symmetry

MDPI

Article

Finding Solutions to the Yang–Baxter-like Matrix Equation for Diagonalizable Coefficient Matrix

Dongmei Chen [1,*] and Xuerong Yong [2]

[1] College of Mathematics and Statistics, Shenzhen University, Shenzhen 518060, China
[2] Department of Mathematical Sciences, The University of Puerto Rico Mayaguez, Mayagüe, PR 00681, USA; xuerong.yong@upr.edu
* Correspondence: cddmmm@szu.edu.cn

Abstract: Let A be a diagonalizable complex matrix. In this paper, we discuss finding solutions to the Yang–Baxter-like matrix equation $AXA = XAX$. We then present a concrete example to illustrate the validity of the results obtained.

Keywords: Yang–Baxter-like matrix equation; diagonalizable matrix; Jordan blocks

MSC: 15A24; 15A18; 15A30

1. Introduction

In past decades, much effort has been put into solving the following Yang–Baxter-like matrix equation

$$AXA = XAX, \tag{1}$$

where A is a given square matrix, and X is the unknown complex matrix. Recent representative articles relevant to this problem can be referenced from [1–22]. Regarding the background, this equation has also been called a star–triangle-like equation in statistical mechanics for many years, and, to the best of our knowledge, it originates from the classical Yang–Baxter equation, which was proposed first by C.N. Yang in December 1967, in his article on simple one-dimensional multi-body problems [21] and, subsequently, in 1972, by R.J. Baxte, who independently discussed this equation when studying some classical two-dimensional statistical mechanics problems [23]. Since then, the Yang–Baxter equation has been transformed into the so-called (simpler) Yang–Baxter-like matrix equation through the appropriate imposition of some restrictions. The Yang–Baxter-like matrix equation is actually the nonparametric form of the classical Yang–Baxter equation in matrix theory. Moreover, in terms of the matrix algebra, finding the relations between A and X turns out to be interesting because it provides information about the commutability and idempotency of matrices [24].

The Yang–Baxter-like matrix equation appears to be simple, but it is generally not easy to find all of its solutions because of the nonlinearity of the equation. Even in the case of a lower-order situation, all of the solutions found thus far have been only for very special cases, and only basic and partial answers have been discovered.

A systematic study of the Yang–Baxter-like matrix equation from the perspective of matrix theory has basically started in the last decade. Ding and Rhee first eastablished a series of important results. For example, when A is nonsingular, and its inverse matrix is a stochastic matrix, the existence of the solution is proven by using Brouwer's fixed-point theorem. They also obtained some numerical solutions to the equation by making use of the mean ergodic theorem for matrices and the direct iterative method [6]. We then see that, when using the spectral projection theorem in matrix analysis and the generalized eigen-subspace technique, a finite number of spectral solutions to the equation are constructed

Citation: Chen, D.; Yong, X. Finding Solutions to the Yang–Baxter-like Matrix Equation for Diagonalizable Coefficient Matrix. *Symmetry* **2022**, *14*, 1577. https://doi.org/10.3390/sym14081577

Academic Editor: Dumitru Baleanu

Received: 28 June 2022
Accepted: 25 July 2022
Published: 31 July 2022

in [7]. In the case that the eigenvalues of the coefficient matrix are semi-simple, and their multiplicity is at least two, Ding and Zhang claimed that all spectral solutions to the equation can be constructed [9], and, in [8], they discussed finding solutions that satisfy $AX = XA = X^2$. The above solutions are partial, special or commutative, that is, the solutions X satisfy the commutative condition $AX = XA$. Some recent results on the commutative solutions to the equation can be seen in [10,12,13,17,18], and more general discussion in finding the general non-commutative solutions and all solutions of Yang–Baxter like matrix equation are found in [1,2,4,5,16,19,20,22]. When A is an idempotent matrix, all of its solutions were obtained in [4,16], where the idea is based on the property of the diagonalization of A. When A has rank one or rank two, or if $A^{-1} = A$, $A^3 = A$, all solutions have been obtained (e.g., see [1,2,20,22]). The construction of all solutions corresponding to diagonalizable matrices with a spectral set $\{1, \alpha, 0\}$ is discussed in [5], and, in [19], the general expressions of X are established for the diagonalizable matrix A with two different eigenvalues.

We notice that all previous discussions are under the assumptions that their coefficient matrices A are either diagonalizable matrices whose number of distinct eigenvalues is no more than three or that the matrices possess special kinds of Jordan blocks. For a more general coefficient matrix, an approach to seek for all of its solutions has not been seen. With this motivation, in this paper, we focus on constructing all of the solutions to the equation when A is a diagonalizable matrix, thereby extending the existing results.

2. All Solutions to the Equation $AXA = XAX$ for Any Diagonalizable Matrices A

Let $A, T \in C^{m\times m}$ and $A = TJT^{-1}$, where J is the Jordan canonical form of A, and let $Z = T^{-1}XT$. Then, the matrix equation $AXA = XAX$ is equivalent to $JZJ = ZJZ$, and, if A is diagonalizable, the orders of all Jordan blocks for A are 1.

Recently, in [19], Equation (1) was discussed for the diagonalizable matrix A with two distinct eigenvalues λ_1 and λ_2, that is, $A = TJT^{-1}$, where $J = diag(\lambda_1 I_n, \lambda_2 I_{m-n})$. The main results of [19] are given as follows:

Theorem 1 ([19])**.** *Suppose that $A \in \mathbf{C}^{m\times m}$ is a diagonalizable matrix with two different eigenvalues λ_1 and $\lambda_2 (\lambda_1\lambda_2 \neq 0)$, that is, $A = TJT^{-1}$, in which T is nonsingular, and $J = diag(\lambda_1 I_n, \lambda_2 I_{m-n})$. Then, all solutions of the Yang–Baxter-like matrix equation $AXA = XAX$ can be expressed as $X = T\begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} T^{-1}$:*

(I) when $\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2 = 0$, Y_1 and Y_4 have the forms

$$Y_1 = P\begin{pmatrix} \lambda_1 I_{t_1} & 0 \\ 0 & 0 \end{pmatrix} P^{-1}, Y_4 = Q\begin{pmatrix} \lambda_2 I_{t_2} & 0 \\ 0 & 0 \end{pmatrix} Q^{-1},$$

P,Q are invertible matrices of appropriate size. Y_2 and Y_3 have the forms

$$Y_2 = P\begin{pmatrix} Y_2^{(1)} & 0 \\ 0 & 0 \end{pmatrix} Q^{-1}, Y_3 = Q\begin{pmatrix} Y_3^{(1)} & 0 \\ 0 & 0 \end{pmatrix} P^{-1},$$

in which $0 \leq t_1 \leq n, 0 \leq t_2 \leq m-n$, $Y_2^{(1)}$ is an arbitrary $t_1 \times t_2$ matrix, and $Y_3^{(1)} = (I - (Y_2^{(1)})^\dagger Y_2^{(1)}) W (I - Y_2^{(1)} (Y_2^{(1)})^\dagger)$, W is an arbitrary $t_2 \times t_1$ matrix.

(II) when $\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2 \neq 0$, for suitable invertible matrices P,Q with appropriate sizes and the given integer $r, 0 \leq r \leq min\{n, m-n\}$, Y_1, Y_2 and Y_3, Y_4 have the forms

$$Y_1 = P\begin{pmatrix} \widetilde{\lambda_1} I_r & 0 & 0 \\ 0 & \lambda_1 I_\mu & 0 \\ 0 & 0 & 0 \end{pmatrix} P^{-1}, Y_4 = Q\begin{pmatrix} \widetilde{\lambda_2} I_r & 0 & 0 \\ 0 & \lambda_2 I_\nu & 0 \\ 0 & 0 & 0 \end{pmatrix} Q^{-1},$$

$$Y_2 = P\begin{pmatrix} Y_2^{(1)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q^{-1}, Y_3 = Q\begin{pmatrix} Y_4^{(1)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} P^{-1},$$

in which $0 \le \mu \le n - r, 0 \le \nu \le m - n - r, \widetilde{\lambda_1} = \frac{\lambda_2^2}{\lambda_2 - \lambda_1}, \widetilde{\lambda_2} = \frac{\lambda_1^2}{\lambda_1 - \lambda_2}$.

Furthermore, $Y_2^{(1)}$ *is an arbitrary* $r \times r$ *invertible matrix, and*

$$Y_3^{(1)} = \frac{-\lambda_1\lambda_2(\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2)}{(\lambda_1 - \lambda_2)^2}(Y_2^{(1)})^{-1}.$$

From the result above, it is readily seen that the structure of the solutions is complicated. When we continue to look for the solutions under the condition that A is diagonalizable and has three different nonzero eigenvalues, it can be seen that the method will appear to be invalid. However, for a general diagonalizable matrix A, by extending the idea and technique introduced in our previous work, we find that setting up the general expression of the solutions turns out to be possible.

Lemma 1. *Suppose that* $ZJZ = JZJ$ *and* $J = diag(\lambda_1 I_{n_1}, \lambda_2 I_{n_2}, \cdots, \lambda_t I_{n_t})$, *with* $\lambda_1\lambda_2\cdots\lambda_t \neq 0$, $n_1 + n_2 + \cdots + n_t = m$. *Then,* (I) Z *is diagonalizable;* (II) *any nonzero eigenvalue* σ *of* Z *satisfies* $\sigma \in \{\lambda_1, \lambda_2, \cdots, \lambda_t\}$, *and, if* $\sigma = \lambda_i, i \in \{1, \cdots, t\}$, *the algebraic multiplicity of* σ *is no more than* n_i.

Proof. Let $Rank(Z) = s, 0 \le s \le m$, and let $Z_{p_i}, i = 1, \cdots, s$ be the linearly independent column vectors of Z. From

$$ZJZ_{p_i} = ZJZe_{p_i} = JZJe_{p_i} = \lambda_{q_i} JZ_{p_i},$$

where λ_{q_i} belongs to the set $\{\lambda_1, \lambda_2, \cdots, \lambda_t\}$. We see that $JZ_{p_i}, i = 1, \cdots, s$ are the linearly independent eigenvectors of Z corresponding to the eigenvalues λ_{q_i}, $i = 1, \cdots, s$. If we write the Jordan decomposition of Z as

$$\begin{pmatrix} J_1(\sigma_1) & & & & & & \\ & J_2(\sigma_2) & & & & & \\ & & \ddots & & & & \\ & & & J_j(\sigma_j) & & & \\ & & & & J_{j+1}(0) & & \\ & & & & & \ddots & \\ & & & & & & J_k(0) \end{pmatrix}$$

where $\sigma_i \neq 0, i = 1, \cdots, j$, then Z has at most j linearly independent eigenvectors corresponding to the nonzero eigenvalues, i.e., $s \le j$. On the other hand, $s = rank(Z) \ge j$. Therefore, there must be $s = j$, and every Jordan block of Z has an order of one. Meanwhile, $\{\sigma_i \neq 0, i = 1, \cdots, s\} = \{\lambda_{q_i}, i = 1, \cdots, s\}$, and the algebraic multiplicity of $\sigma_i = \lambda_{q_j}$ is no more than n_{q_j}. □

Theorem 2. *Given* $A \in \mathbf{C}^{l \times l}$, *if* $A = Tdiag(J_m, 0)T^{-1}$ *for some nonsingular matrices,* T *and* $J_m = diag(\lambda_1 I_{n_1}, \lambda_2 I_{n_2}, \cdots, \lambda_t I_{n_t})$, *then the general solution* X *to the Yang–Baxter-like matrix equation* $XAX = AXA$ *is given by*

$$X = T\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix} T^{-1}, \qquad Z_4 \in \mathbf{C}^{(l-m)\times(l-m)} \text{ is arbitrary,}$$

where $Z_1 = P\begin{pmatrix} 0 & 0 \\ 0 & \Lambda \end{pmatrix}P^{-1} \in \mathbf{C}^{m\times m}$ *for a nonsingular matrix* P*,* Λ *is a diagonal matrix.* $Z_2 = 0$*, and* $Z_3 = 0$ *when* Z_1 *is nonsingular. Otherwise,* $Z_2 = J_m^{-1}P\begin{pmatrix} Q \\ 0 \end{pmatrix}$*,* $Z_3 = (W, 0)P^{-1}J_m^{-1}$*, and* $Q \in C^{(m-s)\times(l-m)}$ *is arbitrary.* $W \in C^{(l-m)\times(m-s)}$ *is any matrix satisfying* $W\widetilde{P}Q = 0$ *where* $\widetilde{P}$ *is the* $m-s$ *order leading principle submatrix of* $P^{-1}J_m^{-1}P$*.*

Proof. Suppose Z, $J \in \mathbf{C}^{l\times l}$ satisfies $JZJ = ZJZ$, in which $J = diag(J_m, 0)$ $(m \le l)$. Let Z be partitioned conformally with J as

$$Z = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}, \qquad Z_1 \in \mathbf{C}^{m\times m}, \quad Z_2 \in \mathbf{C}^{m\times(l-m)}. \tag{2}$$

Then, comparing the two sides of the 2×2 block matrix equation

$$\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}\begin{pmatrix} J_m & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix} = \begin{pmatrix} J_m & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}\begin{pmatrix} J_m & 0 \\ 0 & 0 \end{pmatrix}$$

yields the following system of four equations

$$Z_1J_mZ_1 = J_mZ_1J_m, \qquad Z_1J_mZ_2 = 0, \qquad Z_3J_mZ_1 = 0, \qquad Z_3J_mZ_2 = 0. \tag{3}$$

Notice that $Z_4 \in \mathbf{C}^{(l-m)\times(l-m)}$ can be any matrix.

Therefore, if Z_1, Z_2, and Z_3 are solved, then the solutions X to Equation (1) will be given by the equation $X = T\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}T^{-1}$, where $Z_1 \in \mathbf{C}^{m\times m}$, and $Z_2 \in \mathbf{C}^{m\times(l-m)}$.

Since $J_m = diag(\lambda_1 I_{n_1}, \lambda_2 I_{n_2}, \cdots, \lambda_t I_{n_t})$, the matrix Z_1 in the equation $Z_1J_mZ_1 = J_mZ_1J_m$ of (3) can be determined from Lemma 1.

Below, we discuss solving the matrices Z_2 and Z_3 in (3). Obviously, $Z_2 = 0$, and $Z_3 = 0$ when Z_1 is nonsingular. Otherwise, if Z_1 is singular, there exists a nonsingular matrix P such that $Z_1 = P\begin{pmatrix} 0 & 0 \\ 0 & \Lambda \end{pmatrix}P^{-1}, \Lambda \in C^{s\times s}, 0 \le s < m$, where Λ is a nonsingular diagonal matrix. By the second equation of (3), we obtain $Z_2 = J_m^{-1}P\begin{pmatrix} Q \\ 0 \end{pmatrix}$, and $Q \in C^{(m-s)\times(l-m)}$ is arbitrary. By the third equation of (3), $Z_3 = (W, 0)P^{-1}J_m^{-1}$ where $W \in C^{(l-m)\times(m-s)}$ is arbitrary. However, because of the last equation of (3), we find $(W, 0)P^{-1}J_m^{-1}P\begin{pmatrix} Q \\ 0 \end{pmatrix} = 0$. If $\widetilde{P}$ represents the $m-s$ order leading principle submatrix of $P^{-1}J_m^{-1}P$, we may have $W\widetilde{P}Q = 0$. This means that for $\forall Q \in C^{(m-s)\times(l-m)}$, we can derive $W \in C^{(l-m)\times(m-s)}$ with it. □

Next, we present one numerical example to illustrate our results.

Example 1. *Find the general solution* X *to the equation* $AXA = XAX$*, where* $A = TJT^{-1}$*, and* $J = diag(3, 3, 2, 5, 0, 0)$*.*

By Theorem 2, $l = 6, m = 4$, $J_4 = diag(3, 3, 2, 5)$, Z_1 is the solution of $Z_1J_4Z_1 = J_4Z_1J_4$. From Lemma 1, there exists a nonsingular matrix P such that $P^{-1}Z_1P = \Lambda$, in which Λ is one of the following:

$$\begin{pmatrix} 3 & & & \\ & 3 & & \\ & & 2 & \\ & & & 5 \end{pmatrix}, \begin{pmatrix} 0 & & & \\ & 3 & & \\ & & 2 & \\ & & & 5 \end{pmatrix}, \begin{pmatrix} 0 & & & \\ & 3 & & \\ & & 3 & \\ & & & 2 \end{pmatrix}, \begin{pmatrix} 0 & & & \\ & 3 & & \\ & & 3 & \\ & & & 5 \end{pmatrix},$$

$$\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 3 & \\ & & & 2 \end{pmatrix}, \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 3 & \\ & & & 5 \end{pmatrix}, \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 2 & \\ & & & 5 \end{pmatrix}, \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 3 & \\ & & & 3 \end{pmatrix},$$
$$\begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 3 \end{pmatrix}, \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 2 \end{pmatrix}, \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 5 \end{pmatrix}, \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}.$$

when $\Lambda = \begin{pmatrix} 3 & & & \\ & 3 & & \\ & & 2 & \\ & & & 5 \end{pmatrix}$, Z_1 is nonsingular, and hence, $Z_2 = 0, Z_3 = 0$. Otherwise, $Z_2 = diag(\frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{5})P\begin{pmatrix} Q \\ 0 \end{pmatrix}$, $Z_3 = (W, 0)P^{-1}diag(\frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{5})$, where $Q \in C^{(4-s)\times 2}$, $W \in C^{2\times(4-s)}$ ($s = Rank(\Lambda)$) are any matrices satisfying $W\widetilde{P}Q = 0$, and $\widetilde{P}$ is the $4 - s$ order leading principle submatrix of $P^{-1}diag(\frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{5})P$. Therefore, for $\forall Q \in C^{(4-s)\times 2}$, we can obtain $W \in C^{2\times(4-s)}$ by solving the homogeneous equation $W\widetilde{P}Q = 0$, and vice versa. We obtain all solutions to (1).

By direct calculation, it can be verified that, when Λ is singular,

$$\begin{aligned}
XAX =& T\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix} \cdot T^{-1}AT \cdot \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix} T^{-1} \\
=& T\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}\begin{pmatrix} J_4 & \\ & 0 \end{pmatrix}\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix} T^{-1} \\
=& T\begin{pmatrix} Z_1 & J_4^{-1}P\begin{pmatrix} Q \\ 0 \end{pmatrix} \\ (W,0)P^{-1}J_4^{-1} & Z_4 \end{pmatrix}\begin{pmatrix} J_4 & \\ & 0 \end{pmatrix}\begin{pmatrix} Z_1 & J_4^{-1}P\begin{pmatrix} Q \\ 0 \end{pmatrix} \\ (W,0)P^{-1}J_4^{-1} & Z_4 \end{pmatrix} T^{-1} \\
=& T\begin{pmatrix} Z_1J_4 & 0 \\ (W,0)P^{-1} & 0 \end{pmatrix}\begin{pmatrix} Z_1 & J_4^{-1}P\begin{pmatrix} Q \\ 0 \end{pmatrix} \\ (W,0)P^{-1}J_4^{-1} & Z_4 \end{pmatrix} T^{-1} \\
=& T\begin{pmatrix} Z_1J_4Z_1 & P\Lambda\begin{pmatrix} Q \\ 0 \end{pmatrix} \\ (W,0)\Lambda P^{-1} & (W,0)P^{-1}J_4^{-1}P\begin{pmatrix} Q \\ 0 \end{pmatrix} \end{pmatrix} T^{-1} \\
=& T\begin{pmatrix} Z_1J_4Z_1 & 0 \\ 0 & 0 \end{pmatrix} T^{-1} = T\begin{pmatrix} J_4Z_1J_4 & 0 \\ 0 & 0 \end{pmatrix} T^{-1} \\
=& T\begin{pmatrix} J_4 & \\ & 0 \end{pmatrix} T^{-1} \cdot T\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix} T^{-1} \cdot T\begin{pmatrix} J_4 & \\ & 0 \end{pmatrix} T^{-1} = AXA
\end{aligned}$$

When Λ is nonsingular, we also have $XAX = AXA$.

3. Conclusions

In this paper, we have discussed finding all of the solutions to the Yang–Baxter-like matrix equation $AXA = XAX$ when the orders of all Jordan blocks for A are one. Research on the non-commuting solutions is interesting, but it seems to be hard to find. In the future, we hope to be able to attack the non-commuting solutions to $AXA = XAX$ when the order of Jordan blocks of A is more general.

Author Contributions: Conceptualization, D.C. and X.Y.; methodology, D.C. and X.Y.; original draft preparation and editing, D.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Adam, M.; Ding, J.; Huang, Q.; Zhu, L. Solving a class of quadratic matrix equations. *Appl. Math. Lett.* **2018**, *82*, 58–63. [CrossRef]
2. Adam, M. All Solutions of the Yang-Baxter-like matrix equation when $A^3 = A$. *J. Appl. Anal. Comput.* **2019**, *9*, 1022–1031.
3. Cibotarica, A. An Examination of the Yang-Baxter Equation. Master's Thesis, The University of Southern Mississippi, Hattiesburg, MS, USA, 2011.
4. Cibotarica, A.; Ding, J.; Kolibal, J.; Rhee, N. Solutions of the Yang-Baxter matrix equation for an idempotent. Numer. *Algebra Control Optim.* **2013**, *3*, 347–352. [CrossRef]
5. Chen, D.; Chen, Z.; Yong, X. Explicit solutions of the Yang-Baxter-like matrix equation for a diagonalizable matrix with spectrum $\{1, \alpha, 0\}$. *Appl. Math. Comput.* **2019**, *348*, 523–530. [CrossRef]
6. Ding, J.; Rhee, N. A nontrivial solution to a stochastic matrix equation. *East Asian J. Appl. Math.* **2012**, *2*, 277–284. [CrossRef]
7. Ding, J.; Rhee, N. Spectral solutions of the Yang-Baxter matrix equation. *J. Math. Anal. Appl.* **2013**, *402*, 567–573. [CrossRef]
8. Ding, J.; Rhee, N.; Zhang, C. Further solutions of a Yang-Baxter-like matrix equation. *East Asian J. Appl. Math.* **2013**, *2*, 352–362. [CrossRef]
9. Ding, J.; Zhang, C. On the structure of the spectral solutions of the Yang-Baxter matrix equation. *Appl. Math. Lett.* **2014**, *35*, 86–89. [CrossRef]
10. Ding, J.; Rhee, N.; Zhang, C. Commuting solutions of the Yang-Baxter matrix equation. *Appl. Math. Lett.* **2015**, *44*, 1–4. [CrossRef]
11. Ding, J.; Tian, H. Solving the Yang-Baxter-like matrix equation for a class of elementary matrices. *Comput. Math. Appl.* **2016**, *72*, 1541–1548. [CrossRef]
12. Dong, Q.; Ding, J. Complete commuting solutions of the Yang-Baxter-like matrix equation for diagonalizable matrices. *Comput. Math. Appl.* **2016**, *72*, 194–201. [CrossRef]
13. Dong, Q.; Ding, J.; Huang, Q. Commuting solutions of a quadratic matrix equation for nilpotent matrices. *Algebra. Colloq.* **2018**, *25*, 31–44. [CrossRef]
14. Kumar, A.; Cardoso, J. Iterative methods for finding commuting solutions of the Yang-Baxter-like matrix equation. *Appl. Math. Comput.* **2018**, *333*, 246–253. [CrossRef]
15. Kumar, A.; Cardoso, J.; Singh, G. Explicit solutions of the singular Yang-Baxter-like matrix equation and their numerical computation. *Mediterr. J. Math.* **2022**, *19*, 85. [CrossRef]
16. Mansour, S.; Ding, J.; Huang, Q. Explicit solutions of the Yang-Baxter-like matrix equation for an idempotent matrix. *Appl. Math. Lett.* **2017**, *63*, 71–76. [CrossRef]
17. Ren, H.; Wang, X.; Wang, T. Commuting solutions of the Yang-Baxter-like matrix equation for a class of rank-two updated matrices. *Comput. Math. Appl.* **2018**, *76*, 1085–1098. [CrossRef]
18. Shen, D.; Wei, M.; Jia, Z. On commuting solutions of the Yang-Baxter-like matrix equation. *J. Math. Anal. Appl.* **2018**, *462*, 665–696. [CrossRef]
19. Shen, D.; Wei, M. All solutions of the Yang-Baxter-like matrix equation for diagonalizable coefficient matrix with two different eigenvalues. *Appl. Math. Lett.* **2020**, *101*, 106048. [CrossRef]
20. Tian, H. All solutions of the Yang-Baxter-like matrix equation for rank-one matrices. *Appl. Math. Lett.* **2016**, *51*, 55–59. [CrossRef]
21. Yang, C. Some exact results for the many-body problem in one dimension with repulsive delta-function interaction. *Phys. Rev. Lett.* **1967**, *19*, 1312–1315. [CrossRef]
22. Zhou, D.; Chen, G.; Ding, J. Solving the Yang-Baxter-like matrix equation for rank-two matrices. *J. Comput. Appl. Math.* **2017**, *313*, 142–151. [CrossRef]
23. Baxter, R. Partition function of the eight-vertex lattice model. *Ann. Phys.* **1972**, *70*, 193–228. [CrossRef]
24. Horn, R.; Johnson, C. *Matrix Analysis*; Cambridge University Press: Cambridge, UK, 1991.