Colored vertex models and Iwahori Whittaker functions

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Abstract

We give a recursive method for computing *all* values of a basis of Whittaker functions for unramified principal series invariant under an Iwahori or parahoric subgroup of a split reductive group G over a nonarchimedean local field F. Structures in the proof have surprising analogies to features of certain solvable lattice models. In the case $G = \mathrm{GL}_r$ we show that there exist solvable lattice models whose partition functions give precisely all of these values. Here 'solvable' means that the models have a family of Yang–Baxter equations which imply, among other things, that their partition functions satisfy the same recursions as those for Iwahori or parahoric Whittaker functions. The R-matrices for these Yang–Baxter equations come from a Drinfeld twist of the quantum group $U_q(\widehat{\mathfrak{gl}}(r|1))$, which we then connect to the standard intertwining operators on the unramified principal series. We use our results to connect Iwahori and parahoric Whittaker functions to variations of Macdonald polynomials.

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Contents

1 Introduction

Solvable lattice models [5, 39] are statistical-mechanical systems (usually two-
dimensional) that are amenable to analysis using Yang-Baxter equations, highly
constrained identities whose mysterious nature led to the discovery of quantum groups
[31, 38]. Beyond their origins in statistical mechanics, lattice models are also closely
connected to quantum field theory [4, 6, 30], knot invariants [1, 40, 57] and inte-
grable probability [8, 29]. Most importantly for us, they also give a fruitful method
for studying symmetric function theory and its generalizations by representing poly-
nomials as partition functions of solvable lattice models, as for example in [32, 43,
45, 47, 49, 65, 66]. In this context, the Yang-Baxter equation becomes a powerful tool
for demonstrating identities among partition functions.

Variants of these methods have been used by the authors and their collaborators
[12, 16, 19, 20, 33, 36] to study the representation theory of algebraic groups and
their covers over a p-adic field F. These papers use families of solvable six-vertex
models and their generalizations to produce partition functions giving special values of
Whittaker functions for unramified principal series, particularly for the groups $\mathrm{GL}_r(F)$
and $\mathrm{Sp}(2r, F)$ and their metaplectic covers. The results shed a lot of light on the nature
of these Whittaker functions. However, these prior results have treated only *spherical*
Whittaker functions, those invariant under a maximal compact subgroup. It is very
desirable to have lattice model interpretations for Whittaker functions invariant under
smaller compact subgroups.

Iwahori Whittaker functions for an unramified principal series representation are
fixed by an Iwahori subgroup J that is smaller than the maximal compact subgroup K.
Even if one is mainly interested in spherical (K-fixed) Whittaker functions, the Iwahori

Whittaker functions are needed. The Iwahori Hecke algebra that acts on them has a richer structure and the Iwahori Whittaker functions play a crucial role in evaluating the spherical Whittaker function in the work of Casselman and Shalika [26]. Moreover, the Iwahori Whittaker functions interestingly mirror the geometry of the Schubert varieties in the flag variety [22, 54, 56]. So understanding Iwahori Whittaker functions by bringing them into the lattice model framework is an important goal. Once this connection is made, we find that Whittaker functions invariant under certain *parahoric* subgroups (intermediate between J and K) also fit naturally into these models, and they too should be included in order to have a complete story.

A recent breakthrough by Borodin and Wheeler [9, 10] showed how to refine lattice models using an additional attribute they called 'color.' This led us to wonder whether a similar refinement of the six-vertex model in [20] might produce values of Iwahori Whittaker functions for the general linear group. The surprisingly complete answer to this question for GL_r, and related results about Whittaker functions on *all* split reductive groups, are the subject of this paper.

Let us highlight two of our main results and their implications before a precise accounting of our results in the next section. Given a split, reductive group G, let **J** be a subset of the index set for simple reflections of the Weyl group W of $G(F)$ and $K_{\mathbf{J}}$ the associated standard parahoric subgroup (as in Definition 4.2). Note that $\mathbf{J} = \varnothing$ reduces to the Iwahori case $K_\varnothing = J$. There exists a *standard basis* $\{\psi_w\}$ of $K_{\mathbf{J}}$-fixed Whittaker functions for any irreducible unramified principal series, given explicitly in Sect. 4. In this paper we construct what we will call the *parahoric lattice model* for which we prove the following theorem:

Theorem A *For every g in $GL_r(F)$, any subset $\mathbf{J} \subseteq \{1, 2, \ldots, r-1\}$, and for every ψ_w in a basis of $K_{\mathbf{J}}$-fixed Whittaker functions, there exists a choice of boundary conditions for the parahoric lattice model such that its partition function equals $\psi_w(g)$.*

Conversely, we find that *every* admissible choice of boundary conditions (in particular every assignment of colors to the boundary) has a meaning in terms of Whittaker functions. That is, assume that the colors used along the top boundary are distinct, and that no colors appear on the left or bottom boundary edges. For the partition function to be nonzero, the same distinct colors must appear on the right boundary edges. Then each choice of boundary data corresponds to an element g in $G(F)$ and a w in the Weyl group such that the partition function of this solvable lattice model is the value of the Iwahori Whittaker function $\phi_w(g)$. (See Theorem 7.2, Lemma 3.5 and Remark 3.3.)

If more generally some colors are allowed to repeat on the top boundary, the resulting partition functions correspond precisely to values of Whittaker functions for vectors fixed by a parahoric subgroup (Theorem 8.3). The restriction of the parahoric lattice model to top boundary conditions with distinct colors will therefore be called the *Iwahori lattice model*. If instead the top boundary colors are all the same, we obtain a one-colored lattice model for the spherical Whittaker function, which is equivalent to the uncolored lattice model in [20] that we will here call the *Tokuyama model*.

In order to prove Theorem A, we needed to extend known results expressing Whittaker functions recursively using Demazure-like operators, and these very general

Table 1 Relations between Whittaker functions and special polynomials as detailed in Sect. 9

Whittaker function		Special polynomial	Algebraic	Lattice models
Spherical Whittaker function	=	Schur polynomial	[26, 61]	[20, 35]
Li's Whittaker function	=	Hall-Littlewood polynomial	[50]	Sect. 9 and [64]
Iwahori Whittaker function	=	Non-symmetric Macdonald polynomial	[22], Sect. 3	Sect. 7
Parahoric Whittaker function	=	Macdonald polynomial with prescribed symmetry	Sect. 4	Sects. 8 and 9

In the third column we list references for the algebraic viewpoint on the Whittaker function and in the fourth column we give references for a lattice model interpretation

results (in Sects. 3 and 4) are valid for any split, reductive group. Thus we were led to prove the following result:

Theorem B *Let $G := G(F)$ be any split reductive group defined over a local field F. For a basis $\{\phi_w\}$ of the space of Iwahori Whittaker functions for any irreducible unramified principal series representation of G and any $g \in G$, there is a recursive algorithm using Demazure-like operators to compute $\phi_w(g)$.*

We have similar algorithmic results for parahoric Whittaker functions. (See Theorem 4.7 and Remark 4.8.) Prior to this work, even a conjectural description of all values of the Iwahori and parahoric Whittaker functions was unknown. Reeder [56] and Brubaker, Bump, and Licata [22] computed Iwahori Whittaker functions if g is a torus element. In the introduction to [56], Reeder describes the determination of the remaining values as a difficult problem. Lansky [48] determined the dimensions of the spaces of standard parahoric fixed vectors in unramified principal series.

In addition to the above theorems, the development of these results led us to:

- Prove a Casselman-Shalika formula for certain parahoric Whittaker functions on split, reductive groups in Theorem 4.7.
- Relate several classes of Whittaker functions to special functions that arise in algebraic combinatorics summarized in Table 1 below and further detailed in Table 2 of Sect. 9.

Moreover, in pursuing the connection to solvable lattice models in type A, we were led to:

- Interpret the Demazure recurrence relations in our algorithmic description in terms of the R-matrix for the quantum affine superalgebra $U_{\sqrt{v^{-1}}}(\hat{\mathfrak{gl}}(r|1))$;
- Develop a combinatorial analog of a fusion procedure from quantum groups in Sect. 5, which allows us to prove that our Boltzmann weights solve Yang–Baxter equations for any rank r even though the number of admissible vertex configurations grows exponentially in r.
- Interpret the action of intertwiners of Iwahori or parahoric fixed vectors in the principal series representation with R-matrices acting on evaluation representations of quantum loop groups in Sect. 10.

Table 2 Relations between different Whittaker functions and associated special polynomials

Whittaker function		Special polynomial
Spherical Whittaker function		Schur polynomial
$\sum_{w \in W} \phi_w(\mathbf{z}; \varpi^{-\lambda})$	$=$	$\prod_{\alpha \in \Delta^+}(1 - v\mathbf{z}^{-\alpha}) s_\lambda(\mathbf{z})$
Li's Whittaker function		Hall-Littlewood polynomial
$\sum_{w \in W}(-v)^{-\ell(w)}\phi_w(\mathbf{z}; \varpi^{-\lambda})$	$=$	$\mathbf{z}^{-\rho} P_{\lambda+\rho}(\mathbf{z}, v^{-1})$
Iwahori Whittaker function		Non-symmetric Macdonald polynomial
$\phi_{w_1}(\mathbf{z}; \varpi^{-\lambda})$	$=$	$(-v)^{\ell(w)}\mathbf{z}^{-\rho} w_0 E_{w_0 w(\lambda+\rho)}(\mathbf{z}; \infty, v)$
Parahoric Whittaker function		Macdonald polynomial with prescribed symmetry
$\psi_1^{\mathbf{J}}(\mathbf{z}; \varpi^{-\lambda})$	$=$	$\mathbf{z}^{-\rho} S_{\lambda+\rho}^{(\varnothing, \mathbf{J})}(\mathbf{z}; 0, v^{-1}) a_{\lambda+\rho}^{(\varnothing, \mathbf{J})}$

Potential future applications include new Cauchy identities for these Whittaker functions, which may then find application in local Rankin-Selberg computations. It may also be possible to relate our lattice models to interacting particle processes such as ASEP and TASEP in the spirit of similar results by Borodin and Wheeler [9, Section 12]. Furthermore in [16] we found connections between the metaplectic models in [12] and LLT polynomials (ribbon symmetric functions) and we expect that there may be similar connections here.

Another exciting open question is whether Whittaker functions for even smaller compact subgroups might similarly have representations as partition functions of solvable lattice models. In other words, if L is an arbitrary compact open subgroup, does there exist a basis of L-invariant Whittaker functions whose values may be represented by solvable lattice models? If the answer is affirmative one could also look for representations of Whittaker of other admissible representations, particularly supercuspidals. To what extent are solvable lattice models and their associated quantum groups a tool for studying *all* representations of p-adic groups? These possibilities are purely speculative at this time, but our success in treating Iwahori and parahoric Whittaker functions and the work of Ju-Lee Kim [41] which gives combinatorial descriptions of Whittaker functions for other admissible representations is suggestive in this regard.

Finally, the models in this paper may also be generalized to refine the models representing metaplectic Whittaker functions introduced in [12], called *metaplectic ice*. Whereas for metaplectic ice certain edges are enhanced by adding an attribute called charge, in this paper we enhance the system by decorating certain edges with color. These attributes of charge and color are handled differently, and at first glance the schemes seem different. We show in the papers [15, 18], which are successors to this one, how they may be placed in a unified framework. The R-matrices for the solvable models whose partition functions are Iwahori Whittaker functions on the n-fold metaplectic cover of GL(r) are related to $U_{\sqrt{v^{-1}}}(\widehat{\mathfrak{gl}}(r|n))$, or more precisely a Drinfeld twist thereof which introduces Gauss sums into the R-matrix.

Let us also mention the bosonic models in [24], which represent not Iwahori Whittaker functions, but rather Iwahori fixed vectors in the spherical model. Many aspects of that theory such as the monochrome factorization of the models and the color lifting properties (Sect. 8 below) have exact analogs in that setting. But whereas the models

in this paper are related to the quantum group $U_{v^{-1/2}}(\mathfrak{gl}(r|1))$, the bosonic models in [24] are related to $U_{v^{-1/2}}(\mathfrak{gl}(r+1))$.

2 Outline of the paper

Let us now explain the results of this paper in more detail. We begin with a brief description of spherical Whittaker functions for $GL_r(F)$ and their associated six-vertex model, which we will call the *Tokuyama model*. Let $\mathfrak{o}$ be the ring of integers of the nonarchimedean local field F and let v^{-1} be the cardinality of the residue field. Construct an unramified principal series representation of $GL_r(F)$ from a character of $T(F)/T(\mathfrak{o})$ where T is the maximal split torus (see Sect. 3 for full details). These representations have unique Whittaker functionals and a unique-up-to-constant vector which is right invariant under $K = GL_r(\mathfrak{o})$. The spherical Whittaker function is the image of this vector in the Whittaker functional and it is completely determined by its values on $T(F)/T(\mathfrak{o})$, which we identify with the weight lattice Λ of the Langlands dual group. It is easily seen that the spherical Whittaker function vanishes unless the associated weight is dominant. The remaining values for dominant weights are given by the Shintani-Casselman-Shalika formula in terms of Schur polynomials in the Langlands parameters of the principal series. By Tokuyama's theorem, described in [20, 35, 63] and recalled below as (49), there exists a solvable six-vertex model with boundary conditions indexed by dominant weights whose partition function give the Shintani-Casselman-Shalika formula.

There are multiple ways to describe this six-vertex model. A state of the system attaches a spin $\oplus$ or $\ominus$ to each edge of a grid so that each vertex has adjacent edges in one of six possible configurations seen for example in [20, Table 1]. Alternatively, the state may be described by connecting the edges labeled with $\ominus$ into lines or paths as in Section 8.1 of [5]. The models needed in this paper refine the six vertex model by coloring these paths.

For the Iwahori lattice model in this paper the grid has r rows and boundary conditions that specify the particular GL_r Iwahori Whittaker function ϕ_w and group element g. Then a state of the system representing $\phi_w(g)$ will consist of r paths with distinct colors beginning at the top boundary and traveling downward and rightward until each path exits on a distinct row along the right boundary. The partition function is the sum of the Boltzmann weights for all such configurations of paths. The Boltzmann weights can, for example, be found in the row labeled $S_\Gamma(i)$ of [20, Table 2].

In Sect. 6, we exhibit a set of colored Boltzmann weights and prove Yang–Baxter equations for them in Theorem 6.5. In Theorem 7.2 we use these Yang–Baxter equations to demonstrate that the partition function of the Iwahori lattice model for different boundary conditions is equal to values of Iwahori Whittaker functions in the standard basis.

The Iwahori lattice model never has more than one path of a given color, because the boundary conditions only have two edges of a given color: one on the top boundary and one on the right. In a state of the system, these are connected by a single path, so there is only one path of that color. However, the Yang–Baxter equations, which are highly constrained and overdetermined relations the Boltzmann weights must satisfy,

force us to assign nonzero values to local vertex or path configurations where two paths of the same color cross, even though these configurations can never appear in any state of the Iwahori model. Remarkably, these seemingly unneeded weights that we are forced to include *do* turn out to be needed when we turn to the more general parahoric Whittaker functions discussed in Sect. 4, and the corresponding parahoric lattice model of Sect. 8 allowing for multiple paths of the same colors.

It is precisely these latter, parahoric weights that distinguish our models from models of Borodin and Wheeler [9], which also have configurations of multicolored paths crossing in a grid. In both types of colored models, the horizontal edges can carry only one color, but the vertical edges can carry more than one. In this paper, the vertical edges are 'fermionic' meaning that they satisfy an exclusion principle: no vertical edge may carry more than one instance of a particular color. The models in [9] are 'bosonic' and the vertical edges may carry their colors with a multiplicity. The parahoric lattice model makes clear the need for these fermionic weights in order to produce the correct partition function describing a parahoric Whittaker function.

This distinction may also be observed from the quantum groups that underlie the models. One may check that the R-matrix in Fig. 6 is a Drinfeld twist of the R-matrix for the quantum affine Lie superalgebra $U_{\sqrt{v^{-1}}}(\widehat{\mathfrak{gl}}(r|1))$ from [44]. By contrast the R-matrix for the systems in [9] is a twist of a quantum group $U_q(\widehat{\mathfrak{sl}}_{r+1})$. See also Remark 6.7 for further details on the relationship between our models and those of [9].

While our R-matrix is associated to a quantum superalgebra, we have no such module interpretation for the Boltzmann weights of our colored models. To demonstrate the Yang–Baxter equation, we must instead introduce an equivalent version of the systems in which each vertex is replaced by r vertices, and each vertical edge by r vertical edges. We refer these expanded systems as *monochrome* because each column is assigned a color, and each vertical edge in that column can carry only that color and no other. See Fig. 16 for an example of a colored model and its equivalent expanded monochrome system. The relationship between the Boltzmann weights for the regular colored systems and the expanded monochrome systems is demonstrated in Fig. 8. This relationship is reminiscent of the *fusion* construction for tensor products of quantum group modules (see [46, 55] and Appendix B of [9]) and our results in Sect. 5 may be viewed as a combinatorial substitute for fusion in the absence of a quantum group module interpretation. This link is the key to the solvability of the colored models, for it is with the monochrome vertices and weights that we prove the Yang–Baxter equations.

Apart from our results on solvable lattice models, our independent results on Iwahori and parahoric Whittaker functions (i.e. which are not depending on their relations to lattice model partition functions) are more precise than what is found in the literature. Let us explain this point.

With notation as in Sect. 3, our task is to study so-called 'standard basis' Iwahori Whittaker functions $\phi_{w_1}(g)$ for $w_1 \in W$ and $g \in G$. Using left and right translation properties of ϕ_{w_1}, we may assume that $g = \varpi^{-\lambda} w_2$ where w_2 is again a Weyl group element, λ is a weight and ϖ is a uniformizer of F. Not every pair λ, w_2 needs to be considered, since $\phi_{w_1}(g)$ vanishes for many values of g. For example if $w_2 = 1$

then λ must be dominant, meaning that $\langle \alpha_i^\vee, \lambda \rangle \geqslant 0$ for all simple roots α_i. If w_2 is a general permutation, then we allow $\langle \alpha_i^\vee, \lambda \rangle \geqslant -1$ when i is a descent of w_2^{-1} (see Definition 3.4).

In fact, for $G = \mathrm{GL}_r$, it turns out that the triples w_1, w_2, λ with w_2 and λ as above correspond exactly to the boundary conditions with distinct colors in our lattice model scheme! The permutations w_1 and w_2 describe the order of the colors on the right and top boundaries; λ describes the columns on the top boundary where a colored line has its terminus; and the cases where $\langle \alpha_i^\vee, \lambda \rangle = -1$ is allowed correspond to the possibility that a top vertical edge carries more than one color. This correspondence between the data describing the values of Whittaker functions and available systems in our scheme becomes even more striking when we consider the parahoric systems in Sect. 8.

In order to prove all this we need new results on the values of $\phi_{w_1}(g)$ with $g = \varpi^{-\lambda} w_2$ when $w_2 \neq 1$. In prior work such as [22, 54, 56] the group element g has usually been taken to be diagonal. We are able to give a recursive method of computing the values $\phi_{w_1}(\varpi^{-\lambda} w_2)$ based on Proposition 3.6 and Theorem 3.8. This then is used in Theorem 7.2 to prove that $\phi_{w_1}(\varpi^{-\lambda} w_2)$ equals the partition function of the Iwahori lattice model with certain boundary conditions determined by λ, w_1 and w_2 as explained above. The above methods to compute the Iwahori Whittaker functions, both the recursive algorithm (for any reductive group) and the partition function (for GL_r), are easy to implement on a computer. In the parahoric case, our results are also applicable to all group elements.

In Sect. 9, we use our descriptions for Whittaker functions in terms of Demazure-like divided difference operators, which arise from our Yang–Baxter equations, to relate them to variations of Macdonald polynomials. Indeed, we provide interpretations for spherical, Iwahori, and parahoric Whittaker functions in terms of a specialization or generalization of a Macdonald polynomial. In particular, the parahoric Whittaker functions are expressed as Macdonald polynomials with prescribed symmetry studied in [2, 3, 53]. Moreover, Jian-Shu Li [50] introduced a certain Iwahori Whittaker function to study the unique genuine subquotient of the unramified principal series, and we will show that its values are symmetric Hall–Littlewood polynomials (see Proposition 9.4). See Table 1 for a summary of these connections and relevant references. Details of the exact relations are shown in Table 2 of Sect. 9.

Finally, in Sect. 10, we explain how lattice models shed some further light on p-adic representation theory. Not only are the outputs of both the lattice models and the p-adic representation theory the same, but each tool or technique has a counterpart in this dictionary, as we shall now explain.

Standard intertwining operators on principal series are a basic tool in the representation theory of p-adic groups. Their action on Iwahori fixed vectors and how they interact with the Whittaker functional are the two principal ingredients in the Casselman-Shalika formula [25, 26] and are also the key to Theorem 3.8. Roughly, we show that these two actions of intertwining operators, on Iwahori fixed vectors and for the Whittaker functional, correspond to restrictions of the quantum superalgebra $U_{\sqrt{v^{-1}}}(\widehat{\mathfrak{gl}}(r|1))$ to its $U_{\sqrt{v^{-1}}}(\widehat{\mathfrak{gl}}(r))$ and $U_{\sqrt{v^{-1}}}(\widehat{\mathfrak{gl}}(1))$ pieces, respectively.

In Theorem 10.5 we show that the action of the intertwining integral on the space of Iwahori fixed vectors is the same as the action of the affine R-matrix on a subspace of the tensor product of evaluation representations $V_r(z_1) \otimes \cdots \otimes V_r(z_r)$ of $U_{\sqrt{v^{-1}}}(\widehat{\mathfrak{gl}}(r))$. This result is independent of the Whittaker functional and only the smaller quantum group $U_{\sqrt{v^{-1}}}(\widehat{\mathfrak{gl}}(r)) \subset U_{\sqrt{v^{-1}}}(\widehat{\mathfrak{gl}}(r|1))$ appears due to the fact that the right boundary conditions of our model contain only colored edges (which span a subspace that can be thought of as the tensor product of evaluation representations of $U_{\sqrt{v^{-1}}}(\widehat{\mathfrak{gl}}(r))$). This result can be easily generalized to the parahoric setting.

A result similar to Theorem 10.5 was proved in the case of spherical Whittaker functions on the metaplectic n-cover of GL_r in [12, Theorem 1.1], where the first three authors relate the Kazhdan-Patterson scattering matrix to the $U_{\sqrt{v^{-1}}}(\widehat{\mathfrak{gl}}(n))$ R-matrix. The relation was used in [13] to build finite dimensional representations of the affine Hecke algebra starting from metaplectic Whittaker functionals. Theorem 10.5 now allows for a similar construction starting from Iwahori fixed vectors in an unramified principal series representation.

3 Iwahori Whittaker functions

We will review the constructions of Iwahori Whittaker functions following [22]. There are several differences between choices made here and in [22] with those in Casselman-Shalika [26]. Let us summarize these choices, with notations to be defined more precisely below.

- As in [26], principal series representations are induced from the standard Borel subgroup B. But in contrast with [26], we will take Whittaker functions with respect to the unipotent radical N_- of the opposite Borel subgroup B_-.
- We will take our Iwahori subgroup J to be the preimage in the maximal compact subgroup K of B_- modulo $\mathfrak{p}$.
- We will apply our construction to the contragredient representation of the principal series with Langlands parameters $\mathbf{z}$.
- When restricting to the maximal torus we will evaluate our Whittaker functions at values $\varpi^{-\lambda}$ where $-\lambda$ is antidominant.

The advantage of these unconventional choices is that it keeps the long Weyl group element w_0 out of the formulas. Thus whereas for Casselman and Shalika the simplest Whittaker function is that supported on the double coset Bw_0J, and its value at ϖ^λ is (up to normalization), $\mathbf{z}^{w_0\lambda}$, with our conventions the simplest Whittaker function is supported on $B \cdot 1_W J$, and its value is (up to normalization) $\mathbf{z}^\lambda$.

In more detail, let F be a nonarchimedean local field with ring of integers $\mathfrak{o}$. Let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$ with generator $\varpi \in \mathfrak{p}$. Then, ϖ is a prime element, or uniformizer, of F. We will denote by q the cardinality $q = |\mathfrak{o}/\mathfrak{p}|$ and the residue field itself by $\mathbb{F}_q = \mathfrak{o}/\mathfrak{p}$.

Let G be a split reductive Chevalley group, that is, an affine algebraic group scheme over $\mathbb{Z}$ with a fixed Chevalley basis for its Lie algebra $\mathfrak{g}_\mathbb{Z}$. Let T be the standard maximal split torus of G obtained from our choice of Chevalley basis, and similarly let N be the standard maximal unipotent subgroup whose Lie algebra is the union of the positive

root spaces. Together they form the standard Borel subgroup $B = TN$ and the Weyl group W is defined by $N_G(T)/T$ where $N_G(T)$ is the normalizer of T in G.

Remark 3.1 For each Weyl group element w we will always choose a representative in $K = G(\mathfrak{o})$ that is the maximal compact subgroup of $G(F)$. Strictly speaking this representative is only determined modulo $T(\mathfrak{o})$. However because we are considering representations induced from unramified data, none of the functions we compute ever depend on this choice, nor on the choice of ϖ.

Let B_- be the opposite Borel subgroup and N_- be its unipotent radical generated by the negative root spaces. In the later sections of this paper we will mainly consider $G = \mathrm{GL}_r$ for which B is the subgroup of upper triangular matrices, T the diagonal matrices and B_- the lower triangular matrices.

Let $\hat{G}$ be the Langlands dual group of G. We will denote the root system of $\hat{G}$ by Δ and the simple roots of $\hat{G}$ by $\alpha_1, \ldots, \alpha_r$. The root system of G is the dual root system $\Delta^\vee$. We prefer this notation instead of making Δ the root system of G, because the weight lattice Λ of $\hat{G}$ appears frequently in the sequel.

We will consider an unramified character τ of $T(F)$, that is, a character that is trivial on $T(\mathfrak{o})$. The group of such characters is isomorphic to $\hat{T}(\mathbb{C}) \cong (\mathbb{C}^\times)^r$, where $\hat{T}$ is the standard split maximal torus of $\hat{G}$. To define the unramified character $\tau_{\mathbf{z}}$ for $\mathbf{z} \in \hat{T}(\mathbb{C})$ we will use the following isomorphisms.

The group $X_*(T)$ of rational cocharacters of T is isomorphic to the weight lattice $\Lambda = X^*(\hat{T})$ of rational characters of the dual torus, and we will identify these two groups. But $X_*(T)$ is also isomorphic to the quotient $T(F)/T(\mathfrak{o})$. Indeed, if λ is a cocharacter let ϖ^λ be the image of the uniformizer ϖ in T under λ; then we associate with λ the coset $\varpi^\lambda T(\mathfrak{o})$ in $T(F)/T(\mathfrak{o})$. On the other hand we may regard λ as a rational character and, with $\mathbf{z} \in \hat{T}(\mathbb{C})$, let $\mathbf{z}^\lambda \in F^\times$ be the application of this character to $\mathbf{z}$. Then we define the unramified character $\tau_{\mathbf{z}}$ of $T(F)$ by $\tau_{\mathbf{z}}(t) = \mathbf{z}^\lambda$ when $t \in \varpi^\lambda T(\mathfrak{o})$.

In particular, for $G = \mathrm{GL}_r$ with $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r \cong \Lambda$ and $\mathbf{z} = (z_1, \ldots, z_r) \in (\mathbb{C}^\times)^r \cong \hat{T}(\mathbb{C})$ we let

$$\varpi^\lambda = \begin{pmatrix} \varpi^{\lambda_1} & & & \\ & \varpi^{\lambda_2} & & \\ & & \ddots & \\ & & & \varpi^{\lambda_r} \end{pmatrix} \in \mathrm{GL}_r(F) \text{ and } \tau_{\mathbf{z}}(\varpi^\lambda) = \mathbf{z}^\lambda = \prod_{i=1}^r z_i^{\lambda_i} \,.$$

The Iwahori subgroup $J = J_-$ of $G(F)$ is the subgroup of $K = G(\mathfrak{o})$ defined as the preimage of $B_-(\mathbb{F}_q)$ under the mod $\mathfrak{p}$ reduction map $K \to G(\mathbb{F}_q)$. Let J_+ be the preimage of $B(\mathbb{F}_q)$. For $G = \mathrm{GL}_r$ the Iwahori subgroup J consists of elements in $\mathrm{GL}_r(\mathfrak{o})$ which are lower triangular mod $\mathfrak{p}$.

We trivially extend an unramified character $\tau_{\mathbf{z}}$ of $T(F)$ to $B(F)$ and let $(\pi, I(\mathbf{z}))$ denote the induced representation $I(\mathbf{z}) = \mathrm{Ind}_B^G(\delta^{1/2}\tau_{\mathbf{z}})$ under the right-regular action π of $G(F)$ where $\delta : B(F) \to \mathbb{R}^\times$ is the modular quasicharacter.

Remark 3.2 It is convenient to extend δ to a function on $G(F)$ as follows using the Iwasawa decomposition. If $g \in G(F)$ we may write $g = bk$ where $b \in B(F), k \in K$ and we define $\delta(g) = \delta(b)$. This is well-defined since δ is trivial on $B(F) \cap K$.

Consider the space $I(\mathbf{z})^J$ of Iwahori fixed vectors in $I(\mathbf{z})$ which is of dimension $\dim I(\mathbf{z})^J = |W|$. We will now describe a basis for $I(\mathbf{z})^J$ which will be used throughout the paper. Using the Iwasawa decomposition $G(F) = B(F)K$, the Bruhat decomposition $K = \bigsqcup_{w \in W} J_+ w J$, the Iwahori factorization $J_+ = (B \cap J_+) N_-(\mathfrak{p})$ and the fact that $w^{-1} N_-(\mathfrak{p}) w \subset J$ we obtain the decomposition $G = \bigsqcup_{w \in W} B(F) w J$ [25]. Thus the elements $\Phi_w^{\mathbf{z}} \in I(\mathbf{z})^J$ for $w \in W$ defined by

$$\Phi_w^{\mathbf{z}}(bw'k) := \begin{cases} \delta^{1/2} \tau_{\mathbf{z}}(b) & \text{if } w' = w \\ 0 & \text{otherwise} \end{cases} \qquad b \in B(F), w' \in W, k \in J$$

form a basis of $I(\mathbf{z})^J$, commonly referred to as the 'standard basis.'

For $\alpha \in \Delta$, let $x_\alpha : \mathbb{G}_a \to G$ be the one-parameter subgroup of G corresponding to $\alpha^\vee$. (We recall that Δ is the root system of the dual group $\widehat{G}$, and it is the coroot $\alpha^\vee$ that is a root of G.) Thus $x_\alpha(t) = \exp(t X_\alpha)$ where X_α is the corresponding Chevalley basis element of the Lie algebra. The group $K = G(\mathfrak{o})$ is generated by the unipotent groups $x_\alpha(\mathfrak{o})$. Fix a unitary character ψ on $N_-(F)$ such that, for any simple root α, $\psi \circ x_{-\alpha} : F \to \mathbb{C}^\times$, is a character on F trivial on $\mathfrak{o}$ but no larger fractional ideal. The space of Whittaker functionals, which are linear maps $\Omega_{\mathbf{z}} : I(\mathbf{z}) \to \mathbb{C}$ satisfying $\Omega_{\mathbf{z}}(\pi(n_-)f) = \psi(n_-)\Omega_{\mathbf{z}}(f)$ for $n_- \in N_-(F)$, is one dimensional [58]. We need therefore only consider the following explicit Whittaker functional

$$\Omega_{\mathbf{z}}(f) := \int\limits_{N_-(F)} f(n)\psi(n)^{-1}\, dn \qquad f \in I(\mathbf{z}). \tag{1}$$

The integral is convergent if $|\mathbf{z}^\alpha| < 1$ for positive roots α, and can be extended to all $\mathbf{z}$ by analytic continuation.

The objects of study in this paper are the *Iwahori Whittaker functions* obtained by applying the Whittaker functional (1) to right-translates of standard basis elements $\Phi_w^{\mathbf{z}}$.

Remark 3.3 Any $g \in G(F)$ may be written as $g = n\varpi^{-\lambda} w_2 k$ with $n \in N_-(F)$, $\lambda \in \Lambda$, w_2 a Weyl group representative chosen in K by Remark 3.1, and $k \in J$. Using the left $N_-(F)$ equivariance and the right J invariance, any Iwahori Whittaker function $\mathcal{W}(g)$ then satisfies $\mathcal{W}(g) = \psi(n)\mathcal{W}(\varpi^{-\lambda} w_2)$. Thus we reduce to computing our Whittaker functions at values $\varpi^{-\lambda} w_2$.

We will use the following conventions and normalizations for Whittaker functions of the contragredient $I(\mathbf{z}^{-1})$ of $I(\mathbf{z})$ at these values. For $w_1 \in W$ we consider the Iwahori Whittaker function

$$\phi_{w_1}(\mathbf{z}; g) := \delta^{1/2}(g)\Omega_{\mathbf{z}^{-1}}\left(\pi(g)\Phi_{w_1}^{\mathbf{z}^{-1}}\right) \tag{2}$$

which is determined by its values on $g = \varpi^{-\lambda} w_2$ for a weight $\lambda \in \Lambda$ and $w_2 \in W$. See Remark 3.2 for the extension of δ to $G(F)$. In [22] these Whittaker functions were considered only for torus elements $g = \varpi^{-\lambda}$ and were there denoted as $\mathcal{W}_{\lambda, w_1}(\mathbf{z})$.

In this paper we treat the general case, not only $w_2 = 1$, and will therefore need to compute more generally $\phi_{w_1}(\mathbf{z}; \varpi^{-\lambda} w_2)$.

First, we determine the pairs λ and w_2 for which this J-invariant Whittaker function is non-vanishing.

Definition 3.4 We say that λ is w-*almost dominant* if for all simple roots α_i

$$\begin{cases} \langle \alpha_i^\vee, \lambda \rangle \geqslant 0 & \text{if } w^{-1}\alpha_i \in \Delta^+, \\ \langle \alpha_i^\vee, \lambda \rangle \geqslant -1 & \text{if } w^{-1}\alpha_i \in \Delta^-. \end{cases} \tag{3}$$

Lemma 3.5 *Let* $\mathcal{W}$ *be any* J-*invariant Whittaker function. Then*

$$\mathcal{W}(\varpi^{-\lambda} w_2) = 0$$

unless λ *is* w_2-*almost dominant.*

Proof This is similar to Lemma 5.1 of [26]. Let α_i be a simple root such that (3) fails. We may find $t \in \mathfrak{p}^{-1}$ such that $\psi(u) \neq 1$ where $u = x_{-\alpha_i}(t)$. Now

$$\psi(u)\mathcal{W}(\varpi^{-\lambda} w_2) = \mathcal{W}(u\varpi^{-\lambda} w_2) = \mathcal{W}(\varpi^{-\lambda} w_2 j) \tag{4}$$

where

$$j = w_2^{-1}\varpi^\lambda u \varpi^{-\lambda} w_2 = x_{-w_2^{-1}(\alpha_i)}(\varpi^{-\langle \alpha_i^\vee, \lambda \rangle} t).$$

Our assumption that (3) fails implies that $\varpi^{-\langle \alpha_i^\vee, \lambda \rangle} t \in \mathfrak{o}$ if $w_2^{-1}(\alpha) \in \Delta^+$ and $\varpi^{-\langle \alpha_i^\vee, \lambda \rangle} t \in \mathfrak{p}$ if $w_2^{-1}(\alpha) \in \Delta^-$ and in either case $j \in J$, so $\mathcal{W}(\varpi^{-\lambda} w_2 j)$ equals $\mathcal{W}(\varpi^{-\lambda} w_2)$ which must therefore vanish by (4). □

Next we analyze the special case $w_1 = w_2$.

Proposition 3.6 *Let* $w \in W$ *and* $\lambda \in \Lambda$ *a* w-*almost dominant weight. Then*

$$\phi_w(\mathbf{z}; \varpi^{-\lambda} w) = q^{-\ell(w)} \mathbf{z}^\lambda,$$

where $\ell(w)$ *denotes the length of a reduced expression for* w.

Proof By definition

$$\phi_w(\mathbf{z}; \varpi^{-\lambda} w) = \delta^{-1/2}(\varpi^\lambda) \int_{N_-(F)} \Phi_w^{\mathbf{z}^{-1}}(n\varpi^{-\lambda} w)\psi(n)^{-1} dn.$$

We make the variable change $n \mapsto \varpi^{-\lambda} n \varpi^\lambda$. This multiplies the measure by $\delta(\varpi^\lambda)$ and using $\Phi_w^{\mathbf{z}^{-1}}(\varpi^{-\lambda} g) = \delta^{1/2}(\varpi^{-\lambda})\mathbf{z}^\lambda \Phi_w^{\mathbf{z}^{-1}}(g)$ we get

$$\mathbf{z}^\lambda \int_{N_-(F)} \Phi_w^{\mathbf{z}^{-1}}(nw)\psi(\varpi^{-\lambda} n \varpi^\lambda)^{-1} dn.$$

Let $J_w = wJw^{-1}$. This has the Iwahori factorization $J_w = N_w^- T(\mathfrak{o})N_w^+$ where $N_w^- = J_w \cap N_-(F)$ and similarly for N_w^+. In particular

$$N_w^- = \prod_{\alpha \in \Delta^+} \begin{cases} x_{-\alpha}(\mathfrak{o}) \text{ if } w^{-1}\alpha \in \Delta^+, \\ x_{-\alpha}(\mathfrak{p}) \text{ if } w^{-1}\alpha \in \Delta^-. \end{cases} \tag{5}$$

The integrand is nonzero only if $nw \in BwJ$. We will show that this is true if and only if $n \in N_w^-$. Indeed, write $nw = bwj$ where $j \in J$. Then $n = bj_w$ where $j_w = wjw^{-1} \in J_w$. Using the Iwahori factorization, $j_w = \beta n_w^-$ where $\beta \in B$ and $n_w^- \in N_w^-$. Because $B \cap N_- = \{1\}$, $b = \beta = 1$ and $n = n_w^-$. Therefore the integral equals

$$\mathbf{z}^\lambda \int_{N_w^-} \Phi_w^{\mathbf{z}^{-1}}(nw)\psi(\varpi^{-\lambda}n\varpi^\lambda)^{-1}dn.$$

Now we will show that the value of the integrand is 1 so this is just $\mathbf{z}^\lambda$ times the volume of N_w^-. We have $\Phi_w^{\mathbf{z}^{-1}}(nw) = 1$ since the argument is in wJ. We must show that $\varpi^{-\lambda}n\varpi^\lambda$ is in the kernel of ψ. For this it is sufficient to show that if $\alpha = \alpha_i$ is a simple positive root then

$$\varpi^{-\lambda}x_{-\alpha_i}(t)\varpi^\lambda \in N_-(\mathfrak{o})$$

where using (5) we may assume that $t \in \mathfrak{o}$ if $w^{-1}(\alpha_i) \in \Delta^+$ and $t \in \mathfrak{p}$ otherwise. Now

$$\varpi^{-\lambda}x_{-\alpha_i}(t)\varpi^\lambda = x_{-\alpha_i}(\varpi^{\langle\lambda,\alpha_i^\vee\rangle}t).$$

Because λ is w-almost dominant $\varpi^{\langle\lambda,\alpha_i^\vee\rangle}t$ is indeed in $\mathfrak{o}$.

Hence $\phi_w(\mathbf{z}; \varpi^{-\lambda}w)$ equals $\mathbf{z}^\lambda$ times the volume of N_w^-, which is $q^{-\ell(w)}$. □

In order to determine the values of the Iwahori Whittaker function $\phi_{w_1}(\mathbf{z}; \varpi^{-\lambda}w_2)$ in full generality, we mimic the methods of [22], which used ingredients from earlier papers of Casselman and Shalika [25, 26]. In brief, we will develop a recursion using the Bruhat order in the Weyl group in the w_1 variable above, whose base case is given by Proposition 3.6. The recursion results from computing the function $\Omega_{\mathbf{z}}(\mathcal{A}_{s_i} \cdot \Phi_w)$ in two ways, where $\mathcal{A}_w$ denotes the standard intertwining operator on principal series corresponding to the Weyl group element $w \in W$ and s_i is a simple reflection. Comparing the two methods of computation will give the values of the Whittaker function. We begin by briefly reviewing the basics of intertwining operators. These facts will also be needed in Sect. 10. To avoid technical problems with the poles and zeros of the intertwining integrals, we will assume that $\mathbf{z}^\alpha \neq 1, q^{\pm 1}$ for all α.

The standard intertwining integral $\mathcal{A}_w^{\mathbf{z}} : I(\mathbf{z}) \to I(w\mathbf{z})$ is given by

$$\mathcal{A}_w^{\mathbf{z}}\Phi(g) = \int_{N(F)\cap wN_-(F)w^{-1}} \Phi(w^{-1}ng)dn. \tag{6}$$

The integral converges when $|\mathbf{z}^\alpha| < 1$ for $\alpha \in \Delta^+$ and can be extended to arbitrary $\mathbf{z}$ by meromorphic continuation. The intertwining integral induces a map $\mathcal{A}_w^{\mathbf{z}} : I(\mathbf{z})^J \to I(w\mathbf{z})^J$ and an explicit expression for $\mathcal{A}_{s_i}^{\mathbf{z}}$ on $I(\mathbf{z})^J$ is given by the following formula. See Proposition 3 in [22] for a proof of this fact, which is equivalent to Theorem 3.4 of [25]:

$$\mathcal{A}_{s_i}^{\mathbf{z}}(\Phi_w^{\mathbf{z}}) = \begin{cases} (1 - c_{\alpha_i}(s_i\mathbf{z}))\Phi_w^{s_i\mathbf{z}} + \Phi_{s_iw}^{s_i\mathbf{z}} & \text{if } \ell(s_iw) > \ell(w), \\ (q^{-1} - c_{\alpha_i}(s_i\mathbf{z}))\Phi_w^{s_i\mathbf{z}} + q^{-1}\Phi_{s_iw}^{s_i\mathbf{z}} & \text{if } \ell(s_iw) < \ell(w), \end{cases} \tag{7}$$

where

$$c_{\alpha_i}(\mathbf{z}) = \frac{1 - q^{-1}\mathbf{z}^{\alpha_i}}{1 - \mathbf{z}^{\alpha_i}}. \tag{8}$$

Substituting in the definition of c_{α_i} and using that $(s_i\mathbf{z})^{\alpha_i} = \mathbf{z}^{-\alpha_i}$ we get that equation (7) is equivalent to

$$\bar{\mathcal{A}}_{s_i}^{\mathbf{z}}(\Phi_w^{\mathbf{z}}) = \begin{cases} \frac{1-q^{-1}}{1-q^{-1}\mathbf{z}^{\alpha_i}}\Phi_w^{s_i\mathbf{z}} + \frac{1-\mathbf{z}^{\alpha_i}}{1-q^{-1}\mathbf{z}^{\alpha_i}}\Phi_{s_iw}^{s_i\mathbf{z}} & \text{if } \ell(s_iw) > \ell(w), \\ \mathbf{z}^{\alpha_i}\frac{1-q^{-1}}{1-q^{-1}\mathbf{z}^{\alpha_i}}\Phi_w^{s_i\mathbf{z}} + q^{-1}\frac{1-\mathbf{z}^{\alpha_i}}{1-q^{-1}\mathbf{z}^{\alpha_i}}\Phi_{s_iw}^{s_i\mathbf{z}} & \text{if } \ell(s_iw) < \ell(w), \end{cases} \tag{9}$$

where $\bar{\mathcal{A}}_{s_i}^{\mathbf{z}} := \frac{1-\mathbf{z}^{\alpha_i}}{1-q^{-1}\mathbf{z}^{\alpha_i}}\mathcal{A}_{s_i}^{\mathbf{z}}$. The normalized version of the intertwiner $\bar{\mathcal{A}}_{s_i}^{\mathbf{z}}$ is sometimes preferred because it is a cocycle, i.e.

$$\bar{\mathcal{A}}_{s_js_i}^{\mathbf{z}} = \bar{\mathcal{A}}_{s_j}^{s_i\mathbf{z}} \circ \bar{\mathcal{A}}_{s_i}^{\mathbf{z}}.$$

To any $w \in W$, let Δ_w^+ denote the set of positive roots $\{\alpha \in \Delta^+ \mid w(\alpha) \in \Delta^-\}$. In Proposition 2 of [22], following from Proposition 4.3 of [26], the following result is proven.

Proposition 3.7 *For any $w \in W$,*

$$\Omega_{w\mathbf{z}} \circ \mathcal{A}_w^{\mathbf{z}} = \Big(\prod_{\alpha \in \Delta_w^+} \frac{1 - q^{-1}\mathbf{z}^{-\alpha}}{1 - \mathbf{z}^\alpha}\Big)\Omega_{\mathbf{z}}. \tag{10}$$

In Theorem 3.8 below we will combine the above results to obtain a recursion relation for $\phi_w(\mathbf{z}; g)$ using Demazure type operators that we will define now.

Let $\mathcal{O}(\hat{T})$ be the ring of regular (polynomial) functions on $\hat{T}(\mathbb{C}) \cong (\mathbb{C}^\times)^r$. This ring is isomorphic to the group algebra of $\Lambda = \mathbb{Z}^r$ as follows. If $\mathbf{z} = (z_1, \cdots, z_r) \in (\mathbb{C}^\times)^r$ and $\lambda \in \Lambda$ let $\mathbf{z}^\lambda = \prod z_i^{\lambda_i}$. Then $\mathcal{O}(\hat{T})$ is spanned by the functions $\mathbf{z}^\lambda$. We may now define operators $\mathfrak{T}_i$ on $\mathcal{O}(\hat{T})$ as follows. Let v be a nonzero complex number and, for $f \in \mathcal{O}(\hat{T})$, let

$$\mathfrak{T}_{i,v}f(\mathbf{z}) = \mathfrak{T}_if(\mathbf{z}) = \frac{f(\mathbf{z}) - f(s_i\mathbf{z})}{\mathbf{z}^{\alpha_i} - 1} - v\frac{f(\mathbf{z}) - \mathbf{z}^{-\alpha_i}f(s_i\mathbf{z})}{\mathbf{z}^{\alpha_i} - 1}. \tag{11}$$

These operators satisfy the same braid relations

$$\mathfrak{T}_i \mathfrak{T}_j \cdots = \mathfrak{T}_j \mathfrak{T}_i \cdots$$

as the corresponding simple reflections s_i, s_j in the finite Weyl group (see, for example [22, Proposition 5]). The operators also satisfy the quadratic relation

$$\mathfrak{T}_i^2 = (v - 1)\mathfrak{T}_i + v. \tag{12}$$

This quadratic relation implies that $\mathfrak{T}_i$ is invertible. Indeed its inverse is

$$\mathfrak{T}_i^{-1} f(\mathbf{z}) = \frac{\mathbf{z}^{-\alpha_i} f(s_i \mathbf{z}) - \mathbf{z}^{\alpha_i} f(\mathbf{z})}{\mathbf{z}^{\alpha_i} - 1} - \frac{f(s_i \mathbf{z}) - \mathbf{z}^{\alpha_i} f(\mathbf{z})}{v(\mathbf{z}^{\alpha_i} - 1)}. \tag{13}$$

See [22] Propositions 5 and 6 for proofs of these facts.[1] The operators $\mathfrak{T}_i$ thus generate a finite Iwahori Hecke algebra. They are similar to the well-known *Demazure-Lusztig operators* [51], which by comparison send f to

$$\mathfrak{L}_{i,v} f(\mathbf{z}) = \mathfrak{L}_i f(\mathbf{z}) = \frac{f(\mathbf{z}) - f(s_i \mathbf{z})}{\mathbf{z}^{\alpha_i} - 1} - v \frac{f(\mathbf{z}) - \mathbf{z}^{\alpha_i} f(s_i \mathbf{z})}{\mathbf{z}^{\alpha_i} - 1}. \tag{14}$$

As we will discuss in Sect. 9, the difference is slight but significant. We will refer to the $\mathfrak{T}_i$ operators in (11) as *Demazure-Whittaker operators*.

The following result generalizes Theorem 2 of [22].

Theorem 3.8 *For any $w \in W$, a simple reflection s_i, and with $v = q^{-1}$,*

$$\phi_{s_i w}(\mathbf{z}; g) = \begin{cases} \mathfrak{T}_i \cdot \phi_w(\mathbf{z}; g) & if \ \ell(s_i w) > \ell(w), \\ \mathfrak{T}_i^{-1} \cdot \phi_w(\mathbf{z}; g) & if \ \ell(s_i w) < \ell(w), \end{cases} \tag{15}$$

where the $\mathfrak{T}_i$ and their inverses are as in (11) and (13).

Proof Recall from (2) that the Whittaker functions $\phi_w(\mathbf{z}; g)$ are made with respect to principal series with the Langlands parameter $\mathbf{z}^{-1}$. The result follows by expanding

$$\Omega_{s_i \mathbf{z}}(\mathcal{A}_{s_i}^{\mathbf{z}} \pi(g) \Phi_{w_1}^{\mathbf{z}})$$

in two different ways using the relation (9) and Proposition 3.7, respectively. Here one has to use the fact that $\mathcal{A}_{s_i}^{\mathbf{z}}$ is an intertwiner and therefore commutes with $\pi(g)$. This produces a functional equation, which can be rewritten in the form of equation (14). See [22, Theorem 2] for more details. □

Proposition 3.8 gives a recursion on Iwahori fixed vectors ϕ_w which is independent of the word used to represent w.

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[1] All references to [22] are to the published version; the operators in the arXiv version are slightly different.

Corollary 3.9 *Given any w_1, w_2 and a w_2-almost dominant weight λ, let $(s_{i_1}, \ldots, s_{i_k})$ be any path in the Weyl group from w_2 to w_1, so $w_2 \to s_{i_1} w_2 \to \cdots \to s_{i_k} \cdots s_{i_1} w_2 = w_1$. Set e_{i_j} to be $+1$ or -1 depending on whether s_{i_j} is an ascent or descent, respectively, in the Bruhat order. Then, with $v = q^{-1}$,*

$$\phi_{w_1}(\mathbf{z}; \varpi^{-\lambda} w_2) = v^{\ell(w_2)} \mathfrak{T}_{i_k}^{e_{i_k}} \cdots \mathfrak{T}_{i_1}^{e_{i_1}} \mathbf{z}^\lambda.$$

Proof This corollary follows immediately from Proposition 3.6 and Theorem 3.8. It generalizes Theorem 1 of [22], which evaluates the special case $w_2 = 1$.   □

Corollary 3.10 *For any $g \in G(F)$, the function $\phi_w(\mathbf{z}; g)$ is regular as a function of $\mathbf{z}$ on $\widehat{T}(\mathbb{C})$.*

Proof This is known on other grounds from Proposition 2.1 of [26], but let us show how it follows from our results. By Remark 3.3 we may assume $g = \varpi^{-\lambda} w_2$ and if $w = w_2$, regularity follows from Proposition 3.6. Then for more general $w = w_1$ as in Corollary 3.9, $\phi_w(\mathbf{z}; g)$ may be obtained by applying the $\mathfrak{T}_i$ and $\mathfrak{T}_i^{-1}$, and these preserve regularity.   □

Suppose that $w = yz$ in W, with $\ell(w) = \ell(y) + \ell(z)$. Then we write $z \leqslant_L w$. The partial order $\leqslant_L$ on W is the *left weak* (Bruhat) order. See [7] Chapter 3 for more information about the left and right weak orders.

Corollary 3.11 *Suppose that $w_2 \leqslant_L w_1$ in the left weak order. Let λ be a w_2-almost dominant weight. Then*

$$\phi_{w_1}(\mathbf{z}; \varpi^{-\lambda} w_2) = v^{\ell(w_2)} \phi_{w_1 w_2^{-1}}(\mathbf{z}; \varpi^{-\lambda}). \tag{16}$$

Proof If $w_2 = w_1$, this follow from Proposition 3.6. Therefore we assume that $w_2 <_L w_1$. Let s_i be a left descent of $w_1 w_2^{-1}$ and let $w_1' = s_i w_1$ so that $w_2 \leqslant_L w_1'$. By induction on $\ell(w_1 w_2^{-1})$ we have

$$\phi_{w_1'}(\mathbf{z}; \varpi^{-\lambda} w_2) = v^{\ell(w_2)} \phi_{w_1' w_2^{-1}}(\mathbf{z}; \varpi^{-\lambda}).$$

Applying $\mathfrak{T}_i$ to both sides of this identity and applying Theorem 3.8 gives (16).   □

4 Parahoric Whittaker functions

We now extend the results of the last section to Whittaker functions that are invariant under so-called 'parahoric subgroups' which are intermediate between the Iwahori subgroup J and the maximal compact subgroup K.

Let W be a Coxeter group with generators s_i ($i \in \mathbf{I}$), which we will call *simple reflections*. Let $\mathbf{J}$ be a subset of $\mathbf{I}$, let $W_{\mathbf{J}}$ be the subgroup generated by the s_j with $j \in \mathbf{J}$, and let

$$W^{\mathbf{J}} = \left\{ w \in W \mid w s_j > w \text{ for all } j \in \mathbf{J} \right\}. \tag{17}$$

Here $>$ is with respect to the Bruhat order, but by the Exchange Property for Coxeter groups ([7] Theorem 1.5.1) the condition $ws_j > w$ is equivalent to $\ell(ws_j) = \ell(w)+1$ in terms of the length function on W. By Proposition 2.4.4 of [7] every element of W has a unique factorization $w^{\mathbf{J}}w_{\mathbf{J}}$ with $w^{\mathbf{J}} \in W^{\mathbf{J}}$ and $w_{\mathbf{J}} \in W_{\mathbf{J}}$. Moreover, by the corollary to that proposition, every coset $wW_{\mathbf{J}}$ has a unique representative of shortest length, and $w \in W^{\mathbf{J}}$ if and only if w is that generator.

Lemma 4.1 *Let $w \in W^{\mathbf{J}}$. Suppose that s_i is a simple reflection such that $w^{-1}s_i w \notin W_{\mathbf{J}}$. Then $s_i w \in W^{\mathbf{J}}$. Assume furthermore that $s_i w > w$. Then $s_i wy > wy$ for any $y \in W_{\mathbf{J}}$.*

Proof Let us show that $s_i w \in W^{\mathbf{J}}$. If not, then for some $j \in \mathbf{J}$ we have $s_i ws_j < s_i w$. Since $w \in W^{\mathbf{J}}$ we have $ws_j > w$. Now there are two cases. First assume that $s_i w < w$. Then $ws_j > w > s_i w > s_i ws_j$, and $\ell(ws_j) = \ell(s_i ws_j) + 3$. This is a contradiction since $\ell(s_i y) = \ell(y) \pm 1$ for any $y \in W$. On the other hand suppose that $s_i w > w$. By the Lifting Property of the Bruhat order (Proposition 2.2.7 of [7]) the inequalities $s_i w > w, s_i ws_j < s_i w$ and $ws_j > w$ imply that $w \leqslant s_i ws_j < s_i w$. Since $\ell(s_i w) = \ell(s_i) + 1$ this implies that $w = s_i ws_j$ and therefore $w^{-1}s_i w = s_j \in W_{\mathbf{J}}$, contradicting one of our assumptions. Thus, $s_i w \in W_{\mathbf{J}}$.

For the rest of the proof, we assume that $s_i w > w$, and we will prove that $s_i wy > wy$ for any $y \in W_{\mathbf{J}}$. Arguing by contradiction, assume that $s_i wy < wy$. We claim that either $s_i wy = w'y$ for $w' < w$ or $s_i wy = wy'$ for $y' < y$. Indeed, let $s_{i_1} \cdots s_{i_k}$ be a reduced word for w and $s_{i_{k+1}} \cdots s_{i_l}$ be a reduced word for y. By [7] Proposition 2.4.4 $\ell(wy) = \ell(w)+\ell(y)$, so $s_{i_1} \cdots s_{i_l}$ is a reduced word for wy. By the Exchange Property for Coxeter Groups (Theorem 1.5.1 of [7]) it follows that $s_i wy = s_{i_1} \cdots \hat{s}_{i_m} \cdots s_{i_l}$ for some m. If $m \leqslant k$ we may take $w' = s_{i_1} \cdots \hat{s}_{i_m} \cdots s_{i_k}$, otherwise, we may take $y' = s_{i_{k+1}} \cdots \hat{s}_{i_m} \cdots s_{i_l}$.

First suppose that $s_i wy = w'y$ with $w' < w$. Then $s_i w = w' < w$, contradicting one of our assumptions. On the other hand, suppose that $s_i wy = wy'$ with $y' < y$. Then $y' \in W_{\mathbf{J}}$ since $y \in W_{\mathbf{J}}$ and $y' < y$. Hence $w^{-1}s_i w = y'y^{-1} \in W_{\mathbf{J}}$, also contradicting one of our assumptions. This proves $s_i wy > wy$. $\qquad\square$

Definition 4.2 By a *standard parahoric subgroup* we mean a subgroup of $K = G(\mathfrak{o})$ that arises as the preimage of a standard parabolic under the canonical map $K \longrightarrow G(\mathbb{F}_q)$. In particular, any such group contains the Iwahori subgroup J. We will denote these groups by $K_{\mathbf{J}}$ where $\mathbf{J}$ is the index set of simple roots in the corresponding standard parabolic subgroup P of G. We will denote the Levi subgroup of P by M which has Weyl group $W_{\mathbf{J}}$. With this notation $K_\varnothing = J$ (the Iwahori subgroup) and $K_{\mathbf{I}} = K$ (the maximal compact subgroup).

For example, if $G = \mathrm{GL}_r$, these may be constructed as follows. Let $\mathbf{r} = (r_1, \cdots, r_k)$ be a set of positive integers such that $\sum_i r_i = r$. Let $P := P_{\mathbf{r}}$ be the parabolic subgroup of GL_r containing the opposite Borel B_- with Levi subgroup $M = \mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_k}$ embedded diagonally in GL_r. The Weyl group $W_{\mathbf{J}} = S_{r_1} \times \cdots \times S_{r_k}$ of M is a parabolic subgroup of $W = S_r$ with $\mathbf{J}$ consisting of integers $1 \leqslant j \leqslant r - 1$ such that j is not of the form $r_1, r_1 + r_2, r_1 + r_2 + r_3$, etc. Then the standard parahoric subgroup $K_{\mathbf{J}}$ is the preimage of $P_{\mathbf{r}}(\mathbb{F}_q)$ under the map $K \longrightarrow \mathrm{GL}_r(\mathbb{F}_q)$.

If $w \in W^{\mathbf{J}}$ let

$$\psi_w^{\mathbf{J}} := \sum_{y \in W_{\mathbf{J}}} \phi_{wy}, \tag{18}$$

where the ϕ_w denote the standard basis Iwahori Whittaker functions defined in (2).

Proposition 4.3 *The $\psi_w^{\mathbf{J}}$ with $w \in W^{\mathbf{J}}$ are a basis of the $K_{\mathbf{J}}$-fixed vectors in the Whittaker model of a principal series $I(\mathbf{z})$.*

Proof Let $k = \mathbb{F}_q$ be the residue field. We have

$$G(k) = \bigsqcup_{w \in W^{\mathbf{J}}} B(k) w P(k) \quad \text{(disjoint)}$$

by [11], Remark 2 in Section IV.2.5 and the fact that $W^{\mathbf{J}}$ are a set of coset representatives for $W/W_{\mathbf{J}}$. (The parabolic P actually contains the opposite Borel subgroup to B but the decomposition is still valid.) Pulling this back to $K = G(\mathfrak{o})$ we have

$$K = \bigsqcup_{w \in W^{\mathbf{J}}} J_+ w K_{\mathbf{J}} \tag{19}$$

where J_+ is the upper Iwahori subgroup. We have bijections of coset spaces

$$B(F) \backslash G(F) \longleftrightarrow (B(F) \cap K) \backslash K \longleftrightarrow \bigsqcup_{w \in W^{\mathbf{J}}} N_-(\mathfrak{p}) w K_{\mathbf{J}}$$

where the first step follows from the Iwasawa decomposition $G(F) = B(F)K$ and for the second step we have used (19) and the Iwahori factorization $J_+ = (B(F) \cap K)N_-(\mathfrak{p})$. Therefore

$$G(F) = \bigsqcup_{w \in W^{\mathbf{J}}} B(F)N_-(\mathfrak{p}) w K_{\mathbf{J}} = \bigsqcup_{w \in W^{\mathbf{J}}} B(F) w K_{\mathbf{J}},$$

where for the last step we have used the fact that $w^{-1} N_-(\mathfrak{p}) w \subset K_{\mathbf{J}}$. Hence $I(\mathbf{z})^{K_{\mathbf{J}}}$ has a basis of functions $\Psi_w^{\mathbf{J}}$ ($w \in W^{\mathbf{J}}$) defined by

$$\Psi_w^{\mathbf{J}}(bw'k) = \begin{cases} \delta^{1/2} \tau_{\mathbf{z}}(b) & \text{if } w' \in w W_{\mathbf{J}}, \\ 0 & \text{otherwise}, \end{cases}$$

for $w' \in W$ and $k \in K_{\mathbf{J}}$. Decomposing the support of $\Psi_w^{\mathbf{J}}$ into a union of J-cosets, we see that $\Psi_w^{\mathbf{J}} = \sum_{y \in W_{\mathbf{J}}} \Phi_{wy}$, and projecting this identity into the Whittaker model, the statement follows. $\square$

Remark 4.4 One can deduce from Proposition 4.3 the dimension of the space of parahoric fixed vectors in the unramified principal series. This is in accordance with the work of Lansky [48] (see Theorem 1.1), where these dimensions were first computed.

Proposition 4.5 *Suppose that $w \in W^{\mathbf{J}}$ and that s_i is a simple reflection such that $w^{-1}s_i w \in W_{\mathbf{J}}$. Then $\psi_w^{\mathbf{J}}(\mathbf{z}; g)$ is divisible by $1 - v\mathbf{z}^{-\alpha_i}$ as a Laurent polynomial, and $(1 - v\mathbf{z}^{-\alpha_i})^{-1}\psi_w^{\mathbf{J}}(\mathbf{z}; g)$ is invariant under the reflection s_i. Moreover $s_i w > w$.*

Proof Let $t = w^{-1}s_i w$ be the reflection that is in $W_{\mathbf{J}}$. Assume $y \in W_{\mathbf{J}}$ and $ty > y$. Then $s_i wy = wty$ and by Theorem 3.8 we have $\phi_{wty} = \mathfrak{T}_i \phi_{wy}$. Thus

$$\psi_w^{\mathbf{J}} = \sum_{y \in W_{\mathbf{J}}} \phi_{wy} = \sum_{\substack{y \in W_{\mathbf{J}} \\ ty > y}} (\phi_{wy} + \phi_{wty}) = (1 + \mathfrak{T}_i) \sum_{\substack{y \in W_{\mathbf{J}} \\ ty > y}} \phi_{wy}.$$

Now using the fact that $(\mathfrak{T}_i - v)(\mathfrak{T}_i + 1) = 0$, we have $\mathfrak{T}_i \psi_w^{\mathbf{J}} = v\psi_w^{\mathbf{J}}$. Substituting the definition of $\mathfrak{T}_i$ a small amount of algebra gives

$$(1 - v\mathbf{z}^{\alpha_i})\psi_w^{\mathbf{J}}(\mathbf{z}; g) = (1 - v\mathbf{z}^{-\alpha_i})\psi_w^{\mathbf{J}}(s_i \mathbf{z}; g). \tag{20}$$

The function $\psi_w^{\mathbf{J}}(\mathbf{z}; g)$ is a regular function on $\hat{T}(\mathbb{C})$ by Corollary 3.10. The ring $\mathcal{O}(\hat{T})$ of regular functions is a principal ideal domain; indeed it is a Laurent polynomial ring. The functions $1 - v\mathbf{z}^{\alpha_i}$ and $1 - v\mathbf{z}^{-\alpha_i}$ are coprime. From the right-hand side of (20), $1 - v\mathbf{z}^{-\alpha_i}$ divides the left-hand side, and it therefore divides $\psi_w^{\mathbf{J}}(\mathbf{z}; g)$. Remembering that $w(\alpha_i) = -\alpha_i$, we may rearrange (20) in the form

$$\frac{\psi_w^{\mathbf{J}}(\mathbf{z}; g)}{1 - v\mathbf{z}^{-\alpha_i}} = s_i \left(\frac{\psi_w^{\mathbf{J}}(\mathbf{z}; g)}{1 - v\mathbf{z}^{-\alpha_i}} \right),$$

and we have proved that this is an s_i-invariant regular function.

The last assertion to be proved is that $s_i w > w$. For this we note that with $t = w^{-1}s_i w \in W_{\mathbf{J}}$, wt is the unique factorization of $s_i w$ into a product of elements of $W^{\mathbf{J}}$ and $W_{\mathbf{J}}$ by Proposition 2.4.4 of [7] which was mentioned before, and by part (iii) of that result, $\ell(s_i w) = \ell(w) + \ell(t) > \ell(w)$, as required. $\square$

Proposition 4.6 *Suppose that $w \in W^{\mathbf{J}}$ and that s_i is a simple reflection such that $w^{-1}s_i w \notin W_{\mathbf{J}}$. Then*

$$\psi_{s_i w}^{\mathbf{J}}(\mathbf{z}; g) = \begin{cases} \mathfrak{T}_i \psi_w^{\mathbf{J}}(\mathbf{z}; g) & \text{if } s_i w > w, \\ \mathfrak{T}_i^{-1} \psi_w^{\mathbf{J}}(\mathbf{z}; g) & \text{if } s_i w < w. \end{cases}$$

Proof By Lemma 4.1, $w \in W^{\mathbf{J}}$ if and only if $s_i w \in W^{\mathbf{J}}$. With this in mind, the two cases are equivalent since we may interchange the roles of w and $s_i w$. Therefore we may assume that $s_i w > w$. Let $y \in W_{\mathbf{J}}$. By Lemma 4.1 we have $s_i wy > wy$ and so by Theorem 3.8 we have $\phi_{s_i wy} = \mathfrak{T}_i \phi_{wy}$. Therefore

$$\psi_{s_i w}^{\mathbf{J}} = \sum_{y \in W_{\mathbf{J}}} \phi_{s_i wy} = \mathfrak{T}_i \sum_{y \in W_{\mathbf{J}}} \phi_{wy} = \mathfrak{T}_i \psi_w^{\mathbf{J}}.$$

$\square$

In the following theorem, we show that Proposition 4.5 implies a Casselman-Shalika formula for $\psi_1^{\mathbf{J}} = \sum_{w \in W_{\mathbf{J}}} \phi_w$ for any fixed subset $\mathbf{J}$. If λ is a dominant weight for $\hat{G}(\mathbb{C})$, the complex, connected Langlands dual group of G, then it is also a dominant weight for the Levi subgroup $\hat{M}(\mathbb{C})$, the dual group of M having Weyl group $W_{\mathbf{J}}$. So we may consider the irreducible character $\chi_\lambda^{\mathbf{J}}$ of $\hat{M}(\mathbb{C})$ with highest weight λ. We will denote by $\chi_\lambda^{\mathbf{J}}(\mathbf{z})$ the value of this character on the semisimple element with eigenvalues $(z_1, \ldots, z_r)$. The root system $\Delta_{\mathbf{J}}$ of $\hat{M}(\mathbb{C})$ is a subsystem of the root system Δ of $\hat{G}(\mathbb{C})$.

Theorem 4.7 *Suppose that λ is a dominant weight for G. Then*

$$\psi_1^{\mathbf{J}}(\mathbf{z}; \varpi^{-\lambda}) = \prod_{\alpha \in \Delta_{\mathbf{J}}^+} (1 - v\mathbf{z}^{-\alpha}) \chi_\lambda^{\mathbf{J}}(\mathbf{z}). \tag{21}$$

If $w \in W^{\mathbf{J}}$ then

$$\psi_w^{\mathbf{J}}(\mathbf{z}; g) = \mathfrak{T}_w \psi_1^{\mathbf{J}}(\mathbf{z}; g) \tag{22}$$

Proof We will argue that the expression

$$\frac{\psi_1^{\mathbf{J}}(\mathbf{z}; \varpi^{-\lambda})}{\prod_{\alpha \in \Delta_{\mathbf{J}}^+} (1 - v\mathbf{z}^{-\alpha})} \tag{23}$$

is regular for $\mathbf{z} \in \hat{T}(\mathbb{C})$, symmetric under the action of $W_{\mathbf{J}}$, and independent of v.

If s_i is a simple reflection in $W_{\mathbf{J}}$ then we may write (23) as $(1 - v\mathbf{z}^{-\alpha_i})^{-1} \psi_1^{\mathbf{J}}(\mathbf{z}; \varpi^{-\lambda})$ divided by the remaining factors, which are permuted by s_i. By Proposition 4.5 this shows that (23) is invariant under s_i and so it is invariant under $W_{\mathbf{J}}$.

Next let us show that (23) is regular on all of $\hat{T}(\mathbb{C})$. To see this, note that its potential poles are in the union of the hypersurfaces $\mathbf{z}^\alpha = v$ ($\alpha \in \Delta_{\mathbf{J}}^+$). But by Proposition 4.5 if α_i is a simple root the hyperplane $\mathbf{z}^{\alpha_i} = v$ is not among the poles. As the polar divisor is invariant under $W_{\mathbf{J}}$, neither are any of the other loci $\mathbf{z}^\alpha = v$. Hence (23) is regular.

Now the numerator and the denominator of (23) are polynomials in z_i, z_i^{-1} and v, and the numerator is divisible by the denominator. We will argue that both have degree $|\Delta_{\mathbf{J}}^+| = \ell(w_0^{\mathbf{J}})$ as polynomials in v where $w_0^{\mathbf{J}}$ is the long element of $W_{\mathbf{J}}$ which shows that the ratio is in fact independent of v. For the numerator, we may write $\psi_1^{\mathbf{J}} = \sum_{w \in W_{\mathbf{J}}} \phi_w$ and $\phi_w(\varpi^{-\lambda}) = \mathfrak{T}_w \mathbf{z}^\lambda$; it follows from the definition of $\mathfrak{T}_w$ that its degree in v is $\ell(w)$, which takes its maximum at the long element $w_0^{\mathbf{J}}$, proving that the degree is $\ell(w_0^{\mathbf{J}})$. And clearly the denominator has the same degree.

Then (23) is independent of v and we may therefore take the limiting case $v \to 0$ to evaluate it. Let

$$\partial_i^\circ f(\mathbf{z}) = \frac{f(\mathbf{z}) - f(s_i \mathbf{z})}{\mathbf{z}^{\alpha_i} - 1}, \qquad \partial_i f(\mathbf{z}) = \frac{f(\mathbf{z}) - \mathbf{z}^{-\alpha_i} f(s_i \mathbf{z})}{1 - \mathbf{z}^{-\alpha_i}}.$$

We have $\partial_i^\circ = \partial_i - 1$. These Demazure operators are known to satisfy the braid relations; see for example [23] Propositions 25.1 and 25.3 but note that in Proposition 25.3 there is a typo and D_i should be ∂_i. If $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression define $\partial_w = \partial_{i_1} \cdots \partial_{i_k}$ and similarly for ∂_w°.

When $v \to 0$ the operator $\mathfrak{T}_i$ reduces to ∂_i° and so

$$\lim_{v \to 0} \phi_w(\varpi^{-\lambda}) = \partial_w^\circ \mathbf{z}^\lambda.$$

By Theorem 2.1 of [17], it follows that (23) equals $\sum_{w \in W_{\mathbf{J}}} \partial_w^\circ \mathbf{z}^\lambda = \partial_{w_0^{\mathbf{J}}} \mathbf{z}^\lambda$ and by the Demazure character formula ([23] Theorem 25.3), this is $\chi_\lambda^{\mathbf{J}}(\mathbf{z})$. This proves (21).

Now let us prove (22) by induction on $\ell(w)$. If $w = 1$, there is nothing to prove. Thus suppose that $w = s_i w'$ where s_i is a simple reflection and $\ell(w') < \ell(w)$. Then $w^{-1} s_i w \notin W_{\mathbf{J}}$, since otherwise the last assertion of Proposition 4.5 would be contradicted. Therefore by Lemma 4.1 $w' \in W^{\mathbf{J}}$ and by induction $\psi_{w'}^{\mathbf{J}} = \mathfrak{T}_{w'} \psi_1^{\mathbf{J}}$. Now Proposition 4.6 gives

$$\psi_w^{\mathbf{J}} = \mathfrak{T}_i \mathfrak{T}_{w'} \psi_1^{\mathbf{J}} = \mathfrak{T}_w \psi_1^{\mathbf{J}},$$

proving (22). □

Remark 4.8 For Iwahori Whittaker functions Corollary 3.9 gives an algorithm to compute ϕ_{w_1} at any value of g (which we can take to be on the form $\varpi^{-\lambda} w_2$). This depends on Proposition 3.6, which gives the base case $\phi_w(\varpi^{-\lambda} w)$, i.e. $w_1 = w_2$, for a recursive algorithm using Demazure-Whittaker operators, suitable for implementation on a computer. Our parahoric results are not as general because we only have a direct expression (without Demazure-Whittaker operators) for the base case (21) which is only for $w_1 = w_2 = 1$. For an arbitrary base case $w_1 = w_2 \neq 1$ we need to fall back on (18) which expresses $\psi_{w_1}^{\mathbf{J}}(\varpi^{-\lambda} w_2)$ as a sum of $\phi_{w_1 y}(\varpi^{-\lambda} w_2)$. However since the ϕ_w are computable, this is not an obstacle to an explicit computation of $\psi_w^{\mathbf{J}}$. In both cases we may then compute all $\psi_{w_1}^{\mathbf{J}}(\varpi^{-\lambda} w_2)$ from $\psi_{w_2}^{\mathbf{J}}(\varpi^{-\lambda} w_2)$ recursively, taking Proposition 4.5 and 4.6 into account. Alternatively, knowing all ϕ_w we may compute $\psi_{w_1}^{\mathbf{J}}(\varpi^{-\lambda} w_2)$ directly by the sum (18).

5 Yang–Baxter equations from fusion

The models that we will be concerned with take place on planar graphs. In using the term *graph* to describe these arrays we are deviating from usual terminology, where edges have always two vertices, for we will allow open edges with only a single endpoint. Thus we mean a set of *vertices* which are points in the plane, together with *edges* that are arcs which either join two vertices, or which are attached to only a single vertex. The edges which are only attached to a single vertex are called *boundary edges*. The edges attached to two vertices are called *interior edges*. Every vertex is adjacent to four edges. Edges can only cross at a vertex.
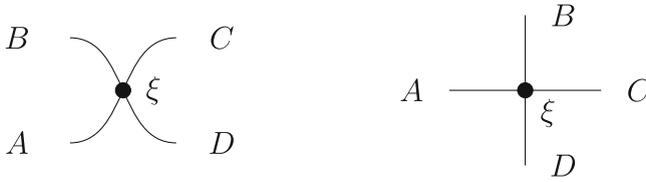
Fig. 1 Left: a vertex adjoining four horizontal edges. Such a vertex will be called an *R-vertex*. Right: A vertex adjoining two horizontal edges and two vertical ones. We will call such vertices *ordinary*. Each vertex receives a label ξ corresponding to its Boltzmann weights

For every edge A in the graph there is a finite set Σ_A of values called *spins* that may be assigned to the edge in a state of the system.

Assumption 5.1 At each vertex, let A, B, C, D be the four adjacent edges, arranged so that A and C are opposite edges, as are B and D. Then $\Sigma_A = \Sigma_C$ and $\Sigma_B = \Sigma_D$.

Each vertex has a *label* ξ, and an associated set of *Boltzmann weights* β_ξ. This is a rule which assigns a complex number to every possible choice of spins at the four adjacent edges of the vertex. Thus if A, B, C, D are the four adjacent edges to a vertex with label ξ, this data consists of a map

$$\beta_\xi : \Sigma_A \times \Sigma_B \times \Sigma_C \times \Sigma_D \to \mathbb{C}.$$

We call the set of spins $(a, b, c, d) \in \Sigma_A \times \Sigma_B \times \Sigma_C \times \Sigma_D$ a *configuration* at the vertex. The configuration is *admissible* if $\beta_\xi(a, b, c, d) \neq 0$.

Assumption 5.2 At each vertex, if three out of four spins in (a, b, c, d) in an admissible configuration are given, the fourth is uniquely determined.

In a system $\mathfrak{S}$, the data specifying the system are the graph itself, the spinsets Σ_A, the Boltzmann weight data β_ξ for each label ξ, and for every boundary edge A a fixed boundary spin $\mathfrak{b}_A \in \Sigma_A$. For example, the labels ξ might be complex numbers and β_ξ are uniformly described as a set of complex-valued functions of ξ for each configuration. Note that the spins of the boundary edges are fixed, and are part of the data specifying the system.

A *state* $\mathfrak{s}$ of the system is an assignment of spins to all edges. That is, for each edge A there is specified a spin $\mathfrak{s}_A \in \Sigma_A$. For boundary edges $\mathfrak{s}_A$ must be the fixed boundary spin $\mathfrak{b}_A$, while the interior edge spins are allowed to vary. We will use the notation $\mathfrak{s} \in \mathfrak{S}$ to mean that $\mathfrak{s}$ is a state of the system $\mathfrak{S}$. The *Boltzmann weight* $\beta(\mathfrak{s})$ of the state is the product of the Boltzmann weights at the labelled vertices and the state is said to be *admissible* if all of its vertices are admissible. The *partition function* $Z(\mathfrak{S})$ is the sum of the Boltzmann weights of all the (admissible) states.

In the systems that we will consider, the edges may all be classified as either *horizontal* or *vertical*. There will be two types of vertices. In one type, the vertex intersects four horizontal edges and will be called an *R-vertex*. In the other, called *ordinary*, it intersects two horizontal and two vertical ones. See Fig. 1.
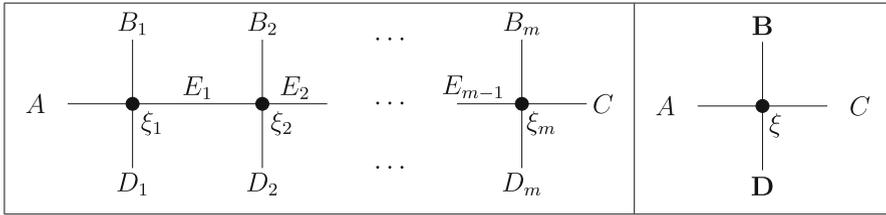
Fig. 2 Fusion. This procedure replaces a sequence of vertices by a single vertex

Next we explain a procedure we refer to as *fusion* for producing new kinds of edges and vertices from given ones. (This is partly inspired by a process of the same name described in Borodin and Wheeler [9], Appendix B that goes back to [46].)

Given a sequence of edges, $A_1, \cdots, A_m$ we may replace these with a single edge **A** such that $\Sigma_{\mathbf{A}} = \prod_k \Sigma_{A_k}$. This edge is called the *fusion* of the edges $\{A_k\}$. Next assume that we have a sequence of m *ordinary* vertices with labels $\xi_1, \cdots, \xi_m$ such that the vertex with label ξ_k is adjoined to the vertex with label ξ_{k+1} by an edge E_k if $1 \leqslant k \leqslant m - 1$. Let the remaining adjacent edges of the vertex with label ξ_k be B_k and D_k and A (if $k = 1$) and C if $k = m$. Thus the configuration is as in Fig. 2 (left).

Now we may construct the fusion **B** of the edges B_i as above, as well as the fusion **D** of the edges D_i. We may then fuse the vertices, replacing the sequence of m vertices with labels $\xi_1, \ldots, \xi_m$ by a single vertex labeled ξ (as in Fig. 2). It remains to discuss the Boltzmann weights. Let spins $(a, b, c, d) \in \Sigma_A \times \Sigma_{\mathbf{B}} \times \Sigma_C \times \Sigma_{\mathbf{D}}$. By definition b and d are sequences of spins $b_k \in \Sigma_{B_k}$ and $d_k \in \Sigma_{D_k}$. Fixing (a, b, c, d), it follows from Assumption 5.2 that the system in Fig. 2 (left) has at most one (admissible) state. We define $\beta_\xi(a, b, c, d)$ to be its partition function. It is clear that Assumption 5.2 remains valid for this fused vertex.

At any vertex, it will be useful to choose a clockwise ordering (A, B, C, D) of the adjoining edges. In our illustrations, we will always choose the ordering as in Fig. 1. If A is an edge, we will denote by V_A the free vector space with basis Σ_A. By Assumption 5.1, we may identify $V_A = V_C$ and $V_B = V_D$. Then the Boltzmann weights at a vertex with label ξ define an element of $\mathrm{End}(V_A \otimes V_B)$ by

$$a \otimes b \mapsto \sum_{(c,d) \in \Sigma_C \times \Sigma_D} \beta_\xi(a, b, c, d)(c \otimes d). \tag{24}$$

If the vertex is an R-vertex we will denote this endomorphism as R_ξ; this endomorphism is called an *R-matrix*. For ordinary vertices, we will denote the endomorphism (24), which is called a *transfer matrix*, as T_ξ.

Definition 5.3 Suppose that, for ordinary vertices labeled ξ, η and R-vertex labeled ζ, there exists Boltzmann weights such that for every choice of boundary spins (a, b, c, d, e, f) the partition functions of the two systems in Fig. 3 are equal. Then we say we have a solution of the *Yang–Baxter equation*.

Let A, B, C, D, E, F be the boundary edges of these configurations, so that $a \in \Sigma_A$, etc. By Assumption 5.1 $V_A = V_D$, $V_B = V_E$ and $V_C = V_F$. Then $R_\zeta \in \mathrm{End}(V_A \otimes V_B)$,
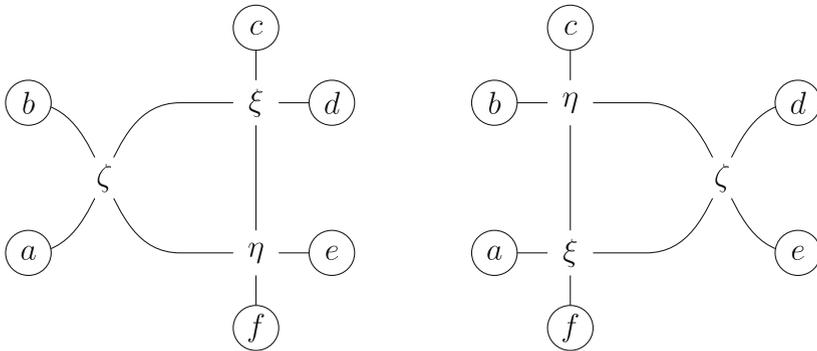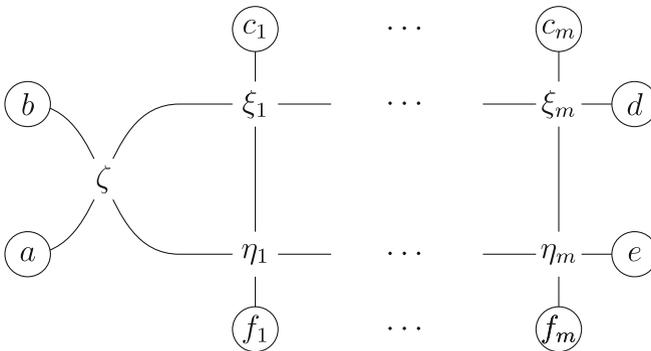
Fig. 3 The Yang–Baxter equation

Fig. 4 Setup fr the Yang–Baxter equation with fused vertices ξ and η

$T_\xi \in \mathrm{End}(V_A \otimes V_C)$ and $T_\eta \in \mathrm{End}(V_B \otimes V_C)$. The Yang–Baxter equation can be expressed in the formula

$$(R_\zeta)_{12}(T_\xi)_{13}(T_\eta)_{23} = (T_\eta)_{23}(T_\xi)_{13}(R_\zeta)_{12}, \tag{26}$$

an identity in $\mathrm{End}(V_A \otimes V_B \otimes V_C)$, where, in the notation common in quantum group theory, $(R_\zeta)_{12}$ denotes R_ζ acting on the first two components of $V_A \otimes V_B \otimes V_C$ and so forth. Note that the left side of (26) corresponds to the right side in Fig. 3, and vice versa. We wish to consider examples of (25) where the ordinary vertices arise from the fusion process described above. Thus the left configuration can be expanded as in Fig. 4.

Lemma 5.4 *Suppose there exists a sequence of R-vertices with labels $\zeta_1, \ldots, \zeta_{m+1}$ such that $\zeta_1 = \zeta_{m+1} = \zeta$ and such that for each $1 \leqslant k \leqslant m$, the two partition functions in Fig. 5 are equal. (Note that the R-vertex of the left-hand side is ζ_k while the one on the right-hand side is ζ_{k+1}.) Then the auxiliary Yang–Baxter equations in (27) induce a solution to the Yang–Baxter equation in (25) for the fused system.*
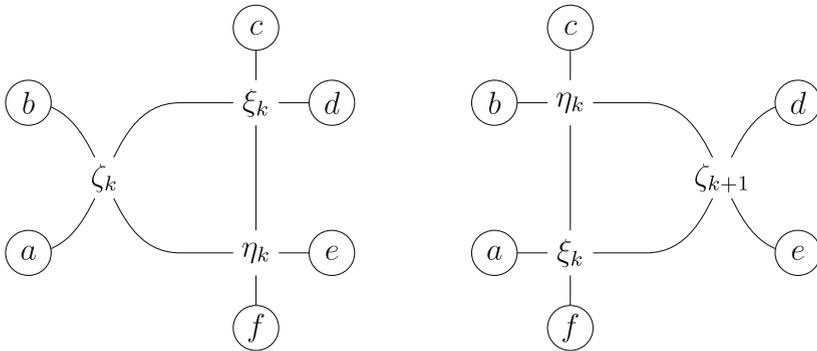
Fig. 5 Auxiliary Yang–Baxter equations. These imply a Yang–Baxter equation for the fusion situation in Fig. 4. In these equations, the R-matrix changes after moving past the vertical edges. After m such changes, it is back to its original form

Proof This follows from the usual train argument. Each time the R-matrix moves to the right, ζ_k is replaced by ζ_{k+1}. Since $\zeta_1 = \zeta_{m+1} = \zeta$, the statement follows. ☐

Remark 5.5 We have chosen to call the method for producing new solutions to the Yang–Baxter equation outlined in this section 'fusion,' despite some differences with the prior notion in the literature (see for example [46] and Appendix B of [9]). Both methods construct new solutions from old by forming new weights using one-row partition functions. The typical fusion construction features two steps: first summing over all one-row systems with given multiset of spins on its vertical edges (which is the graphical manifestation of the R-matrix of a tensor product of quantum group modules) and then taking a further weighted average (which manifests the resulting R-matrix for projection onto irreducible constituents of the tensor product; see for example (B.2.1) of [9]). However, our fusion prescribes a set of labels for each vertex in the one-row system, our weights are allowed to vary based on the label, and we do not require a second summation acting as a projection. Our example of weights for fusion in the next section (see Fig. 7) will have vertices labeled by colors and the weights depend critically on this color.

6 Yang–Baxter equations for colored models

We shall describe Yang–Baxter equations for systems that generalize the Tokuyama model introduced in Sect. 2 by replacing its $\ominus$ spins by a set $\mathfrak{P}$ (called the *palette*) of r different *colors*. The set $\mathfrak{P}$ is ordered, and when convenient we may take $\mathfrak{P}$ to be the set of integers $1 \leqslant c \leqslant r$.

If A is a horizontal edge, the spinset Σ_A is $\{\oplus\} \cup \mathfrak{P}$. On the other hand if A is a vertical edge, the spinset Σ_A is the power set of $\mathfrak{P}$ where it will be convenient to identify the empty set with $\oplus$ as before. Now in a state $\mathfrak{s}$ of such a system, we say that a horizontal edge A *carries* the color c if $\mathfrak{s}_A = c$. If A is a vertical edge, we say that A *carries* the color c if $c \in \mathfrak{s}_A$, remembering that $\mathfrak{s}_A$ is a subset of $\mathfrak{P}$. The vertical edges are thus allowed to carry more than one color. But since Σ_A is a set

Fig. 6 The colored R-vertex weights. The colors c and d are an arbitrary choice of distinct colors in $\mathfrak{P}$. If a configuration does not appear in this table, the Boltzmann weight is zero. The associated R-matrix equals that of evaluation modules $V(z_j) \otimes V(z_i)$ of the quantum supergroup $U_q(\widehat{\mathfrak{gl}}(r|1))$ for $q = 1/\sqrt{v}$

Fig. 7 Boltzmann weights for monochrome ordinary vertices. The weight depends on a pair of labels: a complex number z_i (suppressed in pictures above) and a color (denoted c above). Note that admissible vertical edges adjacent to the monochrome vertex may only carry the color c of the vertex, while adjacent horizontal edges may carry any color. In particular, in the diagrams above, $c = d$ is allowed

(instead of a multiset as in bosonic models) each color appears along a vertical edge with multiplicity at most 1. The horizontal edges may carry at most one color.

Having described the admissible configurations at each vertex, it remains to describe the Boltzmann weights for both the ordinary and the R-vertices in colored systems. The Boltzmann weights of the R-vertex are given in Fig. 6.

Remark 6.1 It may be checked that with the Boltzmann weights in Fig. 6, the R-matrix agrees with a Drinfeld twist of the (ungraded) R-matrix of evaluation modules $V(z_j) \otimes V(z_i)$ for the quantum affine Lie superalgebra $U_{1/\sqrt{v}}(\widehat{\mathfrak{gl}}(r|1))$ (cf. [44, Definition 2.1]). The r colored spins span one graded piece in the super vector space, while the $\oplus$ spin spans the remaining one-dimensional piece.

The Boltzmann weights of the ordinary vertices, which adjoin two horizontal edges and two vertical edges with many coloring possibilities, are harder to describe. We will define these by means of fusion, starting with simpler *monochrome vertices*: vertices that adjoin only *monochrome edges* that are only allowed to carry at most one particular color. Each monochrome vertex is itself assigned a color $c \in \mathfrak{P}$, and

$(1-v)z_i$ if $c > d$,	left: 1	left: $(1-v)z_i$
$(-v)(1-v)z_i$ if $c < d$.	right: $(1-v)z_i$	right: $-v$

Fig. 8 Ordinary fused vertex constructed from monochrome vertices by fusion for $r = 2$. Left: a fused vertex (compare with Fig. 10). Middle: The case $c > d$, using weights from Fig. 7. Right: The case $d > c$

the spinset of a vertical edge attached to that vertex is $\{\oplus, c\}$. For horizontal edges, the spinset is $\{\oplus\} \cup \mathfrak{P}$ just as before. The Boltzmann weights for the monochrome (ordinary) vertices are given in Fig. 7.

Convention 6.2 (*Monochrome vertices*) Now the admissible ordinary vertices and their weights may be described by fusion of monochrome vertices as detailed in Sect. 5. In a model with r colors, we replace each ordinary vertex by a single row of r monochrome vertices with color labels arranged in ascending order from left to right. Recall that a vertical edge A adjacent to an ordinary vertex is decorated by a subset $\mathfrak{s}_A$ of $\mathfrak{P}$. For the corresponding monochrome vertices we color the c-th such edge (with color c) if and only if the color c appears in the set $\mathfrak{s}_A$.

Remark 6.3 Looking ahead to Sect. 7, we will consider systems made from these ordinary, fused vertices. Regarding the vertex as a fusion, we may replace the entire system by an *expanded* or *monochrome system* with monochrome vertices; each column of vertices is replaced by r different vertices. Then we may refer to the system with fused edges as the *fused* system. See Fig. 16 for an example of this procedure. It follows from the definition of the fused weights that the fused and expanded systems have the same partition function. Indeed, there is a bijection between the states of the fused and expanded systems, and corresponding states have the same Boltzmann weight, by definition.

As described in Sect. 5, the Boltzmann weight of the fused vertex is just the partition function of the single row of these ordered r monochrome vertices, which has at most one admissible state. In Fig. 8, we compute an example of a fused Boltzmann weight when $r = 2$ from the corresponding monochrome vertices. In Fig. 9 we give all the fused Boltzmann weights (for any r) in which the vertical edges carry at most one color. The possible cases in which vertical edges carry two colors are shown in Fig. 10. For $r > 2$ one would have to complete these with similar tables for vertical edges carrying more colors. At the end of this section we will give all the fused weights in a closed form in a notation similar to the one used in [9]. See Fig. 12.

It remains to discuss the Yang–Baxter equation for fused vertices, which will result from auxiliary Yang–Baxter equations for the monochrome model according to Lemma 5.4. We first need to define monochrome R-vertices for use in (27), generalizing the R-vertices in Fig. 6. These will play a role of the vertices labeled ζ_k in Lemma 5.4, but now each such R-matrix depends not only on a pair of complex

| 1 | z_i | $\begin{array}{ll} z_i & \text{if } c > d \\ v z_i & \text{if } c < d \end{array}$ | $\begin{array}{ll} (1-v)z_i & \text{if } c > d \\ 0 & \text{if } c < d \end{array}$ |

| $-v$ | z_i | 1 | $(1-v)z_i$ |

Fig. 9 Fused weights (I). These are the Boltzmann weights in which the vertical edges carry no more than one color. Since edges can carry more than one color, this is not a complete list of the possibilities. In this figure, $c \neq d$ except where explicitly allowed

| $\begin{array}{ll} (1-v)z_i & \text{if } c > d, \\ (-v)(1-v)z_i & \text{if } c < d. \end{array}$ | $\begin{array}{ll} v & \text{if } c > d, \\ -v & \text{if } c < d. \end{array}$ | $(-v)^2$ | $\begin{array}{ll} z_i & \text{if } c > d, \\ v z_i & \text{if } c < d. \end{array}$ |

Fig. 10 Fused weights (II). The vertical edges can carry more than one color, with multiplicity at most one (so $c \neq d$ in this figure). These are the extra possibilities when at most two colors appear

parameters z_i, z_j, but also on a color c. The Boltzmann weights for these are given in Fig. 11.

Let $R(z_i, z_j)$ denote the R-matrix constructed with the weights in Fig. 6 according to (24), and if $1 \leqslant c \leqslant r$ is a color, let $R^{(c)}(z_i, z_j)$ denote the colored R-matrix constructed from the Boltzmann weights in Fig. 11 where the vertex is labeled by the color c. Note that $R^{(1)} = R$. Also, let $T^{(c)}(z_i)$ denote the matrix associated with the monochrome (ordinary) vertices labeled by the color c whose Boltzmann weights are described in Fig. 7. We recall that the colors c are identified with the integers $1 \leqslant c \leqslant r$, so there is a next color $c + 1$ unless c is the last color $c = r$, in which case we define $R^{(r+1)} := R$. We may now describe auxiliary Yang–Baxter equations involving the monochrome vertices.

Proposition 6.4 *If $1 \leqslant c \leqslant r$, then*

$$R^{(c)}(z_i, z_j)_{12} T^{(c)}(z_i)_{13} T^{(c)}(z_j)_{23} = T^{(c)}(z_j)_{23} T^{(c)}(z_i)_{13} R^{(c+1)}(z_i, z_j)_{12}. \quad (28)$$

Fig. 11 R-vertices for auxiliary Yang–Baxter equations. These are labeled by a color c and a pair of parameters (z_i, z_j) (suppressed in the pictures above). If the color c is minimal, that is if $c \leqslant d, e$ for all colors that appear in this figure, this agrees with the R-vertices in Fig. 6. In this figure, the colors c, d, e are distinct except when $c = d$ or $c = e$ is explicitly allowed

Proof Note that since we are using monochrome edges, at most three different colors can appear in the boundary (and interior) spins in the equivalent description (27) of (28). There are two more colors c and $c + 1$ which must also be compared with these three in the values for $T^{(c)}$, $R^{(c)}$, $R^{(c+1)}$ from Figs. 6 and 7. All possibilities are covered if we take 5 colors and we conclude that if the Yang–Baxter equation is checked for $r = 5$ then it is true for all r. We checked the Yang–Baxter equation for $r = 5$ using a computer and (28) is proved. □

Theorem 6.5 *The Yang–Baxter equation for colored models is satisfied:*

$$R(z_i, z_j)_{12} T(z_i)_{13} T(z_j)_{23} = T(z_j)_{23} T(z_i)_{13} R(z_i, z_j)_{12}$$

Proof This follows from Proposition 6.4 and Lemma 5.4. □

Remark 6.6 Restricting the palette $\mathfrak{P}$ to a single color, the weights in Figs. 9 recover the weights used to compute Tokuyama's deformation of the Weyl character formula as seen for example in the row labeled $S^{\Gamma}(i)$ of [20, Table 2] if we replace this color with a $\ominus$ spin. The R-matrices for the two models are also equal.

Σ $+ \;—\!\!+\!\!—\; +$ Σ	Σ $c \;—\!\!+\!\!—\; c$ Σ	Σ $+ \;—\!\!+\!\!—\; c$ Σ_c^-
$(-v)^{\lvert\Sigma_{[1,r]}\rvert}$	$z_i v^{\lvert\Sigma_{[c+1,r]}\rvert}$	$(-v)^{\lvert\Sigma_{[1,c-1]}\rvert} v^{\lvert\Sigma_{[c+1,r]}\rvert}$
Σ $c \;—\!\!+\!\!—\; +$ Σ_c^+	Σ $c \;—\!\!+\!\!—\; d$ $\Sigma_{c\,d}^{+-}$	Σ $d \;—\!\!+\!\!—\; c$ $\Sigma_{d\,c}^{+-}$
$(1-v)z_i(-v)^{\lvert\Sigma_{[c+1,r]}\rvert}$	$(1-v)z_i(-v)^{\lvert\Sigma_{[c+1,d-1]}\rvert} v^{\lvert\Sigma_{[d+1,r]}\rvert}$	0

Fig. 12 Colored Boltzmann weights in a style resembling [9] (2.2.2) or (2.2.6), except that we are using $+$ for the 'colorless' horizontal edges, and in place of their multiset **I** of colors, we use a subset Σ of the palette. We are assuming $c < d$

We will now describe the fused weights in a closed form. For comparison with [9], we will choose a notation close to theirs. In [9], vertical edges are labeled by tuples $\mathbf{I} = (I_1, \cdots, I_r) \in \mathbb{N}^r$ representing a state in which the k-th color has multiplicity I_k. The principal difference between their systems and ours is that colors can only occur with multiplicity 0 or 1 in our systems. In other words, if we imitate their setup, each $I_k \in \{0, 1\}$. Hence the same data can be specified by the subset $\Sigma = \{k \mid I_k = 1\}$ of the palette $\mathfrak{P}$.

In [9], an operation adds (resp. removes) a color a to the tuple **I**, that is, increments (resp. decrements) I_a and the resulting tuple is denoted $\mathbf{I}_a^+$ (resp. $\mathbf{I}_a^-$). We therefore introduce the corresponding operations on the set Σ and denote $\Sigma_a^+ = \Sigma \cup \{a\}$, to be used only if $a \notin \Sigma$, and $\Sigma_a^- = \Sigma \setminus \{a\}$, to be used only if $a \in \Sigma$. Finally if $a \in \Sigma$ and $b \notin \Sigma$, we will denote $\Sigma_{ab}^{+-} = \Sigma \cup \{a\} \setminus \{b\}$, also corresponding to the $\mathbf{I}_{ab}^{+-}$ in [9]. If $1 \leqslant a \leqslant b \leqslant r$, we will define $\Sigma_{[a,b]} = \{c \in \Sigma \mid a \leqslant c \leqslant b\}$.

In Fig. 12 we give our Boltzmann weights in closed form using these notations. It is easy to see that these are the correct weights obtained from the monochrome weights by fusion.

Remark 6.7 The weights in Fig. 12 closely resemble weights presented in Section 2.2 of [9]. One important distinction is that our weights are 'fermionic' — we do not allow multiple copies of any given color on an edge — while their weights are 'bosonic' (allowing multiplicities). Nevertheless, we may compare the weights of the multiplicity-free colored vertices in [9] with those in Fig. 12; even allowing for changes of variables and Drinfeld twisting, small differences persist. For example, we

may take the mirror image of the weights in [9] (2.2.2) specialized by setting $s = 0$ and compare to the weights in Fig. 12 by making the substitutions $v = q^{-1}$ and $z_i = x$. Note that if no color occurs more than once on the top boundary then no vertical edge on the interior of the grid will ever have a color with multiplicity larger than one, so that the distinction between bosonic and fermionic weights becomes unimportant. Then both weight schemes have the same admissible vertices grouped into types as in Fig. 12. Upon Drinfeld twisting, the Boltzmann weights agree in their powers of x and $(1 - q^{-1})$ but differ by various factors of -1 and q that cannot be resolved. Furthermore, although we have noted that bosonic weights can be excluded by imposing boundary conditions they are still important for the Yang–Baxter equation, so the R-matrices for [9] and for our system must definitely be different. Indeed, our R-matrix is a Drinfeld twist of the R-matrix for the quantum group for $\widehat{\mathfrak{gl}}(r|1)$ while the R-matrix in [9] is a Drinfeld twist of the one for $\widehat{\mathfrak{sl}}_{r+1}$. For further comparison of our models with [9] see arXiv version 1 of this paper [14].

7 The Iwahori lattice model

We now describe a family of statistical-mechanical systems made from fused vertices whose partition functions may be shown to give values of Iwahori Whittaker functions in the case $G = \mathrm{GL}_r$. Indeed, if $g \in \mathrm{GL}_r(F)$, where F is a nonarchimedean local field, we will see that the Whittaker function $\phi_w(g)$ defined by (2) can be represented as the partition function of such a model. First note that, as mentioned in Remark 3.3, we may assume that g is of the form $\varpi^{-\lambda} w_2$ for some weight λ and Weyl group element w_2. By Lemma 3.5, we may assume that λ is w_2-almost dominant. Finally, multiplying g by $\varpi^N \cdot I_r$ for some N just multiplies $\phi_w(g)$ by $(z_1 \cdots z_r)^N$, which corresponds to adding N to each part of λ. Using this flexibility we may assume with no loss of generality that $g \in \mathrm{Mat}_r(\mathfrak{o})$ and that the entries in λ are nonnegative.

Having already explained how the Boltzmann weights for fused vertices are defined, it remains to explain the boundary conditions for the model and the labels on each of the vertices. For any positive integer r, the boundary conditions and vertex labels depend on three pieces of data: a partition denoted $\lambda + \rho$ with at most r nonzero parts, a pair of permutations $w_1, w_2 \in S_r = W$, the Weyl group of GL_r, and r complex parameters $\mathbf{z} = (z_1, \ldots, z_r)$. The systems we present here, denoted $\mathfrak{S}_{\mathbf{z}, \lambda, w_1, w_2}$ and referred to as *Iwahori systems*, may be considered as simultaneous generalizations of those appearing previously in the Tokuyama model and in the colored systems of [17]. Our goal in this section is to equate the partition function of $\mathfrak{S}_{\mathbf{z}, \lambda, w_1, w_2}$ with the value of the Whittaker function $\phi_{w_1}(\mathbf{z}; \varpi^{-\lambda} w_2)$ of Sect. 3.

With r fixed, let $\rho = (r - 1, \ldots, 1, 0)$ and let $\lambda + \rho = (\lambda_1 + r - 1, \cdots, \lambda_r)$ be a partition, whose parts are written in weakly decreasing order as usual. In the identification of the weight lattice of GL_r with $\mathbb{Z}^r$, the corresponding weight λ satisfies $\langle \alpha, \lambda \rangle \geqslant -1$ for all simple roots α, a necessary condition for the non-vanishing of the Whittaker function according to Lemma 3.5.

Given the partition $\lambda + \rho$, we form a rectangular lattice consisting of $N + 1$ columns and r rows, where N is any integer at least $\lambda_1 + r - 1$. The columns will be numbered from left to right from N to 0 in decreasing order. The rows are numbered 1 to r, in
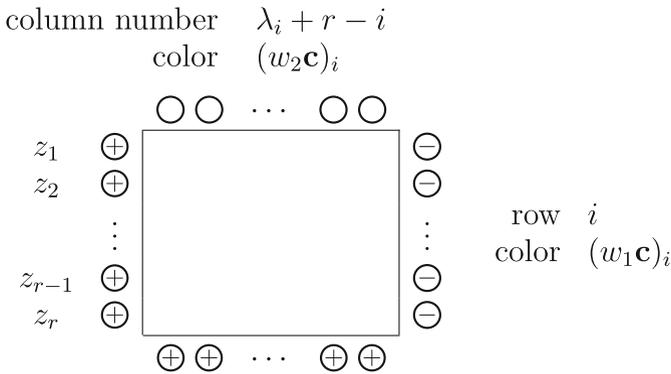
column number $\lambda_i + r - i$
color $(w_2 \mathbf{c})_i$

Fig. 13 Summary of boundary conditions for the colored systems. Let $\mathbf{c} = (c_1, \cdots, c_r)$ be a semistandard flag of colors $c_1 \geqslant \cdots \geqslant c_r$; for the systems $\mathfrak{S}_{\mathbf{z}, \lambda, w_1, w_2}$, if we identify the colors c_i with integers we take $c_i = r + 1 - i$, while in Sect. 8 we will take more general flags. On the top boundary the minus signs are positioned at columns $\lambda_i + r - i$ with color $(w_2 \mathbf{c})_i = c_{w_2^{-1}(i)}$, and on the right boundary the edge in row i is colored $(w_1 \mathbf{c})_i = c_{w_1^{-1}(i)}$

increasing order from top to bottom. Given $\mathbf{z}$, each vertex in the i-th row receives the label z_i. The Boltzmann weights are the fused weights in Fig. 12. Unless otherwise stated we will henceforth assume that the parameter v appearing in the Boltzmann weights (as well as in the Demazure-Whittaker operators among other places) equals q^{-1} with q the cardinality of the residue field of F. We will prefer the use of v to avoid confusion in later sections where, to follow tradition, q will have another meaning.

It remains to describe the boundary spins and colors located around the edge of the rectangular grid. They depend on the choice of the weight λ and the two Weyl group elements w_1, w_2 as follows and summarized in Fig. 13.

We have colors numbered $1, \ldots, r$ at our disposal. For the top boundary, we assign the color $r + 1 - w_2^{-1}(i)$ to the edge in the column labeled $\lambda_i + r - i$ for each $i \in \{1, \ldots, r\}$ and a $\oplus$ spin in the remaining columns. That is, we color each edge whose column index is a part of $\lambda + \rho$ and we have multiple colors on a given top boundary edge according to the multiplicity of parts in the partition. Then, we put a $\oplus$ spin on all the left and bottom boundary edges. This leaves the right boundary edges to be described. These will depend on the choice of permutation $w_1 \in W$ for the system. The right boundary edge in the i-th row is assigned the color $r + 1 - w_1^{-1}(i)$. For $w_2 = 1$, these boundary conditions are exactly as in [17]. A particularly simple admissible state in an Iwahori system is given in Fig. 14.

In any state of the system $\mathfrak{S}_{\mathbf{z}, \lambda, w_1, w_2}$, the edges of any one particular color form a line or path starting at the top boundary and ending at the right boundary. This depiction of admissible states as configurations of lines is present in many works on lattice models, for example Baxter's book [5], Chapter 8. The idea of using *colored* lines and refined systems that specify starting and ending points of each colored line is presented in [9]. We exploited this idea in a prior paper [17] to give a new theory of Demazure atoms, nonsymmetric pieces of Schur functions. The colored weights in this paper specialize to those of [17] by setting $v = 0$, which leads to a vast simplification.
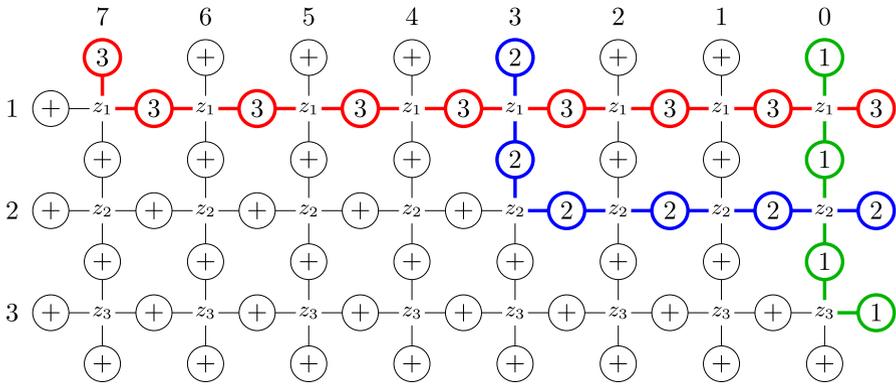
Fig. 14 The ground state for $r = 3$, $\lambda = (5, 2, 0)$ so $\lambda + \rho = (7, 3, 0)$, $w_1 = w_2 = 1$ (the identity permutation), and $\mathbf{z} = (z_1, z_2, z_3)$ is an arbitrary triple of complex numbers. The top row colors read from left to right are are $(3, 2, 1)$. The colors on the right edge, read from top to bottom, are also $(3, 2, 1)$. This is the unique state of the system $\mathfrak{S}_{\mathbf{z}, \lambda, 1, 1}$. Its Boltzmann weight is $\mathbf{z}^{\lambda + \rho}$. Assuming w_2 stays fixed, the monomial partition function of the ground state is the "seed" from which the partition functions for systems corresponding to other w_1 are derived by application of Demazure operators

In particular, every edge in [17] may carry at most one color (even in the fused model) and two colored lines can cross at most once. In this paper, weights and subsequent Yang–Baxter equations are understood via fusion, and two colored lines can cross more than once.

Let $Z(\mathfrak{S}_{\mathbf{z}, \lambda, w_1, w_2})$ denote the partition function of the system $\mathfrak{S}_{\mathbf{z}, \lambda, w_1, w_2}$. We will now demonstrate that this partition function satisfies the same functional equation as the Iwahori Whittaker function ϕ_{w_1} in Theorem 3.8 under Demazure-Whittaker operators using the Yang–Baxter equation. It will be convenient to conjugate the Demazure-Whittaker operators $\mathfrak{T}_i$ of (11) as follows

$$T_i = \mathbf{z}^\rho \mathfrak{T}_i \mathbf{z}^{-\rho}. \tag{29}$$

such that

$$T_i \cdot f(\mathbf{z}) = \frac{\mathbf{z}^{\alpha_i} - v}{1 - \mathbf{z}^{\alpha_i}} f(s_i \mathbf{z}) + \frac{v - 1}{1 - \mathbf{z}^{\alpha_i}} f(\mathbf{z}) \tag{30}$$
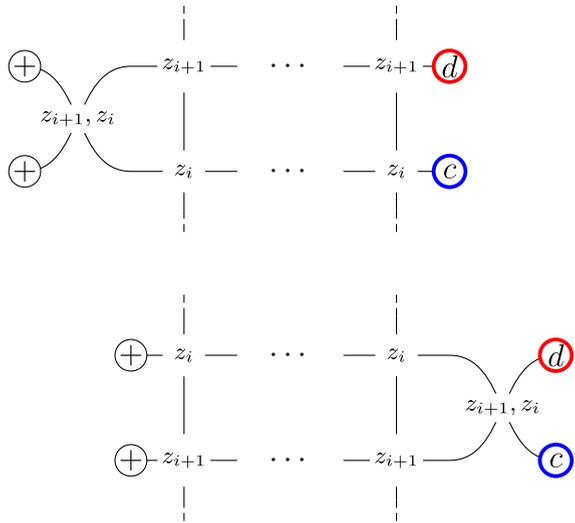
and

$$T_i^{-1} \cdot f(\mathbf{z}) = \frac{\mathbf{z}^{\alpha_i} - v}{v(1 - \mathbf{z}^{\alpha_i})} f(s_i \mathbf{z}) + \frac{(v - 1)\mathbf{z}^{\alpha_i}}{v(1 - \mathbf{z}^{\alpha_i})} f(\mathbf{z}). \tag{31}$$

Proposition 7.1 *For any partition* $\lambda + \rho$, *simple reflection* s_i, *and any pair of Weyl group elements* $w_1, w_2 \in W$,

$$Z(\mathfrak{S}_{\mathbf{z}, s_i w_1, w_2}) = \begin{cases} T_i Z(\mathfrak{S}_{\mathbf{z}, \lambda, w_1, w_2}) & \text{if } \ell(s_i w_1) > \ell(w_1), \\ T_i^{-1} Z(\mathfrak{S}_{\mathbf{z}, \lambda, w_1, w_2}) & \text{if } \ell(s_i w_1) < \ell(w_1). \end{cases} \tag{32}$$

Proof Repeated use of the Yang–Baxter equation gives the equality of the partition functions in Fig. 15.

Using the R-matrix weights from Fig. 6, we obtain the following identity of partition functions:

$$
\begin{aligned}
&(z_i - v z_{i+1}) Z(\mathfrak{S}_{s_i \mathbf{z}, \lambda, w_1, w_2}) \\
&= \begin{cases}
(1 - v) z_{i+1} Z(\mathfrak{S}_{\mathbf{z}, \lambda, w_1, w_2}) + (z_{i+1} - z_i) Z(\mathfrak{S}_{\mathbf{z}, \lambda, s_i w_1, w_2}) & \text{if } \ell(s_i w_1) > \ell(w_1), \\
(1 - v) z_i Z(\mathfrak{S}_{\mathbf{z}, \lambda, w_1, w_2}) + v(z_{i+1} - z_i) Z(\mathfrak{S}_{\mathbf{z}, \lambda, s_i w_1, w_2}) & \text{if } \ell(s_i w_1) < \ell(w_1).
\end{cases}
\end{aligned}
\tag{33}
$$

Consulting the table in Fig. 6, there is one possible configuration for the R-matrix for the top state in Fig. 15, and two possible configurations for the bottom state, accounting for the three terms in the identity (33). The weights for the latter two R-matrix configurations are separated into the two cases shown for the right-hand side of (33) where $\ell(s_i w_1) > \ell(w_1)$ is equivalent to $d > c$ using the color-assignment of Fig. 15. Note that we take (i, j) in Fig. 6 to be $(i + 1, i)$.

Setting $\mathbf{z}^{\alpha_i} = z_i / z_{i+1}$, and rearranging terms in (33) upon division by z_{i+1}, we obtain the desired equality. □

We have noted in Remark 6.3 that we may replace a system such as $\mathfrak{S}_{\mathbf{z}, \lambda, w_1, w_2}$ made with fused Boltzmann weights by an equivalent system with r times as many vertices, using monochrome weights. The expanded monochrome system will appear in the following proof. See Fig. 16 for an example.

Theorem 7.2 *Given any $w_1, w_2 \in W$, let λ be a w_2-almost dominant weight and let $\mathfrak{S}_{\mathbf{z},\lambda,w_1,w_2}$ be the corresponding Iwahori system. Then*

$$Z(\mathfrak{S}_{\mathbf{z},\lambda,w_1,w_2}) = \mathbf{z}^\rho \phi_{w_1}(\mathbf{z}; \varpi^{-\lambda}w_2). \tag{34}$$

Proof Comparing Proposition 7.1 and Theorem 3.8 while bearing in mind (29), both sides of (34) satisfy the same recursive formula, so if (34) is true for one value of w_1, it is true for all w_1. Thus we may assume that $w_1 = w_2$.

We will show that when $w_1 = w_2 = w$ the system has a unique state. We will use the monochrome model, in which each vertical edge has been broken into r distinct vertical edges, and the color c, if we identify c with an integer $1 \leqslant c \leqslant r$ can only be carried by the c-th such vertical edge. The following argument shows that the condition that λ is w_2-dominant implies that the sequence of colors on the top boundary edges are the same as the sequence of colors on the right boundary edges. By definition of $\mathfrak{S}_{\mathbf{z},\lambda,w,w}$ the sequence of colors on the right edge are $r + 1 - w^{-1}(i)$. On the top edge, the color in the $\lambda_i + r - i$ column is also $r + 1 - w^{-1}(i)$, and the sequence of integers $\lambda_i + r - i$ is weakly decreasing. Since columns are labeled in decreasing order, we see that if the $\lambda_i + r - i$ are distinct, then the colors are in the same order on the top boundary and on the right boundary, as claimed. But we must consider what happens if several $\lambda_i + r - i$ are equal, as in Fig. 16. If $\lambda_i + r - i = \lambda_{i+1} + r - (i+1)$ then $\langle \lambda, \alpha_i^\vee \rangle = -1$ so our condition that λ is w-almost dominant implies that $w^{-1}\alpha_i$ is a negative root. Therefore $w^{-1}(i) > w^{-1}(i+1)$ and so the colors on the right edge in rows $i, i+1$ are $r + 1 - w^{-1}(i) < r + 1 - w^{-1}(i+1)$. Now let us see that this agrees with the condition for the top boundary. Indeed, when we split the vertices into monochrome vertices as in Convention 6.2, they are in increasing order.

We have shown that the colors of the top boundary edges of the monochrome model are in the same order as those of the right boundary edges. From this it is easily deduced that there is only one possible state, and that every colored line crosses every other colored line (exactly once). We need to consider the Boltzmann weights that arise from these crossings. Consulting the second case in Fig. 7 we see that when $c > d$, the crossing produces a factor of v, otherwise it does not. The total number of such crossings is the number of inversions of w^{-1}, that is $\ell(w)$. Also in the i-th row, using the fused (non-monochrome) description, the number of factors z_i will be the number of vertices with a colored edge to the left, which will be $z_i^{\lambda_i + r - i}$. Therefore the Boltzmann weight of the state is therefore $v^{\ell(w)}\mathbf{z}^{\lambda+\rho}$. By Proposition 3.6 this equals $\mathbf{z}^\rho \phi_w(\mathbf{z}, \varpi^{-\lambda}w)$, and this concludes the proof. $\square$

Proposition 7.3 *Let $\mu' \in \mathbb{Z}^r$. There exists a unique pair (w, λ), with $w \in W$ and λ a w-almost dominant weight, such that*

$$w(\mu') = \lambda + \rho.$$

Proof We may find w and $\lambda = (\lambda_1, \cdots, \lambda_r)$ such that $w(\mu') = \lambda + \rho$ and $\lambda + \rho$ is dominant. Clearly λ is unique but w may not be if we only require $w(\mu')$ to be dominant. However the stronger condition that $w(\mu') - \rho$ is w-almost dominant will force w to be unique as follows.
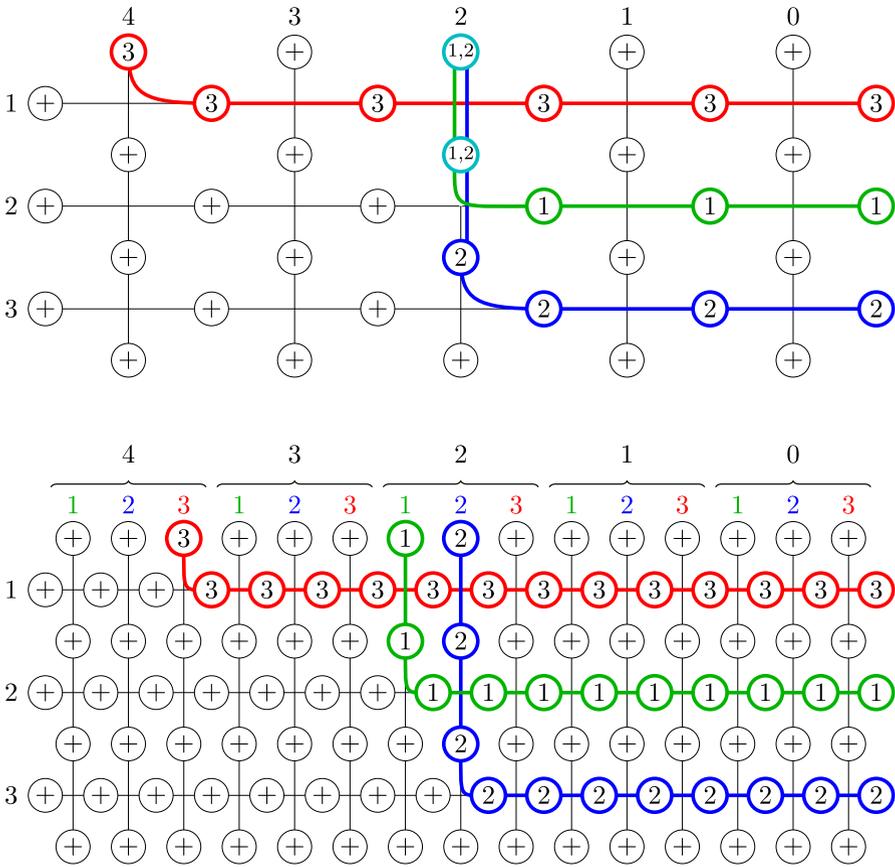
Fig. 16 The unique state of $\mathfrak{S}_{\mathbf{z},\lambda,w_1,w_2}$ for $G = \mathrm{GL}_3$ with $w_1 = w_2 = s_2$ in $W = S_3$ and $\lambda = (2, 1, 2)$ shown with fused vertices (top) and monochrome vertices (bottom). Note that while λ is not dominant, it is w_2-almost dominant. The partition function $vz_1^4z_2^2z_3^2$ of this system equals $\mathbf{z}^\rho\phi_{s_2}(\varpi^{-\lambda}s_2)$

We recall that $w^{-1}\alpha_i \in \Delta^+$ if and only if $\ell(s_iw) > \ell(w)$. The w such that $w(\mu') = \lambda + \rho$ lie in a single left coset of the stabilizer of $\lambda + \rho$, which is a Coxeter group generated by the s_i such that $\lambda_i + 1 = \lambda_{i+1}$. But the condition that $w(\mu') - \rho$ is w-almost dominant is equivalent to the assumption that $\ell(s_iw) < \ell(w)$ whenever $\lambda_i + 1 = \lambda_{i+1}$. So this condition means that any s_i among the generators of this stabilizer is a left descent of w. Thus clearly there is a unique w in this coset such that λ is w-almost dominant, and that is the longest element of W such that $w(\mu') = \lambda + \rho$. □

Remark 7.4 As noted above, the 'standard basis' of Iwahori Whittaker functions ϕ_{w_1} are determined by their values at $\varpi^{-\lambda}w_2$. We have shown in Theorem 7.2 that these values are partition functions of certain systems $\mathfrak{S}_{\mathbf{z},\lambda,w_1,w_2}$. Proposition 7.3 shows that the partition function of every Iwahori system is a value of an Iwahori Whittaker function. Indeed, the data describing the system are colorings of the top and right

boundary edges in the fused model. In other words, the data are two maps from the set of colors to the top boundary edges (labeled by columns) and to the right boundary edges (labeled by rows). The map to rows is bijective but the map to columns can be any map; as in Fig. 16, where colors 1 and 2 (blue and green) map to the same column, it does not need to be injective. Let μ'_i be the column corresponding to the $(r+1-i)$-th color. Applying Proposition 7.3 to $\mu' = (\mu'_1, \cdots, \mu'_r)$ produces a pair (w_2, λ) such that λ is w_2-almost dominant and $w_2\mu' = \lambda + \rho$. In column $\lambda_i + \rho_i = (w_2\mu')_i = \mu'_{w_2^{-1}(i)}$ we then have the color $r + 1 - w_2^{-1}(i)$ exactly as specified for the top boundary edges in Fig. 13. Thus, from every μ' we obtain a system $\mathfrak{S}_{\mathbf{z},\lambda,w_1,w_2}$ with w_1 determined by the permutation of colors on the right edge.

8 The parahoric lattice model

In this section we will generalize the Iwahori lattice model to allow multiple colored lines of the same color. This allows us to represent parahoric Whittaker functions as partition functions. See Fig. 17 for an example.

We will call a sequence of r colors a *flag*. The boundary conditions of the colored systems are represented by two flags: one on the top edge, and one on the right edge, which is a permutation of the former. A flag $\mathbf{c} = (\gamma_1, \ldots, \gamma_r)$ is called *standard* if $\gamma_1 > \cdots > \gamma_r$, and *semistandard* if $\gamma_1 \geqslant \cdots \geqslant \gamma_r$. Since we have a palette of r colors there is a unique standard flag, and if, as in Sect. 6, we identify the colors with integers $1 \leqslant c \leqslant r$, then the unique standard flag is $(r, r-1, \ldots, 1)$. Any flag may be represented as $w\mathbf{c} = (\gamma_{w^{-1}(1)}, \ldots, \gamma_{w^{-1}(r)})$, where $\mathbf{c}$ is a semistandard flag and $w \in W = S_r$. For the Iwahori systems in Sect. 7 every color appeared exactly once on the top boundary and on the right boundary, meaning the boundaries were represented by permutations $w_1, w_2 \in W$ of a standard flag $\mathbf{c}$. For the parahoric systems these boundaries will instead be represented by permutations of a semistandard flag.

If $\mathbf{c}$ is a semistandard flag then we may write

$$\mathbf{c} = (\gamma_1, \cdots, \gamma_r) = (c_1^{r_1}, c_2^{r_2}, \cdots, c_k^{r_k}) \tag{35}$$

with $c_1 > \cdots > c_k$ and $\sum r_i = r$. Here the notation means that we have r_1 copies of c_1 followed by r_2 copies of c_2, and so forth. The stabilizer of this flag in W is the parabolic subgroup $W_{\mathbf{J}} = S_{r_1} \times \cdots \times S_{r_k}$, which is the Weyl group of $\mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_k}$. Here, using the notation of Sect. 4, $\mathbf{J}$ is the index set of simple reflections generating this subgroup of W, that is, all simple reflections except $s_{r_1}, s_{r_1+r_2}, \cdots$. If a general flag is written $w\mathbf{c}$ with $\mathbf{c} = (c_1^{r_1}, c_2^{r_2}, \ldots, c_k^{r_k})$ then we may choose the representative w to be in $W^{\mathbf{J}}$, meaning that it is the shortest element of the coset $wW_{\mathbf{J}}$.

We now explain the parahoric lattice model generalizing the Iwahori lattice model of Sect. 7. Let $\mathbf{c}$ be a semistandard flag parametrized as above and let $w_1, w_2 \in W^{\mathbf{J}}$. Assume that λ is w_2-almost dominant.

To this data we associate a parahoric system $\mathfrak{S}^{\mathbf{J}}_{\mathbf{z},\lambda,w_1,w_2}$ as follows. We start from the same construction as for the Iwahori lattice model, with the only difference being which boundary conditions we allow. We take the top boundary edge spin in column
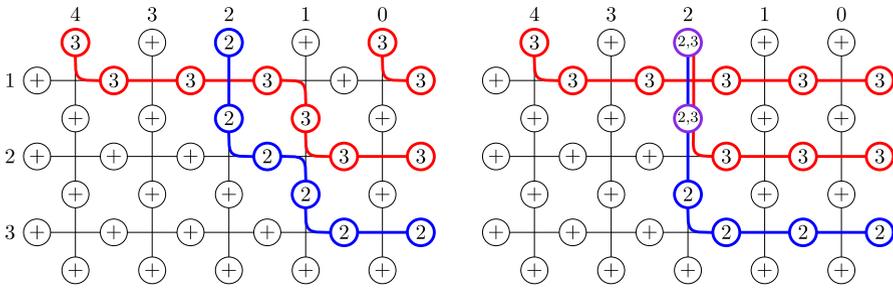
Fig. 17 Two states for parahoric Whittaker functions corresponding to the Levi subgroup $GL_2 \times GL_1 \subset GL_3$. The set of simple reflections in this Levi subgroup indexed by **J** is $\{s_1\}$. On the left we have a state for the system $\mathfrak{S}^{\mathbf{J}}_{\mathbf{z}, \lambda, w_1, w_2}$ with $\lambda = (2, 1, 0)$, $w_1 = 1$ and $w_2 = s_2$, and on the right we have a state for the system with $\lambda = (2, 2, 1)$, $w_1 = 1$ and $w_2 = s_2$

$\lambda_i + r - i$ to be the color $(w_2 \mathbf{c})_i = \gamma_{w_2^{-1}(i)}$; the remaining top boundary edge spins are $\oplus$. The right boundary spin in row i is $(w_1 \mathbf{c})_i = \gamma_{w_1^{-1}(i)}$. The boundary spins on the left and bottom edges are $\oplus$. For the interior, we use the same Boltzmann weights in Fig. 12 as before. Note that if $\mathbf{c}$ is a standard flag we recover the Iwahori lattice model from the parahoric model.

Recall that our 'fermionic' Boltzmann weights do not allow more than one instance of the same color on a given vertical edge. The following proposition implies that this requirement is satisfied for the top boundary edges (and therefore for all vertical edges).

Proposition 8.1 *Suppose that $w_2 \in W^{\mathbf{J}}$, and assume that λ is a w_2-almost dominant weight. If $\lambda_i + r - i = \lambda_j + r - j$ then $\gamma_{w_2^{-1}(i)} \neq \gamma_{w_2^{-1}(j)}$.*

Proof If $1 \leqslant i, j \leqslant r$ let t_{ij} denote the transposition in $W = S_r$ that interchanges i and j. Then obviously

$$\gamma_i = \gamma_j \iff t_{ij} \in W_{\mathbf{J}}. \tag{36}$$

Without loss of generality assume $i < j$. If λ is w_2-almost dominant then $\lambda + \rho$ is dominant, so the sequence $\lambda_i + r - i$ is monotone nonincreasing. Therefore

$$\lambda_i + r - i = \lambda_{i+1} + r - i - 1 = \cdots = \lambda_j + r - j$$

and so $\langle \lambda, \alpha_i^\vee \rangle = \langle \lambda, \alpha_{i+1}^\vee \rangle = \cdots = \langle \lambda, \alpha_{j-1}^\vee \rangle = -1$. Because λ is w_2-almost dominant, it follows that $w_2^{-1}(\alpha_i), \cdots, w_2^{-1}(\alpha_{j-1})$ are all negative roots. Thus $w_2^{-1}(\alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1})$ is a negative root. Now the reflection in the hyperplane orthogonal to the root $\alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1}$ is t_{ij} so by [7] Proposition 4.4.6 we have $\ell(w_2^{-1} t_{ij}) < \ell(w_2^{-1})$, or equivalently $\ell(t_{ij} w_2) < \ell(w_2)$. Now $t_{ij} w_2 = w_2 t_{w_2^{-1}(i), w_2^{-1}(j)}$. We claim that this implies that $t_{w_2^{-1}(i), w_2^{-1}(j)} \notin W_{\mathbf{J}}$. Indeed, $w_2 \in W^{\mathbf{J}}$ so w_2 is the shortest element in the coset $w_2 W_{\mathbf{J}}$. Since we have shown that $\ell(w_2 t_{w_2^{-1}(i), w_2^{-1}(j)}) < \ell(w_2)$, this would be a contradiction if $t_{w_2^{-1}(i), w_2^{-1}(j)} \in W_{\mathbf{J}}$. It now follows from (36) that $\gamma_{w_2^{-1}(i)} \neq \gamma_{w_2^{-1}(j)}$. □

Remark 8.2 We observe that the Boltzmann weights of Fig. 12 are constructed from the monochrome weights of Fig. 7 and that the monochrome weights involving more than one color only depend on the orders of the colors. Therefore if we choose another set of colors such that $c'_1 > \cdots > c'_k$ and replace each color c_i by c'_i, the partition function is unchanged. We refer to this type of transformation as *reparametrization*. Because of this, we are permitted to omit the colors $c_1, \ldots, c_k$ from the notation $\mathfrak{S}^{\mathbf{J}}_{\mathbf{z},\lambda,w_1,w_2}$.

We may now state the main theorem of this section that relates a parahoric Whittaker function $\psi^{\mathbf{J}}_{w_1}$ of Sect. 4 to the partition function of a parahoric system.

Theorem 8.3 *Assume that $w_1, w_2 \in W^{\mathbf{J}}$ and that λ is w_2-almost dominant. Then*

$$Z(\mathfrak{S}^{\mathbf{J}}_{\mathbf{z},\lambda,w_1,w_2}) = \mathbf{z}^{\rho}\psi^{\mathbf{J}}_{w_1}(\mathbf{z}; \varpi^{-\lambda}w_2). \tag{37}$$

Before we prove this we will establish a result about lattice models that implies it. Once this is established, it is easy to see that the two results are equivalent. We will make use of two sets $\mathbf{K}$ and $\mathbf{J}$ of simple reflections such that $\mathbf{K} \subseteq \mathbf{J}$. Then $W_{\mathbf{K}} \subseteq W_{\mathbf{J}}$ and $W^{\mathbf{K}} \supseteq W^{\mathbf{J}}$. We will prove

Proposition 8.4 *Assuming that $\mathbf{K} \subseteq \mathbf{J}$, we have*

$$Z(\mathfrak{S}^{\mathbf{J}}_{\mathbf{z},\lambda,w_1,w_2}) = \sum_{y \in W_{\mathbf{J}}/W_{\mathbf{K}}} Z(\mathfrak{S}^{\mathbf{K}}_{\mathbf{z},\lambda,w_1 y,w_2}), \tag{38}$$

where we may choose the coset representatives y so that $w_1 y \in W^{\mathbf{K}}$.

The proof of Proposition 8.4 and hence of Theorem 8.3 will occupy most of this Section. These proofs will follow from Lemmas 8.5 and 8.6 below. Proposition 8.4 is a *global lifting property* expressing the partition function of a system with fewer colors as a sum over systems with more colors. Lemma 8.6 is a more precise statement whose proof shows how to do this reduction at the level of individual states. The idea is to show that each state of the system with fewer colors can be split up into states of the systems with more colors [Lemma 8.6 (i)]. On the other hand, some states of the system with more colors will cancel, and these are accounted for in Lemma 8.6 (ii). Both these properties depend on *local lifting properties* here called Property A and Property B, which are phenomena concerning the Boltzmann weights at a single vertex. Local lifting properties such as these were called *color-blindness* in [9]. Our proof that the local lifting properties imply the global lifting property will make use of an argument in which states of the system with more colors are arranged in a tree. For a more informal treatment of a similar result see Bump and Naprienko [24], Section 5. See also arXiv Version 1 of this paper ([14]) for further discussion related to the models of this paper.

Lemma 8.5 *Suppose that whenever $\mathbf{J}$ is nonempty, there exists a proper subset $\mathbf{K}$ of $\mathbf{J}$ such that (38) is true. Then Proposition 8.4 is true.*

Proof We may prove (37) by induction. For the base case $\mathbf{J} = \varnothing$, (37) is true by Theorem 7.2 because $\psi_w^\varnothing = \phi_w$ and $\mathfrak{S}_{\mathbf{z},\lambda,w_1,w_2}^\varnothing = \mathfrak{S}_{\mathbf{z},\lambda,w_1,w_2}$. Now assume inductively that (37) is true for proper subsets $\mathbf{K}$ of $\mathbf{J}$. By our assumption there exists such a $\mathbf{K}$ such that (38) is satisfied. Then (37) follows for $\mathbf{J}$ by combining (38), (48) and (37) for $\mathbf{K}$. $\qquad\square$

In preparation for applying Lemma 8.5 assume that $\mathbf{J}$ is nonempty, so $r_i > 1$ for some i. Then, recalling that $W_{\mathbf{J}} = S_{r_1} \times \cdots \times S_{r_k}$, the last simple reflection that is contained in S_{r_i} is s_m where $m = r_1 + \ldots + r_i - 1$. Let $\mathbf{K}$ be obtained by removing m from $\mathbf{J}$ so that $W_{\mathbf{K}} = S_{r_1} \times \cdots \times S_{r_i-1} \times S_1 \times \cdots \times S_{r_k}$ and let the system $\mathfrak{S} = \mathfrak{S}_{\mathbf{z},\lambda,w_1,w_2}^{\mathbf{J}}$ be described by the semistandard flag $(c_1^{r_1}, c_2^{r_2}, \cdots, c_k^{r_k})$ as in (35).

Let $c = c_i$. We wish to insert a color between c_i and c_{i+1}. Reparametrizing by Remark 8.2 if necessary, we may assume that there is a color c' such that

$$c_1 > c_2 > \cdots > c_i = c > c' > c_{i+1} > \cdots > c_k. \tag{39}$$

The color c' corresponds to the color m in the last paragraph, but since we are reparametrizing, we are changing the notation. Note that the colors c and c' are adjacent. We may use these colors to describe $\mathfrak{S}_{\mathbf{z},\lambda,w_1y,w_2}^{\mathbf{K}}$. Let

$$\mathfrak{S}' := \bigsqcup_{y \in W_{\mathbf{J}}/W_{\mathbf{K}}} \mathfrak{S}_{\mathbf{z},\lambda,w_1y,w_2}^{\mathbf{K}}. \tag{40}$$

Note that each state in $\mathfrak{S}'$ has one line of color c' that starts at a fixed location in the top row, replacing one of the top vertical edges colored c in $\mathfrak{S}_{\mathbf{z},\lambda,w_1,w_2}^{\mathbf{J}}$. Since $w_2 \in W^{\mathbf{J}}$, the edge containing the instance of the color c that is replaced by c' is the rightmost such top vertical edge. The c' colored line ends up on the right edge, replacing one of the horizontal vertical edges colored c. There are r_i possible such locations and the decomposition of $\mathfrak{S}'$ into the $r_i = |W_{\mathbf{J}}/W_{\mathbf{K}}|$ ensembles $\mathfrak{S}_{\mathbf{z},\lambda,w_1y,w_2}^{\mathbf{K}}$ corresponds to these r_i possibilities.

We will now instead decompose $\mathfrak{S}'$ into two parts. We call a state $\mathfrak{s}' \in \mathfrak{S}'$ *strict* if it has no vertical edge carrying both colors c and c'. Let $\mathfrak{S}'_{\text{strict}}$ consist of strict states in $\mathfrak{S}'$, and let $\mathfrak{S}'_{\text{ns}}$ be the remaining, nonstrict states.

Lemma 8.6 *Let $\mathfrak{S}'_{\text{strict}}$ and $\mathfrak{S}'_{\text{ns}}$ be as above. Then,*

(i) $Z(\mathfrak{S}'_{\text{strict}}) = Z(\mathfrak{S}_{\mathbf{z},\lambda,w_1,w_2}^{\mathbf{J}})$,
(ii) $Z(\mathfrak{S}'_{\text{ns}}) = 0$.

Note that Lemma 8.6 implies (38) for the above chosen subset $\mathbf{K}$ of $\mathbf{J}$, which together with Lemma 8.5, proves Theorem 8.3. We will prove Lemma 8.6 later in this section, but first we will need to introduce some terminology.

Let $\mathcal{L}$ be the set of vertices for our lattice models as first introduced in Sect. 7. We order the vertices $\mathcal{L}$ lexicographically from left to right, top to bottom. We will denote this total order on vertices $\prec$. Thus $x \prec y$ if and only if x is in a row above y, or x and y are in the same row and x is to the left of y. An initial segment of $\mathcal{L}$ is either

the empty set $\varnothing$ or $I(x) = \{y \in \mathcal{L} \mid y \preccurlyeq x\}$ for some $x \in \mathcal{L}$. Let $\hat{\mathcal{L}}$ be the set of initial segments of $\mathcal{L}$. If $I \in \hat{\mathcal{L}}$ is not $\mathcal{L}$ then I has a unique successor $\mathrm{succ}(I)$ in $\hat{\mathcal{L}}$, which is the unique initial segment of cardinality $|I| + 1$. If x is a vertex of $\mathcal{L}$ we will call the edges above and left of x *inputs*, and the edges below and to the right *outputs*.

For a state $\mathfrak{s}$ in some system with lattice $\mathcal{L}$ and a vertex x in $\mathcal{L}$ we denote by $\mathfrak{s}|_x$ the vertex configuration of $\mathfrak{s}$ at x, that is, the sequence of spins for the four adjacent edges. Similarly, for any sequence I of vertices in $\mathcal{L}$ we denote by $\mathfrak{s}|_I$ the sequence of vertex configurations of $\mathfrak{s}$ for vertices in I. We may extend our previous notation and denote by $\beta(\mathfrak{s}|_I)$ the product of the Boltzmann weights for the state $\mathfrak{s}$ at the vertices in I.

Let Γ be a finite directed graph with no cycles. We call Γ a *tree* if it has a unique initial node, called the *root*; the terminal nodes are called *leaves*. If $X, Y \in \Gamma$ we say that Y is a *child* of X if $X \to Y$ is an edge of Γ. Let R be an abelian group, which in our applications will be $\mathbb{C}(v)$. We call a function $F : \Gamma \longrightarrow R$ *additive* if for every non-leaf $X \in \Gamma$ we have $F(X) = \sum F(Y)$ where the sum is over the children of X (but not over further descendants). If Γ is a tree with root X_{root} and an additive function F then clearly

$$F(X_{\mathrm{root}}) = \sum_{\text{leaves } Y} F(Y).$$

We will now define a tree and an additive function that we will use to prove Lemma 8.6. Let $\mathfrak{S}'$ be defined as in (40). If $I \in \hat{\mathcal{L}}$ define an equivalence relation on states in $\mathfrak{S}'$ where $\mathfrak{s}'_1 \equiv_I \mathfrak{s}'_2$ if $\mathfrak{s}'_1|_I = \mathfrak{s}'_2|_I$, that is, the two states have the same vertex configurations at each vertex in I. Let $[\mathfrak{s}']_I$ be the equivalence class of a state $\mathfrak{s}'$ under this relation. We may define a directed graph Γ whose nodes are pairs $(I, [\mathfrak{s}']_I)$ and with edges of the form $(I, [\mathfrak{s}'_1]_I) \to (J, [\mathfrak{s}'_2]_J)$ where $J = \mathrm{succ}(I)$ and $\mathfrak{s}'_2 \in [\mathfrak{s}'_1]_I$. Note that the equivalence class $[\mathfrak{s}'_1]_I$ is a union of $\equiv_J$ equivalence classes.

We may enumerate the children of a parent node $(I, [\mathfrak{s}'_{\mathrm{parent}}]_I)$ as follows. Let $J = \mathrm{succ}(I) = I \cup \{x\}$ where x is the last vertex in J. A child of $(I, [\mathfrak{s}'_{\mathrm{parent}}]_I)$ has the form $(J, [\mathfrak{s}'_{\mathrm{child}}]_J)$ where the class $[\mathfrak{s}'_{\mathrm{child}}]_J$ is determined by the spins of the four edges adjacent to x in a representative $\mathfrak{s}'_{\mathrm{child}}$. Moreover, the input spins for $\mathfrak{s}'_{\mathrm{child}}$ at x are determined by $\mathfrak{s}'_{\mathrm{parent}}$ since the input edges at x are either output edges for vertices in I or boundary edges. Thus, $[\mathfrak{s}'_{\mathrm{child}}]_J$ is determined by the two output spins at x. We see that the graph Γ is a tree, and its branching at the node $(I, [\mathfrak{s}'_{\mathrm{parent}}]_I)$ is determined by the different ways that the two inputs at the vertex x can be completed with the output spins to an admissible configuration at the vertex x. The root of the tree Γ is $(\varnothing, \mathfrak{S}')$ where all states in $\mathfrak{S}'$ are equivalent under $\equiv_\varnothing$, and each leaf of the tree corresponds to an individual state in $\mathfrak{S}'$.

We define a function F on Γ as follows

$$F(I, [\mathfrak{s}']_I) = \sum_{\mathfrak{s} \in [\mathfrak{s}']_I} \beta(\mathfrak{s}), \tag{41}$$
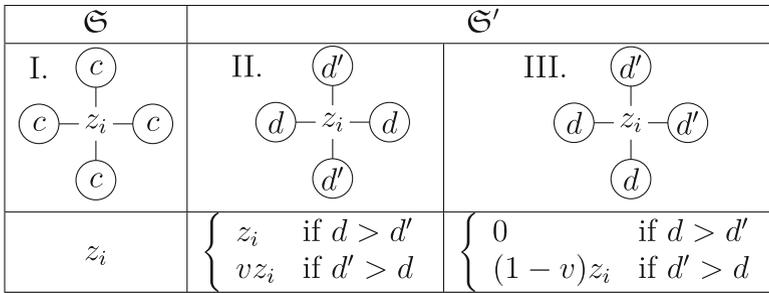
$\mathfrak{S}$	$\mathfrak{S}'$	
I. $\overset{\displaystyle \textcircled{c}}{\underset{\displaystyle \textcircled{c}}{\textcircled{c}-z_i-\textcircled{c}}}$	II. $\overset{\displaystyle \textcircled{d'}}{\underset{\displaystyle \textcircled{d'}}{\textcircled{d}-z_i-\textcircled{d}}}$	III. $\overset{\displaystyle \textcircled{d'}}{\underset{\displaystyle \textcircled{d}}{\textcircled{d}-z_i-\textcircled{d'}}}$
z_i	$\begin{cases} z_i & \text{if } d > d' \\ vz_i & \text{if } d' > d \end{cases}$	$\begin{cases} 0 & \text{if } d > d' \\ (1-v)z_i & \text{if } d' > d \end{cases}$

Fig. 18 Illustrating local lifting Property A in the case where the vertex x carries the colors c and c' and no others. Here d and d' are c and c' in some order

which is additive by the fact that $[\mathfrak{s}']_I$ is a union of its child $\equiv_{\mathrm{succ}(I)}$ equivalence classes. The root value is the partition function, and the leaf values are the Boltzmann weights of the individual states.

We will consider subgraphs of the tree Γ to prove Lemma 8.6. These graphs are used to organize the application of the following two properties of the Boltzmann weights, which are explained in Figs. 18 and 19.

Property A We consider the situation where, for a state $\mathfrak{s}' \in \mathfrak{S}'$ and a vertex x in $\mathcal{L}$, the vertex configuration $\mathfrak{s}'|_x$ has one input edge carrying the color c while the other carries c', but with no vertical edge carrying both colors c and c'. In this case one output edge must carry the color c and the other must carry c'. There are in total four configurations to consider but we group together the configurations that are mapped to each other under the interchange of c and c'. We denote these groups as II and III shown in Fig. 18. Because of the allowed vertex configurations, only one of the two configurations within each group is possible for any given pair of input edges. If the color c' is replaced by c, the four vertex configurations map to a single vertex configuration for $\mathfrak{S}$ that we denote by I. The fact that we need is that the Boltzmann weights satisfy

$$\beta_x(\mathrm{II}) + \beta_x(\mathrm{III}) = \beta_x(\mathrm{I}), \tag{42}$$

for any given input edges on the left-hand side of the equation. This is shown in Fig. 18. It is possible that both vertical edges carry other colors besides c and c' but if c'' is such a color, since c and c' are adjacent (cf. (39)), we have either $c'' > c, c'$ or $c, c' > c''$. In other words, other colors cannot distinguish between c and c'. Using this, we see that (42) remains true even with these extra colors.

Property B We also consider the situation where the vertical edge above vertex x carries both c and c', but one color exits to the right, and the other to the bottom. There are two ways this can happen, as shown in Fig. 19, and the Boltzmann weights of these patterns cancel:

$$\beta_x(\mathrm{IV}) + \beta_x(\mathrm{V}) = 0. \tag{43}$$

Fig. 19 Illustrating Property B, in which the Boltzmann weights of two nonstrict configurations are negatives of each other
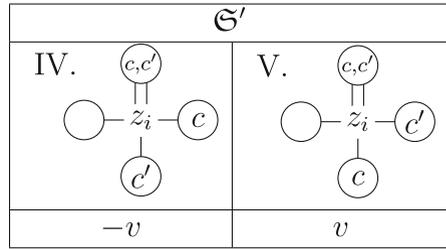
As for Property A, this relation also remains true even if the vertical edges carry one or more additional colors.

If $\mathfrak{s}'$ is a strict state of the system $\mathfrak{S}'$ then there is a corresponding state $\pi(\mathfrak{s}')$ of the system $\mathfrak{S}$ in which every instance of the color c' is replaced by c. For each $\mathfrak{s} \in \mathfrak{S}$ there exists at least one $\mathfrak{s}' \in \mathfrak{S}'_{\text{strict}}$ such that $\pi(\mathfrak{s}') = \mathfrak{s}$ and can be obtained by coloring one c-path in the color c'.

For $I \in \hat{\mathcal{L}}$, let $\bar{I}$ be the complement of I in the set of vertices for the lattice $\mathcal{L}$. We say that a state $\mathfrak{s}'$ is $\bar{I}$-strict if for all $x \in \bar{I}$ no vertical edge of $\mathfrak{s}'|_x$ carries both c and c'. That is, if the vertex configurations $\mathfrak{s}'|_{\bar{I}}$ are strict. In this case (generalizing the above notation) we may define $\pi(\mathfrak{s}'|_{\bar{I}})$ to be the $\bar{I}$-sequence of vertex configurations obtained from the sequence $\mathfrak{s}'|_{\bar{I}}$ by replacing every instance of c' by c. Note that these vertex configurations are admissible only when $\mathfrak{s}'$ is $\bar{I}$-strict.

Lemma 8.7 *Let $\mathfrak{s}'_0 \in \mathfrak{S}'$ and $X_0 = (I_0, [\mathfrak{s}'_0]_{I_0})$ be a node of Γ such that $\mathfrak{s}'_0$ is $\bar{I}_0$-strict. Consider the subtree Γ_{X_0} of Γ with its root at X_0 together with all its descendants. Let $\Gamma_{X_0}^{(\mathfrak{s}'_0)}$ be the tree obtained from Γ_{X_0} by selecting the branches whose leaves $\mathfrak{s}' \in \mathfrak{S}'$ are $\bar{I}_0$-strict and such that $\pi(\mathfrak{s}'|_{\bar{I}_0}) = \pi(\mathfrak{s}'_0|_{\bar{I}_0})$. Then*

$$F_{\mathfrak{s}'_0}(I, [\mathfrak{s}']_I) := \beta(\mathfrak{s}'|_I)\beta(\pi(\mathfrak{s}'_0|_{\bar{I}})) \tag{44}$$

is additive on $\Gamma_{X_0}^{(\mathfrak{s}'_0)}$.

Note that the representative $\mathfrak{s}' \in \mathfrak{S}'$ in (44) need not be a leaf of $\Gamma_{X_0}^{(\mathfrak{s}'_0)}$; $F_{\mathfrak{s}'_0}$ is still well-defined and independent of the representative in $[\mathfrak{s}']_I$.

Proof Let $X = (I, [\mathfrak{s}'_{\text{parent}}]_I)$ be a node of $\Gamma_{X_0}^{(\mathfrak{s}'_0)}$ and let $Y = (J, [\mathfrak{s}'_{\text{child}}]_J)$ be a child of X, where $J = \text{succ}(I) = I \cup \{x\}$. Denote the set of leaves of $\Gamma_{X_0}^{(\mathfrak{s}'_0)}$ by Ξ and choose the representatives $\mathfrak{s}'_{\text{parent}}$ and $\mathfrak{s}'_{\text{child}}$ such that they are in Ξ. That is, $\mathfrak{s}'_{\text{parent}}$ and $\mathfrak{s}'_{\text{child}}$ are $\bar{I}_0$-strict and $\pi(\mathfrak{s}'_{\text{parent}}|_{\bar{I}_0}) = \pi(\mathfrak{s}'_{\text{child}}|_{\bar{I}_0}) = \pi(\mathfrak{s}'_0|_{\bar{I}_0})$.

The fact that $\mathfrak{s}'_{\text{child}}$ is a child of $\mathfrak{s}'_{\text{parent}}$ means that $\mathfrak{s}'_{\text{parent}}|_I = \mathfrak{s}'_{\text{child}}|_I$. Furthermore $\mathfrak{s}'_{\text{parent}}$ and $\mathfrak{s}'_{\text{child}}$ agree on the two input edges of x, but they may differ on the two output edges of x. Since $\pi(\mathfrak{s}'_{\text{parent}}|_x) = \pi(\mathfrak{s}'_{\text{child}}|_x) = \pi(\mathfrak{s}'_0|_x)$, this can happen only if, for both states, one of the input edges carries the color c and the other carries c' while

the output edges for the two states have c and c' interchanged. Note that since $\mathfrak{s}'_{\text{child}}$ is $\bar{I}_0$-strict, no edge of x carries both c and c'. If $\mathfrak{s}'_{\text{parent}}$ and $\mathfrak{s}'_{\text{child}}$ do agree on the two output edges of x, then they are J-equivalent, so the node $(I, [\mathfrak{s}'_{\text{parent}}]_I)$ has at most two children.

Thus, $\Gamma_{X_0}^{(\mathfrak{s}'_0)}$ is a binary tree and to prove that $F_{\mathfrak{s}'_0}$ is additive we need to check two cases: X has one or two children.

Assume first that $X = (I, [\mathfrak{s}'_{\text{parent}}]_I)$ has two children $Y_1 = (J, [\mathfrak{s}'_1]_J)$ and $Y_2 = (J, [\mathfrak{s}'_2]_J)$ where we have chosen the representatives $\mathfrak{s}'_1$ and $\mathfrak{s}'_2$ in Ξ. Then,

$$
\begin{aligned}
F_{\mathfrak{s}'_0}(X) &= \beta(\mathfrak{s}'_{\text{parent}}|_I)\beta(\pi(\mathfrak{s}'_0|_x))\beta(\pi(\mathfrak{s}'_0|_{\bar{j}})) \\
F_{\mathfrak{s}'_0}(Y_i) &= \beta(\mathfrak{s}'_i|_I)\beta(\mathfrak{s}'_i|_x)\beta(\pi(\mathfrak{s}'_0|_{\bar{j}})) = \beta(\mathfrak{s}'_{\text{parent}}|_I)\beta(\mathfrak{s}'_i|_x)\beta(\pi(\mathfrak{s}'_0|_{\bar{j}})).
\end{aligned}
\tag{45}
$$

As argued above, $\pi(\mathfrak{s}'_0|_x)$, $\mathfrak{s}'_1|_x$ and $\mathfrak{s}'_2|_x$ are in the situation of Property A meaning that $\beta(\pi(\mathfrak{s}'_0|_x)) = \beta(\mathfrak{s}'_1|_x) + \beta(\mathfrak{s}'_2|_x)$ and thus $F(X) = F(Y_1) + F(Y_2)$.

For the case where $X = (I, [\mathfrak{s}'_{\text{parent}}]_I)$ has a single child $Y_1 = (J, [\mathfrak{s}'_1]_J)$ equation (45) is still valid and the vertex configuration $\mathfrak{s}'_1|_x$ is uniquely determined by $[\mathfrak{s}'_{\text{parent}}]_I$ and $\mathfrak{s}'_0$. Since $\mathfrak{s}'_0$ is $\bar{I}_0$-strict this means that the vertex configuration does not contain both c and c'. Because other colors cannot differentiate between c and c' we then have that $\beta(\mathfrak{s}'_1|_x) = \beta(\mathfrak{s}'_{\text{parent}}|_x) = \beta(\pi(\mathfrak{s}'_0|_x))$, which concludes the proof. □

If we apply Lemma 8.7 to the full tree $\Gamma_{X_0} = \Gamma$ we get the following corollary.

Corollary 8.8 *If $\mathfrak{s}$ is a state of $\mathfrak{S}$ then*

$$
\beta(\mathfrak{s}) = \sum_{\substack{\mathfrak{s}' \in \mathfrak{S}'_{\text{strict}} \\ \pi(\mathfrak{s}') = \mathfrak{s}}} \beta(\mathfrak{s}').
\tag{46}
$$

Proof Let X_0 be the root $X_{\text{root}} = (\varnothing, \mathfrak{S}')$ of Γ. For each $\mathfrak{s} \in \mathfrak{S}$ there exists $\mathfrak{s}'_0 \in \mathfrak{S}'_{\text{strict}}$ such that $\pi(\mathfrak{s}'_0) = \mathfrak{s}$. Then $\Gamma_{X_0}^{(\mathfrak{s}'_0)}$ contains all branches of Γ for which the leaves $\mathfrak{s}'$ are strict and map to $\mathfrak{s}$ under π independent of the choice of $\mathfrak{s}'_0$. The value of $F_{\mathfrak{s}'_0}$ at the root X_{root} equals $\beta(\mathfrak{s})$ while at a leaf $\mathfrak{s}'$ it equals $\beta(\mathfrak{s}')$ also independent of $\mathfrak{s}'_0$, and the statement follows from the additivity of $F_{\mathfrak{s}'_0}$ on $\Gamma_{X_0}^{(\mathfrak{s}'_0)}$. □

Proof of Lemma 8.6 and Proposition 8.4 We have already noted that Lemma 8.6 implies Proposition 8.4, so we turn to the proof of the Lemma. Statement (i) follows from Corollary 8.8 by summing over $\mathfrak{s} \in \mathfrak{S}$.

For statement (ii) we will start with the tree Γ. Let I be a maximal initial segment such that there is a nonstrict state $\mathfrak{s}'_1$ that is $\bar{I}$-strict. This means that all states in $[\mathfrak{s}'_1]_I$ are strict at the vertices in $\bar{I}$, but if x is the last vertex in I, then the colors c and c' of $\mathfrak{s}'_1|_x$ are in one of the two configurations in Fig. 19 of Property B (disregarding other colors). The node $X_1 = (I, [\mathfrak{s}'_1]_I)$ has a single sibling $X_2 = (I, [\mathfrak{s}'_2]_I)$ for which $\mathfrak{s}'_2|_x$ is in the other configuration of Fig. 19. Indeed, we can construct a representative $\mathfrak{s}'_2$ by starting from $\mathfrak{s}'_1$, apply π on $\mathfrak{s}'_1|_{\bar{I}}$ and then apply the color c' to one of the c-paths

going in the other direction at x. By construction $\pi(\mathfrak{s}_1'|_{\bar{I}}) = \pi(\mathfrak{s}_2'|_{\bar{I}})$ while $\mathfrak{s}_1'|_I$ and $\mathfrak{s}_2'|_I$ only differ at the vertex x with an overall minus sign for their Boltzmann weights. Let Ξ_1 and Ξ_2 be the sets of leaves for $\Gamma_{X_1}^{(\mathfrak{s}_1')}$ and $\Gamma_{X_2}^{(\mathfrak{s}_2')}$ respectively. By Lemma 8.7,

$$\sum_{\mathfrak{s}' \in \Xi_1} \beta(\mathfrak{s}') = F_{\mathfrak{s}_1'}(I, [\mathfrak{s}_1']_I) = \beta(\mathfrak{s}_1'|_I)\beta(\pi(\mathfrak{s}_1'|_{\bar{I}}))$$

$$= -\beta(\mathfrak{s}_2'|_I)\beta(\pi(\mathfrak{s}_2'|_{\bar{I}})) = -F_{\mathfrak{s}_2'}(I, [\mathfrak{s}_2']_I) = -\sum_{\mathfrak{s}' \in \Xi_2} \beta(\mathfrak{s}') \quad (47)$$

Note that the freedom in constructing $\mathfrak{s}_2'$ above is given by Ξ_2. Any other choice of the pair $(\mathfrak{s}_1', \mathfrak{s}_2')$ in $\Xi_1 \times \Xi_2$ would give the same trees $\Gamma_{X_1}^{(\mathfrak{s}_1')}$ and $\Gamma_{X_2}^{(\mathfrak{s}_2')}$ with the sames sets of leaves, as well as the same equation (47) for these leaves. We may thus choose pairs of representatives in $[\mathfrak{s}_1']_I \times [\mathfrak{s}_2']_I$ such that the corresponding sets $\Xi_1 \times \Xi_2$ are disjoint and their union equals $\Xi(\Gamma_{X_1}) \times \Xi(\Gamma_{X_2})$ where $\Xi(\Gamma_{X_i})$ is the set of leaves for the tree Γ_{X_i}.

Hence, we can remove the nodes X_1 and X_2 together with their descendants from the tree Γ without affecting the values and additivity property of the function F on the remaining nodes of Γ. That is, we may remove these nodes without changing the partition function of the leaves of Γ. Repeating the process with a new maximal initial segment and I and siblings X_1 and X_2 of the remaining Γ we have thus shown that we may remove all branches with non-strict states as leaves without changing the partition function. Together with statement (i) that was shown above, this proves statement (ii). □

Proof of Theorem 8.3 From the definition (18) of $\psi_w^{\mathbf{J}}$ we have

$$\psi_{w_1}^{\mathbf{J}}(\mathbf{z}; g) = \sum_{y \in W_{\mathbf{J}}/W_{\mathbf{K}}} \psi_{w_1 y}^{\mathbf{K}}(\mathbf{z}; g), \quad (48)$$

where, as in Proposition 8.4, we are choosing the coset representatives y so that $w_1 y \in W^{\mathbf{K}}$. Comparing (38) with (48), the Whittaker functions have the same restriction property as the partition functions. So if Theorem 8.3 is true for some set $\mathbf{K}$ of simple reflections it is true for any larger set $\mathbf{J}$. But if $\mathbf{K} = \varnothing$ then (37) reduces to the Iwahori case (Theorem 7.2). This completes the proof. □

The case of the maximal compact subgroup $K = K_{\mathbf{I}}$ is a special case of the parahoric system where $\mathbf{J} = \mathbf{I}$ is the set of all simple reflections. As previously mentioned, the Boltzmann weights for the Tokuyama model, shown for example in the row labeled $S_\Gamma(i)$ of [20, Table 2], are a special case of the weights in Figs. 7 (where $\ominus$ is replaced by one fixed color c). The partition function with these Boltzmann weights depends only on the partition λ and is given by the formula

$$Z(\mathbf{z}; \lambda) = \mathbf{z}^\rho \prod_{\alpha \in \Delta^+} (1 - v\mathbf{z}^{-\alpha}) s_\lambda(\mathbf{z}) \quad (49)$$

which is one version of Tokuyama's formula [20, 35, 63]. It also agrees with the Casselman-Shalika formula for the spherical Whittaker function.

9 Whittaker functions and Macdonald polynomials

The purpose of this section is to give a dictionary between values of Whittaker functions on $\mathrm{GL}_r(F)$ and certain specializations and generalizations of Macdonald polynomials, as introduced in Table 1 and further detailed here in Table 2. The second and last of these identities are new to our knowledge and will be proved later in this section. Both sides of the dictionary can be studied either algebraically, usually involving some variations of Demazure-Lusztig operators, or combinatorially which, for the Whittaker functions, can be achieved via the theory of solvable lattice models.

We start with the well-known case of the spherical Whittaker function, which can be expressed as the product of a quantized Weyl denominator and a Schur polynomial. This result is due to Shintani [61] and was generalized to all quasi-split reductive groups by Casselman-Shalika [26]. It may be proved by studying the more refined Iwahori fixed vectors. One can use the Yang–Baxter equation to give a lattice model interpretation of the spherical function; this was done in [20] based on ideas of Tokuyama [63].

Parallel to this work, Li [50] studied certain Iwahori fixed vectors in the unramified principal series and their associated Whittaker functions which can be used to identify the unique genuine subquotient of $I(\mathbf{z})$. Li computed a variation of the Casselman-Shalika formula for these Whittaker functions each of which we will express in terms of a Hall-Littlewood polynomial in Proposition 9.4. A (bosonic) lattice model called the q-boson model exists for Hall–Littlewood polynomials (see [64]); it has successfully been used to study Hall–Littlewood polynomials in both combinatorics [65] and representation theory [45]. By our results we can then associate this lattice model to Li's Whittaker function.

To understand both examples presented above, we need to understand the passage to the Iwahori level. As documented earlier, these Whittaker functions may be described in terms of certain divided difference operators. The definition of the Demazure-Lusztig and Demazure-Whittaker operators $\mathfrak{L}_i$ and $\mathfrak{T}_i$ associated to simple reflections s_i in the Weyl group W were given in (14) and (11), respectively. They also arise naturally in certain induced representations of the affine Hecke algebra made from the trivial and sign characters of the finite Hecke algebra, respectively (see [21, 22]).

Given any $w \in W$ and a reduced expression $w = s_{i_1} \cdots s_{i_k}$, set $\mathfrak{T}_w = \mathfrak{T}_{i_1} \cdots \mathfrak{T}_{i_k}$, which is well-defined because the $\mathfrak{T}_i$ satisfy the braid relations. We will similarly write $\mathfrak{L}_w = \mathfrak{L}_{i_1} \cdots \mathfrak{L}_{i_k}$. Then the following relation holds between these operators, where we add v-dependence to the notation for $\mathfrak{L}_w$ and $\mathfrak{T}_w$.

Proposition 9.1 *For any $w \in W$, as operators on the ring $\mathcal{O}(\hat{T})$ of regular (polynomial) functions on $\hat{T}(\mathbb{C}) \cong (\mathbb{C}^\times)^r$,*

$$\mathfrak{L}_{w,v} = (-v)^{\ell(w)} \mathbf{z}^\rho \mathfrak{T}_{w,v^{-1}} \mathbf{z}^{-\rho}. \tag{50}$$

Proof Using (11) and (14) it follows that $-v\mathfrak{L}_{i,v^{-1}} = \mathbf{z}^\rho \mathfrak{T}_{i,v} \mathbf{z}^{-\rho}$ which implies (50). □

In order to understand the spherical Whittaker functions (which are sums of standard Iwahori Whittaker functions), we must compute how the corresponding spherical idempotents in the Hecke algebra act on a dominant weight. Define

$$\Theta = \mathbf{z}^{-\rho} \prod_{\alpha \in \Delta^+} (1 - \mathbf{z}^{-\alpha})^{-1} \left(\sum_{w \in W} (-1)^{\ell(w)} w \right) \mathbf{z}^\rho \,.$$

The operator Θ was denoted Ω in [21].

Proposition 9.2 *The following identities of operators hold in $\mathcal{O}(\hat{T})$:*

$$\sum_{w \in W} \mathfrak{T}_w = \mathbf{z}^{-\rho} \left(\prod_{\alpha \in \Delta^+} (1 - v\mathbf{z}^{-\alpha}) \right) \Theta \, \mathbf{z}^\rho, \qquad \sum_{w \in W} \mathfrak{L}_w = \Theta \prod_{\alpha \in \Delta^+} (1 - v\mathbf{z}^{-\alpha}).$$

Proof Consider the operator (9) of [21] with the specialization $\pi^\lambda \mapsto \mathbf{z}^{-\lambda}$ and taking q to be our v. If we choose ε to be the character of the Hecke algebra such $\varepsilon(T_i) = q$, the operator (9) becomes our $\mathfrak{L}_{i,v}$. On the other hand if we instead take $\varepsilon(T_i) = -1$, we obtain the operator

$$\mathbf{z}^\rho \mathfrak{T}_{i,v} \mathbf{z}^{-\rho} : f \mapsto (\mathbf{z}^{\alpha_i} - 1)^{-1}(f - \mathbf{z}^{\alpha_i}\,({}^{s_i} f)) - v(\mathbf{z}^{\alpha_i} - 1)^{-1}(f - {}^{s_i} f),$$

that is, the operator of (29). Therefore Theorem 14 of [21] gives both formulas. □

By the Weyl character formula if λ is a partition then $\Theta \mathbf{z}^\lambda = s_\lambda(\mathbf{z})$ is the corresponding Schur function. Thus

$$\sum_w \mathbf{z}^\rho \mathfrak{T}_w \mathbf{z}^{-\rho} \, \mathbf{z}^{\lambda+\rho} = \mathbf{z}^\rho \prod_{\alpha \in \Phi^+} (1 - v \, \mathbf{z}^{-\alpha}) s_\lambda(\mathbf{z}).$$

This identity agrees with Proposition 8.4 with **J** being the set of all simple reflections, **K** $= \varnothing$ and $w_2 = 1$. In this case the sum in (38) is over the entire Weyl group. To see this, note that the right-hand side agrees with the uncolored partition function (49), while each term $\mathbf{z}^\rho \mathfrak{T}_w \mathbf{z}^{-\rho} \, \mathbf{z}^{\lambda+\rho}$ is one of the colored partition functions $Z(\mathfrak{S}_{\mathbf{z},\lambda,w,1})$ by Proposition 32.

On the other hand, the sum over $\mathfrak{L}_w$ produces a (symmetric) Hall–Littlewood polynomial. While Proposition 9.1 gives a relationship between $\mathfrak{L}_w \mathbf{z}^\lambda$ and $\mathfrak{T}_w \mathbf{z}^\lambda$ for any w, it is remarkable that their sums over all $w \in W$ result in such different functions.

Next we explain precisely how these operators $\mathfrak{T}_i$ and $\mathfrak{L}_i$ may be used to compute various specializations of non-symmetric Macdonald polynomials. In general, these polynomials depend on two parameters which are usually denoted q and t. There are differing notations in the literature, but in this paper we will follow the notation $E_\lambda(\mathbf{z}; q, t)$ of Haglund, Haiman and Loehr [34]. Note that this q is not the cardinality

of the residue field of F as in earlier sections; in this section, the notation v^{-1} will be used for this quantity, where the t of [34] equals our v.

If $q = 0$ or ∞ the polynomials $E_\lambda(\mathbf{z}; q, t)$ are non-symmetric variants of Hall–Littlewood polynomials. According to [22], one such specialization arises from successively applying the operators $\mathfrak{T}_i$ to a dominant weight λ. The notation of Haglund, Haiman and Loehr [34] that we follow here differs from the notation in [22] by the variable change $(q, t) \mapsto (q^{-1}, t^{-1})$, so Theorem 7 of [22] will now be written

$$\phi_w(\mathbf{z}; \varpi^{-\lambda}) = \mathfrak{T}_{w,v}(\mathbf{z}^\lambda) = (-v)^{\ell(w)} \mathbf{z}^{-\rho} w_0 E_{w_0 w(\lambda+\rho)}(\mathbf{z}; \infty, v). \tag{51}$$

The next result is an analog of this for the $\mathfrak{L}_{w,v}$ using Proposition 9.1.

Proposition 9.3 *If λ is dominant, then*

$$\mathfrak{L}_{w,v}(\mathbf{z}^{\lambda+\rho}) = w_0 E_{w_0 w(\lambda+\rho)}(\mathbf{z}; \infty, v^{-1}). \tag{52}$$

Proof This follows by comparing (50) and (51). Another proof may be based on the Knop-Sahi recurrence ([28, 42, 59]) and other facts that can be found in [34]. For brevity we will not give this alternative proof. □

Having made this connection, we can now relate Li's Whittaker functions to the Hall–Littlewood polynomials P_λ ([52], Chapter III). If we denote by $\mathcal{W}_{\mathrm{Li}}(\mathbf{z}, \varpi^{-\lambda})$ the Whittaker function described by Jian-Shu Li in [50], we have the following result:

Proposition 9.4 *Let λ be a dominant weight. Then*

$$\mathbf{z}^\rho \mathcal{W}_{Li}(\mathbf{z}, \varpi^{-\lambda}) := \mathbf{z}^\rho \sum_{w \in W} (-v)^{-\ell(w)} \phi_w(\mathbf{z}; \varpi^{-\lambda}) = P_{\lambda+\rho}(\mathbf{z}, v^{-1}).$$

Proof By (1.1) in Chapter III of Macdonald [52], if λ is a partition then

$$\sum_{w \in W} \mathfrak{L}_w \mathbf{z}^\lambda = R_\lambda(\mathbf{z}; v) = v_\lambda(v) P_\lambda(\mathbf{z}; v) \tag{53}$$

where R_λ, v_λ and P_λ are as in [52] Section III.1. Multiplying (51) by $\mathbf{z}^\rho (-v)^{-\ell(w)}$ and summing over w, then using (52) and (53), the left-hand side of the desired identity equals

$$\sum_{w \in W} w_0 E_{w_0 w(\lambda+\rho)}(\mathbf{z}; \infty, v) = \sum_{w \in W} \mathfrak{L}_{w,v^{-1}}(\mathbf{z}^{\lambda+\rho}) = v_{\lambda+\rho}(v^{-1}) P_{\lambda+\rho}(\mathbf{z}; v^{-1}).$$

Because $\lambda + \rho$ is strongly dominant $v_{\lambda+\rho} = 1$ and the statement follows. □

To conclude this section, we relate parahoric Whittaker functions to generalizations of Macdonald polynomials called *Macdonald polynomials with prescribed symmetry*.

They were introduced by Baker, Dunkl and Forrester [2] and studied further by Marshall [53] and Baratta [3]. We shall follow the conventions in [3].

For I, J disjoint subsets of $\{1, \cdots, r-1\}$ such that $i \pm 1 \notin J$ for $i \in I$ and $j \pm 1 \notin I$ for $j \in J$, we define

$$S_{\eta^*}^{(I,J)}(\mathbf{z}; q, t) := \frac{1}{a_\eta^{(I,J)}} \sum_{w \in W_{I \cup J}} (-t)^{-\ell(w)} \mathfrak{L}_{w,t} E_\eta(\mathbf{z}; q, t), \qquad (54)$$

where η^* is a composition such that $\eta_i^* \geqslant \eta_{i+1}^*$ for all $i \in I$ and $\eta_j^* > \eta_{j+1}^*$ for all $j \in J$, η is any composition in $W_{I \cup J}\eta^*$, and $a_\eta^{(I,J)}$ is a normalization factor as described in [3, eq. (16)].

Proposition 9.5 *Let λ be a dominant weight, $I = \varnothing$, $J = \mathbf{J}$ and $t = v^{-1}$. Then*

$$\psi_1^{\mathbf{J}}(\lambda, \mathbf{z}) = \mathbf{z}^{-\rho} S_{\lambda+\rho}^{(\varnothing, \mathbf{J})}(\mathbf{z}; 0, t) a_{\lambda+\rho}^{(\varnothing, \mathbf{J})}.$$

Proof From (51) and (52) with $t = v^{-1}$ we have that

$$\phi_w(\mathbf{z}; \varpi^{-\lambda}) = (-t)^{-\ell(w)} \mathbf{z}^{-\rho} \mathfrak{L}_{w,t}(\mathbf{z}^{\lambda+\rho}). \qquad (55)$$

Since $E_{\lambda+\rho}(\mathbf{z}; 0, t) = \mathbf{z}^{\lambda+\rho}$ for dominant weights λ, the result follows from (18) and (54).   □

The following corollary follows immediately from Theorem 4.7:

Corollary 9.6 *Let λ be a dominant weight and $t = v^{-1}$. Then*

$$S_{\lambda+\rho}^{(\varnothing, \mathbf{J})}(\mathbf{z}; 0, t) = \frac{\mathbf{z}^\rho}{a_{\lambda+\rho}^{(\varnothing, \mathbf{J})}} \prod_{\alpha \in \Delta_{\mathbf{J}}^+} (1 - v\mathbf{z}^{-\alpha}) \chi_\lambda^{\mathbf{J}}(\mathbf{z}).$$

At this point in time, we have a good understanding of the dictionary relating Whittaker functions to special polynomials when the group we are working with is GL_r. It would be interesting for both combinatorial and number theoretic reasons to understanding generalizations of this dictionary to the metaplectic cover of GL_r and to other reductive groups. In the metaplectic setting, earlier results suggest these questions merit further inquiry: in [16], metaplectic spherical Whittaker functions are related to supersymmetric LLT polynomials, while in [60] a new family of special polynomials is introduced that generalizes metaplectic Iwahori Whittaker functions and non-symmetric Macdonald polynomials.

10 Intertwining integrals and *R*-matrices

In this section, we will explore the dictionary between *p*-adic representation theory and R-matrices of quantum groups, using lattice models for a pictorial interpretation of

either side. Roughly speaking, we will show that parts of the R-matrix for the quantum superalgebra $U_q(\widehat{\mathfrak{gl}}(r|1))$ corresponding to the smaller quantum groups $U_q(\widehat{\mathfrak{gl}}(r))$ and $U_q(\widehat{\mathfrak{gl}}(1))$ neatly express the action of intertwining operators on standard Iwahori fixed vectors (Theorem 10.5) and on Whittaker functionals (Remark 10.8), respectively. To prove Theorem 10.5, we identify the R-matrix for $U_q(\widehat{\mathfrak{gl}}(r))$ and the intertwining integral acting on standard Iwahori fixed vectors with a part of the colored R-matrix in Fig. 6. This allows us to give a pictorial interpretation of the functional equations used to prove Theorem 3.8 [see Eqs. (63) and (64)].

Before proving these facts, we will first make several comments related to Theorem 10.5. A common principle in the theory of symmetric functions (related to Schur duality) is to consider the coefficient of $z_1 z_2 \cdots z_r$ in the r variables z_i as having some combinatorial significance. Applying this to Schur functions gives the representation degrees of the irreducible representations of the symmetric group, and this principle was also used by Stanley [62] in counting the number of reduced words for the longest element of S_r.

A somewhat analogous procedure (related to Schur-Jimbo duality [38]) is to consider the space of vectors of the form (60) below in a tensor representation of $U_q(\widehat{\mathfrak{gl}}(r))$. These vectors are like the monomials $z_1 z_2 \cdots z_r$, because there are no repetitions allowed among the indexing set. The R-matrix acts on these vectors and we will relate this fact to the action of the intertwining operators on the Iwahori fixed vectors.

The larger quantum group $U_q(\widehat{\mathfrak{gl}}(r|1))$ will not appear in Theorem 10.5, only $U_q(\widehat{\mathfrak{gl}}(r))$. Concretely, the reason for this is that $\oplus$ spins do not appear on the right boundary of our systems. We relate the $\oplus$ spins with the Whittaker functional in Remark 10.8. We therefore want to think of the boundary condition for the left side of our lattice model as indexing the unique Whittaker functional of the unramified principal series, while the boundary conditions for the right side index the basis of the space of Iwahori fixed vectors. This can be generalized to the parahoric case as well, per Remark 10.7.

In this section, q will *not* denote the cardinality of the residue field; instead it will stand for the quantum parameter q in $U_q(\widehat{\mathfrak{gl}}(r))$ as is customary in the theory of quantum groups. We will continue to denote the cardinality of the residue field of F by v^{-1}. With these conventions, we set $q^2 = 1/v$ for Theorem 10.5, consistent with our relation between quantum groups and residue field cardinalities in earlier sections (where we wrote $U_{1/\sqrt{v}}$).

Consider the quantum loop group $U_q(\widehat{\mathfrak{gl}}(r))$, which is a quantization of a central extension of the loop algebra of $\mathfrak{gl}(r)$; for its formal definition see Section 12.2 in [27]. The quantum loop group acts on the evaluation representation $V_r(z)$ for $z \in \mathbb{C}^\times$. The evaluation representation has a basis $\{v_i(z), 1 \leqslant i \leqslant r\}$. Denote $V_r(\mathbf{z}) := V_r(z_1) \otimes \cdots \otimes V_r(z_r)$.

There is an affine R-matrix, initially due to Jimbo [37], that intertwines between tensor products of evaluation representations. We denote it by $R_q(\mathbf{z}^{\alpha_k}) : V_r(z_k) \otimes V_r(z_{k+1}) \to V_r(z_{k+1}) \otimes V_r(z_k)$ and it is given by the following formula:
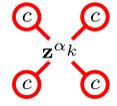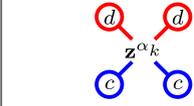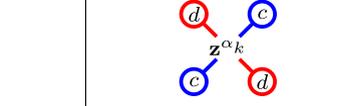
Fig. 20 The colored R-matrix $R_{col}(\mathbf{z}^{\alpha_k})$. The colors $c, d \in \mathfrak{P}$ which we will represent by $\{1, 2, \cdots, r\}$ are always distinct

$$R_q(\mathbf{z}^{\alpha_k}) = \frac{1}{1 - q^{-2}\mathbf{z}^{\alpha_k}} \sum_{1 \leqslant i \leqslant r} (q - \mathbf{z}^{\alpha_k}q^{-1})e_{ii} \otimes e_{ii}$$

$$+ \frac{1}{1 - q^{-2}\mathbf{z}^{\alpha_k}} \sum_{i > j} ((-q^{-1})(1 - \mathbf{z}^{\alpha_k})e_{ij} \otimes e_{ji} + (-q)(1 - \mathbf{z}^{\alpha_k})e_{ji} \otimes e_{ij})$$

$$+ \frac{1}{1 - q^{-2}\mathbf{z}^{\alpha_k}} \sum_{i > j} ((q - q^{-1})e_{jj} \otimes e_{ii} + \mathbf{z}^{\alpha_k}(q - q^{-1})e_{ii} \otimes e_{jj}).$$

$$(56)$$

In the above, e_{ij} stands for the $r \times r$ matrix with a 1 in the (i, j) entry and all other entries equal to 0. It is a map $V_r(z_k) \to V_r(z_{k+1})$ if it is on the left of the tensor product and $V_r(z_{k+1}) \to V_r(z_k)$ if it is on the right of the tensor product.

Remark 10.1 This is not exactly the R-matrix in [37]; it is a Drinfeld twist by $-q$. See [13] for a definition of the Drinfeld twist and details on how it modifies an R-matrix. This particular Drinfeld twist appears very often when one deals with $U_q(\widehat{\mathfrak{gl}}(r))$ lattice models. Let us consider the weights in our Fig. 20 in which we restrict to configurations with all edges colored. This is the $U_q(\widehat{\mathfrak{gl}}(r))$ portion of the larger $U_q(\widehat{\mathfrak{gl}}(r|1))$ R-matrix. These are the same as the weights in equation (2.1.8) of [9] (up to a factor, and their q is our q^{-2}). Both R-matrices come from the same Drinfeld twist of $U_q(\widehat{\mathfrak{gl}}(r))$. Throughout this section, when we write $U_q(\widehat{\mathfrak{gl}}(r))$, we will in fact refer to a Drinfeld twist of the usual affine quantum group that produces the R-matrix $R_q(\mathbf{z}^{\alpha_k})$.

It is a standard fact in the theory of quantum groups that $R_q(\mathbf{z}^{\alpha_k})$ is a $U_q(\widehat{\mathfrak{gl}}(r))$-module homomorphism. We will also denote by $(R_q(\mathbf{z}^{\alpha_k}))_{k,k+1} : V_r(\mathbf{z}) \to V_r(s_k\mathbf{z})$ the map that acts as $R_q(\mathbf{z}^{\alpha_k})$ on the k and $k + 1$ tensor components of $V_r(\mathbf{z})$ and the identity elsewhere.

Consider the R-matrix in Fig. 6 restricted to vertices where all edges are colored. It is preferable to normalize the weights of $R(z_{k+1}, z_k)$ by dividing by z_{k+1} so that they may be expressed in terms of $\mathbf{z}^{\alpha_k} = z_k/z_{k+1}$. We further normalize by a factor of $1 - v\mathbf{z}^{\alpha_k}$. Denote the resulting restricted R-matrix by $R_{col}(\mathbf{z}^{\alpha_k})$, and similarly the restricted version of $R(z_k, z_{k+1})$ is then $R_{col}(\mathbf{z}^{-\alpha_k})$. The vertices and weights are pictured in Fig. 20.

Let $\mathbf{i}_0 := (1, 2, \cdots, r)$ and let $\mathbb{I}$ be the set of all permutations of $\mathbf{i}_0$. Given $w \in S_r$ and $i \in \{1, \cdots, r\}$, let $w_i = w(i)$ which equals the i-th component of the r-tuple

$w(\mathbf{i}_0)$. For example, if $s_1 s_2 \in S_3$, then $(s_1 s_2)(\mathbf{i}_0) = (312)$ and therefore $(s_1 s_2)_3 = 2$. The following lemma is well-known:

Lemma 10.2 $\ell(s_k w) > \ell(w)$ *if and only if* $w_{k+1} > w_k$.

In the notation of Sect. 3, let $\bar{\mathcal{A}}^{\mathbf{z}}_{s_k} := \frac{(1-\mathbf{z}^{\alpha_k})}{1-v\mathbf{z}^{\alpha_k}} \mathcal{A}^{\mathbf{z}}_{s_k}$ where $\mathcal{A}^{\mathbf{z}}_w : I(\mathbf{z}) \to I(w\mathbf{z})$ is the standard intertwining integral (6).

Proposition 10.3 *Equation (9) can be rewritten as*

$$
\bar{\mathcal{A}}^{\mathbf{z}}_{s_k}(\Phi^{\mathbf{z}}_w) = \mathrm{wt}
\begin{pmatrix}
\overset{w_{k+1}}{} \ \overset{w_{k+1}}{} \\
\mathbf{z}^{\alpha_k} \\
\underset{w_k}{} \ \underset{w_k}{}
\end{pmatrix}
\Phi^{s_k \mathbf{z}}_w + \mathrm{wt}
\begin{pmatrix}
\overset{w_k}{} \ \overset{w_{k+1}}{} \\
\mathbf{z}^{\alpha_k} \\
\underset{w_{k+1}}{} \ \underset{w_k}{}
\end{pmatrix}
\Phi^{s_k \mathbf{z}}_{s_k w}. \tag{57}
$$

Proof Let $s_k, w \in S_r$ and let us prove the statement for the case $\ell(s_k w) > \ell(w)$; the opposite case is similar. Lemma 10.2 implies that $w_{k+1} > w_k$. By consulting the weights in Fig. 20, we see that

$$
\mathrm{wt}
\begin{pmatrix}
\overset{w_{k+1}}{} \ \overset{w_{k+1}}{} \\
\mathbf{z}^{\alpha_k} \\
\underset{w_k}{} \ \underset{w_k}{}
\end{pmatrix}
= \frac{1-v}{1-v\mathbf{z}^{\alpha_k}}, \qquad
\mathrm{wt}
\begin{pmatrix}
\overset{w_k}{} \ \overset{w_{k+1}}{} \\
\mathbf{z}^{\alpha_k} \\
\underset{w_{k+1}}{} \ \underset{w_k}{}
\end{pmatrix}
= \frac{1-\mathbf{z}^{\alpha_k}}{1-v\mathbf{z}^{\alpha_k}}.
$$

and the equivalence follows immediately by comparison with equation (9). □

Remark 10.4 We will represent the colors of the palette $\mathfrak{P}$ by the integers $\{1, \ldots, r\}$. In this section, our color ordering will be opposite to the ordering we used in previous sections, so $1 > 2 > \cdots > r$. We use this ordering because we want to match the largest color (in this case 1) with the highest weight vector of a quantum group representation which is customarily denoted by v_1.

Since the edges of the R-matrix $R_{\mathrm{col}}(\mathbf{z}^{\alpha_k})$ are specified by these colors, we can then think of it as a map $R_{\mathrm{col}}(\mathbf{z}^{\alpha_k}) : U_r(z_k) \otimes U_r(z_{k+1}) \to U_r(z_{k+1}) \otimes U_r(z_k)$, where $U_r(z)$ is a vector space with formal basis elements $u_i(z)$ associated to the colors i for $1 \leqslant i \leqslant r$. One can write the R-matrix in Fig. 20 in matrix form as follows:

$$R_{\mathrm{col}}(\mathbf{z}^{\alpha_k}) = \sum_{1 \leqslant i \leqslant r} \mathrm{wt}\left(\begin{smallmatrix} i & & i \\ & \mathbf{z}^{\alpha_k} & \\ i & & i \end{smallmatrix}\right) e_{ii} \otimes e_{ii} + \sum_{i>j} \mathrm{wt}\left(\begin{smallmatrix} j & & i \\ & \mathbf{z}^{\alpha_k} & \\ i & & j \end{smallmatrix}\right) e_{ij} \otimes e_{ji} \qquad (58)$$

$$+ \sum_{i>j} \mathrm{wt}\left(\begin{smallmatrix} i & & j \\ & \mathbf{z}^{\alpha_k} & \\ j & & i \end{smallmatrix}\right) e_{ji} \otimes e_{ij} + \sum_{i>j} \mathrm{wt}\left(\begin{smallmatrix} i & & i \\ & \mathbf{z}^{\alpha_k} & \\ j & & j \end{smallmatrix}\right) e_{jj} \otimes e_{ii}$$

$$+ \sum_{i>j} \mathrm{wt}\left(\begin{smallmatrix} j & & j \\ & \mathbf{z}^{\alpha_k} & \\ i & & i \end{smallmatrix}\right) e_{ii} \otimes e_{jj}.$$

Define $U_r(\mathbf{z}) := U_r(z_1) \otimes \cdots \otimes U_r(z_r)$ and, for $\mathbf{i} = (i_1, \cdots, i_r) \in \mathbb{I}$, let

$$u_{\mathbf{i}}(\mathbf{z}) := u_{i_1}(z_1) \otimes \cdots \otimes u_{i_r}(z_r) \in U_r(\mathbf{z}). \qquad (59)$$

Let $U_r^{\mathrm{alt}}(\mathbf{z})$ be the subspace of $U_r(\mathbf{z})$ with basis $\{u_{\mathbf{i}}(\mathbf{z}), \mathbf{i} \in \mathbb{I}\}$. The R-matrix $R_{\mathrm{col}}(\mathbf{z}^{\alpha_k})_{k,k+1} : U_r(\mathbf{z}) \to U_r(s_k\mathbf{z})$ restricts to $R_{\mathrm{col}}(\mathbf{z}^{\alpha_k})_{k,k+1} : U_r^{\mathrm{alt}}(\mathbf{z}) \to U_r^{\mathrm{alt}}(s_k\mathbf{z})$ by removing the first term of (58). We similarly define

$$v_{\mathbf{i}}(\mathbf{z}) := v_{i_1}(z_1) \otimes \cdots \otimes v_{i_r}(z_r) \in V_r(\mathbf{z}), \qquad (60)$$

and denote by $V_r^{\mathrm{alt}}(\mathbf{z})$ the subspace of $V_r(\mathbf{z})$ with basis $\{v_{\mathbf{i}}(\mathbf{z}), \mathbf{i} \in \mathbb{I}\}$. Note that this is *not* a $U_q(\widehat{\mathfrak{gl}}(r))$ submodule of $V_r(\mathbf{z})$. Even so, the restriction of $R_q(\mathbf{z}^{\alpha_k})_{k,k+1}$ to $V_r^{\mathrm{alt}}(\mathbf{z})$ maps into $V_r^{\mathrm{alt}}(s_k\mathbf{z})$ because $R_r^{\mathrm{alt}}(\mathbf{z}^{\alpha_1})$ maps $v_1(z_1) \otimes v_2(z_2)$ to a linear combination of $v_1(z_2) \otimes v_2(z_1)$ and $v_2(z_2) \otimes v_1(z_1)$ as seen in (56).

Consider the following isomorphisms of vector spaces $\theta_{\mathbf{z}} : I(\mathbf{z})^J \to U_r^{\mathrm{alt}}(\mathbf{z})$ and $\xi_{\mathbf{z}} : U_r^{\mathrm{alt}}(\mathbf{z}) \to V_r^{\mathrm{alt}}(\mathbf{z})$ defined by

$$\theta_{\mathbf{z}}(\Phi_w^{\mathbf{z}}) := u_{w(\mathbf{i}_0)}(\mathbf{z}), \quad \xi_{\mathbf{z}}(u_{w(\mathbf{i}_0)}(\mathbf{z})) := v_{w(\mathbf{i}_0)}(\mathbf{z}). \qquad (61)$$

In this basis we have that

$$\theta_{s_k\mathbf{z}}(\Phi_w^{s_k\mathbf{z}}) = (e_{w_k,w_k} \otimes e_{w_{k+1},w_{k+1}})\theta_{\mathbf{z}}(\Phi_w^{\mathbf{z}}), \quad \theta_{s_k\mathbf{z}}(\Phi_{s_k w}^{s_k\mathbf{z}})$$
$$= (e_{w_{k+1},w_k} \otimes e_{w_k,w_{k+1}})\theta_{\mathbf{z}}(\Phi_w^{\mathbf{z}}),$$

$$(62)$$

where we have suppressed the notation that the operators $(e_{ab} \otimes e_{cd})$ here act on the k and $k+1$ factors of the tensor product of $U_r^{\mathrm{alt}}(\mathbf{z})$. We may use this to rewrite the action of $\bar{\mathcal{A}}_{s_k}^{\mathbf{z}}(\Phi_w^{\mathbf{z}})$ in (57) in terms of the action of $R_{\mathrm{col}}(\mathbf{z}^{\alpha_k})$ in (58), or similarly the action of $R_q(\mathbf{z}^{\alpha_k})$ in (56), to obtain the following result.

Theorem 10.5 *The following diagram commutes assuming $q^2 = 1/v$:*

$$
\begin{array}{ccc}
I(\mathbf{z})^J & \xrightarrow{\ \theta_{\mathbf{z}}\ } U_r^{\text{alt}}(\mathbf{z}) & \xrightarrow{\ \xi_{\mathbf{z}}\ } V_r^{\text{alt}}(\mathbf{z}) \\
\downarrow{\scriptstyle \bar{\mathcal{A}}_{s_k}^{\mathbf{z}}} & \downarrow{\scriptstyle R_{\text{col}}(\mathbf{z}^{\alpha_k})_{k,k+1}} & \downarrow{\scriptstyle q^{-1}R_q(\mathbf{z}^{\alpha_k})_{k,k+1}} \\
I(s_k\mathbf{z})^J & \xrightarrow{\ \theta_{s_k\mathbf{z}}\ } U_r^{\text{alt}}(s_k\mathbf{z}) & \xrightarrow{\ \xi_{s_k\mathbf{z}}\ } V_r^{\text{alt}}(s_k\mathbf{z})
\end{array}
$$

Proof The commutativity of the left side of the diagram follows from Proposition 10.3 noting that after the identification between $I(\mathbf{z})^J$ and $U_r^{\text{alt}}(\mathbf{z})$ via $\theta_{\mathbf{z}}$, the action of $\bar{\mathcal{A}}_{s_k}^{\mathbf{z}}$ and $R_{\text{col}}(\mathbf{z}^{\alpha_k})_{k,k+1}$ are the same as seen from Eqs. (57), (58) and (62). The right half of the diagram commutes because the restriction of the two R-matrices are equal. Indeed we have matching entries of the R-matrices in Eqs. (56) and (58) (we need to match each entry except for the first term which does not occur in the restriction). For example, the last entry in Eq. (58) has weight $\frac{(1-v)\mathbf{z}^{\alpha_k}}{1-v\mathbf{z}^{\alpha_k}}$, while the last entry in Eq. (56) multiplied by q^{-1} has weight $\frac{(1-q^{-2})\mathbf{z}^{\alpha_k}}{1-q^{-2}\mathbf{z}^{\alpha_k}}$. □

Remark 10.6 The theorem above can also be proved if we set $q^2 = v$ (as opposed to $q^2 = v^{-1}$). In that setting the quantum group needs to be Drinfeld twisted by the parameter $-q$ (as opposed to by q^{-1}). We emphasize this as both choices might be useful when considering representation theoretic applications of this result.

Remark 10.7 One can generalize Theorem 10.5 to the parahoric setting by choosing elements of the form in Eq. (60) with repetitions determined by the chosen Levi subgroup or by the possible right boundary conditions of the lattice model associated to the parahoric Whittaker function. For example, if the Levi subgroup is $\text{GL}_2 \times \text{GL}_1$, the space of K_J-fixed vectors in $I(\mathbf{z})$ has a basis indexed by elements in $W^J = \{1, s_2, s_1 s_2\}$ which is in bijection with $\{v_{112}, v_{121}, v_{211}\}$. For each basis element we also have a corresponding right boundary condition for the colored lattice model in Sect. 8. Assume $R > B$. If (in accord with Remark 10.4) we let the integers 1 and 2 be identified with the colors R and B respectively, v_{112} corresponds to right boundary condition (R, R, B), v_{121} to (R, B, R), and v_{211} to (B, R, R).

Remark 10.8 Note that Proposition 3.7 can be rewritten as

$$
\Omega_{s_k\mathbf{z}} \circ \bar{\mathcal{A}}_{s_k}^{\mathbf{z}} = \frac{1 - v\mathbf{z}^{-\alpha_k}}{1 - v\mathbf{z}^{\alpha_k}} \Omega_{\mathbf{z}}.
$$

The factor above agrees up to a scalar with the fully uncolored Boltzmann weight for $R(z_{k+1}, z_k)$ in Fig. 6 and should be thought of as an entry of the R-matrix for the evaluation module of $U_q(\widehat{\mathfrak{gl}}(r|1))$. This remark and the previous theorems realize the action of intertwiners on both the space of Whittaker functionals and Iwahori (or parahoric) fixed vectors as entries in the R-matrix of $U_q(\widehat{\mathfrak{gl}}(r|1))$.

The purpose of this section was to give a dictionary between objects related to different areas of mathematics: intertwiners for p-adic groups, R-matrices for quantum groups, and R-matrices for lattice models. Let us now upgrade this dictionary by

matching two techniques used in the theory of p-adic groups and lattice models. For lattice models we used the train argument to derive functional equations for the partition function. This argument was first used by Baxter to prove commutativity of transfer matrices and in our setting can be summarized as follows:

$$\text{wt}\begin{pmatrix} \\ z^{\alpha_k} \\ \\ \end{pmatrix} Z(\mathfrak{S}_{\mathbf{z},\lambda,w_1,w_2}) = \text{intermediate states appearing in the train argument}$$

$$= \text{wt}\begin{pmatrix} w_{k+1} & w_{k+1} \\ & z^{\alpha_k} & \\ w_k & w_k \end{pmatrix} Z(\mathfrak{S}_{s_k\mathbf{z},\lambda,w_1,w_2}) + \text{wt}\begin{pmatrix} w_{k+1} & w_{k+1} \\ & z^{\alpha_k} & \\ w_k & w_k \end{pmatrix} Z(\mathfrak{S}_{s_k\mathbf{z},\lambda,s_k w_1,w_2}).$$

$$(63)$$

For p-adic groups we used the intertwiner to derive functional equations for Iwahori Whittaker functions based on results and ideas of Casselman and Shalika. This process can be reimagined using Proposition 10.3 and Remark 10.8 as follows (we denote $\varpi^{-\lambda}w_2$ by g):

$$\text{wt}\begin{pmatrix} \\ z^{\alpha_k} \\ \\ \end{pmatrix} \Omega_{\mathbf{z}}(\pi(g)\Phi^{\mathbf{z}}_{w_1}) = \Omega_{s_k\mathbf{z}} \circ \bar{\mathcal{A}}^{\mathbf{z}}_{s_k}(\pi(g)\Phi^{\mathbf{z}}_{w_1}) = \Omega_{s_k\mathbf{z}}(\pi(g)\bar{\mathcal{A}}^{\mathbf{z}}_{s_k}\Phi^{\mathbf{z}}_{w_1})$$

$$= \Omega_{s_k\mathbf{z}}\left(\pi(g)\left(\text{wt}\begin{pmatrix} w_{k+1} & w_{k+1} \\ & z^{\alpha_k} & \\ w_k & w_k \end{pmatrix}\Phi^{s_k\mathbf{z}}_{w_1} + \text{wt}\begin{pmatrix} w_k & w_{k+1} \\ & z^{\alpha_k} & \\ w_{k+1} & w_k \end{pmatrix}\Phi^{s_k\mathbf{z}}_{s_k w_1}\right)\right) \qquad (64)$$

$$= \text{wt}\begin{pmatrix} w_{k+1} & w_{k+1} \\ & z^{\alpha_k} & \\ w_k & w_k \end{pmatrix}\Omega_{s_k\mathbf{z}}(\pi(g)\Phi^{s_k\mathbf{z}}_{w_1}) + \text{wt}\begin{pmatrix} w_{k+1} & w_{k+1} \\ & z^{\alpha_k} & \\ w_k & w_k \end{pmatrix}\Omega_{s_k\mathbf{z}}(\pi(g)\Phi^{s_k\mathbf{z}}_{s_k w_1}).$$

We see that the beginning and the end of the equations (63), (64) correspond to each other by use of Theorem 7.2 and Proposition 10.3. The idea of both arguments is also the same. In the p-adic setting one moves the intertwiner from the 'Whittaker side' to the 'Iwahori side', while in the lattice model setting one moves the R-matrix from the left side (which corresponds to the Whittaker functionals as we argued before) to the right side (corresponding to the space of Iwahori fixed vectors).

This phenomena also appears in the theory of metaplectic spherical Whittaker functions for an n-fold metaplectic cover of $\text{GL}_r(F)$, which can also be realized as partition functions of a solvable lattice model [12]. In that case the action of the intertwining integral on the space of Whittaker functionals is the Kazhdan-Patterson scattering matrix, which has been interpreted (up to a Drinfeld twist) as the $U_q(\widehat{\mathfrak{gl}}(n))$ R-matrix in [12, Theorem 1], while the action of the intertwining integral on the spherical vector is the Gindinkin-Karpelevich factor which can be interpreted as the

spin $\oplus$ part of a larger R-matrix. The train argument and the p-adic argument for producing functional equations work in the same way. This compelling connection between two a priori different methods of argument should be useful in further relating the representation theories of p-adic and quantum groups.

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