QUANTIZED DE SITTER GAUGE THEORY WITH CLASSICAL METRIC AND AXIAL TORSION

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(Received: November 22, 1993)

Abstract. Geometro-stochastically quantized fields are introduced as sections on a first quantized Hilbert bundle, $\mathcal{H}$, over Riemann-Cartan space-time with axial vector torsion representing quantized elementary matter in a gauge theory based on the (4,1)-de Sitter group. $\mathcal{H}$ is a soldered bundle with built-in fundamental length parameter R typical for hadron physics carrying a spin zero phase space representation of $G = SO(4,1)$ belonging to the principal series of unitary irreducible representations. In a nonlinear realization of G the Lorentz subgroup may be related to a gauge formulation of gravitation. Bilinear currents are introduced through G-invariant integration over the local fibers in $\mathcal{H}$, and covariant field equations are set up for the quantum fiber dynamics (QFD) describing the coupling of quantized material sources to the underlying bundle geometry in the presence of gravitation.

1. Introduction

It is well known that Einstein's metric theory of gravitation may be formulated as a Lorentz gauge theory by reducing the original linear frame bundle $P'(B, G' = GL(4,R))$ over the space-time base B, in the presence of a pseudo-Riemannian metric $g_{\mu\nu}(x)$ with Lorentz signature, to the Lorentz frame bundle $P_L(B, H = SO(3,1))$ over $B = V_4$. Also the connection on P' reduces to a connection on P_L provided $g_{\mu\nu}(x)$ is covariant constant, i.e. satisfies $\bar{\nabla}_\rho g_{\mu\nu}(x) = 0$, which just defines the Levi-Civita connection denoted by $\bar{\Gamma}^\rho_{\mu\nu} = \left\{ {\rho \atop \mu\nu} \right\}$. [Purely metric quantities will be denoted by a bar in the following]. The metric $g_{\mu\nu}(x)$ may thus be regarded as a parallel section on a bundle over space-time with ten-dimensional homogeneous fiber $G'/H = GL(4,R)/SO(3,1)$.

The pull back of a connection on P_L with respect to a local section defining a gauge will be called $\bar{\omega}(x)$ which is a Lorentz Lie algebra-valued matrix of one-forms, $\bar{\omega}_{ij}(x) = -\bar{\omega}_{ji}(x)$; $i,j = 0,1,2,3$, $x \in V_4$, with $\bar{\omega}_{ij}(x) = \theta^k \bar{\Gamma}_{kij}(x)$, where $\theta^k = \lambda^k_\mu(x) dx^\mu$ are the fundamental one-forms on the base V_4 of P_L [providing an orthonormal basis for the dual tangent space $T^*_x(V_4)$ at x], and $\bar{\Gamma}_{kij}(x)$ are the Ricci rotation coefficients. A local orthonormal basis of the tangent space $T_x(V_4)$ at x, which is provided by a local section on P_L, will be denoted by e_i; $i = 0, 1, 2, 3$; with $e_i = \lambda_i^\mu(x)\partial_\mu$, where $\lambda_i^\mu(x)$ are the vierbein fields and $\lambda_\mu^k(x)$ their inverse, obeying

$$\lambda_\mu^k(x)\lambda_j^\mu(x) = \delta_j^k \; ; \qquad g_{\mu\nu}(x) = \lambda_\mu^i(x)\lambda_\nu^k(x)\eta_{ik} \, . \tag{1.1}$$

$\eta_{ik} = diag(1, -1, -1, -1)$ is the Minkowski metric, and local Lorentzian indices will be written with Latin letters $(i, k, j \ldots)$, while Greek indices $(\mu, \nu, \rho \ldots)$ refer to a natural basis ∂_μ in $T_x(V_4)$ and dx^μ in $T_x^*(V_4)$, respectively. [For repeated covariant and contravariant Greek and Latin indices the summation convention applies. Greek indices are lowered with $g_{\mu\nu}(x)$ and raised with its inverse $g^{\mu\nu}(x)$; Latin indices are raised and lowered with η^{ik} and η_{ik}, respectively.]

The problem of the theoretical description of atomic, nuclear or subnuclear particles in the presence of gravitational fields raises the question of how to extend classical general relativity – relating the geometry of space-time to the distribution of energy and momentum of *classical macroscopic matter* – to the domain of quantum physics obeying the laws of quantum mechanics for the description of matter at the atomic and "elementary" particle level and requiring a treatment in terms of wave functions and field operators. We are aiming here at a unified geometric formulation of gravitational *and* subnuclear hadronic forces in the presence of classical *as well as* quantized material sources. [The electroweak interaction will, for simplicity, be disregarded in the following discussion. It may be included by enlarging the principal bundles introduced below by an additional $U(1) \otimes SU(2)$ fiber.]

Although gravitational effects are negligibly small in particle physics the structure of Einstein's metric theory of gravitation is so unique in its dualism between the metric of the ambient space and the distribution of matter therein that it may legitimately be asked whether a similar dualism may also be invoked for the theoretical description of interections in the subnuclear world, i.e. being relevant for hadrons at distances of the order of a Fermi or below ($\sim 10^{-13} - 10^{-15}$ cm). With this aim in mind we shall investigate here a model based on a higher dimensional bundle raised over space-time characterized by a structural group G which is bigger than the Lorentz group but, in fact, contains the Lorentz group $H = SO(3,1)$ as a closed subgroup with $SO(3,1) \equiv O(3,1)^{++}$ (proper isochronous Lorentz group) being related to a gauge formulation of gravitation [1]. However, there will appear further contributions at the level of the connection $\bar{\omega}(x)$ on P_L introduced above when one considers the induced Lorentz gauge degrees of freedom appearing in this enlarged bundle formalism; i.e. there may appear torsion or Weyl degrees of freedom related to the additional motions present or possible in the internal spaces [the local fibers] on which the group G acts and which may come into play at small (subnuclear) distances in the space-time base.

We shall base the following discussion on the (4,1)-de Sitter group, $G = SO(4,1)$, as the bigger gauge or structural group containing the Lorentz group as a gauged subgroup, and introduce the de Sitter frame bundle, $P(B = U_4, G = SO(4,1))$, over a Riemann-Cartan space-time U_4 as a geometric arena for the unification of general relativity describing classical (macrophysical) gravitation, and strong subnuclear interactions modifying Einstein's theory at small distances due to the presence of quantized elementary (microphysical) sources. We shall disregard Weyl degrees of freedom in the following and shall specialize later to axial vector torsion (i.e. to a completely antisymmetric torsion tensor). Compare Ref. [2] for a Weyl rescaling of the metric in the fiber in the de Sitter gauge theory. It will be seen in this context that Einstein's metric of general relativity *remains* a classical field describing macroscopic gravitation despite the presence of quantized elementary sources in the geometry. Gravitation need thus not be quantized in this unified theory.

Nonlinear field equations for the additional nonmetric geometric fields are set up establishing a further feed back mechanism between matter (i.e. elementary hadronic matter described in a quantum mechanical manner) and the underlying bundle geometry raised over space-time. These additional source equations have the consequence that despite the presence of quantized matter – represented in the form of generalized wave functions (sections on a Hilbert bundle $\mathcal{H}$) transforming under an irreducible phase space representation of $SO(4,1)$ – which induce in the bundle geometry the additional geometric fields through certain bilinear currents, the metric continues to play a classical rôle as the potential for a classical part of the connection on $P(U_4, SO(4,1))$ [more exactly, its Lorentz part in the so-called nonlinear gauge (see below)]. For details see Ref. [3]. After introducing in the next section the Hilbert bundle $\mathcal{H}$ over space-time and discussing the generalized wave functions representing quantized spinless matter in the theory, we investigate various generalized gauge currents as source currents for the geometry and discuss, finally, two sets of covariant nonlinear field equations (current-curvature and Einstein-type equations) for a gauge dynamics on $\mathcal{H}$ which we call quantum fiber dynamics (QFD).

2. Representation of Quantized Matter

In order to describe quantized elementary matter in the presence of gravitational fields generated by distant macroscopic classical masses one introduces a Hilbert bundle $\mathcal{H}$ over a Riemann-Cartan space-time U_4 carrying a system of covariance of the (4,1)-de Sitter group [4]. $\mathcal{H}$ is a " first quantized" bundle (in the terminology used in [4]), which is associated to $P(U_4, G = SO(4,1))$, possessing a standard fiber, $\mathcal{H}_{\tilde{\eta}}^{(\rho)}$, being a resolution kernel Hilbert space with resolution generator $\tilde{\eta}$ and generalized coherent state basis. The bundle $\mathcal{H}$ carries a spin zero *phase space representation* of $SO(4,1)$ belonging to the principal series of UIR's (unitary irreducible representation) determined by the parameter ρ. The quantum bundle $\mathcal{H}$ with local fiber $\mathcal{H}^{(\rho)}_{\tilde{\eta}}(x)$ provides a geometric arena for the propagation of the de Sitter quantum fields $\Psi^{(\rho)}_x(\xi,\zeta)$, as described below. Moreover, $\mathcal{H}$ possesses a built-in fundamental length parameter R of geometric origin chosen, as mentioned, to be of the order of 10^{-13} cm typical for hadron physics [4,1]. We call $\Psi^{(\rho)}_x(\xi,\zeta)$ a *generalized quantum mechanical wave function* of de Sitter type which is square-integrable, for any $x \in U_4$, with respect to a G-invariant measure $d\tilde{\Sigma}(\xi,\zeta)$ [see below] in the local de Sitter phase space variables $(\xi,\zeta) \in \tilde{\Sigma}^{\pm}_x \subset \mathcal{N}^{\pm}_x$, where

$$\mathcal{N}^{\pm} = V'_4 \times \mathcal{C}^{\pm} \tag{2.1}$$

with $[\eta_{ab} = diag(1,-1,-1,-1,-1)]$:

$$V'_4 : \ [\xi,\xi] = \xi^a \xi^b \eta_{ab} = -R^2 \ ,$$

$$\mathcal{C}^{\pm} : \ [\zeta,\zeta] = \zeta^a \zeta^b \eta_{ab} = 0; \ \ \zeta^5 = \frac{1}{R} \ . \tag{2.2}$$

The summations in (2.2) run over $a,b = 0,1,2,3,5$. Here $\mathcal{N}^{\pm}$ denotes the de Sitter phase space: $V'_4 \simeq G/H = SO(4,1)/SO(3,1)$ is (4,1)-de Sitter space [a single-shell hyperboloid of radius R in a Lorentzian embedding space $R_{4,1}$], and $\mathcal{C}^{\pm}$ is the intersection of the light cone in $R_{4,1}$ with the surface $\zeta^5 = \frac{1}{R}$. The superscript of $\mathcal{C}^{\pm}$ stands for $sign\zeta^0 = \pm$ with the vector $\zeta^a = (\zeta^i, \zeta^5 = \frac{1}{R}); i = 0,1,2,3$, characterizing a so-called horosphere or horocycle [5] through the origin $\xi^o = (0,0,0,0,-R)$ of V'_4. $\zeta \in \mathcal{C}^{\pm}$ plays the rôle of the wave vector or momentum variable for a wave phenomenon in de Sitter space (a space of constant curvature with curvature radius R). $\tilde{\Sigma}^{\pm} = H \times \mathcal{C}^{\pm}$ denotes a six-dimensional horospherical submanifold of $\mathcal{N}^{\pm}$ composed of a horosphere H (a space-like hypersurface) in V'_4 and the cone $\mathcal{C}^{\pm}$.

For later use we, furthermore, introduce the de Sitter phase space bundle over space-time U_4,

$$\tilde{E} = \tilde{E}(U_4, \tilde{F} = V'_4 \times \mathcal{C}^{\pm}, G = SO(4,1)) \tag{2.3}$$

which is a soldered bundle associated to P. [The soldering is performed here through the local subspace $V'_4(x)$ of $\mathcal{N}^{\pm}_x$ being tangent to the space-time base U_4 for each x [6,7].] Moreover, we introduce the de Sitter bundle over U_4,

$$E = E(U_4, F = V'_4 \simeq G/H, G = SO(4,1)) \tag{2.4}$$

which is a soldered bundle associated to P with "curled up" four-dimensional fiber of definite (fixed) radius R which is isomorphic to the noncompact coset space $G/H = SO(4,1)/SO(3,1)$ [7].

We now construct a *phase space representation* of the de Sitter group for spinless particles, denoted by $\tilde{U}(A_g) = \tilde{U}^{(\rho)}(A_g); A_g \in SO(4,1)$, which is related to the spin zero UIR of $SO(4,1)$ of the principal series characterized by the parameter ρ, $0 \leq \rho < \infty$. The value of ρ determines the mass of the particle in question. [The eigenvalue of the Laplace-Beltrami operator, $\Box_\xi$, on V_4' has eigenvalues $\kappa(\kappa+3)/R^2$ with $\kappa = -\frac{3}{2} + i\rho$; $0 \leq \rho < \infty$, leading to the following relation between ρ, the radius R of de Sitter space and the mass m of the particle: $[\frac{mc}{\hbar}]^2 R^2 = \rho^2 + \frac{1}{4}$ (compare Ref. [8]).]

$\mathcal{H}_{\tilde{\eta}}^{(\rho)}$ is the Hilbert space $L^2(\tilde{\Sigma}^\pm)$ of square-integrable functions in the variables $(\xi,\zeta) \in \tilde{\Sigma}^\pm \subset \mathcal{N}^\pm$ with respect to the G-invariant measure [4]

$$d\tilde{\Sigma}(\xi,\zeta) = \frac{1}{R^2}\frac{1}{[\xi,\zeta]^2}\delta(|[\xi,\zeta]| - c)d\mu(\xi)\delta([\zeta,\zeta])d^4\zeta\ , \tag{2.5}$$

where $d\mu(\xi) = \frac{R}{|\xi^5|}d\xi^0 d\xi^1 d\xi^2 d\xi^3$ is the invariant measure on V_4'. $d\tilde{\Sigma}(A_g\xi, A_g\zeta) = d\tilde{\Sigma}(\xi,\zeta)$. In (2.5) c is a positive constant determining a particular horosphere H_ζ^c in V_4' characterized by ζ being parallel to a horosphere H_ζ^1 through the origin ξ^o characterized by the same vector ζ. One can construct a coherent state basis of $\mathcal{H}_{\tilde{\eta}}^{(\rho)}$ in terms of horospherical waves [8,4] (which are analogous to plane waves in flat space) from $SO(3)$-invariant resolution generators $\tilde{\eta}(\zeta')$ yielding a parametrization of the basis of $\mathcal{H}_{\tilde{\eta}}^{(\rho)}$ in terms of the coset space $SO(4,1)/SO(3)$. $\mathcal{H}_{\tilde{\eta}}^{(\rho)}$ is a single-particle resolution kernel Hilbert space with decomposition $\mathcal{H}_{\tilde{\eta}}^{(\rho)} = \mathcal{H}^+ \bigoplus \mathcal{H}^-$, where the superscripts $+$ and $-$ stand for the sign of ζ^0 with $\mathcal{H}^+$ and $\mathcal{H}^-$ denoting the one-particle and one-antiparticle Hilbert spaces, respectively. For the discussion of second quantized Hilbert bundles with Fock space fibers constructed in terms of tensor products of the spaces $\mathcal{H}^+$ and $\mathcal{H}^-$ compare [4] and also the recent book by Prugovečki [9] Here we shall confine the discussion to the spaces $\mathcal{H}^\pm$ and the resulting first quantized Hilbert bundle.

Having introduced the Hilbert space $\mathcal{H}_{\tilde{\eta}}^{(\rho)}$ we now consider as the geometric arena for the description of spinless quantized matter the following soldered (first quantized) Hilbert bundle over Riemann-Cartan space-time with standard fiber $\mathcal{H}_{\tilde{\eta}}^{(\rho)}$ and structural group provided by the unitary irreducible phase space representation $\tilde{U}(A_g)$:

$$\mathcal{H} = \mathcal{H}(U_4, \mathcal{F} = \mathcal{H}_{\tilde{\eta}}^{(\rho)}, \tilde{U}(A_g)). \tag{2.6}$$

$\mathcal{H}$ is associated to $P(U_4, SO(4,1))$ and carries a system of covariance of the $(4,1)$-de Sitter group. The variables (ξ,ζ) in each local fiber $\mathcal{N}_x^\pm$ of $\tilde{E}$

play the rôle of local geometro-stochastic variables on $\mathcal{H}$ determining the quantum-kinematical localization properties of spinless quantum particles (possessing internal de Sitter gauge degrees of freedom) on curved space-time [4]. Denoting now the generalized coherent state basis in the local fibers $\mathcal{H}_{\tilde{\eta}}^{(\rho)}(x)$ of $\mathcal{H}$ which is adapted to a particular choice of gauge $\sigma = \mathbf{u}(x)$ on P by $\Phi_{\xi,\zeta}^{\mathbf{u}(x)}$, where $x \in U_4$ and $(\xi,\zeta) \in \tilde{\Sigma}_x^{\pm}$, one obtains the following resolution of unity at $x \in U_4$:

$$\int_{\tilde{\Sigma}_x^{\pm}} |\Phi_{\xi,\zeta}^{\mathbf{u}(x)}\rangle d\tilde{\Sigma}(\xi,\zeta)\langle\Phi_{\xi,\zeta}^{\mathbf{u}(x)}| = \mathbf{1}_x^{\pm} \; . \tag{2.7}$$

Here $d\tilde{\Sigma}(\xi,\zeta)$ is the measure (2.5) on the local (horospherical) hypersurface $\tilde{\Sigma}_x^{\pm}$ in $\mathcal{N}_x^{\pm}$. Any state vector $\Psi_x^{(\rho)\pm}$ belonging to the principal series of UIR's of $SO(4,1)$ with zero spin may be expanded with respect to the local quantum frame basis, $\Phi_{\xi,\zeta}^{\mathbf{u}(x)}$ according to:

$$\Psi_x^{(\rho)\pm} = \int_{\tilde{\Sigma}_x^{\pm}} d\tilde{\Sigma}(\xi,\zeta)\Psi_x^{(\rho)}(\xi,\zeta)\Phi_{\xi,\zeta}^{\mathbf{u}(x)} \; . \tag{2.8}$$

The coefficient $\Psi_x^{(\rho)}(\xi,\zeta)$ in the expansion (2.8) is the *scalar de Sitter coordinate wave function*, called for short the *generalized wave function*, which may be regarded as a section on the first quantized bundle $\mathcal{H}$ and represents first quantized matter in the theory.

One can adopt a convenient bracket notation for the G-invariant integration with measure (2.5) over the local hypersurface $\tilde{\Sigma}_x^{\pm}$ and solve (2.8) for $\Psi_x^{(\rho)}(\xi,\zeta)$ yielding

$$\Psi_x^{(\rho)}(\xi,\zeta) = \langle\Phi_{\xi,\zeta}^{\mathbf{u}(x)} \mid \Psi_x^{(\rho)\pm}\rangle_{\tilde{\Sigma}_x^{\pm}}. \tag{2.9}$$

$\Psi_x^{(\rho)}(\xi,\zeta)$ has the following transformation property under gauge transformations (i.e. changes of section on $\mathcal{H}$)[4]:

$$\left(\tilde{U}(A_g)\Psi_x^{(\rho)}\right)(\xi,\zeta) = \Psi_x^{(\rho)}\left(A_{g(x)}^{-1}\xi, A_{g(x)}^{-1}\zeta\right) \; . \tag{2.10}$$

A G-invariant scalar product of two sections $\Psi_{1,x}^{(\rho)}(\xi,\zeta)$ and $\Psi_{2,x}^{(\rho)}(\xi,\zeta)$ is defined by

$$\langle\Psi_{1,x}^{(\rho)} \mid \Psi_{2,x}^{(\rho)}\rangle_{\tilde{\Sigma}_x^{\pm}} = \int_{\tilde{\Sigma}_x^{\pm}} \Psi_{1,x}^{(\rho)*}(\xi,\zeta)\Psi_{2,x}^{(\rho)}(\xi,\zeta)d\tilde{\Sigma}(\xi,\zeta). \tag{2.11}$$

The covariant derivative of a section $\Psi_x^{(\rho)}(\xi,\zeta)$ on $\mathcal{H}$ is given by

$$D^R\Psi_x^{(\rho)}(\xi,\zeta) = [d + i\Gamma^R]\Psi_x^{(\rho)}(\xi,\zeta)$$

$$= [d + \frac{i}{2}[\omega^R(x)]_{ab}\tilde{M}^{ab}]\Psi_x^{(\rho)}(\xi,\zeta) \tag{2.12}$$

where $[\omega^R(x)]_{ab} = -[\omega^R(x)]_{ba}$ is the pull back of a connection on P, and $\tilde{M}_{ab}$ denote the generators of the spin zero phase space representation $\tilde{U}(A_g)$ of $SO(4,1)$ given by

$$\tilde{M}_{ab} = -\tilde{M}_{ba} = L_{ab}(\xi) + L_{ab}(\zeta) \tag{2.13}$$

with

$$L_{ab}(\xi) = i\Big(\xi_a\frac{\partial}{\partial\xi^b} - \xi_b\frac{\partial}{\partial\xi^a}\Big), \qquad L_{ab}(\zeta) = i\Big(\zeta_a\frac{\partial}{\partial\zeta^b} - \zeta_b\frac{\partial}{\partial\zeta^a}\Big) . \tag{2.14}$$

Local de Sitter indices $a, b, c, \ldots$ running over $0,1,2,3,5$ are raised and lowered with the de Sitter metric η^{ab} and η_{ab}, respectively.

We, finally, introduce the kernel for the propagation from (ξ,ζ) to (ξ',ζ') in the local fiber over $x \in U_4$ in $\mathcal{H}$ which is determined by the following overlap of the coherent state basis $\mathbf{\Phi}_{\xi,\zeta}^{\mathbf{u}(x)}$ at x:

$$\tilde{K}_{\tilde{\eta},x}^{(\rho)}(\xi',\zeta';\xi,\zeta) = \langle\mathbf{\Phi}_{\xi',\zeta'}^{\mathbf{u}(x)} \mid \mathbf{\Phi}_{\xi,\zeta}^{\mathbf{u}(x)}\rangle_{\tilde{\Sigma}_x^{\pm}} . \tag{2.15}$$

Eq. (2.15) defines a reproducing kernel in $\mathcal{H}_{\tilde{\eta}}^{(\rho)}(x)$ with the reproducing property following from (2.7), i.e.

$$\tilde{K}_{\tilde{\eta},x}^{(\rho)}(\xi',\zeta';\xi,\zeta) = \int\limits_{\tilde{\Sigma}_x^{\pm}} \tilde{K}_{\tilde{\eta},x}^{(\rho)}(\xi',\zeta';\xi'',\zeta'')\tilde{K}_{\tilde{\eta},x}^{(\rho)}(\xi'',\zeta'',\xi,\zeta)d\tilde{\Sigma}(\xi'',\zeta'') . \tag{2.16}$$

The kernel $\tilde{K}_{\tilde{\eta},x}^{(\rho)}(\xi',\zeta';\xi,\zeta)$ determines the propagation of the generalized wave functions $\Psi_x^{(\rho)}(\xi,\zeta)$ in the local fiber variables. For the discussion of the (strongly and weakly) causal geometro-stochastic propagation on the bundle $\mathcal{H}$ we refer to Refs. [4] and [9].

We, moreover, require that the generalized wave function $\Psi_x^{(\rho)}(\xi,\zeta)$ satisfies a de Sitter gauge covariant and U_4-covariant second order wave equation on $\mathcal{H}$ with real eigenvalue α. Specializing to axial vector torsion in the U_4 base this equation may be written, with $D_\rho^R = \partial_\rho + i\Gamma_\rho^R(x)$, as [1]

$$(\Box_{\mathcal{H}} + \alpha)\Psi_x^{(\rho)}(\xi,\zeta) = \Big(\frac{1}{\sqrt{-g}}D_\rho^R\sqrt{-g}g^{\rho\sigma}D_\sigma^R + \alpha\Big)\Psi_x^{(\rho)}(\xi,\zeta) = 0 \tag{2.17}$$

where $g = det\ g_{\mu\nu}(x)$, and α is a constant of dimension L^{-2} (L=length) characterizing the wave motion on $\mathcal{H}$.

Using the operators $D_k^R = \lambda_k^\mu(x)D_\mu^R$ with $D^R = \theta^k D_k^R$ as defined in (2.12) and the generators $\tilde{M}_{ab}$ of the phase space representation $\tilde{U}(A_g)$ of $SO(4,1)$

one can construct, by G-invariant integration over the local fiber variables, the following set of hermitean gauge covariant currents, antisymmetric in a, b, and bilinear in the matter fields $\Psi_x^{(\rho)}(\xi,\zeta)$ and their adjoints for a fixed value of ρ:

$$J_{kab}^{(\rho)}(x) = \frac{i}{2}\int_{\tilde{\Sigma}_x^{\pm}} \Psi_x^{(\rho)}(\xi,\zeta)^* [\overrightarrow{\tilde{M}}_{ab}\overrightarrow{D}_k^R - \overleftarrow{D}_k^R \overleftarrow{\tilde{M}}_{ab}]\Psi_x^{(\rho)}(\xi,\zeta) d\tilde{\Sigma}(\xi,\zeta)\,, \quad (2.18)$$

with

$$\overrightarrow{D}_k^R = \overrightarrow{\partial}_k + i\,\overrightarrow{\Gamma}_k^R(x); \qquad \overleftarrow{D}_k^R = \overleftarrow{\partial}_k - i\,\overleftarrow{\Gamma}_k^R(x)\,, \quad (2.19)$$

and analogously for $\overrightarrow{\tilde{M}}_{ab}$ and $\overleftarrow{\tilde{M}}_{ab} = \overrightarrow{\tilde{M}}_{ab}^{\dagger}$. As a result of (2.17) the currents (2.18) are covariantly conserved. The equations (2.18), (2.12) and (2.13), (2.14) show that the currents $J_{kab}^{(\rho)}(x)$ result from an averaged internal motion taking place in the local fibers on $\mathcal{H}$. For $a, b = i, j$ it is an internal rotational motion (Lorentz rotation); for $a, b = i, 5$ it is a generalized translation (de Sitter boost) in the fiber. It is thus apparent that our formalism describes quantized material objects possessing internal gauge degrees of freedom and extension. We shall use the currents (2.18) as source currents for the bundle geometry tying thereby the quantized motion in the fiber to the geometry of the entire space.

3. Nonlinear gauge and field equations

In order to recover gravitation in a G-invariant manner as a gauge theory of the Lorentz subgroup $H = SO(3,1)$ of $G = SO(4,1)$ we introduce a new Higgs-type field in the formalism given as a section, $\xi(x)$, of the soldered bundle E defined in (2.4) obeying $\xi^a(x)\xi^b(x)\eta_{ab} = -R^2$ [compare (2.2)]. Global sections on E always exist. The "zero section", $\xi(x) = \xi^o$, may be identified with the space-time base of E. The field $\xi(x)$ acts as a symmetry reducing field in the bundle framework: If $\xi(x)$ is *parallel* with respect to $\omega^R(x)$, i.e. satisfies

$$D^R\xi^a(x) = d\xi^a(x) + [\omega^R(x)]_b{}^a\xi^b(x) = 0, \quad (3.1)$$

the $SO(4,1)$ gauge symmetry reduces to the $SO(3,1)$ gauge symmetry describing pure (metric) gravitation. We assume that the full de Sitter gauge symmetry does not reduce everywhere to the Lorentz subsymmetry, but that this reduction of symmetry indeed occurs far outside the quantized material sources present in the geometry. There are, however, regions in space-time, denoted by $D_{(i)}; i = 1, \ldots N$, where the G-symmetry does *not* reduce, i.e. where $D^R\xi^a(x) \neq 0$. In these regions ω^R (or W^R, see (3.2) below) takes values in the Lie algebra $\mathbf{g}$ of $SO(4,1)$ while for regions where (3.1) is true

ω^R (or W^R) reduces to a set of one-forms with values in the Lie algebra $\mathbf{g}'$ of the subgroup $SO(3,1)$. In general $\mathbf{g}$ may be decomposed as $\mathbf{g}=\mathbf{g}'\oplus\mathbf{t}$, where $\mathbf{t}$ is a vector space generating the homogeneous space G/H isomorphic to V_4'.

One may now consider de Sitter boost transformations, $A(\xi(x)) \in SO(4,1)$, transforming the origin ξ^o in V_4' into $\xi(x)$ at $x \in U_4$ and use these transformations to go over to a nonlinear realization of the de Sitter transformations in terms of Lorentz transformations in ξ^o yielding for ω^R the following form:

$$[\omega^R(x)]^a{}_b \overset{A^{-1}(\xi(x))}{\longrightarrow} [W^R(x,\xi(x))]^a{}_b = \begin{pmatrix} [W^R(x,\xi(x))]^i{}_j & [\theta^R(x,\xi(x))]^i \\ [\theta^R(x,\xi(x))]_j & 0 \end{pmatrix}, \tag{3.2}$$

with $$[W^R(x,\xi(x))]^i{}_j = [\bar{\omega}(x)]^i{}_j + [\tau^R(x,\xi(x))]^i{}_j, \tag{3.3}$$

and $$[W^R(x,\xi(x))]^i{}_5 = \frac{1}{R}[A^{-1}(\xi(x))]^i{}_a D^R\xi^a(x). \tag{3.4}$$

We call the form $W^R(x,\xi(x))$ in (3.2) the nonlinearly transforming form of the connection on P (the nonlinear gauge denoted by N.L.). It transforms under gauge transformations, $\xi'(x) = A_{g(x)}\xi(x)$, with a matrix $A(\Lambda(\xi'(x),\xi(x))) \in H$ leaving the form of the r.-h. side of (3.2) unchanged (for details see [1] and [3]). The first term on the r.-h. side of (3.3) is the metric part defining a connection on P_L. However, the Lorentz part (3.3) of (3.2) has a torsion addition denoted by $\tau^R(x,\xi(x))$. (3.4) defining the soldering forms $[\theta^R(x,\xi(x))]^i$ of the de Sitter connection shows explicitly that (3.2) is Lorentz valued for $D^R\xi(x) = 0$, i.e. outside the domains $D_{(i)}$ where the G-gauge symmetry reduces to the H-gauge symmetry. A form analogous to (3.2) is obtained for the curvature two-forms $[\Omega^R(x,\xi(x))]^a{}_b$ in the N.L. gauge.

As field equations for the bundle geometry we now introduce, besides (2.17) for $\Psi_x^{(\rho)}(\xi,\zeta)$, the following two sets of de Sitter gauge covariant and U_4-covariant nonlinear source equations

$$R^R_{ik}(x,\xi(x)) = \frac{1}{2}\eta_{ik}R^R(x,\xi(x)) = \kappa \overset{N.L.}{T_{ik}} (x,\xi(x)) \,, \tag{3.5}$$

$$\overset{N.L.}{D^i} R^R_{ijab}(x,\xi(x)) = \bar{\kappa} \overset{N.L.}{J_{jab}} (x,\xi(x)) \,. \tag{3.6}$$

A further equation for the reducing section $\xi(x)$ is introduced and discussed in [1]. Here κ and $\bar{\kappa}$ are two independent coupling constants; κ is Einstein's gravitational constant, and $\bar{\kappa}$ is a new coupling constant characterizing the quantum fiber dynamics (QFD), i.e. the dynamical relation between quantized matter described on $\mathcal{H}$ and the full uncontracted bundle curvature tensor. The operator $\overset{N.L.}{D^i}$ in (3.6) denotes the full covariant derivative of the Lorentz *and* de Sitter indices [the latter taken with respect to the N.L. form, $W^R(x,\xi(x))$, of the connection]. The r.-h. side of (3.6) is the current (2.18) tranformed to N.L. form with the help of $A^{-1}(\xi(x))$. $R^R_{ijab}(x,\xi(x))$ is the full curvature tensor with the Lorentz part (for $a,b = k,l$ composed of metric, torsion, and quadratic de Sitter boost contributions (see [1]), and with the de Sitter boost part (for $a,b = i,5$). Eqs. (3.5) are of Einstein type involving the contracted Lorentz curvature tensor, $R^R_{ik}(x,\xi(x)) = \eta^{jl} R^R_{ijkl}(x,\xi(x))$, and the corresponding curvature scalar $R^R(x,\xi(x))$ again composed of three parts (metric, torsion and boost). On the r.-h. side of (3.5) appears the total energy-momentum tensor decomposing into the classical symmetric part, $\bar{T}_{ik}(x)$, of general relativity representing classical matter, and a quantum part induced by $\Psi^{(\rho)}_x(\xi,\zeta)$ possessing no symmetry in the indices, i,k :

$$\overset{N.L.}{T_{ik}}(x,\xi(x)) = \bar{T}_{ik}(x) + \overset{N.L.}{T_{ik}}(\Psi) \ . \tag{3.7}$$

A detailed investigation of (3.5) and (3.6) is presented in [1] and [3], in the latter reference with particular emphasis of the rôle played by the metric of Einstein's theory in this context. It is shown there that the $g_{\mu\nu}$-field of classical general relativity survives unchanged in this theory in the presence of quantized matter which, on the other hand, determines the additional fields characterizing the bundle geometry: axial vector torsion in the base and de Sitter boost contributions related to the soldering forms of the $SO(4,1)$ connection in P.

The rôle of axial torsion outside the region $D_{(i)}$ and outside the sources may be studied solving the vacuum torsion equation contained in (3.6). It reads when one neglects classical gravitational forces, i.e. for a metrically flat space-time base,

$$\Box {}^*K^R_s - \frac{1}{12}\partial^*_s P^R - \varepsilon_s{}^{ijl*} K^R_i \partial_j {}^*K^R_l = 0 \tag{3.8}$$

where ${}^*P^R = -6\partial^{i*}K^R_i$, and the axial vector torsion field, ${}^*K^R_s$, is given by [compare (3.3)]

$${}^*K^R_s = -\frac{1}{6}\varepsilon_s{}^{kij} K^R_{kij}(x,\xi(x)) \ ; \qquad \tau^R_{ij}(x,\xi(x)) = \theta^k K^R_{kij}(x,\xi(x)) \ . \tag{3.9}$$

Despite serious efforts no truely nonlinear solution of the equations (3.8) has yet been found except the trivial solution ${}^*K^R_s = k_s exp(\pm ik\cdot x)$ with $k_s k^s = k\cdot k = 0$, for which each term on the l.-h. side of (3.8) is separately zero.

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