

SINGULAR 1-SOLITON SOLUTION OF THE $K(m, n)$ EQUATION WITH GENERALIZED EVOLUTIONS AND ITS SUBSIDIARIES

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(Received November 27, 2012)

This paper addresses the singular 1-soliton solution of the $K(m, n)$ equation that is being considered with generalized evolution. The ansatz method will be used to extract the singular 1-soliton solution of this equation. A couple of constraint conditions will fall out in order for the singular soliton solutions to exist. Subsequently, the special cases of this equation will be studied. They are the KdV and the mKdV equations where the extended (G'/G) -expansion method will be employed to extract a few non-linear wave solutions.

DOI:10.5506/APhysPolB.44.1825

PACS numbers: 02.30.Jr, 04.25.Jb, 04.30.Nk

1. Introduction

The theory of nonlinear evolution equations (NLEEs) is very important in areas of mathematical sciences and theoretical physics [1–20]. These equations form the fabric of nonlinear science. The stupendous analytical developments of these NLEEs have led to a plethora of results that are constantly being applied to various areas of engineering sciences, biological sciences, chemical and geological sciences. Therefore, it is imperative that these NLEEs are dug deeper and so that furthermore new results can be extracted. Consequently, it is important to revise several of the NLEEs studied already earlier to obtain further results carrying unprecedented novelty.

Equations discussed in this paper are all already well studied. However, it is some of several new solutions that make this paper very unique and an extension of various previous works. Initially, this paper apply the ansatz

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method in order to extract the singular 1-soliton solution to the $K(m, n)$ equation that is a generalized version of the KdV equation. In fact, this paper will look into the $K(m, n)$ equation with generalized evolution. The constraint conditions will naturally come out of this equation. Subsequently, the G'/G -expansion method will be applied to extract several solutions to the subsidiaries of the $K(m, n)$ equation. They are the KdV equation and the modified KdV (mKdV) equation. These two special cases, namely the KdV equation and the mKdV equation are studied in the context of shallow water waves that appear in oceans and rivers.

2. Singular 1-soliton solution

The form of the $K(m, n)$ equation with generalized evolution that will be studied in this paper is given by

$$\left(q^l\right)_t + a(q^m)_x + b(q^n)_{xxx} = 0. \quad (2.1)$$

Equation (2.1) is the $K(m, n)$ equation with generalized evolution. The dependent variables is $q(x, t)$ represent the wave profile, while x and t are the spatial and temporal variables respectively. The first term in (2.1) is the evolution term in its generalized format. If $l = 1$, it is the linear evolution. Then, the coefficients of a and b , which are non-zero real-valued constants, are the nonlinear and dispersion terms respectively and both of these terms are in their generalized format. The special cases when $l = n = 1$ and $m = 2$ lead to the KdV equation. Then, if $l = n = 1$ along with $m = 3$, mKdV equation is recovered. This equation has been studied before in several papers [1–4]. It must be stressed that singular soliton solutions of this equation are being reported here for the very first time.

In order to obtain the singular 1-soliton solution to (2.1), the following ansatz is taken into account

$$q(x, t) = A \operatorname{csch}^p[B(x - vt)], \quad (2.2)$$

where A and B are free parameters while v is the velocity of the soliton. The unknown exponent is p whose value will fall out during the course of derivation of the soliton solution. Subsequently, this ansatz into (2.1) leads to

$$\begin{aligned} &lvA^l \operatorname{csch}^{lp}\tau - amA^m \operatorname{csch}^{mp}\tau - bn^3p^2A^nB^2 \operatorname{csch}^{np}\tau \\ &-bn(np+1)(np+2)A^nB^2 \operatorname{csch}^{np+2}\tau = 0, \end{aligned} \quad (2.3)$$

where

$$\tau = B(x - vt). \quad (2.4)$$

Then, by the balancing principle, equating the exponents lp and np leads to

$$lp = np \quad (2.5)$$

which implies

$$l = n. \quad (2.6)$$

Again, equating the exponents mp and $np + 2$ gives

$$mp = np + 2 \quad (2.7)$$

which yields

$$p = \frac{2}{m - n}. \quad (2.8)$$

Now, setting the coefficients of the linearly independent functions $\operatorname{csch}^{np+j}\tau$ for $\tau = 0.2$ to zero gives the velocity of the soliton and the relation between the free parameters as

$$v = \frac{4n^2bB^2}{(m - n)^2} \quad (2.9)$$

and

$$A = \left[-\frac{2n(m + n)bB^2}{a(m - n)^2} \right]^{\frac{1}{m-n}}. \quad (2.10)$$

Therefore, equations (2.9) and (2.10) introduce the constraints

$$m \neq n \quad (2.11)$$

for $m - n$ even and

$$ab < 0. \quad (2.12)$$

This shows that the coefficient of dispersion term and the nonlinear term must be of opposite sign in order for the singular 1-soliton solution to exist. Hence, finally, the $K(m, n)$ equation with generalized evolution that support singular 1-soliton solution is given by

$$(q^n)_t + a(q^m)_x + b(q^n)_{xxx} = 0 \quad (2.13)$$

whose singular 1-soliton solution is

$$q(x, t) = A \operatorname{csch}^{\frac{2}{m-n}} [B(x - vt)], \quad (2.14)$$

where the velocity of the soliton is (2.9) and the relation between the free parameters is related as in (2.10). This singular 1-soliton solution will exist provided the constraint condition given by (2.11) and (2.12) remains valid.

3. Extended (G'/G) -expansion method

This section will address this new version of the G'/G -expansion method that is also known as the extended G'/G -expansion method. The study will be split into two subsections. The first subsection will describe this method in a succinct form. Subsequently, it will be applied to a couple of NLEEs.

3.1. Brief description of the method

Suppose that a nonlinear equation, say in two independent variables x and t , is given by

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \dots) = 0, \quad (3.1)$$

where $u = u(x, t)$ is an unknown complex-valued function, P is a polynomial in $u = u(x, t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

The main steps of the extended $(\frac{G'}{G})$ -expansion method are the following:
Step 1: By taking $u(x, t) = u(\xi)$, $\xi = x - vt$ equation (3.1) can be converted to an ODE

$$P(u, -vu', u', v^2u'', -vu'', u'', \dots) = 0. \quad (3.2)$$

Step 2: Suppose that the solution of ODE (3.2) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows

$$u(\xi) = a_0 + \sum_{i=1}^m a_i \left(\frac{G'}{G}\right)^i + b_i \left(\frac{G'}{G}\right)^{-i}, \quad (3.3)$$

where $G = G(\xi)$ satisfies the second order linear differential equation in the form

$$G'' + \lambda G' + \mu G = 0, \quad (3.4)$$

where $a_0, a_i, b_i, v, \lambda$ and μ are constants to be determined later. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (3.2).

Step 3: Substituting (3.3) into (3.2) and using (3.4), collecting all terms with the same power of $(\frac{G'}{G})$ together and then equating the coefficients of each power of $(\frac{G'}{G})$ to zero gives a system of algebraic equations for $a_0, a_i, b_i, v, \lambda$ and μ .

Step 4: These constants are written in their places in equation (3.3) and if the solutions of second order linear ordinary differential equation (3.4) are used, the soliton solutions of nonlinear partial differential equation (3.1) are obtained.

3.1.1. KdV equation

We begin with the KdV equation in the form

$$u_t + \alpha uu_x + u_{xxx} = 0, \quad (3.5)$$

which arises in many physical problems such as surface water waves and ion-acoustic waves in plasma. To look for the travelling wave solution of Eq. (3.5), we make the transformation $u(x, t) = u(\xi)$, $\xi = x - vt$, where v is the wave speed to be determined later, Eq. (3.5) becomes on ODE for $u = u(\xi)$, $-vu' + \alpha uu' + u''' = 0$, integrating it with respect to ξ once yields

$$c - vu + \frac{\alpha}{2}u^2 + u'' = 0, \quad (3.6)$$

where c is an integration constant that is to be determined later.

Balancing the terms u^2 and u'' in Eq. (3.6) yields the leading order $m = 2$. Therefore, we can write the solution of Eq. (3.6) in the form

$$u(\xi) = a_2 \left(\frac{G'}{G} \right)^2 + a_1 \left(\frac{G'}{G} \right) + a_0 + b_1 \left(\frac{G'}{G} \right)^{-1} + b_2 \left(\frac{G'}{G} \right)^{-2}. \quad (3.7)$$

Substituting (3.4) and (3.7) into (3.6), collecting all terms with the same power of $\left(\frac{G'}{G} \right)$ and setting each coefficient to zero, we obtain the following system of algebraic equations

$$\begin{aligned} \left(\frac{G'}{G} \right)^{-4} : \frac{1}{2}b_2^2\alpha + 6b_2\mu^2 &= 0, \\ \left(\frac{G'}{G} \right)^{-3} : b_1b_2\alpha + 10b_2\lambda\mu + 2b_1\mu^2 &= 0, \\ \left(\frac{G'}{G} \right)^{-2} : -b_2v + \frac{1}{2}b_1^2\alpha + a_0b_2\alpha + 4b_2\lambda^2 + 8b_2\mu + 3b_2\mu + 3b_1\lambda\mu &= 0, \\ \left(\frac{G'}{G} \right)^{-1} : -b_1v + a_0b_1\alpha + 6b_2\lambda + b_1\lambda^2 + 2 + a_1\lambda &= 0, \\ \left(\frac{G'}{G} \right)^0 : 2b_2 + c - a_0v + \frac{1}{2}a_0^2\alpha + a_1b_1\alpha + a_2b_2\alpha + b_1\lambda + a_1\lambda\mu + 2a_2\mu^2 &= 0, \\ \left(\frac{G'}{G} \right)^1 : -a_1v + a_0a_1\alpha + a_2b_1\alpha + a_1\lambda^2 + 2a_1\mu + 6a_2\lambda\mu &= 0, \\ \left(\frac{G'}{G} \right)^2 : -a_2v + \frac{1}{2}a_1^2\alpha + a_0a_2\alpha + 3a_1\lambda + 4a_2\lambda^2 + 8a_2\mu &= 0, \end{aligned}$$

$$\left(\frac{G'}{G}\right)^3 : 2a_1 + a_1a_2\alpha + 10a_2\lambda = 0,$$

$$\left(\frac{G'}{G}\right)^4 : 6a_2 + \frac{1}{2}a_2^2\alpha = 0.$$

From the solutions system, we obtain the following with the aid of Mathematica.

Case 1:

$$\alpha \neq 0, \quad a_1 = -\frac{12\lambda}{\alpha}, \quad a_2 = -\frac{12}{\alpha}, \quad b_1 = 0, \quad b_2 = 0, \quad v = \alpha_0\alpha + \lambda^2 + 8\mu,$$

$$c = \frac{1}{2} (a_0^2\alpha + 2a_0\lambda^2 + 16a_0\mu - 2a_1\lambda\mu - 4a_2\mu^2).$$

Case 2:

$$a_1 = 0, \quad a_2 = 0, \quad \alpha\mu \neq 0, \quad b_1 = -\frac{12\lambda\mu}{\alpha}, \quad b_2 = -\frac{12\mu^2}{\alpha}, \quad b_1 \neq 0,$$

$$v = \frac{1}{b_1} (\alpha_0b_1\alpha + 6b_2\lambda + b_1\lambda^2 + 2b_1\mu), \quad c = \frac{1}{2} (-4b_2 + 2a_0v - a_0^2\alpha - 2b_1\lambda).$$

For Case 1:

(i): When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function travelling wave solution

$$u_1(\xi) = \frac{\left(\frac{-(A^2 - B^2)(a_0\alpha + 6(\lambda^2 - 2\mu)) + (A^2 + B^2)(a_0\alpha + 12\mu)}{\cosh\left[(x - vt)\sqrt{\lambda^2 - 4\mu}\right] + 2AB(a_0\alpha + 12\mu)\sinh\left[(x - vt)\sqrt{\lambda^2 - 4\mu}\right]} \right)}{2\alpha \left(B \cosh\left[\frac{1}{2}(x - vt)\sqrt{\lambda^2 - 4\mu}\right] + A \sinh\left[\frac{1}{2}(x - vt)\sqrt{\lambda^2 - 4\mu}\right] \right)^2}, \quad (3.8)$$

where $\xi = x - (\alpha_0\alpha + \lambda^2 + 8\mu)t$, A and B are arbitrary constants.

(ii): When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function travelling wave solution

$$u_2(\xi) = \frac{\left(\frac{(A^2 + B^2)(a_0\alpha + 6(\lambda^2 - 2\mu)) + (A^2 - B^2)(a_0\alpha + 12\mu)}{\cosh\left[(x - vt)\sqrt{4\mu - \lambda^2}\right] + 2AB(a_0\alpha + 12\mu)\sinh\left[(x - vt)\sqrt{4\mu - \lambda^2}\right]} \right)}{2\alpha \left(A \cos\left[\frac{1}{2}(x - vt)\sqrt{4\mu - \lambda^2}\right] + B \sin\left[\frac{1}{2}(x - vt)\sqrt{4\mu - \lambda^2}\right] \right)^2}, \quad (3.9)$$

where $\xi = x - (\alpha_0\alpha + \lambda^2 + 8\mu)t$, A and B are arbitrary constants.

In particular, if we take $A = 0$, $B \neq 0$, then the solutions (3.8) and (3.9) become

$$u_1(\xi) = \frac{a_0\alpha + 3\lambda^2 - 3(\lambda^2 - 4\mu) \tanh \left[\frac{1}{2} \sqrt{\lambda^2 - 4\mu}(x - vt) \right]^2}{\alpha}, \quad (3.10)$$

$$u_2(\xi) = \frac{1}{\alpha} (a_0\alpha + 3\lambda^2 + 3(\lambda^2 - 4\mu)) \cot \left[\frac{1}{2} \sqrt{4\mu - \lambda^2}(x - vt) \right]^2. \quad (3.11)$$

On the other hand, if we take $A \neq 0$, $B = 0$, then the solutions (3.8) and (3.9) become

$$u_1(\xi) = \frac{a_0\alpha + 3\lambda^2 - 3(\lambda^2 - 4\mu) \coth \left[\frac{1}{2} \sqrt{\lambda^2 - 4\mu}(x - vt) \right]^2}{\alpha}, \quad (3.12)$$

$$u_2(\xi) = \frac{1}{\alpha} (a_0\alpha + 3\lambda^2 + 3(\lambda^2 - 4\mu)) \tan \left[\frac{1}{2} \sqrt{4\mu - \lambda^2}(x - vt) \right]^2. \quad (3.13)$$

For Case 2:

(i): When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function travelling wave solution

$$u_3(\xi) = a_0 + \frac{24 \left(\lambda^2 - p\lambda\sqrt{\lambda^2 - 4\mu} - 2\mu \right) \mu}{\alpha \left(\lambda - p\sqrt{\lambda^2 - 4\mu} \right)^2}, \quad (3.14)$$

where

$$p = \frac{A \cosh \left[\frac{1}{2} \sqrt{\lambda^2 - 4\mu}(x - vt) \right] + B \sinh \left[\frac{1}{2} \sqrt{\lambda^2 - 4\mu}(x - vt) \right]}{A \sinh \left[\frac{1}{2} \sqrt{\lambda^2 - 4\mu}(x - vt) \right] + B \cosh \left[\frac{1}{2} \sqrt{\lambda^2 - 4\mu}(x - vt) \right]},$$

$\xi = x - \left(\frac{1}{b_1} \alpha_0 b_1 \alpha + 6b_2 \lambda + b_1 \lambda^2 + 2b_1 \mu \right) t$, A and B are arbitrary constants.

(ii): When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function travelling wave solution

$$u_4(\xi) = a_0 + \frac{24\mu \left(\lambda^2 - 2\mu - q\lambda\sqrt{4\mu - \lambda^2} \right)}{\alpha \left(\lambda - q\sqrt{4\mu - \lambda^2} \right)^2}, \quad (3.15)$$

where

$$q = \frac{-A \sin \left[\frac{1}{2} \sqrt{4\mu - \lambda^2}(x - vt) \right] + B \cos \left[\frac{1}{2} \sqrt{4\mu - \lambda^2}(x - vt) \right]}{A \cos \left[\frac{1}{2} \sqrt{4\mu - \lambda^2}(x - vt) \right] + B \sin \left[\frac{1}{2} \sqrt{4\mu - \lambda^2}(x - vt) \right]},$$

$\xi = x - \left(\frac{1}{b_1} \alpha_0 b_1 \alpha + 6b_2 \lambda + b_1 \lambda^2 + 2b_1 \mu \right) t$, A and B are arbitrary constants.

In particular, if we take $A = 0$, $B \neq 0$, then the solutions (3.14) and (3.15) become

$$u_3(\xi) = a_0 + \frac{24\mu \left(\lambda^2 - 2\mu - \lambda \sqrt{\lambda^2 - 4\mu} \tanh \left[\frac{1}{2} \sqrt{\lambda^2 - 4\mu} (x - vt) \right] \right)}{\alpha \left(\lambda - \sqrt{\lambda^2 - 4\mu} \tanh \left[\frac{1}{2} \sqrt{\lambda^2 - 4\mu} (x - vt) \right] \right)^2}, \quad (3.16)$$

$$u_4(\xi) = a_0 + \frac{24\mu \left(\lambda^2 - 2\mu - \lambda \sqrt{4\mu - \lambda^2} \cot \left[\frac{1}{2} \sqrt{4\mu - \lambda^2} (x - vt) \right] \right)}{\alpha \left(\lambda - \sqrt{4\mu - \lambda^2} \cot \left[\frac{1}{2} \sqrt{4\mu - \lambda^2} (x - vt) \right] \right)^2}. \quad (3.17)$$

On the other hand, if we take $A \neq 0$, $B = 0$, then the solutions (3.14) and (3.15) become

$$u_3(\xi) = a_0 + \frac{24\mu \left(\lambda^2 - 2\mu - \lambda \sqrt{\lambda^2 - 4\mu} \coth \left[\frac{1}{2} \sqrt{\lambda^2 - 4\mu} (x - vt) \right] \right)}{\alpha \left(\lambda - \sqrt{\lambda^2 - 4\mu} \coth \left[\frac{1}{2} \sqrt{\lambda^2 - 4\mu} (x - vt) \right] \right)^2}, \quad (3.18)$$

$$u_4(\xi) = a_0 + \frac{24\mu \left(\lambda^2 - 2\mu - \lambda \sqrt{4\mu - \lambda^2} \tan \left[\frac{1}{2} \sqrt{4\mu - \lambda^2} (x - vt) \right] \right)}{\alpha \left(\lambda - \sqrt{4\mu - \lambda^2} \tan \left[\frac{1}{2} \sqrt{4\mu - \lambda^2} (x - vt) \right] \right)^2}. \quad (3.19)$$

Remark 1: Solutions (3.10)–(3.13) and (3.16)–(3.19) are similar with Wazwaz's [6] solutions obtained by using sine–cosine, tanh–coth methods. However, we observe that solutions (3.8), (3.9), (3.14) and (3.15) are presented here for the first time.

3.1.2. mKdV equation

We consider the mKdV equation in the form

$$u_t + \alpha u^2 u_x + u_{xxx} = 0. \quad (3.20)$$

To look for the travelling wave solution of Eq. (3.20), we make the transformation $u(x, t) = u(\xi)$, $\xi = x - vt$, where v is the wave speed to be determined later, and integrating it with respect to ξ once yields

$$c - vu + \frac{\alpha}{3} u^3 + u'' = 0, \quad (3.21)$$

where c is an integration constant that is to be determined later. Balancing the terms u^3 and u'' in Eq. (3.21) yields the leading order $m = 1$. Therefore,

we can write the solution of Eq. (3.21) in the form

$$u(\xi) = a_0 + a_1 \left(\frac{G'}{G} \right) + b_1 \left(\frac{G'}{G} \right)^{-1}. \quad (3.22)$$

Substituting (2.4) and (3.22) into (3.21), collecting all terms with the same power of $\left(\frac{G'}{G} \right)$ and setting each coefficient to zero, we obtain the following system of algebraic equations:

$$\begin{aligned} \left(\frac{G'}{G} \right)^{-3} : \frac{1}{3} b_1^3 \alpha + 2b_1 \mu^2 &= 0, \\ \left(\frac{G'}{G} \right)^{-2} : a_0 b_1^2 \alpha + 3b_1 \lambda \mu &= 0, \\ \left(\frac{G'}{G} \right)^{-1} : -b_1 v + a_0^2 b_1 \alpha + a_1 b_1^2 \alpha + b_1 \lambda^2 + 2b_1 \mu &= 0, \\ \left(\frac{G'}{G} \right)^0 : c - a_0 v + \frac{1}{3} a_0^3 \alpha + 2a_0 a_1 b_1 \alpha + b_1 \lambda + a_1 \lambda \mu &= 0, \\ \left(\frac{G'}{G} \right)^1 : -a_1 v + a_0^2 a_1 \alpha + a_1^2 b_1 \alpha + a_1 \lambda^2 + 2a_1 \mu &= 0, \\ \left(\frac{G'}{G} \right)^2 : a_0 a_1^2 \alpha + 3a_1 \lambda &= 0, \\ \left(\frac{G'}{G} \right)^3 : 2a_1 + \frac{1}{3} a_1^3 \alpha &= 0. \end{aligned}$$

From the solutions system, we obtain the following with the aid of **Mathematica**

$$\alpha \neq 0, \quad \left(a_0 = -\frac{1}{\sqrt{\alpha}} i \sqrt{\frac{3}{2}} \lambda, \quad a_1 = \frac{1}{\sqrt{\alpha}} i \sqrt{\frac{3}{2}} \lambda \right), \quad a_0 \neq 0, \quad a_1 = -\frac{3\lambda}{a_0 \alpha},$$

$$\lambda \neq 0, \quad b_1 = \frac{2a_0 \mu}{\lambda}, \quad v = \frac{1}{2} (2a_1 b_1 \alpha - \lambda^2 + 4\mu), \quad c = 4a_0 \mu.$$

(i): When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function travelling wave solution

$$u_1(\xi) = a_0 + \frac{3\lambda \left(\lambda - p\sqrt{\lambda^2 - 4\mu} \right)}{2a_0 \alpha} - \frac{4a_0 \mu}{\lambda^2 - p\lambda \sqrt{\lambda^2 - 4\mu}}, \quad (3.23)$$

where

$$p = \frac{A \cosh \left[\frac{1}{2} \sqrt{\lambda^2 - 4\mu}(x - vt) \right] + B \sinh \left[\frac{1}{2} \sqrt{\lambda^2 - 4\mu}(x - vt) \right]}{A \sinh \left[\frac{1}{2} \sqrt{\lambda^2 - 4\mu}(x - vt) \right] + B \cosh \left[\frac{1}{2} \sqrt{\lambda^2 - 4\mu}(x - vt) \right]},$$

$\xi = x - \frac{1}{2}(2a_1b_1\alpha - \lambda^2 + 4\mu)t$, A and B are arbitrary constants.

(ii): When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function travelling wave solution

$$u_2(\xi) = a_0 + \frac{3\lambda \left(\lambda - q\sqrt{4\mu - \lambda^2} \right)}{2a_0\alpha} - \frac{4a_0\mu}{\lambda^2 - q\lambda\sqrt{4\mu - \lambda^2}}, \quad (3.24)$$

where

$$q = \frac{-A \sin \left[\frac{1}{2} \sqrt{4\mu - \lambda^2}(x - vt) \right] + B \cos \left[\frac{1}{2} \sqrt{4\mu - \lambda^2}(x - vt) \right]}{A \cos \left[\frac{1}{2} \sqrt{4\mu - \lambda^2}(x - vt) \right] + B \sin \left[\frac{1}{2} \sqrt{4\mu - \lambda^2}(x - vt) \right]},$$

$\xi = x - \frac{1}{2}(2a_1b_1\alpha - \lambda^2 + 4\mu)t$, A and B are arbitrary constants.

In particular, if we take $A = 0$, $B \neq 0$, then the solutions (3.23) and (3.24) become

$$u_1(\xi) = a_0 - \frac{4a_0\mu}{\lambda \left(\lambda - \sqrt{\lambda^2 - 4\mu} \tanh \left[\frac{1}{2} \sqrt{\lambda^2 - 4\mu}(x - vt) \right] \right)} + \frac{3\lambda \left(\lambda - \sqrt{\lambda^2 - 4\mu} \tanh \left[\frac{1}{2} \sqrt{\lambda^2 - 4\mu}(x - vt) \right] \right)}{2a_0\alpha}, \quad (3.25)$$

$$u_2(\xi) = a_0 - \frac{4a_0\mu}{\lambda \left(\lambda - \sqrt{4\mu - \lambda^2} \cot \left[\frac{1}{2} \sqrt{4\mu - \lambda^2}(x - vt) \right] \right)} + \frac{3\lambda \left(\lambda - \sqrt{4\mu - \lambda^2} \cot \left[\frac{1}{2} \sqrt{4\mu - \lambda^2}(x - vt) \right] \right)}{3\lambda}. \quad (3.26)$$

On the other hand, if we take $A \neq 0$, $B = 0$, then the solutions (3.23) and (3.24) become

$$u_1(\xi) = a_0 - \frac{4a_0\mu}{\lambda \left(\lambda - \sqrt{\lambda^2 - 4\mu} \coth \left[\frac{1}{2} \sqrt{\lambda^2 - 4\mu} (x - vt) \right] \right)} + \frac{3\lambda \left(\lambda - \sqrt{\lambda^2 - 4\mu} \coth \left[\frac{1}{2} \sqrt{\lambda^2 - 4\mu} (x - vt) \right] \right)}{2a_0\alpha}, \quad (3.27)$$

$$u_2(\xi) = a_0 - \frac{4a_0\mu}{\lambda \left(\lambda - \sqrt{4\mu - \lambda^2} \tanh \left[\frac{1}{2} \sqrt{4\mu - \lambda^2} (x - vt) \right] \right)} + \frac{3\lambda \left(\lambda - \sqrt{4\mu - \lambda^2} \tanh \left[\frac{1}{2} \sqrt{4\mu - \lambda^2} (x - vt) \right] \right)}{2a_0\alpha}. \quad (3.28)$$

Remark 2: We observed that our results, when given special values in equations (3.25)–(3.28), include Wazwaz's [6] results obtained by using sine–cosine, tanh–coth methods. However, solutions (3.23) and (3.24) are presented here for the first time.

4. Conclusion

This paper obtained the singular 1-soliton solution to the $K(m, n)$ equation with generalized evolution. Furthermore, the two natural subsidiaries of this equation were studied and these are the KdV equation and the mKdV equation. Here, the G'/G -expansion method was applied which yielded a range of nonlinear wave solutions that are listed. These solutions are going to be extremely useful in future studies where these results will be expanded, generalized further and reported elsewhere. For example, these equations with time-dependent coefficients will be considered. The perturbation terms will be added and the adiabatic parameter dynamics of the soliton parameters will be determined. Then the stochastic perturbation terms will also be added and the corresponding Langevin equation will be analyzed in order to obtain the mean free velocity of the soliton. These just form the tip of the iceberg.

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