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Analytical solution to DGLAP integro-differential equation via complex maps in domains of contour integrals

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Abstract

A simple model for QCD dynamics in which the DGLAP integro-differential equation may be solved analytically has been considered in our previous papers arXiv:1611.08787 [hep-ph] and arXiv:1906.07924 [hep-ph]. When such a model contains only one term in the splitting function of the dominant parton distribution, then Bessel function appears to be the solution to this simplified DGLAP equation. To our knowledge, this model with only one term in the splitting function for the first time has been proposed by Blümlein in arXiv:hep-ph/9506403. In arXiv:1906.07924 [hep-ph] we have shown that a dual integro-differential equation obtained from the DGLAP equation by a complex map in the plane of the Mellin moment in this model may be considered as the BFKL equation. Then, in arXiv:1906.07924 we have applied a complex diffeomorphism to obtain a standard integral from Gradshteyn and Ryzhik tables starting from the contour integral for parton distribution functions that is usually taken by calculus of residues. This standard integral from these tables appears to be the Laplace transformation of Jacobian for this complex diffeomorphism. Here we write up all the formulae behind this trick in detail and find out certain important points for further development of this strategy. We verify that the inverse Laplace transformation of the Laplace image of the Bessel function may be represented in a form of Barnes contour integral.

1. Introduction

It often happens that a solution to an integro-differential equation is obtained in a form of contour integrals in one or more complex planes. Such integrals may be taken via Cauchy integral formula by calculating residues. Usually they are not the classical Barnes integrals which are a convenient representation for generalized hypergeometric functions. The integrands of the Barnes integrals are ratios of certain products of several Euler Gamma functions. A transformation of a contour integral representing a solution to an integro-differential equation to this form of the Barnes integrals would be helpful because they may be classified in terms of suitable special functions. To find such a transformation is the aim of this paper. The intermediate step will be a representation of these integrals as the Laplace transformations of the Jacobians of some complex maps in the complex variable plane of the contour integral. We have considered such a possibility in the previous paper [1] in which we transformed the contour integral representing the solution to the DGLAP integro-differential equation in a simple model of QCD dynamics from this obtained form of a contour integral in the complex plane of the Mellin moment to the Laplace transformation of the corresponding Jacobian. These Jacobians may appear to be multivalued functions of new complex variables and integration over cuts may be required. We may avoid integration over these cuts if represent the integrals with multivalued Jacobians obtained by the first complex map in the form of the Barnes integrals by applying one complex map more. The holomorphic maps of variables in the complex domains of the contour integrals which we apply in the present paper are based on the standard theory of complex variable which may be found in any textbook on this subject [2]. As to manipulations with closed contours in the complex plane of the integration variable, they were already used in

quantum field theory when integral transformations are involved in calculations [3]. We rectify or curve the integration contours too when it is necessary here.

The contour integral we consider in this paper is a solution to the DGLAP integro-differential equation in a simple model of QCD dynamics considered in [4]. The DGLAP equation was written in the seventies for the structure functions of proton. They may be measured experimentally in the deep inelastic scattering processes [5]. In [6–8] Gribov and Lipatov studied these processes in QED and found that these structure functions satisfy certain integro-differential equations. The discovery of QCD has been marked by the Nobel prize paper [9] in which the renormalization group equations for the Mellin moments of the coefficient functions of Wilson operator product expansion for the matrix element of two currents of the deep inelastic scattering process have been obtained. Then, in [10] Altarelli and Parisi wrote these renormalization group equations for the coefficient functions of the operator product expansion in an integro-differential form in the space of Bjorken x and interpreted them as integro-differential equations for the parton distribution functions. In [11] Dokshitzer developed to the QCD case the Gribov and Lipatov approach used in QED and wrote integro-differential equations similar to Altarelli-Parisi equations. These integro-differential equations became known as Dokshitzer-Gribov-Lipatov-Altarelli-Parisi equation (also known as DGLAP equation).

The splitting functions are the input in the DGLAP equation. They may be found from anomalous dimensions of operators in QCD [12, 13] and are some combinations of several terms [12, 13]. Residue calculus via the Cauchy integral formula for the contour integral in the complex plane of the Mellin moment which represents a solution to the DGLAP equation is straightforward but this calculus is not simple in the real world because many infinite sums are involved in the result and these sums should be classified [14, 15]. However, a simple model with only one term in the splitting function may be considered and the DGLAP equation in this case may be solved in terms of the Bessel function. To our knowledge, this simplification to one Bessel function of the solution to the DGLAP equation in the case of only one term in the splitting function for the first time has been mentioned in 1995 in [16]. We have considered this model in detail in our papers [4] and [1]. In this paper we transform this contour integral of the simple model of [1, 4, 16] via a complex map to the form of the Laplace transformation of the Jacobian of the corresponding complex diffeomorphism and then to the form of the Barnes integrals via another complex map. As the result the contour integral in the plane of the Mellin moment transforms to another contour integral in some complex domain. The integrand after these consequent complex maps transforms to a ratio of certain Gamma functions which is a typical form of the Barnes integrals.

We should defend usefulness of the proposed strategy for the DGLAP community because there is already a long history of many achievements related to this equation. Indeed, the DGLAP integro-differential equation may be converted to a first-order differential equation by taking the Mellin moment of both the sides of the DGLAP equation with respect to Bjorken variable x [10]. The resulting differential equation is the renormalization group equation for the Mellin moments [10] with respect to the scale of momentum transfer in the process of deep inelastic scattering. These differential equations may be combined with the renormalization group equation for the running coupling and solved. In the late nineties in [17, 18] the evolution operator in the case of the running coupling has been constructed and fully analytical solutions of the non-singlet and singlet evolution equations at the next-to-next to leading order with small x resummations included were found. The recent developments of the solution for these first order differential equations may be found in [19]. At the NNLO Mellin space solutions with the running coupling have been worked out in several numerical codes, for example [20], and various later numerical software packages may be found in the citations of [20]. When these first order differential equations for the Mellin moments are solved, the usual way is to convert these moments back to the Bjorken x-space by making the inverse Mellin transformation which may be performed by evaluation of residues on the complex plane of the Mellin moment [3, 21, 22]. At the lowest order in the running coupling the calculation may be done analytically, however even at this level a lot of work is required when the real QCD case is considered instead of simple models. There are different software packages available to do all these steps analytically, at least at the leading order. At higher orders new advanced analytical software tools exist. They are based on using concepts from algebraic geometry like a shuffle product [14, 15]. Shuffle product is used in the construction of single-valued harmonic polylogarithms. Harmonic polylogarithms are described in [23]. There are numerical packages which solve integro-differential equations as they are written without taking the Mellin moments, solving first order differential equations and then transforming the moments back to x space. For example, a numerical software package 'PartonEvolution' has been developed in [24]. Another numerical package QCDNUM has been created later [25, 26].

In the shadow of all these achievements cited in the previous paragraphs our approach is an attempt to look differently at the contour integrals arising in solutions to the DGLAP equation. Here we propose an alternative way in which evaluation of the inverse Mellin transformation reduces to calculation of the inverse Laplace transformation of the Jacobian of the corresponding complex map. These diffeomorphisms may be performed in the complex plane of the Mellin moment. These complex maps make the structure of the integrands uniform reducing it in many of the cases to the standard tables like [27]. Then, we may convert them by one more

transformation to the Barnes integrals. These would allow their systematic classification in the terms of generalized hypergeometric functions. Any systematic classification is useful in construction of computer algorithms.

Due to the significant computational progress of the last decades (see for example [12, 13, 25, 26]) the perturbative solution to the DGLAP equation is already computed up to N^{2}LO for the Mellin moments of parton distribution functions with full inclusion of running coupling and then the corresponding particle distribution functions (quark, gluon and some other combinations) were obtained numerically. However, approximate solutions to the DGLAP equation corresponding to simple models still have a practical value because they capture in a resumed way (in the sense of a compact expression) the behaviour of a given asymptotic regime. In particular, in the present paper it captures the Bessel-like behaviour with respect to square root of the product of logarithm on the Bjorken variable and logarithm of the momentum transfer in the region of the small values of x when the main contribution comes from the gluon part of the matrix DGLAP equation. On the other side, the approximate solutions should not be discarded because they serve as a consistency check for the current manipulations and formulas which lead us to the known obtained results. Another reason in favor of viability of such approximate solutions is that in the low momentum transfer regime of the DGLAP equations the numerical solutions just start to show bad behaviour and one may at least make some estimations in such limits by using these solutions and then obtain novel relations and interpretations.

In the next section we consider the necessary formulas which may be found in the Gradshteyn and Ryzhik tables [27]. All the necessary formulas are related by integral transformations which are given explicitly. We pay some attention to the relation between the Bessel function and the confluent function, and to different integral representations of the generalized hypergeometric functions. In section 3 we convert the contour integral solution to the DGLAP equation to the Laplace transformation of the complex Jacobian. The Jacobian corresponds to a complex map selected for a given transformation. Finally, in section 4 we make a transparent trick with help of which we re-write the Laplace transformation of the complex Jacobians in a form of the Barnes integrals.

2. Preliminary

The only purpose of this section is to collect together from [27] all the formulae necessary for use in the next section. These formulae are not new, each of them is at least one hundred fifty years old, the same book of integral tables [27] is quite old too. However, all the formulas that we have taken from [27] may be related by integral transformations from each one to another. We do all these transformations explicitly in this section but it is probable that we are not first who publish these intermediate steps taking into account the age of [27].

We start in section 2.1 with the confluent function $_1F_1$. It is a particular case of generalized hypergeometric functions and may be written in terms of the Barnes integral. We start from this Barnes integral representation for $_1F_1$ and obtain the corresponding series, then we obtain another integral representation for $_1F_1$, re-write this second integral representation for $_1F_1$ in terms of the same series again and finally prove one useful relation in terms of the same integral representation for $_1F_1$. In section 2.2 two different integral representations of the Bessel function I_0 have been considered. Then, we show by a change of an integration variable that they are equivalent. The Bessel function I_0 is represented in terms of the confluent function $_1F_1$. In section 2.3 we consider the Barnes integral representation for the Gauss hypergeometric function $_2F_1$ and take this integral in terms of a series. Another integral representation is obtained for this Gauss hypergeometric function later. With help of this Euler integral representation for the Gauss hypergeometric function $_2F_1$ and with help of the established in section 2.2 integral representation for the confluent function $_1F_1$ we reproduce the Laplace transform of the Bessel function I_0 in section 2.4.

2.1. Integral representations of hypergeometric function $_1F_1$

This subsection is dedicated to the generalized hypergeometric function $_1F_1(a, c, x)$. Sometimes this function is called a confluent function. As a starting point to work with a hypergeometric function we use the Barnes integral representation for it[3]

$$_1F_1(a, c, x) = \frac{\Gamma(c)}{\Gamma(a)} \oint_C dz \, \frac{\Gamma(a + z)\Gamma(-z)}{\Gamma(c + z)} (-x)^z,$$

which is basically a contour integral in the complex plane z. The contour contains the vertical line which passes a bit to the left of the imaginary axis and should be closed to the right complex infinity in order to guarantee the vanishing of the contribution of the contour C at the complex infinity. Due to this vanishing the series which

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[3] We omit the factor $1/2\pi i$ in front of each contour integral in the complex plane. The inverse factor is generated with the residues according to Cauchy integral formula.

appears due to application of the Cauchy integral formula will be convergent and turns out to be a traditional representation of the confluent function

$$
{}_1F_1(a, c, x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{x^k}{k!},
\tag{1}
$$

where the Pochhammer symbol $(a)_k = \Gamma(a + k)/\Gamma(a)$. Equation (1) is formula **9.210.1** of [27]. In such a case all the residues come form the Gamma function with the negative sign of its argument and we have the series (1) above. The Barnes integrals are the contour integral representation of the generalized hypergeometric functions, however it is a convenient but not a unique integral representation of the hypergeometric functions ${}_qF_p$. There are several integral representations more. For example, in the case of ${}_1F_1(a, c, x)$ we may write

$$
\begin{aligned}
{}_1F_1(a, c, x) &= \frac{\Gamma(c)}{\Gamma(a)} \oint_C dz \, \frac{\Gamma(a + z)\Gamma(-z)}{\Gamma(c + z)}(-x)^z = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \oint_C dz \, \mathrm{B}(a + z, c - a)\Gamma(-z)(-x)^z \\
&= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \oint_C dz \int_0^1 d\tau \, \tau^{a+z-1}(1 - \tau)^{c-a-1}\Gamma(-z)(-x)^z \\
&= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_0^1 d\tau \, \tau^{a-1}(1 - \tau)^{c-a-1} \oint_C dz \Gamma(-z)(-x\tau)^z \\
&= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_0^1 d\tau \, \tau^{a-1}(1 - \tau)^{c-a-1}e^{x\tau},
\end{aligned}
$$

and we may recognize formula **9.211.2** of [27] for the confluent function. Of course, this representation turns out to be the series (1) again,

$$
\begin{aligned}
{}_1F_1(a, c, x) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_0^1 d\tau \, \tau^{a-1}(1 - \tau)^{c-a-1}e^{x\tau} \\
&= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \sum_{k=0}^{\infty} \frac{x^k}{k!} \int_0^1 d\tau \, \tau^{a+k-1}(1 - \tau)^{c-a-1} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \sum_{k=0}^{\infty} \frac{x^k}{k!}\mathrm{B}(a + k, c - a) \\
&= \frac{\Gamma(c)}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{x^k}{k!}\frac{\Gamma(a + k)}{\Gamma(c + k)} = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k}\frac{x^k}{k!}.
\end{aligned}
$$

This integral representation may be useful in calculation. For example, the following property may be proven

$$
\begin{aligned}
{}_1F_1(a, c, x) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_0^1 d\tau \, \tau^{a-1}(1 - \tau)^{c-a-1}e^{x\tau} \\
&= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_0^1 d\tau \, (1 - \tau)^{a-1}\tau^{c-a-1}e^{x(1-\tau)} = e^x {}_1F_1(c - a, c, -x).
\end{aligned}
\tag{2}
$$

2.2. From the Bessel function I_0 to the hypergeometric function ${}_1F_1$

The traditional integral representation for the Bessel function I_0 turns out to be the well-known series in terms of the even powers of its argument,

$$
\begin{aligned}
I_0(x) &= \frac{1}{\Gamma^2(1/2)} \int_{-1}^1 d\tau \, (1 - \tau^2)^{-\frac{1}{2}}e^{x\tau} = \frac{1}{\Gamma^2(1/2)} \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \int_{-1}^1 d\tau \, (1 - \tau^2)^{-\frac{1}{2}}\tau^{2k} \\
&= \frac{2}{\Gamma^2(1/2)} \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \int_0^1 d\tau \, (1 - \tau^2)^{-\frac{1}{2}}\tau^{2k} = \frac{1}{\Gamma^2(1/2)} \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} \int_0^1 dt \, (1 - t)^{-\frac{1}{2}}t^{k-\frac{1}{2}} \\
&= \frac{1}{\Gamma^2(1/2)} \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}\mathrm{B}\left(\frac{1}{2}, k + \frac{1}{2}\right) = \frac{1}{\Gamma(1/2)} \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!k!}\Gamma\left(k + \frac{1}{2}\right) \\
&= \frac{1}{\Gamma(1/2)} \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!k!}\frac{(2k)!}{4^k k!}\Gamma\left(\frac{1}{2}\right) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(k!)^2 4^k} = \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\frac{x^2}{4}\right)^k.
\end{aligned}
$$

This integral representation is **8.431.1** of [27]. This integral representation for the Bessel function I_0 is related to another integral representation which is **9.211.2** of [27] by a change of the integration variable,

$$I_0(x) = \frac{1}{\Gamma^2(1/2)} \int_{-1}^{1} d\tau \ (1 - \tau^2)^{-\frac{1}{2}} e^{x\tau} = \frac{2}{\Gamma^2(1/2)} \int_{0}^{1} dt \ (1 - (1 - 2t)^2)^{-\frac{1}{2}} e^{x(1-2t)}$$

$$= \frac{e^x}{\Gamma^2(1/2)} \int_{0}^{1} dt \ (t(1-t))^{-\frac{1}{2}} e^{-2tx} = e^x {}_1F_1\left(\frac{1}{2}, 1, -2x\right) = e^{-x} {}_1F_1\left(\frac{1}{2}, 1, 2x\right)$$

$$= \frac{e^{-x}}{\Gamma^2(1/2)} \int_{0}^{1} dt \ t^{-\frac{1}{2}} (1-t)^{-\frac{1}{2}} e^{2tx}. \tag{3}$$

This integral representation is **9.238.2** of [27]. Thus, the integral representation (3) allows us to re-write the Bessel function I_0 in terms of the confluent function ${}_1F_1$ and it will be more useful for us in section 2.4 where we calculate the Laplace transform of the Bessel function I_0.

2.3. Integral representations of hypergeometric function ${}_2F_1$

In order to study the Gauss hypergeometric function ${}_2F_1$ we again use as the starting point the representation in terms of the Barnes contour integral, which may be re-written in a form of the well-known series for the Gauss hypergeometric function, equations **9.100** and of **9.113** [27].

$$_2F_1(a, b; c, x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \oint_C dz \ \frac{\Gamma(a+z)\Gamma(b+z)\Gamma(-z)}{\Gamma(c+z)} (-x)^z = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},$$

Again, we have to take this contour containing the vertical line situated a bit to the left from the imaginary axis and close it in a such way that the complex infinity does not contribute. Where to close it depends on the absolute value of the variable x. We suppose that $0 < x < 1$ and in this range we have to close it to the right hand side. The residues come from the Gamma function with the negative sign of its argument and we obtain the series above.

The Barnes integrals are the contour integral representation of the hypergeometric functions, it is not a unique integral representation of the hypergeometric functions ${}_qF_p$, there are other integral representations. In the case of ${}_2F_1(a, b; c, x)$ we may transform

$$_2F_1(a, b; c, x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \oint_C dz \ \frac{\Gamma(a+z)\Gamma(b+z)\Gamma(-z)}{\Gamma(c+z)} (-x)^z$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-b)} \oint_C dz \ B(b+z, c-b)\Gamma(a+z)\Gamma(-z)(-x)^z$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-b)} \oint_C dz \int_{0}^{1} d\tau \ \tau^{b+z-1}(1-\tau)^{c-b-1}\Gamma(a+z)\Gamma(-z)(-x)^z$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-b)} \int_{0}^{1} d\tau \ \tau^{b-1}(1-\tau)^{c-b-1} \oint_C dz \Gamma(a+z)\Gamma(-z)(-x\tau)^z$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_{0}^{1} d\tau \ \tau^{b-1}(1-\tau)^{c-b-1}(1-x\tau)^{-a}. \tag{4}$$

This is formula **9.111** of [27] for ${}_2F_1$. In the next section we will use the integral representation (4) of the Gauss hypergeometric function in order to calculate the Laplace transform of the Bessel function. In complete analogy to equation (2) we may prove identity **9.131** of [27]

$$_2F_1(a, b; c, x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_{0}^{1} d\tau \ \tau^{b-1}(1-\tau)^{c-b-1}(1-x\tau)^{-a}$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_{0}^{1} d\tau \ (1-\tau)^{b-1}\tau^{c-b-1}(1-x+x\tau)^{-a}$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} (1-x)^{-a} \int_{0}^{1} d\tau \ \tau^{c-b-1}(1-\tau)^{b-1}\left(1 - \frac{x}{x-1}\tau\right)^{-a}$$

$$= (1-x)^{-a} {}_2F_1\left(a, c-b; c, \frac{x}{x-1}\right). \tag{5}$$

2.4. The Laplace transform of the Bessel function I_0

In this section we reproduce a result for the Laplace transform of the Bessel function I_0. The result is written in **17.13.109** of [27]. When we do this transformation, we suppose that z is in the corresponding domain of the complex plane, that is, on the right hand side of the critical exponent of the Bessel function [4].

$$\int_0^\infty I_0(x)e^{-xz}dx = \frac{1}{\Gamma^2(1/2)} \int_0^1 dt \ t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}} \int_0^\infty e^{-x}e^{2tx}e^{-xz}dx$$

$$= \frac{1}{\Gamma^2(1/2)} \int_0^1 dt \ t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\frac{1}{1-2t+z}$$

$$=$$

$$\frac{1}{z+1}\frac{1}{\Gamma^2(1/2)}\int_0^1 dt \ t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}\left(1-\frac{2}{z+1}t\right)^{-1}$$

$$= \frac{1}{z+1}{}_2F_1\left(1,\frac{1}{2};1,\frac{2}{z+1}\right) = \frac{1}{z+1}\frac{1}{\Gamma(1/2)}\oint_C dz \ \frac{\Gamma(1+z)\Gamma\left(\frac{1}{2}+z\right)\Gamma(-z)}{\Gamma(1+z)}\left(-\frac{2}{z+1}\right)^z$$

$$= \frac{1}{z+1}\frac{1}{\Gamma(1/2)}\oint_C dz \ \Gamma\left(\frac{1}{2}+z\right)\Gamma(-z)\left(-\frac{2}{z+1}\right)^z$$

$$= \frac{1}{z+1}\left(1-\frac{2}{z+1}\right)^{-\frac{1}{2}} = \frac{1}{z+1}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{z^2-1}} \tag{6}$$

This inverse square root is very known result for the Laplace transform of the Bessel function I_0 and may be found in many tables of integrals.

3. The DGLAP contour integral solution as Laplace transform of the complex Jacobian

This is the main section. Everything we have written in the previous sections was a preparation for this section. Here we calculate the contour integrals of this type

$$\phi(x, u) = \int_{-1+\delta-i\infty}^{-1+\delta+i\infty} dN \frac{x^{-N}}{N+1}u^{1/(N+1)} \tag{7}$$

by making complex maps in the plane of the Mellin moment. Here $x \in [0, 1]$ and $u \in [0, \infty[$ are external variables. This contour integral represents solution to the DGLAP integro-differential equation which plays an important role in Quantum Chromodynamics. We commented on this equation in the Introduction. The traditional way is to calculate this contour integral directly by evaluating residues according to the Cauchy integral formula,

$$\phi(x, u) = \int_{-1+\delta-i\infty}^{-1+\delta+i\infty} dN \frac{x^{-N}}{N+1}u^{1/(N+1)} = x\sum_{k=0}^\infty \frac{1}{k!}(\ln u)^k \int_{-1+\delta-i\infty}^{-1+\delta+i\infty} dN \frac{x^{-N-1}}{(N+1)^{k+1}}$$

$$= x\sum_{k=0}^\infty \frac{1}{k!}(\ln u)^k \frac{(-\ln x)^k}{k!} = xI_0\left(2\sqrt{\ln u \ln\frac{1}{x}}\right).$$

In this example this is the shortest way of getting the result for this integral. However, we would like to reproduce this result by using complex geometry in order to make a map in the complex plane of the Mellin moment. For this purpose we re-write the previous integral

$$\phi(x, u) = \int_{-1+\delta-i\infty}^{-1+\delta+i\infty} dN \frac{x^{-N}}{N+1}u^{1/(N+1)} = \int_{-1+\delta-i\infty}^{-1+\delta+i\infty} dN \frac{1}{N+1}e^{-N\ln x + \ln u/(N+1)} \tag{8}$$

and choose a new complex variable M of integration in such a way that

$$M\sqrt{\ln u \ln\frac{1}{x}} = -N\ln x + \frac{\ln u}{N+1}. \tag{9}$$

This relation defines M as a function of the initial complex variable N [1]. Introducing for the brevity a notation

$$w \equiv \sqrt{\frac{\ln u}{\ln\frac{1}{x}}},$$

we may write

$$M = \frac{N}{w} + \frac{w}{N+1},$$

from which it follows that in the inverse mapping from M to N the initial variable N must satisfy the quadratic equation

$$N^2 + (1-wM)N + (w^2 - Mw) = 0,$$

which has two roots

$$N^{(+)} = \frac{Mw - 1 + \sqrt{(Mw + 1)^2 - 4w^2}}{2}, \quad N^{(-)} = \frac{Mw - 1 - \sqrt{(Mw + 1)^2 - 4w^2}}{2}. \tag{10}$$

If we do not want to change the contour orientation while mapping from N to M according to equation (9) we should use a positive branch of the map (10). Thus, we have chosen the map

$$N(M) = N^{(+)}(M) = \frac{Mw - 1 + \sqrt{(Mw + 1)^2 - 4w^2}}{2}. \tag{11}$$

We could choose the negative branch too, but in this case we should change the sign of the integral because the contour orientation would be changed under this map.

Having the complex map chosen in equation (11) we may re-write the integrand in terms of the new complex variable M. From equation (9) we have

$$dM = \left(\frac{1}{w} - \frac{w}{(N+1)^2} \right) dN \quad \Rightarrow \quad \frac{dN}{dM} \frac{1}{N+1} = \left(\frac{N+1}{w} - \frac{w}{N+1} \right)^{-1} = \frac{1}{\sqrt{(M + 1/w)^2 - 4}}.$$

Now we may proceed the line (8) in terms of the new integration variable C

$$
\begin{aligned}
\phi(x, u) &= \int_{-1+\delta-i\infty}^{-1+\delta+i\infty} dN \frac{x^{-N}}{N+1} u^{1/(N+1)} = \int_{-1+\delta-i\infty}^{-1+\delta+i\infty} dN \frac{1}{N+1} e^{-N \ln x + \ln u/(N+1)} \\
&= \oint_C dM \frac{e^{M\sqrt{\ln u \ln \frac{1}{x}}}}{\sqrt{(M + 1/w)^2 - 4}} = e^{-\sqrt{\ln u \ln \frac{1}{x}}/w} \oint_{C'} dM \frac{e^{M\sqrt{\ln u \ln \frac{1}{x}}}}{\sqrt{M^2 - 4}} \\
&= e^{-\ln \frac{1}{x}} \oint_{C''} dM \frac{e^{2M\sqrt{\ln u \ln \frac{1}{x}}}}{\sqrt{M^2 - 1}} = x I_0 \left(2 \sqrt{\ln u \ln \frac{1}{x}} \right). \tag{12}
\end{aligned}
$$

Under this sequence of the maps, the contour in the complex plane changes its shape from the vertical line parallel to the imaginary axis to a very complicated form C'' at the end of this chain of transformations. We wrote the last equality because we know that C'' may be transformed to the vertical line parallel to the imaginary axis in the complex plane M. In such a way we may write the last equality basing on the Laplace transform of the Bessel function found in equation (6). The direct proof of the last equality will be given in the next section by explicit calculation of the integral.

In general, our purpose is to obtain the result for the contour integrals that represent solution to DGLAP integro-differential equation by making complex diffeomorphism in the plane of the Mellin moment N related to the Bjorken variable x. Why do we do complex diffeomorphisms and represent the contour integral of this type (7) as a Laplace transformation of a Jacobian? We find this way more systematic in order to classify the obtained results in terms of special functions. One reason is that standard table integrals may appear as it happened in the case of equation (12). Another reason is that the Laplace transformation may be represented in a form of the Barnes integrals (we consider this second reason in detail in the next section). These two reasons are fundamental in construction of computer algorithm [1] in the real world case of QCD in which the functions of anomalous dimension are highly complicated and contain many terms At the three loop level these functions can be found in [12, 13].

The suggested solution to the DGLAP equation given in equations (7) and (12) corresponds to the kinematic region of small x in which this approximate solution makes a sense. Strictly speaking, we need to have as the result a function singular at $x = 0$. This corresponds to the singularity at a point $N = 1$ in the complex N plane. The terms in the matrix of anomalous dimensions singular at this point correspond to the dominance of the gluon distribution function in the small x region. The matrix form may be reduced to the DGLAP equation for the dominant parton distribution in the region of the small x. This would be a quite good approximation to the matrix DGLAP integro-differential equation. To have such a behaviour we would do a small modification and take another model for the anomalous dimension $\gamma(N)$ which is singular at the point $N = 1$. In general, the kinematic region of the DGLAP equation is given by the Bjorken limit in which the ratio of the momentum transfer to the Bjorken x is large. By increasing or decreasing the momentum transfer u one has to increase or decrease the $\phi(x, u)$ distribution by a double logarithm law found in equations (7) and (12) as a result of the pure gluonic DGLAP dynamics.

As we have mentioned in the Introduction, the practical value of this simple model is that its solution to the DGLAP equation may be used to estimate qualitatively the behaviour of the dominant distribution function nearby $x = 0$, that is for the very small x. However, this model may appear to be even more useful than a simple approximation for the dominant distribution or a consistency check for the numerical or analytical calculation based on a powerful software are. As it is known, there are several groups in the world which make global analysis of the parton distribution functions taking into account recent data from the LHC [28–32]. These analysis allows

them to fix many parameters of initial parton distribution functions from data only because they cannot be computed from first principles. Despite such a fitting procedure is rather arbitrary on how to choose the form of parton distribution function as well as number of free parameters, they tend to be some combination of Euler beta functions, which parametrize the parton distribution at some scale [33–36] and than evolve according to the DGLAP integro-differential equation. Due to big amount of data the software for the fitting of the PDF parameters and for the PDF evolution is created on the principles of neural networks [30, 31]. This simple model may be used to train the neural networks[4]. We also note that it is not a unique simple model which may be used for this purpose. We may create several simple models that solve the DGLAP integro-differential equation with different splitting functions by adjusting the shape functions at some given scale in order to combine them with Jacobians of complex diffeomorphisms in the plane of the Mellin moment variable in easily integrable factors.

4. From inverse Laplace transformations of Jacobians to the Barnes integrals

In the previous section we have done complex diffeomorphisms in order to represent the initial form (7) of the solution to the DGLAP equation in terms of contour integral to the Jacobian form (12) of this solution. This Jacobian form may be a standard integral from the well-known integral tables [27], for example the Bessel function I_0 in the case of our model. We have proved that this is Bessel function by the direct Laplace transformation in section 2.4. Now we evaluate the inverse Laplace transformation,

$$\oint_{C''} dz \frac{e^{zx}}{\sqrt{z^2 - 1}} = \frac{1}{(-1)^{1/2}} \frac{1}{\Gamma(1/2)} \oint_{C''} dz \; e^{zx} \oint_C du \; \Gamma\left(u + \frac{1}{2}\right)\Gamma(-u)(-z^2)^u. \tag{13}$$

At this moment we change the order of integration and we have

$$\int_0^\infty x^{-2u-1}e^{-xz}dx = z^{2u}\Gamma(-2u) \quad \Rightarrow \quad \oint_{C''} dz \; e^{zx}z^{2u} = \frac{x^{-2u-1}}{\Gamma(-2u)} \tag{14}$$

In such a way we may say that we have Hankel contour C''. Now we may proceed line (13)

$$\oint_{C''} dz \frac{e^{zx}}{\sqrt{z^2 - 1}} = \frac{1}{(-1)^{1/2}} \frac{1}{\Gamma(1/2)} \oint_C du \; \Gamma\left(u + \frac{1}{2}\right)\Gamma(-u)\frac{x^{-2u-1}(-1)^u}{\Gamma(-2u)}$$

$$= \frac{1}{(-1)^{1/2}} \frac{1}{\Gamma(1/2)} \oint_C du \; \Gamma\left(u + \frac{1}{2}\right)\Gamma(-u)\frac{x^{-2u-1}(-1)^u 2^{1+2u}\Gamma(1/2)}{\Gamma(-u)\Gamma(-u + 1/2)}$$

$$= \frac{1}{(-1)^{1/2}} \frac{1}{\Gamma(1/2)} \oint_C du \; \Gamma\left(u + \frac{1}{2}\right)\Gamma(-u)\frac{x^{-2u-1}(-1)^u 2^{1+2u}\Gamma(1/2)}{\Gamma(-u)\Gamma(-u + 1/2)}$$

$$= \frac{1}{(-1)^{1/2}}\left(\frac{2}{x}\right) \oint_C du \; \frac{\Gamma(u + 1/2)}{\Gamma(-u + 1/2)}\left(-\frac{4}{x^2}\right)^u = \left(\frac{2}{x}\right)^2 \oint_{C_1} du \; \frac{\Gamma(u + 1)}{\Gamma(-u)}\left(-\frac{4}{x^2}\right)^u$$

$$= \left(\frac{2}{x}\right)^2 \oint_{C_2} du \; \frac{\Gamma(1 - u)}{\Gamma(u)}\left(-\frac{x^2}{4}\right)^u = \left(\frac{2}{x}\right)^2 \sum_{k=1}^\infty \frac{(-1)^k}{((k-1)!)^2}\left(-\frac{x^2}{4}\right)^k$$

$$= \sum_{k=0}^\infty \frac{1}{(k!)^2}\left(\frac{x^2}{4}\right)^k = I_0(x).$$

By comparing this equation and equation (12) we may write this identity

$$\oint_{C''} dM \frac{e^{Mx}}{\sqrt{M^2 - 1}} = \left(\frac{2}{x}\right)^2 \oint_{C_2} du \; \frac{\Gamma(1 - u)}{\Gamma(u)}\left(-\frac{x^2}{4}\right)^u,$$

in which on the left hand side we have the Jacobian form (12) of the contour integral (7) and on the right hand side we have the Barnes integral. We may compare these two different forms and observe that the Barnes integral form is more useful in order to classify the result in terms of the special functions. Roughly speaking, in order to write the Bessel function on the left hand side it is better to keep Gradshteyn and Ryzhik tables in hands and on the right hand side we do not need them. The second reason to represent the initial contour integral (7) in terms of the Barnes integrals is to have a uniform well-studied representation for all the contour integrals involved in the calculation. The third reason is that the Jacobian form on the left hand side will frequently contain multivalued functions, in our case it is a square root in the denominator. We avoid the integration over cuts which is necessary in the complex plane of the integration variable M in the Jacobian form. We do not have any cuts in the Barnes integral representation on the right hand side. The fourth reason to prefer the result written in the form of the Barnes integrals is that the Jacobians will be more and more complicated in higher orders while the contour integrals of ratios of some products of several Gamma functions are well-studied, their structure is

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[4] IK is grateful to Andrei Arbuzov for attracting attention to this point.

more uniform and better analyzable. Ratios of Gamma functions in the integrands of the contour integrals appear frequently in quantum field theory calculation [37–44].

5. Conclusion

In the previous paper [1] we have communicated that the BFKL equation [45–49] may be obtained from the DGLAP equation via a complex map in the domain of the contour integral (7) that represents the solution to this DGLAP integro-differential equation. This is why the BFKL integro-differential equation may be considered as a dual equation to the DGLAP integro-differential equation. The question of duality between the BFKL equation and the DGLAP equation has been arisen in [21]. Also, in [1] we have proposed that a complex map in the plane of the Mellin moment may be used to transform the solution to the DGLAP integro-differential equation obtained in the form of contour integral of the type (7) to the form which represents the Laplacian transformation of the Jacobian of this complex map. In the present paper we have shown that the Jacobian form of the contour integral may be further transformed to the Barnes contour integrals. The Barnes integrals are contour integrals in which the integrand is a ratio of products of several Gamma functions and the result of their integration may be written in terms of special functions in a traditional well-studied way. The representation in terms of the Barnes integrals is useful in construction of computational algorithms [1].

Acknowledgments

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