string solution parameters $a = -1$ and $p = 1$. The worldvolume action (1.10) has no dilaton dependence, and the target space action (1.20) has an overall $e^{-2\phi}$ factor. This guarantees that the combined action transforms homogeneously as λ^2 under

$$g^{\mathrm{S}}_{\mu\nu} \to \lambda^2 g^{\mathrm{S}}_{\mu\nu}\,, \quad B_{\mu\nu} \to \lambda^2 B_{\mu\nu}\,, \quad e^{\phi} \to \lambda^3 e^{\phi}\,. \tag{1.83}$$

Finally, there is another important frame – the dual frame. This frame is defined as the sigma-model frame of the magnetically dual brane. The dual frame is related to the Einstein frame by

$$g^{\mathrm{D}}_{\mu\nu} = e^{\omega_{\mathrm{D}}\phi} g^{\mathrm{E}}_{\mu\nu}\,, \quad \omega_{\mathrm{D}} = \frac{a}{\tilde{d}}\,. \tag{1.84}$$

In the dual frame, the magnetically dual brane's tension is independent of the dilaton, and the magnetic formulation of the target space action (1.57) has an overall dilaton factor and a modified kinetic term for the dilaton

$$\mathcal{L}^{\mathrm{D}}_{(D,\,\tilde{p})} = e^{\delta_{\mathrm{D}} \phi}\left(R \star \mathbb{1} + \gamma_{\mathrm{D}} \star \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{2} \star F_{(\tilde{p}+2)} \wedge F_{(\tilde{p}+2)}\right) , \tag{1.85}$$

which are given by

$$\delta_{\mathrm{D}} = -\frac{(D-2)a}{2\tilde{d}}\,, \quad \gamma_{\mathrm{D}} = \frac{D-1}{D-2}\delta_{\mathrm{D}}^2 - \frac{4}{D-2}\,. \tag{1.86}$$

In chapter 3, we will see that the near-horizon geometry of a brane takes on a simplified form in the dual frame.

We can now calculate the dilaton-dependence of a p-brane in the string frame from

$$\mathcal{L}^{\mathrm{S}}_{p-\mathrm{brane}} = -\tau_p\left(\sqrt{|g^{\mathrm{S}}|} + C_{(p+1)}\right) , \tag{1.87}$$

and using the connections between the various frames given in (1.23) and (1.80) we find

$$\tau_p = e^{-k\phi} T_p\,, \quad k = \frac{a}{2} + \frac{2d}{D-2}\,. \tag{1.88}$$

Comparing with (1.51), we see that the dilaton coupling for the p-brane worldvolume action in the string frame is equal to the dilaton coupling for the electric field-strength in the string frame.

Remembering that the exponential of the dilaton corresponds to the string coupling, we can compute the coupling dependence of the various brane tensions in the string frame. For the string, we take $a = -1$ and $p = 1$; for the Neveu-Schwarz five-brane, we take $a = 1$ and $p = 5$; and finally for the Dp-branes, we have $a = \frac{3-p}{2}$

$$\frac{\tau_{\rm F1}}{T_{\rm F1}} = 1\,, \qquad \frac{\tau_{{\rm D}p}}{T_{{\rm D}p}} = \frac{1}{g_s}\,, \qquad \frac{\tau_{\rm NS5}}{T_{\rm NS5}} = \frac{1}{g_s^2}\,. \tag{1.89}$$

So, we see that the NS5-brane and the Dp-branes are really solitonic objects: they become massive compared to fundamental strings for small values of the string couplings.

In the next chapters, we will need expressions for the tension of Dp-branes. In [58] this was calculated by comparing the worldsheet and the target space calculations of RR-field and gravitational interactions. This gave the same ℓ_s-dependence as derived from dimensional analysis, but the precise numerical factor can also be fixed by demanding that

$$\frac{\tau_{F1}}{\tau_{D1}} = g_s\,. \tag{1.90}$$

Using (1.3) and (1.89) gives as the answer

$$T_{{\rm D}p} = \frac{2\pi}{(2\pi\ell_s)^{p+1}}\,. \tag{1.91}$$

Chapter 2

The AdS/CFT correspondence

In chapter 1, we have seen that the search for a theory of Quantum Gravity has led to string theory, and that string theory has a rich structure and many exotic features, some of which do not appear in quantum field theories. Nevertheless, as we have argued in the introduction, there are many connections between theories of gravity and field theories. In the last few decades of the previous century, more, seemingly unrelated, conjectures and discoveries were made in this direction, which we will briefly summarize now.

In the nineteen seventies, it was noted by ’t Hooft [68] that gauge theories like QCD behave as string theories when the gauge group becomes large: the Feynman diagram series becomes dominated by planar diagrams. Such diagrams are in a one-to-one correspondence with two-dimensional surfaces, a feature characteristic of string theories. However, the precise description of such a string theory in terms of a worldsheet action was never found.

On grounds of entropy considerations, it was argued by ’t Hooft [69] and by Susskind [70] that any gravitational theory in a spacetime with length scales of the order of the Planck scale should be described by a quantum field theory living on the boundary of that spacetime. This idea is called the holographic principle; the gravitational theory is said to be holographically dual to the quantum field theory. Again, specific examples proved to be hard to find.

In many ways, the discovery of D-branes was a breakthrough for string theory. D-branes provide non-perturbative solutions to the theory. They also couple naturally to both open strings, which have gauge fields in their spectrum; and to closed strings, which have gravitons as vibration modes. We have displayed these two aspects in figure 2.1. This complementary nature of D-branes makes for a powerful framework for calculating black hole entropies [71, 72].

The connections between gauge theory and gravity described above led Maldacena to his conjecture [73] of the AdS/CFT correspondence. Inspired by the properties of D3-branes, he conjectured that Type IIB string theory on an Anti-de-Sitter (AdS) spacetime is holographically dual to a conformal field theory (CFT), namely $\mathcal{N} = 4$ supersymmetric, large N,

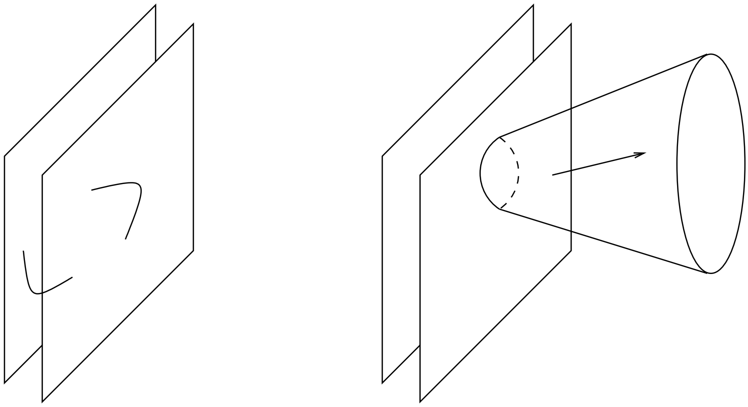

Figure 2.1: D-branes as open string boundary conditions and closed string sources.

$\mathrm{SU}(N)$ Yang-Mills theory in four dimensions. Specific proposals for quantitatively checking this conjecture were soon put forward [74, 75].

In this chapter, we will start with describing the basic arguments leading to the Maldacena conjecture. After a description of some properties of Anti-de-Sitter spacetime, we will indicate how one arrives at a scheme for computing correlation functions for both the gauge theory and gravity. We will finish with a summary of the enormous body of evidence that has accumulated over the years. For more details, we refer to the Physics Report [76] and to the more elementary reviews [77–80].

2.1 The D3-brane

In this section, we will look in more detail into aspects of the D3-brane. We will start with describing the interaction between a spacetime supergravity theory and a worldvolume gauge theory. We will then take two particular limits of the system and argue that these limits are equivalent. This is the reasoning that led Maldacena to his conjecture of the AdS/CFT correspondence.

2.1.1 Interacting theories

The D3-brane is a four-dimensional BPS-solution of Type IIB string theory preserving half of the 32 supersymmetries. The form of the solution is obtained by taking $p = 3$ in the expression for the general Dp-brane solution (1.61) and implementing the self-duality constraint

(1.29)

$$\text{D3-brane} = \begin{cases} \mathrm{d}s^2 &= H^{-\frac{1}{2}}\mathrm{d}x^2_{(4)} + H^{\frac{1}{2}}\left(\mathrm{d}y^2 + y^2\mathrm{d}\Omega^2_{(5)}\right)\,, \\ e^{\Phi} &= g_s\,, \\ G_{(5)} &= \mathrm{d}^4x \wedge \mathrm{d}H^{-1} + \star\,\mathrm{d}^4x \wedge \mathrm{d}H^{-1}\,, \\ H(r) &= 1 + \left(\frac{R}{y}\right)^4\,. \end{cases} \tag{2.1}$$

The constant R can be determined from (1.69)

$$R^4 = 4\pi g_s \ell_s^4\,. \tag{2.2}$$

The action describing (2.1) is given by the combined system

$$S_{\mathrm{D3}} = \frac{1}{2\kappa^2}\int_M \mathrm{d}^{10}x\,\mathcal{L}^{\mathrm{S}}_{(10,\,3)} + \int_\Sigma \mathrm{d}^4\sigma\,\mathcal{L}^{\mathrm{S}}_{\mathrm{D3}} + S_{\mathrm{int}}\,. \tag{2.3}$$

For completeness, we have also indicated the action S_{int} which describes the interactions between the worldvolume and target space actions; it contains the higher-derivative and higher order α' corrections to the worldvolume and target-space actions. Hence, one can view S_{int} as the action parameterizing all the string-theory corrections to the supergravity plus worldvolume action approximation.

The target space action is a truncation of the Type IIB supergravity pseudo-action in the string frame (1.27) to the metric $G_{\mu\nu}$, the dilaton ϕ, and the self-dual RR-field $G_{(5)}$

$$\mathcal{L}^{\mathrm{S}}_{(10,\,3)} = e^{-2\phi}\left(R\star\mathbb{1} - \frac{1}{2}\star\mathrm{d}\phi\wedge\mathrm{d}\phi\right) - \frac{1}{4}\star G_{(5)}\wedge G_{(5)}\,. \tag{2.4}$$

The worldvolume theory is given by a Dirac-Born-Infeld (DBI) action [22]

$$\mathcal{L}^{\mathrm{S}}_{\mathrm{D3}} = -T_{\mathrm{D3}}\,e^{-\phi}\sqrt{\det(g^{\mathrm{S}}_{ab} + 2\pi\alpha' F_{ab})} + \dots\,. \tag{2.5}$$

This DBI-theory can be seen as a non-linear generalization of electromagnetism: F_{ab} is the field-strength of a vector field living on the worldvolume of the D3-brane. The complete set of degrees of freedom describing the fluctuations around the static solution (2.1) also contains spinors and scalars; they can be seen as Goldstone modes corresponding to the broken ten-dimensional supersymmetry and translational symmetry. For simplicity, we have not included them in (2.5), and we have also omitted the Wess-Zumino terms and higher order corrections.

The solution describing N overlapping D3-branes is also given by (2.1), but where the constant R in this case is given by

$$R^4 = 4\pi g_s N \ell_s^4\,. \tag{2.6}$$

For the DBI-action (2.5), no such generalization to $N > 1$ is known, but instead one has to expand (2.5) as a series in α', and generalize each term individually[1]. The lowest order terms are given by supersymmetric $\mathrm{SU}(N)$ Yang-Mills theory.

[1]For recent progress in finding higher-order terms in this expansion, see for instance [81].

The target space action (2.4) in the Einstein frame is given by

$$\mathcal{L}^{\mathrm{E}}_{(10,\,3)} = R \star \mathbb{1} - \frac{1}{2} \star \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{4} \star G_{(5)} \wedge G_{(5)}\,. \tag{2.7}$$

Comparing this with (1.81) and (1.85), we see that the Einstein frame, the sigma-model frame, and the dual frame all coincide for the D3-brane . Since the dilaton vanishes for the D3-brane, the form of the D3-brane solution is the same in any frame. When we will study more general p-branes, we will see that the dual frame is the preferred frame for studying the geometry.

2.1.2 Decoupling limits

In the solution (2.1), we can take the near-horizon limit

$$\frac{y}{R} \to 0\,. \tag{2.8}$$

The metric then takes on the form

$$\begin{aligned} \mathrm{d}s^2 &= \left(\frac{y}{R}\right)^2 \mathrm{d}x^2_{(4)} + \left(\frac{R}{y}\right)^2 \mathrm{d}y^2 + R^2 \mathrm{d}\Omega^2_{(5)} && (2.9)\\ &\equiv AdS_5(R) \times S^5(R)\,. && (2.10) \end{aligned}$$

This geometry is a five-dimensional Anti-de-Sitter spacetime times a five-dimensional sphere. In section 2.2, we will be more detailed about the geometry of these spaces. On the other hand, if we look at the asymptotic geometry by taking

$$\frac{y}{R} \to \infty\,, \tag{2.11}$$

then the harmonic function becomes constant. The metric therefore describes Minkowski space $\mathbb{R}^{1,9}$. We have sketched the D3-brane geometry[2] in figure 2.2. In particular, the flat ten-dimensional asymptotic limit is separated from the near-horizon region by an infinitely long "throat".

Both geometries are believed to be exact vacua of string theory, which solve the full equations of motion of string theory to all orders in α'. Moreover, even though the complete D3-brane solution breaks half the supersymmetry, both the near-horizon limit (2.8) and the asymptotic limit (2.11) preserves all 32 supersymmetries of the Type IIB supergravity action [82–84]. This can be seen from taking either the near-horizon limit (2.8) or the asymptotic (2.11) directly in the supersymmetry variations (1.75): in both cases one finds that the supersymmetry variations vanish identically and that the projection condition (1.76) is not needed anymore. Hence, we can view the D3-brane as a string theory soliton that interpolates between two string theory vacua with unbroken supersymmetry.

[2] We have suppressed several extra dimensions, the figure only attempts to indicate the separation into two regions.

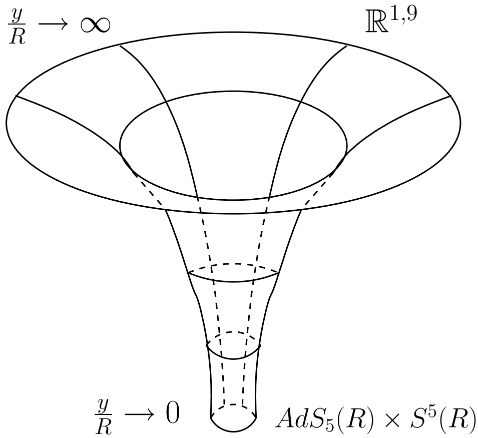

Figure 2.2: The interpolating D3-brane geometry.

The gravitational dynamics in the presence of a stack of N D3-branes separates into two regimes. Far away from the branes, the dynamics is given in terms of fluctuations around flat Minkowski spacetime, but near the branes, the dynamics is given in terms of fluctuations around an Anti-de-Sitter spacetime times sphere geometry. These two regions decouple, since a physical process of energy E_{emitted} near the brane is observed with an infinitely red-shifted energy E_{observed} far away from the brane

$$E_{\text{observed}} = \frac{\sqrt{g_{00}}|_{\frac{y}{R}\to 0}}{\sqrt{g_{00}}|_{\frac{y}{R}\to\infty}} E_{\text{emitted}} \,. \tag{2.12}$$

At the level of the actions, there is also a limit in which the near-brane and asymptotic regions decouple, namely the low-energy limit

$$\frac{E}{E_s} \to 0\,, \quad E_s = \frac{\hbar c}{\ell_s}\,. \tag{2.13}$$

Since the massive modes of strings have energies in the order of E_s, the low-energy limit is obtained by considering processes which involve only the massless modes, which is the supergravity approximation to superstring theory.

The effect of the low-energy limit on the action (2.3) is that the interaction part of the action S_{int} becomes negligible. Moreover, the DBI-action can be approximated by a $\mathrm{SU}(N)$ Yang-Mills theory

$$\mathcal{L}^{\mathrm{S}}_{\mathrm{D3}} = -\frac{1}{4g^2_{\mathrm{YM}}} \mathrm{Tr}\, F_{ab}F^{ab} + \dots\,. \tag{2.14}$$

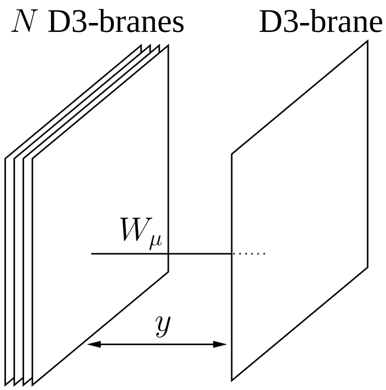

Figure 2.3: A stack of D3-branes probed by another D3-brane.

The Yang-Mills coupling constant can be obtained from the expression for the effective D3-brane tension and expanding the action (2.5)

$$g_{\mathrm{YM}}^2 = 2\pi g_s \,. \tag{2.15}$$

In the low-energy limit (2.13), the action (2.3) describes two decoupled systems. Far away from the branes, it is given by the fluctuations of Type IIB string theory around Minkowski spacetime, but the fluctuations are governed by a supersymmetric $\mathrm{SU}(N)$ Yang-Mills theory near the branes. By calculating the absorption cross-sections of scalar fields by the D3-branes, it was shown that the two systems indeed decouple in the low-energy limit [85, 86].

2.1.3 The Maldacena conjecture

From both the solution and the action perspective, the dynamics far away from the branes coincides and is given by Type IIB string theory in a Minkowski spacetime. However, near the branes, there are two different descriptions: a supersymmetric $\mathrm{SU}(N)$ gauge theory and Type IIB string theory around an Anti-de-Sitter spacetime times a sphere, respectively

To relate these two descriptions, it is useful to connect the near-horizon limit with the low-energy limit. Below the string scale E_s, a natural energy scale is given by the energy of an open string stretched between a stack of N D3-branes and a single D3-brane probe. We have indicated this setup in figure 2.3. Such a string behaves as a W-boson in the Yang-Mills theory on the D3-brane worldvolume, and its energy is given by

$$E_{\mathrm{W}} \equiv U = \frac{y}{\ell_s^2} \,. \tag{2.16}$$

If we keep g_s and N fixed and substitute the W-boson energy (2.16) into the low-energy limit (2.13), we obtain the near-horizon limit (2.8). We can then write the near-horizon metric

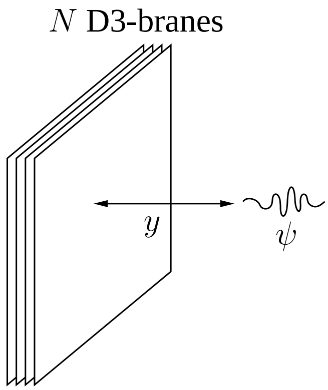

Figure 2.4: A stack of D3-brane probed by a supergravity field ψ.

(2.9) in terms of this energy scale

$$\frac{\mathrm{d}s_{\mathrm{E}}^2}{\ell_s^2} = \left(\frac{U}{(4\pi g s N)^{\frac{1}{4}}}\right)^2 \mathrm{d}x_{(4)}^2 + \left(\frac{(4\pi g_s N)^{\frac{1}{4}}}{U}\right)^2 \mathrm{d}U^2 + (4\pi g_s N)^{\frac{1}{2}} \mathrm{d}\Omega_{(5)}^2 \, . \tag{2.17}$$

Another natural energy scale can be obtained by considering a supergravity field[3] ψ probing a stack of N D3-branes. We have indicated this setup in figure 2.4. From an analysis of the wave-equation for ψ in the background described by the metric (2.17), it was shown in [87] that this field has the characteristic energy

$$E_\psi \equiv u = \frac{y}{R^2} \, . \tag{2.18}$$

Such a relation where the energy of a gauge theory is proportional to a distance scale in gravity is called a UV/IR-relation [87] since large energies (UV) in one theory map to low energies (IR) in the other, and vice versa.

This energy scale is a holographic energy: for a certain class of black holes, it can be shown [88] that this energy gives the same entropy as can be deduced from the holographic principle [69, 70]. The near-horizon metric (2.9) in the so-called holographic coordinates is

$$\frac{\mathrm{d}s_{\mathrm{E}}^2}{R^2} = u^2 \mathrm{d}x_{(4)}^2 + \frac{\mathrm{d}u}{u}^2 + \mathrm{d}\Omega_{(5)}^2 \, . \tag{2.19}$$

We have seen that the near-horizon limit of the D3-brane geometry corresponds to the low-energy limit in the action describing this D3-brane solution. Moreover, far away from the brane, the system describes Type IIB string theory around flat Minkowski spacetime.

[3]We consider an s-wave: the field ψ has no angular momentum related to the sphere S^5.

This led Maldacena to his conjecture [73] that also the near-brane descriptions should be equivalent. In other words, Type IIB string theory around $AdS_5 \times S^5$ should be equivalent to the $\mathrm{SU}(N)$ Yang-Mills theory on the four-dimensional worldvolume of the D3-branes.

This is not as absurd as it sounds. First of all, the precise worldvolume theory is $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. This is a special theory: its beta-function vanishes to all orders, meaning that it is a superconformal field theory [89]. The extra superconformal symmetries correspond to the supersymmetry enhancement found in the near-horizon limit. The superconformal group in four dimensions is $\mathrm{SU}(2, 2|4)$. Its bosonic subgroup is the conformal group $\mathrm{SO}(2, 4)$ times the $\mathrm{SU}(4)$ R-symmetry group. These groups are isomorphic to the $\mathrm{SO}(2, 4)$ isometry group of AdS_5 and to the $\mathrm{SO}(6)$ isometry group of S^5.

Many more kinematic properties of both theories are in a one-to-one correspondence [76]. We saw in chapter 1 that Type IIB string theory has an $\mathrm{S}\,\ell(2, \mathbb{Z})$-duality symmetry; $\mathcal{N} = 4$ Yang-Mills theory also has such a duality. It is known as Montonen-Olive duality [90] in which the θ-parameter of the gauge theory is mixed with the gauge coupling constant

$$\tau \equiv \frac{\theta}{2\pi} + \frac{2\pi\,\mathrm{i}}{g_{\mathrm{YM}}^2}\,. \tag{2.20}$$

Furthermore, we can define the ’t Hooft coupling constant

$$\lambda = 2g_{\mathrm{YM}}^2 N\,, \tag{2.21}$$

after which the size of the Anti-de-Sitter spacetime and the sphere becomes

$$\left(\frac{R}{\ell_s}\right)^4 = \lambda\,. \tag{2.22}$$

The string coupling constant can be expressed in terms of λ and N

$$g_s = \frac{\lambda}{4\pi N}\,. \tag{2.23}$$

The ratio of the two energy scales is also given by the ’t Hooft coupling constant

$$\frac{U}{u} = \lambda^{\frac{1}{2}}\,. \tag{2.24}$$

Many computations in field theory can only be done in perturbation theory, where the dimensionless coupling constant is small. Similarly, string theory on curved spacetimes such as an AdS times sphere geometry is rather complicated, especially at the quantum level [91]. There are three regimes of the parameters N and λ for which one side of the correspondence becomes computationally feasible, which we have displayed in table 2.1.

From the above, we see that gravity and gauge theory are valid in different regimes, and if this were the whole story, the conjectured duality would be hard to verify. However,

Regime	Gravity	Gauge theory
Perturbative gauge theory	$\frac{R}{\ell_s} \ll 1$	$\lambda \ll 1$
Classical string theory	$e^{\Phi} = g_s \ll 1$	$\frac{\lambda}{N} = g_{\rm YM}^2 \ll 1$
Supergravity	$\frac{R}{\ell_s} \gg 1$	$\lambda \gg 1$

Table 2.1: Regimes of the AdS/CFT correspondence.

since $\mathcal{N} = 4$ supersymmetric Yang-Mills theory is a conformal field theory, some correlation functions are independent of the coupling.

The Maldacena conjecture relates large N and large λ Yang-Mills theory in four dimensions to classical supergravity on a five-dimensional Anti-de-Sitter spacetime times a sphere. Such a correspondence is an example of both holography and of the string-like behavior of large N gauge theory, since the boundary of Anti-de-Sitter spacetime is Minkowski spacetime. In the following sections, we will make this more precise.

2.2 Anti-de-Sitter spacetime

In this section, we will discuss some elementary geometrical aspects of Anti-de-Sitter spacetime: we will derive several forms of its metric from an embedding equation, we will show that it solves Einstein's equations with a negative cosmological constant, and we will show that it has a projective boundary given by Minkowski spacetime. For more details, we refer to [92].

2.2.1 Embedding and metric

The $(d+1)$-dimensional Anti-Sitter spacetime[4] AdS_{d+1} may be realized as the hypersurface

$$AdS_{d+1}: \quad -X_0^2 - X_{d+1}^2 + X_1^2 + \ldots + X_d^2 = -L^2\,, \tag{2.25}$$

in flat $\mathbb{R}^{2,d}$, where L is a parameter with dimensions of length called the Anti-de-Sitter radius. The minus sign on the right-hand side of (2.25) is essential: it ensures that AdS_{d+1} is a spacetime of negative curvature. There are two closely related spacetimes: namely the $(d+1)$-dimensional version of the sphere S^{d+1}, and of the de Sitter spacetime dS_{d+1}; they have the embeddings

$$\begin{aligned} S^{d+1}: &\qquad X_0^2 + \ldots + X_{d+1}^2 &= \quad L^2\,, \\ dS_{d+1}: &\qquad -X_0^2 + X_1^2 + \ldots + X_{d+1}^2 &= \quad L^2\,. \end{aligned} \tag{2.26}$$

[4] A subscript rather than a superscript denoting the dimension is conventional.

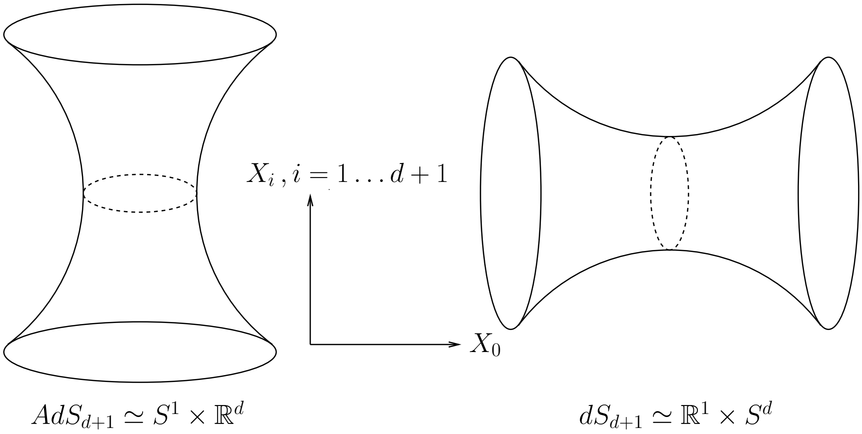

Figure 2.5: AdS_{d+1} and dS_{d+1} as hyperboloids in $\mathbb{R}^{2,d}$.

Both these spacetimes have positive curvatures, as can be inferred from the right-hand side of (2.26). The relative sign between the X_0^2 and L^2 terms determines if the spacetime has closed timelike curves. For Anti-de-Sitter spacetime, this is indeed the case: AdS_{d+1} has topology $S^1 \times \mathbb{R}^d$; de Sitter spacetime has no such closed timelike curves, and the topology is $\mathbb{R}^1 \times S^d$. In figure 2.5, we have schematically indicated the difference between the hyperbolic embeddings of AdS_{d+1} and dS_{d+1}.

The metric of AdS_{d+1} can be written as

$$\mathrm{d}s^2 = -\mathrm{d}X_0^2 - \mathrm{d}X_{d+1}^2 + \mathrm{d}X_1^2 + \ldots + \mathrm{d}X_d^2\,, \tag{2.27}$$

which is manifestly invariant under $\mathrm{SO}(2,d)$. Coordinate systems covering the entire hyperboloid (2.25) exactly once have a periodic timelike coordinate: in order to obtain a causal spacetime it is necessary to go to the universal covering space by unwrapping the timelike coordinate. Whenever we refer to AdS_{d+1} in the remainder of this thesis, we mean this universal covering space.

A convenient coordinate system which solves the embedding equation (2.25) is given by so-called horospherical coordinates

$$X^\mu = \left(\frac{U}{L}\right) x^\mu\,, \quad X^\pm = \frac{-1}{\sqrt{2}}\left(X^d \pm X^{d+1}\right)\,:\quad X^- = U\,,\quad X^+ = \frac{X^\mu X_\mu + L^2}{X^-}\,. \tag{2.28}$$

Calculating the differentials of (2.28), substituting them into the line element (2.27), and using the embedding equation (2.25), one obtains the induced metric

$$\mathrm{d}s^2 = \left(\frac{U}{L}\right)^2 \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \left(\frac{L}{U}\right)^2 \mathrm{d}U^2\,. \tag{2.29}$$

Different forms of the metric are used to emphasize different aspects of Anti-de-Sitter space: a form of the metric convenient for studying correlation functions is [75]

$$\mathrm{d}s^2 = \frac{\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}z^2}{z^2}\,, \quad z = \frac{L}{U}\,. \tag{2.30}$$

We will frequently use another form of the metric where we take

$$\mathrm{d}s^2 = e^{-2r/L}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}r^2\,, \quad e^{-r/L} = \frac{U}{L}\,. \tag{2.31}$$

This particular form of the metric is known as the Poincaré coordinate-system. In this case, we can let the radial coordinate U take on only positive values: this means that the Poincaré-coordinates cover only half of the hyperboloid (2.25).

2.2.2 Curvature and cosmological constant

The physical significance of Anti-de-Sitter spacetime lies in the fact that it is a vacuum solution to the gravitational field equations with a negative cosmological constant. In $d+1$ dimensions and in the absence of matter, the Einstein-Hilbert action with a cosmological constant Λ is given by

$$S_{d+1} = \frac{1}{2\kappa^2}\int \mathrm{d}^{d+1}x\,\sqrt{|g|}\,(R - 2\Lambda)\,. \tag{2.32}$$

The field equations that follow from the action (2.32) are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0\,. \tag{2.33}$$

Taking the trace of this equation yields

$$\Lambda = \frac{d-1}{2(d+1)}R\,, \tag{2.34}$$

from which we deduce that the curvature scalar has the same sign as the cosmological constant. Substituting (2.34) back into (2.33) yields

$$R_{\mu\nu} = \frac{2\Lambda}{d-1}g_{\mu\nu}\,. \tag{2.35}$$

Spaces for which the Ricci tensor is proportional to the metric are called Einstein spaces. The particular class of Einstein spaces called maximally symmetric spaces satisfies a stronger constraint

$$R_{\mu\nu\lambda\rho} = \frac{2\Lambda}{d(d-1)}\left(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}\right)\,. \tag{2.36}$$

Taking the trace of (2.36) shows that any maximally symmetric space is indeed an Einstein space.

We will now show that AdS_{d+1} is such a maximally symmetric space, and indeed a solution to (2.33). We start with writing a slightly more general Ansatz than (2.31)

$$\mathrm{d}s^2 = e^{2A(r)}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}r^2\,, \tag{2.37}$$

after which we rewrite this metric in terms of vielbeins

$$\mathrm{d}s^2 = g_{AB}\mathrm{d}x^A\mathrm{d}x^B\,, \quad g_{AB} = \eta_{ab}e_A{}^a e_B{}^b\,. \tag{2.38}$$

A convenient way of doing calculations in General Relativity is by working with differential forms: we introduce vielbein 1-forms

$$e^a = e_A{}^a\mathrm{d}x^A \rightarrow \left\{ \begin{array}{rcl} e^m &=& e^{A(r)}\mathrm{d}x^m\,, \quad m=0,\ldots d-1\,, \\ e^d &=& \mathrm{d}r\,, \end{array}\right. \tag{2.39}$$

and from the Cartan structure equations we obtain the spin-connection 1-form and the curvature 2-form

$$\begin{array}{rcl} \mathrm{d}e^a + \omega^a{}_b\wedge e^b &=& 0 \\ \mathrm{d}\omega^a{}_b + \omega^a{}_c\wedge\omega^c{}_b &=& R^a{}_b \end{array} \rightarrow \left\{ \begin{array}{rcl} \omega^m{}_d &=& A'(r)e^m\,, \\ R^m{}_n &=& -A'(r)^2 e^m\wedge e^n\,, \\ R^m{}_d &=& -\left[A''(r)+A'(r)^2\right] e^m\wedge e^d\,. \end{array}\right. \tag{2.40}$$

We can now read off the components of the Riemann tensor in the vielbein basis, and transform it into the standard form

$$\begin{array}{rcl} R^a{}_b &=& \frac{1}{2}R^a{}_{bcd}e^c\wedge e^d \\ R_{ABCD} &=& e_{Aa}R^a{}_{bcd}e_B{}^b e_C{}^c e_D{}^d \end{array} \rightarrow \left\{ \begin{array}{rcl} R_{\mu\nu\mu\nu} &=& -A'(r)^2 g_{\mu\mu}g_{\nu\nu}\,, \\ R_{\mu r\mu r} &=& -\left[A''(r)+A'(r)^2\right] g_{\mu\mu}g_{rr}\,. \end{array}\right. \tag{2.41}$$

For future reference, we also give the Ricci tensor and the Ricci scalar

$$\begin{array}{rcl} R_{AB} &=& g^{CD}R_{CADB} \\ R &=& g^{AB}R_{AB} \end{array} \rightarrow \left\{ \begin{array}{rcl} R_{\mu\mu} &=& -\left[A''(r)+dA'(r)^2\right] g_{\mu\mu}\,, \\ R_{rr} &=& -d\left[A''(r)+A'(r)^2\right] g_{rr}\,, \\ R &=& -2dA''(r) - d(d+1)A'(r)^2\,. \end{array}\right. \tag{2.42}$$

Comparing (2.31) with our Ansatz (2.37) we see that if we take

$$A(r) = \pm\frac{r}{L} \rightarrow A'(r) = \pm\frac{1}{L}\,, \quad A''(r) = 0\,, \tag{2.43}$$

then we regain Anti-de-Sitter spacetime in Poincaré coordinates. The choice of sign is arbitrary: comparing with the Poincaré-coordinates (2.31), we choose the minus sign here. Substituting (2.43) into the curvature expressions (2.41) and (2.42), and comparing this with (2.36), we see that AdS_{d+1} is indeed a maximally symmetric space with a negative cosmological constant given by

$$\Lambda = -\frac{d(d-1)}{2L^2}\,. \tag{2.44}$$

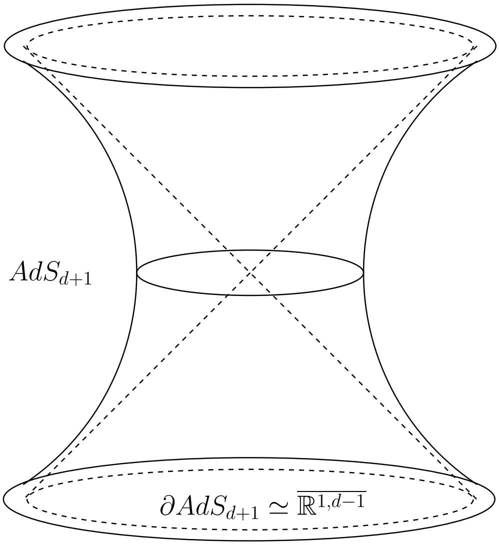

Figure 2.6: The projective boundary of Anti-de-Sitter spacetime.

2.2.3 Boundary and conformal structure

Rescaling the embedding coordinates of AdS_{d+1} by a large factor

$$X_i \to X_i' = tX_i\,, \quad t \gg 1\,, \tag{2.45}$$

changes the hyperbolic embedding equation (2.25) to

$$-X_0^2 - X_{d+1}^2 + \left(X_0^2 + \ldots + X_d^2\right) = 0\,. \tag{2.46}$$

The scaled embedding equation (2.46) describes a cone lying inside the Anti-de-Sitter space. Since all coordinates have been scaled to large values, any additional scalings have no further effect; this is expressed by the equivalence relation

$$X_i \simeq \lambda X_i\,. \tag{2.47}$$

The cone-embedding (2.46) modulo the scale equivalence relation (2.47) describes the two projective boundaries[5] of AdS_{d+1}, which are topologically equivalent to conformally compactified Minkowski space $\overline{\mathbb{R}^{1,d-1}}$. Minkowski space is conformally compactified by adding a point at infinity, analogously to how the Riemann sphere S^2 is obtained from the complex plane $\mathbb{C}$. We have indicated the projective boundary of AdS_{d+1} in figure 2.6.

[5]In order to consider only a single boundary, one should also mod out by a $\mathbb{Z}_2$-symmetry [75].

The $\mathrm{SO}(2,d)$ isometry group of AdS_{d+1} is linearly realized on the coordinates of the embedding space as

$$\tfrac{1}{2}(d+2)(d+1) \quad \text{rotations} \quad : \quad X^i \to \Lambda^i{}_j X^j \,. \tag{2.48}$$

On the projective boundary $\overline{\mathbb{R}^{1,d-1}}$, the isometry group splits up into

$$\begin{array}{cll} \tfrac{1}{2}d(d-1) & \text{Lorentz transformations} & : \quad x^\mu \to \Lambda^\mu{}_\nu x^\nu \,, \\ d & \text{translations} & : \quad x^\mu \to x^\mu + a^\mu \,, \\ 1 & \text{dilatation} & : \quad x^\mu \to \lambda x^\mu \,, \\ d & \text{conformal transformations} & : \quad \frac{x^\mu}{x^2} \to \frac{x^\mu}{x^2} + k^\mu \,, \end{array} \tag{2.49}$$

which means that the isometry group of AdS_{d+1} acts as the conformal group on its boundary $\overline{\mathbb{R}^{1,d-1}}$. One therefore expects that if a $\mathrm{SO}(2,d)$ invariant gravitational theory in AdS_{d+1} is to have any holographically dual description at all, then this dual theory should be given in terms of a conformal field theory on the projective boundary $\overline{\mathbb{R}^{1,d-1}}$. Moreover, since the boundary corresponds to large radial coordinate, or large energy U in the language of the previous section, the holographic dual is a conformal field theory in the UV limit.

Instead of giving the precise connection between the coordinate transformations (2.48) and (2.49), we will give the connection between the generators of the AdS_{d+1} isometry group and the d-dimensional conformal group

$$M_{ij} = \begin{pmatrix} M_{\mu\nu} & \frac{1}{4}\left(P_\mu - K_\mu\right) & \frac{1}{4}\left(P_\mu + K_\mu\right) \\ -\frac{1}{4}\left(P_\mu - K_\mu\right) & 0 & -\frac{1}{2}D \\ -\frac{1}{4}\left(P_\mu + K_\mu\right) & \frac{1}{2}D & 0 \end{pmatrix} \,. \tag{2.50}$$

The commutation relations of M_{ij} are given by

$$\left[M_{ij}, M^{kl}\right] = -2\delta_{[i}{}^{[k} M_{j]}{}^{l]} \,. \tag{2.51}$$

In chapter 5, we will come back to the algebraic structure of the conformal group and its supersymmetric extension. In particular, we will give the the commutation relations that result when one substitutes (2.50) into (2.51).

2.3 Conformal field theory

In this section, we will make more precise how the AdS/CFT correspondence is realized. We will analyze a toy model example and show that it has many of the qualitative features of more realistic models. After that, we will describe the various approximations that have to be made in practice. We will finish with a brief summary of the evidence in favor of the AdS/CFT correspondence.

2.3.1 A toy model example

In order to clarify how a conformal field theory can give a holographically dual description to a gravitational theory, we will consider a toy model example of a $d+1$-dimensional scalar field and its potential coupled to gravity

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^{d+1}x\,\sqrt{|g|}\left(R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right) . \tag{2.52}$$

The equations of motion for the scalar and the metric in this model are given by

$$\begin{aligned} \tfrac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi\right) &= \tfrac{\partial V}{\partial\phi} , \\ R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} &= \tfrac{1}{2}\partial_\mu\phi\partial_\nu\phi - \left(\tfrac{1}{4}(\partial\phi)^2 + \tfrac{1}{2}V(\phi)\right)g_{\mu\nu} . \end{aligned} \tag{2.53}$$

For generic values of the scalar field, the equations of motion are complicated to solve: some simplifications occur if we look for perturbations around certain critical points of the potential

$$\varphi = \phi - \phi_c\,, \quad \frac{\partial V}{\partial\phi}|_{\phi=\phi_c} = 0\,, \quad V(\phi_c) < 0\,, \tag{2.54}$$

and introduce new parameters Λ and M^2

$$\Lambda = \frac{1}{2}V(\phi_c)\,, \quad M^2 = \frac{\partial^2 V}{\partial\phi^2}|_{\phi=\phi_c} . \tag{2.55}$$

The equations of motion then take on the form

$$\frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}g^{\mu\nu}\partial_\nu\varphi\right) = M^2\varphi\,, \tag{2.56}$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0\,. \tag{2.57}$$

So, we see that each critical point of the scalar potential in (2.54) corresponds to an Anti-de-Sitter spacetime with a negative cosmological constant given in terms of the value of the potential; the masses of the fluctuations around such critical points are given in terms of the Hessian matrix of the potential[6]. We can associate a length scale L to the Anti-de-Sitter spacetime and express the mass in units of this length scale

$$\Lambda = -\frac{d(d-1)}{2L^2}\,, \quad \frac{m}{L} = M\,. \tag{2.58}$$

Since we expect that the dynamics of a gravitational theory in Anti-de-Sitter spacetime is described by a conformal field theory on its boundary, we are particularly interested in the

[6] A slightly negative m^2 does not imply instability, as long as the bound $m^2 \geq -\frac{d^2}{4}$ is satisfied [93] .

boundary conditions of the scalar field. If we take the metric of the form (2.30) then scaling arguments determine the behavior of solutions of the wave equation (2.56) near the boundary $z = 0$

$$\varphi(\vec{x}, z) = z^{d-\Delta^+}\varphi_+(\vec{x}) + z^{d-\Delta^-}\varphi_-(\vec{x}), \quad \Delta^\pm = \frac{d}{2} \pm \frac{\sqrt{d^2+4m^2}}{2}. \tag{2.59}$$

In general, only one of the two solutions will give a finite-energy solution[7]: in particular, if the mass does not saturate the Breitenlohner-Freedman bound [93], we can only select Δ^+

$$m^2 > 1 - \frac{d^2}{4}, \quad \Delta \equiv \Delta^+, \quad \varphi_0(\vec{x}) \equiv \varphi_+(\vec{x}). \tag{2.60}$$

The bound (2.60) corresponds to the unitarity bound on scaling dimensions of operators in a conformal field theory

$$\Delta \geq \frac{d-2}{2}. \tag{2.61}$$

Standard Green's functions techniques then determine the complete solution to the equations of motion (2.56) in terms of the bulk-to-boundary propagator $K_\Delta(z, \vec{x}, \vec{x}')$

$$\varphi(\vec{x}, z) = \int \mathrm{d}^d x' K_\Delta(z, \vec{x}, \vec{x}')\varphi_0(\vec{x}'), \quad K_\Delta(z, \vec{x}, \vec{x}') \simeq \frac{z^\Delta}{(z^2 + |\vec{x} - \vec{x}'|^2)^\Delta}. \tag{2.62}$$

Substituting the solution (2.62) into the Euclidean version of the action (2.52) and performing the z-integral yields the on-shell action up to a constant factor

$$S[\varphi_0] \simeq \int \mathrm{d}\vec{x} \int \mathrm{d}\vec{x}' \frac{\varphi_0(\vec{x})\varphi_0(\vec{x}')}{|\vec{x} - \vec{x}'|^{2\Delta}}. \tag{2.63}$$

If we now view this action as a functional of the boundary data and differentiate its exponential with respect to the scalar fields

$$\frac{\delta^2}{\delta\varphi_0(\vec{x})\delta\varphi_0(\vec{x}')} e^{-S[\varphi_0]} \simeq \frac{1}{|\vec{x} - \vec{x}'|^{2\Delta}} \tag{2.64}$$

$$\equiv \langle \mathcal{O}_\Delta(\vec{x})\mathcal{O}_\Delta(\vec{x}')\rangle_{\mathrm{CFT}}, \tag{2.65}$$

then we observe that we have obtained the two-point correlation function for a conformal field theory operator $\mathcal{O}_\Delta$ of scaling dimension Δ. Analogous formulae exist for the higher-point correlation functions. One can view the scalar field $\phi_0(\vec{x})$ as a source term or generalized coupling to the operator $\mathcal{O}_\Delta$

$$e^{-S[\varphi_0]} = \left\langle e^{\int \mathrm{d}^d\vec{x}\varphi_0(\vec{x})\mathcal{O}_\Delta(\vec{x})} \right\rangle_{\mathrm{CFT}}. \tag{2.66}$$

[7] For $-\frac{d^2}{4} \leq m^2 \leq 1 - \frac{d^2}{4}$, both Δ^+ and Δ^- are admissible [94].

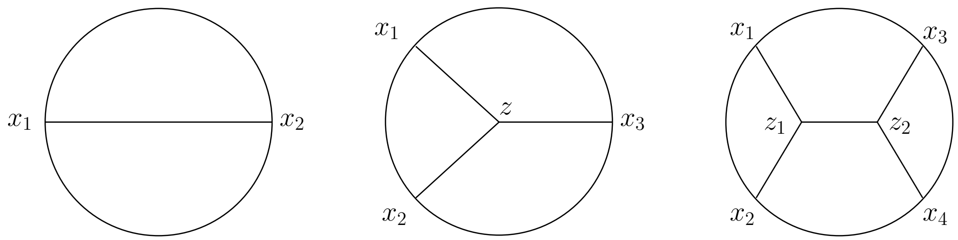

Figure 2.7: Witten diagrams of 2-, 3- and 4-point correlation functions.

The above equation can be made more precise: in particular, the precise regularizations which need to be performed on both sides of (2.66) and the connection with the conformal anomaly were discussed in [95].

There is also a diagrammatic way of displaying the equations involved, as we have indicated in figure 2.7. The points in the Witten diagram labeled x_i are positioned at the boundary of AdS, whereas the points denoted by z_i are located in the bulk of the Anti-de-Sitter spacetime. To each vertex one assigns a propagator for the corresponding field, and one integrates the bulk coordinates z_i over the entire Anti-de-Sitter spacetime, which can be quite complicated in practice [96].

2.3.2 Approximations of the correspondence

In its strongest form, the AdS/CFT correspondence relates the partition function of a gravitational theory on a manifold $\mathcal{M}$ to the partition function for a conformal field theory on the boundary $\partial\mathcal{M}$

$$\mathcal{Z}_{\text{gravity}}(\mathcal{M}) = \mathcal{Z}_{\text{CFT}}(\partial\mathcal{M})\,, \tag{2.67}$$

the canonical example being the equivalence of IIB string theory on $AdS_5 \times S^5$ to $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in $3+1$-dimensions

$$\mathcal{Z}_{\text{IIB}}(AdS_5 \times S^5) = \mathcal{Z}_{\text{SYM}}(\overline{\mathbb{R}^{1,3}})\,. \tag{2.68}$$

Since string theory on Anti-de-Sitter spaces is not well enough understood even at the classical level, a weaker but more manageable form of the correspondence is to approximate the full quantum string theory on $AdS_5 \times S^5$ in (2.68) with its effective classical supergravity action which translates in the field theory to the regime of large gauge group and large 't Hooft coupling

$$e^{-S_{\text{IIB}}(AdS_5 \times S^5)} = \mathcal{Z}_{\text{SYM}}(\overline{\mathbb{R}^{1,3}})\,, \quad N, \lambda \gg 1\,. \tag{2.69}$$

Since the classical supergravity computations have to be compared with strong coupling results for the field theory, and since only for conformal field theory such calculations can be performed, the AdS/CFT correspondence has not yet been applied to theories without superconformal symmetry such as pure QCD in four dimensions.

Combining the original ten-dimensional equations of motion [39] with the complete Kaluza-Klein mass-spectrum of Type IIB supergravity on S^5 [97, 98] would in principle give the complete dynamics for fluctuations around the $AdS_5 \times S^5$ background. Since the expansion in spherical harmonics on S^5 is quite complicated, one would like to eliminate the higher Kaluza-Klein modes.

Since the radius of the sphere is proportional to the Anti-de-Sitter radius, a normal Kaluza-Klein reduction (i.e. taking the radius to zero) will not solve this problem: instead one needs to make a consistent truncation to the zero-modes. However, finding the correct reduction Ansatz is already complicated at the linearized level, and the non-linear interactions are even more daunting: they have been worked out in only a few sectors of the theory [99].

On the other hand, there is a known complete non-linear five-dimensional supergravity theory, the $\mathrm{SO}(6)$ gauged $\mathcal{N} = 8$ theory of [100, 101], which has the same graviton multiplet as Type IIB supergravity and is invariant under the same superalgebra. It is widely believed, but nevertheless still unproven, that this $D = 5$, $\mathcal{N} = 8$ theory is a consistent truncation of Type IIB theory on $AdS_5 \times S^5$, meaning that any classical solution of the five-dimensional theory can be lifted to ten dimensions.

For practical calculations, the form of the AdS/CFT correspondence is therefore

$$e^{-S_{D=5}^{\mathcal{N}=8}(AdS_5)} = \mathcal{Z}_{\mathrm{SYM}}(\overline{\mathbb{R}^{1,3}}) , \quad N, \lambda \gg 1 , \tag{2.70}$$

which is similar to the relation (2.66) for our toy model example (2.52) if we keep in mind that every fluctuating field on the Anti-de-Sitter side appears as a term in the Lagrangian on the conformal field theory side as a coupling to some composite conformal operator.

2.3.3 Evidence for the AdS/CFT correspondence

We can summarize the kinematic evidence for the AdS/CFT correspondence with the dictionary given in table 2.2.

There is also a large body of dynamical evidence in favor of the AdS/CFT correspondence. Soon after Maldacena's conjecture [73] and the concrete proposals [74, 75] for calculating amplitudes, a whole class of correlators was calculated [102] from the supergravity side. In most cases, perfect agreement with the known field theory results was found. In other cases, computations from the supergravity point of view yielded new and unexpected non-renormalization theorems for certain classes of field theory correlators [103, 104].

Many other calculations have been performed: instanton corrections to perturbative results [105, 106], relations between Wilson loops in gauge theory [107] and minimal surfaces in string theory [108], and thermal properties of black holes [109] in relation with the field theory free energy [110]. In chapter 3, we will discuss some of the generalizations of the AdS/CFT correspondence.

Concept	Gravity	Gauge theory
UV/IR	length y	energy $U = \frac{y}{\ell_s^2}$
Decoupling	near-horizon	low-energy
Regime	curvature radius $\frac{R}{\ell_s}$	’t Hooft coupling λ
Coupling constant	g_s	g_{YM}^2
Stringy corrections	$\mathcal{O}(\alpha')$	$\mathcal{O}(\frac{1}{\lambda})$
Quantum corrections	$\mathcal{O}(g_s)$	$\mathcal{O}(\frac{1}{N})$
Isometry/symmetry	$\mathrm{SO}(2,4) \times \mathrm{SO}(6)$	$\mathrm{SU}(2,2\vert 4)$
$\mathrm{S}\,\ell(2,\mathbb{Z})$-duality	$\tau = C_{(0)} + \frac{\mathrm{i}}{g_s}$	$\tau = \frac{\theta}{2\pi} + \frac{2\pi\,\mathrm{i}}{g_{\mathrm{YM}}^2}$
Scalar	field $\varphi(\vec{x}, z)$	coupling $\varphi_0(\vec{x})$
Dimension	mass m	scaling Δ
Bound	$m^2 \geq 1 - \frac{d^2}{4}$	$\Delta \geq \frac{(d-2)}{2}$

Table 2.2: A gravity/gauge theory dictionary.

We conclude by remarking that it is fair to say that the AdS/CFT correspondence is no longer a mere conjecture, but that it is a firmly established gravity/gauge theory correspondence. For more details, we refer to the review [76].

Chapter 3

The DW/QFT correspondence

It has been known for some time that the geometry of a large class of p-branes interpolates between the near-horizon geometry $AdS_{p+2} \times S^{D-p-2}$ and the asymptotic geometry $\mathbb{R}^{1,D-1}$ [82]. This interpolating structure becomes apparent when one studies the geometry in the sigma-model frame of the magnetically dual brane – the so-called dual frame [83].

Soon after the discovery of the AdS/CFT correspondence, the connection between the geometry and worldvolume theory was therefore also investigated for other Dp-branes [111]. In contrast to the D3-brane, Dp-branes generically have a non-vanishing dilaton; this breaks the conformal invariance possessed by the AdS near-horizon geometry. The worldvolume theory of generic Dp-branes is also not a conformal field theory, but rather a more general quantum field theory (QFT).

It was shown in [112] that p-brane solutions having an AdS_{p+2} near-horizon geometry and a non-vanishing dilaton fall in a specific class of domain-walls, which we will denote by DW_{p+2}. Anti-de-Sitter spacetime then becomes the special case that the dilaton vanishes. Domain-wall spacetimes naturally occur in massive supergravities; theories with a mass parameter or a cosmological constant [113].

The developments above inspired the authors of [114] to conjecture a DW/QFT correspondence for ten-dimensional Dp-branes. They conjectured that gravity on a domain-wall spacetime should be holographically dual to a quantum field theory on a slice of that spacetime. We have generalized this correspondence for general p-branes in arbitrary dimensions [15]. The mapping between classical supergravity and a strongly coupled field theory persists for more general Dp-branes. However, the lack of conformal invariance forms an obstruction for making quantitative checks on the DW/QFT correspondence in this case.

Instead of considering the holographic duals of more general brane solutions, one can also study the gravity duals of more general quantum field theories. In particular, relevant deformations of conformal field theories turn out to be of interest. Such deformations induce a renormalization group (RG) flow in the space of coupling constants. The gravity duals of

RG-flows between fixed points of the field theory beta-function, can be seen as domain-walls interpolating between Anti-de-Sitter vacua of the scalar potential of a supergravity theory.

We will start this chapter with describing the near-horizon geometry of a class of two-block p-brane solutions. After that, we will present a class of domain-wall solutions that contain Anti-de-Sitter spacetime as a special case. We will indicate how a sphere reduction of the supergravity action supporting the original p-brane gives rise to a domain-wall solution of a gauged supergravity. These results have been published in [15], and an abridged version appeared as a proceedings in [115].

We will finish this chapter with indicating how RG-flows of conformal field theories are related to supergravity solutions that interpolate between different Anti-de-Sitter vacua; this will provide the transition to chapter 4, where we will describe brane world scenarios.

3.1 Near-horizon geometries of p-branes

In this section we will look in more detail into the geometrical properties of the class of two-block p-brane solutions of section 1.4.1.

3.1.1 Two-block solutions

Our starting point is the magnetic formulation of the generic supergravity action in the Einstein frame in (1.57)

$$\mathcal{L}^{\mathrm{E}}_{(D,\tilde{p})} = R\star\mathbb{1} - \frac{4}{D-2}\star\mathrm{d}\phi\wedge\mathrm{d}\phi - \frac{1}{2}e^{-a\phi}g_s^{2-4k}\star F_{(\tilde{p}+2)}\wedge F_{(\tilde{p}+2)}\,. \tag{3.1}$$

We recall that this action supports an electric p-brane solution given by (1.53)

$$\text{electric } p\text{-brane} = \begin{cases} \mathrm{d}s^2_{\mathrm{E}} &= H^{\frac{-4\tilde{d}}{(D-2)\Delta}}\mathrm{d}x^2_{(d)} + H^{\frac{4d}{(D-2)\Delta}}\mathrm{d}y^2_{(\tilde{d}+2)}\,,\\ e^{\Phi} &= g_s H^{\frac{(D-2)a}{4\Delta}}\,,\\ F_{(p+2)} &= g_s^{-k}\sqrt{\frac{4}{\Delta}}\,\mathrm{d}^d x\wedge\mathrm{d}H^{-1}\,,\\ H(y) &= 1+\left(\frac{R}{y}\right)^{\tilde{d}}\,. \end{cases} \tag{3.2}$$

The parameter Δ is defined as

$$\Delta = \frac{(D-2)a^2}{8} + \frac{2d\tilde{d}}{(D-2)}\,. \tag{3.3}$$

We also recall that we restrict the worldvolume dimension of the dual brane to be strictly positive. The case $\tilde{d}=0$ corresponds to $(D-3)$-branes, which have a logarithmic harmonic function. The case $\tilde{d}=-2$ corresponds to spacetime filling branes. We will not consider such branes in our subsequent analysis. The branes with $\tilde{d} = -1$ are $(D-2)$-branes; their asymptotic geometries are not given by flat spacetime. They can be viewed as domain-walls, and we will discuss them in the next section.

For the D3-brane, the Einstein frame coincides with both the sigma-model frame and the dual frame, but for more general branes this is not the case. The dual frame is the most useful for our purposes. Recall that after the rescaling (1.84)

$$g^{\mathrm{D}}_{\mu\nu} = e^{\omega_{\mathrm{D}}\phi} g^{\mathrm{E}}_{\mu\nu}\,, \quad \omega_{\mathrm{D}} = \frac{a}{\tilde{d}}\,, \tag{3.4}$$

the action will simplify to the form (1.85)

$$\mathcal{L}^{\mathrm{D}}_{(D,\,\tilde{p})} = e^{\delta_{\mathrm{D}}\phi}\left(R \star \mathbb{1} + \gamma_{\mathrm{D}} \star \mathrm{d}\phi \wedge \mathrm{d}\phi - \frac{1}{2} \star F_{(\tilde{p}+2)} \wedge F_{(\tilde{p}+2)}\right)\,. \tag{3.5}$$

The overall dilaton factor and the modified kinetic term are given by (1.86)

$$\delta_{\mathrm{D}} = -\frac{(D-2)a}{2\tilde{d}}\,, \quad \gamma_{\mathrm{D}} = \frac{D-1}{D-2}\delta^2_{\mathrm{D}} - \frac{4}{D-2}\,. \tag{3.6}$$

In this dual frame, the metric is given by

$$\mathrm{d}s^2_{\mathrm{D}} = H^{\left(\frac{2}{\tilde{d}} - \frac{4}{\Delta}\right)} \mathrm{d}x^2_{(d)} + H^{\frac{2}{\tilde{d}}}\left(\mathrm{d}y^2 + y^2 \mathrm{d}\Omega^2_{(\tilde{d}+1)}\right)\,. \tag{3.7}$$

3.1.2 The near-horizon limit

If we now take the near-horizon limit

$$\frac{y}{R} \to 0\,, \tag{3.8}$$

then we find for the near-horizon geometry and dilaton dependence of the electric p-brane in the dual frame

$$\mathrm{d}s^2_{\mathrm{D}} = \left(\frac{R}{y}\right)^{\left(2-\frac{4\tilde{d}}{\Delta}\right)} \mathrm{d}x^2_{(d)} + \left(\frac{R}{y}\right)^2 \mathrm{d}y^2 + R^2 \mathrm{d}\Omega^2_{(\tilde{d}+1)}\,, \quad e^{\Phi(y)} = g_s \left(\frac{R}{y}\right)^{\frac{(D-2)\tilde{d}a}{4\Delta}}\,. \tag{3.9}$$

This looks similar to Anti-de-Sitter spacetime in Poincaré coordinates (2.31). Specifically, we take

$$e^{-r/R} = \frac{y}{R}\,. \tag{3.10}$$

In these coordinates, the metric and the dilaton take on the form

$$\mathrm{d}s^2_{\mathrm{D}} = e^{(2-\frac{4\tilde{d}}{\Delta})r/R} \mathrm{d}x^2_{(d)} + \mathrm{d}r^2 + R^2 \mathrm{d}\Omega^2_{(\tilde{d}+1)}\,, \quad \phi(r) = \frac{(D-2)\tilde{d}ar}{4\Delta R}\,. \tag{3.11}$$

The analog of the horospherical coordinates (2.29) is given by

$$\frac{U}{L} = e^{-r/L}\,, \quad L = \frac{R}{\left(\frac{2\tilde{d}}{\Delta} - 1\right)}\,. \tag{3.12}$$

In these coordinates, the metric takes on the form

$$\begin{aligned} \mathrm{d}s^2_{\mathrm{D}} &= \left(\frac{U}{L}\right)^2 \mathrm{d}x^2_{(d+1)} + \left(\frac{L}{U}\right)^2 \mathrm{d}U^2 + R^2 \mathrm{d}\Omega^2_{(\tilde{d}+1)} \\ &\equiv AdS_{d+1}\,(L) \times S^{\tilde{d}+1}(R)\,. \end{aligned} \tag{3.13}$$

3.1.3 Interpolating solitons

From this, we deduce that the near-horizon geometry for the two-block p-branes is given by $AdS_{p+2} \times S^{D-p-3}$ in the background of a dilaton depending linearly on the radial AdS-coordinate. Anticipating the discussion on domain-walls in the following section, we will call such geometries $DW_{p+2} \times S^{D-p-3}$. The size of the Anti-de-Sitter is proportional to the size of the sphere, as can be seen from (3.12).

There are two special case to be considered: $a = 0$, and $\tilde{d} = \frac{\Delta}{2}$. The first case corresponds to branes having no dilaton. Examples of this case are the D3-brane in ten dimensions and the eleven-dimensional M2-brane and M5-brane. These branes have a pure Anti-de-Sitter spacetime in their near-horizon geometry. In [15], a table of all cases with $a = 0$ was given.

The second case corresponds to branes with an infinite AdS-radius. For such a radius, the cosmological constant (2.44) vanishes, and the spacetime becomes conformally flat. The near-horizon becomes $\mathbb{R}^{1,p+1} \times S^{D-p-3}$. Examples of such spaces are five-branes in ten-dimensions, which have $\mathbb{R}^{1,6} \times S^3$ as their near-horizon geometry. For more details and examples, we refer to [15].

On the other hand, taking

$$\frac{y}{R} \to \infty\,, \tag{3.14}$$

the harmonic function becomes constant, and the metric describes Minkowski space $\mathbb{R}^{1,D-1}$. This means that we can view a p-brane as a gravitational soliton with a geometry that interpolates between the near-horizon geometry $DW_{p+2} \times S^{D-p-3}$ and the asymptotic geometry $\mathbb{R}^{1,D-1}$.

3.2 Domain-walls

Domain-walls can be defined as topological defects of co-dimension one. They separate a spacetime (or a phase space) into several domains along a single coordinate. In the presence of domain-walls, physical parameters can generically taken to be piecewise smooth. However, on an intersection of two domains – the domain-wall – such a parameter is generically not differentiable or even continuous, and its derivatives can have delta-function singularities. These properties make domain-walls useful for describing physical processes such as phase transitions.

In this section, we will describe a class of such domain-walls that occur in supergravity theories. We will make a distinction between “thin” domain-walls and “thick” domain-walls. The former class can be viewed as a single $(D-2)$-brane placed in the origin of the y-coordinate, separating the spacetime into two regions. In each region, a characteristic magnetic field-strength can be defined that changes its sign across the brane. At $y=0$, there is a curvature singularity.

In section 3.3.3, we will describe “thick” domain-walls; they can be viewed as smoothly interpolating solitons between different supergravity vacua, without having singularities. They have no direct brane-interpretation, but they can sometimes be related to intersecting branes in a higher-dimensional spacetime.

3.2.1 Solution Ansatz

We will now discuss domain-walls which support only a single scalar and a d-dimensional gauge potential. The action in the Einstein frame is

$$\mathcal{L}^{\mathrm{E}}_{\mathrm{domain}} = R \star \mathbb{1} - \frac{4}{d-1} \star \mathrm{d}\varphi \wedge \mathrm{d}\varphi - \frac{1}{2} e^{b\varphi} g_s^{2\bar{k}} \star F_{(d+1)} \wedge F_{(d+1)} \,, \tag{3.15}$$

where the dilaton exponential factor is obtained from (1.88)

$$\bar{k} = \frac{b}{2} + \frac{2d}{d-1} \,. \tag{3.16}$$

The domain-wall solution is analogous to the general electric p-brane

$$\text{domain-wall} = \begin{cases} \mathrm{d}s^2_{\mathrm{E}} &= H^{\frac{-4\varepsilon}{(d-1)\Delta_{\mathrm{dw}}}} \mathrm{d}x^2_{(d)} + H^{\frac{-4d\varepsilon}{(d-1)\Delta_{\mathrm{dw}}} - 2(\varepsilon+1)} \mathrm{d}y^2 \,, \\ e^{\varphi} &= H^{\frac{-(d-1)b\epsilon}{4\Delta_{\mathrm{dw}}}} \,, \\ F_{(d+1)} &= g_s^{-\bar{k}} \sqrt{\frac{4}{\Delta_{\mathrm{dw}}}}\, \mathrm{d}^d x \wedge \mathrm{d}H^{\varepsilon} \,, \\ H(y) &= 1 + Q|y| \,. \end{cases} \tag{3.17}$$

with the parameter Δ_{dw} given by

$$\Delta_{\mathrm{dw}} = \frac{(d-1)b^2}{8} - \frac{2d}{d-1} \,. \tag{3.18}$$

If the dilaton vanishes, the parameter Δ_{dw} reduces to

$$\Delta_{\mathrm{AdS}} = -\frac{2d}{d-1} \,. \tag{3.19}$$

This is a lower-bound on Δ_{dw}, and we can classify the domain-wall solutions into four classes, depending on whether

1. $\Delta_{\rm dw} = \Delta_{\rm AdS}$,
2. $\Delta_{\rm AdS} < \Delta_{\rm dw} < 0$,
3. $\Delta_{\rm dw} = 0$,
4. $\Delta_{\rm dw} > 0$.

We will not consider the third category. It will turn out that domain-walls which are reductions of branes in higher dimensions fall into categories 1 or 2. Elementary domain-walls fall into category 4.

The domain-wall (3.17) is a one-parameter class of solutions: the parameter ε cannot be determined, in contrast with normal p-branes which have $\varepsilon = -1$. Even though there exists no magnetically dual for the domain-wall, we can still define a magnetically dual field-strength

$$F_{(0)} \equiv e^{b\varphi} g_s^{\bar{k}} \star F_{(d+1)} \, . \tag{3.20}$$

We can eliminate ε if we define a mass parameter as $m = Q\varepsilon$. Using the form of $F_{(d+1)}$ given in (3.17), we see that the magnetic field-strength changes its sign across the point $y = 0$; this is the position where the brane is located.

It is straightforward to check that the invariant volume form is given by

$$e^{2b\varphi} H^{2(\varepsilon-1)} \mathrm{d}^d x \wedge \mathrm{d}y = \star \mathbb{1} \, . \tag{3.21}$$

Using this, we can express the action in the magnetic formulation as

$$\mathcal{L}^{\rm E}_{\rm domain} = R \star \mathbb{1} - \frac{4}{d-1} \star \mathrm{d}\varphi \wedge \mathrm{d}\varphi - 2e^{-b\varphi} \Lambda \star \mathbb{1} \, . \tag{3.22}$$

The cosmological constant is given by

$$\Lambda = \frac{m^2}{\Delta_{\rm dw}} \, . \tag{3.23}$$

3.2.2 Asymptotic geometry

Even though domain-walls do not have a magnetically dual brane, it is again it useful to transform to the dual frame. This frame is defined by (1.84), with $\tilde{d} = -1$

$$g^{\rm D}_{\mu\nu} = e^{-b\varphi} g^{\rm E}_{\mu\nu} \, . \tag{3.24}$$

The action (3.22) now has an overall dilaton factor and a modified kinetic term

$$\mathcal{L}^{\rm D}_{\rm domain} = e^{\bar{\delta}_{\rm D}\varphi} \left(R \star \mathbb{1} + \bar{\gamma}_{\rm D} \star \mathrm{d}\varphi \wedge \mathrm{d}\varphi - 2\Lambda \right) \, , \tag{3.25}$$

with $\bar{\delta}_{\rm D}$ and $\bar{\gamma}_{\rm D}$ given by

$$\bar{\delta}_{\rm D} = \frac{(d-1)b}{2}\,, \quad \bar{\gamma}_{\rm D} = \frac{d}{d-1}\bar{\delta}_{\rm D}^2 - \frac{4}{d-1} \tag{3.26}$$

The metric in the dual frame reads

$$\mathrm{d}s_{\rm D}^2 = H^{\frac{2(\Delta_{\rm dw}+2)\varepsilon}{\Delta_{\rm dw}}}\mathrm{d}x_{(d)}^2 + H^{-2}\mathrm{d}y^2\,. \tag{3.27}$$

In the previous section, we have shown that generic p-branes could be interpreted as solutions which interpolate between two different supergravity vacua: Minkowski spacetime $\mathbb{R}^{1,D-1}$ asymptotically away from the brane and $AdS_{d+1}\times S^{\tilde{d}+1}$ near the brane. For domain-walls, this is not the case: they are not asymptotically flat. To discover what asymptotic geometry they have, we take the limit

$$Q|y| \to \infty\,. \tag{3.28}$$

The metric and the dilaton then take the form

$$\mathrm{d}s_{\rm D}^2 = (Qy)^{\frac{2(\Delta_{\rm dw}+2)\varepsilon}{\Delta_{\rm dw}}}\mathrm{d}x_{(d)}^2 + (Qy)^{-2}\mathrm{d}y^2\,, \quad e^{\varphi(y)} = (Qy)^{\frac{-(d-1)b\epsilon}{4\Delta_{\rm dw}}}\,. \tag{3.29}$$

We can now exponentiate the y-coordinate

$$e^{-Qr} = Qy\,, \tag{3.30}$$

after which the metric has the form of Anti-de-Sitter spacetime in Poincaré coordinates, and the dilaton now has a linear dependence on the radial AdS-coordinate

$$\mathrm{d}s_{\rm D}^2 = e^{-\frac{2(\Delta_{\rm dw}+2)}{\Delta_{\rm dw}}mr}\mathrm{d}x_{(d)}^2 + \mathrm{d}r^2\,, \quad \varphi(r) = \frac{(d-1)mb}{4\Delta_{\rm dw}}r\,. \tag{3.31}$$

After going to horospherical coordinates

$$\frac{U}{L} = e^{-r/L}\,, \quad L = \frac{\Delta_{\rm dw}}{(\Delta_{\rm dw}+2)m}\,, \tag{3.32}$$

we get for the metric

$$\begin{aligned} \mathrm{d}s_{\rm D}^2 &= \left(\frac{U}{L}\right)^2 \mathrm{d}x_{(d+1)}^2 + \left(\frac{L}{U}\right)^2 \mathrm{d}U^2 \\ &\equiv AdS_{d+1}\,(L)\,. \end{aligned} \tag{3.33}$$

From the above, we see that $(D-2)$-branes are different from generic p-branes. They do not interpolate between flat spacetime and a product of Anti-de-Sitter spacetime times a sphere. Instead, they form an interpolation between two asymptotic Anti-de-Sitter spacetimes, with a dilaton depending linearly on the radial AdS-coordinate.

3.2.3 Sphere reductions

A p-brane in $(p+2)$ dimensions can be seen as a domain-wall. Hence, if we reduce the action (1.85) over the sphere $S^{\tilde{d}+1}$, we expect to find a domain-wall described by the action

$$\mathcal{L}^{\mathrm{D}}_{(d+1,\,p)} = e^{\delta_{\mathrm{D}}\phi}\left(R\star\mathbb{1} + \gamma_{\mathrm{D}}\star\mathrm{d}\phi\wedge\mathrm{d}\phi - 2\Lambda\star\mathbb{1}\right)\,. \tag{3.34}$$

Up to a dilaton rescaling, this is of the same form as (3.25). We can determine the scale factor from[1]

$$\phi = c\varphi \rightarrow c^2 = \frac{\bar{\gamma}_{\mathrm{D}}}{\gamma_{\mathrm{D}}} = \left(\frac{\bar{\delta}_{\mathrm{D}}}{\delta_{\mathrm{D}}}\right)^2\,. \tag{3.35}$$

Combining (1.86) and (3.26) with either (1.54) or with (3.18), we can express the dilaton rescaling in two ways

$$c^2 = \frac{2\tilde{d}^2}{\Delta + (\Delta-2)\tilde{d}} = -\frac{\Delta_{\mathrm{dw}} + (\Delta_{\mathrm{dw}}+2)\tilde{d}}{2}\,. \tag{3.36}$$

This means that we can express the parameter Δ_{dw} of the $(d+1)$-dimensional domain-wall in terms of the parameter Δ of the original D-dimensional p-brane solution

$$\Delta_{\mathrm{dw}} = \frac{-2\tilde{d}\Delta}{\Delta + (\Delta-2)\tilde{d}}\,. \tag{3.37}$$

The dilaton couplings a and b in the Einstein frame actions (1.50) and (3.15) are then related by

$$b = -a\frac{c(D-2)}{\tilde{d}(d-1)}\,. \tag{3.38}$$

Furthermore, comparing the sizes of the Anti-de-Sitter spacetime given in (3.12) and (3.32), we deduce

$$m = \frac{\tilde{d}}{R}\,. \tag{3.39}$$

Finally, we can also relate the cosmological constant of the reduced brane solution in terms of parameters of the original brane solution

$$\Lambda = -\frac{\tilde{d}}{2R^2}\left((\tilde{d}+1) - \frac{2\tilde{d}}{\Delta}\right)\,. \tag{3.40}$$

So far, we have shown that the near-horizon geometry of a generic p-brane is given by $AdS_{p+2}\times S^{D-p-3}$, and that the reduction of this geometry over the sphere can be related

[1] This is a refinement w.r.t. [15] where the same scale factor was obtained by transforming the reduced action (3.34) back to the Einstein frame, and comparing this with (3.15). However, such a rescaling is singular for $d=0$. We avoid this slight complication by the method sketched above.

to a domain-wall in $(p+2)$-dimensions. Since the action (1.50) is a consistent truncation of a more general supergravity action, this would suggest that a sphere-reduction of this more general supergravity action leads to a lower-dimensional supergravity action, of which (3.15) should be a consistent truncation.

In section 2.3.2, we saw that the AdS/CFT formulation was most conveniently formulated as a duality between $\mathrm{SO}(6)$ gauged $\mathcal{N}=8$ supergravity in $D=5$, and $\mathcal{N}=4$ Yang-Mills theory in $D=4$. The former theory is conjectured to be a consistent truncation of the S^5-reduction of Type IIB supergravity in $D=10$. A natural form of the DW/QFT correspondence would then be in terms of a duality between an $\mathrm{SO}(\tilde{d}+2)$ gauged supergravity in $d+1$ dimensions, and the worldvolume theory of the corresponding p-brane in d dimensions [114].

The underlying assumption of such a scheme is that it is possible to consistently truncate the $S^{\tilde{d}+1}$-reduction of the higher-dimensional supergravity to only the massless Kaluza-Klein modes. The Anti-de-Sitter spacetime and the sphere are of comparable radius, as we showed in (3.12). This is fundamentally different from, say, Calabi-Yau compactifications of string theory, where the consistency is at least approximately guaranteed by taking the compactification radius to zero, thereby automatically decoupling all the higher Kaluza-Klein modes. Consistent truncations of sphere reductions are in general hard to find. Until recently, only the gauged maximally supersymmetric supergravities in $D=4$ [116] and $D=7$ [117] were shown to be consistent truncations of the compactifications of eleven-dimensional supergravity on S^7 [118] and S^4 [119].

For sphere reductions, if one wants to keep the massless gauge fields generating the $\mathrm{SO}(\tilde{d}+2)$ symmetry, one generically also needs to keep most, if not all, scalar fields coming from the reduction Ansatz of the metric and the $(p+1)$-form gauge potential. This complicated matters enormously: e.g. the S^5 reduction of Type IIB supergravity results in 42 scalars in $D=5$ which interact in a non-linear fashion. Moreover, the Killing vectors on the sphere need to satisfy certain consistency conditions. These conditions turn out to be hard to satisfy, precisely only for the examples mentioned above does a maximally supersymmetric gauged supergravity form a consistent truncation of a sphere-compactification [120].

Nevertheless, many new results on consistent sphere reductions have been obtained in recent years following the AdS/CFT correspondence. In particular, it is possible to consider truncations of the complete massless Kaluza-Klein sector to only the subset of the lower-dimensional scalars that transform in the Cartan-subalgebra of the gauge group. These truncated gauged supergravities have solutions that can be lifted to solutions of the original supergravity theory. There, they correspond to an infinite stack of overlapping branes [121].

3.3 Quantum field theory

In this section, we will explore what field theory information can be extracted from the DW/QFT correspondence. In particular for the class of Dp-branes and intersections thereof, we will derive the scaling dependence of the corresponding worldvolume theory coupling constants. The end of this section will review how the renormalization-group flow induced by relevant deformations of conformal theories give rise to domain-wall solutions that interpolate between different supergravity vacua.

3.3.1 Dual worldvolume theories

The geometrical structure of a large class of p-branes in the dual frame is rather similar. However, the worldvolume theories of these branes are much more diverse, as we saw in section 1.4.2. We will parameterize the worldvolume action to first approximation as a theory described by a q-form gauge field. The cases $q = 0$, $q = 1$, and $q = 2$ then correspond to a an action describing a scalar, a vector, and a tensor multiplet, respectively.

$$\begin{aligned} S_{\rm brane} &= -\tau_p \int \mathrm{d}^{p+1}\sigma \,(\ell_s^{q+1} F_{(q+1)})^2 + \dots && (3.41) \\ &\equiv -\frac{1}{g^2_{\rm gauge}} \int \mathrm{d}^{p+1}\sigma \, F^2_{(q+1)} \,. && (3.42) \end{aligned}$$

The mass-dimension of the field-theory coupling constant can be obtained from the general expression for a p-brane tension

$$g^2_{\rm gauge} = g_s^k \ell_s^\alpha \,, \quad \alpha = p - 2q - 1 \,. \tag{3.43}$$

At a given energy scale E, a dimensionless coupling constant is defined as

$$\lambda(E) \equiv g^2_{\rm gauge} E^\alpha \,. \tag{3.44}$$

In the case of the D3-brane, we saw that there were two natural energy scales: the energy $E_{\rm W}$ of open strings stretching between the stack of N D3-branes and a single D3-brane probe, and the energy E_ψ of a supergravity field ψ probing the N D3-branes.

For general p-branes, the holographic energy scale can be obtained from an analysis of the wave equation for a supergravity scalar field ψ. The analog of (2.18) for a general p-brane is

$$E_\psi \equiv u = \frac{y^\beta}{R^{\beta+1}} \,, \quad \beta = \frac{2\tilde{d}}{\Delta} - 1 \,. \tag{3.45}$$

Of all the brane solutions in string theory, Dp-branes have been studied most. In particular, they have a exact description in conformal field theory as boundary states [122]. Other branes, such as the NS5-brane, are also believed to form coherent states in the conformal field theory description of string theory, but a precise understanding is lacking. This suggests that we should consider p-branes which are closely related to Dp-branes.

If we also consider Dp-brane probes of the p-brane, then we have an additional energy scale equivalent to (2.16)

$$E_W \equiv U = \frac{y}{\ell_s^2} \,. \tag{3.46}$$

For general p-branes, it is not possible to choose a Dp-brane as a sensible probe. The reason is that we would like the dimensionless coupling constants of both energy scales to be related independently of the near-horizon limit. In particular, looking at the y-dependence of the dimensionless coupling constants constructed from the two energy scales we expect that

$$\lambda(u) = \lambda(U)^{\beta} \,. \tag{3.47}$$

For this to happen, the g_s and N dependence on both sides will also have to match. This gives two restrictions

$$k = 1 \,, \quad \alpha = \Delta - \tilde{d} \,. \tag{3.48}$$

The first constraint has an obvious interpretation; it says that the dilaton dependence of the p-brane tension is the same as for a Dp-brane, namely $\tau_p = \frac{1}{g_s}$. The second constraint is more surprising, it gives the mass dimension of the coupling constant on the p-brane worldvolume. If we combine the constraint (3.48) with the expression for Δ (1.54), then we find

$$a = \frac{2(D - 2(2+p))}{D-2} \,, \quad \Delta = \frac{D-2}{2} \,, \quad \alpha = -a\frac{D-2}{4} \,. \tag{3.49}$$

This means that the scale dependence of the worldvolume coupling constant is proportional to the dilaton dependence of the gauge field kinetic term in the action. In particular, p-branes that do not couple to the dilaton have a scale-independent coupling constant in their worldvolume theory.

The supergravity approximation is valid as long as the string tension in the dual frame is large. We can calculate this with the same scaling arguments as we used in deriving the effective brane tension in the string frame

$$\tau_s^{\mathrm{D}} = \frac{\lambda(U)^{\frac{2}{\Delta}}}{\ell_s^2} \,. \tag{3.50}$$

Quantum corrections in string theory are controlled through the dilaton which is now not a constant g_s, as in (2.23), but is instead given by

$$e^{\Phi} = \frac{\lambda(U)^{\frac{\tilde{d}}{\Delta}}}{N} \,. \tag{3.51}$$

The ratio of the two different energy scales can be expressed in a similar form as (2.24)

$$\frac{U}{u} = \lambda(U)^{\frac{2}{\Delta}} \,. \tag{3.52}$$

The generalization of table 2.1 is given in table 3.1. It gives the relations between the ranges of the various parameters on both sides of the theory for which one side becomes computationally feasible.

In the remainder of this section, we will discuss some specific examples of worldvolume theories of the p-branes discussed in this chapter. For more details see [15].

Regime	Gravity	Gauge theory
Perturbative field theory	$\tau_s^{\mathrm{D}}\ell_s^2 \ll 1$	$\lambda(U) \ll 1$
Classical string theory	$e^{\Phi} \ll 1$	$\frac{\lambda(U)^{\frac{\tilde{d}}{\Delta}}}{N} \ll 1$
Supergravity	$\tau_s^{\mathrm{D}}\ell_s^2 \gg 1$	$\lambda(U)^{\frac{2}{\Delta}} \gg 1$

Table 3.1: Regimes of the DW/QFT correspondence.

Ten-dimensional Dp-branes

The first class of examples is formed by the ten-dimensional Dp-branes. They have $a = \frac{3-p}{2}$ and $\Delta = 4$, in accordance with (3.49), from which we also deduce that $\alpha = p-3$. Comparing this with (3.43), we deduce that $q = 1$. In other words, the coupling constant on the Dp-brane worldvolume scales consistently with the vector multiplet description of the worldvolume theory.

The regime where perturbative field theory is possible is when $\lambda(U) \ll 1$. The sign of α is positive for $p > 3$ and negative for $p < 3$. This means that, for the Dp-branes with $p < 3$, the perturbative field theory description is valid for large U – the gauge theory is UV-free. The field theories of Dp-branes with $p > 3$ can be treated perturbatively for small U – the IR regime.

Since the conformal symmetry of the D3-brane worldvolume theory does not extend to the Dp-brane worldvolume theories, there are hardly any quantitative tests available. However, the qualitative structure of the phase diagram of these theories as a function of N, U and λ has been investigated in [111], and the relation to gauged supergravities has been studied in [114].

Six-dimensional dp-branes

Intersections of a Dp-brane with a D$(p+4)$-brane in which the smaller brane lies entirely inside the larger brane, are denoted as $(p|\mathrm{D}p, \mathrm{D}(p+4))$. These intersections preserve half the supersymmetries of the constituent D-branes and this means that they have $\Delta = 2$.

Generically, a Dp-brane has an $\mathcal{N} = 4$ vector-multiplet on its worldvolume. In the presence of D$(p+4)$-branes, one can split these degrees of freedom into an $\mathcal{N} = 2$ vector multiplet parallel to the D$(p+4)$-brane and an $\mathcal{N} = 2$ hypermultiplet transverse to both D-branes [123]. The strings stretching between branes of different dimension have the interpretation of quarks on the worldvolume of the Dp-branes, whereas the strings starting and ending on Dp-branes have the usual interpretation of gauge fields.

If there are N_{c} Dp-branes and N_{f} D$(p+4)$-branes, then $\mathrm{U}(N_{\mathrm{c}})$ acts as the color group and $\mathrm{U}(N_{\mathrm{f}})$ as the flavor group [123]. After a dimensional reduction of the four transverse coordinates of both branes, they form a stack of $N = N_{\mathrm{c}} + N_{\mathrm{f}}$ six-dimensional p-branes called dp-branes [15]. Comparing with (3.49), we see that such branes indeed have $\Delta = 2$ as well as $a = 1 - p$ and $\alpha = p - 1$. From (3.43), we deduce that $q = 0$ for such branes; their worldvolume theory should consist of a hypermultiplet.

This result is not too surprising. First of all, the vector multiplet corresponding to the fluctuations parallel to the D$(p+4)$-brane is lost in the dimensional reduction process. Moreover, scalar multiplets have spins in a range which is twice as small as that of vector multiplets. This is consistent with the ratio of the amounts of supersymmetries preserved by dp-branes and Dp-branes.

For $p = 1$, the worldvolume is a two-dimensional conformal field theory and the near-horizon geometry is $AdS_3 \times S^3$ without a dilaton background. This is the most studied example [124]; it corresponds to the $(1|\mathrm{D1}, \mathrm{D5})$ system in ten dimensions compactified on a four-dimensional torus [125]. In this case, there is also some progress in the area of treating string theory on the curved AdS_3 spacetime [126].

3.3.2 Deformations and renormalization

Up to now, we have discussed the most obvious deformation of the D3-brane system: Dp-branes, and intersections thereof. However, the worldvolume theories of these branes do not lend themselves for a computationally feasible extension of the AdS/CFT correspondence, as we have seen in the previous section.

Another way of generalizing the AdS/CFT correspondence is to look at deformations of the conformal field theory that is a dual description of gravity around an AdS spacetime. In general, such deformations will break the conformal invariance and not much information can be obtained. However, as will be made precise below, for so-called relevant deformations, the theory can flow to another conformal theory.

The AdS/CFT correspondence provides the field theory with two natural energy scales: the Dp-brane probe energy U, and the holographic energy u. The formalism which deals most efficiently with field theories having an energy scale is called effective field theory. For a good review, we refer to [127].

In a field theory with an energy scale Λ, one can make a distinction between the momentum modes of a field into high-frequency and low-frequency modes

$$\{\phi(\omega)\} = \{\phi(\omega)_{\mathrm{L}}\} + \{\phi(\omega)_{\mathrm{H}}\} . \tag{3.53}$$

The obvious definitions are given by

$$\begin{aligned} \{\phi(\omega)_{\mathrm{L}}\} &= \{\phi(\omega) : \omega < \Lambda\} , \\ \{\phi(\omega)_{\mathrm{H}}\} &= \{\phi(\omega) : \omega > \Lambda\} . \end{aligned} \tag{3.54}$$

An effective field theory is obtained by integrating out the high-frequency modes in the (Euclidean) path integral

$$\int \mathcal{D}\phi_{\mathrm{L}} \int \mathcal{D}\phi_{\mathrm{H}} e^{-S(\phi_{\mathrm{L}},\phi_{\mathrm{H}})} \equiv \int \mathcal{D}\phi_{\mathrm{L}} e^{-S_\Lambda(\phi_{\mathrm{L}})} . \tag{3.55}$$

This defines the low-energy effective action as

$$e^{-S_\Lambda(\phi_\mathrm{L})} \equiv \int \mathcal{D}\phi_\mathrm{H} e^{-S(\phi_\mathrm{L},\phi_\mathrm{H})} \,. \tag{3.56}$$

The effective action can be expanded in a complete set of local operators $\mathcal{O}_{\Delta_i}$ which consist of powers of the low-energy fields and their derivatives

$$S_\Lambda = S_\mathrm{CFT}(\Lambda, g^*) + \sum_i \int \mathrm{d}^d x\, g^i \mathcal{O}_{\Delta_i} \,. \tag{3.57}$$

The action $S_\mathrm{CFT}(\Lambda, g^*)$ is the free action around a fixed point of the beta-function (see below). Normally one takes this fixed point to be the trivial one $\{g^* = 0\}$ which implies that free action is just the kinetic part of the low-energy effective action. For non-trivial fixed points, S_CFT can describe an interacting conformal field theory.

Simple dimensional analysis gives for the scaling dimensions

$$\begin{aligned} [\mathcal{O}_{\Delta_i}] &= \Delta_i \,, \\ [g^i] &= d - \Delta_i \,. \end{aligned} \tag{3.58}$$

It is important to note that these scaling dimensions are defined relative to the fixed point of the couplings $\{g^*\}$, and that the value of this dimension can be changed by the interactions. One can then introduce dimensionless couplings by defining

$$\lambda^i = g^i \Lambda^{\Delta_i - d} \,. \tag{3.59}$$

Around energy scales E, we have the following order of magnitude for a typical operator

$$\int \mathrm{d}^d x \mathcal{O}_{\Delta_i} \simeq E^{\Delta_i - d} \,. \tag{3.60}$$

This means that the i-th term in the action is of the size

$$\lambda^i \left(\frac{E}{\Lambda}\right)^{\Delta_i - d} \,. \tag{3.61}$$

The sign of the exponent will determine the relevance of an operator at a given energy scale E compared to the natural energy scale Λ, as we have indicated in table 3.3.2. If the exponent is negative, then for energies much smaller than Λ, the term in the action will become large, and the operator is called relevant. For positive exponents, the term in the action will vanish at low energies – the operator is irrelevant for the low-energy theory. The case of vanishing exponent corresponds to a marginal operator.

A familiar example of theories with an energy scale Λ appears in the orthodox renormalization of quantum field theory. There, Λ is introduced as a regulator, or cut-off, to calculate a

$\Delta_i - d$	Size as $E \to 0$	Type	Theory
< 0	Grows	Relevant	Super-renormalizable
$= 0$	Constant	Marginal	Strictly renormalizable
> 0	Decays	Irrelevant	Non-renormalizable

Table 3.2: Classification of operators in effective field theory.

divergent path-integral. After obtaining a finite answer and renormalizing certain quantities, the cut-off is send to infinity. This is precisely the opposite limit considered in effective field theories.

It therefore follows that irrelevant operators correspond to non-renormalizable theories since they yield infinite terms at high energies. One can nevertheless still make sense of non-renormalizable theories, such as General Relativity, by considering them as low-energy effective theories and only using them at energies far below the cut-off Λ. The dependence of a low-energy effective theory on the high energy physics is only through the marginal and relevant operators.

The scaling derived from simple power counting is modified by interactions in the effective theory; these effects are controlled by the beta-functions. They are defined as follows

$$\beta^i(g) \equiv E \frac{\partial g^i(E)}{\partial E} \,. \tag{3.62}$$

The beta-functions can be calculated in perturbation theory around the fixed point $\{g^*\}$

$$\beta^i(g) = \eta^i g^i + C_{jk}{}^i g^j g^k + \ldots \,. \tag{3.63}$$

The constants η^i are called the anomalous scaling dimensions, they measure the deviation from the canonical scaling dimension derived from the free action. The coefficients $C_{jk}{}^i$ appear in operator product expansion of local operators $\mathcal{O}_{\Delta_i}$

$$\left\langle \mathcal{O}_{\Delta_i}(x_i)\mathcal{O}_{\Delta_j}(x_j)\right\rangle_{\rm CFT} = C_{ij}{}^k(x_i - x_j)\left\langle \mathcal{O}_{\Delta_k}\right\rangle_{\rm CFT} \,. \tag{3.64}$$

Of particular interest are the points for which the beta-function vanishes; for these values of the coupling constants, the theory is invariant under a change in scale, and such points are therefore called fixed points. The sign of the derivative of the beta-function at the fixed-points determines whether the fixed-point will be reached for increasing or decreasing energy scale.

We have plotted a typical example of a beta-function in figure 3.1. The arrows on the g-axis indicate in which direction the couplings will flow for increasing energy. The fixed-points at which the slope of the graph is negative are called UV-fixed points since the beta-function will drive the couplings to these values for increasing energies. On the other hand, the fixed-point having a positive slope is reached in the IR.

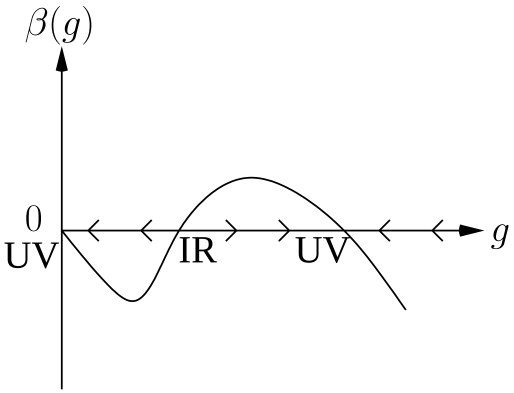

Figure 3.1: A beta-function with UV and IR fixed points.

In coupling space, one can raise and lower indices with the Zamolodchikov metric [128]

$$G_{ij} = |x_i - x_j|^{2d-\eta_i-\eta_j} \left\langle \mathcal{O}_{\Delta_i}(x_i)\mathcal{O}_{\Delta_j}(x_j)\right\rangle_\Lambda \,, \tag{3.65}$$

where the expectation value is now computed with the full effective action S_Λ. The beta-functions are related to a gradient-flow in coupling space [128] also known as a renormalization group flow, or RG-flow

$$\frac{\partial C(g)}{\partial g_i} = -G_{ij}\beta^j(g)\,. \tag{3.66}$$

The C-function is invariant under a change of scale. In particular, in two-dimensions, the C-function is related to the central charge c of the conformal field theory [128], which is proportional to the trace of the energy-momentum tensor

$$C \simeq \langle T^\mu{}_\mu\rangle \,. \tag{3.67}$$

The scale-invariance of $C(g)$ implies that

$$E\frac{\mathrm{d}C(g)}{\mathrm{d}E} = 0 \to E\frac{\partial C(g)}{\partial E} = -E\frac{\partial C(g)}{\partial g_i}\frac{\partial g^i(E)}{\partial E} = G_{ij}\beta^i(g)\beta^j(g) \geq 0\,. \tag{3.68}$$

The last inequality has been proven in two dimensions [128], but no such proof is available in higher-dimensions [129]. The interpretation is that the C-function decreases monotonically form the UV to the IR.

The formalism described above can be applied to the AdS/CFT correspondence in the following way. We saw that, at the boundary of the AdS spacetime at large U, there was a dual description in terms of the UV regime of a conformal field theory. Moreover, fluctuations around the AdS solution corresponded to a correlation function in the conformal field theory of the form

$$\left\langle e^{\int \mathrm{d}^d\vec{x}\, \varphi_0(\vec{x})\mathcal{O}_\Delta(\vec{x})} \right\rangle_{\mathrm{CFT}} . \tag{3.69}$$

This implies that the conformal field theory action is modified with a local operator

$$\int \mathrm{d}^d\vec{x}\, \varphi_0(\vec{x})\mathcal{O}_\Delta(\vec{x}) . \tag{3.70}$$

The coupling $\phi_0(\vec{x})$ correspond to a scalar field $\phi_0(z,\vec{x})$ which has two eigenmodes under rescalings with eigenvalues Δ_+ and Δ_-. The eigenvalue Δ_+ correspond to a relevant perturbation of the conformal field theory inducing a UV-IR flow in the CFT. On the other hand, the Δ_- eigenvalue has the interpretation of deforming the conformal field theory with the vacuum expectation value [130]

$$\left\langle \mathcal{O}_{\Delta_-} \right\rangle_{\mathrm{CFT}} . \tag{3.71}$$

Following the AdS/CFT correspondence, the possible deformations of $\mathcal{N} = 4$ Yang-Mills theory gained new interest [131]. As we will now see, the possible RG-flows that these deformations induce will correspond to interpolating domain-walls in the dual gravity theory [132, 133].

3.3.3 Domain-walls as RG-flows

This section follows to a large extent the treatment of the papers [80, 134]. Recall the toy model from section 2.3.1 of a scalar field with a potential coupled to gravity

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^{d+1}x \sqrt{|g|} \left(R - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right) . \tag{3.72}$$

We will be particularly interested in potentials of the form

$$V(\phi) = \frac{(d-1)^2}{2} \left(\frac{\partial W}{\partial \phi} \right)^2 - \frac{d(d-1)}{4} W(\phi)^2 . \tag{3.73}$$

The function $W(\phi)$ will be called the superpotential since supergravity theories generically have a scalar potential of the above form. Moreover, it can be shown [135] that potentials of the form (3.73) have stable minima. These minima are related to the following conditions on the superpotential

$$\frac{\partial V}{\partial \phi} = 0 \rightarrow \frac{\partial W}{\partial \phi} = 0 , \quad \text{or} \quad \frac{\partial^2 W}{\partial \phi^2} = \frac{d}{2(d-1)} W(\phi) . \tag{3.74}$$

For more realistic models, such as $\mathcal{N} = 8$ gauged supergravity in $D = 5$ which has a potential for no less than 42 scalars, finding the minima of the superpotential is non-trivial.

Nevertheless, for truncations of the total set of scalars, several exact minima have been found for this theory [136].

We saw in chapter 2 that minima of the scalar potential corresponds to Anti-de-Sitter spacetimes. Since gravity in an AdS spacetime should have a holographically dual CFT description, and since deformations of conformal field theories induce RG-flows, we will consider a class of solutions that can be thought of as deformations of Anti-de-Sitter spacetime.

Specifically, we generalize the metric (2.31) to the following form

$$\mathrm{d}s^2 = e^{2A(r)}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}r^2\,, \quad \phi = \phi(r)\,. \tag{3.75}$$

In the case $A(r) = -\frac{r}{L}$ we regain Anti-de-Sitter space. The analog of the horospherical coordinates (2.29) is given by

$$\mathrm{d}s^2 = U^2\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \left(\frac{U}{A'(r)}\right)^2 \mathrm{d}U^2\,, \quad U = e^{A(r)}\,. \tag{3.76}$$

For the Ansatz (3.75), the equations of motion (2.53) take on the form

$$\begin{aligned} \phi''(r) + dA'(r)\phi'(r) &= \tfrac{\partial V}{\partial \phi}\,, \\ (d-1)A''(r) + \tfrac{d(d-1)}{2}A'(r)^2 &= -\tfrac{1}{4}\phi'(r)^2 - \tfrac{1}{2}V(\phi)\,, \\ \tfrac{d(d-1)}{2}A'(r)^2 &= \tfrac{1}{4}\phi'(r)^2 - \tfrac{1}{2}V(\phi)\,. \end{aligned} \tag{3.77}$$

The equations of motion (3.77) are the Euler-Lagrange equations for the functional

$$E = \int_{-\infty}^{\infty} \mathrm{d}r \frac{e^{dA(r)}}{d-1}\left(-d(d-1)A'(r)^2 + \frac{1}{2}\phi'(r)^2 + V(\phi)\right)\,. \tag{3.78}$$

If the scalar potential is of the form (3.73), then we can use the Bogomol'nyi trick

$$\begin{aligned} E &= \int_{-\infty}^{\infty} \mathrm{d}r \frac{e^{dA(r)}}{d-1}\left(\frac{1}{2}\left[\phi'(r) \mp (d-1)\frac{\partial W}{\partial \phi}\right]^2 - d(d-1)\left[A'(r) \pm \frac{1}{2}W(\phi)\right]^2\right) \\ &\quad \pm \left[e^{dA(r)}W(\phi)\right]_{-\infty}^{\infty}\,. \end{aligned} \tag{3.79}$$

The extrema of this functional are given by

$$\begin{aligned} \phi'(r) &= \mp(d-1)\frac{\partial W}{\partial \phi}\,, \\ A'(r) &= \pm\frac{1}{2}W(\phi)\,. \end{aligned} \tag{3.80}$$

This means that for scalar potentials of the form (3.73), the second order differential equations (3.77) reduce to a pair gradient flow equations that can be solved by successive quadrature. The equations (3.80) are the same as the ones one would obtain from demanding that the supersymmetry variations of the fermions in the theory vanish. In particular, the supersymmetry variations of the gravitino and the dilatino will take on a schematic form that is similar to (1.75)

$$\begin{aligned}\delta\psi_\mu &= \partial_\mu\epsilon - \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} + W(\phi)\gamma_\mu\epsilon\,,\\ \delta\lambda &= \partial\!\!\!/\phi - (d-1)\frac{\partial W}{\partial\phi}\epsilon\,.\end{aligned} \tag{3.81}$$

Substituting the spin-connection $\omega_\mu{}^{ab}$ for the metric Ansatz (3.75) into the supersymmetry transformations (3.81), and demanding that these transformations vanish, gives the same pair of first-order equations (3.80) as was derived from the action (2.52). In other words, the flow equations (3.80) actually describe supersymmetric flows.

The scalar field $\phi(r)$ has the dual interpretation of a coupling in the CFT, and from (3.76) we see that $A(r)$ corresponds to the logarithmic energy scale in the field theory. We can then define the analog of the beta-function as

$$\begin{aligned}\beta(\phi) &\equiv U\frac{\partial\phi}{\partial U}\\ &= \frac{\phi'(r)}{A'r}\\ &= -\frac{2(d-1)}{W(\phi)}\frac{\partial W}{\partial\phi}\,.\end{aligned} \tag{3.82}$$

From (3.76), we also deduce that

$$A''(r) = -\frac{1}{2(d-1)}\phi'(r)^2\,. \tag{3.83}$$

We can then define a C-function [132]

$$C(U) = \frac{C_0}{A'(r)^{2(d-1)}}\,, \tag{3.84}$$

that satisfies monotonicity, something that is not possible to prove from field theory alone [129]

$$\begin{aligned}U\frac{\partial C}{\partial U} &= -2(d-1)C\frac{A''(r)}{A'(r)^2}\\ &= C\left(\frac{\phi'(r)}{A'(r)}\right)^2\\ &\geq 0\,.\end{aligned} \tag{3.85}$$

To summarize, the minima of the superpotential, corresponding to AdS spacetimes, are in correspondence with the fixed points of the beta-function. These fixed points are related to

Concept	Domain-wall	RG-flow
Scale	U	E
Log-scale	$A(r)$	$\log E$
Coupling constant	$\phi(r)$	$g(E)$
Beta-function	$\beta(\phi) = \frac{\phi'(r)}{A'r}$	$\beta(g) = E\frac{\partial g(E)}{\partial E}$
Fixed point	AdS spacetime	CFT
C-function	$C \simeq A'(r)^{-2(d-1)}$	$C \simeq \langle T^{\mu}{}_{\mu}\rangle$
C-theorem	$U\frac{\partial C}{\partial U} \geq 0$	only in $d = 2$

Table 3.3: A domain-wall/RG-flow dictionary.

conformal field theories. The induced RG-flow from the UV to the IR between two conformal field theories corresponds in this picture to a domain-wall that interpolates between two AdS spacetimes. We have summarized this domain-wall/RG-flow dictionary in table 3.3.

Using the newly found vacua of $\mathcal{N} = 8$ gauged supergravity in $D = 5$ [136], several interpolating domain-wall solutions were found [137]. These supersymmetric domain-walls correspond to deformations of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory by relevant operators. The induced RG-flows generically have IR fixed-points preserving a smaller amount of supersymmetry, creating hope that also non-supersymmetric gauge theories such as QCD might be described by a dual gravitational theory.

Chapter 4

Brane world scenarios

In the previous two chapters, we described various aspects of the correspondence between gravity and gauge theories. The material in the final two chapters concerns the structure of conformal supergravity and its couplings to matter. This chapter aims to provide the connection between these two, at first sight, rather different subjects.

We will start with giving a short review of two old problems, the hierarchy problem and the cosmological constant problem, and we will indicate how recent developments have shed some new light on these subjects. In particular, we will show how brane world scenarios, such as those of Randall and Sundrum [138, 139], provide a new framework in which various phenomenological aspects of elementary particle physics can be treated. The various techniques which were introduced in the previous chapters, in particular the geometrical properties of supergravity brane solutions, can be re-used in this description.

Attempts to supersymmetrize the brane world scenarios and to embed them in a natural way into string theory have so far met with many obstacles. In order to be able to address this in detail, the structure of five-dimensional supergravity and the various possible couplings to supersymmetric matter need to be taken into account. These problems will be the topic of the following two chapters where they will be studied in a superconformal context.

4.1 Fine-tuning problems

Two longstanding problems in theoretical physics, the hierarchy problem and the cosmological constant problem, are fine-tuning problems. In both cases, there are two fundamental scales, an experimentally observed scale and a theoretically expected scale, which are many orders of magnitude apart. We will first briefly review these problems.

4.1.1 The hierarchy problem

The electroweak scale is defined to be the energy scale in the Standard Model description of elementary particle physics at which the electromagnetic interaction unifies with the weak interaction. Since the Higgs-particle hypothesized to be responsible for the breaking of the electroweak $\mathrm{U}(1)_Y \times \mathrm{SU}(2)_\mathrm{L}$ gauge symmetry into the the electromagnetic gauge symmetry $\mathrm{U}(1)_\mathrm{em}$ has not been observed yet, an accurate number for this energy scale is not available. Instead, we will take the upper-limit[1] of 10^3 GeV, or 10^{-19} m.

The Planck energy scale is theoretically calculated to lie at $\lambda_\mathrm{Planck} = \sqrt{\frac{\hbar c}{G}} = 10^{19}$ GeV or at 10^{-35} m. At the Planck scale, a theory of Quantum Gravity should be revealed, and it is hoped that the gravitational interaction unifies with the remaining three interactions described by the Standard Model. The hierarchy of sixteen orders of magnitude between these two scales, and in particular the difficulty in explaining the radiative stability of electroweak scale masses in a theory with the Planck scale as the fundamental scale is called the hierarchy problem.

There is a sharp distinction between the two scales: the electroweak interactions have been (or will be in the near future) accurately probed up to scales of λ_weak, but the gravitational interaction has only been probed for distances up to the sub-millimeter range [140]. This opens up the possibility for a qualitatively different picture of gravity already far below the Planck scale. In particular, it is possible that new gravitational effects might appear already just above the electroweak scale. Indeed, in recent years, there was speculation on the existence of TeV scale strings [141].

A related direction that explored was the possible relation between the hierarchy problem and the existence of n extra compact dimensions [142–144]. The essential feature of these models is that in the higher-dimensional theory there is only a single scale: the four-dimensional weak scale λ_weak. The size of the extra dimensions generates the hierarchy between the weak scale and the Planck scale in four dimensions: from the standard relation (1.48) between the gravitational couplings in theories related by dimensional reduction, we find

$$\lambda^2_\mathrm{Planck} = \lambda^{n+2}_\mathrm{weak} \left(\frac{R}{\hbar c}\right)^n . \tag{4.1}$$

The characteristic radius R of such extra dimensions is then given by

$$R = \frac{\hbar c}{\lambda_\mathrm{weak}} \left(\frac{\lambda_\mathrm{Planck}}{\lambda_\mathrm{weak}}\right)^{\frac{2}{n}} . \tag{4.2}$$

Taking $n = 2$, we find $R \approx 2$ mm, whereas $n = 3$ yields $R \approx 9.3$ nm. However, recent measurements have verified that the gravitational interactions follows Newton's law for distances down to 0.2 mm [140]. In order to explain the hierarchy problem, one needs at least

[1] This is one order of magnitude larger than the expected Higgs-mass of around 10^2 GeV, for our discussion this will not make a difference

three nanometer-sized, or more and even smaller, extra dimensions: this will make it harder to measure possible deviations from standard gravitational physics at such distances.

Another problem of these models is that the hierarchy between the Planck and the weak scale is replaced by the hierarchy between the weak scale and the compactification scale. Even for $n = 6$, representing a ten-dimensional compactification scheme suitable for string theory, this ratio of the weak scale and the compactification scale is still about five orders of magnitude. In the next section, we will see how a specific brane world setup might solve this subtlety in terms of only a single compact dimension.

4.1.2 The cosmological constant problem

The cosmological constant problem is the puzzle that the bound on a cosmological energy scale coming from Hubble's constant is of much smaller value than can be explained by any effective field theory [4].

The cosmological constant can be attributed to a fluid form of matter with a negative pressure that equals minus its density. The associated energy is usually called the vacuum energy density

$$\Lambda \simeq 8\pi G_4 \left\langle \rho_{\text{vac}} \right\rangle \, . \tag{4.3}$$

On the other hand, the Friedman equations – describing the cosmological evolution of a Robertson-Walker like universe – give a bound on the cosmological constant Λ in terms of the Hubble parameter H_0

$$\Lambda \leq 3H_0^2 \, . \tag{4.4}$$

Hubble's constant measures the relative rate of expansion of the universe; recent astronomical data [145] give the value

$$H_0 \simeq 71 \text{ km s}^{-1}\text{Mpc}^{-1} \, . \tag{4.5}$$

This gives an upper-bound on the energy density of the vacuum

$$\langle \rho_{\text{vac}} \rangle \leq 4.0 \cdot 10^{-47} \text{ GeV}^4 \, . \tag{4.6}$$

A naive quantum field theory calculation of summing the zero-point energies of all normal modes of some field of mass m up to a cut-off $\lambda_{\text{cut-off}} \gg m$ yields a vacuum energy [4]

$$\begin{aligned} \langle \rho_{\text{vac}} \rangle &\simeq \int_0^{\lambda_{\text{cut-off}}} \frac{4\pi k^2 \mathrm{d}k}{(2\pi)^3} \cdot \frac{1}{2}\sqrt{k^2 + m^2} \\ &\approx \frac{\lambda_{\text{cut-off}}^4}{16\pi^2} \, . \end{aligned} \tag{4.7}$$

Depending on one's confidence in field theory, one can take the cut-off at either the weak scale or at the Planck scale. In these cases, the value of the vacuum energy-density would be 56, respectively 121 orders of magnitude above any reasonable cosmological value

$$\langle \rho_{\text{vac}} \rangle_{\text{weak}} \approx 6.3 \cdot 10^9 \text{ GeV}^4 \, , \quad \langle \rho_{\text{vac}} \rangle_{\text{Planck}} \approx 1.3 \cdot 10^{74} \text{ GeV}^4 \, . \tag{4.8}$$

The only viable field theory reason for why the cosmological constant is very small, is that it should be exactly zero through some sort of cancellation mechanism. Supersymmetry is a candidate for such a mechanism since there are fermionic and bosonic contributions to the zero-point energies which can cancel each other. However, since supersymmetry is broken at low-energy scales, the cosmological constant problem is shifted to the problem of finding a mechanism for breaking supersymmetry that protects a vanishing energy from blowing up. At present, no such mechanism has been found. In the next section, we will see how this might nevertheless be circumvented.

4.2 The Randall-Sundrum scenarios

In 1999, Randall and Sundrum published two papers in which they studied three-brane solutions in a five-dimensional Anti-de-Sitter space [138, 139]. We will now briefly review these papers. The Randall-Sundrum (RS) scenarios are by no means the only brane world models: we refer to the reviews [146, 147] for more information on this subject.

4.2.1 Two-brane setup

The first paper of Randall and Sundrum [138] discusses a two-brane setup also known as the RS1-scenario. They discussed two three-branes in a five-dimensional Anti-de-Sitter spacetime for which the radial coordinate first compactified to a circumference of $2r_c$, and then was acted upon by a $S^1/\mathbb{Z}_2$ orbifold projection.

$$r \equiv r + 2r_c\,, \quad r \equiv -r\,. \tag{4.9}$$

The two three-branes are located at the two fixed-points $r = 0$ and $r = r_c$ of the $\mathbb{Z}_2$-reflection. The brane located at the origin $r = 0$ is called the Planck brane (or hidden brane in [138]), the brane located at the edge $r = r_c$ is called the Standard Model (SM) brane (or visible brane in [138]). It is at this latter brane where our four-dimensional physics takes place. We have displayed this setup (including the mirror-brane at $r = -r_c$) in figure 4.1.

An action which has this setup as a solution is given by

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}^5x\,\sqrt{|g_{(5)}|}\,\big(R_{(5)} - 2\Lambda\big) + S_{\rm Planck} + S_{\rm SM}\,. \tag{4.10}$$

The actions for the Planck and the Standard Model brane consist of a tension part and higher order corrections that will not be specified further

$$S_{\rm Planck} = \int \mathrm{d}^4x\,\sqrt{|g_{(4)}|}\,(\mathcal{L}_{\rm Planck} - T_{\rm Planck})\,, \tag{4.11}$$

$$S_{\rm SM} = \int \mathrm{d}^4x\,\sqrt{|g_{(4)}|}\,(\mathcal{L}_{\rm SM} - T_{\rm SM})\,. \tag{4.12}$$

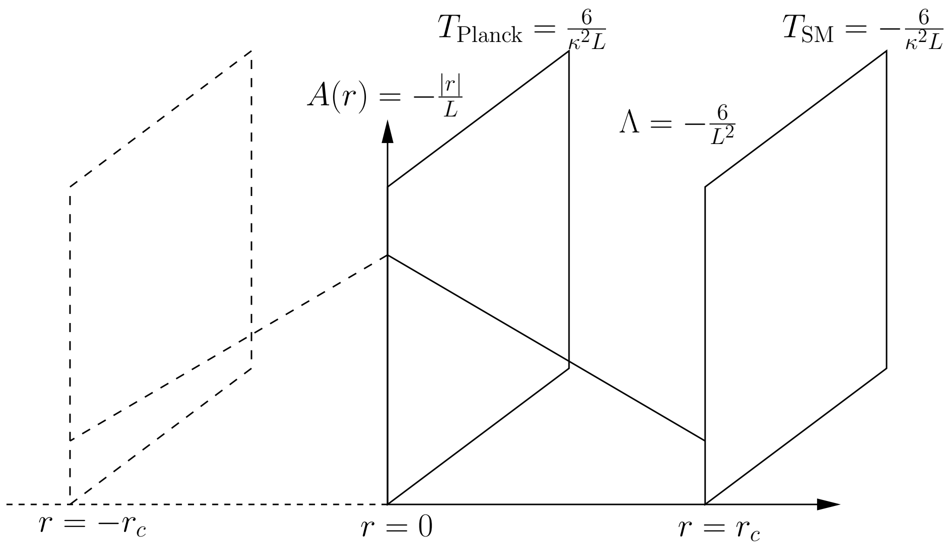

Figure 4.1: The two-brane Randall-Sundrum setup.

We will try the following Ansatz for the solution

$$\mathrm{d}s^2 = e^{2A(r)}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}r^2\,. \tag{4.13}$$

The equations of motion for this Ansatz are given by a generalization of (1.67) and (2.53)

$$\begin{aligned} 3A''(r) &= -\kappa^2 \left(T_{\text{Planck}}\,\delta(r) + T_{\text{SM}}\,\delta(r - r_c)\right)\,, \\ 6A'(r)^2 &= -\Lambda\,. \end{aligned} \tag{4.14}$$

In order to generate the appropriate delta-functions, $A(r)$ has to depend on the absolute value of the radial coordinate. Using (4.9), this gives the following expressions

$$A(r) = -\frac{|r|}{L}\,, \quad \Lambda = -\frac{6}{L^2}\,, \quad T_{\text{Planck}} = -T_{\text{SM}} = \frac{6}{\kappa^2 L}\,. \tag{4.15}$$

We will now look at the fluctuations around this solution. First, we note that the off-diagonal fluctuations of the metric correspond to those isometries of the five-dimensional space that are broken by the three-branes. Hence, these Kaluza-Klein vectors $A_\mu(x^\mu)$ are massive and can be ignored in a linearized analysis.

The remaining fluctuations are described by a symmetric tensor $h_{\mu\nu}(x^\mu)$ and a scalar field $T(x^\mu)$. The tensor $h_{\mu\nu}(x^\mu)$ generates four-dimensional gravity, and the effective gravitational action can be obtained from substituting (4.13) and (4.15) into the action (4.10) and integrating out the radial coordinate

$$\begin{aligned}\frac{1}{16\pi G_5}\int \mathrm{d}^5x\,\sqrt{|g_{(5)}|}R_{(5)} &= \frac{1}{16\pi G_5}\int_{-r_c}^{r_c}\mathrm{d}r\,e^{-2|r|/L}\int \mathrm{d}^4x\,\sqrt{|g_{(4)}|}R_{(4)}\\ &\equiv \frac{1}{16\pi G_4}\int \mathrm{d}^4x\,\sqrt{|g_{(4)}|}R_{(4)}\,. \end{aligned} \tag{4.16}$$

From this, we deduce that the effective four-dimensional gravitational constant depends only weakly on r_c

$$\begin{aligned} G_4 &= \frac{G_5}{L}\left(1-e^{-2r_c/L}\right)\\ &\approx \frac{G_5}{L}\,,\quad r_c \gg L\,. \end{aligned} \tag{4.17}$$

The second observation is that the effective four-dimensional metric, describing the gravitational fluctuations, is given by the metric localized on the Planck brane

$$\begin{aligned} g^{(4)}_{\mu\nu}(x^\mu) &= g^{\mathrm{Planck}}_{\mu\nu}(x^\mu)\\ &\equiv g^{(5)}_{\mu\nu}(x^\mu, r=0)\\ &= \eta_{\mu\nu}+h_{\mu\nu}(x^\mu)\,. \end{aligned} \tag{4.18}$$

It is with respect to this metric that physical quantities will have to be measured with. If we postulate that the Standard Model matter content is located on the brane at $r = r_c$, then all terms, in particular the mass terms, in the Lagrangian $\mathcal{L}_{\mathrm{SM}}$ will couple to the metric

$$\begin{aligned} g^{\mathrm{SM}}_{\mu\nu}(x^\mu) &\equiv g^{(5)}_{\mu\nu}(x^\mu, r=r_c)\\ &= e^{-2r_c/L}g^{\mathrm{Planck}}_{\mu\nu}(x^\mu)\,. \end{aligned} \tag{4.19}$$

From this we deduce that a mass m_0 in $\mathcal{L}_{\mathrm{SM}}$ corresponds to a physical mass m which is shifted by an exponential factor

$$m = m_0\,e^{-r_c/L}\,. \tag{4.20}$$

So we see that the energy scales on both branes are related by an exponential factor which gives a method of solving the hierarchy problem. In particular, if $r_c \approx 35L$ then we can obtain TeV scale masses on the Standard Model brane from Planck scale masses on the hidden Planck brane.

Some problems with the above model are that the fine-tuning between the weak scale and the Planck scale is replaced by the fine-tuning between the Anti-de-Sitter radius L and the brane separation r_c. This is related to the problem of treating the scalar field $T(x^\mu)$ that describes the relative motion between the branes. For consistency reasons, this so-called radion has to be a massive field with the correct expectation value in order to maintain stability of the solution.

Another problem is that the Standard Model brane has a negative tension which makes it quite hard to embed the whole setup into string theory in a natural way. We refer to the literature [146, 147] for a more thorough discussion of these subtleties.

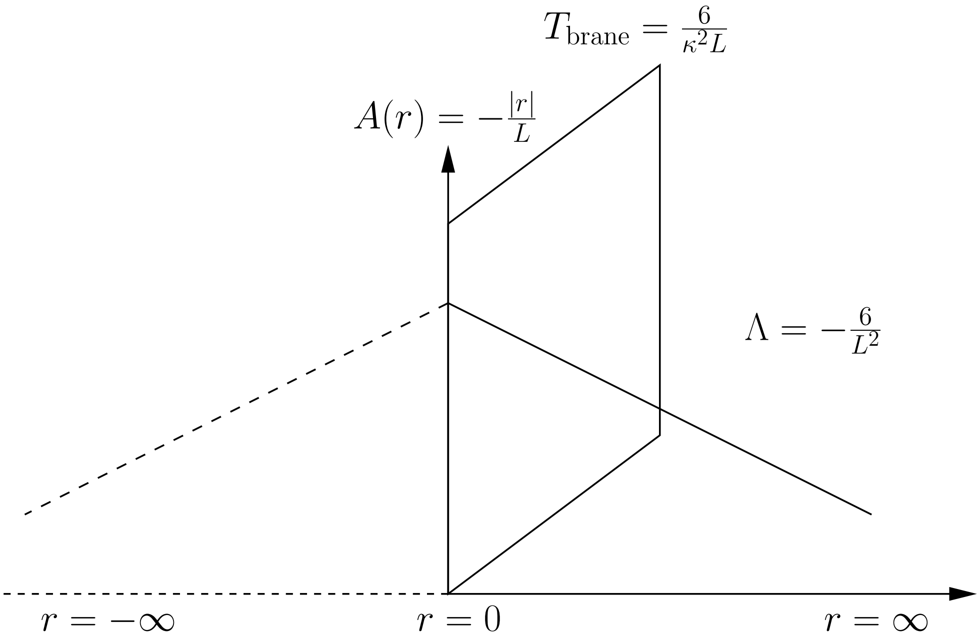

Figure 4.2: The single-brane Randall-Sundrum setup.

4.2.2 Single-brane setup

In the previous section, we made the observation that the four-dimensional Newton's constant (4.17) does not depend on the brane separation if r_c is large compared to the Anti-de-Sitter radius L. We also noted that the relevant four-dimensional metric was equal to the metric located on the brane in the origin, the Planck brane. This suggests that gravity might be effectively localized on the Planck brane.

This reasoning led Randall and Sundrum to consider a modified scenario in which the Standard Model brane was pushed to infinity [139]. The resulting one-brane scenario (or RS2-scenario[2]) is shown in figure 4.2.

The action supporting this configuration is given by

$$S = \frac{1}{2\kappa^2} \int \mathrm{d}^5 x \sqrt{|g_{(5)}|} \left(R_{(5)} - 2\Lambda\right) - T_{\text{brane}} \int \mathrm{d}^4 x \sqrt{|g_{(4)}|} \,. \tag{4.21}$$

We will again look for solutions which preserve four-dimensional Poincaré invariance

$$\mathrm{d}s^2 = e^{2A(r)} \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}r^2 \,. \tag{4.22}$$

[2]The number here indicates chronology, not the number of branes.

The equations of motion for this Ansatz are analogous to (4.14)

$$\begin{aligned} 3A''(r) &= -\kappa^2 T_{\text{brane}}\, \delta(r)\,, \\ 6A'(r)^2 &= -\Lambda\,. \end{aligned} \tag{4.23}$$

We find the following solution

$$A(r) = -\frac{|r|}{L}\,, \quad \Lambda = -\frac{6}{L^2}\,, \quad T_{\text{brane}} = \frac{6}{\kappa^2 L}\,. \tag{4.24}$$

Next, we want to analyze the effective gravitational dynamics for this solution. A standard Kaluza-Klein reduction over a compact fifth dimension of size R_5 relates Newton's constant G_5 in five dimensions to the four-dimensional gravitational constant G_4 through the volume of the compact dimension (c.f. (1.48))

$$G_4 = \frac{G_5}{R_5}\,. \tag{4.25}$$

For the RS2 scenario, where the fifth dimension is infinite, such a mechanism would imply that the effective gravitational interaction would have a vanishing strength on the brane. However, when we compare (4.25) with (4.17), then we see that something remarkable has happened: the warp-factor in the metric ensures that the infinite fifth dimension effectively behaves as a region of finite size L.

In order to determine the effective four-dimensional action, we substitute the solution given in (4.22) and (4.24) into the action (4.21) and integrate out the radial coordinate to find for the effective four-dimensional action

$$S = \frac{1}{16\pi G_4} \int \mathrm{d}^4 x \sqrt{|g_{(4)}|}\,\left(R_{(4)} - 2\Lambda_{\text{eff}}\right)\,, \tag{4.26}$$

where the effective cosmological constant Λ_{eff} on the brane actually vanishes

$$\begin{aligned} \Lambda_{\text{eff}} &\equiv \Lambda + \frac{\kappa^2 T_{\text{brane}}}{L} \\ &= 0\,. \end{aligned} \tag{4.27}$$

In the last equality we made once again use of (4.24). This is a nice result: the cosmological constant in the bulk and the tension on the brane cancel each other, and observers on the brane experience a vanishing effective cosmological constant.

However, recent astronomical observations [148,149] indicate that the universe is not only expanding, but that it is actually accelerating. This implies that the cosmological constant is not only very small, but also positive which undermines the relevance of the RS2 model. For more details concerning these problems, we refer to the literature [150].

4.2.3 Localization of gravity on the brane

The most remarkable feature of the Randall-Sundrum brane world is that gravity in the five-dimensional bulk is effectively localized on the four-dimensional brane. This is surprising since no elementary branes have gravitational degrees of freedom on their worldvolume. Instead, the fluctuations around the static solutions are described by scalar, vector or tensor multiplets.

Randall and Sundrum calculated the effective Schrödinger-like equation which the four-dimensional graviton modes have to satisfy and showed that the graviton is indeed localized near the brane. Another check is to calculate the corrections to Newton's law on the brane. In general, one-loop corrections to the graviton propagator induce $1/r^3$ corrections to the gravitational potential [151, 152]

$$V(r) = \frac{G_4 m_1 m_2}{r}\left(1 + \frac{\alpha G_4}{r^2}\right). \tag{4.28}$$

If only spins ≤ 1 contribute, then the coefficient α is given by the following expression in terms of the numbers N_s of particles of spin s

$$45\pi\alpha = 12N_1 + 3N_{\frac{1}{2}} + N_0 . \tag{4.29}$$

In order to calculate these coefficients, we first clarify the interpretation of the RS2 scenario in the AdS/CFT correspondence. Recall that in the Poincaré coordinates for Anti-de-Sitter spacetime the dual CFT is located at $r = -\infty$. The RS2 scenario has a three-brane located in the origin which acts as an UV cutoff in the dual CFT. Moving the brane to $r = -\infty$ removes the cutoff and gives back the Anti-de-Sitter solution without the absolute value function in the exponential.

Hence, from the analogy with the AdS/CFT correspondence, we expect that the theory on the brane is given by $\mathcal{N} = 4$ supersymmetric $\mathrm{SU}(N)$ Yang-Mills theory. This theory has a single vector, four spinors, and six scalars that all transform in the N^2-dimensional[3] adjoint representation

$$(N_1, N_{\frac{1}{2}}, N_0) = (N^2, 4N^2, 6N^2) . \tag{4.30}$$

Substituting this into (4.28), we obtain

$$V(r) = \frac{G_4 m_1 m_2}{r}\left(1 + \frac{2N^2 G_4}{3\pi r^2}\right). \tag{4.31}$$

In order to eliminate N, we recall the relations

$$4\pi g_s N = \left(\frac{L}{\ell_s}\right)^4 , \qquad \frac{1}{16\pi G_{10}} = \frac{2\pi}{g_s^2 (2\pi\ell_s)^8} . \tag{4.32}$$

[3] The adjoint representation of $\mathrm{SU}(N)$ has dimension $N^2 - 1$, which scales as N^2 for large N.

The gravitational coupling constants in ten and five dimensions are related to each other by the volume of the five sphere S^5

$$\begin{aligned} G_5 &= \frac{G_{10}}{\Omega_{(5)} L^5} \\ &= \frac{G_{10}}{\pi^3 L^5} \, . \end{aligned} \tag{4.33}$$

Substituting this into (4.31), we finally obtain

$$V(r) = \frac{G_4 m_1 m_2}{r} \left(1 + \frac{2L^2}{3r^2} \right) \, . \tag{4.34}$$

This is the same result as was found in [153] from a canonical one-loop calculation. With this illustrative example, we conclude this section. More discussion on this topic can be found in [139].

4.3 Supersymmetric brane worlds

The brane world scenarios described in the previous section have a very rich structure, and one would like to embed them in a natural way into string theory. As a first step, one would like to have a supersymmetric version of a brane world scenario. It turns out that finding such a simple extension is nontrivial. In this section, we will summarize the current status of the search for a supersymmetric brane world scenario.

4.3.1 Conditions on the scalar potential

As we remarked in the previous chapter, one can make a distinction between "thin" branes and "thick" branes. The "thin" brane approach has as an advantage that it is more similar to the original RS scenario. In [154] a method was developed to formulate supersymmetric theories in the presence of delta function singularities in general, and in the presence of brane sources in particular. A possible embedding of such a supersymmetric RS2 scenario with singular sources into string theory might be a suitable Calabi-Yau compactification of the eleven-dimensional Hořava-Witten model [53].

The "thick" brane approach would be to search for supersymmetric interpolating soliton solutions of five-dimensional supergravity, in much the same way as was described in the previous chapter. For such supersymmetric branes, the amount of supersymmetry preserved on the brane is generically half of the supersymmetry of the bulk theory. In particular, in order to have $\mathcal{N} = 1$ supergravity on the four-dimensional brane, one would have to start with $\mathcal{N} = 2$ supergravity in the five-dimensional bulk.

As we saw in the previous chapter, in order to find interpolating soliton solutions, the critical points of the scalar potential need to be analyzed. The toy model example that we discussed in chapter 3 only had a single scalar field, but the matter multiplets that can couple to five-dimensional models have a set of scalars ϕ^i.

The structure of such supergravity matter-couplings is complicated. A useful tool in studying matter-couplings is the geometry induced by the scalar fields present in the various matter multiplets. These scalars can be viewed as coordinates on a manifold. The extensive mathematical literature on the various manifolds can then be used to analyze the structure of the various matter-couplings. In particular, the generalization of (3.73) in the case of multiple scalar fields ϕ^i is given by

$$V(\phi^i) = \frac{(d-1)^2}{2} g^{ij} \frac{\partial W}{\partial \phi^i} \frac{\partial W}{\partial \phi^j} - \frac{d(d-1)}{4} W(\phi)^2 \,, \tag{4.35}$$

where g^{ij} is the metric on the manifold spanned by the scalars. The flow equations (3.80) are then given by

$$\begin{aligned} \phi^{i\prime}(r) &= \mp (d-1) g^{ij} \frac{\partial W}{\partial \phi^j} \,, \\ A'(r) &= \pm \frac{1}{2} W(\phi^i) \,. \end{aligned} \tag{4.36}$$

For a supersymmetric brane world scenario exhibiting localization of gravity to exist, the following conditions need to be fulfilled.

1. The scalar potential $V(\phi^i)$ needs to have two different stable[4] critical points, ϕ_1^i and ϕ_2^i corresponding to fixed points of the holographically dual beta-function
$$V'(\phi_1^i) = V'(\phi_2^i) = 0 \,, \quad \beta(\phi_1^i) = \beta(\phi_2^i) = 0 \,. \tag{4.37}$$

2. At these critical points, the value of the scalar potential needs to be equal and negative in order to have an Anti-de-Sitter background with the same cosmological constant on both sides of the brane
$$V(\phi_1^i) = V(\phi_2^i) < 0 \,. \tag{4.38}$$

3. In order to have a decreasing warp factor on both sides of the brane, the holographically dual beta-function needs to have two IR fixed points: i.e its derivative at the critical points needs to be positive
$$\beta'(\phi_1^i) = \beta'(\phi_2^i) > 0 \,. \tag{4.39}$$

4. Finally, one needs to find an explicit solution that interpolates between the two critical points of the scalar potential.

[4]Stable in the sense of satisfying the Breitenlohner-Freedman bound [93] defined in section 2.3.1.

4.3.2 Overview of $\mathcal{N} = 2$ supergravity in $D = 5$

We will now review the status of the search for supersymmetric brane world scenarios in the context sketched above by giving a short overview of the literature on $\mathcal{N} = 2$ supergravity in five dimensions.

The pure, ungauged, $\mathcal{N} = 2$ supergravity in five dimensions was developed in 1981 by Cremmer [155]. A few years later, Günaydin, Sierra and Townsend [156–158] used the $\mathcal{N} = 2$ vector multiplets to construct the gauged $\mathcal{N} = 2$ supergravity. In recent years, the couplings to other matter multiplets have also been constructed. In particular, Günaydin and Zagermann [159–161] constructed the couplings of $\mathcal{N} = 2$ tensor multiplets to supergravity, and Ceresole and Dall'Agata [162–164] did the same for $\mathcal{N} = 2$ hypermultiplets. The total Lagrangian of five-dimensional $\mathcal{N} = 2$ supergravity coupled to an arbitrary number of vector multiplets, tensor multiplets and hypermultiplets was given in [162].

It was shown in [165] that five-dimensional $\mathcal{N} = 2$ supergravity coupled only to vector multiplets could not fulfill the above criteria, since all critical points of the scalar potential were UV fixed points. Shortly thereafter, it was also shown that adding tensor multiplets did not improve the situation [162]. Adding hypermultiplets yields more possibilities: it was even shown that IR fixed points could exist [166]. For vacua having the same value of the cosmological constant, only UV-IR flows have been found so far.

As a final remark, we mention that if a supersymmetric RS scenario will be found, it is still not directly related to string theory. A possible embedding into a higher-dimensional theory might be given by the compactification of eleven-dimensional supergravity on a three-dimensional Calabi-Yau manifold[5] to five-dimensional supergravity coupled to matter. In particular, the number of vector multiplets and hypermultiplets in five-dimensional supergravity realizing a supersymmetric brane world should be related to the Hodge numbers of the Calabi-Yau manifold.

In four dimensions, conformal supergravity has turned out to be a very effective tool in analyzing matter-couplings to ordinary supergravity. The next two chapters will describe the conformal approach to five-dimensional supergravity matter-couplings. In chapter 5, we will introduce conformal supersymmetry and construct the so-called Weyl multiplet: the smallest irreducible supermultiplet containing the graviton. In chapter 6, we will couple vector, tensor and hypermultiplets in a superconformal manner to this Weyl multiplet.

[5] Such complex manifolds have six real dimensions.

Chapter 5

Weyl multiplets of conformal supergravity

Superconformal methods are an elegant way to construct general couplings of Poincaré supergravities to matter [167, 168]. This so-called superconformal tensor calculus uses the basic superconformal multiplets as a starting point for a gauge-fixing procedure in which the superconformal symmetry is broken down to Poincaré supersymmetry.

Conformal supergravities have been constructed in various dimensions (for a review, see [38]), but not yet in five dimensions. In the five-dimensional case, these matter coupled supergravities have recently attracted renewed attention due to the important role they play in the Randall–Sundrum (RS) scenarios [138, 139] and the AdS_6/CFT_5 [169, 170] and AdS_5/CFT_4 [171] correspondences.

Moreover, it has turned out in the past that superconformal constructions lead to new insights and results in the structure of matter-couplings. A recent example is the insight in relations between hyper-Kähler cones and quaternionic manifolds, based on the study of superconformal invariant matter-couplings with hypermultiplets [172]. For all these reasons, a superconformal construction of general matter-couplings of $\mathcal{N} = 2\,, D = 5$ supergravity is useful.

The superconformal multiplet that contains all the (independent) gauge fields of the superconformal algebra is called the Weyl multiplet. In this chapter, we take the first step in the superconformal program by constructing the Weyl multiplets of $\mathcal{N} = 2\,, D = 5$ conformal supersymmetry. In our construction, we use the methods developed first for $\mathcal{N} = 1\,, D = 4$ [173, 174]. They are based on gauging the conformal superalgebra [175], which in our case is $\mathrm{F}^2(4)$.

In general, one needs to include matter fields to have an equal number of bosons and fermions. We will see that in five dimensions there are two possible sets of matter fields one can add, yielding two versions of the Weyl multiplet: the Standard Weyl multiplet and the

Dilaton Weyl multiplet. This result is similar to what was found for the $(1,0)\,, D=6$ Weyl multiplet [176].

In [169], the field content and transformation rules for the Standard Weyl multiplet were constructed from the $\mathrm{F}(4)$-gauged six-dimensional supergravity [177] using the AdS_6/CFT_5 correspondence. Another attempt was undertaken in [178] by reducing the six-dimensional result [176] to five dimensions. However, by gauge-fixing some symmetries of the superconformal algebra during the reduction process, they found a multiplet that is larger than the Weyl multiplet that we will construct.

We will start this chapter by giving an introduction to the algebraic structure of rigid conformal (super)symmetry. In section 5.2, we will discuss local conformal supersymmetry and the gauging of the superconformal algebra. In section 5.3, we will construct the supercurrent as well as the improved supercurrent in order to determine the field content and the linearized transformation rules of the two Weyl multiplets. The linearized results will be used in section 5.4 to construct the fully non-linear transformation rules of the two Weyl multiplets as well as the modified superconformal algebra. Finally, in section 5.5, we will clarify the connection between the Weyl multiplets by showing that the coupling of an off-shell vector multiplet to the Standard Weyl multiplet gives rise to the Dilaton Weyl multiplet.

This chapter is based on the work published in [16]. A similar paper with overlapping results appeared somewhat later [179]. For a more extensive background on conformal supergravity, we refer to the reviews [180, 181].

5.1 Rigid superconformal symmetry

In this section, we will start with deriving the rigid superconformal transformations. After that, we will clarify the algebraic structure of the superconformal transformations by giving the (anti-)commutation relations of the superconformal algebra. Finally, we discuss some aspects of the corresponding representation theory.

5.1.1 Conformal Killing vectors

We will first introduce conformal symmetry, and in a second step we will extend this to conformal supersymmetry. Given a spacetime with a metric tensor $g_{\mu\nu}(x)$, the conformal transformations are defined as the class of general coordinate transformations that leaves "angles" invariant. The parameters of these coordinate transformations define a conformal Killing vector $k^\mu(x)$

$$\delta_{\rm gct} x^\mu = -k^\mu(x)\,. \tag{5.1}$$

The defining equation for this conformal Killing vector is given by

$$\delta_{\rm gct}(k) g_{\mu\nu}(x) \equiv \nabla_\mu k_\nu(x) + \nabla_\nu k_\mu(x) = \omega(x) g_{\mu\nu}(x)\,, \tag{5.2}$$

where $\omega(x)$ is an arbitrary function, $k_\mu = g_{\mu\nu}k^\nu$, and the covariant derivative is given by $\nabla_\mu k_\nu = \partial_\mu k_\nu - \Gamma^\rho_{\mu\nu}k_\rho$. Taking the trace of (5.2), and making the restriction to flat D-dimensional Minkowski spacetime yields

$$\partial_{(\mu}k_{\nu)}(x) - \tfrac{1}{D}\eta_{\mu\nu}\partial^\rho k_\rho(x) = 0\,. \tag{5.3}$$

Taking the derivative of (5.3), we obtain

$$\Box k_\mu(x) + \left(1 - \tfrac{2}{D}\right)\partial_\mu\partial^\rho k_\rho(x) = 0\,. \tag{5.4}$$

In $D = 2$, the solutions to this are given by the infinite-dimensional group of analytic coordinate transformations[1]. In dimensions $D > 2$, the group of conformal transformations is finite-dimensional, and the most general solution to (5.3) is given by

$$k^\mu(x) = \xi^\mu + \Lambda_M^{\mu\nu}x_\nu + \Lambda_D x^\mu + (x^2\Lambda_K^\mu - 2x^\mu x\cdot\Lambda_K)\,. \tag{5.5}$$

Corresponding to the parameters ξ^μ are the translations P_μ, the parameters $\Lambda_M^{\mu\nu}$ correspond to Lorentz rotations $M_{\mu\nu}$, to Λ_D are associated the dilatations D, and Λ_K^μ are the parameters of 'special conformal transformations' K_μ. Thus, the full set of conformal transformations δ_C can be expressed as follows:

$$\delta_C = \xi^\mu P_\mu + \Lambda_M^{\mu\nu}M_{\mu\nu} + \Lambda_D D + \Lambda_K^\mu K_\mu\,. \tag{5.6}$$

5.1.2 Conformal Killing spinors

We next consider the extension to conformal supersymmetry. In D-dimensional Minkowski spacetime, the conformal supersymmetry transformations are defined as the supersymmetry transformations that satisfy

$$\partial_\mu\epsilon^i(x) - \tfrac{1}{D}\gamma_\mu\partial\!\!\!/\epsilon^i(x) = 0\,. \tag{5.7}$$

The solution to this equation is given by

$$\epsilon^i(x) = \epsilon^i + \mathrm{i}\,x^\mu\gamma_\mu\eta^i\,, \tag{5.8}$$

where the (constant) parameters ϵ^i correspond to "ordinary" supersymmetry transformations Q^i_α and the parameters η^i define special conformal supersymmetries generated by S^i_α.

The conformal transformations (5.5) and the supersymmetries (5.8) do not form a closed algebra. To obtain closure, one must introduce additional R-symmetry generators. In particular, in the case of 8 supercharges Q^i_α in $D = 5$, there is an additional $\mathrm{SU}(2)$ R-symmetry

[1]Recall that the Cauchy-Riemann equations for a complex function reduce to the wave equation for the real and imaginary part of that function.

Generators	P_a	$M_{[ab]}$	D	K_a	$U_{(ij)}$	$Q_{\alpha i}$	$S_{\alpha i}$
Parameters	ξ^a	$\Lambda_M^{[ab]}$	Λ_D	Λ_K^a	$\Lambda_U^{(ij)}$	ϵ^i	η^i
# symmetries	5	10	1	5	3	8	8

Table 5.1: The generators of the superconformal algebra $\mathrm{F}^2(4)$.

with generators $U_{ij} = U_{ji} (i = 1, 2)$. Thus, the full set of superconformal transformations δ_C is given by:

$$\delta_C = \xi^\mu P_\mu + \Lambda_M^{\mu\nu} M_{\mu\nu} + \Lambda_D D + \Lambda_K^\mu K_\mu + \Lambda_U^{ij} U_{ij} + \mathrm{i}\,\bar{\epsilon} Q + \mathrm{i}\,\bar{\eta} S\,. \tag{5.9}$$

The extra factor of i in the last two terms is necessary because of the reality properties of five-dimensional spinors.

We have summarized the generators and parameters of the five-dimensional superconformal algebra $\mathrm{F}^2(4)$ in table 5.1. Also indicated here are number symmetries associated to each generator: in total, there are 24+16 bosonic plus fermionic symmetries.

5.1.3 The superconformal algebra $\mathrm{F}^2(4)$

When one allows for central charges, there exist many varieties of superconformal algebras [182, 183]. However, so far a suitable superconformal Weyl multiplet has only been constructed from those superconformal algebras[2] that appear in Nahm's classification [185]. The particular real form that we need here is the five-dimensional algebra denoted by $\mathrm{F}^2(4)$, see tables 5 and 6 in [186].

The commutation relations defining the $\mathrm{F}^2(4)$ algebra are given by

$$\begin{array}{rclrcl}
[M_{bc}, P_a] &=& -\eta_{a[b} P_{c]}\,, & [D, P_a] &=& P_a\,, \\
{[M_{bc}, K_a]} &=& -\eta_{a[b} K_{c]}\,, & [D, K_a] &=& -K_a\,, \\
\left[M_{ab}, M^{cd}\right] &=& -2\delta_{[a}{}^{[c} M_{b]}{}^{d]}\,, & \left[U_{ij}, U^{kl}\right] &=& 2\delta_{(i}{}^{(k} U_{j)}{}^{l)}\,, \\
{[P_a, K_b]} &=& 2(\eta_{ab} D + 2M_{ab})\,, & & & \\
& & & & & \\
{[M_{ab}, Q_{i\alpha}]} &=& -\frac{1}{4}(\gamma_{ab} Q_i)_\alpha\,, & [D, Q_{i\alpha}] &=& \frac{1}{2} Q_{i\alpha}\,, \\
{[M_{ab}, S_{i\alpha}]} &=& -\frac{1}{4}(\gamma_{ab} S_i)_\alpha\,, & [D, S_{i\alpha}] &=& -\frac{1}{2} S_{i\alpha}\,, \\
{[U_{kl}, Q_{i\alpha}]} &=& -\varepsilon_{i(k} Q_{l)\alpha}\,, & [K_a, Q_{i\alpha}] &=& \mathrm{i}\,(\gamma_a S_i)_\alpha\,, \\
{[U_{kl}, S_{i\alpha}]} &=& -\varepsilon_{i(k} S_{l)\alpha}\,, & [P_a, S_{i\alpha}] &=& -\mathrm{i}\,(\gamma_a Q_i)_\alpha\,, \\
& & & & & \\
\{Q_{i\alpha}, Q_{j\beta}\} &=& -\frac{1}{2}\varepsilon_{ij} (\gamma^a)_{\alpha\beta} P_a\,, & \{S_{i\alpha}, S_{j\beta}\} &=& -\frac{1}{2}\varepsilon_{ij} (\gamma^a)_{\alpha\beta} K_a\,, \\
\{Q_{i\alpha}, S_{j\beta}\} &=& \multicolumn{4}{l}{-\frac{1}{2}\,\mathrm{i}\,\left(\varepsilon_{ij} C_{\alpha\beta} D + \varepsilon_{ij} (\gamma^{ab})_{\alpha\beta} M_{ab} + 3 C_{\alpha\beta} U_{ij}\right)\,.}
\end{array} \tag{5.10}$$

[2]An exception is the ten-dimensional Weyl multiplet [184], which is not based on a known algebra.

The first seven commutation relations between the bosonic generators form the bosonic subalgebra $\mathrm{SO}(2,5)\times\mathrm{SU}(2)$: the conformal algebra times the R-symmetry group. In particular, as we referred to in chapter 2, the $\mathrm{SO}(2,5)$ commutation relations are obtained by substituting (2.50) into (2.51). The commutation relations of the (conformal) supercharges with the bosonic generators indicates that the supercharges are spinorial $\mathrm{SU}(2)$-doublets with dilatational weight of $\pm\frac{1}{2}$, respectively.

We will also give the form of the anti-commutators in terms of commutators of infinitesimal transformations on the fields. We can write the algebra (5.10) as

$$[T_A,T_B]=f_{AB}{}^C T_C\,,\qquad \{T_A,T_B\}=-f_{AB}{}^C T_C\,, \tag{5.11}$$

where the minus sign in the second equation is due to the factor of i in the last two terms of (5.9). We then have for all commutators of infinitesimal transformations

$$\left[\delta_A(\Lambda_1^A),\delta_B(\Lambda_2^B)\right]=\delta_C(\Lambda_3^C)\,,\qquad \Lambda_3^C=\Lambda_2^B\Lambda_1^A f_{AB}{}^C\,. \tag{5.12}$$

In particular, the anti-commutation relations for the conformal supercharges translate into the following infinitesimal commutators

$$[\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)] = \delta_P\left(\tfrac{1}{2}\bar\epsilon_2\gamma_\mu\epsilon_1\right)\,, \tag{5.13}$$

$$[\delta_S(\eta),\delta_Q(\epsilon)] = \delta_D(\tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\eta)+\delta_M(\tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\gamma^{ab}\eta)+\delta_U(-\tfrac{3}{2}\,\mathrm{i}\,\bar\epsilon^{(i}\eta^{j)})\,, \tag{5.14}$$

$$[\delta_S(\eta_1),\delta_S(\eta_2)] = \delta_K(\tfrac{1}{2}\bar\eta_2\gamma^a\eta_1)\,. \tag{5.15}$$

As a final note, we remark that the superconformal algebra is equipped with two gradings. First of all, there is a $\mathbb{Z}_2$-grading that separates the generators into bosonic and fermionic operators: this dictates the kind of bracket (commutator or anti-commutator) that has to be specified for two particular generators. There is also a $\mathbb{Z}_5$-grading given by the dilatational weights of the various generators (the numbers on the right-hand side of the commutator with the dilatational generator D): this determines what specific operators can appear on the right-hand side given the left-hand side of an algebraic relation. The coefficients in the superconformal algebra are fixed (up to an overall normalization) by imposing the generalized Jacobi-identities.

5.1.4 Representation theory

We wish to consider representations of the conformal algebra on fields $\phi^\alpha(x)$ where α stands for a collection of internal indices referring to the stability subalgebra of $x^\mu=0$. From the expression for the conformal Killing vector (5.5), we deduce that this algebra is isomorphic to the algebra generated by $M_{\mu\nu}, D$ and K_μ.

We denote this stability subalgebra by H, and the generators of H by $\Sigma_{\mu\nu}\,,\Delta$ and κ_μ. Denoting the complete conformal algebra by G, we can write rigid conformal transformations as

$$G_{\mathrm{rigid}}=(P\otimes H)_{\mathrm{rigid}}\,. \tag{5.16}$$

Applying the theory of induced representations, it follows that any representation $(\Sigma\,,\Delta\,,\kappa)$ of the stability subalgebra H will induce a representation $(P\,,M\,,D\,,K)$ of the full conformal algebra G with the following transformation rules (we suppress any internal indices)

$$\begin{aligned}
\delta_P(\xi)\phi(x) &= \xi^\mu\partial_\mu\phi(x)\,,\\
\delta_M(\Lambda_M)\phi(x) &= \tfrac{1}{2}\Lambda_M^{\mu\nu}(x_\nu\partial_\mu - x_\mu\partial_\nu)\phi(x) + \delta_\Sigma(\Lambda_M)\phi(x)\,,\\
\delta_D(\Lambda_D)\phi(x) &= \Lambda_D x^\lambda\partial_\lambda\phi(x) + \delta_\Delta(\Lambda_D)\phi(x)\,,\\
\delta_K(\Lambda_K)\phi(x) &= \Lambda_K^\mu(x^2\partial_\mu - 2x_\mu x^\lambda\partial_\lambda)\phi(x) +\\
&\quad \big(\delta_\delta(-2x\cdot\Lambda_K) + \delta_\Sigma(-4x_{[\mu}\Lambda_{K\nu]}) + \delta_\kappa(\Lambda_K)\big)\,\phi(x)\,.
\end{aligned} \tag{5.17}$$

Lorentz-transformations

We now look at the non-trivial representation (Σ, Δ, κ) that we use in this thesis. Concerning the Lorentz representations, we will encounter anti-symmetric tensors $\phi_{a_1\cdots a_n}(x)(n = 0, 1, 2, \cdots)$, and spinors $\psi_\alpha(x)$:

$$\delta_\Sigma(\Lambda_M)\phi_{a_1\cdots a_n}(x) = -n\,(\Lambda_M)_{[a_1}{}^b\phi_{|b|a_2\cdots a_n]}(x)\,, \tag{5.18}$$

$$\delta_\Sigma(\Lambda_M)\psi_\alpha(x) = -\tfrac{1}{4}\Lambda_M^{ab}(\gamma_{ab})_\alpha{}^\beta\psi_\beta(x)\,. \tag{5.19}$$

Dilatations

Secondly, we consider the dilatations. For most fields, the Δ-transformation is determined by a single number w, which is called the Weyl weight of ϕ^α

$$\delta_\Delta(\Lambda_D)\phi^\alpha(x) = w\Lambda_D\phi^\alpha(x)\,. \tag{5.20}$$

For scalar fields, it is often convenient to consider the set of fields $\phi^\alpha(x)$ as the coordinates of a scalar manifold with affine connection $\Gamma^\gamma_{\alpha\beta}$. With this understanding, the transformation of ϕ^α under dilatations can be characterized by

$$\delta_\Delta(\Lambda_D)\phi^\alpha = \Lambda_D k^\alpha(\phi)\,, \tag{5.21}$$

for some arbitrary function $k^\alpha(\phi)$.

Special conformal transformations

All[3] fields that we will discuss in this thesis are invariant under the internal special conformal transformations

$$\delta_\kappa(\Lambda_K)\phi^\alpha(x) = 0\,. \tag{5.22}$$

[3] An exception is formed by some of the gauge fields in the Weyl multiplet.

However, this does not mean that special conformal transformations do not play a role in constructing superconformal theories. Indeed, the derivative of a scalar field $\phi^\alpha(x)$ that transforms under Δ- and κ-transformations according to (5.21) and (5.22), will transform under special conformal transformations according to

$$\delta_\kappa(\Lambda_K)\partial_\mu\phi^\alpha(x) = -2\Lambda_{K\mu}k^\alpha(\phi)\,. \tag{5.23}$$

Scalar manifold geometry

In the next chapter, we will construct superconformal field theories. In particular, we will also construct superconformal actions. Constructing an action for a set of scalar fields $\phi^\alpha(x)$ corresponds to taking a scalar manifold with a metric $g_{\alpha\beta}(\phi)$

$$\mathcal{L} = -\tfrac{1}{2}g_{\alpha\beta}(\phi)\partial^\mu\phi^\alpha(x)\partial_\mu\phi^\beta(x)\,. \tag{5.24}$$

Such a Lagrangian describes a sigma model with the D-dimensional spacetime as "worldsheet" and the scalar manifold as target space. Requiring dilatational invariance of this kinetic term yields that the vector $k^\alpha(\phi)$ should be a homothetic Killing vector: namely it should satisfy the conformal Killing equation (5.3) for *constant* $\omega(x)$:

$$\mathfrak{D}_\alpha k_\beta + \mathfrak{D}_\beta k_\alpha = (D-2)g_{\alpha\beta}\,, \tag{5.25}$$

where D is the dimension of the "worldsheet", and where the covariant derivative on the scalar manifold is given by $\mathfrak{D}_\alpha k_\beta = \partial_\alpha k_\beta - \Gamma^\gamma_{\alpha\beta}k_\gamma$.

Demanding invariance of (5.24) under the special conformal transformations (5.23) restricts $k^\alpha(\phi)$ even further to be an *exact* homothetic Killing vector

$$k_\alpha = \partial_\alpha\chi\,, \tag{5.26}$$

for some function $\chi(\phi)$. One can show that the restrictions (5.25) and (5.26) are equivalent to

$$\mathfrak{D}_\alpha k^\beta \equiv \partial_\alpha k^\beta + \Gamma^\beta_{\alpha\gamma}k^\gamma = w\delta^\beta_\alpha\,. \tag{5.27}$$

The Weyl weight w of ϕ^α has to be $w = \frac{1}{2}(D-2)$, or $w = \frac{3}{2}$ in $D = 5$. The proof of the necessity of (5.27) can be extracted from [187], see also [188, 189]. In these papers the conditions for conformal invariance of a sigma model with gravity are investigated. Note that the condition (5.27) can be formulated *independently* of a metric. Only an affine connection is necessary.

For the special case of a zero affine connection, the solution to (5.27) is given by

$$k^\alpha(\phi) = w\phi^\alpha\,, \tag{5.28}$$

and the transformation rule (5.21) reduces to the form (5.20). Note that the homothetic Killing vector (5.28) is indeed exact with χ given by

$$\chi = \tfrac{1}{(D-2)}g_{\alpha\beta}k^\alpha k^\beta\,. \tag{5.29}$$

Supersymmetric generalization

To construct field representations of the superconformal algebra, one can again apply the method of induced representations. In this case, one must use superfields $\Phi^\alpha(x^\mu, \theta^i_\alpha)$, where α stands for a collection of internal indices referring to the stability subalgebra of $x^\mu = \theta^i_\alpha = 0$. This algebra is isomorphic to the algebra generated by $M_{\mu\nu}\,, D\,, K_\mu\,, U_{ij}$ and S^i_α.

An additional complication, not encountered in the bosonic case, is that the representation one obtains is reducible. To obtain an irreducible representation, one must impose constraints on the superfield. It is at this point that the transformation rules become nonlinear in the fields. In this thesis, we will follow a different approach. Instead of working with superfields, we will work with the component fields. The nonlinear transformation rules are obtained by imposing the superconformal algebra.

$\mathrm{SU}(2)$-transformations

In the supersymmetric case, we must specify the $\mathrm{SU}(2)$-properties of the different fields as well as the behavior under S-supersymmetry. Concerning the $\mathrm{SU}(2)$, we will only encounter scalars ϕ, doublets ψ^i and triplets $\phi^{(ij)}$ whose transformations are given by

$$\begin{aligned} \delta_{\mathrm{SU}(2)}(\Lambda_U^{ij})\phi &= 0\,, \\ \delta_{\mathrm{SU}(2)}(\Lambda_U^{ij})\psi^i(x) &= -\Lambda_U{}^i{}_j\psi^j(x)\,, \\ \delta_{\mathrm{SU}(2)}(\Lambda_U^{ij})\phi^{ij}(x) &= -2\Lambda_U{}^{(i}{}_k\phi^{j)k}(x)\,. \end{aligned} \tag{5.30}$$

S-transformations

This leaves us with specifying how a given field transforms under the special supersymmetries generated by S^i_α. In superfield language the full S-transformation is given by a combination of an x-dependent translation in superspace, with parameter $\epsilon^i(x) = \mathrm{i}\, x^\mu\gamma_\mu\eta^i$, and an internal S-transformation. This is in perfect analogy to the bosonic case. In terms of component fields, the same holds true. The x-dependent contribution is obtained by making the substitution

$$\epsilon^i \to \mathrm{i}\,\not{x}\eta^i \tag{5.31}$$

in the Q-supersymmetry rules. The internal S-transformations can be deduced by imposing the superconformal algebra.

5.2 Local superconformal symmetry

In this section, we will discuss local superconformal symmetry. We will first introduce the various gauge fields, their transformation rules and their covariant curvatures. After that, we will discuss the emergence of curvature constraints.

Generators	P_a	$M_{[ab]}$	D	K_a	$U_{(ij)}$	$Q_{\alpha i}$	$S_{\alpha i}$
Gauge fields	$e_\mu{}^a$	$\omega_\mu{}^{[ab]}$	b_μ	$f_\mu{}^a$	$V_\mu^{(ij)}$	ψ_μ^i	ϕ_μ^i
# d.o.f.	9	<u>50</u>	0	<u>25</u>	12	24	<u>40</u>

Table 5.2: The gauge fields of the superconformal algebra $\mathrm{F}^2(4)$.

5.2.1 Gauge fields and curvatures

The procedure for gauging the superconformal algebra proceeds along similar ways as for any other local symmetry algebra. We assign to every generator of the superconformal algebra T^A a gauge field $h_\mu{}^A$. We have indicated in table 5.2 the various gauge fields for the superconformal algebra $\mathrm{F}^2(4)$. Since every gauge field has an extra spacetime index μ, the gauge fields have 120+80 bosonic plus fermionic field components: five times as large as the number of gauge symmetries. The number of degrees of freedom is the difference of these two numbers: there are only 96+64 independent gauge field components.

For example, the gauge field $e_\mu{}^a$ can be acted upon by local Lorentz transformations, translations and dilatations to reduce its 25 field components to 9 degrees of freedom. Similar considerations apply to the gauge fields b_μ , V_μ^{ij} and ψ_μ^i. The three gauge fields $\omega_\mu{}^{ab}$, $f_\mu{}^a$ and ϕ_μ^i do not have such a restriction on their field components, and they also have their degrees of freedom underlined, since they will become dependent gauge fields, as we will explain in more detail in the next section.

Gauge fields transformation rules

We can determine the transformation rules for the gauge fields using the general rules for gauge theories

$$\delta h_\mu{}^A = \partial_\mu \epsilon^A + \epsilon^C h_\mu{}^B f_{BC}{}^A \,. \tag{5.32}$$

From the algebra (5.10), we read off

$$\begin{aligned}
\delta e_\mu{}^a &= \mathcal{D}_\mu \xi^a - \Lambda_M^{ab} e_{\mu b} - \Lambda_D e_\mu{}^a + \tfrac{1}{2}\bar\epsilon\gamma^a\psi_\mu \,,\\
\delta f_\mu{}^a &= \mathcal{D}_\mu \Lambda_K^a - \Lambda_M^{ab} f_{\mu b} + \Lambda_D f_\mu{}^a + \tfrac{1}{2}\bar\eta\gamma^a\phi_\mu \,,\\
\delta \omega_\mu{}^{ab} &= \mathcal{D}_\mu \Lambda_M^{ab} - 4\xi^{[a} f_\mu{}^{b]} - 4\Lambda_K^{[a} e_\mu{}^{b]} + \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\gamma^{ab}\phi_\mu - \tfrac{1}{2}\,\mathrm{i}\,\bar\eta\gamma^{ab}\psi_\mu \,,\\
\delta b_\mu &= \partial_\mu \Lambda_D - 2\xi^a f_{\mu a} + 2\Lambda_K^a e_{\mu a} + \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\phi_\mu + \tfrac{1}{2}\,\mathrm{i}\,\bar\eta\psi_\mu \,,\\
\delta V_\mu^{ij} &= \partial_\mu \Lambda_U^{ij} - 2\Lambda_U{}^{(i}{}_\ell V_\mu^{j)\ell} - \tfrac{3}{2}\,\mathrm{i}\,\bar\epsilon^{(i}\phi_\mu^{j)} + \tfrac{3}{2}\,\mathrm{i}\,\bar\eta^{(i}\psi_\mu^{j)} \,,\\
\delta \psi_\mu^i &= \mathcal{D}_\mu \epsilon^i - \tfrac{1}{4}\Lambda_M^{ab}\gamma_{ab}\psi_\mu^i - \tfrac{1}{2}\Lambda_D\psi_\mu^i - \Lambda_U{}^i{}_j\psi_\mu^j + \mathrm{i}\,\xi^a\gamma_a\phi_\mu^i - \mathrm{i}\,e_\mu^a\gamma_a\eta^i \,,\\
\delta \phi_\mu^i &= \mathcal{D}_\mu \eta^i - \tfrac{1}{4}\Lambda_M^{ab}\gamma_{ab}\phi_\mu^i + \tfrac{1}{2}\Lambda_D\phi_\mu^i - \Lambda_U{}^i{}_j\phi_\mu^j - \mathrm{i}\,\Lambda_K^a\gamma_a\psi_\mu^i + \mathrm{i}\,f_\mu^a\gamma_a\epsilon^i \,,
\end{aligned} \tag{5.33}$$

where $\mathcal{D}_\mu$ is the covariant derivative with respect to Lorentz rotations, dilatations, and $\mathrm{SU}(2)$-transformations:

$$
\begin{aligned}
\mathcal{D}_\mu \xi^a &= \partial_\mu \xi^a + \omega_\mu{}^{ab}\xi_b + b_\mu \xi^a \,, \\
\mathcal{D}_\mu \Lambda_K^a &= \partial_\mu \Lambda_K^a + \omega_\mu{}^{ab}\Lambda_{Kb} - b_\mu \Lambda_K^a \,, \\
\mathcal{D}_\mu \Lambda_M^{ab} &= \partial_\mu \Lambda_M^{ab} + 2\omega_{\mu c}{}^{[a}\Lambda_M^{b]c} \,, \\
\mathcal{D}_\mu \epsilon^i &= \partial_\mu \epsilon^i + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\epsilon^i + \tfrac{1}{2} b_\mu \epsilon^i - V_\mu^{ij}\epsilon_j \,, \\
\mathcal{D}_\mu \eta^i &= \partial_\mu \eta^i + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\eta^i - \tfrac{1}{2} b_\mu \eta^i - V_\mu^{ij}\eta_j \,.
\end{aligned}
\tag{5.34}
$$

In addition to these local superconformal transformations, the gauge fields also transform as vectors under general coordinate transformations

$$
\delta_{\mathrm{gct}}(\xi) h_\mu{}^A = \xi^\nu(x)\partial_\nu h_\mu{}^A + \partial_\mu \xi^\nu(x) h_\nu{}^A \,. \tag{5.35}
$$

Covariant curvatures

The gauge fields introduced above transform to derivatives on the parameters of the superconformal algebra. A curvature for each gauge field is defined by

$$
R_{\mu\nu}{}^A(T^A) = 2\partial_{[\mu} h_{\nu]}{}^A + h_\nu{}^C h_\mu{}^B f_{BC}{}^A \,. \tag{5.36}
$$

Such curvatures transform covariantly according to

$$
\delta R_{\mu\nu}{}^A(T^A) = \epsilon^C R_{\mu\nu}{}^B(T^B) f_{BC}{}^A \,. \tag{5.37}
$$

Using the commutator expressions (5.10) we obtain the following expressions for the curvatures

$$
\begin{aligned}
R_{\mu\nu}{}^a(P) &= 2\partial_{[\mu} e_{\nu]}{}^a + \underline{2\omega_{[\mu}{}^{ab} e_{\nu]b}} + \underline{2 b_{[\mu} e_{\nu]}{}^a} - \tfrac{1}{2}\bar\psi_{[\mu}\gamma^a\psi_{\nu]} \,, \\
R_{\mu\nu}{}^a(K) &= 2\partial_{[\mu} f_{\nu]}{}^a + 2\omega_{[\mu}{}^{ab} f_{\nu]b} - 2 b_{[\mu} f_{\nu]}{}^a - \tfrac{1}{2}\bar\phi_{[\mu}\gamma^a\phi_{\nu]} \,, \\
R_{\mu\nu}{}^{ab}(M) &= 2\partial_{[\mu}\omega_{\nu]}{}^{ab} + 2\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b + \underline{8 f_{[\mu}{}^{[a} e_{\nu]}{}^{b]}} + \mathrm{i}\,\bar\phi_{[\mu}\gamma^{ab}\psi_{\nu]} \,, \\
R_{\mu\nu}(D) &= 2\partial_{[\mu} b_{\nu]} - \underline{4 f_{[\mu}{}^a e_{\nu]a}} - \mathrm{i}\,\bar\phi_{[\mu}\psi_{\nu]} \,, \\
R_{\mu\nu}{}^{ij}(V) &= 2\partial_{[\mu} V_{\nu]}{}^{ij} - 2 V_{[\mu}{}^{k(i} V_{\nu]\,k}{}^{j)} - 3\,\mathrm{i}\,\bar\phi^{(i}_{[\mu}\psi^{j)}_{\nu]} \,, \\
R_{\mu\nu}{}^i(Q) &= 2\partial_{[\mu}\psi^i_{\nu]} + \tfrac{1}{2}\omega_{[\mu}{}^{ab}\gamma_{ab}\psi^i_{\nu]} + b_{[\mu}\psi^i_{\nu]} - 2V_{[\mu}{}^{ij}\psi_{\nu]\,j} + \underline{2\,\mathrm{i}\,\gamma_a\phi^i_{[\mu} e_{\nu]}{}^a} \,, \\
R_{\mu\nu}{}^i(S) &= 2\partial_{[\mu}\phi^i_{\nu]} + \tfrac{1}{2}\omega_{[\mu}{}^{ab}\gamma_{ab}\phi^i_{\nu]} - b_{[\mu}\phi^i_{\nu]} - 2V_{[\mu}{}^{ij}\phi_{\nu]\,j} - 2\,\mathrm{i}\,\gamma_a\psi^i_{[\mu} f_{\nu]}{}^a \,.
\end{aligned}
\tag{5.38}
$$

We have underlined all terms proportional to vielbeins for purposes to be explained shortly.

In addition to curvatures, we can also define a covariant derivative as the partial derivative minus the sum over all transformations with as parameter the corresponding gauge field

$$
\nabla_\mu \equiv \partial_\mu - \delta_A(h_\mu{}^A) \,. \tag{5.39}
$$

This definition can also be applied to the curvatures to derive the Bianchi identities

$$\nabla_{[\mu} R_{\nu\lambda]}{}^{A}(T^A) = 0 \,. \tag{5.40}$$

5.2.2 Curvature constraints

Making the superconformal algebra G a local symmetry algebra is a subtle procedure. Taking spacetime dependent parameters in (5.5) makes it impossible to distinguish translations, Lorentz transformations, dilatations, and special conformal transformations, since they are all included in the general coordinate transformations (5.1).

Moreover, in section 5.1, we saw that global conformal transformations on fields $\phi^\alpha(x)$ could be split up as a product of translations and global transformations of $\phi^\alpha(0)$ generated by the stability sub-algebra H. The local analog of (5.16) is a product of general coordinate transformations and local transformations at $x = 0$ generated by H

$$G_{\rm local} = (\mathrm{GCT} \otimes H)_{\rm local} \ . \tag{5.41}$$

However, we have so far only considered a gauge theory of general coordinate transformations and local $\mathrm{SO}(2,5)$-transformations (and its supersymmetric extension $\mathrm{F}^2(4)$). Somehow, we should be able to make the truncation

$$(\mathrm{GCT} \otimes \mathrm{SO}(2,5))_{\rm local} \to (\mathrm{GCT} \otimes H)_{\rm local} \ . \tag{5.42}$$

Covariant general coordinate transformations

In particular, we would like to identify the field $e_\mu{}^a$ as the fünfbein field and not just as the gauge field for translations. To see how this can be resolved, let us rewrite (5.35) on $e_\mu{}^a$ as

$$\begin{aligned} \delta_{\rm gct}(\xi) e_\mu{}^a &= \xi^\nu(x)\partial_\nu e_\mu{}^a + (\partial_\mu \xi^\nu(x))\, e_\nu{}^a \\ &= \partial_\mu \left(\xi^\nu(x) e_\nu{}^a\right) + \xi^\nu(x)\left(\partial_\nu e_\mu{}^a - \partial_\mu e_\nu{}^a\right) \\ &= \big(\delta_P(\xi^a) + \delta_M(\xi^\mu \omega_\mu{}^{ab}) + \delta_Q(\xi^\mu \psi_\mu^i)\big)\, e_\mu{}^a - \xi^\nu R_{\mu\nu}{}^a(P) \,. \end{aligned} \tag{5.43}$$

So, the local translations can be expressed as general coordinate transformations covariantized with respect to all symmetries except translations

$$\delta_P(\xi) \to \delta_{\rm cgct} \equiv \delta_{\rm gct}(\xi) - \delta_I(\xi^\mu h_\mu{}^I) \quad (I \neq P_a) \,, \tag{5.44}$$

if we impose the following curvature constraint

$$R_{\mu\nu}{}^a(P) = 0 \,. \tag{5.45}$$

The curvature constraint (5.45) has as an additional effect that the gauge field $\omega_\mu{}^{ab}$ can be identified with the spin-connection since it can be solved for from (5.45). This is the reason why we underlined the second term in the first line of (5.38): it contains the spin-connection multiplied with the invertible fünfbein.

Conventional constraints

Constraints from which a gauge field can be solved are called conventional constraints, and they have been applied previously in the formulations of conformal supergravity in four dimensions for the $\mathcal{N} = 1$ [173, 175], $\mathcal{N} = 2$ [190] and $\mathcal{N} = 4$ [191], as well as in six dimensions for the $(1, 0)$ [176] and $(2, 0)$ [192] Weyl multiplets.

A comparison with the underlined lines in (5.38) suggests that we constrain the curvatures $R_{\mu\nu}{}^a(P)\,, R_{\mu\nu}{}^{ab}(M)\,, R_{\mu\nu}(D)$ and $R_{\mu\nu}{}^i(Q)$. However, they are not all independent: applying the Bianchi identity (5.40) to the constraint (5.45) gives

$$e_{[\mu}{}^a R_{\nu\lambda]}(D) = R_{[\mu\nu\lambda]}{}^a(M)\,. \tag{5.46}$$

We choose to impose the following constraints[4]

$$\begin{aligned} R_{\mu\nu}{}^a(P) &= 0 \quad (\underline{50})\,, \\ e^\nu{}_b \widehat{R}_{\mu\nu}{}^{ab}(M) &= 0 \quad (\underline{25})\,, \\ \gamma^\mu \widehat{R}_{\mu\nu}{}^i(Q) &= 0 \quad (\underline{40})\,. \end{aligned} \tag{5.47}$$

Matter fields

Before analyzing these constraints any further, we note that (5.47) contains 75+40 bosonic plus fermionic restrictions which leaves us with 21+24 degrees of freedom in the independent gauge fields. So even though we solved the problem of distinguishing local superconformal transformations from general coordinate transformations, the independent gauge fields do not form a supermultiplet with an equal number of bosonic and fermionic degrees of freedom.

The solution is to add matter fields (i.e. fields that do not gauge a superconformal symmetry) to supplement this mismatch. Which matter fields to add, and to determine their transformation rules is the subject of the next section. The effect of these extra fields will be that the transformation rules for the gauge fields (5.33) will be modified (we ignore the translations from all indices ranging over $I\,, J$)

$$\delta_J(\epsilon^J) h_\mu^I = \partial_\mu \epsilon^I + \epsilon^J h_\mu{}^A f_{AJ}{}^I + \epsilon^J M_{\mu J}{}^I\,. \tag{5.48}$$

The last term in (5.48) also modifies the definition of the curvatures (5.38) to

$$\widehat{R}_{\mu\nu}{}^I = 2\partial_{[\mu} h_{\nu]}{}^I + h_\nu{}^B h_\mu{}^A f_{AB}{}^I - 2h_{[\mu}{}^J M_{\nu]J}{}^I\,. \tag{5.49}$$

This is the origin of the hats on the curvatures in the last two equations of (5.47): it anticipates the corrections that will come from matter terms.

In general, one can add extra matter terms to the constraints (5.47), which just amounts to redefinitions of the composite fields. By choosing suitable terms simplifications were

[4]Note that the third constraint implies that $\gamma_{[\mu\nu}\widehat{R}_{\rho\sigma]}{}^i(Q) = 0$.

obtained in four and six dimensions. In these cases, one could add a term to the second constraint which rendered all the constraints invariant under S-supersymmetry, but in five dimensions this turns out to be impossible. Therefore we keep the constraints as written above.

Dependent gauge fields

We underlined the number of restrictions each constraint imposes, and a comparison with table 5.2 and with the underlined lines in (5.38) shows that they have the same number of restrictions as the degrees of freedom of the gauge fields $\omega_\mu{}^{ab}$, $f_\mu{}^a$ and ϕ^i_μ. Therefore, these fields are no longer independent.

In order to write down the explicit solutions of these constraints, it is useful to extract the terms which have been underlined in (5.38). We define $\widehat{R}'$ as the curvatures without these terms. Formally,

$$\widehat{R}'_{\mu\nu}{}^I = \widehat{R}_{\mu\nu}{}^I + 2h_{[\mu}{}^J e_{\nu]}{}^a f_{aJ}{}^I\,, \tag{5.50}$$

where $f_{aJ}{}^I$ are the structure constants in the $\mathrm{F}^2(4)$ algebra that define commutators of translations with other gauge transformations. Then, an explicit solution for them is given by

$$\begin{aligned}
\omega_\mu{}^{ab} &= 2e^{\nu[a}\partial_{[\mu}e_{\nu]}^{\;b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\partial_\nu e_\sigma^{\;c} + 2e_\mu{}^{[a}b^{b]} - \tfrac{1}{2}\bar\psi^{[b}\gamma^{a]}\psi_\mu - \tfrac{1}{4}\bar\psi^b\gamma_\mu\psi^a\,,\\
\phi^i_\mu &= -\tfrac{1}{12}\,\mathrm{i}\left(\gamma^{ab}\gamma_\mu - \tfrac{1}{2}\gamma_\mu\gamma^{ab}\right)\widehat{R}'_{ab}{}^i(Q)\,,\\
f_\mu{}^a &= -\tfrac{1}{6}\mathcal{R}_\mu{}^a + \tfrac{1}{48}e_\mu{}^a\mathcal{R}\,, \qquad \mathcal{R}_{\mu\nu} \equiv \widehat{R}'_{\lambda\mu}{}^{ab}(M)e_a{}^\lambda e_{\nu b}\,, \qquad \mathcal{R} \equiv \mathcal{R}_\mu{}^\mu\,.
\end{aligned} \tag{5.51}$$

The non-invariance of the constraints under Q- and S-supersymmetry has to be compensated by extra Q- and S-supersymmetry transformations of the dependent gauge fields. In section 5.4, we will give these extra supersymmetry transformation rules.

5.3 The supercurrent method

We will now present an elegant method to derive the field content and transformation rules for the matter fields that have to be added to the independent gauge fields to obtain a supermultiplet.

The Noether method

Consider a Lagrangian consisting of the kinetic term for a set of N spinor fields $\vec\psi$

$$\mathcal{L}_{\text{matter}} = \bar{\vec\psi}\cdot\partial\!\!\!/\vec\psi\,. \tag{5.52}$$

This action is invariant under global $\mathrm{U}(N)$-rotations of the form

$$\delta_G(\Lambda)\vec\psi = \Lambda^A(T_A)\cdot\vec\psi\,, \quad \delta_G(\Lambda)\mathcal{L}_{\text{matter}} = 0\,. \tag{5.53}$$

However, under local transformations the kinetic term transforms to the derivative on the gauge parameters

$$\begin{aligned} \delta_G(\Lambda(x))\mathcal{L}_{\rm matter} &= -\tfrac{1}{2}\left(\partial^\mu\Lambda^A(x)\right)\vec{\bar{\psi}}\gamma_\mu T_A\vec{\psi} \\ &\equiv -\left(\partial_\mu\Lambda^A(x)\right)J^\mu{}_A\,, \end{aligned} \tag{5.54}$$

where $J^\mu{}_A$ is the set of Noether currents corresponding to the global symmetries. These currents are divergence-less (using the equations of motion) and transform covariantly

$$\partial^\mu J_\mu{}^A = 0\,,\quad \delta_G(\Lambda(x))J_\mu{}^A = \Lambda^C(x)J_\mu^B f_{BC}{}^A\,. \tag{5.55}$$

The standard procedure is then to introduce a gauge field $h_\mu{}^A$ in a Noether-action that compensates this extra variation

$$\delta(\Lambda)h_\mu{}^A = \partial_\mu\Lambda^A(x) + \Lambda^C(x)h_\mu{}^B f_{BC}{}^A\,,\quad \mathcal{L}_{\rm Noether}\equiv h_\mu{}^A J^\mu{}_A\,, \tag{5.56}$$

such that the combined action is invariant under local gauge transformations

$$\mathcal{L} = \mathcal{L}_{\rm matter} + \mathcal{L}_{\rm Noether}\,,\quad \delta_G(\Lambda(x))\mathcal{L} = 0\,. \tag{5.57}$$

The total action can also be rewritten as a manifestly invariant action in terms of a covariant derivative

$$\mathcal{L} = \vec{\bar{\psi}}\cdot\not{\mathcal{D}}\vec{\psi}\,,\quad \mathcal{D}_\mu \equiv \partial_\mu - \delta_A(h_\mu{}^A)\,. \tag{5.58}$$

Writing out the definition of the covariant derivative, we regain (5.57).

Superconformal currents

We will now mimic this procedure for the superconformal algebra $\mathrm{F}^2(4)$. Also in this case, the currents for superconformal symmetry will themselves not be invariant under the superconformal algebra, instead they form a complete supermultiplet. To each current in this current multiplet, we can then assign a field of the Weyl multiplet. From the precise index structure and comparing with the independent gauge fields of the superconformal algebra, we derive which matter fields have to be added as well as their transformation rules.

The multiplet of currents in a superconformal context has been discussed before in the literature: the current multiplet corresponding to the vector multiplet in four dimensions for $\mathcal{N}=1$ [174], $\mathcal{N}=2$ [193, 194], and $\mathcal{N}=4$ [191]; and the current multiplet of the six-dimensional $(2,0)$ self-dual tensor multiplet [192][5]. In addition to these cases, the $\mathcal{N}=4\,, D=5$ supercurrent [195] has also been constructed before but not in a superconformal context. Moreover, for $\mathcal{N}=2$, the five-dimensional current found by the authors of [195] becomes reducible, as we have shown in the appendix of [16].

[5]The Weyl multiplets of $(1,0)$ in $D=6$ [176] were derived without the use of a current multiplet.

In all these cases, after adding local improvement terms, one obtains a supercurrent multiplet containing an energy-momentum tensor $\theta_{\mu\nu} = \theta_{\nu\mu}$ and a supercurrent J^i_μ which are both conserved and (gamma-)traceless

$$\partial^\mu \theta_{\mu\nu} = \theta_\mu{}^\mu = \partial^\mu J^i_\mu = \gamma^\mu J^i_\mu = 0. \tag{5.59}$$

These improved and conserved currents correspond to the invariance under rigid superconformal symmetry for the various vector and tensor multiplets in four and six dimensions, the analog of the globally invariant $\mathcal{L}_{\text{matter}}$ discussed previously.

However, the standard kinetic term of the $D = 5$ vector field

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{5.60}$$

is not scale invariant, i.e. the energy–momentum tensor is not traceless:

$$\theta_{\mu\nu} = -F_{\mu\lambda} F_\nu{}^\lambda + \tfrac{1}{4}\eta_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}\,, \quad \theta_\mu{}^\mu = \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \neq 0\,. \tag{5.61}$$

Moreover, there do not exist gauge-invariant local improvement terms.

Fortunately, there is a remedy for this problem. Whenever there is a compensating scalar field present, i.e. a scalar with mass dimension zero but non-zero Weyl weight, then the kinetic term (5.60) can be made scale invariant by introducing a scalar coupling of the form

$$\mathcal{L} = -\tfrac{1}{4} e^\phi F_{\mu\nu} F^{\mu\nu}\,. \tag{5.62}$$

This compensating scalar is called the dilaton. In general, there are three possible origins for a dilaton coupling to a non-conformal matter multiplet: the dilaton is part of

1. the matter multiplet itself (the multiplet is then called an “improved” multiplet);

2. the conformal supergravity multiplet;

3. another matter multiplet.

The $\mathcal{N} = 2$ vector multiplet in five dimensions contains precisely such a scalar. We could therefore use it to compensate the broken scale invariance of the kinetic terms. This leads to the so-called improved vector multiplet. This is the first possibility, which will be further discussed in section 5.5.

The second possibility will be considered here (the third possibility is included for completeness). This possibility thus occurs when the Weyl multiplet itself contains a dilaton. We will see that there indeed exists a version of the Weyl multiplet containing a dilaton. This version is called the Dilaton Weyl multiplet. It turns out that there exists another version of the Weyl multiplet without a dilaton. This other version is very similar to the four- and six-dimensional Weyl multiplet and will be called the Standard Weyl multiplet.

When coupling to the Standard Weyl multiplet, one needs to add *non-local* improvement terms to the current multiplet, a feature that was first implemented for the current multiplet

Field	Equation of motion	SU(2)	w	# d.o.f.
A_μ	$\partial_\mu F^{\mu\nu} = 0$	1	0	3
σ	$\Box\sigma = 0$	1	1	1
ψ^i	$\partial\!\!\!/\psi^i = 0$	2	$\frac{3}{2}$	4

Table 5.3: The on-shell Maxwell multiplet.

coming from the $D = 10$ vector multiplet [196]. In that case, the non-local improvement terms that were added required the use of auxiliary fields satisfying differential constraints in order to make the transformation rules local[6].

The current multiplet needs to be improved only when coupled to the Standard Weyl multiplet. In the case of the Dilaton Weyl multiplet, it is not necessary to do so, since in that case the dilaton of the Weyl multiplet can be used to compensate for the lack of scale invariance. In particular, the dilaton will couple directly to the trace of the energy-momentum tensor. We will present both the conventional and the improved current multiplet corresponding to the $\mathcal{N} = 2$ vector multiplet and in this way determine the field content and linearized transformation rules of the Dilaton and the Standard Weyl multiplet.

For matter multiplets having a traceless energy–momentum tensor, no compensating scalar is needed. To see the difference between the various cases, it is instructive to consider $(1, 0)\,, D = 6$ conformal supergravity theory [176]. In that case, also a Standard and a Dilaton Weyl multiplet were found. Even though neither of those Weyl multiplets were derived using the current multiplet method, we expect that both versions can be constructed in that way: the Standard Weyl multiplet starting from the conformal $(1, 0)$ tensor multiplet (being a truncation of the $(2, 0)$ case [192]), and the Dilaton Weyl multiplet by starting from the non-conformal $D = 6$ vector multiplet (which upon reduction should produce our results in $D = 5$).

5.3.1 The supercurrent of the Maxwell multiplet

Our starting point is the on-shell $D = 5$ Abelian vector multiplet, also known as the Maxwell multiplet. Its field content is given by a massless vector A_μ, a symplectic Majorana spinor ψ^i in the fundamental of $\mathrm{SU}(2)$ and a real scalar σ. See table 5.3 for additional information. Our conventions are given in appendix A.

The action for the $D = 5$ Maxwell multiplet is given by

$$\mathcal{L}_{\mathrm{matter}} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{2}\bar{\psi}\partial\!\!\!/\psi - \tfrac{1}{2}(\partial\sigma)^2 \,. \tag{5.63}$$

[6] Note also that in $D = 10$ the trace-part and the traceless part of the energy-momentum tensor are not contained in the same multiplet which necessitates the addition of the non-local improvement terms to project out the trace-part.

Current	Noether	SU(2)	w	# d.of.
$\theta_{(\mu\nu)}$	$\partial^\mu \theta_{\mu\nu} = 0$	1	2	9
$v_\mu^{(ij)}$	$\partial^\mu v_\mu^{ij} = 0$	3	2	12
$b_{[\mu\nu]}$	$\partial^\mu b_{\mu\nu} = 0$	1	2	6
a_μ	$\partial^\mu a_\mu = 0$	1	3	4
${\theta_\mu}^\mu$	–	1	4	1
J_μ^i	$\partial^\mu J_\mu^i = 0$	2	$\frac{5}{2}$	24
$\gamma^\mu J_\mu^i$	–	2	$\frac{7}{2}$	8

Table 5.4: The current multiplet: ${\theta_\mu}^\mu$ and $\gamma^\mu J_\mu^i$ form separate currents.

This action is invariant under the following global supersymmetries

$$\begin{aligned} \delta A_\mu &= \tfrac{1}{2}\bar\epsilon\gamma_\mu\psi\,, \\ \delta\psi^i &= -\tfrac{1}{4}\gamma\cdot F\epsilon^i - \tfrac{1}{2}\,\mathrm{i}\,\partial\!\!\!/\sigma\epsilon^i\,, \\ \delta\sigma &= \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\psi\,, \end{aligned} \tag{5.64}$$

as well as under the standard gauge transformation

$$\delta_\Lambda A_\mu = \partial_\mu\Lambda\,. \tag{5.65}$$

Under local supersymmetry transformations, the action (5.63) transforms to the supercurrent J_μ^i

$$\delta(\epsilon(x))\mathcal{L}_{\rm matter} = \mathrm{i}\,\left(\partial^\mu\bar\epsilon(x)\right)J_\mu\,, \tag{5.66}$$

where the explicit form of the supercurrent is given in (5.67).

The various global symmetries of the Lagrangian (5.63) lead to a number of other Noether currents: the energy–momentum tensor $\theta_{\mu\nu}$ and the $\mathrm{SU}(2)$-current v_μ^{ij}. The supersymmetry variations of these currents lead to a closed multiplet of $32 + 32$ degrees of freedom (see table 5.4). It is convenient to include these trace parts as separate currents since, as it turns out, they couple to independent fields of the Weyl multiplet.

We find the following expressions for the Noether currents and their supersymmetric partners in terms of bilinears of the vector multiplet fields:

$$\begin{aligned}
\theta_{\mu\nu} &= -F_{\mu\lambda}F_\nu{}^\lambda + \tfrac{1}{4}\eta_{\mu\nu}F^2 - \partial_\mu\sigma\partial_\nu\sigma + \tfrac{1}{2}\eta_{\mu\nu}\left(\partial\sigma\right)^2 - \tfrac{1}{2}\bar\psi\gamma_{(\mu}\partial_{\nu)}\psi\,,\\
J_\mu^i &= -\tfrac{1}{4}\,\mathrm{i}\,\gamma\cdot F\gamma_\mu\psi^i - \tfrac{1}{2}(\partial\!\!\!/\sigma)\gamma_\mu\psi^i\,,\\
v_\mu^{ij} &= \tfrac{1}{2}\bar\psi^i\gamma_\mu\psi^j\,,\\
a_\mu &= \tfrac{1}{8}\varepsilon_{\mu\nu\lambda\rho\sigma}F^{\nu\lambda}F^{\rho\sigma} + (\partial^\nu\sigma)F_{\nu\mu}\,,\\
b_{\mu\nu} &= \tfrac{1}{2}\varepsilon_{\mu\nu\lambda\rho\sigma}(\partial^\lambda\sigma)F^{\rho\sigma} + \tfrac{1}{2}\bar\psi\gamma_{[\mu}\partial_{\nu]}\psi\,,\\
\gamma^\mu J_\mu^i &= -\tfrac{1}{4}\,\mathrm{i}\,\gamma\cdot F\psi^i + \tfrac{3}{2}\partial\!\!\!/\sigma\psi^i\,,\\
\theta_\mu{}^\mu &= \tfrac{1}{4}F^2 + \tfrac{3}{2}\left(\partial\sigma\right)^2\,.
\end{aligned} \tag{5.67}$$

Applying the supersymmetry transformation rules (5.64) to the currents (5.67), using the Bianchi identities and equations of motion of the vector multiplet fields, one can calculate the supersymmetry transformations of the currents. A straightforward calculation yields

$$\begin{aligned}
\delta\theta_{\mu\nu} &= \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\gamma_{\lambda(\mu}\partial^\lambda J_{\nu)}\,,\\
\delta J_\mu^i &= -\tfrac{1}{2}\,\mathrm{i}\,\gamma^\nu\theta_{\mu\nu}\epsilon^i - \,\mathrm{i}\,\gamma_{[\lambda}\partial^\lambda v_{\mu]}^{ij}\epsilon_j - \tfrac{1}{2}a_\mu\epsilon^i + \tfrac{1}{2}\,\mathrm{i}\,\gamma^\nu b_{\mu\nu}\epsilon^i\,,\\
\delta v_\mu^{ij} &= \mathrm{i}\,\bar\epsilon^{(i}J_\mu^{j)}\,,\\
\delta a_\mu &= -\bar\epsilon\partial^\lambda\gamma_{[\lambda}J_{\mu]} - \tfrac{1}{4}\bar\epsilon\gamma_{\mu\nu}\gamma^\lambda\partial^\nu J_\lambda + \tfrac{1}{4}\bar\epsilon\gamma_{\mu\nu}\partial^\nu\left(\gamma^\lambda J_\lambda^i\right)\,,\\
\delta b_{\mu\nu} &= \tfrac{3}{4}\,\mathrm{i}\,\bar\epsilon\gamma_{[\lambda\mu}\partial^\lambda J_{\nu]} - \tfrac{1}{8}\,\mathrm{i}\,\bar\epsilon\gamma_{\mu\nu\lambda}\gamma^\rho\partial^\lambda J_\rho + \tfrac{1}{8}\,\mathrm{i}\,\bar\epsilon\gamma_{\mu\nu\lambda}\partial^\lambda\left(\gamma^\rho J_\rho^i\right)\,,\\
\delta\left(\gamma^\mu J_\mu^i\right) &= -\tfrac{1}{2}\,\mathrm{i}\,\theta_\mu{}^\mu\epsilon^i + \tfrac{1}{2}\,\mathrm{i}\,\partial\!\!\!/ v\!\!\!/^{ij}\epsilon_j - \tfrac{1}{2}a\!\!\!/\epsilon^i + \tfrac{1}{2}\,\mathrm{i}\,\gamma\cdot b\epsilon^i\,,\\
\delta\theta_\mu{}^\mu &= \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\partial\!\!\!/\left(\gamma^\mu J_\mu\right)\,.
\end{aligned} \tag{5.68}$$

Note that we have added to the transformation rules for a_μ and $b_{\mu\nu}$ terms that are identically zero: the first term at the r.h.s. contains the divergence of the supercurrent and the last two terms are identical, but we have chosen not to explicitly evaluate the gamma-trace. Similarly, the second term in the variation of the supercurrent contains a term that is proportional to the divergence of the $\mathrm{SU}(2)$ current.

The reason why we added these terms is that in this way we obtain below the linearized Weyl multiplet in a conventional form. Alternatively, we could not have added these terms and later have brought the Weyl multiplet into the same conventional form by redefining the Q-transformations via a field-dependent S- and $\mathrm{SU}(2)$-transformation.

5.3.2 The improved supercurrent

We now add terms to $\theta_{\mu\nu}$ and J_μ^i such that they become (gamma-)traceless while remaining divergence-less. This requires the introduction of non-local projection operators. First of all, we add the following term to the energy-momentum tensor

$$\begin{aligned}\widehat{\theta}_{\mu\nu} &= \theta_{\mu\nu} - \tfrac{1}{4}\left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right)\theta_\mu{}^\mu \\ &\equiv \theta_{\mu\nu} - \tfrac{1}{16}\left(\Box\eta_{\mu\nu} - \partial_\mu\partial_\nu\right)d\,, \end{aligned} \tag{5.69}$$

where the current d has to satisfy the differential constraint

$$\Box d = 4\theta_\mu{}^\mu\,. \tag{5.70}$$

Similarly, we can add the following term to the supercurrent

$$\begin{aligned}\widehat{J}^i_\mu &= J^i_\mu - \tfrac{1}{4}\left(\gamma_\mu - \frac{\partial_\mu\partial\!\!\!/}{\Box}\right)\gamma\cdot J^i \\ &\equiv J^i_\mu + \tfrac{1}{32}\gamma_{\mu\nu}\partial^\nu\lambda\,, \end{aligned} \tag{5.71}$$

where the current λ^i satisfies the differential constraint

$$\partial\!\!\!/\lambda^i = -8\gamma\cdot J^i\,. \tag{5.72}$$

For the supercurrent coming from a vector multiplet in $D \neq 4$, there are no local gauge-invariant improvement terms. However, the improved energy-momentum tensor transforms to the improved supercurrent, and by varying the improved supercurrent we find a new constrained current t_{ab}. This current satisfies the following constraint in terms of the previously found currents a_μ and $b_{\mu\nu}$

$$\Box\, t_{ab} = -8\partial_{[a}a_{b]} - 4\varepsilon_{abcde}\partial^e b^{dc}\,. \tag{5.73}$$

The currents t_{ab} , λ^i and d do not generate any more currents, and we have summarized the improved current multiplet and the differential constraints in table 5.5.

We then find for the improved current multiplet the following transformation rules

$$\begin{aligned}\delta\widehat{\theta}_{\mu\nu} &= \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\partial^\lambda\gamma_{\lambda(\mu}\widehat{J}_{\nu)}\,,\\ \delta\widehat{J}^i_\mu &= -\tfrac{1}{2}\,\mathrm{i}\,\gamma^\nu\widehat{\theta}_{\mu\nu}\epsilon^i - \tfrac{1}{4}\,\mathrm{i}\,\left(\gamma^\lambda\gamma_{\mu\nu} - \tfrac{1}{2}\gamma_{\mu\nu}\gamma^\lambda\right)\partial^\nu v_\lambda^{ij}\epsilon_j\,,\\ &\quad -\tfrac{3}{16}\partial^\nu\left(t_{\mu\nu} - \tfrac{1}{12}\gamma_{\mu\nu}\gamma\cdot t + \tfrac{2}{3}\gamma_{[\mu}\gamma^b t_{\nu]b}\right)\epsilon^i\,,\\ \delta v^{ij}_\mu &= \mathrm{i}\,\bar\epsilon^{(i}\widehat{J}^{j)}_\mu - \tfrac{1}{32}\,\mathrm{i}\,\bar\epsilon^{(i}\gamma_{\mu\nu}\partial^\nu\lambda^{j)}\,,\\ \delta t_{ab} &= -\bar\epsilon\gamma^\mu\gamma_{ab}\widehat{J}_\mu + \tfrac{3}{32}\bar\epsilon\partial\!\!\!/\gamma_{ab}\lambda + \tfrac{1}{32}\bar\epsilon\gamma_{ab}\partial\!\!\!/\lambda\,,\\ \delta\lambda^i &= \mathrm{i}\,\partial\!\!\!/ d\epsilon^i - 4\,\mathrm{i}\,\psi\!\!\!/^{ij}\epsilon_j - \tfrac{1}{2}\gamma\cdot t\epsilon^i\,,\\ \delta d &= -\tfrac{1}{4}\,\mathrm{i}\,\bar\epsilon\lambda\,.\end{aligned} \tag{5.74}$$

Note that these transformations rules are perfectly well-defined, all non-localities have been absorbed in the differentially constrained currents.

Current	Restrictions	SU(2)	w	# d.of.
$\widehat{\theta}_{(\mu\nu)}$	$\partial^\mu \widehat{\theta}_{\mu\nu} = 0\,, \quad \widehat{\theta}_\mu{}^\mu = 0$	1	2	9
$v_\mu^{(ij)}$	$\partial^\mu v_\mu^{ij} = 0$	3	2	12
$t_{[ab]}$	$\Box\, t_{ab} = -8\partial_{[a} a_{b]} - 4\varepsilon_{abcde}\partial^e b^{dc}$	1	2	10
d	$\Box d = 4\theta_\mu{}^\mu$	2	2	1
$\widehat{J}_\mu^i$	$\partial^\mu \widehat{J}_\mu^i = 0\,, \quad \gamma^\mu \widehat{J}_\mu^i = 0$	2	$\frac{5}{2}$	24
λ^i	$\partial\!\!\!/ \lambda^i = -8\gamma^\mu J_\mu^i$	2	$\frac{5}{2}$	8

Table 5.5: The improved current multiplet with constrained currents.

5.3.3 The linearized Weyl multiplets

From the field content of the two current multiplets, we can immediately read off the field content of the two Weyl multiplets. They both have the same 21+24 independent gauge fields, which we have displayed in table 5.6. The dependent gauge fields do not couple to any current but are also displayed. The linearized form of the fünfbein $e_\mu{}^a$ is denoted by $h_{\mu\nu}$.

For both Weyl multiplets, the total number of degrees of freedom becomes 32+32 bosonic plus fermionic fields. The difference between the two Weyl multiplets lies in the set of matter fields. The Standard Weyl multiplet couples to the improved current multiplet and has matter fields $T_{ab}\,, \chi^i$ and D.

The Dilaton multiplet couples to the conventional current multiplet and has matter fields $B_{\mu\nu}\,, A_\mu\,, \psi^i$ and φ. This scalar field φ is the linearized form of the dilaton $\sigma \equiv e^\varphi$. There are also two extra gauge symmetries: a $\mathrm{U}(1)$ gauge symmetry and a two-form tensor gauge symmetry, the explicit form of which we have also indicated in table 5.6.

Note that some of the matter fields of the Dilaton Weyl multiplet have the same names as the fields of the vector multiplet. The reason for doing so will become clear in section 5.5 where we explain the connection between the two Weyl multiplets. From now on, until section 5.5, we will be only dealing with the Weyl multiplets and not with the vector multiplet.

The Standard Weyl multiplet

To derive the linearized transformation rules of the Standard Weyl multiplet, we introduce the following Noether term in the action

$$\mathcal{L}_{\mathrm{Noether}} = \tfrac{1}{2} h_{\mu\nu}\widehat{\theta}^{\mu\nu} + \mathrm{i}\,\bar{\psi}_\mu \widehat{J}^\mu + V_\mu^{ij} v_{ij}^\mu + T_{ab}t^{ab} + \mathrm{i}\,\bar{\chi}\lambda + Dd\,. \tag{5.75}$$

Demanding that the combined action of (5.63) and (5.75) is invariant under the supersymmetry transformations (5.64) and (5.74) results in the following linearized supersymmetry

Field	#	Gauge	SU(2)	w	Field	#	Gauge	SU(2)	w
		Elementary gauge fields					Dependent gauge fields		
$e_\mu{}^a$	9	P^a	1	-1	$\omega_\mu^{[ab]}$	$-$	$M^{[ab]}$	1	0
b_μ	0	D	1	0	$f_\mu{}^a$	$-$	K^a	1	1
$V_\mu^{(ij)}$	12	$\mathrm{SU}(2)$	3	0					
ψ_μ^i	24	Q_α^i	2	$-\frac{1}{2}$	ϕ_μ^i	$-$	S_α^i	2	$\frac{1}{2}$
		Dilaton Weyl multiplet					Standard Weyl multiplet		
$B_{[\mu\nu]}$	6	$\delta B_{\mu\nu} = 2\partial_{[\mu}\Lambda_{\nu]}$	1	0	$T_{[ab]}$	10		1	1
A_μ	4	$\delta A_\mu = \partial_\mu\Lambda$	1	0					
σ	1		1	1	D	1		1	2
ψ^i	8		2	$\frac{3}{2}$	χ^i	8		2	$\frac{3}{2}$

Table 5.6: Gauge fields and matter field of the Weyl multiplets.

transformation rules for the Standard Weyl multiplet

$$\begin{aligned}
\delta h_{\mu\nu} &= \bar\epsilon\gamma_{(\mu}\psi_{\nu)}\,, \\
\delta\psi_\mu^i &= \partial_\mu\epsilon^i + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - V_\mu^{ij}\epsilon_j + \mathrm{i}\,\gamma\cdot T\gamma_\mu\epsilon^i\,, \\
\delta V_\mu{}^{ij} &= -\tfrac{3}{2}\,\mathrm{i}\,\bar\epsilon^{(i}\phi_\mu^{j)} + 4\bar\epsilon^{(i}\gamma_\mu\chi^{j)}\,, \\
\delta T_{ab} &= \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\gamma_{ab}\chi - \tfrac{3}{32}\,\mathrm{i}\,\bar\epsilon R_{ab}(Q)\,, \\
\delta\chi^i &= \tfrac{1}{4}D\epsilon^i - \frac{1}{64}\gamma\cdot R(V)^{ij}\epsilon_j + \tfrac{3}{32}\,\mathrm{i}\,\gamma\cdot T\overleftarrow{\partial\!\!\!/}\,\epsilon^i + \tfrac{1}{32}\,\mathrm{i}\,\partial\!\!\!/\gamma\cdot T\epsilon^i\,, \\
\delta D &= \bar\epsilon\partial\!\!\!/\chi\,,
\end{aligned} \tag{5.76}$$

where we have used the linearized form of the expressions for the curvatures (5.38) and the dependent gauge fields (5.51)

$$\begin{aligned}
\omega_\mu{}^{ab} &= -\partial_{[a}h_{b]\mu} + \dots\,, \\
\phi_\mu^i &= -\tfrac{1}{12}\,\mathrm{i}\,\left(\gamma^{ab}\gamma_\mu - \tfrac{1}{2}\gamma_\mu\gamma^{ab}\right)\psi_{ab}^{j)} + \dots\,, \\
R_{\mu\nu}{}^{ij}(V) &= 2\partial_{[\mu}V_{\nu]}^{ij} + \dots\,, \\
R_{ab}(Q) &= \left(\psi_{ab} - \tfrac{1}{12}\gamma_{ab}\gamma^{cd}\psi_{cd} + \tfrac{2}{3}\gamma_{[a}\gamma^c\psi_{b]c}\right) + \dots\,, \\
\psi_{ab} &= 2\partial_{[a}\psi_{b]}^i\,.
\end{aligned} \tag{5.77}$$

Note that the transformation rules for ψ_μ^i and ${V_\mu}^{ij}$ differ from the original transformation rules (5.33) by matter fields as was explained in (5.48).

The Dilaton Weyl multiplet

To derive the linearized transformation rules of the Dilaton Weyl multiplet, we introduce the following Noether term in the action (note that in particular the trace of the energy-momentum tensor couples to the dilaton φ)

$$\mathcal{L}_{\text{Noether}} = \tfrac{1}{2} h_{\mu\nu}\theta^{\mu\nu} + \mathrm{i}\,\bar{\psi}_\mu J^\mu + V_\mu^{ij} v_{ij}^\mu + B_{\mu\nu} b^{\mu\nu} + A_\mu a^\mu + \mathrm{i}\,\bar{\psi}\,(\,\mathrm{i}\,\gamma^\mu J_\mu) + \varphi\,\theta_\mu^{\ \mu}\,. \quad (5.78)$$

Demanding that the combined action of (5.63) and (5.75) is invariant under supersymmetry transformations (5.64) and (5.68) results in the following linearized supersymmetry transformation rules for the Dilaton Weyl multiplet

$$\begin{aligned}
\delta h_{\mu\nu} &= \bar{\epsilon}\gamma_{(\mu}\psi_{\nu)}\,,\\
\delta\psi_\mu^i &= \partial_\mu\epsilon^i + \tfrac{1}{4}{\omega_\mu}^{ab}\gamma_{ab} - V_\mu^{ij}\epsilon_j + \mathrm{i}\,\gamma\cdot\underline{T}\gamma_\mu\epsilon^i\,,\\
\delta {V_\mu}^{ij} &= -\tfrac{3}{2}\,\mathrm{i}\,\bar{\epsilon}^{(i}\phi_\mu^{j)} + 4\bar{\epsilon}^{(i}\gamma_\mu\underline{\chi}^{j)}\,,\\
\delta B_{\mu\nu} &= \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\gamma_{\mu\nu}\psi + \tfrac{1}{2}\bar{\epsilon}\gamma_{[\mu}\psi_{\nu]}\,,\\
\delta A_\mu &= \tfrac{1}{2}\bar{\epsilon}\gamma_\mu\psi - \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\psi_\mu\,,\\
\delta\psi^i &= -\tfrac{1}{4}\gamma\cdot F\epsilon^i - \tfrac{1}{2}\,\mathrm{i}\,\partial\!\!\!/\varphi\epsilon^i + \gamma\cdot\underline{T}\epsilon^i\,,\\
\delta\varphi &= \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\psi\,,
\end{aligned} \quad (5.79)$$

where we have again used the expressions (5.77). The underlined fields in (5.79) are not independent fields here, but are used as a shorthand notation for the following expressions in terms of fields of the Dilaton Weyl multiplet

$$\begin{aligned}
T_{ab} &= \tfrac{1}{8}\left(F_{ab} - \tfrac{1}{6}\varepsilon_{abcde}H^{edc}\right)\,,\\
\chi^i &= \tfrac{1}{8}\,\mathrm{i}\,\partial\!\!\!/\psi^i + \frac{1}{64}\gamma^{ab}\psi_{ab}\,,\\
D &= \tfrac{1}{4}\Box\varphi - \tfrac{1}{32}\partial^\mu\partial^\nu h_{\mu\nu} + \tfrac{1}{32}\Box h_\mu^\mu\,.
\end{aligned} \quad (5.80)$$

The justification for using the expressions (5.80) is that they transform exactly as the fields T_{ab} , χ^i and D in the Standard Weyl multiplet. Finally, we have defined the curvatures for the gauge fields $B_{\mu\nu}$ and A_μ to be

$$F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}\,, \quad H_{\mu\nu\lambda} = 3\partial_{[\mu}B_{\nu\lambda]}\,. \quad (5.81)$$

Note that the last three lines in (5.79) are similar to the transformation rules of the Maxwell multiplet (5.64). In section 5.5, we will clarify this similarity in more detail.

5.4 The Weyl multiplets

In this section, we will present the full nonlinear supersymmetry transformations rules of the two Weyl multiplets as well as the modifications to the superconformal algebra. We will first briefly outline the iterative algorithm for obtaining the complete transformation rules for the two Weyl multiplets. For more details see [176].

Covariantization

The procedure to obtain these results is straightforward. Starting from the transformation rules of the linearized Weyl multiplets (5.76) and (5.79), the first step is to replace curvatures by hatted curvatures as defined in (5.49). In the transformation rules of T_{ab} and χ^i for instance, we will need

$$\begin{aligned} \widehat{R}_{\mu\nu}{}^i(Q) &= R_{\mu\nu}{}^i(Q) + 2\,\mathrm{i}\,\gamma\cdot T\gamma_{[\mu}\psi^i_{\nu]}\,, \\ \widehat{R}_{\mu\nu}{}^{ij}(V) &= R_{\mu\nu}{}^{ij}(V) - 8\bar{\psi}^{(i}_{[\mu}\gamma_{\nu]}\chi^{j)} - \mathrm{i}\,\bar{\psi}^{(i}_{[\mu}\gamma\cdot T\psi^{j)}_{\nu]}\,, \end{aligned} \tag{5.82}$$

Similarly, we use the bosonic transformation rules given in table 5.6 as well as the supersymmetry transformations to replace derivatives on fields by derivatives which are covariantized with respect to all symmetries except translations

$$\partial_\mu \to \mathcal{D}_\mu \equiv \partial_\mu - \delta_I(h_\mu{}^I)\,. \tag{5.83}$$

Using these new transformation rules, the superconformal algebra (5.10) is imposed on all fields. This will enable us to determine the S-supersymmetry rules for all the fields. An additional effect is that the transformation rules for some fields will need nonlinear modifications in order to satisfy the algebra.

These new nonlinear transformation rules, as well as the S-supersymmetry transformations, will have to be accounted for in the curvatures and covariant derivatives, and the algebra will have to be imposed again with these modified transformation rules, until no new modifications are necessary.

Dependent gauge fields

The dependent gauge fields defined in (5.51) depend on the covariant curvatures (5.49). The non-invariance of the constraints (5.47) and the modifications to the curvatures will induce extra transformations for the dependent gauge fields. We find the following extra transformations for ${\omega_\mu}^{ab}$ and ϕ_μ^i

$$\begin{aligned}
\delta_{\rm extra}{\omega_\mu}^{ab} &= -\,\mathrm{i}\,\bar\epsilon\gamma^{[a}\gamma\cdot T\gamma^{b]}\psi_\mu - \tfrac12\bar\epsilon\gamma^{[a}{\widehat R_\mu}{}^{b]}(Q) \\
&\quad -\tfrac14\bar\epsilon\gamma_\mu\widehat R^{ab}(Q) - 4{e_\mu}^{[a}\bar\epsilon\gamma^{b]}\chi\,. \qquad (5.84)\\
\delta_{\rm extra}\phi_\mu^i &= \tfrac1{12}\,\mathrm{i}\,\left(\gamma^{ab}\gamma_\mu - \tfrac12\gamma_\mu\gamma^{ab}\right)\widehat R_{ab}{}^{ij}(V)\epsilon_j \\
&\quad -\tfrac16\left(\not{D}\gamma_\mu\gamma\cdot T - 2\not{D}\gamma\cdot T\gamma_\mu + \tfrac12\gamma_\mu\not{D}\gamma\cdot T + \tfrac12\gamma_\mu\gamma\cdot T\overleftarrow{\not{D}}\right)\epsilon^i \\
&\quad +\mathrm{i}\,\left(-\gamma_\mu(\gamma\cdot T)^2 + 4\gamma_c{T_\mu}^c\gamma\cdot T + 16\gamma_c T^{cd}T_{\mu d} - 4\gamma_\mu T^2\right)\epsilon^i \\
&\quad -\tfrac23\,\mathrm{i}\,\left(\gamma^{ab}\gamma_\mu - \tfrac12\gamma_\mu\gamma^{ab}\right)T_{ab}\eta^i\,. \qquad (5.85)
\end{aligned}$$

We will not need the transformations for the field ${f_\mu}^a$, except the extra transformations of ${f_\mu}^\mu$ under S-supersymmetry

$$\delta_{\rm extra,S}{f_\mu}^\mu = -5\,\mathrm{i}\,\bar\eta\chi\,. \qquad (5.86)$$

5.4.1 The modified superconformal algebra

It turns out that the commutators of the various transformation rules of the nonlinear Weyl multiplets do not lead to the original algebra (5.10). Instead, the algebra closes modulo terms that can be reorganized in terms of field dependent superconformal transformations.

For instance, the full commutator of two supersymmetry transformations acquires the following terms with respect to (5.13)

$$\begin{aligned}
[\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)] &= \delta_{\rm cgct}(\xi_3^\mu) + \delta_M(\Lambda_M{}_3^{ab}) + \delta_S(\eta_3) + \delta_U(\Lambda_U{}_3^{ij}) \\
&\quad +\delta_K(\Lambda_{K3}^a) + \delta_{\mathrm{U}(1)}(\Lambda_3) + \delta_B(\Lambda_{3\mu})\,, \qquad (5.87)
\end{aligned}$$

where the covariant general coordinate transformations have been defined in (5.44). The last two terms appear only in the Dilaton Weyl multiplet formulation, where we use the notation δ_B for the two-form tensor gauge symmetry. The parameters appearing in (5.87) are

$$\begin{aligned}
\xi_3^\mu &= \tfrac12\bar\epsilon_2\gamma_\mu\epsilon_1\,, \\
\Lambda_M{}_3^{ab} &= -\,\mathrm{i}\,\bar\epsilon_2\gamma^{[a}\gamma\cdot T\gamma^{b]}\epsilon_1\,, \\
\Lambda_U{}_3^{ij} &= \mathrm{i}\,\bar\epsilon_2^{(i}\gamma\cdot T\epsilon_1^{j)}\,, \\
\eta_3^i &= -\tfrac94\,\mathrm{i}\,\bar\epsilon_2\epsilon_1\chi^i + \tfrac74\,\mathrm{i}\,\bar\epsilon_2\gamma_c\epsilon_1\gamma^c\chi^i \\
&\quad +\tfrac14\,\mathrm{i}\,\bar\epsilon_2^{(i}\gamma_{cd}\epsilon_1^{j)}\left(\gamma^{cd}\chi_j + \tfrac14\widehat R^{cd}{}_j(Q)\right)\,, \qquad (5.88)\\
\Lambda_{K3}^a &= -\tfrac12\bar\epsilon_2\gamma^a\epsilon_1 D + \tfrac1{96}\bar\epsilon_2^i\gamma^{abc}\epsilon_1^j\widehat R_{bcij}(V) \\
&\quad +\tfrac1{12}\,\mathrm{i}\,\bar\epsilon_2\left(-5\gamma^{abcd}D_bT_{cd} + 9D_bT^{ba}\right)\epsilon_1 \\
&\quad +\bar\epsilon_2\left(\gamma^{abcde}T_{bc}T_{de} - 4\gamma^cT_{cd}T^{ad} + \tfrac23\gamma^aT^2\right)\epsilon_1\,, \\
\Lambda_3 &= -\tfrac12\,\mathrm{i}\,\sigma\bar\epsilon_2\epsilon_1\,, \\
\Lambda_{3\mu} &= -\tfrac12\sigma^2\xi_{3\mu} - \tfrac12A_\mu\Lambda_3\,.
\end{aligned}$$

The commutator between Q- and S-supersymmetry also gains modifications with respect to (5.14)

$$\begin{aligned} [\delta_S(\eta), \delta_Q(\epsilon)] &= \delta_D(\tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\eta) + \delta_M(\tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\gamma^{ab}\eta) + \delta_U(-\tfrac{3}{2}\,\mathrm{i}\,\bar{\epsilon}^{(i}\eta^{j)}) \\ &\quad + \delta_K(\Lambda^a_{3K})\,, \end{aligned} \tag{5.89}$$

with the field-dependent special conformal transformation given by

$$\Lambda^a_{3K} = \tfrac{1}{6}\bar{\epsilon}\left(\gamma\cdot T\gamma_a - \tfrac{1}{2}\gamma_a\gamma\cdot T\right)\eta\,. \tag{5.90}$$

The commutator of Q- and $\mathrm{U}(1)$-transformations is given by

$$\left[\delta_Q(\epsilon), \delta_{\mathrm{U}(1)}(\Lambda)\right] = \delta_B\left(-\tfrac{1}{2}\Lambda\delta(\epsilon)A_\mu\right)\,. \tag{5.91}$$

In the next chapter, we will discuss matter fields coupled to conformal supergravity. These matter multiplets will have to obey the modified superconformal algebra given above in (5.87), (5.89) and (5.91), apart from possible additional transformations under which the fields of the Weyl multiplets do not transform, and possibly field equations if these matter multiplets are on-shell.

5.4.2 The Standard Weyl multiplet

Applying the rules of covariantization and the extra transformations of the dependent gauge fields and imposing the modified superconformal algebra, we find the following Q- and S-supersymmetry and K-transformation rules for the independent fields of the Standard Weyl multiplet

$$\begin{aligned} \delta e_\mu{}^a &= \tfrac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu\,, \\ \delta\psi^i_\mu &= \mathcal{D}_\mu\epsilon^i + \mathrm{i}\,\gamma\cdot T\gamma_\mu\epsilon^i - \mathrm{i}\,\gamma_\mu\eta^i\,, \\ \delta V_\mu{}^{ij} &= -\tfrac{3}{2}\,\mathrm{i}\,\bar{\epsilon}^{(i}\phi^{j)}_\mu + 4\bar{\epsilon}^{(i}\gamma_\mu\chi^{j)} + \mathrm{i}\,\bar{\epsilon}^{(i}\gamma\cdot T\psi^{j)}_\mu + \tfrac{3}{2}\,\mathrm{i}\,\bar{\eta}^{(i}\psi^{j)}_\mu\,, \\ \delta b_\mu &= \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\phi_\mu - 2\bar{\epsilon}\gamma_\mu\chi + \tfrac{1}{2}\,\mathrm{i}\,\bar{\eta}\psi_\mu + 2\Lambda_{K\mu}\,, \\ \delta T_{ab} &= \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\gamma_{ab}\chi - \tfrac{3}{32}\,\mathrm{i}\,\bar{\epsilon}\widehat{R}_{ab}(Q)\,, \\ \delta\chi^i &= \tfrac{1}{4}D\epsilon^i - \tfrac{1}{64}\gamma\cdot\widehat{R}^{ij}(V)\epsilon_j + \tfrac{3}{32}\,\mathrm{i}\,\gamma\cdot T\overleftarrow{\not{D}}\epsilon^i + \tfrac{1}{32}\,\mathrm{i}\,\not{D}\gamma\cdot T\epsilon^i \\ &\quad + \tfrac{1}{24}T^2\epsilon^i - \tfrac{1}{4}\left(\gamma\cdot T\right)^2\epsilon^i + \tfrac{1}{4}\gamma\cdot T\eta^i\,, \\ \delta D &= \bar{\epsilon}\not{D}\chi - \tfrac{5}{3}\,\mathrm{i}\,\bar{\epsilon}\gamma\cdot T\chi - \mathrm{i}\,\bar{\eta}\chi\,. \end{aligned} \tag{5.92}$$

The covariant derivative $\mathcal{D}_\mu\epsilon$ is given in (5.34). For other covariant derivatives, see the general rule (5.83). The covariant curvatures $\widehat{R}(Q)$ and $\widehat{R}(V)$ are given explicitly in (5.82). We also used the following transformations for these curvatures:

$$
\begin{aligned}
\delta\widehat{R}_{ab}{}^{i}(Q) &= -\tfrac{1}{6}\left(\gamma_{ab}{}^{cd} - \gamma^{cd}\gamma_{ab} - \tfrac{1}{2}\gamma_{ab}\gamma^{cd}\right)\widehat{R}_{cd}{}^{ij}(V)\epsilon_j + \tfrac{1}{4}\widehat{R}_{ab}{}^{cd}(M)\gamma_{cd}\epsilon^i \\
&\quad +2\,\mathrm{i}\,\Big(D_{[a}\gamma\cdot T\gamma_{b]} - \tfrac{1}{3}D_{[a}\gamma_{b]}\gamma\cdot T \\
&\quad - \tfrac{1}{3}\gamma_{[a}\not{D}\gamma\cdot T\gamma_{b]} - \tfrac{1}{3}\gamma_{ab}D_c\gamma_d T^{cd}\Big)\epsilon^i\,, \qquad (5.93)\\
\delta\widehat{R}_{ab}{}^{ij}(V) &= -\tfrac{3}{2}\,\mathrm{i}\,\bar\epsilon^{(i}\widehat{R}_{ab}{}^{j)}(S) - 8\bar\epsilon^{(i}\gamma_{[a}D_{b]}\chi^{j)} + 8\,\mathrm{i}\,\bar\epsilon^{(i}\gamma_{[a}\gamma\cdot T\gamma_{b]}\chi^{j)} \\
&\quad + \mathrm{i}\,\bar\epsilon^{(i}\gamma\cdot T\widehat{R}_{ab}{}^{j)}(Q) + \tfrac{3}{2}\,\mathrm{i}\,\bar\eta^{(i}\widehat{R}_{ab}{}^{j)}(Q) + 8\,\mathrm{i}\,\bar\eta^{(i}\gamma_{ab}\chi^{j)}\,.
\end{aligned}
$$

Note that we also have given the transformation rules for the gauge field of dilatations b_μ, which did not appear in the linearized Weyl multiplet. In the nonlinear Weyl multiplet it is hidden in the covariant derivatives and curvatures.

5.4.3 The Dilaton Weyl multiplet

To obtain the complete Dilaton Weyl multiplet, we first replace the scalar φ by the dilaton $\sigma = e^{\varphi}$. We then introduce appropriate powers of the dilaton in the various terms of (5.79) such that all terms have the same Weyl weight. This will make the Dilaton Weyl multiplet much more nonlinear than the Standard Weyl multiplet.

The Dilaton Weyl multiplet also contains two extra gauge transformations: the gauge transformations of A_μ with parameter Λ and those of $B_{\mu\nu}$ with parameter Λ_μ. The transformation of the fields are given by

$$
\begin{aligned}
\delta e_\mu{}^a &= \tfrac{1}{2}\bar\epsilon\gamma^a\psi_\mu\,, \\
\delta\psi_\mu^i &= \mathcal{D}_\mu\epsilon^i + \mathrm{i}\,\gamma\cdot\underline{T}\gamma_\mu\epsilon^i - \mathrm{i}\,\gamma_\mu\eta^i\,, \\
\delta V_\mu{}^{ij} &= -\tfrac{3}{2}\,\mathrm{i}\,\bar\epsilon^{(i}\phi_\mu^{j)} + 4\bar\epsilon^{(i}\gamma_\mu\underline{\chi}^{j)} + \mathrm{i}\,\bar\epsilon^{(i}\gamma\cdot\underline{T}\psi_\mu^{j)} + \tfrac{3}{2}\,\mathrm{i}\,\bar\eta^{(i}\psi_\mu^{j)}\,, \\
\delta b_\mu &= \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\phi_\mu - 2\bar\epsilon\gamma_\mu\underline{\chi} + \tfrac{1}{2}\,\mathrm{i}\,\bar\eta\psi_\mu + 2\Lambda_{K\mu}\,, \qquad (5.94)\\
\delta B_{\mu\nu} &= \tfrac{1}{2}\,\mathrm{i}\,\sigma\bar\epsilon\gamma_{\mu\nu}\psi + \tfrac{1}{2}\sigma^2\bar\epsilon\gamma_{[\mu}\psi_{\nu]} + A_{[\mu}\delta(\epsilon)A_{\nu]} + 2\partial_{[\mu}\Lambda_{\nu]} - \tfrac{1}{2}\Lambda F_{\mu\nu}\,, \\
\delta A_\mu &= \tfrac{1}{2}\bar\epsilon\gamma_\mu\psi - \tfrac{1}{2}\,\mathrm{i}\,\sigma\bar\epsilon\psi_\mu + \partial_\mu\Lambda\,, \\
\delta\psi^i &= -\tfrac{1}{4}\gamma\cdot\widehat{F}\epsilon^i - \tfrac{1}{2}\,\mathrm{i}\,\not{D}\sigma\epsilon^i + \sigma\gamma\cdot\underline{T}\epsilon^i - \tfrac{1}{4}\,\mathrm{i}\,\sigma^{-1}\bar\psi^i\psi^j\epsilon_j + \sigma\eta^i\,, \\
\delta\sigma &= \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\psi\,.
\end{aligned}
$$

We have again underlined some fields to indicate that they are not independent fields but merely short-hand notations. The explicit expression for these fields in terms of fields of the

Dilaton Weyl multiplet are

$$\begin{aligned}
T_{ab} &= \tfrac{1}{8}\sigma^{-2}\left(\sigma\widehat{F}_{ab} - \tfrac{1}{6}\varepsilon_{abcde}\widehat{H}^{edc} + \tfrac{1}{4}\,\mathrm{i}\,\bar{\psi}\gamma_{ab}\psi\right)\,,\\
\chi^i &= \tfrac{1}{8}\,\mathrm{i}\,\sigma^{-1}\not{D}\psi^i + \tfrac{1}{16}\,\mathrm{i}\,\sigma^{-2}\not{D}\sigma\psi^i - \tfrac{1}{32}\sigma^{-2}\gamma\cdot\widehat{F}\psi^i\\
&\quad +\tfrac{1}{4}\sigma^{-1}\gamma\cdot\underline{T}\psi^i + \tfrac{1}{32}\,\mathrm{i}\,\sigma^{-3}\psi_j\bar{\psi}^i\psi^j\,,\\
D &= \tfrac{1}{4}\sigma^{-1}\Box^c\sigma + \tfrac{1}{8}\sigma^{-2}(D_a\sigma)(D^a\sigma) - \tfrac{1}{16}\sigma^{-2}\widehat{F}^2\\
&\quad -\tfrac{1}{8}\sigma^{-2}\bar{\psi}\not{D}\psi - \tfrac{1}{64}\sigma^{-4}\bar{\psi}^i\psi^j\bar{\psi}_i\psi_j - 4\,\mathrm{i}\,\sigma^{-1}\bar{\psi}\underline{\chi}\\
&\quad +\left(2\sigma^{-1}\widehat{F}_{ab} - \tfrac{26}{3}\underline{T_{ab}} + \tfrac{1}{4}\,\mathrm{i}\,\sigma^{-2}\bar{\psi}\gamma_{ab}\psi\right)\underline{T^{ab}}\,,
\end{aligned} \tag{5.95}$$

The conformal D'Alembertian $\Box^c$ is defined by

$$\begin{aligned}
\Box^c\sigma \equiv D^aD_a\sigma &= \left(\partial^a - 2b^a + \omega_b{}^{ba}\right)D_a\sigma - \tfrac{1}{2}\,\mathrm{i}\,\bar{\psi}_aD^a\psi - 2\sigma\bar{\psi}_a\gamma^a\underline{\chi}\\
&\quad +\tfrac{1}{2}\bar{\psi}_a\gamma^a\gamma\cdot\underline{T}\psi + \tfrac{1}{2}\bar{\phi}_a\gamma^a\psi + 2f_a{}^a\sigma\,.
\end{aligned} \tag{5.96}$$

The justification for using the expressions (5.95) is that they transform exactly as the fields $T_{ab}\,, \chi^i$ and D in the Standard Weyl multiplet. Finally, we have defined the curvatures for the gauge fields $B_{\mu\nu}$ and A_μ to be

$$\begin{aligned}
\widehat{F}_{\mu\nu} &= 2\partial_{[\mu}A_{\nu]} - \bar{\psi}_{[\mu}\gamma_{\nu]}\psi + \tfrac{1}{2}\,\mathrm{i}\,\sigma\bar{\psi}_{[\mu}\psi_{\nu]}\,,\\
\widehat{H}_{\mu\nu\rho} &= 3\partial_{[\mu}B_{\nu\rho]} - \tfrac{3}{2}\,\mathrm{i}\,\sigma\bar{\psi}_{[\mu}\gamma_{\nu\rho]}\psi - \tfrac{3}{4}\sigma^2\bar{\psi}_{[\mu}\gamma_\nu\psi_{\rho]} + \tfrac{3}{2}A_{[\mu}F_{\nu\rho]}\,.
\end{aligned} \tag{5.97}$$

For the convenience of the reader we give their transformation rules

$$\begin{aligned}
\delta\widehat{F}_{ab} &= -\bar{\epsilon}\gamma_{[a}D_{b]}\psi - \tfrac{1}{2}\,\mathrm{i}\,\sigma\bar{\epsilon}\widehat{R}_{ab}(Q) + \mathrm{i}\,\bar{\epsilon}\gamma_{[a}\gamma\cdot\underline{T}\gamma_{b]}\psi + \mathrm{i}\,\bar{\eta}\gamma_{ab}\psi\,,\\
\delta\widehat{H}_{abc} &= \tfrac{3}{2}\,\mathrm{i}\,\sigma\bar{\epsilon}\gamma_{[ab}D_{c]}\psi + \tfrac{3}{2}\,\mathrm{i}\,D_{[a}\sigma\bar{\epsilon}\gamma_{bc]}\psi - \tfrac{3}{4}\sigma^2\bar{\epsilon}\gamma_{[a}\widehat{R}_{bc]}(Q)\\
&\quad -\tfrac{3}{2}\sigma\bar{\epsilon}\gamma_{[a}\gamma\cdot\underline{T}\gamma_{bc]}\psi - \tfrac{3}{2}\bar{\epsilon}\gamma_{[a}\widehat{F}_{bc]}\psi - \tfrac{3}{2}\sigma\bar{\eta}\gamma_{abc}\psi\,.
\end{aligned} \tag{5.98}$$

Finally, we give the Bianchi identities for these two curvatures

$$\begin{aligned}
D_{[a}\widehat{F}_{bc]} &= \tfrac{1}{2}\bar{\psi}\gamma_{[a}\widehat{R}_{bc]}(Q)\,,\\
D_{[a}\widehat{H}_{bcd]} &= \tfrac{3}{4}\widehat{F}_{[ab}\widehat{F}_{cd]}\,.
\end{aligned} \tag{5.99}$$

5.5 Connection between the Weyl multiplets

In the previous section, we have shown that the Standard and Dilaton Weyl multiplets can be related to each other by expressing the fields of the Standard Weyl multiplet in terms of those of the Dilaton Weyl multiplet (see (5.95)). It is known that in six dimensions the coupling of an on-shell self-dual tensor multiplet to the Standard Weyl multiplet leads to a Dilaton Weyl multiplet [176].

Since in five dimensions a tensor multiplet is dual to a vector multiplet, it is natural to consider the coupling of a vector multiplet to the Standard Weyl multiplet. Since the Standard Weyl multiplet has no dilaton, we must consider the improved vector multiplet. We will take the vector multiplet off-shell to simplify the higher-order fermion terms.

5.5.1 The improved Maxwell multiplet

We will first consider the improved vector multiplet in a flat background, i.e. no coupling to conformal supergravity. Our starting point is the Lagrangian corresponding to an off-shell vector multiplet:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}(\partial\sigma)^2 + Y^{ij}Y_{ij}\,. \tag{5.100}$$

The action corresponding to this Lagrangian is invariant under the off-shell supersymmetries

$$\begin{aligned}
\delta A_\mu &= \tfrac{1}{2}\bar{\epsilon}\gamma_\mu\psi\,,\\
\delta Y^{ij} &= -\tfrac{1}{2}\bar{\epsilon}^{(i}\partial\!\!\!/\psi^{j)}\,,\\
\delta\psi^i &= -\tfrac{1}{4}\gamma\cdot F\epsilon^i - \tfrac{1}{2}\,\mathrm{i}\,\partial\!\!\!/\sigma\epsilon^i - Y^{ij}\epsilon_j\,,\\
\delta\sigma &= \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\psi\,.
\end{aligned} \tag{5.101}$$

The action has the wrong Weyl weight to be scale invariant. We therefore improve it by multiplying all terms with the dilaton. This requires additional cubic terms in the action to keep it invariant under supersymmetry. We thus obtain the Lagrangian for the improved vector multiplet:

$$\begin{aligned}
\mathcal{L} \;=\; &\left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}\bar{\psi}\partial\!\!\!/\psi - \tfrac{1}{2}(\partial\sigma)^2 + Y^{ij}Y_{ij}\right)\sigma\\
&-\tfrac{1}{24}\varepsilon^{\mu\nu\lambda\rho\sigma}A_\mu F_{\nu\lambda}F_{\rho\sigma} - \tfrac{1}{8}\,\mathrm{i}\,\bar{\psi}\gamma\cdot F\psi - \tfrac{1}{2}\,\mathrm{i}\,\bar{\psi}^i\psi^j Y_{ij}\,.
\end{aligned} \tag{5.102}$$

The equations of motion and the Bianchi identity corresponding to this Lagrangian are given by

$$0 = L^{ij} = \varphi^i = E_a = N = G_{abc}\,. \tag{5.103}$$

where we have defined

$$\begin{aligned}
L^{ij} &\equiv 2\sigma Y^{ij} - \tfrac{1}{2}\,\mathrm{i}\,\bar{\psi}^i\psi^j\,,\\
\varphi^i &\equiv \mathrm{i}\,\sigma\partial\!\!\!/\psi^i + \tfrac{1}{2}\,\mathrm{i}\,\partial\!\!\!/\sigma\psi^i - \tfrac{1}{4}\gamma\cdot F\psi^i + Y^{ij}\psi_j\,,\\
E_a &\equiv \partial^b\left(\sigma F_{ba} + \tfrac{1}{4}\,\mathrm{i}\,\bar{\psi}\gamma_{ba}\psi\right) - \tfrac{1}{8}\varepsilon_{abcde}F^{bc}F^{de}\,,\\
N &\equiv -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}\bar{\psi}\partial\!\!\!/\psi + \sigma\Box\sigma + \tfrac{1}{2}(\partial\sigma)^2 + Y^{ij}Y_{ij}\,,\\
G_{abc} &\equiv \partial_{[a}F_{bc]}\,.
\end{aligned} \tag{5.104}$$

5.5.2 Coupling to the Standard Weyl multiplet

Next, we consider the coupling of the improved vector multiplet to the Standard Weyl multiplet. The transformation rules for the fields of the off-shell vector multiplet can be found by imposing the superconformal algebra (5.87). We thus find the following Q- and S-transformation rules:

$$\begin{aligned}
\delta A_\mu &= \tfrac{1}{2}\bar{\epsilon}\gamma_\mu\psi - \tfrac{1}{2}\,\mathrm{i}\,\sigma\bar{\epsilon}\psi_\mu\,,\\
\delta Y^{ij} &= -\tfrac{1}{2}\bar{\epsilon}^{(i}\not{D}\psi^{j)} + \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}^{(i}\gamma\cdot T\psi^{j)} - 4\,\mathrm{i}\,\sigma\bar{\epsilon}^{(i}\chi^{j)} + \tfrac{1}{2}\,\mathrm{i}\,\bar{\eta}^{(i}\psi^{j)}\,,\\
\delta\psi^i &= -\tfrac{1}{4}\gamma\cdot\widehat{F}\epsilon^i - \tfrac{1}{2}\,\mathrm{i}\,\not{D}\sigma\epsilon^i + \sigma\gamma\cdot T\epsilon^i - Y^{ij}\epsilon_j + \sigma\eta^i\,,\\
\delta\sigma &= \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\psi\,,
\end{aligned} \tag{5.105}$$

where the covariant curvature is

$$\widehat{F}_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} - \bar{\psi}_{[\mu}\gamma_{\nu]}\psi + \tfrac{1}{2}\,\mathrm{i}\,\sigma\bar{\psi}_{[\mu}\psi_{\nu]}\,. \tag{5.106}$$

The supercovariant extension of the Bianchi identity reads

$$0 = G_{abc} \;=\; D_{[a}\widehat{F}_{bc]} - \tfrac{1}{2}\bar{\psi}\gamma_{[a}\widehat{R}_{bc]}(Q)\,. \tag{5.107}$$

The second term in the transformation of A_μ, reflected also in the curvature, signals a modification of the supersymmetry algebra, as can be seen by comparing with the general rule (5.87)

$$[\delta(\epsilon_1),\delta(\epsilon_2)] = \ldots + \delta_{\mathrm{U}(1)}\left(\Lambda_3 = -\tfrac{1}{2}\,\mathrm{i}\,\sigma\bar{\epsilon}_2\epsilon_1\right), \tag{5.108}$$

where the dots indicate all the terms present for the fields of the Standard Weyl multiplet and where the last term is the gauge transformation of A_μ.

Our next goal is to find the equations of motion for the vector multiplet coupled to conformal supergravity. These equations of motion should be an extension of the flat spacetime results given in (5.105). One way to proceed is to first find the curved background extension of the flat spacetime action defined by (5.102) and next derive the equations of motion from this action. However, for our present purposes, it is sufficient to find the equations of motion only.

We want to identify the spinor ψ^i of the vector multiplet with the spinor ψ^i of the Dilaton Weyl multiplet. This is why we have given these two spinors the same name in the first place. Comparing the $\mathrm{SU}(2)$ triplet term in the supersymmetry transformations of the two spinors, see (5.94) and (5.105), we deduce that the constraint L^{ij} does not get any corrections, and we must have

$$L^{ij} = 2\sigma Y^{ij} - \tfrac{1}{2}\,\mathrm{i}\,\bar{\psi}^i\psi^j\,. \tag{5.109}$$

There are now two ways to proceed. One way is to make the transition to an on-shell vector multiplet by using (5.109) to eliminate the auxiliary field Y^{ij} from the transformation rules (5.105). The commutator of two supersymmetry transformations would then only close modulo the equations of motion.

A more elegant way is to note that the equations of motion must transform into each other. By varying (5.109) under (5.105) we find

$$\delta L^{ij} = \mathrm{i}\,\bar{\epsilon}^{(i}\varphi^{j)}\,, \tag{5.110}$$

where the supercovariant extension of φ^i is now given by

$$\begin{aligned}\varphi^i \;=\;& \mathrm{i}\,\sigma \not{D}\psi^i + \tfrac{1}{2}\,\mathrm{i}\,\not{D}\sigma\psi^i - \tfrac{1}{4}\gamma\cdot\widehat{F}\psi^i + Y^{ij}\psi_j \\ & +2\sigma\gamma\cdot T\psi^i - 8\sigma^2\chi^i\,.\end{aligned} \tag{5.111}$$

Varying this expression under (5.105) and using (5.107) leads to the other equations of motion. We find

$$\delta\varphi^i = -\tfrac{1}{2}\,\mathrm{i}\,\not{D}L^{ij}\epsilon_j - \tfrac{1}{2}\,\mathrm{i}\,\gamma^a E_a\epsilon^i + \tfrac{1}{2}N\epsilon^i - \gamma\cdot T L^{ij}\epsilon_j\,. \tag{5.112}$$

The supercovariant generalizations of (5.105) are given by

$$\begin{aligned}E_a \;=\;& D^b\left(\sigma\widehat{F}_{ba} - 8\sigma^2 T_{ba} + \tfrac{1}{4}\,\mathrm{i}\,\bar{\psi}\gamma_{ba}\psi\right) - \tfrac{1}{8}\varepsilon_{abcde}\widehat{F}^{bc}\widehat{F}^{de}\,, \\ N \;=\;& -\tfrac{1}{4}\widehat{F}_{ab}\widehat{F}^{ab} - \tfrac{1}{2}\bar{\psi}\not{D}\psi + \sigma\Box^c\sigma + \tfrac{1}{2}D^a\sigma D_a\sigma + Y^{ij}Y_{ij} \\ & +\mathrm{i}\,\bar{\psi}\gamma\cdot T\psi - 16\,\mathrm{i}\,\sigma\bar{\psi}\chi - \tfrac{104}{3}\sigma^2 T_{ab}T^{ab} + 8\sigma\widehat{F}_{ab}T^{ab} - 4\sigma^2 D\,,\end{aligned} \tag{5.113}$$

where we have used the expression for the conformal D'Alembertian given in (5.96). The supercovariant equations of motion and Bianchi identity are then given by

$$0 = L^{ij} = \varphi^i = E_a = N = G_{abc}\,. \tag{5.114}$$

5.5.3 Solving the equations of motion

In six dimensions, the equations of motion for an on-shell tensor multiplet coupled to the Standard Weyl multiplet can be used to eliminate the matter fields of the latter in terms of the matter fields of the Dilaton Weyl multiplet.

Precisely the same happens here. First of all, the equations of motion for Y^{ij} can be used to eliminate this auxiliary field. Next, the equations of motion for ψ^i and σ can be used to solve for the fields χ^i and D, respectively. The expressions for these fields exactly coincide with the ones we found in (5.95).

The solution for the matter field T_{ab} in terms of the fields of the Dilaton Weyl multiplet is more subtle. It requires that we first reinterpret the equation of motion for the vector field as the Bianchi identity for a two-form antisymmetric tensor gauge field $B_{\mu\nu}$. To be precise, we rewrite $E_a = 0$ from (5.113) as a Bianchi identity

$$D_{[a}\widehat{H}_{bcd]} \;=\; \tfrac{3}{4}\widehat{F}_{[ab}\widehat{F}_{cd]}\,, \tag{5.115}$$

where the three-form curvature $\widehat{H}_{abc}$ is defined by

$$-\tfrac{1}{6}\varepsilon_{abcde}\widehat{H}^{edc} = 8\sigma^2 T_{ab} - \sigma\widehat{F}_{ab} - \tfrac{1}{4}\,\mathrm{i}\,\bar{\psi}\gamma_{ab}\psi\,. \tag{5.116}$$

Note that the latter equation is just a rewriting of the relation (5.95).

The Bianchi identity (5.115) can be solved in terms of an antisymmetric two-form gauge field $B_{\mu\nu}$. The superconformal algebra (5.87) imposes that such a field transforms under supersymmetry as follows:

$$\delta_Q B_{\mu\nu} = \tfrac{1}{2}\,\mathrm{i}\,\sigma\bar{\epsilon}\gamma_{\mu\nu}\psi + \tfrac{1}{2}\sigma^2\bar{\epsilon}\gamma_{[\mu}\psi_{\nu]} + A_{[\mu}\delta(\epsilon)A_{\nu]}\,. \tag{5.117}$$

In addition, one finds that the field $B_{\mu\nu}$ transforms under a $\mathrm{U}(1)$ and a vector gauge transformation as follows

$$\delta B_{\mu\nu} = 2\partial_{[\mu}\Lambda_{\nu]} - \tfrac{1}{2}\Lambda F_{\mu\nu}\,. \tag{5.118}$$

Furthermore, the commutator of two Q-transformations picks up a vector gauge transformation δ_B for the field $B_{\mu\nu}$:

$$\begin{aligned}
&[\delta(\epsilon_1),\delta(\epsilon_2)] = \ldots + \delta_{\mathrm{U}(1)}\left(\Lambda_3\right) + \delta_B\left(\Lambda_{3\mu}\right)\,,\\
&\Lambda_3 = -\tfrac{1}{2}\,\mathrm{i}\,\sigma\bar{\epsilon}_2\epsilon_1\,, \qquad \Lambda_{3\mu} = -\tfrac{1}{4}\sigma^2\bar{\epsilon}_2\gamma_\mu\epsilon_1 - \tfrac{1}{2}A_\mu\Lambda_3\,.
\end{aligned} \tag{5.119}$$

From the transformation rules (5.118) for $B_{\mu\nu}$ it follows that the supercovariant field strength $\widehat{H}_{\mu\nu\rho}$ is given by

$$\widehat{H}_{\mu\nu\rho} \;=\; 3\partial_{[\mu}B_{\nu\rho]} - \tfrac{3}{2}\,\mathrm{i}\,\sigma\bar{\psi}_{[\mu}\gamma_{\nu\rho]}\psi - \tfrac{3}{4}\sigma^2\bar{\psi}_{[\mu}\gamma_\nu\psi_{\rho]} + \tfrac{3}{2}A_{[\mu}F_{\nu\rho]}\,. \tag{5.120}$$

This field strength indeed satisfies the Bianchi identity (5.115).

We conclude that the connection between the Standard and Dilaton Weyl multiplets can be obtained by first coupling an improved vector multiplet to the Standard Weyl multiplet and, next, solving the equations of motion. To solve the equation of motion for the vector field in terms of the matter field T_{ab}, one must first reinterpret this equation of motion as the Bianchi identity for an antisymmetric two-form gauge field.

Chapter 6

Matter-couplings of conformal supergravity

The basic supergravities in ten and and eleven dimensions are the low energy limits of string theory and M-theory, as we have seen in the introduction and in chapter 1. Matter-coupled supergravity theories in lower dimensions [38] have played an important role in our understanding of the low-energy limit of string theory compactifications.

For phenomenological reasons, much work has been done related to compactifications to four dimensions and the corresponding four-dimensional matter-coupled supergravities [197, 198]. The supergravities in the intermediate dimensions have also played a role in the understanding of string theory. For instance, the structure of nine-dimensional supergravity is important for the understanding of T-duality [40], whereas six-dimensional supergravity plays a crucial role in the understanding of string-string duality [199].

As we have discussed in chapter 4, a lot of attention has recently been given to five-dimensional matter-coupled supergravity theories [159, 162], thereby generalizing the earlier results of [156–158]. In this chapter, we will take the superconformal approach [167, 173–175] to construct a framework from which one can independently derive and study the possible five-dimensional matter-couplings to Poincaré supergravity.

An advantage of the superconformal construction is that, by past experience, it leads to insights into the kinematical and geometrical structure of matter-coupled Poincaré supergravity. For instance, the vector fields of superconformal vector multiplets split into the graviphoton of the Poincaré supergravity multiplet and the photons of ordinary vector multiplets by gauge-fixing the superconformal symmetries. A more recent example is the insight in relations between hyper-Kähler cones and quaternionic manifolds, based on the study of superconformal hypermultiplets [172, 200].

There is a more general interest in the $\mathcal{N} = 2\,, D = 5$ matter-coupled supergravities: they belong to the class of theories with eight supersymmetries [201]. Such theories are especially interesting since the geometry, determined by the kinetic terms of the scalars, contains undetermined functions. Theories with thirty-two supersymmetries have no matter multiplets, whereas the geometry of those with sixteen supersymmetries is completely determined by the number of matter multiplets. Theories with four supersymmetries allow for even more general geometries: continuous deformations of the manifold are allowed. Theories with eight supersymmetries have a more restricted class of geometries, which makes them more manageable.

The geometry related to supersymmetric theories with eight real supercharges is called "special geometry". Special geometry was first found in [202, 203] for local supersymmetry and in [204, 205] for rigid supersymmetry. It occurs in Calabi-Yau compactifications as the moduli space of these manifolds [206–211]. Special geometry was also a very useful tool in the investigation of supersymmetric black holes [212, 213], the work of Seiberg and Witten [214, 215], and the AdS/CFT correspondence [76].

So far, special geometry had been mainly investigated in the context of four dimensions. With the advent of the brane-world scenarios [138, 139], also the $D = 5$ variant of special geometry [156], called "very special geometry", received a lot of attention. The connection to special geometry was made in [216]. Finally, there is a connection between special geometry and and another class of geometries called "quaternionic geometry" [207], which has lead to new results on the classification of homogeneous quaternionic spaces [217, 218].

Superconformal matter multiplets with eight supersymmetries have already been introduced in [178, 179, 219]. However, there are still some ingredients missing. For instance, we will not only introduce vector multiplets in the adjoint representation but in arbitrary representations. The resulting multiplet in this case is called the "vector-tensor" multiplet. We will also construct vector-tensor multiplets in reducible, but not completely reducible representations: they are related to non-compact, non-semi-simple gaugings of Poincaré supergravity, a class of gauged supergravities that have not been considered for $\mathcal{N} = 2$ supersymmetry.

Some of the superconformal matter multiplets are on-shell: the algebra closes only modulo equations of motion. However, this does not imply that these equations of motion have to follow from an action. Indeed, this is a familiar feature of e.g. IIB supergravity and other theories with self-dual antisymmetric tensor fields. In particular for the vector-tensor multiplet, the absence of an action allows for couplings with an *odd* number of tensor multiplets, which generalizes the analysis made in [220]. For the hypermultiplets, we will introduce more general geometries than hyper-Kähler for rigid supersymmetry, or quaternionic-Kähler for local supersymmetry: we will consider hyper-Kähler manifolds without a metric: such manifolds are called "hyper-complex" manifolds.

We will start this chapter with constructing and discussing the possible matter-couplings in the absence of a Lagrangian by giving the rigid transformation rules for the vector-tensor multiplet and the hypermultiplet. We will emphasize the geometrical interpretation of the emerging algebraic structure. Next, in section 6.3, we construct the rigid superconformal Lagrangians for each of the superconformal matter multiplets. We discuss the restrictions on the possible geometries that follow from the requirement of a Lagrangian. Finally, in section 6.4, we extend the rigid superconformal symmetry to local superconformal symmetry, making use of the Weyl multiplet constructed in chapter 5.

This chapter is based on the work to be published in [17].

6.1 The vector-tensor multiplet

In this section, we will discuss superconformal vector multiplets that transform in arbitrary representations of the gauge group. From work on $\mathcal{N} = 2\,, D = 5$ Poincaré matter-couplings [159], it is known that vector multiplets transforming in representations other than the adjoint have to be dualized to tensor fields. We define a vector-tensor multiplet to be a vector multiplet transforming in a reducible representation that contains the adjoint representation as well as another, arbitrary, representation.

We will show that the analysis of [159] can be extended to superconformal vector multiplets. Moreover, we will generalize the gauge transformations for the tensor fields given in [159] by allowing them to transform into the field-strengths of the adjoint gauge fields. These more general gauge transformations are consistent with supersymmetry, even after breaking the conformal symmetry.

The vector-tensor multiplet contains *a priori* an arbitrary number of tensor fields. The restriction to an even number of tensor fields is not imposed by the closure of the algebra. However, one can only construct an action for an even number of tensor multiplets as we will see in section 6.3.

To make contact with other results in the literature, we will break the rigid conformal symmetry by using a vector multiplet as a compensating multiplet for the superconformal symmetry. The adjoint fields of the vector-tensor multiplet are given constant expectation values, and the scalar expectation values will play the role of a mass parameter. If one demands that the field equations do not contain tachyonic modes, an even number of tensor multiplet is required. For the case of two tensor multiplets, this will reduce the superconformal vector-tensor multiplet to the massive self-dual complex tensor multiplet of [221].

6.1.1 Adjoint representation

We will start with giving the transformation rules for a vector multiplet in the adjoint representation [179]. Such an off-shell vector multiplet has $8 + 8$ real degrees of freedom whose $\mathrm{SU}(2)$ labels and Weyl weights we have indicated in table 6.1. If the rank of the gauge group is n_{V}, then we have $I = 1, \ldots, n_{\mathrm{V}}$, and the scalars of the vector multiplet span a n_{V}-dimensional real vector space which is isomorphic to the manifold $\mathbb{R}^{n_{\mathrm{V}}}$.

We consider gauge fields A_μ^I and general matter fields of the vector multiplet X^I that transform under gauge transformations with parameters Λ^I according to

$$\delta_G(\Lambda^J) A_\mu^I = \partial_\mu \Lambda^I + g A_\mu^J f_{JK}{}^I \Lambda^K\,, \qquad \delta_G(\Lambda^J) X^I = -g \Lambda^J f_{JK}{}^I X^K\,, \tag{6.1}$$

Field	SU(2)	w	# d.o.f.
A_μ^I	1	0	$4\,n_{\rm V}$
Y^{ijI}	3	2	$3\,n_{\rm V}$
σ^I	1	1	$1\,n_{\rm V}$
ψ^{iI}	2	$3/2$	$8\,n_{\rm V}$

Table 6.1: The off-shell Yang-Mills multiplet.

where g is the coupling constant of the gauge group. The gauge transformations that we consider satisfy the commutation relations

$$[\delta_G(\Lambda_1^I),\delta_G(\Lambda_2^J)]=\delta_G(\Lambda_3^K)\,,\qquad \Lambda_3^K=g\Lambda_1^I\Lambda_2^J f_{IJ}{}^K\,. \tag{6.2}$$

The expression for the gauge-covariant derivative of X^I and the field-strengths are given by

$$\mathcal{D}_\mu X^I=\partial_\mu X^I+gA_\mu^J f_{JK}{}^I X^K\,,\qquad F_{\mu\nu}^I=2\partial_{[\mu}A_{\nu]}^I+gf_{JK}{}^I A_\mu^J A_\nu^K\,. \tag{6.3}$$

The field-strength satisfies the Bianchi identity

$$\mathcal{D}_{[\mu}F_{\nu\lambda]}^I=0\,. \tag{6.4}$$

The rigid Q- and S-supersymmetry transformation rules for the off-shell Yang-Mills multiplet are given by [179]

$$\begin{aligned}
\delta A_\mu^I &= \tfrac12\bar\epsilon\gamma_\mu\psi^I\,,\\
\delta Y^{ijI} &= -\tfrac12\bar\epsilon^{(i}\not{\mathcal{D}}\psi^{j)I}-\tfrac12\,\mathrm{i}\,g\bar\epsilon^{(i}f_{JK}{}^I\sigma^J\psi^{j)K}+\tfrac12\,\mathrm{i}\,\bar\eta^{(i}\psi^{j)I}\,,\\
\delta\psi^{iI} &= -\tfrac14\gamma\cdot F^I\epsilon^i-\tfrac12\,\mathrm{i}\not{\mathcal{D}}\sigma^I\epsilon^i-Y^{ijI}\epsilon_j+\sigma^I\eta^i\,,\\
\delta\sigma^I &= \tfrac12\,\mathrm{i}\,\bar\epsilon\psi^I\,.
\end{aligned} \tag{6.5}$$

The field-strength transforms according to

$$\delta F_{\mu\nu}^I=-\bar\epsilon\gamma_{[\mu}\mathcal{D}_{\nu]}\psi^I+\,\mathrm{i}\,\bar\eta\gamma_{\mu\nu}\psi^I\,. \tag{6.6}$$

The commutator of two Q-supersymmetry transformations yields a translation with an extra G-transformation

$$[\delta(\epsilon_1),\delta(\epsilon_2)]=\delta_P\left(\tfrac12\bar\epsilon_2\gamma_\mu\epsilon_1\right)+\delta_G\left(-\tfrac12\,\mathrm{i}\,\sigma\bar\epsilon_2\epsilon_1\right)\,. \tag{6.7}$$

Note that even though we are considering rigid superconformal symmetry, the algebra (6.7) contains a field-dependent term on the right hand side. Such soft terms are commonplace in local superconformal symmetry, but here they already appear at the rigid level. In Hamiltonian language, it means that the algebra is satisfied modulo constraints.

Field	SU(2)	w	# d.o.f.
$B_{\mu\nu}^M$	1	0	$3\,n_{\rm T}$
ϕ^M	1	1	$1\,n_{\rm T}$
λ^{iM}	2	$3/2$	$4\,n_{\rm T}$

Table 6.2: The on-shell tensor multiplet.

6.1.2 Reducible representations

Instead of the set of field-strengths $F_{\mu\nu}^I$, we will now consider a more general set of tensor fields $\mathcal{H}_{\mu\nu}^{\widetilde{I}} = \{F_{\mu\nu}^I, B_{\mu\nu}^M\}$ with $\widetilde{I} = (I, M)$ $(I = 1, \cdots, n_{\rm V}; M = n_{\rm V} + 1, \cdots n_{\rm V} + n_{\rm T})$. The set of $n_{\rm T}$ tensor fields $B_{\mu\nu}^M$ are accompanied by spinors λ^{iM} and scalars ϕ^M as we have indicated in table 6.2.

The representation matrices take on the form

$$(t_I)_{\widetilde{J}}{}^{\widetilde{K}} = \begin{pmatrix} (t_I)_J{}^K & (t_I)_J{}^N \\ (t_I)_M{}^K & (t_I)_M{}^N \end{pmatrix}, \qquad \left\{ \begin{array}{rcl} I, J, K &=& 1, \ldots, n_{\rm V} \\ M, N &=& n_{\rm V}+1, \ldots, n_{\rm V}+n_{\rm T}\,. \end{array} \right. \tag{6.8}$$

It is understood that the $(t_I)_J{}^K$ are in the adjoint representation, i.e.

$$(t_I)_J{}^K = f_{IJ}{}^K\,. \tag{6.9}$$

If $n_{\rm T} \neq 0$, then the representation is reducible. We will see that this representation can be more general than assumed so far in treatments of vector-tensor multiplet couplings. The requirement that $n_{\rm T}$ is even will only appear when we demand the existence of an action in section 6.3.2, or if we require absence of tachyonic modes. The matrices t_I satisfy commutation relations

$$[t_I, t_J] = -f_{IJ}{}^K t_K\,, \qquad \text{or} \qquad t_{I\widetilde{N}}{}^{\widetilde{M}} t_{J\widetilde{M}}{}^{\widetilde{L}} - t_{J\widetilde{N}}{}^{\widetilde{M}} t_{I\widetilde{M}}{}^{\widetilde{L}} = -f_{IJ}{}^K t_{K\widetilde{N}}{}^{\widetilde{L}}\,. \tag{6.10}$$

If the index $\widetilde{L}$ is a vector index, then this relation is satisfied using the matrices as in (6.9).

Requiring the closure of the superconformal algebra, we find Q- and S-supersymmetry transformation rules for the vector-tensor multiplet

$$
\begin{aligned}
\delta A_\mu^I &= \tfrac{1}{2}\bar\epsilon\gamma_\mu\psi^I\,,\\
\delta\mathcal{H}_{\mu\nu}^{\widetilde I} &= -\bar\epsilon\gamma_{[\mu}\mathcal{D}_{\nu]}\psi^{\widetilde I} + \mathrm{i}\,g\bar\epsilon\gamma_{\mu\nu}t_{(\widetilde J\widetilde K)}{}^{\widetilde I}\sigma^{\widetilde J}\psi^{\widetilde K} + \mathrm{i}\,\bar\eta\gamma_{\mu\nu}\psi^{\widetilde I}\,,\\
\delta Y^{ij\widetilde I} &= -\tfrac{1}{2}\bar\epsilon^{(i}\not{\mathcal{D}}\psi^{j)\widetilde I} - \tfrac{1}{2}\,\mathrm{i}\,g\bar\epsilon^{(i}\left(t_{[\widetilde J\widetilde K]}{}^{\widetilde I} - 3t_{(\widetilde J\widetilde K)}{}^{\widetilde I}\right)\sigma^{\widetilde J}\psi^{j)\widetilde K} + \tfrac{1}{2}\,\mathrm{i}\,\bar\eta^{(i}\psi^{j)\widetilde I}\,,\\
\delta\psi^{i\widetilde I} &= -\tfrac{1}{4}\gamma\cdot\mathcal{H}^{\widetilde I}\epsilon^i - \tfrac{1}{2}\,\mathrm{i}\,\not{\mathcal{D}}\sigma^{\widetilde I}\epsilon^i - Y^{ij\widetilde I}\epsilon_j + \tfrac{1}{2}gt_{(\widetilde J\widetilde K)}{}^{\widetilde I}\sigma^{\widetilde J}\sigma^{\widetilde K}\epsilon^i + \sigma^{\widetilde I}\eta^i\,,\\
\delta\sigma^{\widetilde I} &= \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\psi^{\widetilde I}\,.
\end{aligned}
\tag{6.11}
$$

Note that (6.11) differs from (6.5) and (6.6) only in terms at $\mathcal{O}(g)$ in the gauge coupling constant, and that the difference is always proportional to the tensor $t_{(\widetilde J\widetilde K)}{}^{\widetilde I}$.

The curly derivatives denote gauge-covariant derivatives as in (6.3) with the replacement of structure constants by general matrices t_I according to (6.9). We have extended the range of the generators from I to $\widetilde I$ in order to simplify the transformation rules with the understanding that

$$(t_M)_{\widetilde J}{}^{\widetilde K} = 0\,. \tag{6.12}$$

We find that the supersymmetry algebra (6.7) is satisfied provided the representation matrices are restricted to

$$t_{(\widetilde J\widetilde K)}{}^{I} = 0\,. \tag{6.13}$$

If $n_{\rm T}\neq 0$, then the algebra only closes provided the following two equations of motion on the fields are satisfied

$$
\begin{aligned}
L^{ij\widetilde I} &\equiv t_{(\widetilde J\widetilde K)}{}^{\widetilde I}\left(2\sigma^{\widetilde J}Y^{ij\widetilde K} - \tfrac{1}{2}\,\mathrm{i}\,\bar\psi^{i\widetilde J}\psi^{j\widetilde K}\right) && (6.14)\\
&= 0\,,\\
E_{\mu\nu\lambda}^{\widetilde I} &\equiv \frac{3}{g}\mathcal{D}_{[\mu}\mathcal{H}_{\nu\lambda]}{}^{\widetilde I} - \varepsilon_{\mu\nu\lambda\rho\sigma}t_{(\widetilde J\widetilde K)}{}^{\widetilde I}\left(\sigma^{\widetilde J}\mathcal{H}^{\rho\sigma\widetilde K} + \tfrac{1}{4}\,\mathrm{i}\,\bar\psi^{\widetilde J}\gamma^{\rho\sigma}\psi^{\widetilde K}\right) && (6.15)\\
&= 0\,.
\end{aligned}
$$

For $\widetilde I = I$, we can use (6.13) to satisfy (6.14) and to reduce (6.15) to the Bianchi identity (6.4). The tensor $F_{\mu\nu}^I$ can therefore be seen as the curl of a gauge vector A_μ^I. We conclude that the fields with indices $\widetilde I = I$ form an off-shell vector multiplet in the adjoint representation of the gauge group. In particular, this means that for $n_{\rm T}=0$ we find back the transformation rules for the vector multiplet.

The constraints (6.14) and (6.15), with $\widetilde I = M$, do not form a supersymmetric set: they are invariant under S-supersymmetry, but under Q-supersymmetry they lead to a constraint on the spinors ψ^{iM} which we will call φ^{iM}:

$$\delta L^{ijM} = \mathrm{i}\,\bar\epsilon^{(i}\varphi^{j)M}\,,\quad \delta E_{\mu\nu\rho}^M = \bar\epsilon\gamma_{\mu\nu\rho}\varphi^M\,. \tag{6.16}$$

The expression for this constraint is given by

$$\begin{aligned}\varphi^{iM} &\equiv t_{(\widetilde{J}\widetilde{K})}{}^{M}\Big[\mathrm{i}\,\sigma^{\widetilde{J}}\not{D}\psi^{i\widetilde{K}} + \tfrac{1}{2}\,\mathrm{i}\,(\not{D}\sigma^{\widetilde{J}})\psi^{i\widetilde{K}} + Y^{ik\widetilde{J}}\psi_k^{\widetilde{K}} - \tfrac{1}{4}\gamma\cdot\mathcal{H}^{\widetilde{J}}\psi^{i\widetilde{K}}\Big] \\ &\quad -g\left(\Big[t_{[\widetilde{J}\widetilde{K}]}{}^{\widetilde{L}} - 3t_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{L}}\Big]\,t_{(\widetilde{I}\widetilde{L})}{}^{M} + \tfrac{1}{2}t_{\widetilde{I}\widetilde{J}}{}^{\widetilde{L}}t_{(\widetilde{K}\widetilde{L})}{}^{M}\right)\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\psi^{i\widetilde{K}} \\ &= 0\,.\end{aligned} \tag{6.17}$$

Varying the new constraint φ^{iM} under Q-and S-supersymmetry, one finds at first sight two more constraints, E_a^M and N^M, of which the first one turns out to be dependent (see below):

$$\begin{aligned}\delta\varphi^{i\,M} &= -\tfrac{1}{2}\,\mathrm{i}\,\not{D}L^{ij\,M}\epsilon_j - \tfrac{1}{2}\,\mathrm{i}\,\gamma^a E_a^M\epsilon^i + \tfrac{1}{2}N^M\epsilon^i - \tfrac{1}{2}gt_{\widetilde{J}\widetilde{K}}{}^{M}\sigma^{\widetilde{J}}L^{ij\,\widetilde{K}}\epsilon_j \\ &\quad -\tfrac{1}{12}\,\mathrm{i}\,gt_{(\widetilde{J}\widetilde{K})}{}^{M}\gamma^{abc}\sigma^{\widetilde{J}}E_{abc}^{\widetilde{K}}\epsilon^i + 3L^{ijM}\eta_j\,.\end{aligned} \tag{6.18}$$

The constraint N^M is given by

$$\begin{aligned}N^M &\equiv t_{(\widetilde{J}\widetilde{K})}{}^{M}\Big(\sigma^{\widetilde{J}}\Box\sigma^{\widetilde{K}} + \tfrac{1}{2}\mathcal{D}^a\sigma^{\widetilde{J}}\mathcal{D}_a\sigma^{\widetilde{K}} - \tfrac{1}{4}\mathcal{H}_{ab}^{\widetilde{J}}\mathcal{H}^{ab\widetilde{K}} - \tfrac{1}{2}\bar{\psi}^{\widetilde{J}}\not{D}\psi^{\widetilde{K}} + Y^{ij\widetilde{J}}Y_{ij}{}^{\widetilde{K}}\Big) \\ &\quad -\mathrm{i}\,g\Big[-\tfrac{1}{2}\,t_{[\widetilde{J}\widetilde{K}]}{}^{\widetilde{L}}t_{(\widetilde{I}\widetilde{L})}{}^{M} + 2\,t_{(\widetilde{I}\widetilde{J})}{}^{\widetilde{L}}t_{(\widetilde{K}\widetilde{L})}{}^{M}\Big]\sigma^{\widetilde{I}}\bar{\psi}^{\widetilde{J}}\psi^{\widetilde{K}} \\ &\quad +\tfrac{1}{2}g^2\,(t_It_Jt_K)_{\widetilde{L}}{}^{M}\sigma^I\sigma^J\sigma^K\sigma^{\widetilde{L}} \\ &= 0\,,\end{aligned} \tag{6.19}$$

and for E_a^M we find

$$\begin{aligned}E_a^M &\equiv t_{(\widetilde{J}\widetilde{K})}{}^{M}\Big(\mathcal{D}^b\Big(\sigma^{\widetilde{J}}\mathcal{H}_{ba}{}^{\widetilde{K}} + \tfrac{1}{4}\,\mathrm{i}\,\bar{\psi}^{\widetilde{J}}\gamma_{ba}\psi^K\Big) - \tfrac{1}{8}\varepsilon_{abcde}\mathcal{H}^{bc\widetilde{J}}\mathcal{H}^{de\widetilde{K}}\Big) \\ &= 0\,.\end{aligned} \tag{6.20}$$

This last constraint is not an independent condition, but it is related to E_{abc}^M

$$E_a^M = -\tfrac{1}{12}\varepsilon_{abcde}\mathcal{D}^bE^{cdeM}\,. \tag{6.21}$$

Subsequent supersymmetry variations do not lead to any new constraints. On a technical note, we made use of identities as

$$t_{K\widetilde{I}}{}^{\widetilde{L}}t_{(\widetilde{J}\widetilde{L})}{}^{M} + t_{K\widetilde{J}}{}^{\widetilde{L}}t_{(\widetilde{I}\widetilde{L})}{}^{M} - t_{(\widetilde{I}\widetilde{J})}{}^{\widetilde{L}}t_{K\widetilde{L}}{}^{M} = 0\,, \tag{6.22}$$

which follow from the commutator relation (6.10), and the restrictions (6.12) and (6.13).

To summarize, the superconformal algebra closes on the vector-tensor multiplet modulo the set of constraints (6.14), (6.14), (6.17) and (6.19). Under Q– and S–supersymmetry, they transform to each other, but they do not form a multiplet by themselves.

6.1.3 Completely reducible representations

Using (6.13), we have reduced the representation matrices t_I to the following block-upper-triangular form:

$$(t_I)_{\widetilde{J}}{}^{\widetilde{K}} = \begin{pmatrix} f_{IJ}{}^K & (t_I)_J{}^N \\ 0 & (t_I)_M{}^N \end{pmatrix} . \tag{6.23}$$

In the case that $(t_I)_J{}^N = 0$, the representation is called *completely reducible*: the field-strengths $F^I_{\mu\nu}$ and the tensor fields $B^M_{\mu\nu}$ do not mix under gauge transformations.

Recall that every *unitary* reducible representation of a Lie group is also completely reducible, and that every representation of a *compact* Lie group is equivalent to a unitary representation. Hence, every reducible representation of a compact Lie group is also completely reducible. Non-compact Lie groups, on the other hand, have no non-trivial and finite-dimensional unitary representations. However, every reducible representation of a *connected*, *semi-simple*, non-compact Lie group or a semi-simple, non-compact Lie *algebra* is also completely reducible. See [222] for an exposition of these theorems.

Hence, we need to consider the class of non-compact Lie algebras that contain an Abelian invariant subalgebra. In $\mathcal{N} = 8$ gauged supergravity in $D = 5$ [223], the algebras $\mathrm{CSO}(p, q, r)$ were studied: they are defined as the set of matrices that leave invariant the metric

$$\eta_{IJ} = (\mathbb{1}_p, -\mathbb{1}_q, 0_r) . \tag{6.24}$$

These algebras contain the following subalgebra

$$\mathrm{SO}(p, q) \oplus \mathrm{SO}(1, 1)^{\frac{r(r-1)}{2}} \subset \mathrm{CSO}(p, q, r) . \tag{6.25}$$

In [220], a classification of possible compact gaugings of $\mathcal{N} = 2$ supergravity in five dimensions was given, but the class of non-compact gaugings was not investigated. An additional remark is that, for non-semi-simple Lie groups, we need to take the vector fields in the co-adjoint representation, rather than the adjoint representation [224]. For semi-simple Lie groups these two representations coincide.

Reducible but not completely reducible representations are then given by n_{V} Abelian vector multiplets and n_{T} tensor multiplets transforming in the following representation

$$(t_I)_{\widetilde{J}}{}^{\widetilde{K}} = \begin{pmatrix} 0 & (t_I)_J{}^N \\ 0 & 0 \end{pmatrix} . \tag{6.26}$$

The literature on five-dimensional tensor multiplets [159] states that, to write down an action, one must assume that the representation is completely reducible, meaning that gauge transformations do not mix the pure Yang–Mills field-strengths and the tensor fields. We, however, find that off-diagonal generators *are* allowed, both when requiring closure of the superconformal algebra and when writing down an action. Thus, we have found more general vector-tensor multiplets.

6.1.4 The massive self-dual tensor multiplet

To obtain the massive self-dual tensor multiplet of [221], we consider a vector-tensor multiplet for general $n_{\rm V}$ and $n_{\rm T}$. Our purpose is to use the vector multiplet as a compensating multiplet for the superconformal symmetry. Thus, we impose conditions on the fields that break the conformal symmetry and preserve Q–supersymmetry. We give the fields of the vector multiplets the following vacuum expectation values

$$F^I_{\mu\nu} = Y^{ijI} = \psi^{iI} = 0\,, \qquad \sigma^I = \frac{2m^I}{g}\,, \tag{6.27}$$

where m^I are constants. Note that these conditions break the conformal group to the Poincaré group, and break S-supersymmetry ($\eta = 0$). This is an example of a compensating multiplet in rigid supersymmetry. The breaking of conformal symmetry is characterized by the mass parameters m^I in (6.27). If we substitute (6.27) into the expression (6.14) for L^{ijM}, then we find that we can eliminate the field Y^{ijM}

$$Y^{ijM} = 0\,. \tag{6.28}$$

Moreover, we can also substitute (6.27) into the constraints $E^M_{\mu\nu\lambda}$, φ^{iM} and N^M to obtain

$$\begin{aligned} 3\partial_{[\mu}B^M_{\nu\lambda]} &= \tfrac{1}{2}\varepsilon_{\mu\nu\lambda\rho\sigma}\mathcal{M}_N{}^M B^{\rho\sigma N}\,, \\ \partial\!\!\!/\psi^{iM} &= \mathrm{i}\,\mathcal{M}_N{}^M\psi^{iN}\,, \\ \Box\sigma^M &= -\left(\mathcal{M}^2\right)_N{}^M\sigma^N - \tfrac{4}{g}t_{IJ}{}^N m^I m^J \mathcal{M}_N{}^M\,. \end{aligned} \tag{6.29}$$

The mass-matrix $\mathcal{M}_N{}^M$ is defined as

$$\mathcal{M}_N{}^M \equiv g\,\sigma^I(t_I)_N{}^M = 2m^I(t_I)_N{}^M\,. \tag{6.30}$$

The last term of (6.29) can be eliminated by redefining σ^M with a constant shift. In order for the tensor fields to have no tachyonic modes, the mass-matrix needs to satisfy a symplectic condition which can only be satisfied if the number of tensor fields is even [221]. We denote the number of tensor multiplets by $n_{\rm T} = 2k$.

The exception is when the representation matrices are purely upper-diagonal: i.e when they take on the form (6.26). For that specific representation, the mass matrix vanishes identically and no tachyonic modes are present. However, in that case the self-duality condition reduces to the Bianchi identity so that we are dealing with $n_{\rm T}$ extra vector multiplets in disguise.

To obtain the massive self-dual tensor multiplet of [221] we consider the case of $n_{\rm V} = 1\,, n_{\rm T} = 2$, i.e. two (real) tensor multiplets $\{B^M_{\mu\nu}, \lambda^{iM}, \phi^M\}$ $(M, N = 2, 3)$ in the background of one (Abelian) vector multiplet $\{F_{\mu\nu}\,\psi^i, \sigma\}$ that has been given the vacuum expectation value (6.27). In what follows we will use a complex notation

$$B_{\mu\nu} = B^2_{\mu\nu} + \mathrm{i}\,B^3_{\mu\nu}\,, \quad \overline{B}_{\mu\nu} = B^2_{\mu\nu} - \mathrm{i}\,B^3_{\mu\nu}\,. \tag{6.31}$$

The generators $(t_1)_{\widetilde{I}}{}^{\widetilde{J}}$ must form a representation of $\mathrm{U}(1) \simeq \mathrm{SO}(2)$. Under a $\mathrm{U}(1)$ transformation the field-strength $F_{\mu\nu}$ is invariant and the complex tensor field gets a phase

$$B'_{\mu\nu} = e^{\,\mathrm{i}\,\theta} B_{\mu\nu} \to \begin{pmatrix} B^2_{\mu\nu} \\ B^3_{\mu\nu} \end{pmatrix}' = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} B^2_{\mu\nu} \\ B^3_{\mu\nu} \end{pmatrix} . \tag{6.32}$$

From this, we obtain the generator

$$(t_1)_{\widetilde{I}}{}^{\widetilde{J}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} . \tag{6.33}$$

After substituting the conditions (6.27) into the transformation rules, we obtain

$$\begin{aligned} \delta B_{\mu\nu} &= -\bar{\epsilon}\gamma_{[\mu}\partial_{\nu]}\lambda - m\bar{\epsilon}\gamma_{\mu\nu}\lambda\,, \\ \delta\lambda^i &= -\tfrac{1}{4}\gamma\cdot B\epsilon^i - \tfrac{1}{2}\,\mathrm{i}\,\partial\!\!\!/\phi\epsilon^i - \mathrm{i}\,m\phi\epsilon^i\,, \\ \delta\phi &= \tfrac{1}{2}\,\mathrm{i}\,\bar{\epsilon}\lambda\,, \end{aligned} \tag{6.34}$$

and

$$3\partial_{[\mu}B_{\nu\lambda]} = \mathrm{i}\,m\varepsilon_{\mu\nu\lambda\rho\sigma}B^{\rho\sigma} . \tag{6.35}$$

This reproduces the massive self-dual tensor multiplet of [221]. Note that the commutator of two Q-supersymmetries yields a translation plus a (rigid) U(1)-transformation whose parameter can be obtained from the general G-transformation in the superconformal algebra, see (6.7), by making the substitution (6.27).

From a six-dimensional point of view, the interpretation of the mass parameter m is that it is the label of the m-th Kaluza-Klein mode in the reduction of the $D = 6$ self-dual tensor multiplet. The zero-mode of the reduced tensor multiplet corresponds to a vector multiplet as can be seen from (6.35) which becomes a Bianchi identity for a field-strength when $m = 0$.

6.2 The hypermultiplet

In this section, we will discuss superconformal hypermultiplets in five dimensions. We will follow the approach of [225], which discussed four-dimensional superconformal hypermultiplets, but we will extend it to the case where an action is not needed, in the spirit explained in [201]. As for the tensor multiplets, there is no off-shell formulation with a finite number of auxiliary fields, and the supersymmetry algebra closes modulo equations of motion.

A single hypermultiplet contains four real scalars and two spinors subject to a symplectic Majorana reality condition. We take the number of hypermultiplets n_H equal to r, which means that we introduce $4r$ real scalars q^X, with $X = 1, \ldots, 4r$, and $2r$ spinors ζ^A with $A = 1, \ldots, 2r$. We have indicated these fields and their relevant properties in table 6.2.

Field	SU(2)	w	# d.o.f.
q^X	2	$\frac{3}{2}$	$4r$
ζ^A	1	2	$4r$

Table 6.3: The on-shell hypermultiplet.

Analogous to the equivalence of the real vector space $\mathbb{R}^{2n}$ to the complex vector space $\mathbb{C}^n$, the real vector space $\mathbb{R}^{4r}$ spanned by the scalars q^X is isomorphic to the quaternionic vector space $\mathbb{H}^r$. Recall that the field of quaternions $\mathbb{H}$ is defined as all four-tuples of the form $a + b\,\mathrm{i} + c\,\mathrm{j} + d\,\mathrm{k}$, with $\{a, b, c, d\} \in \mathbb{R}$ and $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{i}\,\mathrm{j}\,\mathrm{k} = -1$.

To formulate the symplectic Majorana condition, we introduce two matrices $\rho_A{}^B$ and $E_i{}^j$, with

$$\rho\rho^* = -\mathbb{1}_{2r}\,, \qquad EE^* = -\mathbb{1}_2\,. \tag{6.36}$$

This defines symplectic Majorana conditions for the fermions and the supersymmetry transformation parameters [226]:

$$\alpha\mathcal{C}\gamma_0\zeta^B\rho_B{}^A = \left(\zeta^A\right)^*\,, \qquad \alpha\mathcal{C}\gamma_0\epsilon^j E_j{}^i = (\epsilon^i)^*\,, \tag{6.37}$$

where $\mathcal{C}$ is the charge conjugation matrix, and α is an irrelevant number of modulus 1. We can always adopt the basis where $E_i{}^j = \varepsilon_{ij}$, and we will further restrict to that.

We will start this section with describing the rigid supersymmetry transformation rules and their geometrical interpretation. After that, we will realize the superconformal symmetries on the hypermultiplet. Finally, we will discuss how to gauge the isometries of the scalar manifold by coupling the hypermultiplet to a vector multiplet.

6.2.1 Rigid supersymmetry

We will show how the closure of the supersymmetry transformation rules on the scalars leads to equations defining a "hyper-complex" manifold. The scalars can then be regarded as coordinates on this hyper-complex manifold, whereas the fermions take their values in the tangent-space of the manifold. Furthermore, the closure of the algebra on the fermions leads to equations of motion.

Hyper-complex geometry

The rigid supersymmetry transformation rules for the hypermultiplet are given by

$$\begin{aligned} \delta(\epsilon)q^X &= -\mathrm{i}\,\bar{\epsilon}^i\zeta^A f_{iA}^X\,, \\ \delta(\epsilon)\zeta^A &= \tfrac{1}{2}\,\mathrm{i}\,\partial\!\!\!/ q^X f_X^{iA}\epsilon_i - \zeta^B\omega_{XB}{}^A\left(\delta(\epsilon)q^X\right)\,. \end{aligned} \tag{6.38}$$

The functions f_X^{iA} and $\omega_{XA}{}^B$ satisfy reality properties consistent with reality of q^X and the symplectic Majorana conditions

$$\left(f_X^{iA}\right)^* = f_X^{jB} E_j{}^i \rho_B{}^A \,, \qquad \left(\omega_{XA}{}^B\right)^* = \left(\rho^{-1}\omega_X\rho\right)_A{}^B \,. \tag{6.39}$$

A priori, the functions f_{iA}^X and f_X^{iA} are independent, but the commutator of two supersymmetries on the scalars only gives a translation if one imposes

$$\begin{aligned} f_Y^{iA} f_{iA}^X &= \delta_Y^X \,, \qquad f_X^{iA} f_{jB}^X = \delta_j^i \delta_B^A \,, \\ \mathfrak{D}_X f_Y^{iA} &\equiv \partial_X f_Y^{iA} - \Gamma_{XY}^Z f_Z^{iA} + \left(\omega_{Xj}{}^i \delta_B^A + \omega_{XB}{}^A \delta_j^i\right) f_Y^{jB} = 0 \,, \end{aligned} \tag{6.40}$$

where Γ_{XY}^Z is symmetric its lower indices.

The tensors given above have the following geometrical interpretation: f_{iA}^X and f_X^{iA} are invertible vielbeins on the scalar manifold, Γ_{XY}^Z can be interpreted as an affine torsionless connection, and $\omega_{Xj}{}^i$ and $\omega_{XB}{}^A$ are the $\mathrm{SU}(2)$-valued and $\mathrm{G}\,\ell(r,\mathbb{H})$-valued spin-connection one-forms, respectively. The constraint (6.40) then expresses that the vielbeins are covariantly constant with respect to these connections.

The scalar manifold is also endowed with a triplet of complex structures called the hypercomplex, which are constructed from the vielbeins and the Pauli-matrices σ^α

$$J_X{}^{Y\alpha} \equiv -\,\mathrm{i}\, f_X^{iA} (\sigma^\alpha)_i{}^j f_{jA}^Y \,. \tag{6.41}$$

For these complex structures, and other $\mathrm{SU}(2)$-valued quantities, we also use a doublet notation, for which

$$J_X{}^Y{}_i{}^j \equiv \mathrm{i}\, J_X{}^{Y\alpha} (\sigma^\alpha)_i{}^j = 2 f_X^{jA} f_{iA}^Y - \delta_i^j \delta_X^Y \,. \tag{6.42}$$

Using (6.40), the complex structures are covariantly constant and satisfy the quaternion algebra

$$J^\alpha J^\beta = -\mathbb{1}_{4r} \delta^{\alpha\beta} + \varepsilon^{\alpha\beta\gamma} J^\gamma \,. \tag{6.43}$$

The resulting geometry defined by the connections and complex structure goes under the name of *hyper-complex* geometry. The notion of a hyper-complex manifold appeared in the mathematics literature in [227], and various aspects have been treated in two workshops [228, 229].

Note that we do not require the existence of a metric: hyper-complex manifolds possessing a metric are called hyper-Kähler manifolds, and we will encounter them in section 6.3 when we discuss superconformal actions. Examples of (homogeneous) hyper-complex manifolds that are not hyper-Kähler were constructed in [230–232].

As a final remark, the associated curvature tensor for the $\mathrm{SU}(2)$-valued spin-connection one-form $\omega_{Xj}{}^i$ vanishes for all hyper-complex and hyper-Kähler manifolds

$$\mathcal{R}_{XYi}{}^j \equiv 2\partial_{[X}\omega_{Y]i}{}^j + 2\omega_{[X|k|}{}^j \omega_{Y]i}{}^k = 0 \,. \tag{6.44}$$

The connection is therefore always pure gauge and can be set to zero. Manifolds for which this curvature does not vanish are called quaternionic manifolds and quaternionic-Kähler manifolds, respectively. Quaternionic geometry generically arises after gauge-fixing superconformal hypermultiplets.

Reparametrizations

The supersymmetry transformation rules (6.38) are covariant with respect to two different kinds of reparametrizations. The first ones are the target space diffeomorphisms, $q^X \rightarrow \widetilde{q}^X(q)$, under which f_{iA}^X transforms as a vector, $\omega_{XA}{}^B$ as a one-form, and Γ_{XY}^Z as a connection. We can then define a variation $\widehat{\delta}$ which is covariantized with respect to these diffeomorphisms: e.g. for a quantity Δ^X we define

$$\widehat{\delta}\Delta^X = \delta\Delta^X + \Delta^Y \Gamma_{ZY}{}^X\, \delta q^Z\,. \tag{6.45}$$

Furthermore, there are reparametrizations of the tangent space, under which $f_X^{iA}(q)$ transforms as a vector, $\omega_{XA}{}^B$ as a connection,

$$\omega_{XA}{}^B \rightarrow \widetilde{\omega}_{XA}{}^B = \left[\left(\partial_X U^{-1}\right) U + U^{-1}\omega_X U\right]_A{}^B\,, \tag{6.46}$$

and the fermions as

$$\zeta^A \rightarrow \widetilde{\zeta}^A(q) = \zeta^B U_B{}^A(q)\,, \tag{6.47}$$

where $U(q)_A{}^B$ is any invertible matrix.

In general, such a transformation brings us into a basis where the fermions depend on the scalars q^X. In this sense, the hypermultiplet is written in a special basis where q^X and ζ^A are independent fields. These considerations lead us to define the covariant variation of the fermions

$$\widehat{\delta}\zeta^A \equiv \delta\zeta^A + \zeta^B\omega_{XB}{}^A\, \delta q^X\,, \tag{6.48}$$

Two models related by either target space diffeomorphisms or fermion reparametrizations of the form (6.47) are equivalent: they are different coordinate descriptions of the same system. Thus, in a covariant formalism, the fermions can be functions of the scalars. However, the expression $\partial_X\zeta^A$ makes only sense if one compares different bases. But in the same way also the expression $\zeta^B\omega_{XB}{}^A$ makes only sense if one compares different bases, as the connection has no absolute value. The only invariant object is the covariant derivative

$$\mathcal{D}_X\zeta^A \equiv \partial_X\zeta^A + \zeta^B\omega_{XB}{}^A\,. \tag{6.49}$$

Holonomy

Recall that the holonomy group of a manifold is defined as the group of transformations by which a vector can be rotated after parallel transport along a closed curve on the manifold. The holonomy group of a hyper-complex manifold is contained in $\mathrm{G}\,\ell(r,\mathbb{H}) = \mathrm{SU}^*(2r)\times \mathrm{U}(1)$, the group of transformations acting on the tangent-space.

This follows from the integrability conditions on the covariantly constant vielbeins f_X^{iA}, which relates the curvatures of the affine connection Γ_{XY}^Z and the spin-connection $\omega_{XA}{}^B$

$$R_{XYZ}{}^W = f_{iA}^W f_Z^{iB}\mathcal{R}_{XYB}{}^A\,, \qquad \delta_j^i\,\mathcal{R}_{XYB}{}^A = f_W^{iA} f_{jB}^Z\, R_{XYZ}{}^W\,, \tag{6.50}$$

	no metric	Hermitian metric
no $\mathrm{SU}(2)$ curvature	hyper-complex $\mathrm{G}\,\ell(r,\mathbb{H})$	hyper-Kähler $\mathrm{USp}(2r)$
non-zero $\mathrm{SU}(2)$ curvature	quaternionic $\mathrm{SU}(2)\cdot\mathrm{G}\,\ell(r,\mathbb{H})$	quaternionic-Kähler $\mathrm{SU}(2)\cdot\mathrm{USp}(2r)$

Table 6.4: The holonomy groups of the family of quaternionic-like manifolds.

where the curvatures are defined by

$$\begin{aligned} R^W{}_{ZXY} &\equiv 2\partial_{[X}\Gamma^W_{Y]Z} + 2\Gamma^W_{V[X}\Gamma^V_{Y]Z} \\ \mathcal{R}_{XYB}{}^A &\equiv 2\partial_{[X}\omega_{Y]B}{}^A + 2\omega_{[X|C|}{}^A\omega_{Y]B}{}^C\,. \end{aligned} \tag{6.51}$$

A consequence of (6.50) is that the Riemann curvature is purely $\mathrm{G}\,\ell(r,\mathbb{H})$-valued. Moreover, from the cyclicity properties of the Riemann tensor, it follows that

$$\begin{aligned} f^X_{Ci}f^Y_{jD}\mathcal{R}_{XYB}{}^A &= -\tfrac{1}{2}\varepsilon_{ij}W_{CDB}{}^A\,, \\ W_{CDB}{}^A &\equiv f^{iX}_C f^Y_{iD}\mathcal{R}_{XYB}{}^A \\ &= \tfrac{1}{2}f^{iX}_C f^Y_{iD}f^Z_{jB}f^{Aj}_W R_{XYZ}{}^W\,, \end{aligned} \tag{6.52}$$

where W is symmetric in all its three lower indices.

There are two possible modifications for the holonomy group of a hyper-complex manifold: when there is a metric (i.e. for hyper-Kähler manifolds), the holonomy group is reduced to $\mathrm{USp}(2r)$; and when the $\mathrm{SU}(2)$-valued curvature $\mathcal{R}_{XYi}{}^j$ is non-zero(i.e. for quaternionic manifolds), the holonomy group has an extra factor of $\mathrm{SU}(2)$. We have displayed these possibilities in table 6.2.1

As an additional remark, the Ricci tensor for hyper-complex manifolds with vanishing $\mathrm{SU}(2)$-curvature is anti-symmetric, whereas it is symmetric for hyper-complex manifolds equipped with a metric. In particular, hyper-Kähler manifolds (which fall in both classes) have a vanishing Ricci tensor. However, the Ricci-tensor for a hyper-complex manifold defines a non-vanishing but closed two-form. For a more detailed discussion on hyper-complex manifolds and their curvature relations, we refer to [17].

Nijenhuis condition

The covariant constancy condition (6.40) of the vielbein contains the affine connection Γ^Z_{XY} and the $\mathrm{G}\,\ell(r,\mathbb{H})$-valued spin-connection one-form $\omega_{XA}{}^B$. We will now indicate how these two objects can be determined from the vielbeins if and only if the ("diagonal") Nijenhuis condition

$$N_{XY}{}^Z \equiv J^\alpha{}_X{}^W\partial_{[W}J^\alpha{}_{Y]}{}^Z - (X\leftrightarrow Y) = 0\,, \tag{6.53}$$

is satisfied. In this case, the affine connection Γ^Z_{XY} is given by the Obata connection [233]

$$\Gamma^Z_{XY} = -\tfrac{1}{6}\varepsilon^{\alpha\beta\gamma} J^\alpha{}_W{}^Z J^\beta{}_{(X}{}^U \partial_{|U|} J^\gamma{}_{Y)}{}^W - \tfrac{1}{3} J^\beta{}_W{}^Z \partial_{(Y} J^\beta{}_{X)}{}^W \,, \tag{6.54}$$

which leads to covariantly constant complex structures. Moreover, one can show that any torsionless connection that leaves the complex structures invariant is equal to this Obata connection. This is similar to the way that a connection that leaves a metric invariant is the Levi-Civita connection.

With this connection one can then construct the $\mathrm{G}\,\ell(r,\mathbb{H})$ valued spin-connection

$$\omega_{XA}{}^B = \tfrac{1}{2} f^{iB}_Y \left(\partial_X f^Y_{iA} + \Gamma^Y_{XZ} f^Z_{iA}\right) \,, \tag{6.55}$$

such that the vielbeins are covariantly constant.

Equations of motion

Using (6.40), (6.50) and (6.52), we compute the commutator of two supersymmetry transformations on the fermions, and find

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\zeta^A = \tfrac{1}{2}\bar{\epsilon}_2\gamma^a\epsilon_1\partial_a\zeta^A + \tfrac{1}{4}\Gamma^A\bar{\epsilon}_2\epsilon_1 - \tfrac{1}{4}\gamma_a\Gamma^A\bar{\epsilon}_2\gamma^a\epsilon_1 \,. \tag{6.56}$$

The algebra only closes if we set the Γ^A to zero: this defines the equations of motion for the fermions,

$$\begin{aligned} \Gamma^A &\equiv \not{\mathfrak{D}}\zeta^A + \tfrac{1}{2} W_{CDB}{}^A \zeta^B \bar{\zeta}^D \zeta^C \\ &= 0 \,, \end{aligned} \tag{6.57}$$

where we have introduced the covariant derivative, consistent with (6.48)

$$\mathfrak{D}_\mu \zeta^A \equiv \partial_\mu \zeta^A + (\partial_\mu q^X)\zeta^B \omega_{XB}{}^A \,. \tag{6.58}$$

By varying the fermion equation of motion under supersymmetry, we derive the corresponding equation of motion for the scalar fields

$$\widehat{\delta}(\epsilon)\Gamma^A = \tfrac{1}{2}\,\mathrm{i}\, f^{iA}_X \epsilon_i \Delta^X \,, \tag{6.59}$$

where

$$\begin{aligned} \Delta^X &\equiv \Box q^X - \tfrac{1}{2}\bar{\zeta}^B\gamma_a\zeta^D\partial^a q^Y f^{iC}_Y f^X_{iA} W_{BCD}{}^A \\ &\quad -\tfrac{1}{4}\mathfrak{D}_Y W_{BCD}{}^A \bar{\zeta}^E \zeta^D \bar{\zeta}^C \zeta^B f^{iY}_E f^X_{iA} \\ &= 0 \,, \end{aligned} \tag{6.60}$$

and the covariant D'Alembertian is given by

$$\Box q^X = \partial_a\partial^a q^X + \left(\partial_a q^Y\right)\left(\partial^a q^Z\right)\Gamma_{YZ}{}^X \,. \tag{6.61}$$

There are no more constraints since Γ^A and Δ^X form a closed set under supersymmetry

$$\widehat{\delta}(\epsilon)\Delta^X = -\,\mathrm{i}\,\bar{\epsilon}^i \not{\mathcal{D}}\Gamma^A f^X_{iA} + 2\,\mathrm{i}\,\bar{\epsilon}^i\Gamma^B\bar{\zeta}^C\zeta^D f^Y_{Bi}\mathcal{R}^X{}_{YCD}\,, \tag{6.62}$$

where the covariant derivative of Γ^A is defined similar to (6.58).

To summarize, the supersymmetry algebra imposes the hyper-complex constraints (6.40) and the equations of motion (6.57) and (6.60).

6.2.2 Superconformal symmetry

We will now derive further constraints on the target space geometry from requiring the presence of superconformal symmetry. The scalars do not transform under special conformal transformations and special supersymmetry, but under dilatations and $\mathrm{SU}(2)$ transformations, we parameterize

$$\begin{aligned} \delta_D(\Lambda_D)q^X &= \Lambda_D k^X(q)\,, \\ \delta_{\mathrm{SU}(2)}(\Lambda^{ij})q^X &= \Lambda^{ij}k^X_{ij}(q)\,, \end{aligned} \tag{6.63}$$

for some unknown functions $k^X(q)$ and $k^X_{ij}(q)$.

To derive the appropriate transformation rules for the fermions, we first note that the hyperinos should be invariant under special conformal symmetry. This is due to the fact that this symmetry changes the Weyl weight with one. The special supersymmetry transformations of the fermions are determined by calculating the commutator of special conformal and supersymmetry transformations

$$\delta_S(\eta^i)\zeta^A = -k^X f^{iA}_X \eta_i\,. \tag{6.64}$$

Next, we consider the commutator of regular and special supersymmetry (5.14). Realizing this on the scalars, we determine the expression for the generator of $\mathrm{SU}(2)$ transformations in terms of the dilatations and complex structures,

$$k^X_{ij} = \tfrac{1}{3}k^Y J_Y{}^X{}_{ij}\,. \tag{6.65}$$

Realizing (5.14) on the hyperinos, we determine the covariant variations

$$\begin{aligned} \widehat{\delta}_D(\Lambda_D)\zeta^A &= 2\Lambda_D\zeta^A\,, \\ \widehat{\delta}_{\mathrm{SU}(2)}(\Lambda^{ij})\zeta^A &= 0\,. \end{aligned} \tag{6.66}$$

Furthermore, the commutator (5.14) only closes if we impose

$$\mathfrak{D}_Y k^X = \tfrac{3}{2}\delta_Y{}^X\,, \tag{6.67}$$

which also implies

$$\mathfrak{D}_Y k^X_{ij} = \tfrac{1}{2}J_Y{}^X{}_{ij}\,. \tag{6.68}$$

We note that (6.67) determines the Weyl weight of the scalars to be $\frac{3}{2}$, as indicated in table 6.2. Note that (6.67) is imposed by supersymmetry and not, as in the usual derivations, from the dilatation invariance of an action, as we have explained in chapter 6.

The relations (6.67) and (6.65) further restrict the geometry of the target space, and it is easy to derive that the Riemann tensor has four zero eigenvectors,

$$k^X R_{XYZ}{}^W = 0 \,, \qquad k_{ij}^X R_{XYZ}{}^W = 0 \,. \tag{6.69}$$

Under dilatations and $\mathrm{SU}(2)$ transformations, the hyper-complex structure is scale invariant and rotated into itself,

$$\begin{aligned} &\Lambda_D \left(k^Z \partial_Z J_X^{\alpha Y} - \partial_Z k^Y J_X^{\alpha Z} + \partial_X k^Z J_Z^{\alpha Y} \right) = 0 \,, \\ &\Lambda^\beta \left(k_\beta^Z \partial_Z J_X^{\alpha Y} - \partial_Z k_\beta^Y J_X^{\alpha Z} + \partial_X k_\beta^Z J_Z^{\alpha Y} \right) = -\epsilon^\alpha{}_{\beta\gamma} \Lambda^\beta J_X^{\gamma Y} \,. \end{aligned} \tag{6.70}$$

All these properties are similar to those derived from superconformal hypermultiplets in four dimensions [225, 234]. There, the $\mathrm{Sp}(1) \times \mathrm{G}\,\ell(r, \mathbb{H})$ sections, or simply, hyper-complex sections, were introduced

$$A^{iB}(q) \equiv k^X f_X^{iB} \,, \qquad (A^{iB})^* = A^{jC} E_j{}^i \rho_C{}^B \,, \tag{6.71}$$

which allow for a coordinate independent description of the target space. This means that all equations and transformation rules for the sections can be written without the occurrence of the q^X fields. For example, the hyper-complex sections are zero eigenvectors of the $\mathrm{G}\,\ell(r, \mathbb{H})$ curvature

$$A^{iB} W_{BCD}{}^E = 0 \,, \tag{6.72}$$

and have supersymmetry, dilatation and $\mathrm{SU}(2)$ transformation laws given by

$$\widehat{\delta} A^{iB} = \tfrac{3}{2} f_X^{iB} \delta q^X = -\tfrac{3}{2}\, \mathrm{i}\, \bar{\epsilon}^i \zeta^B + \tfrac{3}{2} \Lambda_D A_i{}^B - \Lambda^i{}_j A^{jB} \,, \tag{6.73}$$

where $\widehat{\delta}$ is understood as a covariant variation, in the sense of (6.48).

6.2.3 Gauging symmetries

We will now discuss how to gauge a symmetry group G of the scalar manifold by coupling the hypermultiplet to a vector multiplet. The symmetry algebra must commute with the (conformal) supersymmetry algebra. The symmetries are parametrized by

$$\delta_G q^X = -g \Lambda_G^I k_I^X(q) \,, \tag{6.74}$$

$$\widehat{\delta}_G \zeta^A = -g \Lambda_G^I t_{IB}{}^A(q) \zeta^B \,. \tag{6.75}$$

The vectors k_I^X depend on the scalars and their Poisson brackets generate the algebra of G with structure constants $f_{IJ}{}^K$,

$$k_{[I|}^Y \partial_Y k_{|J]}^X = -\tfrac{1}{2} f_{IJ}{}^K k_K^X \,. \tag{6.76}$$

The commutator of two gauge transformations (6.2) on the fermions requires the following constraint on the field-dependent matrices $t_I(q)$,

$$[t_I, t_J]_B{}^A = -f_{IJ}{}^K t_{KB}{}^A - 2k_{[I|}^X \mathfrak{D}_X t_{|J]B}{}^A + k_I^X k_J^Y \mathcal{R}_{XYB}{}^A \,. \tag{6.77}$$

Requiring the gauge transformations to commute with supersymmetry leads to further relations between the quantities k_I^X and $t_{IB}{}^A$. In particular, the representation matrices $t_{IB}{}^A$ are determined by the vielbeins f_X^{iA} and the vectors k_I^X

$$t_{IA}{}^B = \tfrac{1}{2} f_{iA}^Y \mathfrak{D}_Y k_I^X f_X^{iB} \tag{6.78}$$

if the following constraint on the vectors k_I^X holds

$$f_A^{Y(i} f_X^{j)B} \mathfrak{D}_Y k_I^X = 0 \,. \tag{6.79}$$

Equation (6.79) can be expressed as the vanishing of the commutator of $\mathfrak{D}_Y k_I^X$ with the complex structures

$$(\mathfrak{D}_X k_I^Y) J^\alpha{}_Y{}^Z = J^\alpha{}_X{}^Y (\mathfrak{D}_Y k_I^Z) \,. \tag{6.80}$$

This says that all the symmetries are embedded in $\mathrm{G}\ell(r, \mathbb{H})$. Equivalently, (6.80) can be written as the Lie derivative of the complex structure in the direction of the vector k_I

$$(\mathcal{L}_{k_I} J^\alpha)_X{}^Y \equiv k_I^Z \partial_Z J_X^{\alpha Y} - \partial_Z k_I^Y J_X^{\alpha Z} + \partial_X k_I^Z J_Z^{\alpha Y} = 0 \,. \tag{6.81}$$

Thus, this is the statement that the gauge transformations act *tri-holomorphic*, i.e. they leave the hyper-complex structure invariant.

Vanishing of the gauge-supersymmetry commutator on the fermions requires a new constraint

$$\mathfrak{D}_X \mathfrak{D}_Y k_I^Z = R_{XWY}{}^Z k_I^W \,. \tag{6.82}$$

Note that this equation is in general true for any Killing vector of a metric. As we are only considering hyper-complex manifolds without a metric so far, we could not rely on this fact, but the superconformal algebra alone imposes this equation. A consequence of (6.82) is that

$$\mathfrak{D}_Y t_{IA}{}^B = k_I^X \mathcal{R}_{YXA}{}^B \,, \tag{6.83}$$

which in turn allows for a simplification of (6.77)

$$[t_I, t_J]_B{}^A = -f_{IJ}{}^K t_{KB}{}^A - k_I^X k_J^Y \mathcal{R}_{XYB}{}^A \,. \tag{6.84}$$

The group of gauge symmetries should also commute with the superconformal algebra, in particular with dilatations and $\mathrm{SU}(2)$-transformations. This leads to

$$\begin{aligned} k^Y \mathfrak{D}_Y k_I^X &= \tfrac{3}{2} k_I^X \,, \\ k_\alpha^Y \mathfrak{D}_Y k_I^X &= \tfrac{1}{2} k_I^Y J_Y{}^X{}_\alpha \,, \end{aligned} \tag{6.85}$$

and there are no new constraints from the fermions or from other commutators. Since $\mathfrak{D}_Y k_I^X$ commutes with $J_Y{}^X{}_\alpha$, the second equation in (6.85) is a consequence of the first one.

In the above analysis, we have taken the parameters Λ^I to be constants. In the following, we also allow for local gauge transformations. The gauge coupling is done by introducing vector multiplets and defining the covariant derivatives

$$\begin{aligned}\mathfrak{D}_\mu q^X &\equiv \partial_\mu q^X + g A_\mu^I k_I^X \,,\\ \mathfrak{D}_\mu \zeta^A &\equiv \partial_\mu \zeta^A + \partial_\mu q^X \omega_{XB}{}^A \zeta^B + g A_\mu^I t_{IB}{}^A \zeta^B \,.\end{aligned} \tag{6.86}$$

The commutator of two supersymmetries should now also contain a local gauge transformation, in the same way as for the multiplets of the previous sections, see (6.7). This requires an extra term in the supersymmetry transformation law of the fermions,

$$\widehat{\delta}(\epsilon)\zeta^A = \tfrac{1}{2}\,\mathrm{i}\,\not{\mathfrak{D}} q^X f_X^{iA} \epsilon_i + \tfrac{1}{2} g \sigma^I k_I^X f_{iX}^A \epsilon^i \,. \tag{6.87}$$

With this additional term, the commutator on the scalars closes. However, the fermion equation of motion Γ^A is modified with terms at $\mathcal{O}(g)$ in the gauge coupling constant

$$\begin{aligned}\Gamma^A &\equiv \not{\mathfrak{D}}\zeta^A + \tfrac{1}{2} W_{BCD}{}^A \bar{\zeta}^C \zeta^D \zeta^B - \mathrm{i}\, g (k_I^X f_{iX}^A \psi^{iI} + \zeta^B \sigma^I t_{IB}{}^A)\\ &= 0\,,\end{aligned} \tag{6.88}$$

with the same conventions as in (6.56). The subsequent variation of Γ^A under supersymmetry determines the modified equation of motion for the scalars: it receives modifications at both $\mathcal{O}(g)$ and $\mathcal{O}(g^2)$

$$\begin{aligned}\Delta^X = \;& \Box q^X - \tfrac{1}{2}\bar{\zeta}^B \gamma_a \zeta^D \mathfrak{D}^a q^Y f_Y^{iC} f_{iA}^X W_{BCD}{}^A\\ & -\tfrac{1}{4}\mathfrak{D}_Y W_{BCD}{}^A \bar{\zeta}^E \zeta^D \bar{\zeta}^C \zeta^B f_E^{iY} f_{iA}^X\\ & -g\left(2\,\mathrm{i}\,\bar{\psi}^{iI}\zeta^B t_{IB}{}^A f_{iA}^X - k_I^Y J_Y{}^X{}_{ij} Y^{ijI}\right)\\ & +g^2 \sigma^I \sigma^J \mathfrak{D}_Y k_I^X k_J^Y \,.\end{aligned} \tag{6.89}$$

The gauge-covariant D'Alembertian is given by

$$\Box q^X = \partial_a \mathfrak{D}^a q^X + g \mathfrak{D}_a q^Y \partial_Y k_I^X A^{aI} + \mathfrak{D}_a q^Y \mathfrak{D}^a q^Z \Gamma_{YZ}^X \,. \tag{6.90}$$

The equations of motions Γ^A and Δ^X still transform into each other according to (6.59) and (6.62).

6.3 Superconformal actions

In this section, we will present rigid superconformal actions for the multiplets discussed in the previous sections. We will see that demanding the existence of an action is more restrictive than only considering equations of motion. For the different multiplets, we find that new geometric objects have to be introduced.

6.3.1 The Yang-Mills multiplet

The rigid superconformal invariant action describing $n_{\rm V}$ Abelian vector multiplets can be obtained by taking the cubic action of the improved vector multiplet (5.102), adding indices I, J, K on the fields, and multiplying this with a completely symmetric tensor C_{IJK}. The existence of the tensor C_{IJK} has the geometrical significance that it endows the scalar manifold $\mathbb{R}^{n_{\rm V}}$ with a metric g_{IJ}

$$g_{IJ} \equiv -\frac{1}{3}\frac{\partial^2 \ln N}{\partial\sigma^I \partial\sigma^J}\,, \qquad N \equiv C_{IJK}\sigma^I\sigma^J\sigma^K\,. \tag{6.91}$$

At leading order in the gauge coupling constant, multiplying (5.102) with C_{IJK} also gives the action for $n_{\rm V}$ non-Abelian vector multiplets. However, because the rigid transformation rules for the non-Abelian vector multiplet (6.5) differ from the transformation rules (5.102) and (5.105) of the Abelian vector multiplet at $\mathcal{O}(g)$ in the gauge coupling constant, the tensor C_{IJK} has to satisfy the following constraint

$$f_{I(J}{}^H C_{KL)H} = 0\,. \tag{6.92}$$

Furthermore, the $A \wedge F \wedge F$ Chern-Simons (CS) term has to be modified at $\mathcal{O}(g)$ and $\mathcal{O}(g^2)$. To obtain this Yang-Mills CS term, it is convenient to rewrite the CS term as an integral over a six-dimensional manifold which has a boundary given by the five-dimensional Minkowski spacetime. The six-form appearing in the integral is (in differential form notation) given by

$$I_{\rm V} = C_{IJK} F^I \wedge F^J \wedge F^K\,. \tag{6.93}$$

This six-form is both gauge-invariant and closed, by virtue of (6.92) and the Bianchi identities (6.4). It can therefore be written as the exterior derivative of a five-form which is gauge-invariant up to a total derivative. The spacetime integral over this five-form is the Yang-Mills CS-term.

Finally, there is also an extra fermion bilinear at $\mathcal{O}(g)$ in the action called the Yukawa term. This leads to the action obtained in [219] using an intermediate linear multiplet

$$\begin{aligned}\mathcal{L}_{\rm V} \;=\; & \Big[\big(-\tfrac{1}{4}F^I_{\mu\nu}F^{\mu\nu J} - \tfrac{1}{2}\bar\psi^I \not{\mathcal{D}}\psi^J - \tfrac{1}{2}\mathcal{D}_a\sigma^I\mathcal{D}^a\sigma^J + Y^I_{ij}Y^{ijJ}\big)\,\sigma^K \\ & -\tfrac{1}{24}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}A^I_\mu\left(F^J_{\nu\lambda}F^K_{\rho\sigma} + \tfrac{1}{2}g[A_\nu, A_\lambda]^J F^K_{\rho\sigma} + \tfrac{1}{10}g^2[A_\nu,A_\lambda]^J[A_\rho,A_\sigma]^K\right) \\ & -\tfrac{1}{8}\,{\rm i}\,\bar\psi^I\gamma\cdot F^J\psi^K - \tfrac{1}{2}\,{\rm i}\,\bar\psi^{iI}\psi^{jJ}Y^K_{ij} + \tfrac{1}{4}\,{\rm i}\,g\bar\psi^L\psi^H\sigma^I\sigma^J f_{LH}{}^K\Big]\,C_{IJK}\,. \end{aligned} \tag{6.94}$$

The equations of motion for the fields of the vector multiplet following from the action (6.94) are

$$0 = L^{ij}_I = \varphi^i_I = E^a_I = N\,, \tag{6.95}$$

where we have defined

$$
\begin{aligned}
L_I^{ij} &\equiv C_{IJK}\left(2\sigma^J Y^{ijK} - \tfrac{1}{2}\,\mathrm{i}\,\bar\psi^{iJ}\psi^{jK}\right)\,,\\
\varphi_I^i &\equiv C_{IJK}\left(\mathrm{i}\,\sigma^J \not{\mathcal{D}}\psi^{iK} + \tfrac{1}{2}\,\mathrm{i}\,(\not{\mathcal{D}}\sigma^J)\psi^{iK} + Y^{ikJ}\psi_k^K - \tfrac{1}{4}\gamma\cdot F^J\psi^{iK}\right)\\
&\quad -g\,C_{IJK}\,f_{LH}{}^K\sigma^J\sigma^L\psi^{iH}\,,\\
E_{aI} &\equiv C_{IJK}\left(D^b\left(\sigma^J F_{ba}{}^K + \tfrac{1}{4}\,\mathrm{i}\,\bar\psi^J\gamma_{ba}\psi^K\right) - \tfrac{1}{8}\varepsilon_{abcde}F^{bcJ}F^{deK}\right)\\
&\quad -\tfrac{1}{2}g\,C_{JKL}f_{IH}{}^J\sigma^K\bar\psi^L\gamma_a\psi^H - g\,C_{JKH}f_{IL}{}^J\sigma^K\sigma^L D_a\sigma^H\,,\\
N_I &\equiv C_{IJK}\left(\sigma^J\Box\sigma^K + \tfrac{1}{2}\mathcal{D}^a\sigma^J\mathcal{D}_a\sigma^K - \tfrac{1}{4}F_{ab}^J F^{abK} - \tfrac{1}{2}\bar\psi^J\not{\mathcal{D}}\psi^K + Y^{ijJ}Y_{ij}{}^K\right)\\
&\quad +\tfrac{1}{2}\,\mathrm{i}\,g\,C_{IJK}\,f_{LH}{}^K\sigma^J\bar\psi^L\psi^H\,.
\end{aligned}
\tag{6.96}
$$

We have given these equations of motion the names $L_I^{ij}, \phi_I^i, E_{aI}, N_I$ since they form a linear multiplet in the adjoint representation of the gauge group for which the transformation rules have been given in [17].

6.3.2 The vector-tensor multiplet

We will now generalize the vector action (6.94) to an action for the vector-tensor multiplets (with $n_{\rm V}$ vector multiplets and $n_{\rm T}$ tensor multiplets) discussed in section 6.1.2.

The supersymmetry transformation rules for the vector-tensor multiplet (6.11) were obtained from those for the vector multiplet (6.5) by replacing all contracted indices by the extended range of tilde indices. In addition, extra terms of $\mathcal{O}(g)$ had to be added to the transformation rules. Similar considerations apply to the generalization of the action, as we will see below.

We will first generalize the CS term (6.93) to the case of vector-tensor multiplets. It turns out that this generalization is somewhat surprising: it will involve the inclusion of derivative terms. We find the following expression for the unique closed and gauge-invariant six-form

$$
I_{\rm VT} = C_{\widetilde I\widetilde J\widetilde K}\mathcal{H}^{\widetilde I}\wedge\mathcal{H}^{\widetilde J}\wedge\mathcal{H}^{\widetilde K} - \tfrac{3}{g}\Omega_{MN}\mathcal{D}B^M\wedge\mathcal{D}B^N\,,
\tag{6.97}
$$

The tensor Ω_{MN} is antisymmetric and invertible, and it will restrict the number of tensor multiplets to be *even*: $n_{\rm T}=2k$ and

$$
\Omega_{MN} = -\Omega_{NM}\,,\qquad \Omega_{MP}\Omega^{MR} = \delta_P{}^R\,,
\tag{6.98}
$$

The covariant derivative $\mathcal{D}B^M$ is given by

$$
\begin{aligned}
\mathcal{D}_\lambda B^M_{\rho\sigma} &= \partial_\lambda B^M_{\rho\sigma} + g\,A^I_\lambda t_{I\widetilde J}{}^M\mathcal{H}^{\widetilde J}_{\rho\sigma}\\
&= \partial_\lambda B^M_{\rho\sigma} + g\,A^I_\lambda t_{IJ}{}^M F^J_{\rho\sigma} + g\,A^I_\lambda t_{IN}{}^M B^N_{\rho\sigma}\,.
\end{aligned}
\tag{6.99}
$$

To see why the first term of (6.97) is not a closed six-form by itself, we write it out explicitly as

$$
C_{\widetilde I\widetilde J\widetilde K}\mathcal{H}^{\widetilde I}\mathcal{H}^{\widetilde J}\mathcal{H}^{\widetilde K} = C_{IJK}F^I F^J F^K + 3C_{IJM}F^I F^J B^M + 3C_{IMN}F^I B^M B^N\,.
\tag{6.100}
$$

Since the B^M tensors in (6.100) do not satisfy a Bianchi identity, we also need the second term in (6.97) to obtain a closed six-form. This leads to the following relations between the C and Ω tensors:

$$C_{IJM} = t_{(IJ)}{}^N \Omega_{NM}\,, \qquad C_{IMN} = \tfrac{1}{2} t_{IM}{}^P \Omega_{PN}\,. \tag{6.101}$$

As additional remark, the components of C can have only three different forms: C_{IJK}, C_{IJM} and C_{IMN} (and permutations). The reason is that when the first term of (6.97) is reduced to five dimensions, one of the $\mathcal{H}^{\widetilde{I}}$ factors should correspond to a vector field strength F^I. Only then can the corresponding five-form be written as $A^I \wedge \mathcal{H}^{\widetilde{J}} \wedge \mathcal{H}^{\widetilde{K}}$.

Gauge invariance of the first term of (6.97) requires that the tensor C satisfies a modified version of (6.92)

$$f_{I(J}{}^H C_{KL)H} = t_{I(J}{}^M t_{KL)}{}^N \Omega_{MN}\,. \tag{6.102}$$

In addition to this, the second term of (6.97) is only gauge invariant if the tensor Ω satisfies

$$t_{I[M}{}^P \Omega_{N]P} = 0\,, \tag{6.103}$$

such that the last one of (6.101) is consistent with the symmetry (MN).

Finally, there are extra Yukawa couplings at $\mathcal{O}(g)$ and there is a scalar potential term at $\mathcal{O}(g^2)$ in the vector-tensor multiplet action. The superconformal action for the combined system of $n_{\rm T} = 2k$ tensor multiplets in the background of $n_{\rm V}$ vector multiplets is given by

$$\begin{aligned}
\mathcal{L}_{\rm VT} \;=\;& \Big(-\tfrac{1}{4}\mathcal{H}^{\widetilde{I}}_{\mu\nu}\mathcal{H}^{\mu\nu\widetilde{J}} - \tfrac{1}{2}\bar\psi^{\widetilde{I}}\not{D}\psi^{\widetilde{J}} - \tfrac{1}{2}\mathcal{D}_a\sigma^{\widetilde{I}}\mathcal{D}^a\sigma^{\widetilde{J}} + Y^{\widetilde{I}}_{ij}Y^{ij\widetilde{J}}\Big)\,\sigma^{\widetilde{K}} C_{\widetilde{I}\widetilde{J}\widetilde{K}} \\
&+\tfrac{1}{16g}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{MN}B^M_{\mu\nu}\left(\partial_\lambda B^N_{\rho\sigma} + 2g\,t_{IJ}{}^N A^I_\lambda F^J_{\rho\sigma} + g\,t_{IP}{}^N A^I_\lambda B^P_{\rho\sigma}\right) \\
&-\tfrac{1}{24}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}C_{IJK}A^I_\mu\left(F^J_{\nu\lambda}F^K_{\rho\sigma} + f_{FG}{}^J A^F_\nu A^G_\lambda\left(-\tfrac{1}{2}g\,F^K_{\rho\sigma} + \tfrac{1}{10}g^2 f_{HL}{}^K A^H_\rho A^L_\sigma\right)\right) \\
&-\tfrac{1}{8}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{MN}t_{IK}{}^M t_{FG}{}^N A^I_\mu A^F_\nu A^G_\lambda\left(-\tfrac{1}{2}g\,F^K_{\rho\sigma} + \tfrac{1}{10}g^2 f_{HL}{}^K A^H_\rho A^L_\sigma\right) \\
&+\Big(-\tfrac{1}{8}\,\mathrm{i}\,\bar\psi^{\widetilde{I}}\gamma\cdot\mathcal{H}^{\widetilde{J}}\psi^{\widetilde{K}} - \tfrac{1}{2}\,\mathrm{i}\,\bar\psi^{i\widetilde{I}}\psi^{j\widetilde{J}}Y^{\widetilde{K}}_{ij}\Big)\,C_{\widetilde{I}\widetilde{J}\widetilde{K}} \\
&+\tfrac{1}{4}\,\mathrm{i}\,g\bar\psi^{\widetilde{I}}\psi^{\widetilde{J}}\sigma^{\widetilde{K}}\sigma^{\widetilde{L}}\left(t_{[\widetilde{I}\widetilde{J}]}{}^{\widetilde{M}}C_{\widetilde{M}\widetilde{K}\widetilde{L}} - 4t_{(\widetilde{I}\widetilde{K})}{}^{\widetilde{M}}C_{\widetilde{M}\widetilde{J}\widetilde{L}}\right) \\
&-\tfrac{1}{2}g^2\sigma^K\sigma^I\sigma^{\widetilde{L}}\sigma^J\sigma^{\widetilde{P}}C_{KMN}t_{I\widetilde{L}}{}^M t_{J\widetilde{P}}{}^N\,,
\end{aligned} \tag{6.104}$$

We stress that the tensor $C_{\widetilde{I}\widetilde{J}\widetilde{K}}$ is not a fundamental object: the essential data for the vector-tensor multiplet are the representation matrices $t_{I\widetilde{J}}{}^{\widetilde{K}}$, the Yang-Mills components C_{IJK}, and the symplectic matrix Ω_{MN}. The tensor components of the C tensor are derived quantities, and we can summarize (6.101) as

$$C_{M\widetilde{J}\widetilde{K}} = t_{(\widetilde{J}\widetilde{K})}{}^P \Omega_{PM}\,. \tag{6.105}$$

The two conditions (6.102) and (6.103) combined with the definition (6.105) imply the following generalization of (6.92)

$$t_{I(\widetilde{J}}{}^{\widetilde{M}} C_{\widetilde{K}\widetilde{L})\widetilde{M}} = 0\,. \tag{6.106}$$

To check the supersymmetry of the action (6.104), one needs all the relations between the various tensors given above. Another useful identity implied by the previous definitions is

$$t_{(\widetilde{I}\widetilde{J})}{}^{M} C_{\widetilde{K}\widetilde{L}M} = -t_{(\widetilde{K}\widetilde{L})}{}^{M} C_{\widetilde{I}\widetilde{J}M}\,. \tag{6.107}$$

The action with fields of the tensor multiplets can also be obtained from the field equations (6.17). They are now related to the action by

$$\frac{\delta \mathcal{L}_{\rm VT}}{\delta \bar{\psi}^{iM}} = \mathrm{i}\,\varphi_i^N \Omega_{NM}\,, \tag{6.108}$$

and the remaining bosonic terms can be obtained from a comparison with N^M in (6.19). One may then further check that also the field equations (6.15) and (6.17) follow from this action.

Note, however, that the equations of motion for the vector multiplet fields, obtained from this action, are similar to those given in (6.96), but with the contracted indices running over the extended range of vector and tensor components. Furthermore, the A_μ^I equation of motion gets corrected by a term proportional to the self-duality equation for $B_{\mu\nu}^M$

$$\frac{\delta \mathcal{L}_{\rm VT}}{\delta A_a^I} = E_I^a + \tfrac{1}{12} g \varepsilon^{abcde} A_b^J E_{cde}^M t_{JI}{}^N \Omega_{MN}\,. \tag{6.109}$$

To summarize: in order to write down a rigid superconformal action for the vector-tensor multiplet, we need to introduce a gauge-invariant, anti-symmetric, invertible tensor Ω_{MN}, which restricts the number of tensor multiplets to be even. We can still allow the transformations to have off-diagonal terms between vector and tensor multiplets, if we adapt (6.92) to (6.102).

In this way, we have constructed more general matter-couplings than were known so far: with our extension to allow for the off-diagonal term in (6.23), we also get CS-terms induced by the C_{IJM} components, which were not present in [159]. In particular, in [159] it was found that such $A \wedge F \wedge B$ terms "appear impossible to supersymmetrize (except possibly in very special cases)". However, we see that such terms appear generically in our Lagrangian by allowing for off-diagonal gauge transformations that mix the tensor fields with the Yang-Mills field-strengths.

6.3.3 The hypermultiplet

Let us recapitulate the geometrical setting for the hypermultiplet: the scalar manifold was seen to be a hyper-complex manifold possessing a triplet of complex structures that satisfied the Nijenhuis conditions (6.53). From this integrability condition, it was possible to construct an affine torsionless Obata connection Γ^Z_{XY} and the $\mathrm{G}\,\ell(r,\mathbb{H})$-valued spin-connection one-form $\omega_{XA}{}^B$. Furthermore, the $\mathrm{SU}(2)$-valued spin-connection one-form $\omega_{Xi}{}^j$ had a vanishing curvature.

Using this algebraic description of the hyper-complex geometry, the constraints that were needed to close the superconformal algebra on the on-shell hypermultiplet were seen to be equations of motion. These equations of motion, Γ^A and Δ^X, transformed covariantly with respect to diffeomorphisms on the scalar manifold and to transformations on its tangent space. However, these equations of motion were not derived from an action.

When we introduce an action, the kinetic term takes on the following generic form

$$\mathcal{L}_{\mathrm{H}} = -\tfrac{1}{2} g_{XY}(\phi)\partial_\mu\phi^X\partial^\mu\phi^Y \,, \tag{6.110}$$

where the tensor g_{XY} is interpreted as the metric on the scalar manifold. The field equations for the scalars should now also be covariant with respect to coordinate transformations on the target manifold. This implies that the connection on the tangent bundle should be the Levi-Civita connection. Only in that particular case, the field equations for the scalars will be covariant.

We will now see what the consequences are of introducing the extra input of a metric on the geometry of the scalar manifold.

Hyper-Kähler geometry

We take the fermion equation of motion Γ^A to be proportional to the field equations following from an action

$$\frac{\delta S}{\delta\bar\zeta^A} = 2C_{AB}\Gamma^B \,. \tag{6.111}$$

In general, the tensor C_{AB} could be a function of the scalars and bilinears of the fermions. If we try to construct an action with the above Ansatz, it turns out that the tensor has to be anti-symmetric in AB and

$$\begin{aligned} \frac{\delta C_{AB}}{\delta\zeta^C} &= 0\,, & (6.112)\\ \mathfrak{D}_X C_{AB} &= 0\,. & (6.113)\end{aligned}$$

In other words, the tensor does not depend on the fermions and is covariantly constant[1].

This tensor C_{AB} will be used to raise and lower tangent space indices according to the NW–SE convention similar to ε_{ij}:

$$A_A = A^B C_{BA}\,, \qquad A^A = C^{AB}A_B\,, \tag{6.114}$$

where ε^{ij} and C^{AB} are defined for consistency by

$$\varepsilon_{ik}\varepsilon^{jk} = \delta_i{}^j\,, \qquad C_{AC}C^{BC} = \delta_A{}^B\,. \tag{6.115}$$

[1]This can be derived using the Batalin-Vilkovisky formalism.

From the integrability condition for (6.113) we find

$$[\mathfrak{D}_X,\mathfrak{D}_Y]C_{AB}=0=-2\mathcal{R}_{XY[A}{}^{C}C_{B]C}\,, \tag{6.116}$$

which implies that the anti-symmetric part of the connection $\omega_{XAB}\equiv\omega_{XA}{}^{C}C_{CB}$ is pure gauge, and can be chosen to be zero. If we do so, the covariant constancy condition for C_{AB} reduces to the equation that C_{AB} is just constant.

We can construct the metric g_{XY} on the scalar manifold by multiplying the metric on the tangent space with the vielbeins

$$g_{XY}=f_X^{iA}f_Y^{jB}C_{AB}\varepsilon_{ij}\,. \tag{6.117}$$

Since the connection ω_{XAB} is symmetric, the original holonomy group $\mathrm{G}\,\ell(r,\mathbb{H})$ is reduced to $\mathrm{USp}(2r-2p,2p)$: its signature is the signature of d_{CB}. The tensor d_{AB} is defined as $C_{AB}=\rho_A{}^{C}d_{CB}$ where $\rho_A{}^{C}$ was given in (6.36). These restrictions on the hyper-complex geometry reduce the scalar manifold to a hyper-Kähler manifold.

Furthermore, the affine connection used in the covariant derivative in (6.113) is now given by the Levi-Civita connection constructed from the metric g_{XY}. Indeed, this guarantees that the metric is covariantly constant. On the other hand, we have already seen already that, to have covariantly constant complex structures, we have to use the Obata connection. Hence, the Levi-Civita and Obata connection coincide for hyper-Kähler manifolds.

The action for rigid hypermultiplet takes on the form

$$\mathcal{L}_{\mathrm{H}}=-\tfrac{1}{2}g_{XY}\partial_a q^X\partial^a q^Y+\bar{\zeta}_A\not{\mathfrak{D}}\zeta^A-\tfrac{1}{4}W_{ABCD}\bar{\zeta}^A\zeta^B\bar{\zeta}^C\zeta^D\,. \tag{6.118}$$

where the tensor W_{ABCD} can be proven to be completely symmetric in all of its indices [17]. The field equations derived from this action are

$$\begin{aligned}\frac{\delta S}{\delta\bar{\zeta}^A}&=2C_{AB}\Gamma^B\,,\\ \frac{\delta S}{\delta q^X}&=g_{XY}\Delta^Y-2\bar{\zeta}_A\Gamma^B\omega_{XB}{}^{A}\,,\end{aligned} \tag{6.119}$$

Also remark that due to the introduction of the metric, the expression of Δ^X simplifies to

$$\Delta^X=\Box q^X-\bar{\zeta}^A\not{\partial}q^Y\zeta^B\mathcal{R}^X{}_{YAB}-\tfrac{1}{4}\mathfrak{D}^X W_{ABCD}\bar{\zeta}^A\zeta^B\zeta^C\zeta^D\,. \tag{6.120}$$

Superconformal symmetry

Due to the presence of the metric, the condition for the homothetic Killing vector (6.67) implies that k_X is the derivative of a scalar function as in (5.26). This scalar function $\chi(q)$ is called the hyper-Kähler potential [188, 225, 235]. It determines the metric

$$\mathfrak{D}_X\mathfrak{D}_Y\,\chi=\tfrac{3}{2}g_{XY}\,, \tag{6.121}$$

as well as the homothetic Killing vector

$$k_X = \partial_X \chi\,, \qquad \chi = \tfrac{1}{3} k_X k^X\,. \tag{6.122}$$

Note that this implies that, when χ and the complex structures are known, one can compute the metric with (6.121), using the formula for the Obata connection (6.54).

Gauging isometries

In the presence of a metric, the symmetries of section 6.2.3 should also be symmetries of the metric, i.e they should be *isometries*. This means that the vectors ${k_I}^X$ are now Killing vectors of the metric g_{XY}

$$\mathfrak{D}_X k_{YI} + \mathfrak{D}_Y k_{XI} = 0\,. \tag{6.123}$$

This makes the requirement (6.82) superfluous, but we will still have to impose the tri-holomorphicity expressed by either (6.79), (6.80) or (6.81).

From the tri-holomorphicity condition (6.81) we find that, in order to integrate the equations of motion to an action, we have to define (locally) triplets of "moment maps" P^α_X that satisfy

$$\partial_X P^\alpha_I = {J_{XY}}^\alpha k^Y_I\,. \tag{6.124}$$

The field equations have the same form as in (6.119), except that all derivatives are now covariantized with respect to the new transformations. The same covariantization takes place in the action but here there are now modifications at $\mathcal{O}(g)$ and $\mathcal{O}(g^2)$ in the gauge coupling constant

$$\begin{aligned}\mathcal{L}_{\rm H} \;=\; & -\tfrac{1}{2} g_{XY}\mathfrak{D}_a q^X \mathfrak{D}^a q^Y + \bar\zeta_A \not{\mathfrak{D}}\zeta^A - \tfrac{1}{4} W_{ABCD}\bar\zeta^A\zeta^B\bar\zeta^C\zeta^D \\ & -g\left(P_{Iij}Y^{Iij} + 2\,\mathrm{i}\, k^X_I f^A_{iX}\bar\zeta_A\psi^{iI} + \mathrm{i}\,\sigma^I {t_{IB}}^A\bar\zeta_A\zeta^B\right) \\ & -\tfrac{1}{2} g^2\sigma^I\sigma^J k^X_I k_{JX}\,. \end{aligned} \tag{6.125}$$

Supersymmetry of the action leads to the constraint

$$k^X_I J^\alpha_{XY} k^Y_J = -{f_{IJ}}^K P^\alpha_K\,. \tag{6.126}$$

As only the derivative of P appears in the defining equation (6.124), one may add an arbitrary constant to P. However, this changes the right-hand side of (6.126). One should then consider whether there is a choice of these coefficients such that (6.126) is satisfied [236]. For simple groups there is always a solution[2], whereas for Abelian theories the constant remains undetermined. This free constant is the so-called Fayet–Iliopoulos term [237].

In a superconformally invariant theory, the Fayet–Iliopoulos term is not possible. Indeed, dilatation invariance of the action needs

$$3P^\alpha_I = k^X\partial_X P^\alpha_I\,. \tag{6.127}$$

[2]We thank Gary Gibbons for a discussion on this subject.

Using (6.124) or (6.85), P_{Iij} is completely determined to be

$$3P_I^\alpha = k^X J_{XY}{}^\alpha k_I^Y = -\tfrac{2}{3}k^X k^Z J_Z{}^{Y\alpha}\mathfrak{D}_Y k_{IX}\,. \tag{6.128}$$

The proof of the invariance of the action under the complete superconformal group, uses the equation obtained from (6.85) and (6.124)

$$k^{X\alpha}\mathfrak{D}_X k_I^Y = -\tfrac{1}{2}\partial^Y P_I^\alpha\,. \tag{6.129}$$

If the moment map P_I^α has the value that it takes in the conformal theory, then (6.126) is satisfied due to (6.76). Indeed, one can multiply that equation with $k_X k^Z J^\alpha{}_Z{}^W\mathfrak{D}_W$ and use (6.69), (6.80) and (6.82). Thus, in the superconformal theory, the moment maps are completely determined, and there is no further relation to be obeyed: i.e. the Fayet–Iliopoulos terms of the rigid theories are absent in this case.

6.4 Coupling to the Weyl multiplet

We are now ready to perform the last step in our program, i.e. make the extension to local superconformal supersymmetry. We will make use here of the off-shell $32+32$ Standard Weyl multiplet constructed in chapter 5. Since in the previous sections we have explained most of the subtleties concerning the possible geometrical structures, we can be brief here.

We will obtain our results in two steps. First, we require that the local superconformal commutator algebra (5.87) - (5.91) of the Weyl multiplet is also realized on the matter multiplets, keeping in mind possible additional transformations under which the fields of the standard Weyl multiplet do not transform, and possibly field equations if the matter multiplet is on-shell. Next, we apply a standard Noether procedure to extend the rigid superconformal actions to local superconformal actions.

It is important to note that we do not construct an action for conformal supergravity itself: there will be no kinetic terms for the fields of the Weyl multiplets. Instead, the Weyl multiplet is seen as a fixed background of conformal supergravity to which the various matter multiplets couple. In section 6.5.2, we will indicate how local superconformal matter multiplets nevertheless lead to dynamical theories of Poincaré supergravity coupled to matter.

6.4.1 Vector-tensor multiplet

For brevity, we will present the transformation rules for n_T tensor multiplets in the background of $n_{\rm V}$ vector multiplets. The transformation rules for the vector multiplet itself can be obtained from it by making the restriction to $n_{\rm T}=0$. The local superconformal transformations rules for the vector-tensor multiplet are the following generalization of the transformation rules (5.105) of an Abelian vector multiplet coupled to the Weyl multiplet

$$
\begin{aligned}
\delta A_\mu^I &= \tfrac{1}{2}\bar\epsilon\gamma_\mu\psi^I - \tfrac{1}{2}\,\mathrm{i}\,\sigma^I\bar\epsilon\psi_\mu\,,\\
\delta B_{ab}^M &= -\bar\epsilon\gamma_{[a}D_{b]}\psi^M + \mathrm{i}\,g\bar\epsilon\gamma_{ab}t_{(\widetilde{J}\widetilde{K})}{}^M\sigma^{\widetilde{J}}\psi^{\widetilde{K}} + \mathrm{i}\,\bar\eta\gamma_{ab}\psi^M\,,\\
\delta Y^{ij\widetilde{I}} &= -\tfrac{1}{2}\bar\epsilon^{(i}\not{D}\psi^{j)\widetilde{I}} + \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon^{(i}\gamma\cdot T\psi^{j)\widetilde{I}} - 4\,\mathrm{i}\,\sigma^{\widetilde{I}}\bar\epsilon^{(i}\chi^{j)}\\
&\quad -\tfrac{1}{2}\,\mathrm{i}\,g\bar\epsilon^{(i}\left(t_{[\widetilde{J}\widetilde{K}]}{}^{\widetilde{I}} - 3t_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{I}}\right)\sigma^{\widetilde{J}}\psi^{j)\widetilde{K}} + \tfrac{1}{2}\,\mathrm{i}\,\bar\eta^{(i}\psi^{j)\widetilde{I}}\,,\\
\delta\psi^{i\widetilde{I}} &= -\tfrac{1}{4}\gamma\cdot\widehat{\mathcal{H}}^{\widetilde{I}}\epsilon^i - \tfrac{1}{2}\,\mathrm{i}\,\not{D}\sigma^{\widetilde{I}}\epsilon^i - Y^{ij\widetilde{I}}\epsilon_j + \sigma^{\widetilde{I}}\gamma\cdot T\epsilon^i + \tfrac{1}{2}gt_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}\epsilon^i + \sigma^{\widetilde{I}}\eta^i\,,\\
\delta\sigma^{\widetilde{I}} &= \tfrac{1}{2}\,\mathrm{i}\,\bar\epsilon\psi^{\widetilde{I}}\,.
\end{aligned}
\tag{6.130}
$$

The covariant derivatives are defined by

$$
\begin{aligned}
D_\mu\sigma^{\widetilde{I}} &= \mathcal{D}_\mu\sigma^{\widetilde{I}} - \tfrac{1}{2}\,\mathrm{i}\,\bar\psi_\mu\psi^{\widetilde{I}}\,,\\
\mathcal{D}_\mu\sigma^{\widetilde{I}} &= (\partial_\mu - b_\mu)\sigma^{\widetilde{I}} + gt_{J\widetilde{K}}{}^{\widetilde{I}}A_\mu^J\sigma^{\widetilde{K}}\,,\\
D_\mu\psi^{i\widetilde{I}} &= \mathcal{D}_\mu\psi^{i\widetilde{I}} + \tfrac{1}{4}\gamma\cdot\widehat{\mathcal{H}}^{\widetilde{I}}\psi_\mu^i + \tfrac{1}{2}\,\mathrm{i}\,\not{D}\sigma^{\widetilde{I}}\psi_\mu^i + Y^{ij\widetilde{I}}\psi_{\mu\,j} - \sigma^{\widetilde{I}}\gamma\cdot T\psi_\mu^i\\
&\quad -\tfrac{1}{2}gt_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}\psi_\mu^i - \sigma^{\widetilde{I}}\phi_\mu^i\,,\\
\mathcal{D}_\mu\psi^{i\widetilde{I}} &= (\partial_\mu - \tfrac{3}{2}\,b_\mu + \tfrac{1}{4}\,\gamma_{ab}\,\omega_\mu{}^{ab})\psi^{i\widetilde{I}} - V_\mu^{ij}\psi_j^{\widetilde{I}} + gt_{J\widetilde{K}}{}^{\widetilde{I}}A_\mu^J\psi^{i\widetilde{K}}\,.
\end{aligned}
\tag{6.131}
$$

The covariant curvature $\widehat{\mathcal{H}}_{\mu\nu}^{\widetilde{I}}$ should be understood as having components $(\widehat{F}_{\mu\nu}^I, B_{\mu\nu}^M)$, where the covariantized Yang-Mills field-strength is given by

$$
\widehat{F}_{\mu\nu}^I = 2\partial_{[\mu}A_{\nu]}^I + gf_{JK}{}^IA_\mu^JA_\nu^K - \bar\psi_{[\mu}\gamma_{\nu]}\psi^I + \tfrac{1}{2}\,\mathrm{i}\,\sigma^I\bar\psi_{[\mu}\psi_{\nu]}\,.
\tag{6.132}
$$

In order to close the superconformal algebra on the vector-tensor multiplet, the fields of the tensor multiplet need to satisfy equations of motion. The extensions of (6.14) and (6.15) (which are non-zero only for $\widetilde{I}$ in the tensor multiplet range) are given by

$$
\begin{aligned}
L^{ijM} &\equiv t_{(\widetilde{J}\widetilde{K})}{}^M\left(2\sigma^{\widetilde{J}}Y^{ij\widetilde{K}} - \tfrac{1}{2}\,\mathrm{i}\,\bar\psi^{i\widetilde{J}}\psi^{j\widetilde{K}}\right)\\
&= 0\,,\\
E_{\mu\nu\lambda}^M &\equiv \frac{3}{g}D_{[\mu}B_{\nu\lambda]}{}^M - \varepsilon_{\mu\nu\lambda\rho\sigma}t_{(\widetilde{J}\widetilde{K})}{}^M\left(\sigma^{\widetilde{J}}\widehat{\mathcal{H}}^{\rho\sigma\widetilde{K}} - 8\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}T^{\rho\sigma} + \tfrac{1}{4}\,\mathrm{i}\,\bar\psi^{\widetilde{J}}\gamma^{\rho\sigma}\psi^{\widetilde{K}}\right)\\
&\quad -\tfrac{3}{2}\bar\psi^M\gamma_{[a}\widehat{R}_{bc]}(Q)\\
&= 0\,.
\end{aligned}
\tag{6.133}
$$

Analogously to section 6.1.2, subsequent variation of these constraints gives the superconformal extensions of the equations of motion for the rest of the fields of the tensor multiplet. We will not give them here explicitly, since they can be derived from the action which we will give below.

The local generalization of the action (6.104) for the vector-tensor multiplet is rather involved. A long but straightforward calculation leads us to the following expression

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\rm VT} \;=\;& \Big[\Big(-\tfrac14\widehat{\mathcal{H}}^{\tilde I}_{\mu\nu}\widehat{\mathcal{H}}^{\mu\nu\tilde J}-\tfrac12\bar\psi^{\tilde I}\not{D}\psi^{\tilde J}+\tfrac13\sigma^{\tilde I}\Box^c\sigma^{\tilde J}+\tfrac16 D_a\sigma^{\tilde I}D^a\sigma^{\tilde J}+Y^{\tilde I}_{ij}Y^{ij\tilde J}\Big)\,\sigma^{\tilde K}\\
&-\tfrac43\sigma^{\tilde I}\sigma^{\tilde J}\sigma^{\tilde K}\left(D+\tfrac{26}{3}T_{ab}T^{ab}\right)+4\sigma^{\tilde I}\sigma^{\tilde J}\widehat{\mathcal{H}}^{\tilde I}_{ab}T^{ab}+\Big(-\tfrac18\,{\rm i}\,\bar\psi^{\tilde I}\gamma\cdot\widehat{\mathcal{H}}^{\tilde J}\psi^{\tilde K}\\
&-\tfrac12\,{\rm i}\,\bar\psi^{i\tilde I}\psi^{j\tilde J}Y^{\tilde K}_{ij}+{\rm i}\,\sigma^{\tilde I}\bar\psi^{\tilde J}\gamma\cdot T\psi^{\tilde K}-8\,{\rm i}\,\sigma^{\tilde I}\sigma^{\tilde J}\bar\psi^{\tilde K}\chi\Big)+\tfrac16\sigma^{\tilde I}\bar\psi_\mu\gamma^\mu\\
&\times\Big({\rm i}\,\sigma^{\tilde J}\not{D}\psi^{\tilde K}+\tfrac12\,{\rm i}\,\not{D}\sigma^{\tilde J}\psi^{\tilde K}-\tfrac14\gamma\cdot\widehat{\mathcal{H}}^{\tilde J}\psi^{\tilde K}+2\sigma^{\tilde J}\gamma\cdot T\psi^{\tilde K}-8\sigma^{\tilde J}\sigma^{\tilde K}\chi\Big)\\
&-\tfrac16\bar\psi_a\gamma_b\psi^{\tilde I}\Big(\sigma^{\tilde J}\widehat{\mathcal{H}}^{ab\tilde K}-8\sigma^{\tilde J}\sigma^{\tilde K}T^{ab}\Big)-\tfrac1{12}\sigma^{\tilde I}\bar\psi_\lambda\gamma^{\mu\nu\lambda}\psi^{\tilde J}\widehat{\mathcal{H}}^{\tilde K}_{\mu\nu}\\
&+\tfrac1{12}\,{\rm i}\,\sigma^{\tilde I}\bar\psi_a\psi_b\Big(\sigma^{\tilde J}\widehat{\mathcal{H}}^{ab\tilde K}-8\sigma^{\tilde J}\sigma^{\tilde K}T^{ab}\Big)+\tfrac1{48}\,{\rm i}\,\sigma^{\tilde I}\sigma^{\tilde J}\bar\psi_\lambda\gamma^{\mu\nu\lambda\rho}\psi_\rho\widehat{\mathcal{H}}^{\tilde K}_{\mu\nu}\\
&-\tfrac12\sigma^{\tilde I}\bar\psi^i_\mu\gamma^\mu\psi^{j\tilde J}Y^{\tilde K}_{ij}+\tfrac16\,{\rm i}\,\sigma^{\tilde I}\sigma^{\tilde J}\bar\psi^i_\mu\gamma^{\mu\nu}\psi^j_\nu Y^{\tilde K}_{ij}-\tfrac1{24}\,{\rm i}\,\bar\psi_\mu\gamma_\nu\psi^{\tilde I}\bar\psi^{\tilde J}\gamma^{\mu\nu}\psi^{\tilde K}\\
&+\tfrac1{12}\,{\rm i}\,\bar\psi^i_\mu\gamma^\mu\psi^{j\tilde I}\bar\psi^{\tilde J}_i\psi^{\tilde K}_j-\tfrac1{48}\sigma^{\tilde I}\bar\psi_\mu\psi_\nu\bar\psi^{\tilde J}\gamma^{\mu\nu}\psi^{\tilde K}+\tfrac1{24}\sigma^{\tilde I}\bar\psi^i_\mu\gamma^{\mu\nu}\psi^j_\nu\bar\psi^{\tilde J}_i\psi^{\tilde K}_j\\
&-\tfrac1{12}\sigma^{\tilde I}\bar\psi_\lambda\gamma^{\mu\nu\lambda}\psi^{\tilde J}\bar\psi_\mu\gamma_\nu\psi^{\tilde K}+\tfrac1{24}\,{\rm i}\,\sigma^{\tilde I}\sigma^{\tilde J}\bar\psi_\lambda\gamma^{\mu\nu\lambda}\psi^{\tilde K}\bar\psi_\mu\psi_\nu\\
&+\tfrac1{48}\,{\rm i}\,\sigma^{\tilde I}\sigma^{\tilde J}\bar\psi_\lambda\gamma^{\mu\nu\lambda\rho}\psi_\rho\bar\psi_\mu\gamma_\nu\psi^{\tilde K}+\tfrac1{96}\sigma^{\tilde I}\sigma^{\tilde J}\sigma^{\tilde K}\bar\psi_\lambda\gamma^{\mu\nu\lambda\rho}\psi_\rho\bar\psi_\mu\psi_\nu\Big]\,C_{\tilde I\tilde J\tilde K}\\
&+\tfrac1{16g}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{MN}B^M_{\mu\nu}\left(\partial_\lambda B^N_{\rho\sigma}+2g\,t_{IJ}{}^N A^I_\lambda F^J_{\rho\sigma}+g\,t_{IP}{}^N A^I_\lambda B^P_{\rho\sigma}\right)\\
&-\tfrac18 e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{MN}t_{IK}{}^M t_{FG}{}^N A^I_\mu A^F_\nu A^G_\lambda\left(-\tfrac12 g\,F^K_{\rho\sigma}+\tfrac1{10}g^2 f_{HL}{}^K A^H_\rho A^L_\sigma\right)\\
&-\tfrac1{24}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}C_{IJK}A^I_\mu\left(F^J_{\nu\lambda}F^K_{\rho\sigma}+f_{FG}{}^J A^F_\nu A^G_\lambda\left(-\tfrac12 g\,F^K_{\rho\sigma}\right.\right.\\
&\left.\left.+\tfrac1{10}g^2 f_{HL}{}^K A^H_\rho A^L_\sigma\right)\right)-\tfrac12 g^2\sigma^I\sigma^J\sigma^K\sigma^{\tilde M}\sigma^{\tilde N}t_{J\tilde M}{}^P t_{K\tilde N}{}^Q C_{IPQ}\\
&+\tfrac1{10}\,{\rm i}\,g\bar\psi_\mu\gamma^\mu\psi^{\tilde I}\sigma^{\tilde J}\sigma^{\tilde K}\sigma^{\tilde L}\Big(\Big[t_{[\tilde I\tilde J]}{}^{\tilde M}-2t_{(\tilde I\tilde J)}{}^{\tilde M}\Big]C_{\tilde M\tilde K\tilde L}-\tfrac12 t_{(\tilde J\tilde K)}{}^{\tilde M}C_{\tilde M\tilde I\tilde L}\Big)\\
&+\tfrac14\,{\rm i}\,g\bar\psi^{\tilde I}\psi^{\tilde J}\sigma^{\tilde K}\sigma^{\tilde L}\Big(t_{[\tilde I\tilde J]}{}^{\tilde M}C_{\tilde M\tilde K\tilde L}-4t_{(\tilde I\tilde K)}{}^{\tilde M}C_{\tilde M\tilde J\tilde L}\Big)\,,
\end{aligned}
\tag{6.134}
$$

where the superconformal D'Alembertian is defined as

$$
\begin{aligned}
\Box^c\sigma^{\tilde I} \;=\;& D^aD_a\sigma^{\tilde I}\\
=\;& \left(\partial^a-2b^a+\omega_b{}^{ba}\right)D_a\sigma^{\tilde I}+g t_{\tilde J\tilde K}{}^{\tilde I}A^J_a D^a\sigma^K-\frac{\rm i}{2}\bar\psi_\mu D^\mu\psi^{\tilde I}-2\sigma^{\tilde I}\bar\psi_\mu\gamma^\mu\chi\\
&+\frac12\bar\psi_\mu\gamma^\mu\gamma\cdot T\psi^{\tilde I}+\frac12\bar\phi_\mu\gamma^\mu\psi^{\tilde I}+2f_\mu{}^\mu\sigma^{\tilde I}-\tfrac12 g\bar\psi_\mu\gamma^\mu t_{\tilde J\tilde K}{}^{\tilde I}\psi^{\tilde J}\sigma^{\tilde K}\,.
\end{aligned}
\tag{6.135}
$$

Varying this action with respect to the fields of the tensor multiplet and the vector multiplet, we can obtain their covariant equations of motion.

6.4.2 The hypermultiplet

The local superconformal transformation rules for the hypermultiplet with gauged isometries is given by

$$\begin{aligned} \delta q^X &= -\mathrm{i}\,\bar\epsilon^i\zeta^A f^X_{iA}\,, \\ \widehat\delta\zeta^A &= \tfrac12\,\mathrm{i}\,\not{D}q^X f^{iA}_X\epsilon_i - \tfrac13\gamma\cdot T k^X f^A_{iX}\epsilon^i - \tfrac12 g\sigma^I k^X_I f^A_{iX}\epsilon^i + k^X f^A_{iX}\eta^i\,. \end{aligned} \tag{6.136}$$

The covariant derivatives are defined by

$$\begin{aligned} D_\mu q^X &= \mathcal{D}_\mu q^X + \mathrm{i}\,\bar\psi^i_\mu\zeta^A f^X_{iA}\,, \\ \mathcal{D}_\mu q^X &= \partial_\mu q^X - b_\mu k^X - V^{jk}_\mu k^X_{jk} + gA^I_\mu k^X_I\,, \\ D_\mu\zeta^A &= \mathcal{D}_\mu\zeta^A - k^X f^A_{iX}\phi^i_\mu + \tfrac12\,\mathrm{i}\,\not{D}q^X f^A_{iX}\psi^i_\mu + \tfrac13\gamma\cdot T k^X f^A_{iX}\psi^i_\mu\,, \\ &\quad + g\tfrac12\sigma^I k^X_I f^A_{iX}\psi^i_\mu \\ \mathcal{D}_\mu\zeta^A &= \partial_\mu\zeta^A + \partial_\mu q^X\omega_{XB}{}^A\zeta^B + \tfrac14\omega_\mu{}^{bc}\gamma_{bc}\zeta^A - 2b_\mu\zeta^A + gA^I_\mu t_{IB}{}^A\zeta^B\,. \end{aligned} \tag{6.137}$$

Similar to section 6.2, requiring closure of the commutator algebra on these transformation rules yields the equation of motion for the fermions

$$\begin{aligned} \Gamma^A &= \not{D}\zeta^A + \tfrac12 W_{CDB}{}^A\zeta^B\bar\zeta^D\zeta^C - \tfrac83\,\mathrm{i}\,k^X f^A_{iX}\chi^i + 2\,\mathrm{i}\,\gamma\cdot T\zeta^A \\ &\quad - g(\,\mathrm{i}\,k^X_I f^A_{iX}\psi^{iI} + \mathrm{i}\,\sigma^I t_{IB}{}^A\zeta^B)\,. \end{aligned} \tag{6.138}$$

The scalar equation of motion can be obtained from varying (6.138)

$$\widehat\delta_Q\Gamma^A = \tfrac12\,\mathrm{i}\,f^{iA}_X\Delta^X\epsilon_i + \tfrac14\gamma^\mu\Gamma^A\bar\epsilon\psi_\mu - \tfrac14\gamma^\mu\gamma^\nu\Gamma^A\bar\epsilon\gamma_\nu\psi_\mu\,, \tag{6.139}$$

from which we obtain

$$\begin{aligned} \Delta^X &= \Box^c q^X - \bar\zeta^B\gamma^a\zeta^C D_a q^Y\mathcal{R}^X{}_{YBC} + \tfrac89 T^2 k^X \\ &\quad + \tfrac43 D k^X + 8\,\mathrm{i}\,\bar\chi^i\zeta^A f^X_{iA} - \tfrac14\mathcal{D}^X W_{ABCD}\bar\zeta^A\zeta^B\bar\zeta^C\zeta^D \\ &\quad - g\big(2\,\mathrm{i}\,\bar\psi^{iI}\zeta^B t_{IB}{}^A f^X_{iA} - k^Y_I J_Y{}^X{}_{ij}Y^{ij}\big) \\ &\quad + g^2\sigma^I\sigma^J\mathfrak{D}_Y k^X_I k^Y_J\,, \end{aligned} \tag{6.140}$$

The superconformal D'Alembertian acting on the hyper-scalars is given by

$$\begin{aligned} \Box^c q^X &\equiv D_a D^a q^X \\ &= \partial_a D^a q^X - \tfrac52 b_a D^a q^X - \tfrac12 V^{jk}_a J_Y{}^X{}_{jk} D^a q^Y + \mathrm{i}\,\bar\psi^i_a D^a\zeta^A f^X_{iA} \\ &\quad + 2 f_a{}^a k^X - 2\bar\psi_a\gamma^a\chi k^X + 4\bar\psi^{(j}_a\gamma^a\chi^{k)} k^X_{jk} - \bar\psi^i_a\gamma^a\gamma\cdot T\zeta^A f^X_{iA} \\ &\quad - \bar\phi^i_a\gamma^a\zeta^A f^X_{iA} + \omega_a{}^{ab} D_b q^X - \tfrac12 g\bar\psi^a\gamma_a\psi^I k^X_I - D_a q^Y\partial_Y k^X_I A^{aI} \\ &\quad + D_a q^Y D^a q^Z\Gamma^X_{YZ}. \end{aligned} \tag{6.141}$$

The generalization of (6.125) to the case of local superconformal symmetry is given by

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\rm H} \;=\;& -\tfrac{1}{2}g_{XY}\mathcal{D}_a q^X\mathcal{D}^a q^Y + \bar{\zeta}_A \not{D}\zeta^A + \tfrac{2}{3}f_a{}^a k^2 + \tfrac{4}{9}Dk^2 + \tfrac{8}{27}T^2k^2\\
&+2\,\mathrm{i}\,\bar{\zeta}_A\gamma\cdot T\zeta^A - \tfrac{16}{3}\,\mathrm{i}\,\bar{\zeta}_A\chi^i k^X f_{iX}^A - \tfrac{1}{4}W_{ABCD}\bar{\zeta}^A\zeta^B\bar{\zeta}^C\zeta^D\\
&-\tfrac{2}{9}\bar{\psi}_a\gamma^a\chi k^2 + \tfrac{1}{3}\bar{\zeta}_A\gamma^a\gamma\cdot T\psi_a^i k^X f_{iX}^A + \tfrac{1}{2}\,\mathrm{i}\,\bar{\zeta}_A\gamma^a\gamma^b\psi_a^i\mathcal{D}_b q^X f_{iX}^A\\
&-\tfrac{1}{6}\,\mathrm{i}\,\bar{\psi}_a\gamma^{ab}\phi_b k^2 - \bar{\zeta}_A\gamma^a\phi_a^i k^X f_{iX}^A\\
&+\tfrac{1}{12}\bar{\psi}_a^i\gamma^{abc}\psi_b^j\mathcal{D}_c q^Y J_Y{}^X{}_{ij}k_X - \tfrac{1}{9}\,\mathrm{i}\,\bar{\psi}^a\psi^b T_{ab}k^2 + \tfrac{1}{18}\,\mathrm{i}\,\bar{\psi}_a\gamma^{abcd}\psi_b T_{cd}k^2\\
&-g\Big(P_{ij}^I Y_I^{ij} + 2\,\mathrm{i}\,k_I^X f_{iX}^A\bar{\zeta}_A\psi^{iI} + \mathrm{i}\,\sigma^I t_{IB}{}^A\bar{\zeta}_A\zeta^B\\
&+\tfrac{1}{2}\sigma^I k_I^X f_{iX}^A\bar{\zeta}_A\gamma^a\psi_a^i - \tfrac{1}{2}\bar{\psi}_a^i\gamma^a\psi^{jI}P_{Iij} + \tfrac{1}{4}\,\mathrm{i}\,\bar{\psi}_a^i\gamma^{ab}\psi_b^j\sigma^I P_{Iij}\Big)\\
&-\tfrac{1}{2}g^2\sigma^I\sigma^J k_I^X k_{JX}\,. \qquad (6.142)
\end{aligned}
$$

The field equations can be obtained from this action according to

$$
\begin{aligned}
\frac{\delta\mathcal{S}}{\delta\bar{\zeta}^A} &= 2\,C_{AB}\Gamma^B\,,\\
\frac{\delta\mathcal{S}}{\delta q^X} &= g_{XY}\left(\Delta^Y - 2\bar{\zeta}_A\Gamma^B\omega^Y{}_B{}^A - \mathrm{i}\,\bar{\psi}_a^i\gamma^a\Gamma^A f_{iA}^Y\right)\,. \qquad (6.143)
\end{aligned}
$$

6.5 Discussion and outlook

We will conclude this chapter with an overview of the results that we obtained, and a discussion of possible future research based on these results.

6.5.1 Summary of geometrical objects

In table 6.5, we have collected the essential geometrical data that is needed to construct superconformal matter multiplets. We indicate which are the essential geometrical objects that determine the theory and the independent constraints imposed on them. The symmetries of the objects are indicated by brackets on their indices. All equations are also valid for the theories in the columns next to and rows below its entry, apart from the entries “hyper + gauging” and “hyper + conformal” entry, which are mutually independent.

However, the symbol ▼ indicates that these equations or symbols are not to be taken over below. E.g. the moment map P_I^α itself is completely determined in the superconformal theory, and it should thus not be given as an independent quantity anymore. For the rigid theory without conformal invariance, only constant pieces can be undetermined by the given equations, and are the Fayet–Iliopoulos terms. Furthermore, the equations and symbols indicated by ► are not to be taken over for the theories with an action, as they are then satisfied due to the Killing equation or are defined by χ.

	ALGEBRA (no action)		ACTION	
Multiplet	Object	Restriction	Object	Restriction
Vector	$f_{[IJ]}{}^K$	Jacobi identities	$C_{(IJK)}$	$f_{I(J}{}^H C_{KL)H} \stackrel{\blacktriangledown}{=} 0$
Vector- tensor	$(t_I)_{\widetilde{J}}{}^{\widetilde{K}}$ $\widetilde{I} = (I, M)$	$[t_I, t_J] = -f_{IJ}{}^K t_K$ $t_{IJ}{}^K = f_{IJ}{}^K$ $t_{IM}{}^J = 0$	$\Omega_{[MN]}$	invertible $f_{I(J}{}^H C_{KL)H} = t_{I(J}{}^M t_{KL)}{}^N \Omega_{MN}$ $t_{I[M}{}^P \Omega_{N]P} = 0$
Hyper	f_X^{iA}	invertible and real Nijenhuis tensor $N_{XY}{}^Z = 0$	$C_{[AB]}$	$\mathfrak{D}_X C_{AB} = 0$
Hyper + conformal	k^X $\blacktriangleright$	$\mathfrak{D}_Y k^X \stackrel{\blacktriangleright}{=} \frac{3}{2}\delta_Y{}^X$	χ	$\mathfrak{D}_X \mathfrak{D}_Y \chi = \frac{3}{2} g_{XY}$
Hyper + gauging	k_I^X	$k_{[I\|}^Y \partial_Y k_{\|J]}^X = -\frac{1}{2} f_{IJ}{}^K k_K^X$ $\mathfrak{D}_X \mathfrak{D}_Y k_I^Z \stackrel{\blacktriangleright}{=} R_{XWY}{}^Z k_I^W$ $\mathcal{L}_{k_I} J^\alpha \stackrel{\blacktriangleright}{=} 0$	P_I^α $\blacktriangledown$	$\mathfrak{D}_X k_{YI} + \mathfrak{D}_Y k_{XI} = 0$ $\partial_X P_I^\alpha \stackrel{\blacktriangledown}{=} J_{XY}^\alpha k_I^Y$ $k_I^X J_{XY}^\alpha k_J^Y \stackrel{\blacktriangledown}{=} -f_{IJ}{}^K P_K^\alpha$
Hyper + conformal + gauging		$k^Y \mathfrak{D}_Y k_I^X = \frac{3}{2} k_I^X$		

Table 6.5: The superconformal matter multiplets and their essential geometrical data.

6.5.2 Gauge-fixing the conformal symmetry

In this chapter, we have discussed superconformal matter multiplets coupled to the Weyl multiplet. As mentioned in the introduction of this chapter, as well as at the end of chapter 4, the main motivation for this lengthy program was to construct matter-coupled Poincaré supergravities. We have not performed the complete gauge-fixing procedure to obtain such theories. Nevertheless, we will now briefly indicate how locally superconformal matter multiplets can lead to matter-coupled Poincaré supergravity theories.

The key observation is that the gauge field for special conformal transformations $f_\mu{}^a$ is related to the Ricci tensor $R_\mu{}^a$ of the spacetime manifold. From the constraint (5.47) and its explicit solution (5.51), we find that the trace of the special conformal gauge field is related to the Ricci scalar

$$f_a{}^a = -\tfrac{1}{16}R + \text{gravitino terms}\,. \tag{6.144}$$

This gauge field appears in the conformal D'Alembertian of scalar fields, e.g. for a five-dimensional scalar field ϕ of Weyl weight $\frac{3}{2}$, we have

$$\Box^c\phi = \left(\partial^a - \tfrac{5}{2}b^a + \omega_b{}^{ba}\right)\left(\partial_a - \tfrac{3}{2}b_a\right)\phi + 3f_a{}^a\phi\,. \tag{6.145}$$

With this definition, an action that is invariant under local conformal transformations is given by

$$e^{-1}\mathcal{L} = -\tfrac{1}{2}\phi\Box^c\phi\,. \tag{6.146}$$

We now fix the special conformal and the dilatational symmetry by imposing the Poincaré-gauge

$$b_\mu = 0\,, \qquad \phi^2 = \frac{16}{3\kappa^2}\,. \tag{6.147}$$

In this gauge, we can partially integrate (6.146) and use the solution for the spin-connection (5.51) to obtain

$$e^{-1}\mathcal{L} = \frac{1}{2\kappa^2}R\,. \tag{6.148}$$

So, we see that, in the Poincaré-gauge (6.147), the action for a local conformal scalar field (6.146) reduces to the Einstein-Hilbert action for ordinary gravity. In particular, the scale invariance of the action (6.146) is broken by the length-scale of the gravitational coupling constant κ in the dilatational gauge (6.147). Note also that the scalar field action (6.146) has the wrong sign for its kinetic term (we are using the mostly plus convention): the scalar field is therefore not a physical degree of freedom. Instead, it is a compensating scalar field for the broken conformal symmetry.

The above mechanism can also be applied to the local superconformal action for the $n_{\rm H} = r$ hypermultiplets, $n_{\rm T}$ tensor multiplets coupled to $n_{\rm V}$ vector multiplets in the background of the Standard Weyl multiplet. An additional subtlety here is that one also needs to solve the equation of motion for the scalar field D of the Weyl multiplet. In particular, we demand that all terms multiplying the gauge field $f_a{}^a$ yield a canonical Einstein-Hilbert term, and we impose the equation of motion for the scalar field D. Collecting all the relevant terms from (6.134) and (6.142), we impose the following gauge for the dilatation symmetry

$$-\tfrac{1}{24}\left(C_{\widetilde{I}\widetilde{J}\widetilde{K}}\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}} + g_{XY}k^X k^Y\right)R = \frac{1}{2\kappa^2}R\,, \tag{6.149}$$

$$-\tfrac{4}{3}C_{\widetilde{I}\widetilde{J}\widetilde{K}}\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}} + \tfrac{4}{9}g_{XY}k^X k^Y = 0 \tag{6.150}$$

These equations can be rewritten as

$$C_{\widetilde{I}\widetilde{J}\widetilde{K}}\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}} = -\frac{3}{\kappa^2}\,, \tag{6.151}$$

$$g_{XY}k^X k^Y = -\frac{9}{\kappa^2}\,. \tag{6.152}$$

We can interpret the dilatational gauge (6.152) on the scalars of the hypermultiplet as the definition of a hypersurface within the hyper-complex scalar manifold. In particular, a metric of signature $(1,r)$ on the hyper-complex manifold induces a metric of signature $(0,r)$ on the hypersurface (6.152). This follows from the resemblance of (6.152) to the embedding equation (2.25) of $(d+1)$-dimensional Anti-de-Sitter space, which is a hypersurface of signature $(1,d-1)$ in an ambient space of signature $(2,d)$.

The analysis of the dilatational gauge (6.151) on the scalars of the vector-tensor multiplets goes along similar ways. It induces a metric $g_{\widetilde{I}\widetilde{J}}$ on the vector space $\mathbb{R}^{n_{\rm V}+n_{\rm T}}$ spanned by the scalars

$$g_{\widetilde{I}\widetilde{J}} \equiv -\frac{1}{3}\frac{\partial^2 \ln C}{\partial\sigma^{\widetilde{I}}\partial\sigma^{\widetilde{J}}}\Big|_{C=-\frac{3}{\kappa^2}}\,, \qquad C \equiv C_{\widetilde{I}\widetilde{J}\widetilde{K}}\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}\,. \tag{6.153}$$

If we define scalars ϕ^x (with $x = 1,\ldots,n_{\rm V}+n_{\rm T}-1$), then the metric g_{IJ} induces a metric g_{xy} on the hypersurface (6.151) according to

$$g_{xy} \equiv g_{\widetilde{I}\widetilde{J}}\frac{\partial\sigma^{\widetilde{I}}}{\partial\phi^x}\frac{\partial\sigma^{\widetilde{J}}}{\partial\phi^y} \tag{6.154}$$

The metric g_{xy} on the manifold spanned by the scalars ϕ^x, defines the $D=5$ variant of special geometry [156], called “very special geometry”.

It will be interesting to see in what way the future analysis of the metrics $g_{\widetilde{I}\widetilde{J}}$ and g_{xy} in the case when there are non-vanishing C_{IJM} components will modify the analysis from [156]. The expected result is that, together with one hypermultiplet, one vector multiplet plays the role of a compensating multiplet for the broken conformal symmetries.

The additional conformal symmetries will also have to be gauge-fixed. The special conformal transformation can again be gauge-fixed by imposing $b_\mu = 0$, and the $\mathrm{SU}(2)$-transformations will be gauge-fixed by three of the remaining scalars of the compensating hypermultiplet. Gauge-fixing the S-supersymmetry and imposing the equation of motion for the spinor χ^i of the Weyl multiplet eliminates the spinors of the compensating vector multiplet and the hypermultiplet.

The remaining gauge field of the compensating vector multiplet will play the role of the graviphoton of the Poincaré multiplet. The equation of motion for the $\mathrm{SU}(2)$ gauge field $V_\mu{}^{ij}$ introduces a non-zero $\mathrm{SU}(2)$-valued curvature on the manifold: this promotes the hyper-complex or hyper-Kähler manifold to a quaternionic of quaternionic-Kähler manifold, respectively. Finally, the equation of motion for the tensor T_{ab} of the Weyl multiplet can be used to express T_{ab} in terms of the Yang-Mills field-strengths.

The overall result is that gauge-fixing the conformal symmetries of $n+1$ superconformal vector-tensor multiplets and $r+1$ superconformal hypermultiplets coupled to the Weyl multiplet of conformal supergravity leads to the theory of n vector-tensor multiplets and r hypermultiplets coupled to Poincaré supergravity.

6.5.3 The scalar potential

We will now present the scalar potential of the combined action for n_{T} on-shell tensor multiplets and $n_{\mathrm{H}} = r$ on-shell hypermultiplets in the background of n_{V} off-shell vector multiplets coupled to the Standard Weyl multiplet.

First, we collect all terms of $\mathcal{O}(g^2)$ in (6.134) and (6.142). However, this is not the final answer since the auxiliary field Y_I^{ij} has an algebraic equation of the form

$$2C_{I\widetilde{J}\widetilde{K}}\sigma^{\widetilde{J}}Y^{ij\widetilde{K}} = gP^{ij} + \text{fermion bilinears}\,. \tag{6.155}$$

Solving this equation and substituting the result into the term $-gP^I_{ij}Y^{ij}_I$ of (6.142) will generate an additional term of $\mathcal{O}(g^2)$.

The gauge-fixing of the superconformal symmetries has an additional effect: the corresponding parameters can be expressed in terms of the non-conformal parameters. In particular, the parameter η^i of S-transformations will be expressed in terms of the parameter ϵ^i of Q-supersymmetry. This will make the resulting Poincaré-supersymmetry transformation rules much more complicated: a complication that the conformal approach avoids until the final step in the calculations.

The expression for η^i will also involve the auxiliary field Y^{ij} of the vector multiplet, and by using (6.155) we see that e.g. the Poincaré-supersymmetry transformation for the gravitino will contain a term proportional to P_I^{ij}, and the scalar potential contains the square of that term. The other terms of the scalar potential can also be written in terms of squares of these so-called "fermion-shifts": they are defined as the terms of $\mathcal{O}(g)$ in the supersymmetry transformations of ζ^A and $\psi^{i\widetilde{I}}$, respectively

$$\begin{aligned}\delta\zeta^A &\sim -\tfrac{1}{2}g\sigma^I k_I^X f^A_{iX}\epsilon^i \equiv \mathcal{N}_i^A\epsilon^i\,,\\ \delta\psi^{i\widetilde{I}} &\sim \tfrac{1}{2}gt_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}\epsilon^i \equiv \mathcal{P}^{\widetilde{I}}\epsilon^i\,,\end{aligned} \tag{6.156}$$

Finally, we find for the scalar potential

$$V(\sigma^{\widetilde{I}}, q^X) = g^2 C^{-1}_{IJ\widetilde{K}}P^{ijI}P^J_{ij} + 2\mathcal{N}^{iA}\mathcal{N}_{iA} + 2\sigma^I\mathcal{P}^P\mathcal{P}^Q C_{IPQ}\,, \tag{6.157}$$

where there will be some small modifications due to the constraints on C_{IJK} by the dilatational gauge-fixing. After the gauge-fixing program will have been performed, the analysis of the critical points of this potential, and the Hessian matrix of the corresponding superpotential will give more insight in its possible applications. We stress that, even though this analysis has not been performed yet, the potential (6.157) contains new ingredients which have not been discussed in the literature before.

First of all, we have also considered reducible, but not completely reducible, representations for the vector-tensor multiplet. This opens the possibility of non-compact gauge groups and the existence of new Chern-Simons terms in the action of the form $A \wedge B \wedge F$ that were not constructed before. Algebraically, this is reflected in the non-zero components of the tensor C_{IMN} that appears in the potential (6.157).

Furthermore, without an action, we also allow an odd number of tensor multiplets, which is more general than all analyses so far, which all started from an action. For the hypermultiplets, the same argument applies: here, we have considered hyper-Kähler manifold without a metric, the so-called hyper-complex manifolds.

To conclude, we expect that these new results on superconformal matter multiplets will also lead to more general matter-couplings to Poincaré supergravity. Whether such new matter-couplings will drastically modify the structure of the scalar potential in such a way that supersymmetric Randall-Sundrum scenarios become possible, remains an open question.

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Appendix A

Conventions

In this appendix, we will summarize our conventions. Furthermore, we will give some useful identities that have been used in the previous chapters.

A.1 Indices

The last two chapters have used a large amount of different indices. Below we will summarize the different ranges and meanings of these indices. First of all, the metric that we use is mostly plus: i.e. in five dimensions, we have $g_{\mu\nu} = (- + + + +)$. In chapter 5, we have used the following notations

$$\begin{array}{rcl} \mu, \nu & 0, 1, \ldots, 4 & \text{spacetime}\,, \\ a, b & 0, 1, \ldots, 4 & \text{tangent space}\,, \\ \alpha, \beta & 1, \ldots, 4 & \text{spinor}\,, \\ i, j & 1, 2 & \mathrm{SU}(2)\,, \end{array} \tag{A.1}$$

In chapter 6, we have furthermore used indices labelling the components of matter multiplet. In particular, we have used

$$\begin{array}{rcl} \widetilde{I}, \widetilde{J} & 1, 2, \ldots n_{\mathrm{V}} + n_{\mathrm{T}} & \text{vector-tensor multiplet}\,, \\ I, J & 1, 2, \ldots, n_{\mathrm{V}} & \text{vector multiplet}\,, \\ M, N & 1, 2, \ldots, n_{\mathrm{T}} & \text{tensor multiplet}\,, \\ X, Y & 1, 2, \ldots, 4n_{\mathrm{H}} & \text{hypermultiplet target space}\,, \\ A, B & 1, 2, \ldots, 2n_{\mathrm{H}} & \text{hypermultiplet tangent space}\,, \\ i, j & 1, 2 & \mathrm{SU}(2)\,. \end{array} \tag{A.2}$$

In all cases, we denote symmetrizations with parentheses around the indices, and anti-symmetrizations with brackets around the indices. Furthermore, we (anti-)symmetrize with weight one

$$X_{(ab)} \equiv \tfrac{1}{2}\left(X_{ab} + X_{ba}\right)\,, \quad X_{[ab]} \equiv \tfrac{1}{2}\left(X_{ab} - X_{ba}\right)\,. \tag{A.3}$$

A.2 Tensors

Our conventions for the D-dimensional Levi–Civita tensor are

$$\varepsilon_{a_1\ldots a_D} = -\varepsilon^{a_1\ldots a_D} = 1\,. \tag{A.4}$$

The Levi-Civita tensor with spacetime indices can be obtained from (A.4 by using vielbeins to convert the tangent space indices to spacetime indices, and multiplying the result with the vielbein determinant gives

$$\varepsilon_{\mu_1\ldots\mu_D} = e^{-1} e_{\mu_1}{}^{a_1}\cdots e_{\mu_d}{}^{a_D}\varepsilon_{a_1\ldots a_D}\,, \quad \varepsilon^{\mu_1\ldots\mu_D} = e\, e^{\mu_1}{}_{a_1}\cdots e^{\mu_D}{}_{a_D}\varepsilon^{a_1\ldots a_D}\,, \tag{A.5}$$

where we have used the Einstein summation convention in which repeated indices are summed over.

Note that raising and lowering the indices of the Levi-Civita tensor with spacetime indices is done with the metric, which for the Levi-Civita tensor with tangent space indices is done by using the definition (A.4). Contractions of the Levi-Civita tensor give products of delta-functions which are normalized as

$$\varepsilon_{a_1\ldots a_p b_1\ldots b_q}\varepsilon^{a_1\ldots a_p c_1\ldots c_q} = -p!q!\delta^{[c_1}_{[b_1}\ldots\delta^{c_q]}_{b_q]}\,, \tag{A.6}$$

We have defined the dual of five-dimensional tensors as

$$\widetilde{A}^{a_1\ldots a_{5-n}} = \tfrac{1}{n!}\,\mathrm{i}\,\varepsilon_{a_1\ldots a_{5-n}b_1\ldots b_n}A^{b_n\ldots b_1}\,. \tag{A.7}$$

Using (A.6), one finds the following identities

$$\widetilde{\widetilde{A}} = A\,, \qquad \frac{1}{n!}A^{a_1\ldots a_n}B_{a_1\ldots a_n} = \frac{1}{n!}A\cdot B = \frac{1}{(n-5)!}\widetilde{A}\cdot\widetilde{B}\,, \tag{A.8}$$

where we have introduced the generalized inner product notation $A\cdot B$ that we use throughout this thesis.

We use the same conventions for the Riemann tensor and its contractions as [92]. In particular, we define the Riemann tensor as

$$R^\mu{}_{\nu\lambda\rho} = \partial_\lambda\Gamma^\mu_{\rho\nu} - \partial_\rho\Gamma^\mu_{\lambda\nu} + \Gamma^\mu_{\sigma\lambda}\Gamma^\sigma_{\rho\nu} - \Gamma^\mu_{\sigma\rho}\Gamma^\sigma_{\lambda\nu}\,. \tag{A.9}$$

The Ricci tensor and Ricci scalar in this thesis are given by

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}\,, \quad R = g^{\mu\nu}R_{\mu\nu}\,. \tag{A.10}$$

With these conventions, the Einstein-Hilbert action has a positive sign.

A.3 Differential forms

In chapter 1, we have used differential form notation to simplify the supergravity actions. A p-form is related to a rank-p anti-symmetric tensor according to

$$F_{(p)} = \frac{1}{p!}\mathrm{d}x^{\mu_1}\dots\mathrm{d}x^{\mu_p}F_{\mu_1\dots\mu_p}\,. \tag{A.11}$$

The analog of the dual of an anti-symmetric tensor (A.7), is given by the Hodge-dual: i.e a differential p-form A has a $D-p$-form $B=\star A$ as its dual with components

$$B_{\mu_1\dots\mu_q} = \frac{1}{p!}e\,\varepsilon_{\mu_1\dots\mu_q}{}^{\nu_1\dots\nu_p}A_{\nu_1\dots\nu_p}\,,\quad q=D-p\,. \tag{A.12}$$

Note in particular the different order in which the indices in (A.12) are contracted with respect to (A.7). With this definition, we have the usual identity

$$\star\star A_{(p)} = (-)^{pq+1}A_{(p)}\,,\quad q=D-p\,. \tag{A.13}$$

Furthermore, the D-dimensional invariant volume element can then be written as the star of the unit number

$$\star\mathbb{1}\equiv \mathrm{d}^Dx\,\sqrt{|g|}\,. \tag{A.14}$$

A.4 Spinors

Our five-dimensional spinors are symplectic-Majorana spinors that transform in the $(4,2)$ of $\overline{\mathrm{SO}(5)}\otimes\mathrm{SU}(2)$. The generators U_{ij} of the R-symmetry group $\mathrm{SU}(2)$ are defined to be anti-Hermitian and symmetric, i.e.

$$(U_i{}^j)^* = -U_j{}^i\,,\qquad U_{ij}=U_{ji}\,. \tag{A.15}$$

A symmetric traceless $U_i{}^j$ corresponds to a symmetric U^{ij} since we lower or raise $\mathrm{SU}(2)$ indices using the ε-symbol contracting the indices in a northwest-southeast (NW–SE) convention

$$X^i=\varepsilon^{ij}X_j\,,\qquad X_i = X^j\varepsilon_{ji}\,,\qquad \varepsilon_{12}=-\varepsilon_{21}=\varepsilon^{12}=1\,. \tag{A.16}$$

The actual value of ε is here given as an example. It is in fact arbitrary as long as it is antisymmetric, $\varepsilon^{ij} = (\varepsilon_{ij})^*$ and $\varepsilon_{jk}\varepsilon^{ik} = \delta_j{}^i$. When the $\mathrm{SU}(2)$ indices on spinors are omitted, NW-SE contraction is understood

$$\bar\lambda\psi = \bar\lambda^i\psi_i\,, \tag{A.17}$$

The charge conjugation matrix $\mathcal{C}$ and $\mathcal{C}\gamma_a$ are antisymmetric. The matrix $\mathcal{C}$ is unitary and γ_a is Hermitian apart from the timelike one, which is anti-Hermitian. The bar is the Majorana bar

$$\bar\lambda^i = (\lambda^i)^T\mathcal{C}\,. \tag{A.18}$$

We define the charge conjugation operation on spinors as

$$(\lambda^i)^C \equiv \alpha^{-1} B^{-1} \varepsilon^{ij} (\lambda^j)^* \,, \qquad \bar\lambda^{iC} \equiv \overline{(\lambda^i)^C} = \alpha^{-1} \left(\bar\lambda^k\right)^* B \varepsilon^{ki} \,, \tag{A.19}$$

where $B = \mathcal{C}\gamma_0$, and $\alpha = \pm 1$ when one uses the convention that complex conjugation does not interchange the order of spinors, or $\alpha = \pm\,\mathrm{i}$ when it does. Symplectic Majorana spinors satisfy $\lambda = \lambda^C$. Charge conjugation acts on gamma-matrices as $(\gamma_a)^C = -\gamma_a$, does not change the order of matrices, and works on matrices in $\mathrm{SU}(2)$ space as $M^C = \sigma_2 M^* \sigma_2$. Complex conjugation can then be replaced by charge conjugation, if for every bi-spinor one inserts a factor -1. Then, e.g., the expressions

$$\bar\lambda^i \gamma_\mu \lambda^j \,, \qquad \mathrm{i}\,\bar\lambda^i \lambda_i \tag{A.20}$$

are real for symplectic Majorana spinors. For more details, see [186].

A.5 Gamma-matrices

The gamma-matrices γ_a are defined as matrices that satisfy the Clifford-algebra

$$\{\gamma_a, \gamma_b\} \equiv \gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab} \tag{A.21}$$

Completely anti-symmetrized products of gamma-matrices are denoted in three different ways

$$\gamma_{(n)} = \gamma_{a_1\cdots a_n} = \gamma_{[a_1} \cdots \gamma_{a_n]} \,. \tag{A.22}$$

The product of all gamma-matrices is proportional to the unit matrix in odd dimensions. We use

$$\gamma^{abcde} = \mathrm{i}\,\varepsilon^{abcde} \,. \tag{A.23}$$

This implies that the dual of a $(5-n)$-antisymmetric gamma-matrix is the n-antisymmetric gamma-matrix given by

$$\gamma_{a_1\ldots a_n} = \tfrac{1}{(5-n)!}\,\mathrm{i}\,\varepsilon_{a_1\ldots a_n b_1\ldots b_{5-n}} \gamma^{b_{5-n}\ldots b_1} \,. \tag{A.24}$$

For convenience, we will give the values of gamma-contractions like

$$\gamma^{(m)}\gamma_{(n)}\gamma_{(m)} = c_{n,m}\gamma_{(n)} \,, \tag{A.25}$$

where the constants $c_{n,m}$ are given in table A.1. The constants for $n,m > 2$ can easily be obtained from (A.24) and table A.1.

Changing the order of spinors in a bilinear leads to the following signs

$$\bar\psi^{(1)}\gamma_{(n)}\chi^{(2)} = t_n\, \bar\chi^{(2)}\gamma_{(n)}\psi^{(1)} \qquad \left\{ \begin{array}{l} t_n = +1 \text{ for } n = 0,1 \\ t_n = -1 \text{ for } n = 2,3 \end{array} \right. \tag{A.26}$$

where the labels (1) and (2) denote any $\mathrm{SU}(2)$ representation.

$c_{n,m}$	$m=1$	$m=2$
$n=0$	5	-20
$n=1$	-3	-4
$n=2$	1	4

Table A.1: Coefficients used in contractions of gamma-matrices.

A.6 Fierz-identities

The sixteen different gamma-matrices $\gamma_{(n)}$ for $n = 0, 1, 2$ form a complete basis for four-dimensional matrices. Similarly, the identity matrix $\mathbb{1}_2$ and the three Pauli-matrices σ^i for $i = 1, 2, 3$ form a basis for two-dimensional matrices. A change of basis in a product of two pseudo-Majorana spinors will give rise to so-called Fierz-rearrangement formulae, which in their simplest form are given by

$$\psi_j \bar{\lambda}^i = -\tfrac{1}{4}\bar{\lambda}^i \psi_j - \tfrac{1}{4}\bar{\lambda}^i \gamma^a \psi_j \gamma_a + \tfrac{1}{8}\bar{\lambda}^i \gamma^{ab} \psi_j \gamma_{ab}\,, \qquad \bar{\psi}^{[i}\lambda^{j]} = -\tfrac{1}{2}\bar{\psi}\lambda\varepsilon^{ij}\,. \tag{A.27}$$

Using such Fierz-rearrangements, other useful identities can be deduced for working with cubic fermion terms

$$\begin{aligned} &\lambda_j \bar{\lambda}^j \lambda^i = \gamma^a \lambda_j \bar{\lambda}^j \gamma_a \lambda^i = \tfrac{1}{8}\gamma^{ab}\lambda^i \bar{\lambda}\gamma_{ab}\lambda\,, \\ &\gamma^{cd}\gamma_{ab}\lambda^i \bar{\lambda}\gamma^{cd}\lambda = 4\lambda^i \bar{\lambda}\gamma^{ab}\lambda\,, \\ &\gamma_a \lambda \bar{\lambda}\gamma^{ab}\lambda = 0\,. \end{aligned} \tag{A.28}$$

When one multiplies three spinor doublets, one should be able to write the result in terms of $\binom{8}{3} = 56$ independent structures. From analyzing the representations, one can obtain that these are in the $(4, 2) + (4, 4) + (16, 2)$ representations of $\overline{\mathrm{SO}(5)} \times \mathrm{SU}(2)$. They are

$$\begin{aligned} &\lambda_j \bar{\lambda}^j \lambda^i = \gamma^a \lambda_j \bar{\lambda}^j \gamma_a \lambda^i = \tfrac{1}{8}\gamma^{ab}\lambda^i \bar{\lambda}\gamma_{ab}\lambda\,, \\ &\lambda^{(k}\bar{\lambda}^i \lambda^{j)}\,, \\ &\lambda_j \bar{\lambda}^j \gamma_a \lambda^i\,. \end{aligned} \tag{A.29}$$

As a final Fierz-identity, we give a three-spinor identity which is needed to prove the invariance under supersymmetry of the action for a vector multiplet

$$\psi^i_{[I}\bar{\psi}_J \psi_{K]} = \gamma^a \psi^i_{[I}\bar{\psi}_J \gamma_a \psi_{K]}\,. \tag{A.30}$$

Samenvatting

De elementaire-deeltjesfysica probeert de fundamentele bouwstenen van de Natuur en hun onderlinge wisselwerkingen te beschrijven. Uit experimenten is gebleken dat de elementaire deeltjes in twee klassen zijn onder te brengen: de leptonen, waaronder het elektron en het neutrino; en de quarks, de bouwstenen van protonen en neutronen. De vier bekende wisselwerkingen tussen deze bouwstenen zijn de zwaartekracht, de elektromagnetische, de zwakke, en de sterke wisselwerking.

Op kleine lengteschalen is de zwaartekracht vele orden van grootte zwakker dan alle andere krachten[1], en zij kan dan ook rustig verwaarloosd worden. De resterende drie wisselwerkingen kunnen beschreven worden door een elegante theorie die het Standaard Model wordt genoemd. Deze theorie is een ijktheorie – zij heeft een interne lokale symmetriegroep waardoor elke wisselwerking beschreven kan worden als een uitwisseling van ijkdeeltjes. Deze ijkdeeltjes worden het foton, de W-bosonen en het Z-boson, en de gluonen genoemd voor respectievelijk de elektromagnetische, de zwakke, en de sterke wisselwerking. IJkdeeltjes verschillen in meerdere opzichten van materiedeeltjes: de ijkdeeltjes vallen in de klasse van bosonen, deeltjes met heeltallige spin en commuterende statistiek; de materiedeeltjes daarentegen vallen in de klasse van fermionen, deeltjes met halftallige spin en anticommuterende statistiek. De interne symmetriegroepen van het Standaard Model beelden bosonen op bosonen en fermionen op fermionen af.

De natuurkunde wordt op microscopisch niveau beschreven door de quantummechanica, die kan worden gezien als een verfijning van de klassieke mechanica. Zij heeft verscheidene tegen-intuïtieve eigenschappen: zo kan men niet tegelijkertijd alle waarneembare grootheden met oneindige nauwkeurigheid meten, en veel grootheden kunnen slechts worden uitgedrukt in waarschijnlijkheden. Het Standaard Model is volledig in overeenstemming met de quantummechanica, en alle experimenten tot nu toe hebben de theorie tot op grote nauwkeurigheid bevestigd.

Op grote lengteschalen worden de wisselwerkingen van het Standaard Model vrijwel verwaarloosbaar: de sterke kracht is beperkt tot heel kleine afstanden; de zwakke kracht neemt exponentieel af met de afstand; en hoewel de elektromagnetische kracht een oneindig bereik heeft, is alle materie ruwweg elektrisch neutraal. Hierdoor wordt de zwaartekracht de

[1]De verhouding tussen de zwaartekracht en de elektrische kracht tussen een proton en een elektron is 10^{-40}.

overheersende kracht op grote afstanden.

De zwaartekracht wordt beschreven door de Algemene Relativiteitstheorie. De Algemene Relativiteitstheorie is erop gebaseerd dat ruimte en tijd samengaan in de zogenaamde ruimte-tijd, dat materie een gekromde meetkunde geeft aan de ruimte-tijd, en dat deze krommingen op hun beurt bepalen hoe materie zich door de ruimte-tijd voortbeweegt. Men kan ook proberen de Algemene Relativiteitstheorie te formuleren als een ijktheorie: in dit geval een ijktheorie van ruimte-tijdsymmetrieën, algemene coördinatentransformaties geheten, in plaats van interne symmetrieën. Het bijbehorende ijkveld heet in dit geval het graviton. De Algemene Relativiteitstheorie is een volledig klassieke theorie, en zij is tot nu toe goed in overeenstemming gebleken met alle experimentele waarnemingen, van planetaire tot en met kosmologische lengteschalen.

Aan de hierboven geschetste tweedeling van de natuurkunde in de macroscopische Algemene Relativiteitstheorie en het microscopische Standaard Model zit een aantal haken en ogen. Zo heeft de Algemene Relativiteitstheorie enkele merkwaardige eigenschappen. Allereerst zijn er oplossingen van de klassieke veldvergelijkingen die zwarte gaten worden genoemd. Deze zwarte gaten hebben als generieke eigenschap dat de ruimte-tijd singulariteiten bevat in de buurt waarvan het zwaartekrachtsveld oneindig sterk wordt. Dit ondergraaft de aanname dat de zwaartekracht verwaarloosd zou kunnen worden op kleine lengteschalen: hierdoor wordt het noodzakelijk om de zwaartekracht quantummechanisch te beschrijven.

Men veronderstelt dat de meeste van de ruimte-tijdsingulariteiten verborgen zijn achter zogenaamde gebeurtenishorizonnen: dit zijn oppervlakken van waarachter het zelfs voor licht onmogelijk is om terug te keren. Er wordt dan ook vermoed dat singulariteiten niet direct waarneembaar zijn. Het gedrag van deeltjes in de buurt van zulke gebeurtenishorizonnen is quantummechanisch gezien echter problematisch, aangezien het eenrichtingskarakter van een gebeurtenishorizon de waarschijnlijkheidsinterpretatie van de quantummechanica verstoort. Een karakteristiek gevolg hiervan zijn de zogenaamde informatieparadoxen.

Hoewel de energieschalen die noodzakelijk zijn om het microscopische gedrag van de zwaartekracht te kunnen onderzoeken niet in laboratoriumexperimenten gerealiseerd worden, traden zulke energieschalen in het vroege universum wel degelijk op . Voor een goed begrip van de kosmologie is dan ook een betere beschrijving van de zwaartekracht op kleine lengteschalen noodzakelijk. Hieraan gerelateerd is het probleem van de kosmologische constante, een parameter van de Algemene Relativiteitstheorie waarvoor het Standaard Model een waarde voorspelt die vele orden van grootte hoger ligt dan de experimenteel vastgestelde waarde.

Om bovengenoemde problemen op te kunnen lossen is het noodzakelijk om een theorie voor de quantumzwaartekracht te ontwikkelen. Het quantiseren van een klassieke wisselwerkingstheorie is echter een ingewikkelde zaak. Een eerste stap in de goede richting is de quantummechanische beschrijving van deeltjes in een klassiek krachtenveld. Dit wordt vaak de eerste quantisatie genoemd. Voor de elektromagnetische kracht ontstond zo'n semi-klassieke theorie in de jaren twintig van de vorige eeuw toen onder andere de aard van zwartelichamenstraling en de oorsprong van de energieniveaus van het waterstofatoom werden ontdekt. In de laatste decennia van de vorige eeuw werd de quantummechanische beschrijving van deeltjes in een zwaartekrachtsveld beschreven. In het bijzonder werd het proces van Hawking-straling en de microscopische verklaring van de wanorde van zwarte gaten ontdekt.

De volgende stap is om het krachtenveld zélf op een quantummechanische manier te beschrijven: dit wordt ook wel de tweede quantisatie genoemd. Dit behelst een ingewikkelde procedure waarbij in tussenberekeningen allerlei oneindigheden opduiken. Halverwege de vorige eeuw lukte het met de formulering van de quantumelektrodynamica om een eenduidige manier te vinden om zinvolle antwoorden te verkrijgen uit de berekeningen. Deze procedure wordt ook wel renormalisatie genoemd. De zwaartekracht blijkt niet op dezelfde manier te kunnen worden gequantiseerd als de elektromagnetische kracht: de oorsprong ligt uiteindelijk in de energieafhankelijkheid van de zwaartekrachtsconstante. Hierdoor wordt de zwaartekracht zeer sterk bij hoge energieën, en kunnen de oneindigheden niet meer worden weggewerkt.

In de jaren zeventig van de vorige eeuw werd er een verfijning van de Algemene Relativiteitstheorie ontdekt die zich bij hoge energieën beter gedroeg. Deze theorie wordt ook wel superzwaartekracht genoemd omdat zij een symmetrie heeft die bosonen en fermionen met elkaar weet te verbinden, een zogenaamde supersymmetrie. Supersymmetrische theorieën geven normaliter aanleiding tot minder oneindigheden en zijn makkelijker te renormaliseren. Helaas is superzwaartekracht uiteindelijk ook niet renormeerbaar gebleken. De huidige inzichten zijn dat superzwaartekracht een lage-energielimiet zou kunnen zijn van de quantumzwaartekracht, net zoals bijvoorbeeld Fermi's theorie van het beta-verval een lage-energielimiet is van het Standaard Model.

Om voorspellingen buiten het lage-energiegebied te kunnen doen, is er een methode vereist om botsingen bij hogere energieën te beschrijven. Voor de sterke wisselwerking werd deze zogenaamde S-matrixtheorie in de jaren zestig van de vorige eeuw ontwikkeld. Zij gebruikt een storingsreeks van zogenaamde Feynman-diagrammen om botsingsamplitudes te berekenen. Uiteindelijk bleken de heuristische Feynman-regels te volgen uit een actieprincipe: de Lagrangiaan van het Standaard Model. Omdat de zwakke en sterke wisselwerkingen gebaseerd zijn op grotere symmetriegroepen is de tweede quantisatie aanzienlijk ingewikkelder dan die voor de elektromagnetische kracht: de noodzakelijke wiskundige technieken werden pas in de jaren zeventig van de vorige eeuw ontwikkeld.

Een soortgelijk raamwerk voor de zwaartekracht blijkt een veralgemening van het concept van een elementair deeltje te behelzen. De gedachte is dat elementaire deeltjes zich op hele kleine lengteschalen[2] gedragen als snaren: het spectrum van trillingstoestanden dient dan de bekende elementaire deeltjes te bevatten. In het bijzonder bevat het spectrum van gesloten snaren een deeltje dat veel lijkt op het graviton. Omdat snaren een ruimtelijke dimensie hebben, worden de Feynman-diagrammen uit de deeltjesfysica nu vervangen door zogenaamde Riemann-oppervlakken. In de jaren tachtig van de vorige eeuw werd aangetoond dat een storingsreeks van zulke oppervlakken eindige antwoorden geeft voor botsingen van snaren bij hoge energieën. Dit kan intuïtief begrepen worden uit het feit dat de wissel-

[2]De karakteristieke schaal voor snaren ligt in de buurt van 10^{-33} cm.

werking tussen snaren zich niet op een gelokaliseerd punt afspeelt, maar verdeeld is over een glad oppervlak. De snaartheorie doet een aantal verrassende voorspellingen: de natuur is supersymmetrisch, en naast de bekende drie ruimterichtingen en één tijdrichting bestaan er nog zes andere ruimtelijke dimensies.

Een Lagrangiaan waaruit de botsingsregels voor snaren kunnen worden afgeleid wordt ook wel een snaarveldentheorie genoemd. Voor de eenvoudigste snaarmodellen geven de bijbehorende snaarveldentheorieën de correcte botsingsamplitudes, maar voor meer ingewikkelde snaarmodellen zijn er technische complicaties bij het formuleren van zulke Lagrangianen. Bovendien hebben dergelijke snaarveldentheorieën een veel grotere symmetriegroep dan de tot nu toe bekende quantumveldentheorieën voor elementaire deeltjes: dit leidt tot aanzienlijk grotere wiskundige complicaties bij het quantiseren van snaarveldentheorieën. Het ligt dan ook in de lijn der verwachting dat de huidige quantisatiemethoden nog verder verfijnd zullen moeten worden om tot een theorie van de quantumzwaartekracht te komen.

De klassieke veldvergelijkingen van ijktheorieën hebben veelal een aantal eindige-energieoplossingen, ook wel solitonen genoemd. Zo hebben enigszins aangepaste versies van de zwakke wisselwerking oplossingen die magnetische monopolen worden genoemd. De aanwezigheid van zulke magnetische monopolen geeft aanleiding tot een nieuwe klasse van symmetrieën, de zogenaamde dualiteiten: deze geven vaak verrassende verbanden tussen ogenschijnlijk ongerelateerde grootheden binnen een theorie, of zelfs verbanden tussen verschillende theorieën. In het laatste decennium is ook in de snaartheorie een groot aantal soliton-oplossingen gevonden. Deze oplossingen beschrijven een soort zwarte gaten met extra ruimtelijke dimensies: zo zijn er behalve solitondeeltjes ook nog snaren, membranen, en objecten van nog hogere dimensionaliteit, branen genaamd. Deze branen vormen het middelpunt van een enorm web aan dualiteiten die allerlei aspecten van snaartheorie met elkaar in verband brengen.

Tot begin jaren negentig waren er vijf snaartheorieën in omloop die op een aantal subtiele punten van elkaar verschillen. De vondst van braanoplossingen en de bijbehorende dualiteiten hebben laten zien dat deze snaartheorieën naar alle waarschijnlijkheid verschillende aspecten zijn van een allesomvattende theorie. Het totale raamwerk van snaartheorie, superzwaartekracht, alle solitonen en dualiteiten wordt ook wel M-theorie genoemd. Deze theorie staat nog in de kinderschoenen: zelfs over de betekenis van de letter M wordt nog gedebatteerd[3]. Veel van het huidige onderzoek naar een theorie van de quantumzwaartekracht speelt zich dan ook af rond de vraag hoe dit geschetste raamwerk verder onderbouwd zou kunnen worden.

In dit proefschrift hebben we in hoofdstuk 1 het huidige raamwerk van de snaartheorie verder uitgelegd. Vervolgens hebben we in hoofdstuk 2 een recent ontdekte dualiteit besproken: het Anti-de-Sitter/conforme veldentheorie verband. Dit is een verband tussen theorieën van de zwaartekracht binnen een bepaalde klasse van gekromde ruimten enerzijds en een speciale klasse van quantumveldentheorieën anderzijds. Dit is een zeer opmerkelijke dualiteit omdat allerlei grootheden binnen de quantumzwaartekracht kunnen worden uitgerekend

[3]Een veel-gemaakte grap is dat de M zowel Membraan, Mysterie als Magie kan betekenen.

met methoden die bekend zijn vanuit de elementaire-deeltjesfysica. Centraal in dit verband staat een speciale braanoplossing van snaartheorie: de zogenaamde D3-braan.

In hoofdstuk 3 hebben we ons eigen werk besproken, dat het bovengenoemde verband veralgemeniseert. We hebben laten zien dat er een soortgelijk verband bestaat tussen zwaartekrachtstheorieën in een meer algemene klasse van gekromde ruimten enerzijds en meer algemene quantumveldentheorieën anderzijds. In het bijzonder hebben we een grotere klasse van braanoplossingen bestudeerd die de D3-braan als speciaal geval bevat. Door in een geschikt gekozen coördinatenstelsel, het zogenaamde duale stelsel, de meetkunde van deze braanoplossingen in de buurt van hun gebeurtenishorizon te bestuderen, hebben we informatie over het gedrag van de duale veldentheorie weten te verkrijgen.

De gekromde ruimten die bij de bovengenoemde analyse tevoorschijn komen, worden ook wel domeinvlakken genoemd: zij beschrijven ruimten die bestaan uit verschillende domeinen die gescheiden zijn door een grensvlak waarop bepaalde grootheden op een discontinue manier van grootte veranderen. Worden de bovengenoemde discontinue oplossingen ook wel dunne domeinvlakken genoemd, oplossingen die op een continue manier interpoleren tussen verschillende grondtoestanden van de onderliggende zwaartekrachtstheorie worden dikke domeinvlakken genoemd. Aan het einde van hoofdstuk 3 hebben we uitgelegd wat de interpretatie van deze dikke domeinvlakken is binnen de duale quantumveldentheorie.

Domeinvlakken hebben zeer recentelijk nog een andere toepassing gekregen: zij maken deel uit van de in hoofdstuk 4 besproken klasse van modellen die zogenaamde braanwerelden beschrijft. De kerngedachte van deze modellen is dat ons vierdimensionale universum met zijn drie ruimterichtingen en één tijdrichting in werkelijkheid een hypervlak binnen een vijfdimensionale ruimte is. De grootte van de vijfde ruimterichting die loodrecht op deze zogenaamde braanwereld staat, kan worden gebruikt om een aantal onnatuurlijke verhoudingen binnen de vierdimensionale natuurkunde beter te begrijpen: zo is er meer inzicht gekomen in de oorsprong van de grootte van de kosmologische constante en in de onnatuurlijk grote verhouding tussen de sterktes van de zwaartekracht enerzijds en de resterende wisselwerkingen anderzijds, het zogenaamde hiërarchieprobleem.

Het is tot nu toe niet gelukt om deze braanwereldmodellen op een goede manier in een snaartheoriekader te plaatsen. Het voornaamste obstakel dat hierbij optreedt, is het realiseren van supersymmetrie in de vierdimensionale braanoplossing: dit is gerelateerd aan het vinden van de stabiele grondtoestanden van de onderliggende vijfdimensionale superzwaartekrachtstheorie. Hiervoor moeten echter eerst alle wisselwerkingen van vijfdimensionale materiemodellen met superzwaartekracht worden geclassificeerd. De scalaire velden van de verschillende materiemodellen kunnen worden geïnterpreteerd als coördinaten van een abstracte ruimte. Veel eigenschappen van deze modellen kunnen dan worden uitgedrukt in meetkundige eigenschappen van de bijbehorende ruimten van velden.

In het bijzonder geven de scalaire velden aanleiding tot een potentiaal die de vacuümstructuur bepaalt. Om supersymmetrische braanwerelden mogelijk te maken, moet deze scalaire potentiaal over een tweetal stabiele minima beschikken dat aan een aantal verdere randvoorwaarden moet voldoen. Daarnaast moet er een geschikte braanoplossing gevonden worden die op een continue manier interpoleert tussen de twee minima. Een dergelijke analyse waarmee men (overigens vanwege andere redenen) reeds was begonnen in de jaren tachtig van de vorige eeuw, is recentelijk nog verder uitgediept, maar omvat nog steeds niet de meest algemene gevallen.

We hebben een systematische methode gebruikt om tot een startpunt te komen van waaruit de meest algemene vijfdimensionale materiekoppelingen aan superzwaartekracht kunnen worden afgeleid. Dit zogenaamde superconforme programma neemt als uitgangspunt de meest algemene groep van ruimte-tijdsymmetrieën, de superconforme groep, wat de constructie van materiekoppelingen aan superzwaartekracht aanzienlijk vereenvoudigt. De verschillende modellen die superconforme symmetrie bezitten, worden ook wel multipletten genoemd. Enerzijds is er het Weyl-multiplet: dit is het kleinste multiplet van de superconforme groep dat het graviton bevat. Anderzijds zijn er de materiemultipletten: zij wisselwerken met het Weyl-multiplet dat een vaste achtergrond van conforme superzwaartekracht realiseert. Materiekoppelingen met niet-conforme superzwaartekracht kunnen worden verkregen door de conforme symmetrieën te breken.

In hoofdstuk 5 hebben we onze resultaten omtrent de structuur van de vijfdimensionale Weyl-multipletten besproken. Het blijkt dat er twee versies van het Weyl-multiplet bestaan: het standaard Weyl-multiplet en het dilaton Weyl-multiplet. Multipletten als het standaard Weyl-multiplet komen ook in vier en zes dimensies voor, maar het dilaton Weyl-multiplet was tot nog toe alleen bekend in zes dimensies. We gebruiken een klassieke methode om de transformatieregels voor de verschillende velden af te leiden: de zogenaamde Noether-methode. In het bijzonder hebben we de multipletten van behouden Noether-stromen voor de corresponderende conforme symmetrieën geconstrueerd. Een opmerkelijk detail is dat het stromenmultiplet dat koppelt aan het standaard Weyl-multiplet stromen bevat die aan differentiaalvergelijkingen voldoen, een mechanisme dat tot nu toe alleen in tiendimensionale conforme superzwaartekracht bekend was.

Tot slot hebben we in hoofdstuk 6 onze bevindingen met betrekking tot vijfdimensionale superconforme materiemultipletten gepresenteerd. Zo hebben we zogenaamde vectormultipletten besproken: dit zijn multipletten die transformeren onder een ijkgroep en het bijbehorende ijkdeeltje bevatten. We hebben vectormultipletten geanalyseerd die op de meest algemene manier transformeren onder hun ijkgroep: de zogenaamde vector-tensormultipletten. In het bijzonder hebben we reducibele representaties beschouwd die niet volledig reducibel zijn. Deze geven aanleiding tot nog niet eerder gevonden koppelingen tussen ijkvelden en tensorvelden. De conforme symmetrieën kunnen alleen gerealiseerd worden op tensorvelden als deze voldoen aan bewegingsvergelijkingen. Door de gebruikelijke eis te laten vallen dat zulke vergelijkingen uit een actieprincipe behoren te volgen, hebben we ook vector-tensormultipletten met een oneven aantal tensorvelden kunnen formuleren.

Naast vector-tensormultipletten hebben we in hoofdstuk 6 ook hypermultipletten beschreven. Deze multipletten bevatten geen ijkdeeltjes maar wel scalaire velden. De scalaire velden spannen een vectorruimte over de quaternionen[4] op die door het realiseren van de super-

[4] Een uitbreiding van de complexe getallen.

conforme algebra een niet-triviale meetkunde krijgt, een zogenaamde hypercomplexe meetkunde. Evenals op de tensorvelden kan de superconforme algebra alleen met behulp van bewegingsvergelijkingen op de velden van een hypermultiplet gerealiseerd worden. We hebben ook hier de gevallen beschouwd waarin deze bewegingsvergelijkingen niet volgen uit een actieprincipe. De speciale gevallen waarin er wel een actie is komen overeen met een ruimte van scalaire velden waarop een afstandsfunctie (ook wel metriek genoemd) gedefinieerd is. Hypercomplexe ruimten die een metriek bezitten worden ook wel hyper-Kähler-ruimten genoemd. Daarnaast hebben we ook de wisselwerking van hypermultipletten met de al eerder genoemde vectormultipletten geanalyseerd, waarbij ook hier gebruik is gemaakt van de meetkundige eigenschappen van de ruimte van scalaire velden. Bovendien hebben we een overzicht gegeven van de veelheid aan meetkundige grootheden die in het laatste hoofdstuk gebruikt zijn.

De door ons geconstrueerde materiekoppelingen aan conforme superzwaartekracht kunnen worden gebruikt als een startpunt om nieuwe materiekoppelingen aan niet-conforme superzwaartekracht af te leiden. Aan het einde van hoofdstuk 6 hebben we geschetst hoe dat in zijn werk gaat. Of de zo te verkrijgen nieuwe versies van vijfdimensionale superzwaartekracht ook daadwerkelijk zodanig gewijzigde scalaire potentialen bezitten dat supersymmetrische braanoplossingen gevonden kunnen worden, blijft een vraag die door toekomstig onderzoek beantwoord zal moeten worden.

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