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Weak second Bianchi identity for static, spherically symmetric spacetimes with timelike singularities

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Abstract

The (twice-contracted) second Bianchi identity is a differential curvature identity that holds on any smooth manifold with a metric. In the case when such a metric is Lorentzian and solves Einstein's equations with an (in this case inevitably smooth) energy–momentum–stress tensor of a 'matter field' as the source of spacetime curvature, this identity implies the physical laws of energy and momentum conservation for the 'matter field'. The present work inquires into whether such a Bianchi identity can still hold in a weak sense for spacetimes with curvature singularities associated with timelike singularities in the 'matter field'. Sufficient conditions that establish a distributional version of the twice-contracted second Bianchi identity are found. In our main theorem, a large class of spherically symmetric static Lorentzian metrics with timelike one-dimensional singularities is identified, for which this identity holds. As an important first application we show that the well-known Reissner–Weyl–Nordström spacetime of a point charge does not belong to this class, but that Hoffmann's spacetime of a point charge with negative bare mass in the Born–Infeld electromagnetic vacuum does.

Keywords: second Bianchi identity, general relativity, energy–momentum conservation, naked singularities, particles, electromagnetism

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1. Introduction and main results

1.1. Motivation

Einstein's equations for the spacetime metric[3] $\mathbf{g} = (g_{\mu\nu})$ of a $3+1$-dimensional Lorentzian manifold $(\mathcal{M}, \mathbf{g})$ read

$$\mathbf{R} - \frac{1}{2}R\mathbf{g} = \frac{8\pi G}{c^4}\mathbf{T}[\mathbf{F}]; \tag{1.1}$$

here, $\mathbf{R} = (R_{\mu\nu})$ denotes the Ricci curvature tensor of the metric $\mathbf{g}$, $R = g^{\mu\nu}R_{\mu\nu}$ is its scalar curvature, G is Newton's constant of universal gravitation, and c is the speed of light in vacuum. Moreover, $\mathbf{T} = (T_{\mu\nu})$ is the *energy–momentum–stress tensor* of any 'matter field' $\mathbf{F}$ in (or associated with) the spacetime. By using this hybrid terminology of 'matter field' to cover models of continuum fluids, elastic solids, etc., as well as the electromagnetic field, we follow common practice in the general relativity community; from now on, we will drop the scare quotes. Einstein's equation (1.1) often need to be complemented by evolution equations for the matter field $\mathbf{F}$.

For any sufficiently regular Lorentzian metric (classically, $\mathbf{g} \in C^3$), the (twice-contracted) *second Bianchi identity*

$$\nabla_\mu \left(R^\mu{}_\nu - \frac{1}{2}Rg^\mu{}_\nu \right) = 0 \tag{1.2}$$

holds; here, ∇ denotes the Levi-Civita connection associated with $\mathbf{g}$. As a consequence, for any solution $(\mathbf{g}, \mathbf{F})$ of (1.1) which is regular enough so that this equation, as well as the equation obtained by differentiating both sides of (1.1), is satisfied pointwise, the identity (1.2) implies the matter field's local conservation laws of energy–momentum

$$\nabla_\mu T^\mu{}_\nu = 0. \tag{1.3}$$

Matter field equations must be compatible with (1.3), therefore.

If $\mathbf{F}$ represents a perfect fluid with barotropic equation of state, then for sufficiently regular evolutions (e.g. prior to any shock formation) the space component of (1.3) is part of the equations of the fluid evolution, to be complemented merely by the continuity equation for the fluid; the time component of (1.3) is then redundant. On the other hand, if $\mathbf{F}$ represents a source-free electromagnetic field, then (1.3) does not furnish field evolution equations; they need to be stated separately, compatible with (1.3).

In a series of influential papers, [14–16], Einstein and Infeld (EI), originally joined by Hoffmann (EIH), claimed that the field equations of general relativity theory, (1.1), coupled with the Maxwell–Lorentz evolution equations for the electromagnetic fields, determine the equations of motion of matter modeled atomistically as composed of charged point particles, which they identified with point singularities in spacelike slices of a spacetime. They actually discussed mostly the special case of uncharged particles; a follow-up paper by Infeld's student Wallace [45] supplied more details about their claim concerning charged point singularities.

[3] The signature of a Lorentzian metric $\mathbf{g}$ is $(-,+,+,+)$. Greek indices μ, ν etc denote the components $0, 1, 2, 3$ of a tensor defined on the spacetime, with respect to a local coordinate system $(y^\mu)_{\mu=0}^3$; however, Cartesian coordinates are denoted $(x^\mu)_{\mu=0}^3$. The coordinate vector fields are written $\partial_\mu = \frac{\partial}{\partial y^\mu}$. We use the Einstein summation convention. To facilitate discerning the physical meaning of our results, we retain G and c.

The idea that the world lines of point particles should be replaced by one-dimensional timelike singularities of spacetime seems to go back to Weyl [46]. Already in Weyl's writing, it is clear that such singularities are not subsets of the spacetime[4]. Thus, in this setup the 'world line of a particle' is not a path *in* spacetime but a timelike one-dimensional singularity *of* spacetime—or put differently: an interior boundary of the spacetime—which needs to be determined along with the spacetime. *If* this setup can be consistently implemented into general relativity, it produces spacetimes with one-dimensional timelike curvature singularities that have the appearance of world lines of charged point particles, which are the sources and sinks of the electromagnetic fields living in this spacetime. This is different from the usual textbook story of 'test particle' motion which, when uncharged, is given by a timelike geodesic in a spacetime that is defined independently of the particle's existence.

Although non-rigorous and full of questionable assumptions, and with conclusions which cannot possibly be true in the sweeping generality in which they were stated, the EIH papers have become the template for many formal follow-up calculations (for a survey see, e.g. [38]), in particular the computation of gravitational wave signals and their feedback on the motion of the sources (binary neutron stars or black holes) used for the interpretation of the LIGO and VIRGO gravitational wave data [4]. As far as we can see, existing rigorous works on the problem of motion for 'small bodies' in general relativity (we mention in particular [11, 17, 19–22, 39, 42]) do not yet allow a definitive assessment of the merits of some of the key ideas of the papers by EIH, and by Wallace, on the motion of what nowadays would be called naked singularities. The purpose of the present paper is to take one step further toward this goal.

A rigorous assessment would require one to consistently formulate an at least locally well-posed joint evolution problem for spacelike slices of a spacetime, the electromagnetic field defined on these slices, and the point singularities (in the spatial curvature tensor) that represent the sources and sinks of that field. Moreover, if the EIH and EI claims have any merit, then the equations of motion for the point singularities in the spacelike slices must be a consequence of (1.1), coupled with Maxwell's evolution equations for the electromagnetic fields $\mathbf{F}$ which, however, need to be supplemented by a suitable law of the electromagnetic vacuum. It is clear that such an electromagnetic law must be different from the usual Maxwell vacuum law, for the latter leads to infinite electromagnetic field energy of the point singularity, i.e. non-integrable singularities in the electromagnetic $\mathbf{T}$ that cause non-integrable curvature singularities of the metric, as per Einstein's field equation (1.1) (see section 3 for more details).

To get an idea of the mathematical subtleties that could be involved, suppose that such a spacetime $\mathcal{M}$ with timelike one-dimensional singularities can be continuously extended (just the manifold, not the metric) into the location of these singularities. In such a situation these one-dimensional timelike singularities become proper particle world lines in this extended spacetime, and it is then meaningful to express the energy–momentum–stress tensor $\mathbf{T}$ as a sum of a regular and a singular part,

$$\mathbf{T} = \mathbf{T}^{\text{reg}} + \mathbf{T}^{\text{sing}}, \tag{1.4}$$

with $\mathbf{T}^{\text{reg}}$ sufficiently regular away from the world lines of the point-charges, and $\mathbf{T}^{\text{sing}}$ the usual energy–momentum–stress tensor of a system of point particles which is supported only on these world lines in a weak sense as a measure. Now, if (1.2 holds in a weak sense it then follows that (1.3) must hold in a weak sense as well, and hence

$$\nabla \cdot \mathbf{T}^{\text{sing}} = -\nabla \cdot \mathbf{T}^{\text{reg}} \tag{1.5}$$

[4] Some care is thus required when one talks about causal properties of singularities such as them being timelike, spacelike, or null, since these notions can only be meaningful in a limiting sense.

in the sense of distributions. In [29–31] it has been shown that in special-relativistic electromagnetic spacetimes (i.e. Newton's constant $G \to 0$ in (1.1)) with suitable electromagnetic vacuum laws the total (electromagnetic) force on a charged point singularity, and its classical equation of motion, can be extracted from (1.5). Furthermore, for the Bopp–Landé–Thomas–Podolsky (BLTP) vacuum law, the special-relativistic joint initial value problem for point charges coupled with the electromagnetic Maxwell–BLTP fields is locally well-posed in time [31]. Hence it is reasonable to expect, possibly under further conditions on the behavior of $\mathbf{F}$, that at least some of the well-posed special-relativistic joint initial value problems that can be extracted from (1.5) at $G = 0$, can be continuously extended to the general-relativistic domain when $G > 0$. Which matter field models $\mathbf{F}$ qualify in this sense is an important open problem; we offer some remarks in the last section.

Independent of the inquiry into suitable matter field models, the following is now clear: For the establishment of the energy–momentum conservation law (1.5) when $G > 0$ that could pave the ground toward a well-posed joint initial value problem for the spacelike slices of spacetime, the electromagnetic and perhaps other matter fields in it, and their charged point singularities, along similar lines as in the special-relativistic formulation mentioned above, it is necessary that the second Bianchi identity (1.2) holds in a weak sense. Thus, the key question is:

Under which conditions on the metric of the spacetime does the weak second Bianchi identity

$$\int_{\mathcal{M}} \left(R^{\mu}{}_{\nu} - \frac{1}{2} R g^{\mu}{}_{\nu} \right) \nabla_{\mu} \psi^{\nu} \, \mathrm{dvol}_{\mathbf{g}} = 0 \tag{1.6}$$

hold for all smooth, compactly supported vector fields ψ defined on the spacetime?

Answering this question, in all its generality, is a big challenge, because a complete classification of singularities of solutions of Einstein's equations seems currently out of reach. For example, two timelike singularities in a given spacetime can be vastly different in terms of strength, in the sense that a curvature invariant may blow up at two very different rates for them.

Our strategy is to begin by restricting the key question to special families of spacetimes, incrementally becoming more general. There are many explicit solutions of Einstein's equations where the causal structure is simple enough that everything can be worked out explicitly and the singular behavior can be fully analyzed; in particular, we mention distributional approaches in [18, 23, 24, 32–34, 40, 41, 44]. Such model cases can give us clues as to what the sufficient conditions are for a spacetime singularity to represent the world line of a particle, and which type of 'atomic matter' models can accommodate such singularities. We are particularly interested in 'electromagnetic matter', whose electromagnetic field satisfies the pre-metric Maxwell's equations, complemented with a suitable electromagnetic vacuum law, and with charged sources given by a finite number of one-dimensional timelike singularities that are assigned an energy–momentum–stress tensor in the spirit of EIH, and Wallace.

1.2. Setting

While a number of ideas developed below are clearly adaptable to more general situations, in the present paper we focus our efforts on *static spherically symmetric* spacetimes that feature a single timelike singularity, with special emphasis given to electrostatic spacetimes of a single point charge at the center of symmetry. These are four-dimensional Lorentzian manifolds $(\mathcal{M}, \mathbf{g})$ on which there exists a *global* system of coordinates $(t, r, \vartheta, \varphi)$ such that the line element of the metric $\mathbf{g}$ can be written as

$$ds_{\mathbf{g}}^2 = -e^{2\alpha(r)}c^2\,dt^2 + e^{2\beta(r)}\,dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2). \tag{1.7}$$

Thus $\partial/\partial t$ is a timelike Killing field, $r > 0$ is the area-radius coordinate, and (ϑ, φ) are spherical coordinates on the standard sphere $\mathbb{S}^2$.

It is common knowledge that many of the known solutions of Einstein's equation (1.1) have analytical extensions that feature geometric singularities associated with geodesic incompleteness and/or curvature blow-up. Famous examples are the Schwarzschild solution, both in the positive mass (black hole) as well as in the negative mass (naked singularity) sector, and the charged Reissner–Weyl–Nordström (RWN) solution in the superextremal (naked) as well as extremal and subextremal sectors (black holes). It is in fact a theorem [43] that there are no static, spherically symmetric, asymptotically RWN electrovacuum spacetimes whose maximal extension is devoid of singularities.

The negative mass Schwarzschild (nmS) solution and, in the superextremal (naked) sector, also the RWN spacetimes possess global coordinate systems in which their metric has the form (1.7). In these coordinates, there is a severe curvature singularity on the timelike line $r = 0$, so that $\mathcal{M}$ as a Lorentzian manifold is diffeomorphic to $\mathbb{R}^4$ minus a line, or $\mathbb{R} \times (\mathbb{R}^3\backslash\{0\})$. The fact that the singular set of these and many other spacetimes is of codimension three creates complications for the study of the geometry of these manifolds in the neighborhood of their singularities, including the question of whether or not it is possible to formulate a weak version of the second Bianchi identity for these singular spacetimes.

One approach taken by differential geometers that has been fruitful in this regard is the use of a different coordinate system on these manifolds, one which 'blows up' a neighborhood of the singular set in such a way that in the new coordinates, one has a manifold with a codimension-one boundary, with its own intrinsic smooth geometry, thereby allowing for tools of geometric analysis to be applied to it. Examples of this approach can be found in the Riemannian setting in the works of Bray [8], Bray–Jauregui [9] and others. We describe this geometric approach in section 1.3 and a corresponding notion of mass for these codimension one boundaries in section 1.4. Finally, in section 1.5 of this introduction we state and discuss the main results of this paper regarding the weak second Bianchi identity and applications, which makes use of this geometric formulation.

1.3. Spatial conformally flat coordinates and zero-area singularities (ZAS)

Let $(\mathcal{M}, \mathbf{g})$ be a four-dimensional static, spherically symmetric Lorentzian manifold, diffeomorphic to $\mathbb{R} \times (\mathbb{R}^3\backslash\{0\})$, which has a global spherical coordinate system $(t, r, \vartheta, \varphi)$ defined on it in which the metric $\mathbf{g}$ has the form (1.7). Suppose we can transfer to a new coordinate $\rho \in (\rho_0, \infty)$, $\rho_0 > 0$ such that

$$ds_{\mathbf{g}}^2 = -e^{2\gamma(\rho)}c^2\,dt^2 + \phi^4(\rho)\left[d\rho^2 + \rho^2(d\vartheta^2 + \sin^2\vartheta\,d\varphi^2)\right]. \tag{1.8}$$

We call the coordinate system $(t, \rho, \vartheta, \varphi)$ in (1.8) *spatially conformally flat coordinates*. Note that the metric inside the brackets is simply the Euclidean metric on $\mathbb{R}^3$ in spherical coordinates. Comparing to (1.7) we see that it is necessary that

$$\phi^2 = \frac{r}{\rho}, \qquad e^{\beta}\,dr = \phi^2\,d\rho. \tag{1.9}$$

Solving the differential equation, we thus require

$$\rho(r) := \rho_0\,\exp\left(\int_0^r e^{\beta(r')}\frac{dr'}{r'}\right), \tag{1.10}$$

assuming that the integral exists (we will see that it does for the manifolds of interest to us). Clearly, $\rho(0) = \rho_0$. Moreover, ρ as defined in (1.10) is an increasing function of r, and hence invertible, which determines ϕ and γ as

$$\phi(\rho) := \sqrt{r(\rho)/\rho}, \qquad \gamma(\rho) := \alpha(r(\rho)). \tag{1.11}$$

The manifold $\mathcal{M}$ in these coordinates is diffeomorphic to the exterior of the solid cylinder, i.e. $\mathbb{R} \times (\mathbb{R}^3 \backslash \overline{B_{\rho_0}(0)})$. We note that $\phi(\rho_0) = 0$ and therefore by (1.8) the interior boundary $\mathcal{T} = \partial\mathcal{M}$ is a singular boundary. In particular, the intrinsic area of the sphere $\partial B_{\rho_0}(0)$ (which can be computed by a limiting process, see [8]) is zero. The constant-t slices of $\mathcal{M}$ are spacelike hypersurfaces diffeomorphic to the exterior Σ of the open ball $B_{\rho_0}(0)$ in $\mathbb{R}^3$. We can write $\mathcal{M} = \mathbb{R} \times \Sigma$, and $\mathcal{S}$ for the interior boundary $\partial\Sigma$. The surface $\mathcal{S}$ is therefore an example of a *zero area singularity* (ZAS) for the Riemannian manifold Σ.

In his pioneering work [8] on ZAS, Bray defined a notion of mass for such singularities, and studied its properties. In particular he showed that this mass is coordinate invariant and always *negative*. In this work we will connect Bray's notion of the mass of a ZAS with our notion of the (negative) bare mass of the central singularity in the spacetimes discussed in this paper.

In the rest of this section we derive sufficient conditions for obtaining a coordinate transformation of the form (1.8). Let us recall that for spherically symmetric spacetimes, with r denoting the area-radius, one can define the *cumulative mass function* $m(r)$ via the relation

$$1 - \frac{2Gm(r)}{c^2\, r} = g^{\mu\nu}\partial_\mu r \partial_\nu r. \tag{1.12}$$

Thus in terms of the metric coefficients in (1.7),

$$m(r) := \frac{c^2}{2G} r\left(1 - \mathrm{e}^{-2\beta(r)}\right). \tag{1.13}$$

Using the mass function we can rewrite (1.10) as

$$\rho(r) = \rho_0\, \exp\left(\int_0^r \frac{\mathrm{d}r'}{\sqrt{r'^2 - \frac{2G}{c^2} r' m(r')}}\right). \tag{1.14}$$

Let us assume that $m : (0, \infty) \to \mathbb{R}$ is a C^1 function, with the asymptotics

$$m(r) \sim \begin{cases} m_0 + m_1 r & r \to 0, \\ M - \dfrac{M_1}{r} & r \to \infty, \end{cases} \tag{1.15}$$

for constants $m_0 < 0$, $M \in \mathbb{R}$, $m_1, M_1 > 0$. Thus $m_0 = \lim_{r\to 0} m(r)$ and $M = \lim_{r\to\infty} m(r)$. Moreover, for the denominator in (1.14) we have

$$r^2 - \frac{2G}{c^2} r m(r) \sim Ar + Br^2 \quad \text{as } r \to 0, \tag{1.16}$$

with

$$A := \frac{-2Gm_0}{c^2} > 0, \qquad B := 1 - \frac{2Gm_1}{c^2}, \tag{1.17}$$

where $B \in (0, 1)$ if m_1 is sufficiently small. Under these assumptions the integral in (1.14) is seen to be finite and ρ is well-defined. We will mostly be interested in spacetimes where (1.15) holds, with the stated range of the parameters.

1.4. Mass of a regular ZAS

Let $(\Sigma, \boldsymbol{\sigma})$ be a three-dimensional Riemannian manifold with boundary, and let $\mathcal{S}_0 \subseteq \partial\Sigma$ be a ZAS for Σ. According to Bray, a ZAS is called *regular* if it can be conformally extended, i.e. if there exists a smooth nonnegative function $\bar{\phi}$ defined in a neighborhood of $\mathcal{S}_0$ in Σ, and a smooth metric $\bar{\boldsymbol{\sigma}}$ such that

(a) $\bar{\phi} = 0$ on $\mathcal{S}_0$.
(b) $\bar{\mathbf{n}}(\bar{\phi}) > 0$ where $\bar{\mathbf{n}}$ is the unit normal of $\mathcal{S}_0$ with respect to $\bar{\boldsymbol{\sigma}}$.
(c) $\boldsymbol{\sigma} = \bar{\phi}^4 \bar{\boldsymbol{\sigma}}$.

The prototype example of a regular ZAS is the singularity at the center of the spacelike time-slices of the nmS spacetime, which is the unique static, spherically symmetric, asymptotically flat solution of vacuum Einstein equations whose central singularity is not shielded by a horizon. For this manifold, $m(r) \equiv m_0 < 0$, and it is easy to see that the coordinate ρ as defined in section 1.3 above is a global coordinate: if we set

$$\rho_0 := \frac{G|m_0|}{2c^2} \tag{1.18}$$

then the change of coordinate $r \leftrightarrow \rho$ is given by

$$\rho = \frac{1}{2}\left(r + 2\rho_0 + \sqrt{(r + 2\rho_0)^2 - 4\rho_0^2}\right), \qquad r = \frac{(\rho - \rho_0)^2}{\rho}. \tag{1.19}$$

Moreover, the function $\bar{\phi}$ in the definition of regular ZAS exists and is

$$\bar{\phi}(\rho) = 1 - \frac{\rho_0}{\rho}, \tag{1.20}$$

while the metric $\bar{\boldsymbol{\sigma}}$ is the Euclidean metric on $\mathbb{R}^3 \backslash B_{\rho_0}(0)$.

For ZAS that are regular, Bray [8] defined a notion of mass by

$$m_{\mathrm{reg}}(\mathcal{S}_0) := -\frac{1}{4}\left(\frac{1}{\pi}\int_{\mathcal{S}_0} \bar{\mathbf{n}}(\bar{\phi})^{4/3}\, \mathrm{d}S_{\bar{\sigma}}\right)^{3/2}, \tag{1.21}$$

where $\mathrm{d}S_{\bar{\sigma}}$ denotes the surface element with respect to $\bar{\boldsymbol{\sigma}}$. Note that this mass is always negative. Let $(\mathcal{S}_n)_n$ be a sequence of closed C^2 surfaces in Σ, each one being a graph over $\mathcal{S}_0$, that shrink down to $\mathcal{S}_0$ as $n \to \infty$. It can be shown [8] that

$$m_{\mathrm{reg}}(\mathcal{S}_0) = \lim_{n\to\infty} m_{\mathrm{H}}(\mathcal{S}_n), \tag{1.22}$$

where m_{H} denotes the *Hawking mass*

$$m_{\mathrm{H}}(\mathcal{S}) := \sqrt{\frac{|\mathcal{S}|_\sigma}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{\mathcal{S}} H^2\, \mathrm{d}S_\sigma\right). \tag{1.23}$$

Here H denotes the mean curvature of the surface $\mathcal{S}$ in Σ. Note that for the Hawking mass to be well-defined the metric $\boldsymbol{\sigma}$ and the surface $\mathcal{S}$ need to be at least C^2.

If the three-dimensional manifold Σ is spherically symmetric, it is not hard to see that the Hawking mass of the sphere S_r of area-radius $r > 0$ in the manifold is equal to $m(r)$ obtained in (1.13). It thus follows that for the spacetimes $(\mathcal{M}, \mathbf{g})$ of interest in this paper, which admit a metric of the form (1.8) such that the interior boundary $\mathcal{S}_0$ of the time-slices is a ZAS, the bare mass of the singularity is equal to its Hawking mass and to its Bray mass, i.e.

$$m_0 = \lim_{r\to 0} m(r) = \lim_{\rho\to\rho_0} m_{\mathrm{H}}(S_\rho) = m_{\mathrm{reg}}(\mathcal{S}_0). \tag{1.24}$$

Besides the nmS spacetime that contains such a ZAS singularity, we introduce in section 3.1 a particular asymptotically flat, electrovacuum spacetime. This prototype spacetime represents the vacuum outside a static point charge in a nonlinear electromagnetic theory with an admissible reduced Hamiltonian as discussed in some detail also in section 1.5 below. Most importantly, we will show that the central singularity of this electrovac spacetime is of the same strength as the singularity of the nmS, i.e. it has finite negative bare mass, and is therefore a regular ZAS.

1.5. Main results

We are ready to state our results about the weak analogue of the twice-contracted second Bianchi identity (1.2), which in a suitable weak sense also makes sense on certain spacetimes with a single timelike singularity. In this paper, we prove the following result.

Theorem 1.1. *Let $\mathcal{M} \cong \mathbb{R} \times (\mathbb{R}^3 \backslash \overline{B_{\rho_0}(0)})$ be equipped with a static, spherically symmetric Lorentzian metric $\mathbf{g}$ of the form*

$$\mathrm{d}s^2_{\mathbf{g}} = -\mathrm{e}^{2\gamma(\rho)} c^2\, \mathrm{d}t^2 + \phi^4(\rho) \left[\mathrm{d}\rho^2 + \rho^2 (\mathrm{d}\vartheta^2 + \sin^2\vartheta\, \mathrm{d}\varphi^2)\right],$$

for spatially conformally flat coordinates $(t, \rho, \vartheta, \varphi)$. Assume furthermore that, as $\rho \downarrow \rho_0$,

(*a*) $\phi(\rho) = O(\rho - \rho_0)$, $\phi'(\rho) = O(1)$,
(*b*) $\mathrm{e}^{\gamma(\rho)} = O((\rho - \rho_0)^{-1})$, $\gamma'(\rho) = O((\rho - \rho_0)^{-1})$, *and*
(*c*) $G^\mu{}_\nu(\rho) = O((\rho - \rho_0)^{-5+\kappa})$, *for some* $\kappa > 0$.

Then the second Bianchi identity is satisfied weakly, in the sense that

$$\int_{\mathcal{M}} G^\mu{}_\nu \nabla_\mu \psi^\nu \,\mathrm{dvol}_{\mathbf{g}} = 0, \tag{1.25}$$

for any compactly supported vector field[5] $\psi \in \mathfrak{X}_c(\overline{\mathcal{M}})$.

Theorem 1.1 is our starting point for an in-depth analysis of well-known static, spherically symmetric solutions of the Einstein equations with singularities. Our condition (c) is a mild condition suggested by our method of proof, but may seem a bit unnatural because it involves the independent metric functions γ and ϕ and their first and second derivatives. In corollary 2.2 in section 2 we replace condition (c) in theorem 1.1 by stronger conditions (a′) on ϕ and its first and second derivatives, and (b′) on γ and its first and second derivatives. In corollary 2.3 a special result is also obtained for the simpler case when the metric exponents satisfy $\beta = -\alpha$

[5] Note that the vector field can have support on the inner boundary of the manifold.

in (1.7) (cf [26]) and the cumulative mass function m has the Taylor expansion assumed in (1.15) as $r \downarrow 0$.

Interestingly, we find that the superextremal RWN solution, i.e. the well-known spherically symmetric, asymptotically flat solution of the Einstein–Maxwell–Maxwell (EMM)[6] system with a timelike central singularity, does *not* satisfy the second Bianchi identity weakly at the center. This is not surprising since the mass function of the RWN solution goes to $-\infty$ as $r \downarrow 0$ (the conditions (1.15) are of course only sufficient and not necessary, but we will also rigorously establish that (1.25) does not hold).

In section 3 of this paper we investigate which properties of the EMM system are problematic by comparing RWN to spacetimes of a point charge in different electromagnetic vacua, in particular the Hoffmann spacetime solution of the Einstein–Maxwell–Born–Infeld (EMBI) system, for which we show that the second Bianchi identity is satisfied weakly at the singularity. We next discuss an application of theorem 1.1 to suitable electromagnetic vacua informally (full details are contained in section 3).

In [43] a particular subclass of admissible[7] electromagnetic Lagrangians was identified with the property that the corresponding spherically symmetric, asymptotically flat, electrostatic spacetime metrics have the *mildest* possible singularity at their center, namely, a conical singularity on the time axis. In the setting of [43] this is the case only if the bare rest mass vanishes, i.e. $m_0 = 0$.

In the present work we drop this restriction and allow a nonvanishing bare mass m_0. In fact, since with EIH and Wallace we are interested in a timelike naked singularity at the center, we need to admit *negative* m_0. Note that in the special-relativistic electrodynamical setting of [31] the problem is overdetermined when the bare mass of the particles vanishes, but is well-posed with nonzero bare mass of either sign. In the general-relativistic setting we expect that a naked singularity with strictly positive bare mass to be impossible, though (see section 3.2). Put differently, we expect that a timelike singularity with strictly positive bare mass can only exist inside a black hole. Although timelike singularities with negative bare mass can exist in a black hole, too, they can also be naked.

This generalization opens the door to much more severe than conical, but nevertheless much weaker singularities than the one at the center of superextremal RWN spacetime. One key quantity to measure the different degrees of severity of singularities is the Kretschmann scalar. We know that, as $r \to 0$, the Kretschmann scalar is proportional to r^{-4} in the case of conical singularities studied in [43]. We will see that it is of order r^{-6} in the case of *admissible* reduced Hamiltonians (as defined in section 3), and that it blows up like r^{-8} for the RWN solution. Due to the behavior of the cumulative mass function $m(r)$ at the singularity obtained in proposition 3.6 and confirming our assumption (1.15), the restrictions on the reduced Hamiltonian guarantee that the second Bianchi identity holds weakly everywhere, including at the singularity.

Theorem 1.2. *Suppose* $(\mathcal{M}, \mathbf{g}, \mathbf{F})$ *is an electrostatic spherically symmetric spacetime for an admissible reduced Hamiltonian*[8]*, and that the bare mass of the central singularity is negative. Then the twice-contracted second Bianchi identity is satisfied weakly.*

[6] The first ‘Maxwell’ here stands for the pre-metric Maxwell field equations, the second ‘Maxwell’ for Maxwell’s electromagnetic vacuum law (which he called ‘law of the pure ether’). In the same vein we will speak of EMBI system, etc.

[7] This particular notion of admissibility is fully laid out in section 3.

[8] Rigorously defined and motivated in section 3, see definition 3.1 on page 14.

This result is in stark contrast to the RWN solution for which not only the cumulative mass function (and hence the energy inside a sphere of area $4\pi r^2$) diverges to minus infinity when $r \downarrow 0$, but even the weak version of the second Bianchi identity fails. As such, the nonlinear electromagnetic theories obtained through a Lagrangian formulation are better suited to model static spacetimes of a charged point particle. We expect that these results can be extended also to non-symmetric, non-static solutions with several point charges, or ring singularities, etc.

1.6. Outline

The rest of this paper is organized as follows.

In section 2, we prove our main theorem 1.1. We derive our sufficient criterion for when the twice-contracted second Bianchi identity holds weakly. The assumption of strictly negative bare mass is required for corollary 2.3 of theorem 1.1, which leads the way to theorem 1.2.

In section 3, we investigate the Einstein–Maxwell system for a large family of (nonlinear) electromagnetic vacuum laws. For that we give a precise formulation of the admissible reduced Hamiltonians and their application to theorem 1.2. We will also prove rigorously that the weak second Bianchi identity does not hold for the well-known RWN metric.

In section 4, we conclude with a summary and an outlook on possible extensions of our results. We in particular show by direct computation that the weak second Bianchi identity holds for a family of singular fluid solutions with vanishing bare mass, showing that our conditions of our main theorem are not necessary.

2. The weak second Bianchi identity on a spacetime with timelike singularity

Throughout this section we assume that $\overline{\mathcal{M}}$ is a manifold (with an interior boundary $\mathcal{T}$) that is diffeomorphic to $\mathbb{R}^4$, equipped with a Lorentzian metric $\mathbf{g}$ on a part of $\overline{\mathcal{M}}$ that is diffeomorphic to $\mathbb{R} \times (\mathbb{R}^3 \backslash \{0\})$. The part of $\overline{\mathcal{M}}$ where we are given a Lorentzian metric is denoted by $\mathcal{M}$. In other words, we assume $(\mathcal{M}, \mathbf{g})$ to be extendible to a manifold $\overline{\mathcal{M}}$ with an interior one-dimensional timelike boundary $\mathcal{T}$ that is diffeomorphic to $\mathbb{R} \times \{0\}$ for $0 \in \mathbb{R}^3$, and assume that $\mathbf{g}$ is a (sufficiently) smooth metric tensor on the interior $\mathcal{M} = \overline{\mathcal{M}} \backslash \mathcal{T}$. In our cases of interest, $(\mathcal{M}, \mathbf{g})$ will be generally inextendible to $\overline{\mathcal{M}}$ as a sensible Lorentzian manifold (e.g. due to curvature blow-up at a naked singularity) but our results in this section apply more broadly also to scenarios where $\mathbf{g}$ is extendible in some low-regularity fashion (e.g. as continuous metric).

Recall that the classical second Bianchi identity holds on any smooth semi-Riemannian manifold and thus implies the standard twice-contracted second Bianchi identity (1.2) for the Einstein tensor pointwise *away* from the singularity $\mathcal{T} \cong \mathbb{R} \times \{0\}$. In this section we derive a distributional Bianchi identity involving the behavior *at* the singularity/boundary $\mathcal{T}$ in a manifold-with-boundary setting following the coordinate blow-up approach introduced already in section 1.3. In theorem 1.1 and corollaries 2.2 and 2.3 we provide conditions when this weak second Bianchi identity, defined in definition 2.1, holds in the static spherically symmetric setting of our interest.

As discussed in section 1.3 we assume that we can write a four-dimensional static, spherically symmetric Lorentzian manifold $(\mathcal{M}, \mathbf{g})$, which is diffeomorphic to $\mathbb{R} \times (\mathbb{R}^3 \backslash \{0\})$, using coordinates that are spatially conformally flat. For $\rho_0 > 0$ and the new coordinate $\rho \in (\rho_0, \infty)$ we obtain

$$\mathrm{d}s^2_{\mathbf{g}} = -\mathrm{e}^{2\gamma(\rho)} c^2\, \mathrm{d}t^2 + \phi^4(\rho)[\mathrm{d}\rho^2 + \rho^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\, \mathrm{d}\varphi^2)]. \tag{2.1}$$

In particular, we view $\mathcal{M}$ as $\mathbb{R} \times (\mathbb{R}^3 \backslash \overline{B_{\rho_0}(0)})$ and $\mathcal{T}$ as $\mathbb{R} \times \partial B_{\rho_0}(0)$.

To formulate the second Bianchi identity at the singularity $\{r = 0\} \cong \{\rho = \rho_0\}$ in a meaningful way, we first note that *if* $\mathbf{G}$ were a smooth tensor field defined on the manifold $\overline{\mathcal{M}} \cong \mathbb{R} \times (\mathbb{R}^3 \backslash B_{\rho_0}(0))$ *including* the interior boundary $\mathcal{T} \cong \mathbb{R} \times \partial B_{\rho_0}(0)$, then (1.2), together with integration by parts and Stokes' theorem, implies that for any smooth compactly supported vector field ψ we have

$$\begin{aligned} 0 &= \int_{\mathcal{M}} \psi^\nu \nabla_\mu G^\mu{}_\nu \, \mathrm{dvol}_{\mathbf{g}} = \int_{\mathcal{M}} \nabla_\mu(\psi^\nu G^\mu{}_\nu) \mathrm{dvol}_{\mathbf{g}} - \int_{\mathcal{M}} G^\mu{}_\nu \nabla_\mu \psi^\nu \mathrm{dvol}_{\mathbf{g}} \\ &= \int_{\mathcal{T}} \mathrm{i}_{\psi^\nu G^\mu{}_\nu}(\mathrm{dvol}_{\mathbf{g}}) - \int_{\mathcal{M}} G^\mu{}_\nu \nabla_\mu \psi^\nu \mathrm{dvol}_{\mathbf{g}} = -\int_{\mathcal{M}} G^\mu{}_\nu \nabla_\mu \psi^\nu \mathrm{dvol}_{\mathbf{g}}, \end{aligned} \tag{2.2}$$

where the last equality follows from the fact that ψ and $\mathbf{G}$ are smooth, hence constant on ∂B_{ρ_0}, and ∂B_{ρ_0} is a ZAS. Note that Stokes can be applied here, since the support of ψ is also bounded in t. This motivates us to define the weak formulation of the twice contracted second Bianchi identity in terms of spacetime integration against test vector fields.

In more singular situations where $\mathbf{G}$ and thus the interior boundary integral $\int_{\mathcal{T}}$ in (2.2) may not be well-defined (e.g. in the case of curvature blow-up) we thus seek an inhomogeneous version of the identity, namely

$$\int_{\mathcal{M}} G^\mu{}_\nu \nabla_\mu \psi^\nu \mathrm{dvol}_{\mathbf{g}} = \lim_{\varepsilon \to 0} \int_{\mathcal{T}_\varepsilon} \mathrm{i}_{\psi^\nu G^\mu{}_\nu}(\mathrm{dvol}_{\mathbf{g}}),$$

that should furthermore equal zero in the case of a ZAS. Here, $(\mathcal{T}_\varepsilon)_{\varepsilon > 0}$ with $\mathcal{T}_\varepsilon \cong \mathbb{R} \times \partial B_{\rho_0 + \varepsilon}(0)$ is the net converging to the singularity (compare this to the notation of the Riemannian ZAS $\mathcal{S}_0 \subseteq \partial\Sigma$ and its mass in section 1.4).

We thus rigorously define a weak version of the second Bianchi identity (for $\mathbf{G}$) as follows using an approximation of $\mathcal{T}$. While in this work we only focus on the situation where $\mathcal{T}$ consists of a singular one-dimensional timelike singularity in the center, it is clear that an analogous definition can be used for multiple such singularities, or other more general types of interior boundaries.

Definition 2.1. Let $(\mathcal{M}, \mathbf{g})$ be a smooth four-dimensional static, spherically symmetric Lorentzian manifold, which is diffeomorphic to $\mathbb{R} \times (\mathbb{R}^3 \backslash \overline{B_{\rho_0}(0)})$ with spatially conformally flat coordinates $(t, \rho, \vartheta, \varphi)$ (see section 1.3). We say that the *inhomogeneous twice-contracted second Bianchi identity holds weakly on* $\mathcal{M}$ if for the Einstein tensor $\mathbf{G} = (G^\mu{}_\nu)$ of $\mathbf{g}$ and for any compactly supported vector field $\psi \in \mathfrak{X}_c(\overline{\mathcal{M}})$ the integral

$$\int_{\mathcal{M}} G^\mu{}_\nu \nabla_\mu \psi^\nu \, \mathrm{dvol}_{\mathbf{g}} \tag{2.3}$$

exists, and equals

$$\lim_{\varepsilon \to 0} \int_{\mathcal{T}_\varepsilon} \mathrm{i}_{\psi^\nu G^\mu{}_\nu}(\mathrm{dvol}_{\mathbf{g}}), \tag{2.4}$$

where $\mathcal{T}_\varepsilon \cong \mathbb{R} \times \partial B_{\rho_0 + \varepsilon}(0)$. We say that the *twice-contracted second Bianchi identity holds weakly* if (2.3) = (2.4) = 0.

We can now prove the main result of this paper.

Theorem 1.1. *Let* $\mathcal{M} \cong \mathbb{R} \times (\mathbb{R}^3 \backslash \overline{B_{\rho_0}(0)})$ *be equipped with a static, spherically symmetric Lorentzian metric* $\mathbf{g}$ *of the form* (2.1), *i.e. for spatially conformally flat coordinates* $(t, \rho, \vartheta, \varphi)$

the metric tensor $\mathbf{g}$ *is given by*

$$ds^2_{\mathbf{g}} = -e^{2\gamma(\rho)} c^2\, dt^2 + \phi^4(\rho)\left[d\rho^2 + \rho^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)\right].$$

Assume furthermore that, as $\rho \downarrow \rho_0$,

(*a*) $\phi(\rho) = O(\rho - \rho_0)$, $\phi'(\rho) = O(1)$,
(*b*) $e^{\gamma(\rho)} = O((\rho - \rho_0)^{-1})$, $\gamma'(\rho) = O((\rho - \rho_0)^{-1})$, *and*
(*c*) $G^\mu{}_\nu(\rho) = O((\rho - \rho_0)^{-5+\kappa})$ *for some* $\kappa > 0$.

Then

$$\int_{\mathcal{M}} G^\mu{}_\nu \nabla_\mu \psi^\nu\, \mathrm{dvol}_{\mathbf{g}} = -4\pi\rho_0^2 c \int_{-\infty}^{\infty} \psi^\rho(t, \rho_0) dt \cdot \lim_{\varepsilon\to 0} G^\rho{}_\rho(\rho_0 + \varepsilon) e^{\gamma(\rho_0+\varepsilon)}\, \phi(\rho_0 + \varepsilon)^6 = 0, \tag{2.5}$$

for all vector fields $\psi \in \mathfrak{X}_c(\overline{\mathcal{M}})$ *with* $\psi(t, \rho_0)$ *being independent*[9] *of angular components* (ϑ, φ), *and thus the second Bianchi identity is satisfied weakly in the sense of definition 2.1.*

Proof. We first show that for any compactly supported vector field $\psi \in \mathfrak{X}_c(\overline{\mathcal{M}})$ the integral

$$\int_{\mathcal{M}} G^\mu{}_\nu \nabla_\mu \psi^\nu\, \mathrm{dvol}_{\mathbf{g}} \tag{2.6}$$

exists. Note that in spatially conformally flat coordinates $(t, \rho, \vartheta, \varphi)$, the volume element reads

$$\mathrm{dvol}_{\mathbf{g}} = c\, e^\gamma \phi^6 \rho^2\, \sin\vartheta dt \wedge d\rho \wedge d\vartheta \wedge d\varphi = c\, e^\gamma \phi^6\, dV^4,$$

where dV^n denotes the Euclidean volume form on $\mathbb{R}^n$. The nonvanishing Christoffel symbols are

$$\begin{aligned} &\Gamma^t_{\rho t} = \gamma' && \Gamma^\rho_{tt} = \frac{c^2 e^{2\gamma}\gamma'}{\phi^4} && \Gamma^\rho_{\rho\rho} = \frac{2\phi'}{\phi} && \Gamma^\vartheta_{\vartheta\rho} = \Gamma^\varphi_{\varphi\rho} = \frac{1}{\rho} + \Gamma^\rho_{\rho\rho} \\ &\Gamma^\rho_{\vartheta\vartheta} = -\rho^2\Gamma^\vartheta_{\vartheta\rho} && \Gamma^\rho_{\varphi\varphi} = \sin^2\vartheta\Gamma^\rho_{\vartheta\vartheta} && \Gamma^\varphi_{\varphi\vartheta} = -\cot\vartheta && \Gamma^\vartheta_{\varphi\varphi} = -\sin^2\vartheta\,\Gamma^\varphi_{\varphi\rho}. \end{aligned} \tag{2.7}$$

For proving the existence of the integral (2.6) consider each of the summands in $\phi^6 G^\mu{}_\nu \nabla_\mu \psi^\nu$ separately. Since ψ is smooth on $\bar{\mathcal{M}}$, all derivatives are bounded, and it remains to consider the contributions of the Christoffel symbols in $\nabla_\mu \psi^\nu$. Now,

$$G^\mu{}_\nu \nabla_\mu \psi^\nu = G^t{}_t \nabla_t \psi^t + G^\rho{}_\rho \nabla_\rho \psi^\rho + G^\vartheta{}_\vartheta \nabla_\vartheta \psi^\vartheta + G^\varphi{}_\varphi \nabla_\varphi \psi^\varphi$$

with $\phi^6 G^\mu{}_\nu = O((\rho - \rho_0)^{1+\kappa})$ by assumption (c), and by assumptions (a) and (b) furthermore

$$\begin{aligned} \nabla_t \psi^t &= \partial_t \psi^t + \psi^\alpha \Gamma^t_{t\alpha} = \partial_t \psi^t + \gamma' \psi^\rho = O((\rho - \rho_0)^{-1}), \\ \nabla_\rho \psi^\rho &= \partial_\rho \psi^\rho + \frac{2\phi'}{\phi}\psi^\rho = O((\rho - \rho_0)^{-1}), \\ \nabla_\vartheta \psi^\vartheta &= \partial_\vartheta \psi^\vartheta + \left(\frac{1}{\rho} + \frac{2\phi'}{\phi}\right)\psi^\rho = O((\rho - \rho_0)^{-1}), \end{aligned}$$

[9] While this restriction is not necessary, it makes sense because we are just artificially blowing up the one-dimensional singularity $\mathcal{T} \cong \mathbb{R} \times \{0\}$.

$$\nabla_\varphi \psi^\varphi = \partial_\varphi \psi^\varphi + \left(\frac{1}{\rho} + \frac{2\phi'}{\phi}\right)\psi^\rho + \cot\vartheta\,\psi^\vartheta = O((\rho-\rho_0)^{-1}).$$

Hence $c\,\mathrm{e}^\gamma \phi^6 G^\mu{}_\nu \nabla_\mu \psi^\nu = O((\rho-\rho_0)^{-1+\kappa})$ and (2.6) exists.

Next we show that the integral is the same as the limit of the boundary integral on $\mathcal{T}_\varepsilon \subseteq \partial\mathcal{M}_\varepsilon$ as defined in (2.5), where by $\mathcal{M}_\varepsilon$ we denote the interior that is diffeomorphic to $\mathbb{R}\times(\mathbb{R}^3\backslash\overline{B_{\rho_0+\varepsilon}(0)})$. Since the integral exists,

$$\int_{\mathcal{M}} G^\mu{}_\nu \nabla_\mu \psi^\nu \,\mathrm{dvol}_{\mathbf{g}} = \lim_{\varepsilon\to 0}\int_{\mathcal{M}_\varepsilon} G^\mu{}_\nu \nabla_\mu \psi^\nu \,\mathrm{dvol}_{\mathbf{g}}.$$

Because $\mathbf{g}$ is smooth on $\mathcal{M}_\varepsilon$, the classical second Bianchi identity immediately implies, as in (2.2), that for the vector field $X = X^\mu\partial_\mu$, defined by $X^\mu = G^\mu{}_\nu\psi^\nu$, we obtain by Stokes' theorem

$$\begin{aligned}\int_{\mathcal{M}_\varepsilon} G^\mu{}_\nu \nabla_\mu \psi^\nu \,\mathrm{dvol}_{\mathbf{g}} &= \int_{\mathcal{M}_\varepsilon} (\mathrm{div}\, X)\mathrm{dvol}_{\mathbf{g}} \\ &= \int_{\partial\mathcal{M}_\varepsilon} \mathrm{i}_X(\mathrm{dvol}_{\mathbf{g}}) \\ &= \int_{\mathcal{T}_\varepsilon} \mathrm{i}_X(\mathrm{dvol}_{\mathbf{g}}).\end{aligned}$$

Here, $\mathrm{i}_X(\mathrm{dvol}_{\mathbf{g}})$ denotes the interior product of the volume form with X. Recall that only the component of X normal to $\mathcal{T}_\varepsilon = \mathbb{R}\times\partial B_{\rho_0+\varepsilon}(0)$ contributes. In coordinates $(t,\rho,\vartheta,\varphi)$ the outward-pointing[10] unit normal to $\mathcal{T}_\varepsilon$ is $N = -(g_{\rho\rho})^{-\frac{1}{2}}\partial_\rho$. Since $\mathcal{T}_\varepsilon$ is Lorentzian,

$$\begin{aligned}\mathrm{i}_X(\mathrm{dvol}_{\mathbf{g}})|_{\mathcal{T}_\varepsilon} &= g(X,N)\mathrm{i}_N(\mathrm{dvol}_{\mathbf{g}}) \\ &= g_{\rho\rho}(g_{\rho\rho})^{-\frac{1}{2}}X^\rho \mathrm{i}_{(g_{\rho\rho})^{-\frac{1}{2}}\partial_\rho}(\mathrm{dvol}_{\mathbf{g}}) \\ &= X^\rho \mathrm{i}_{\partial_\rho}(\mathrm{dvol}_{\mathbf{g}}).\end{aligned}$$

Since $X^\rho = G^\rho{}_\rho\psi^\rho$ does not contain off-diagonal terms, this implies

$$\mathrm{i}_X(\mathrm{dvol}_{\mathbf{g}})|_{\mathcal{T}_\varepsilon} = G^\rho{}_\rho\psi^\rho \mathrm{i}_{\partial_\rho}(\mathrm{dvol}_{\mathbf{g}}),$$

with

$$\mathrm{i}_{\partial_\rho}(\mathrm{dvol}_{\mathbf{g}})|_{\mathcal{T}_\varepsilon} = -c\,\mathrm{e}^{\gamma(\rho_0+\varepsilon)}\,\phi(\rho_0+\varepsilon)^6(\rho_0+\varepsilon)^2\,\sin\vartheta\,\mathrm{d}t\wedge\mathrm{d}\vartheta\wedge\mathrm{d}\varphi.$$

Hence the integrand of the boundary integral reduces to

$$\begin{aligned}\mathrm{i}_X(\mathrm{dvol}_{\mathbf{g}})|_{\mathcal{T}_\varepsilon} = &-G^\rho{}_\rho(\rho_0+\varepsilon)\psi^\rho(t,\rho,\vartheta,\varphi)c\,\mathrm{e}^{\gamma(\rho_0+\varepsilon)}\phi(\rho_0+\varepsilon)^6 \\ &\times(\rho_0+\varepsilon)^2\,\sin\vartheta\,\mathrm{d}t\wedge\mathrm{d}\vartheta\wedge\mathrm{d}\varphi,\end{aligned}$$

[10] Recall that for a unit normal vector N to $\partial\Omega$ we have $\mathrm{i}_X\omega|_{\partial\Omega} = \mathrm{i}_{X^\perp}\omega = X^0\mathrm{i}_N\omega$ with $X^\perp = \pm g(X,N)N$. Thus, if N is spacelike (and $\partial\Omega$ Lorentzian) it must be chosen outward-pointing and if N is timelike (and $\partial\Omega$ Riemannian) inward-pointing.

so that

$$\int_{\mathcal{M}_\varepsilon} G^\mu{}_\nu \nabla_\mu \psi^\nu \,\mathrm{dvol}_{\mathbf{g}} = -cG^\rho{}_\rho(\rho_0+\varepsilon)\mathrm{e}^{\gamma(\rho_0+\varepsilon)}\phi(\rho_0+\varepsilon)^6 \int_{\mathcal{T}_\varepsilon} \psi^\rho \,\mathrm{d}S,$$

where $\mathrm{d}S = (\rho_0+\varepsilon)^2 \sin\vartheta \,\mathrm{d}t\wedge\mathrm{d}\vartheta\wedge\mathrm{d}\varphi$. Since ψ^ρ is smooth (and compactly supported)

$$\lim_{\varepsilon\to 0}\int_{\mathcal{T}_\varepsilon}\psi^\rho\,\mathrm{d}S = \lim_{\varepsilon\to 0} 4\pi(\rho_0+\varepsilon)^2 \fint_{\partial B_{\rho_0+\varepsilon}(0)}\int_{-\infty}^{\infty}\psi^\rho\,\mathrm{d}t\,\mathrm{d}\tilde{S},$$

where $\mathrm{d}\tilde{S} = \mathrm{d}S_{\partial B_{\rho_0+\varepsilon}(0)}$ denotes the surface element of $\partial B_{\rho_0+\varepsilon}(0)$ in flat $\mathbb{R}^3$. Since ψ^ρ and therefore $\Psi^\rho := \int_{-\infty}^{\infty}\psi^\rho\,\mathrm{d}t$ is smooth, we observe that $\lim_{\varepsilon\to 0}\fint \Psi^\rho\,\mathrm{d}\tilde{S} = \Psi^\rho(\rho_0)$. We thus obtain that

$$\begin{aligned}\int_{\mathcal{M}} G^\mu{}_\nu\nabla_\mu\psi^\nu\,\mathrm{dvol}_{\mathbf{g}} &= \lim_{\varepsilon\to 0}\int_{\mathcal{M}_\varepsilon} G^\mu{}_\nu\nabla_\mu\psi^\nu\,\mathrm{dvol}_{\mathbf{g}}\\ &= -4\pi\rho_0^2 c\Psi^\rho(\rho_0)\lim_{\varepsilon\to 0} G^\rho{}_\rho(\rho_0+\varepsilon)\mathrm{e}^{\gamma(\rho_0+\varepsilon)}\phi(\rho_0+\varepsilon)^6.\end{aligned}$$

Due to the assumptions (a)–(c) it follows that

$$G^\rho{}_\rho(\rho_0+\varepsilon)\mathrm{e}^{\gamma(\rho_0+\varepsilon)}\phi(\rho_0+\varepsilon)^6 \sim |\rho-\rho_0|^{-5+\kappa-1+6}\to 0 \quad \text{as } \varepsilon\to 0$$

□

Even though condition (c) of theorem 1.1 is close to being optimal for the conclusion of that theorem to hold, the condition may seem somewhat unnatural in view of the fact that the components of G^μ_ν generally involve both metric coefficients $\phi(\rho)$ and $\gamma(\rho)$ together with their first and second derivatives. We now show that it is possible to eliminate condition (c) entirely by strengthening conditions (a) and (b) to include suitable assumptions on the second derivatives of the independent metric coefficients $\phi(\rho)$ and $\gamma(\rho)$.

Corollary 2.2. *With the same setup as in theorem 1.1, assume that the metric coefficients ϕ and γ satisfy the following stronger assumptions as $\rho_0\downarrow\rho$:*

(i') *Assumptions on ϕ:*

- (a) $\phi(\rho) = O(\rho-\rho_0)$
- (b) $\frac{\phi'}{\phi}(\rho) = (\rho-\rho_0)^{-1} - \frac{\rho+\rho_0}{2\rho\rho_0} + O(\rho-\rho_0)$
- (c) $\frac{\phi''}{\phi}(\rho) = \frac{-2}{\rho}(\rho-\rho_0)^{-1} + O(1)$

(ii') *Assumptions on γ:*

- (a) $\mathrm{e}^{\gamma(\rho)} = O((\rho-\rho_0)^{-1})$,
- (b) $\gamma'(\rho) = -(\rho-\rho_0)^{-1} + \frac{1}{2\rho_0} + O(\rho-\rho_0)$,
- (c) $\gamma''(\rho) = (\rho-\rho_0)^{-2} + O(1)$.

Then the same conclusion as theorem 1.1 holds, namely, the twice contracted second Bianchi identity is satisfied weakly in the sense of definition 2.1.

Proof. A computation shows that for the metric $\mathbf{g}$ of theorem 1.1, the Einstein tensor $\mathbf{G}$ is diagonal, and we have

$$G^\rho{}_\rho = \frac{2}{\rho\phi^4}\left(\gamma' + (1+\rho\gamma')\frac{2\phi'}{\phi} + 2\rho\left(\frac{\phi'}{\phi}\right)^2\right). \tag{2.8}$$

Hence, by (i′b) and (ii′b), $G^{\rho}{}_{\rho} = O((\rho-\rho_0)^{-4})$ as $\rho \downarrow \rho_0$, while by (i′b) and (i′c) we have

$$G^{t}{}_{t} = \frac{4}{\rho\phi^4}\left(\frac{2\phi'}{\phi} + \frac{\rho\phi''}{\phi}\right) = O((\rho-\rho_0)^{-4}). \tag{2.9}$$

Finally, the above two, together with (i′b), (ii′b) and (ii′c) imply that

$$\begin{aligned} G^{\vartheta}{}_{\vartheta} = G^{\varphi}{}_{\varphi} &= \frac{1}{2}(G^{t}{}_{t} - G^{\rho}{}_{\rho}) + \frac{1}{\rho\phi^4}\left(\gamma'\left(2+\rho\left(\gamma' + \frac{2\phi'}{\phi}\right)\right) + \rho\gamma''\right) \\ &= O((\rho-\rho_0)^{-4}), \end{aligned} \tag{2.10}$$

so that hypothesis (c) of theorem 1.1 holds with $\kappa = 1$. □

Sufficient conditions in the usual spherically symmetric coordinates can also be derived:

Corollary 2.3. *Consider* $(\mathcal{M}, \mathbf{g})$ *as in theorem 1.1. Suppose* $\mathbf{g}$ *in area-radius coordinates* $(t, r, \vartheta, \varphi)$ *is of the form*

$$\mathrm{d}s^2_{\mathbf{g}} = -\mathrm{e}^{2\alpha(r)}c^2\,\mathrm{d}t^2 + \mathrm{e}^{-2\alpha(r)}\,\mathrm{d}r^2 + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2),$$

with

$$\mathrm{e}^{2\alpha(r)} = 1 - \frac{2Gm(r)}{c^2\,r} \tag{2.11}$$

and such that the cumulative mass function has an absolutely converging power series expansion

$$m(r) \;=\; m_0 + m_1 r + m_2 r^2 + O(r^3), \quad \text{as } r\downarrow 0, \tag{2.12}$$

with $m_0 < 0$*. Then the second Bianchi identity holds weakly in the sense of definition 2.1.*

Proof. By the discussion in section 1.3 the transformation to spatially conformally flat coordinates $(t, \rho, \vartheta, \varphi)$ is possible. It remains to be checked that the assumptions (i′) and (ii′) in corollary 2.2 are also satisfied.

By assumption (2.12) we have

$$r^2 - \frac{2G}{c^2} r m(r) = O_1(r), \quad r\downarrow 0,$$

hence the new coordinate $\rho(r)$ given by (1.14) is well-defined for small $r > 0$ and satisfies

$$\begin{aligned} \rho(r) &= \rho_0\left(1 + \sqrt{-\frac{2c^2}{Gm_0}}\,r^{\frac{1}{2}} - \frac{c^2}{Gm_0}r - \frac{3c^2 + Gm_1}{6\left(Gm_0\sqrt{-\frac{2Gm_0}{c^2}}\right)}r^{\frac{3}{2}} + \frac{c^2 m_1}{3Gm_0^2}r^2 + O\left(r^{\frac{5}{2}}\right)\right) \\ &= \rho_0 + O_2(\sqrt{r}). \end{aligned}$$

Hence the inverse function $r = r(\rho)$ has an expansion of the form

$$
\begin{aligned}
r(\rho) &= \frac{Gm_0}{2c^2\rho_0^2}(\rho-\rho_0)^2\left(-1+\frac{1}{\rho_0}(\rho-\rho_0)\right.\\
&\quad\left.+\left(-\frac{1}{4\rho_0^2}+\frac{2Gm_1-9c^2}{12c^2\rho_0^2}\right)(\rho-\rho_0)^2+O((\rho-\rho_0)^3)\right)\\
&= O_2((\rho-\rho_0)^2),
\end{aligned}
$$

and thus

$$
\phi(\rho)^2 = \frac{r(\rho)}{\rho} = O_2(|\rho-\rho_0|^2), \quad \rho\downarrow\rho_0,
$$

with

$$
\frac{\phi'}{\phi} = -\frac{1}{2\rho}+\frac{r'(\rho)}{2r(\rho)} = (\rho-\rho_0)^{-1}-\frac{1}{2}\left(\frac{1}{\rho_0}+\frac{1}{\rho}\right)+O(\rho-\rho_0), \tag{2.13}
$$

which establishes (i′a) and (i′b) of corollary 2.2. Moreover, (ii′a) holds since (2.12) implies

$$
\begin{aligned}
\mathrm{e}^{2\gamma(\rho)} &= \mathrm{e}^{2\alpha(r(\rho))}\\
&= 1-\frac{2Gm(r(\rho))}{r(\rho)}\\
&= 4c^2\rho_0^2(\rho-\rho_0)^{-2}+4c^2\rho_0(\rho-\rho_0)^{-1}+1-2Gm_1+O(\rho-\rho_0)\\
&= O_2((\rho-\rho_0)^{-2}),
\end{aligned}
$$

and, in particular, using (2.13),

$$
\begin{aligned}
\gamma'(\rho) &= \frac{1}{2}(\mathrm{e}^{2\gamma(\rho)})'\mathrm{e}^{-2\gamma(\rho)}\\
&= G\,\mathrm{e}^{-2\gamma(\rho)}\frac{r(\rho)'}{r(\rho)}\left(-m'(r(\rho))+\frac{m(r(\rho))}{r(\rho)}\right)\\
&= -\frac{1}{2}\left(1-\frac{1}{\rho_0}(\rho-\rho_0)+\cdots\right)\left(2\frac{\phi'}{\phi}+\frac{1}{\rho}\right)\left(1+\frac{1}{\rho_0}(\rho-\rho_0)+\cdots\right)\\
&= -\left(\frac{\phi'}{\phi}+\frac{1}{2\rho}\right)\left(1+O((\rho-\rho_0)^2)\right)\\
&= -(\rho-\rho_0)^{-1}+(2\rho_0)^{-1}+O(\rho-\rho_0)
\end{aligned}
$$

which establishes (ii′b), and upon differentiation, (ii′c) of corollary 2.2. Finally, using (2.13) we have

$$
\begin{aligned}
\frac{\phi''}{\phi} &= \left(\frac{\phi'}{\phi}\right)'+\left(\frac{\phi'}{\phi}\right)^2\\
&= -\left(\frac{1}{\rho}+\frac{1}{\rho_0}\right)(\rho-\rho_0)^{-1}+O(1)\\
&= \frac{-2}{\rho}(\rho-\rho_0)^{-1}+O(1)
\end{aligned}
$$

which establishes (i′c) of corollary 2.2, and thus the second twice-contracted Bianchi identity holds weakly. □

Remark 2.4. The condition that $\beta = -\alpha$ for the coordinate coefficients in corollary 2.3 is not necessary for the coordinate transformation. If $\beta \neq -\alpha$ then the cumulative mass function is given as usual by $m(r) := \frac{c^2}{2G} r \left(1 - \mathrm{e}^{-2\beta(r)}\right)$ and m needs to define the same conditions in order for the coordinate transformation to spatially conformally flat coordinates to go through. However, whether the conditions (b) and (c) in theorem 1.1 hold then not only depends on the behavior of m but also on the asymptotic behavior on α as $r \downarrow 0$ since $\gamma(\rho) := \alpha(r(\rho))$.

3. Spherically symmetric electrostatic spacetimes

The Einstein–Maxwell equations, i.e. (1.1) together with

$$\mathrm{d}\mathbf{F} = 0, \quad \mathrm{d}\mathbf{M} = 0, \tag{3.1}$$

valid pointwise away from any singularities of spacetime, are part of any field theory of electromagnetism in general relativity. The electromagnetic field is represented by the Faraday tensor **F** and the Maxwell tensor **M**. To arrive at a field theory of electromagnetism the tensors **F** and **M** need to be related by a 'law of the electromagnetic vacuum' ('ether law' for short), which also fixes the corresponding electromagnetic energy-momentum-stress tensor **T**.

Using Eiesland's theorem [12, 13], which is a generalized and, in fact, preceding version of the well-known Birkhoff theorem, one of us [43, theorem 6.2] showed that the metric **g** of any electrostatic, spherically symmetric spacetime with an electromagnetic vacuum law determined by a field Lagrangian which depends only on the two invariants of **F**, viz $\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and $\frac{1}{4}F_{\mu\nu} \star F^{\mu\nu}$, must, in spherical coordinates $(t, r, \vartheta, \varphi)$, be given by

$$\mathrm{d}s^2_{\mathbf{g}} = -\mathrm{e}^{2\alpha(r)}c^2\mathrm{d}t^2 + \mathrm{e}^{-2\alpha(r)}\,\mathrm{d}r^2 + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2). \tag{3.2}$$

The function $\alpha(r)$ is smooth for $r > 0$ and depends on the ether law.

The simplest law of the electromagnetic vacuum is Maxwell's 'law of the pure ether',

$$\mathbf{M} = -\star\mathbf{F}, \tag{3.3}$$

where $\star$ is the Hodge star operator[11] with respect to **g**. In this case the set of coupled equations (1.1), (3.1) and (3.3) is called the EMM system. The unique static, spherically symmetric, asymptotically flat EMM spacetime is the RWN solution with metric component

$$\mathrm{e}^{2\alpha(r)} = 1 - \frac{2G}{c^4\,r}\left(Mc^2 - \frac{Q^2}{2r}\right). \tag{3.4}$$

One can show that M is the ADM mass of the spacetime, while Q is its charge. The ratio[12] $\frac{GM^2}{Q^2}$ determines the causal structure of the RWN spacetime ($\frac{GM^2}{Q^2} > 1$: subextremal with black

[11] The Hodge $\star$ dual of a k-form is a $(n-k)$-form, where n is the number of dimensions. In our setting, $\star$ takes a two-form to the dual two-form.

[12] In a Newtonian theory, the fraction $\frac{GM_1M_2}{|Q_1Q_2|}$ is the ratio of the coupling constants of the gravitational and electrical pair interaction energies of any two interacting point charges. Inserting empirical values, for two interacting electrons one finds the tiny value $\frac{Gm_e^2}{e_e^2} \approx 2.4 \times 10^{-43}$. If one has only one point charge, it is tempting to think of $\frac{GM^2}{Q^2}$ as the ratio of the gravitational and electrical self-energy coupling constants, but in a Newtonian theory there is no such thing, and in special-relativistic electromagnetic Maxwell–Lorentz field theory of point charges, the self energies are infinite. This does not improve in general relativity, so the meaning of $\frac{GM^2}{Q^2}$ lies elsewhere.

hole region; $\frac{GM^2}{Q^2} = 1$: extremal with black hole region; $\frac{GM^2}{Q^2} < 1$: superextremal with a naked singularity). While one would expect the superextremal RWN spacetime to represent the simplest realistic charged-particle spacetime, some troubling divergence behaviors occur. More precisely, the cumulative mass function

$$m(r) \;=\; \frac{c^2}{2G} r \left(1 - \mathrm{e}^{2\alpha(r)}\right),$$

(cf (1.13)) which for RWN reads

$$m_{\mathrm{RWN}}(r) = M - \frac{Q^2}{2c^2 r}, \tag{3.5}$$

diverges when $r \downarrow 0$, together with the Kretschmann scalar for RWN (see the discussion in [43, section 1] and section 3.2 below).

One way to overcome the divergence of the cumulative mass function is to consider a non-linear electromagnetic theory, for instance the Born–Infeld theory [6, 7] (for more historical context in a modern language, see [27, 28]). This is done by choosing a Lorentz- and (Weyl) gauge-invariant Lagrangian density $\mathbf{L}$ for the electromagnetic action

$$\mathcal{S}[\mathbf{A}] = \int_{\mathcal{M}} \mathbf{L}(\mathrm{d}\mathbf{A})$$

such that in the weak field limit it reduces to the Lagrangian of the Maxwell–Maxwell system (3.1) and (3.3), and has *finite* total energy for a point charge.

In the spherically symmetric electrostatic case the above formulation boils down to finding a suitable *reduced Hamiltonian*[13] ζ that yields a solution to (1.1) and (3.1) having an ADM mass $M = \lim_{r\to\infty} m(r)$, and a charge Q. Such a reduced Hamiltonian is subject to the following admissibility criteria.

Definition 3.1. A function $\zeta : \mathbb{R}_0^+ \to \mathbb{R}$ is called an *admissible reduced Hamiltonian* if it satisfies

(R1) $\lim_{\mu\to 0} \frac{\zeta(\mu)}{\mu} = 1$.
(R2) $\zeta'(\mu) > 0$ and $\zeta(\mu) - \mu\zeta'(\mu) \geqslant 0$ for all $\mu > 0$.
(R3) $\zeta'(\mu) + 2\mu\zeta''(\mu) \geqslant 0$ for all $\mu > 0$.
(R4) $I_\zeta = 2^{-\frac{11}{4}} \int_0^\infty y^{-\frac{7}{4}} \zeta(y) \mathrm{d}y < \infty$.
(R5) There exist constants $J_\zeta, K_\zeta, L_\zeta > 0$ such that

$$J_\zeta\sqrt{\mu} - K_\zeta \leqslant \zeta(\mu) \leqslant J_\zeta\sqrt{\mu}, \quad \text{and} \quad J_\zeta\sqrt{\mu} - 2L_\zeta \leqslant 2\zeta'(\mu)\mu \leqslant J_\zeta\sqrt{\mu}.$$

These admissibility criteria are derived and motivated in detail in section 3.2 below. Note that in terms of the reduced Hamiltonian ζ the cumulative mass function is

$$m(r) = M - \frac{1}{c^2}\int_r^\infty \zeta\left(\frac{Q^2}{2s^4}\right) s^2\,\mathrm{d}s, \tag{3.6}$$

and the electric potential is $\mathbf{A} = \varphi(r)c\,\mathrm{d}t$ with

$$\varphi(r) = Q\int_r^\infty \zeta'\left(\frac{Q^2}{2s^4}\right)\frac{1}{s^2}\,\mathrm{d}s. \tag{3.7}$$

[13] We now largely follow the notation of [43], with only smaller deviations.

One can easily check that the Born–Infeld reduced Hamiltonian

$$\zeta_{\mathrm{BI}}(\mu) = \sqrt{1+2\mu} - 1$$

is an admissible Hamiltonian, and it leads to the Hoffmann solution [25]. Here, μ is a dimensionless $|D|^2$, where D is Maxwell's displacement field (w.r.t. a Lorentz frame).

In section 3.1 we give another prototype example with a ZAS (with asymptotic behavior as discussed in section 1.3) which is obtained from a particular admissible reduced Hamiltonian and hence satisfies the second twice-contracted Bianchi identity weakly.

In section 3.2 we study reduced Hamiltonians ζ more systematically. In particular, we discuss the admissibility conditions (R1)–(R5) from definition 3.1 and its consequences. One such important consequence is that given an electrostatic spacetime solution with charge $Q \in \mathbb{R}\backslash\{0\}$, by rescaling the reduced Hamiltonian ζ we can find another electrostatic spacetime solution that corresponds to the new, rescaled vacuum law, has the same charge Q, and has any desired *bare rest mass* $m_0 \leqslant 0$ and ADM mass $M > m_0$. We prove this in proposition 3.2 in section 3.2. Subsequently we revisit the second Bianchi identity and show that an admissible reduced Hamiltonian implies that the second Bianchi identity is satisfied weakly (cf theorem 1.2 in the introduction) based on our general results in section 2.

3.1. A prototype electrovac spacetime with finite negative bare mass

Let $m_0 < 0$, $M > 0$ and $Q \in \mathbb{R}\backslash\{0\}$ be given, and assume that

$$\xi^2 := \frac{G(M-m_0)^2}{Q^2} < 1. \tag{3.8}$$

Let $(\mathcal{M}_0, \mathbf{g}_0)$ denote the static, spherically symmetric, asymptotically flat, electromagnetic spacetime that corresponds to the following reduced Hamiltonian

$$\zeta(\mu) = \min\{\mu, \sqrt{\mu_0\mu}\}, \qquad \mu_0 := \frac{(M-m_0)^4 c^8}{2Q^6}. \tag{3.9}$$

It is not hard to see that the mass function of this spacetime will be

$$m(r) = m_0 + \frac{M-m_0}{2}\begin{cases} \dfrac{r}{r_0} & r < r_0 \\ 2 - \dfrac{r_0}{r} & r > r_0, \end{cases} \qquad r_0 := \frac{Q^2}{(M-m_0)c^2}. \tag{3.10}$$

Thus $m(0) = m_0 < 0$, $m(\infty) = M > 0$. One can also verify that the total charge of the spacetime is Q, and that the singularity at $r = 0$ is not shielded by a horizon.

We observe that on this three-parameter family of electrovac spacetimes, indexed by m_0, M, Q, the asymptotic behavior (3.10) together with corollary 2.3 immediately implies that the twice-contracted second Bianchi identity is satisfied weakly.

3.2. Nonlinear electrostatic spacetimes with naked singularities

We defined admissible reduced Hamiltonians in definition 3.1 via the properties (R1)–(R5). The reason for these requirements are discussed extensively in [43, section 4]. Let us briefly mention that (R1) implies that the weak field limit is the same as for classical linear electromagnetics, (R2) implies the dominant energy condition is satisfied, and (R3) guarantees that ζ is the Legendre–Fenchel transform of a Lagrangian density. Note that (R3) together with (R1) implies the strong energy condition.

If m denotes the cumulative mass function (3.6), then $\lim_{r\to\infty} m(r) = M$, and the metric components in coordinates $(t, r, \vartheta, \varphi)$ in (3.2) are given by $\mathrm{e}^{2\alpha(r)} = 1 - \frac{2Gm(r)}{c^2 r}$ as in (2.11).

If $\lim_{r\downarrow 0} m(r) =: m_0$ is finite and nonzero, the Kretschmann scalar

$$\begin{aligned}\mathcal{K} &:= R_{\mu\nu\lambda\eta}R^{\mu\nu\lambda\eta}\\ &= \frac{4G^2}{c^4 r^6}\left(12m^2 + 4rm(-4m' + rm'') + r^2(8m'^2 - rm'm'' + r^2 m''^2)\right)\end{aligned} \tag{3.11}$$

blows up at least like r^{-6} as $r \downarrow 0$, indicating the presence of a true singularity as opposed to a mere coordinate singularity. In the case of the negative-mass Schwarzschild metric the Kretschmann scalar $\mathcal{K} = \frac{48G^2M^2}{c^4r^6}$ is moreover proportional to the square of the mass M. For the RWN metric (3.4) with ADM mass M and charge Q (the Maxwell law $\zeta(\mu) = \mu$ satisfies the criteria (R1)–(R3) but not more), the Kretschmann scalar

$$\mathcal{K}_{\mathrm{RWN}}(r) = \frac{48G^2M^2}{c^4r^6}\left(1 + \frac{2Q^2}{c^2Mr} + 7\frac{Q^4}{48c^4M^2r^2}\right) \tag{3.12}$$

blows up like r^{-8} as $r \downarrow 0$.

Thus the above conditions alone already imply that the center at $r = 0$ must be nonregular. In [43] the mildest possible naked singularity was studied and found to be a conical singularity with $m_0 = 0$ and blow-up rate r^{-4} of the Kretschmann scalar. More generally *naked* singularities require $m_0 \leqslant 0$; otherwise black holes will occur. In other words, there are two cases to consider:

Case 1: $m_0 > 0$. We show that, so long as $M < \infty$, the spacetime will have a horizon. In this vein, assume to the contrary that there is no horizon, so that the coordinate chart $(t, r, \vartheta, \varphi) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{S}^2$ is global, with r spacelike and t timelike throughout. Set

$$f(r) := c^2 r - 2Gm(r).$$

Then, since $m_0 > 0$ by assumption, $f(0) = -2Gm_0 < 0$, while $M < \infty$ clearly implies $f(\infty) > 0$. Since $m(r)$ is a continuous function for $r > 0$, it now follows that there exists an $r_0 > 0$ such that $f(r_0) = 0$. But this implies that $\mathrm{e}^{2\alpha(r_0)} = \frac{1}{c^2}\frac{f(r_0)}{r_0} = 0$. In fact the metric coefficient g_{00} generally changes sign across $r = r_0$, which means that the area-radius coordinate r becomes timelike for $r < r_0$, in contradiction to the hypothesis that r is spacelike throughout. Therefore $r = r_0$ is a Killing horizon, and the coordinate chart only covers the region $r > r_0$ of the spacetime.

Case 2: $m_0 \leqslant 0$. In the superextremal RWN spacetime one has $\lim_{r\downarrow 0} m(r) = -\infty$ and a severe curvature singularity at $r = 0$. This is due to a non-integrable electrostatic field energy density about $r = 0$. However, for admissible Hamiltonians with finite electrostatic field energy, one can compute the total field energy to be

$$\int_0^\infty \zeta\left(\frac{Q^2}{2s^4}\right)s^2\,\mathrm{d}s = |Q|^{\frac{3}{2}} I_\zeta, \qquad I_\zeta := 2^{-\frac{11}{4}}\int_0^\infty y^{-\frac{7}{4}}\zeta(y)\,\mathrm{d}y \ < \infty,$$

as demanded in (R4), and we therefore have $M - m_0 = \frac{|Q|^{\frac{3}{2}} I_\zeta}{c^2}$; i.e. the difference between the accumulated mass of the spacetime and the bare rest mass is entirely due to the electrostatic field. For the ADM mass M we then have, in general,

$$M = m_0 + \frac{|Q|^{\frac{3}{2}} I_\zeta}{c^2}. \tag{3.13}$$

Thus we have $\lim_{r\downarrow 0} m(r) =: m_0 \in (-\infty, 0]$, yielding a less singular behavior at $r = 0$.

From now on we assume that $m_0 \leqslant 0$ is finite and consider the behavior of the spacetime near the center. If we assume that there exists a positive constant J_ζ such that

(R5′) $\zeta(\mu) \leqslant J_\zeta \sqrt{\mu}$,

then the integral term of the cumulative mass function $m(r)$ in (3.6) can be estimated using

$$\int_0^r \zeta\left(\frac{Q^2}{2s^4}\right) s^2\,\mathrm{d}s \leqslant J_\zeta |Q| \frac{r}{\sqrt{2}},$$

which implies that in a neighborhood of the center, $m(r)$ is bounded from above by

$$m(r) \leqslant m_0 + J_\zeta |Q| \frac{r}{\sqrt{2}c^2}. \tag{3.14}$$

Since by assumption $m_0 \leqslant 0$, this shows that the metric coefficient g_{00} is bounded away from zero,

$$\mathrm{e}^{2\alpha(r)} = 1 - \frac{2Gm(r)}{c^2 r} \geqslant 1 - \frac{2Gm_0}{c^2 r} - \sqrt{2}J_\zeta |Q| \frac{G}{c^4} > 1 - \sqrt{2}J_\zeta |Q| \frac{G}{c^4} > 0,$$

as long as

$$\frac{|Q| G J_\zeta}{c^4} < \frac{1}{\sqrt{2}}. \tag{3.15}$$

(Note that the left-hand-side is dimensionless.) Thus, (3.15) implies the absence of a horizon, which means that a *naked singularity* occurs whenever the charge is sufficiently small. Of course, (3.15) is only a sufficient condition for absence of a horizon.

We now show that if an electrostatic spacetime solution exists for prescribed total charge Q, then for the same charge Q one can generate such a spacetime with any bare rest mass $m_0 \leqslant 0$ and ADM mass $M > m_0$. This is achieved by a rescaling of the associated reduced Hamiltonian.

Proposition 3.2. *Let ζ be an admissible Hamiltonian that satisfies (R1)–(R3). We additionally assume that ζ satisfies*

(R4) $I_\zeta = 2^{-\frac{11}{4}} \int_0^\infty y^{-\frac{7}{4}} \zeta(y)\mathrm{d}y < \infty$.

Suppose there exists an electrostatic spacetime metric $\mathbf{g}$ *with charge* $Q \in \mathbb{R}\backslash\{0\}$ *satisfying the Einstein–Maxwell equations for the ether law generated by* ζ. *Let* $m_0 \leqslant 0$ *and* $M > m_0$ *be given. Then for the dimensionless number*

$$\lambda := \frac{|Q|^{\frac{3}{2}} I_\zeta}{(M - m_0)c^2}$$

the λ*-scaled version of* ζ, *defined by*

$$\zeta_\lambda(\mu) = \lambda^{-4} \zeta(\lambda^4 \mu), \tag{3.16}$$

is itself an admissible reduced Hamiltonian, and there exists a corresponding electrostatic spacetime metric $\mathbf{g}_\lambda$ *which has charge* Q, *ADM mass* $M = \lim_{r\to\infty} m(r)$, *and bare rest mass*

$$m_0 := \lim_{r\downarrow 0} m(r). \tag{3.17}$$

Proof. Note that ζ_λ satisfies (R1)–(R4) because ζ does. Furthermore, (3.16) implies that I_{ζ_λ} as defined in (R4) transforms as

$$I_{\zeta_\lambda} = 2^{-\frac{11}{4}} \int_0^\infty y^{-\frac{7}{4}} \lambda^{-4} \zeta(\lambda^4 y) \mathrm{d}y = \lambda^{-1} I_\zeta. \tag{3.18}$$

Therefore, by (3.13),

$$M = m_0 + \frac{|Q|^{\frac{3}{2}} I_{\zeta_\lambda}}{c^2}$$

as desired. □

Remark 3.3. The borderline case $m_0 = 0$ was treated already in [43]. In this case there is no bare mass at the center, and the geometric ADM mass M is entirely due to the electrostatic field energy $|Q|^{\frac{3}{2}} I_\zeta$, more precisely, $M = \frac{|Q|^{\frac{3}{2}} I_\zeta}{c^2}$. In [43] it was also shown that given any charge Q, also *any* positive ADM mass $M > 0$ can be achieved in this case via an appropriate choice of a scaling parameter: For $\zeta_\lambda(\mu) = \lambda^{-4}\zeta(\lambda^4\mu)$ we have $I_{\zeta_\lambda} = \lambda^{-1} I_\zeta$, and the ADM mass becomes $M = \frac{1}{\lambda}\frac{|Q|^{\frac{3}{2}} I_\zeta}{c^2}$, with Q still the charge of the spacetime. By a suitable choice of λ, any value of $M > 0$ can be generated. Clearly, this is a special case of our proposition 3.2. These solutions are asymptotically flat with a conical singularity at the center if the ratio $\frac{GM^2}{Q^2}$ is sufficiently small; see [43, section 5.1].

The sufficient condition (3.15) for obtaining a naked singularity can be reformulated in the λ-scaled setting. Note that $J_{\zeta_\lambda} = \lambda^{-2} J_\zeta$. Hence (3.15) translates to

$$\frac{G(M - m_0)^2}{|Q|^2} < \frac{I_\zeta^2}{\sqrt{2} J_\zeta}. \tag{3.19}$$

From now on we always assume that (3.19) is satisfied.

Together with $m_0 \leqslant 0$, condition (3.19) guarantees that there is no horizon and r is a spacelike coordinate on $(0, \infty)$. In fact, we have

Proposition 3.4. *Suppose ζ is an admissible reduced Hamiltonian satisfying (R1)–(R5′) and λ etc. is given as in proposition 3.2. If the dimensionless ratio*

$$\epsilon^2 := \frac{G(M - m_0)^2}{|Q|^2}$$

is sufficiently small (as in (3.19)), then $\mathbf{g}_\lambda$ features a naked singularity at the center. □

Note that ϵ^2 is a *dimensionless* quantity in Gaussian units (see also [3, p 5] for a discussion).

Example 3.5 (Born–Infeld model). In the setting of the Born–Infeld theory, where $\zeta(\mu) = \sqrt{1 + 2\mu} - 1$, we can choose $J_\zeta = \sqrt{2}$ in (R5′). Moreover,

$$I_\zeta = -\frac{\Gamma\left(-\frac{3}{4}\right)\Gamma\left(\frac{5}{4}\right)}{2\sqrt{\pi}} \approx 1.236\,0498,$$

and thus

$$\frac{I_\zeta^2}{\sqrt{2} J_\zeta} \approx 0.763\,90954.$$

If we consider the mass and charge of an electron, i.e.

$$M_{\mathrm{e}} = 9.109\,383\,56 \times 10^{-31}\ (\mathrm{kg}), \qquad Q_{\mathrm{e}} = 1.6021765 \times 10^{-19}\ (\mathrm{C}) \cdot \sqrt{k_{\mathrm{e}}},$$

where $k_{\mathrm{e}} = 8.987\,551\,79 \times 10^{9}$ (kg m^3 s^{-2} C^{-2}) is the Coulomb constant, then for $m_0 = 0$, and gravitational constant $G = 6.674\,08 \times 10^{-11}$ (m^3 kg^{-1} s^{-2}), we have

$$\epsilon^2 = \frac{GM^2}{|Q|^2} \approx 2.400\,53 \times 10^{-43},$$

so we are far in the naked singularity regime due to (3.19) being satisfied. Since gravitational effects ($\propto G$) are small, for $m_0 < 0$ we are guaranteed a naked singularity so long as $m_0 > -\varsigma M$, where ς is a large positive constant.

Next, let us consider the behavior of the spacetime near the center of the symmetry, for $m_0 < 0$. The singularity at $r = 0$ will no longer be conical, but exhibit a stronger blow-up behavior. If, in addition to (R5′) we assume that there is also $J_\zeta, K_\zeta > 0$ such that

(R5″)3.1) $J_\zeta\sqrt{\mu} - K_\zeta \leqslant \zeta(\mu)$,
then we also obtain an estimate of $m(r)$ from below. More precisely,

$$m(r) = m_0 + \frac{1}{c^2}\int_0^r \zeta\left(\frac{Q^2}{2s^4}\right) s^2\,\mathrm{d}s \geqslant m_0 + J_\zeta|Q|\frac{r}{\sqrt{2}c^2} - K_\zeta\frac{r^3}{3c^2}, \tag{3.20}$$

which together with the upper bound (3.14) implies that

$$m(r) = m_0 + J_\zeta|Q|\frac{r}{\sqrt{2}c^2} + O(r^3)$$

as $r \downarrow 0$.

If we in addition assume that there is a positive constant $L_\zeta > 0$ such that

(R5″)3.2) $J_\zeta\sqrt{\mu} - 2L_\zeta \leqslant 2\zeta'(\mu)\mu \leqslant J_\zeta\sqrt{\mu}$,
then we can also infer something about the decay of the derivatives of $m(r)$. We combine all properties (R5′)–(R5‴) in (R5).

Proposition 3.6. *If $\zeta : \mathbb{R}_0^+ \to \mathbb{R}$ is an admissible reduced Hamiltonian, that is, it satisfies the properties (R1)–(R4) as well as*

(*R5*) *There exist positive constants $J_\zeta, K_\zeta, L_\zeta$ such that*

$$J_\zeta\sqrt{\mu} - K_\zeta \leqslant \zeta(\mu) \leqslant J_\zeta\sqrt{\mu}, \quad \text{and} \quad J_\zeta\sqrt{\mu} - 2L_\zeta \leqslant 2\zeta'(\mu)\mu \leqslant J_\zeta\sqrt{\mu},$$

then the cumulative mass function is of the form

$$\begin{aligned} m(r) &= m_0 + \frac{1}{2c^2}\int_0^r \zeta\left(\frac{Q^2}{2s^4}\right) s^2\,\mathrm{d}s \\ &= m_0 + \frac{J_\zeta|Q|}{\sqrt{2}c^2}r - \frac{K_\zeta}{3c^2}r^3 + O_2(r^3), \quad \text{as } r \downarrow 0, \end{aligned}$$

where we say that $f(r) = O_k(r^\alpha)$ as $r \downarrow 0$ if $r^{j-\alpha}\frac{d^j f}{\mathrm{d}r^j}$ is bounded for $j = 0, \ldots, k$ as $r \downarrow 0$.

Remark 3.7. By the first part of (R2), in particular, $\zeta'(\mu) \geqslant 0$ and thus also $\zeta(\mu) \geqslant 0$ by the second part for all $\mu \geqslant 0$. Next, consider $f(\mu) = \log \frac{\zeta(\mu)}{\mu}$. Then $f(0) = \log 1 = 0$ by (R1) and by (R2)

$$f'(\mu) = \frac{\mu\zeta'(\mu) - \zeta(\mu)}{\mu\zeta(\mu)} \leqslant 0.$$

Hence by integration also $f(\mu) \leqslant 0$ and therefore $\zeta(\mu) \leqslant \mu$ for all $\mu \geqslant 0$. Together with the first part of (R5) we thus obtain for $\mu \geqslant 0$ that

$$\max\{0, J_\zeta\sqrt{\mu} - K_\zeta\} \leqslant \zeta(\mu) \leqslant \min\{\mu, J_\zeta\sqrt{\mu}\}. \tag{3.21}$$

Similarly, (R2) implies that $0 \leqslant \zeta'(\mu)\mu \leqslant \zeta(\mu)$ so that together with the second part of (R5) we have for $\mu \geqslant 0$ that

$$\max\{0, J_\zeta\sqrt{\mu} - 2L_\zeta\} \leqslant 2\zeta'(\mu)\mu \leqslant \min\{2\mu, J_\zeta\sqrt{\mu}\}. \tag{3.22}$$

We will use these inequalities in our proof of proposition 3.6 below.

Proof. As we have already seen in (3.14)–(3.20) the first part of (R5) implies that

$$m_0 + \frac{J_\zeta|Q|}{\sqrt{2}c^2}r - \frac{K_\zeta}{3c^2}r^3 \leqslant m(r) \leqslant m_0 + \frac{J_\zeta|Q|}{\sqrt{2}c^2}r,$$

and thus shows that

$$0 \leqslant r^{-3}\left[m(r) - \left(m_0 + \frac{J_\zeta|Q|}{\sqrt{2}c^2}r - \frac{K_\zeta}{3c^2}r^3\right)\right] \leqslant \frac{K_\zeta}{3c^2},$$

remains bounded. Since $m'(r) = \zeta\left(\frac{Q^2}{2r^4c^2}\right)r^2$, using the first part of (R5) we again obtain that $\frac{J_\zeta|Q|}{\sqrt{2}c^2} - \frac{K_\zeta r^2}{c^2} \leqslant m'(r) \leqslant \frac{J_\zeta|Q|}{\sqrt{2}c^2}$, hence

$$0 \leqslant r^{-2}\frac{\mathrm{d}}{\mathrm{d}r}\left[m(r) - \left(m_0 + \frac{J_\zeta|Q|}{\sqrt{2}c^2}r - \frac{K_\zeta}{3c^2}r^3\right)\right] \leqslant \frac{K_\zeta}{c^2}.$$

The second derivative of $m(r)$ satisfies

$$\begin{aligned} -\frac{2rK_\zeta}{c^2} \leqslant m''(r) &= \frac{2r}{c^2}\zeta\left(\frac{Q^2}{2r^4}\right) - 4\zeta'\left(\frac{Q^2}{2r^4}\right)\frac{Q^2}{2r^4}\frac{r}{c^2} \\ &\leqslant \frac{\sqrt{2}J_\zeta|Q|}{rc^2} - \frac{\sqrt{2}J_\zeta|Q|}{rc^2} + \frac{4L_\zeta r}{c^2} = \frac{4L_\zeta r}{c^2}, \end{aligned}$$

and thus

$$\begin{aligned} 0 &\leqslant r^{-1}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left[m(r) - \left(m_0 + \frac{J_\zeta|Q|}{\sqrt{2}c^2}r - \frac{K_\zeta}{3c^2}r^3\right)\right] = r^{-1}\left[m''(r) + \frac{2K_\zeta}{c^2}r\right] \\ &\leqslant \frac{4L_\zeta + 2K_\zeta}{c^2} \end{aligned}$$

is bounded as well. Therefore, by definition of $O_2(r^3)$,

$$m(r) = m_0 + \frac{J_\zeta|Q|}{\sqrt{2}c^2}r - \frac{K_\zeta}{3c^2}r^3 + O_2(r^3), \quad \text{as } r \downarrow 0.$$

□

With the results obtained for admissible nonlinear theories we are now in a position to show that the weak second Bianchi identity does hold for spherically symmetric electrostatic spacetimes where the reduced Hamiltonian ζ satisfies (R1)–(R5).

Theorem 1.2. *Suppose* $(\mathcal{M}, \mathbf{g}, \mathbf{F})$ *is an electrostatic spherically symmetric spacetime considered in proposition 3.6 such that* $m_0 < 0$ *and there is a naked singularity at the center. Then the twice-contracted second Bianchi identity holds weakly on* $\mathcal{M}$.

Proof. By proposition 3.6, $m(r) = m_0 + \frac{J_\zeta |Q|}{\sqrt{2}c^2} r - \frac{K_\zeta}{3c^2} r^3 + O_2(r^3)$, as $r \downarrow 0$. Hence the second Bianchi identity is satisfied weakly also at the singularity due to corollary 2.3. □

Remark 3.8. Note that even though the value of m_0 is not relevant for whether the Bianchi identity holds or does not hold weakly, its sign does matter, since we use the radial variable r all the way down to $r = 0$, which is not possible in the presence of a horizon.

3.3. The RWN spacetime does not satisfy the weak second Bianchi identity

In the previous subsection we identified a whole class of electrostatic spacetimes for which the second Bianchi identity *does* hold weakly. Using spatially conformally flat coordinates we now show that the RWN spacetime *does not* satisfy the weak second Bianchi identity. Since this coordinate transformation is rather involved in practice, we also include a heuristic explanation of this ‘too singular’ behavior of the RWN spacetime in terms of the blow-up rate of the Kretschmann scalar.

We recall that the RWN metric is of the form

$$\mathrm{d}s_{\mathbf{g}}^2 = -\mathrm{e}^{2\alpha(r)} c^2\, \mathrm{d}t^2 + \mathrm{e}^{-2\alpha(r)}\, \mathrm{d}r^2 + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\, \mathrm{d}\varphi^2),$$

where

$$\mathrm{e}^{2\alpha(r)} = 1 - \frac{2GM}{c^2\, r} + \frac{GQ^2}{c^4 r^2} =: 1 - \frac{2}{r} A + \frac{1}{r^2} B^2$$

with ADM mass M and charge Q. We are interested in the superextremal case $\frac{A^2}{B^2} = \frac{GM^2}{Q^2} < 1$ which has a naked singularity in the center, and will show that in this case the second Bianchi identity does not hold weakly.

Note that the cumulative mass function

$$m(r) = M - \frac{Q^2}{2c^2 r}$$

blows up at the center and hence does *not* have the required asymptotics (1.15) discussed earlier. However, the integral appearing in the definition of ρ in (1.14), i.e.

$$\int_0^r \frac{\mathrm{d}r'}{\sqrt{r'^2 - \frac{2G}{c^2} r' m(r')}} = \int_0^r \frac{\mathrm{d}r'}{\sqrt{r'^2 - 2Ar' + B^2}}$$

does converge for $r < B = \frac{\sqrt{G}|Q|}{c^2}$ since the denominator satisfies

$$0 < (r' - B)^2 < r'^2 - 2Ar' + B^2 < (r' - A)^2$$

due to superextremality. Hence a coordinate transformation to spatially conformally flat coordinates $(t, \rho, \vartheta, \varphi)$ is possible near the singularity.

If we set

$$\rho_0 := B - A = \frac{\sqrt{G}|Q| - GM}{c^2} > 0$$

then the change of coordinates (1.14) is given by (compare to the calculation of the nmS case in section 1.4)

$$\rho(r) = (r - A) + \sqrt{(r - A)^2 + (B^2 - A^2)}, \qquad r = \frac{(\rho + A)^2 - B^2}{2\rho}.$$

The conformal factor, ϕ, is then

$$\phi(\rho)^2 = \frac{r}{\rho} = \frac{(\rho + A)^2 - B^2}{2\rho^2} = O(\rho - \rho_0),$$

and

$$\begin{aligned} e^{2\gamma(\rho)} &= e^{2\alpha(r(\rho))} \\ &= 1 - \frac{2}{\rho\phi^2}A + \frac{1}{\rho^2\phi^4}B^2 \\ &= \frac{\rho^2\phi^4 - 2\rho\phi A + B^2}{\rho^2\phi^4} \\ &= O((\rho - \rho_0)^{-2}). \end{aligned}$$

where the blow-up rate follows from the above since $A < B$. Note that some of these decay/blow-up rates already differ from the requirement in theorem 1.1 (a) and (b). Without looking into all the requirements of theorem 1.1 separately, we directly jump to investigate the critical quantity $G^{\rho}{}_{\rho} e^{\gamma}\phi^6$ appearing in (2.5). To this end we observe that

$$\gamma' = \frac{(e^{2\gamma})'}{2\, e^{2\gamma}} = (e^{-2\gamma} - 1)\left(\frac{1}{\rho} + 2\frac{\phi'}{\phi}\right) = O((\rho - \rho_0)^{-1}),$$

since

$$\frac{\phi'}{\phi} = \frac{(\phi^2)'}{2\phi^2} = \frac{(B^2 - A^2) - 2A\rho}{2\rho((\rho + A)^2 - B^2)} = O((\rho - \rho_0)^{-1}).$$

Thus by (2.8)

$$G^{\rho}{}_{\rho} = \frac{1}{\rho^2\phi^4 - 2\rho\phi A + B^2}\left(2 + 8\rho\frac{\phi'}{\phi} + 8\rho^2\left(\frac{\phi'}{\phi}\right)^2\right) - \frac{1}{\rho^2\phi^4} - \frac{4}{\phi^4}\left(\frac{\phi'}{\phi}\right)^2,$$

hence $G^{\rho}{}_{\rho} = O((\rho - \rho_0)^{-4})$ and $G^{\rho}{}_{\rho} e^{\gamma}\phi^6 = O((\rho - \rho_0)^{-2})$. Therefore, the second term in (2.5) does not converge to zero, and the second Bianchi identity does not hold weakly. In fact, even the inhomogeneous second Bianchi identity does not hold weakly.

4. Summary and outlook

In this paper we have considered the following question: under what conditions is the twice-contracted second Bianchi identity satisfied in a weak sense in a neighborhood of a singular line of a spacetime $\mathcal{M}$ with the metric $\mathbf{g}$? We were able to answer this question in case $(\mathcal{M}, \mathbf{g})$ is both static and spherically symmetric, by finding sufficient conditions on the metric that, if satisfied, guarantee that the weak second Bianchi identity holds everywhere, the location of a timelike singularity included.

The main application of this result is to electrovacuum spacetimes with timelike singularities. We have shown that the Einstein–Maxwell equations, complemented with a nonlinear vacuum law which satisfies certain admissibility conditions, have spherically symmetric, static solutions describing the electrostatic spacetime of a point charge with weakly satisfied twice-contracted second Bianchi identity. We also found that the Bianchi identity is not weakly satisfied by the RWN solution, which is obtained when complementing the Einstein–Maxwell equations with the standard linear vacuum law of Maxwell. The favorable electrostatic spacetimes turn out to be less singular than RWN, a fact that is evident from the blow-up behavior of their curvature invariants. In our setting, for example, the Kretschmann scalar of an electrostatic spacetime with weakly satisfied twice-contracted second Bianchi identity blows up at most like r^{-6} as $r \downarrow 0$, while it blows up as r^{-8} in the RWN solution. In the case of a vanishing bare rest mass, i.e. $m_0 = 0$, the blow-up rate is only r^{-4} as $r \downarrow 0$, leading to the mildest possible (a conical) singularity.

Our findings add another argument to the many that have already been offered for why Maxwell's linear vacuum law (3.3) should be replaced by a nonlinear law that reduces to (3.3) in the weak-field limit (for in this limit the Maxwell–Maxwell electromagnetic field equations are indisputably successful), see [36], and which furnishes finite field energies of point charges (unlike Maxwell's law of the vacuum), see [6, 7] for the most prominent earliest voices in this regard. The family of possible laws that allow for a weak twice-contracted second Bianchi identity is huge, so that one has to look elsewhere for arguments that could help narrowing down the list of potential candidates. The Born–Infeld law [7] stands out in this regard because it follows from a handful of compelling principles, see [6, 37], each of which seemingly capturing some aspect of nature. Since all these models depend on at least one extra parameter, and reduce to Maxwell's law in the limit where this parameter vanishes, experimental results can restrict the realm of possible parameter values for each model, and possibly rule out specific models, but it is difficult to see how empirical results alone could hint at the 'right' nonlinear model. To find the right one—if this is indeed the message—one needs to argue based on deeper plausible principles, as Born and Infeld tried, and Plebanski [37] and Boillat [5] did.

The conditions that yield a weak second Bianchi identity, which we derived in theorem 1.1 and several corollaries, can be applied more generally, to static spherically symmetric Lorentzian manifolds with a singularity in the center. We note that the proof of our corollary 2.3 does, in general, *not* extend to the case $m_0 = 0$, because the coordinate transformation from r to ρ would involve the infinite integral $\int_0^r \frac{\mathrm{d}r'}{r'}$. However, we believe a result similar to corollary 2.3 can be obtained as long as $m_0 \leqslant 0$ and $m(r) = O(r^{\kappa})$, $0 \leqslant \kappa < 1$, as $r \downarrow 0$. Moreover, there are cases with zero bare mass $m_0 = 0$ and conical singularities that do admit a transformation to spatially conformally flat coordinates (2.1) if we allow $\rho_0 = 0$ and possibly interpret the Bray mass using the more general definition of [9, section 3.2] via approximation of regular ones. Examples are an explicit singular solution of the astrophysically important Tolmann–Oppenheimer–Volkoff equation, studied already by Chandrasekhar [10] and others [1, 35] in view of its asymptotics and discussed below, and also the Hoffmann spacetime discussed in [25, 43].

Example 4.1 (A singular static spherically symmetric fluid with vanishing bare mass that satisfies the weak second Bianchi identity). For solutions of the Einstein–Euler equations with linear equation of state $p = (kc)^2\mu, k \in (0, 1)$, with conical singularity described in [1, section 3.2.1] the cumulative mass function is of the form

$$m(r) = \frac{Kc^2}{2G}r, \qquad K := \frac{4k^2}{4k^2 + (1 + k^2)^2}, \qquad r \in (0, \infty),$$

and hence goes to zero when $r \downarrow 0$. Thus $e^{-2\beta} = 1 - \frac{2Gm}{c^2} = 1 - K < 1$. Hence the transformation to spatially conformally flat coordinates (2.1) is given by $\rho = r^{\frac{1}{\sqrt{1-K}}}$ and $\phi^2 = \rho^{\sqrt{1-K}-1}$ if we allow $\rho_0 = 0$. In this case the weak version of the second Bianchi identity can be obtained directly in the (t, r, ϑ, ϕ) coordinates since $r^2\,\mathrm{e}^{\beta(r)}\,\mathrm{d}r = \rho^2\phi^6\,\mathrm{d}\rho$ and $\alpha(r) = \gamma(\rho)$ etc. In particular, the limit in (2.5) translates to

$$\lim_{\varepsilon\to 0}\varepsilon^2 G^r{}_r(\varepsilon)\mathrm{e}^{\alpha(\varepsilon)+\beta(\varepsilon)} = \lim_{\varepsilon\to 0}\mathrm{e}^{\alpha(\varepsilon)-\beta(\varepsilon)}\left(1 - \mathrm{e}^{2\beta(\varepsilon)} + \frac{4k^2}{1+k^2}\right)$$
$$= C_k\lim_{\varepsilon\to 0}\varepsilon^{\frac{2k^2}{1+k^2}} = 0,$$

where C_k is a constant depending on k. This ends our example.

Furthermore, it is clear that the main ideas developed in this paper are not restricted to static spherically symmetric spacetimes and are adaptable to more general situations. In particular, we expect that our results can be extended to non-static, non-symmetric spacetimes with finitely many timelike singularities, which appear as point-type, or perhaps (st)ring-type singularities in the spacelike leaves of any foliation of the spacetime into 'evolving spaces'; for a study of the equation of motion of singularities of the latter type, see [2]. We expect that the less severe blow-up behavior demanded by the weak Bianchi identity will point the way to the formulation of a well-posed dynamical theory for charged timelike singularities and the electromagnetic spacetime structures around them. By requiring compatible singularities in the electromagnetic energy–momentum–stress tensor this in turn should lead to the identification of an admissible class of electromagnetic vacuum laws. Our preliminary inquiry in this direction also indicates that the admixture of a scalar field that modulates the gravitational coupling of the electromagnetic field energy–momentum–stress tensor to the spacetime curvature may be needed. Moreover, due to the occurrence of off-diagonal components in the Einstein tensor, it is reasonable to expect that more restrictions on the metric may be required in order to obtain a broadly applicable result analogous to theorem 1.1.

In all these cases we also expect the bare mass of the singularity to be strictly negative. We recall that it had to be strictly negative in the spacetimes studied in the present paper for the weak twice-contracted second Bianchi identity to hold with a rigorous geometric interpretation of the bare rest mass as Bray's ZAS mass. Moreover, in the general-relativistic setting we expect positive bare mass to imply a black hole (cf the discussion in [19]), not a naked singularity, and as such could not serve as a suitable point-charge model of physical 'particles' like nuclei, or electrons.

To this we add the following thought: given the spectacular high precision agreement of quantum-mechanical computations of atomic spectra with the empirical ones, modifications of

Maxwell's vacuum law would have to be significant only in the immediate vicinity of a point charge. This in turn suggests that the electromagnetic self-field energies, though finite, will still be huge, and this in turn implies that the bare mass would have to be negative, so that the total mass agrees with the empirical mass as obtained in scattering experiments. This is an argument for why even in special relativity a consistent electrodynamical theory of fields and their point charge sources that also agrees with observations would have to be formulated with negative bare mass. Our finding that we were able to establish the weak second twice-contracted Bianchi identity only for naked singularities with negative bare mass seems fitting.

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Data availability statement

No new data were created or analysed in this study.

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