SUPERGRAVITY, STRING THEORY AND BLACK HOLES

by

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Table of Contents

List of Figures

Abstract

In this thesis we study a large class of three-charge solutions of supergravity in various dimensions and duality frames. The solutions have the same asymptotic structure, supersymmetry and conserved charges as black holes and black rings with macroscopic horizons and finite entropy. These supergravity solutions have neither horizons nor singularities and can be viewed as examples of the "geometric transition" mechanism of string theory.

We also study the gravitational analog of spectral flow in conformal field theory and use it as an efficient solution-generating technique in supergravity. This provides a useful relation between two-charge and three-charge supergravity solutions which we study in detail. By studying two-charge supertubes in the background of regular three-charge solutions with dipole charges we uncover a novel mechanism for entropy enhancement and discuss its implications for black-hole physics.

We generalize the well-known class of three-charge supersymmetric solutions in five dimensions, based on Gibbons-Hawking spaces, by studying supergravity solutions with less global symmetry. This is computationally challenging but we manage to present the first examples of such solutions and discuss their relation to black holes and black rings.

We also study non-supersymmetric asymptotically flat supergravity solutions with a four-dimensional Ricci-flat or Einstein-Maxwell base. These are solutions of string or M-theory compactified to five dimensions and provide some of the very few examples of regular non-supersymmetric gravity solutions.

Introduction

The existence of black holes and cosmological singularities are some of the most dramatic consequences of General Relativity. They were first developed as theoretical predictions but recent astronomical observations provide strong evidence that the Big Bang model of the Universe is correct and that black holes exist in Nature. Black holes and Big Bang models have singularities - regions of space time with infinite curvature and tidal forces. These singularities cannot be explained by General Relativity and point to the existence of a more fundamental theory of gravity which should resolve them.

In the early 1970's it was found that black holes have thermodynamic properties and can emit thermal radiation [14, 18, 136]. Thus, in addition to the singularity behind the horizon of a black hole, there are two more puzzles of black hole physics:

1) Understanding the origin of black hole thermodynamics from the point of view of statistical mechanics and explaining why the entropy is proportional to the area of the black hole.

2) Resolution of the information paradox - the fact that unitarity of quantum mechanics seems to be violated by black holes, since a pure state can be absorbed by the black hole and emitted as thermal radiation.

A fundamental theory of gravity should address and eventually resolve these puzzles. String theory is a theory of quantum gravity and in the last fifteen years has led to a better understanding of the physics of black holes and indeed, in some cases, provides resolutions to some of the above problems.

The entropy of a black hole is proportional to the area of its horizon. This is puzzling since entropy is an extensive quantity and one would expect it to scale with the volume. Somehow the degrees of freedom of the black hole, and thus of gravity, are encoded in a surface with one dimensions less than the dimension of the dynamical space-time. This is the basis of the holographic principle [1, 200] - the degrees of freedom of a gravitational theory can be described by a theory without gravity living in one less dimension. This idea was made more precise by the AdS/CFT [163, 128, 208] correspondence and its further generalizations [2]. AdS/CFT also suggests that there is no information loss since the theory on the boundary is unitary, however there is no detailed and satisfactory explanation how the information is restored within the gravitational (bulk) theory.

For a class of supersymmetric black holes in string theory Stominger and Vafa showed that the entropy of the black hole can be accounted for by counting bound states of D-branes at vanishing gravitational coupling using results from string theory and conformal field theory [198, 52]. At strong gravitational coupling these D-branes form a black hole with finite entropy in supergravity. Supersymmetry then protects the number of states as one changes the coupling and it was shown that the weak coupling counting matches with the Bekenstein-Hawking entropy of the black hole at strong coupling. This result was later generalized for other black holes in various dimensions and for some non-supersymmetric black holes. This shows that indeed gravitational entropy has statistical origin, however it provides no clue what is the fate of each individual microstate as one increases the gravitational coupling.

Another way to understand the Strominger-Vafa entropy counting is via the AdS/CFT correspondence[1] [163, 128, 208]. One can make a black hole with macroscopic horizon in string theory by putting together N_5 D5 branes and N_1 D1 branes and turning on N_p units of momentum along the direction of the D1's. If one takes a near horizon limit of this system, one finds a bulk that is asymptotic to $AdS_3 \times S^3 \times T^4$. The dual boundary theory is the two-dimensional conformal field theory that lives on the intersection of the D1 branes and the D5 branes and is known as the D1-D5-P CFT

[1]Historically the AdS/CFT correspondence was found later.

[73]. If one counts the states with momentum N_p in this conformal field theory [199], one obtains the entropy

$$S = 2\pi\sqrt{N_1 N_5 N_p}\,, \tag{1}$$

which precisely matches the entropy of the dual black hole [69] in the bulk.

A very important question, with deep implications for the physics of black holes, is: “What is the fate of these microscopic brane configurations as the effective coupling becomes large?” Alternatively, the question can be rephrased in AdS/CFT language as: “What is the gravity dual of individual microstates of the D1-D5-P CFT?” More physically, “What do the black-hole microstates look like in a background that a relativist would recognize as a black hole?”

In the last few years there has been substantial progress in answering these questions and this will be the main topic of this thesis. We will by no means provide comprehensive answers but we hope that the results of the research presented here will provide some clues to the structure of the black hole microstates at strong gravitational coupling and details of the resolution of the information paradox.

Chapters 1 and 2 of this dissertation are devoted to reviews of black hole physics, supergravity, D-branes and the construction of regular three-charge geometries. Chapters 3, 4, 5 and 6 are based on research that I did in collaboration with Iosif Bena, Clément Ruef and Nick Warner and which was published in [31, 33, 34, 35, 45].

The dissertation is organized as follows:

In Chapter 1 we give a short review of black holes and black hole thermodynamics in General Relativity. We discuss in some detail the Reissner-Nordström black hole, which will serve as a toy model for other black holes studied in the dissertation. We also provide a brief overview of some aspects of supergravity, string theory and D-branes which are of relevance for black holes. At the end of this Chapter we outline the counting of black hole microstates in string theory.

Chapter 2 is devoted to a summary of three-charge solutions of eleven-dimensional supergravity from intersecting M2 and M5 branes. We discuss how to construct black hole and black ring solutions in eleven-dimensional supergravity, but the main emphasis is on the construction on regular asymptotically flat solutions with non-trivial topology and with no horizons. We also discuss the implications of the existence of these solutions on the physics of black holes.

In Chapter 3, based on [33] and [35], we use string dualities to recast the eleven-dimensional solutions of Chapter 2 in type IIA and IIB supergravity. We also discuss "spectral flow" - a solution generating technique in supergravtiy, which we use to relate two and three-charge supergravity solutions and study which of these solutions are true bound states.

Using the results of Chapter 3 we continue in Chapter 4 with the study of probe supertubes in the background of various three-charge supergravity solutions. We uncover a novel mechanism for entropy enhancement due to the presence of non-trivial dipole charges of the supergravity solutions [34, 35]. We discuss how this mechanism may provide a better understanding of the black hole entropy and microstates in supergravity and string theory.

In Chapter 5 we relax the very special $U(1)$ isometry of the well-known class of explicit supergravity solutions discussed in Chapter 2 and present a new more general class of three-charge supersymmetric solutions found in [31]. We present an explicit example of such solution based on the Atiyah-Hitchin manifold. We also discuss some interesting solutions based on the irregular Eguchi-Hanson manifold.

In the final Chapter we discuss two possible ways to break supersymmetry and study a large class of non-supersymmetric supergravity solutions. We also present one of the very few regular non-supersymmetric solutions based on the Euclidean Reissner-Nordström manifold [45].

In the Conclusions we discuss some open problems and directions for possible future work based on the results of this thesis.

Chapter 1

Black Holes, Supergravity and D-branes

1.1 Black holes

Black holes are one of the most striking predictions of General Relativity [134, 203, 57]. In four-dimensional general relativity coupled to electro-magnetism there are uniqueness theorems that state that the only axially symmetric asymptotically flat solution to Einstein-Maxwell theory is the Kerr-Newman black hole [148, 172]. Thus, under some reasonable assumptions, the black hole is uniquely specified by its mass, M, electric and magnetic charges Q and P and angular momentum J. This on the other hand implies that if one is given the thermodynamic state functions of the black hole there is a unique gravitational solution, i.e. there is only one corresponding state in General Relativity. This uniqueness theorem is not valid in higher dimensional gravity and supergravity and this fact will be discussed later in this dissertation. For the moment we will stick to four space-time dimensions and illustrate the physics of black holes by studying the toy example of the charged black hole in Einstein-Maxwell theory.

1.1.1 The Reissner-Nordström black hole

In this section we will present some details about the Reissner-Nordström solution of Einstein-Maxwell theory [185, 173]. This solution will serve as an example of a black hole and will capture the essential physics of the more complicated black holes that will be discussed later in this thesis.

The equations of motion of general relativity coupled to matter are given by the Einstein equations[1]

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_4 T_{\mu\nu} \ , \tag{1.1}$$

where G_4 is the four-dimensional Newton constant, $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar of the dynamical metric. These equations can be derived from the action

$$S = \frac{1}{8\pi G_4} \int d^4x \sqrt{-g} R + S_M \ , \tag{1.2}$$

where S_M is the action of the matter fields. Note that we will use conventions in which $c = \hbar = k_B = 1$ The energy-momentum tensor of the matter fields can be derived from S_M and is given by

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} \ . \tag{1.3}$$

Here we will be interested in gravity coupled to electro-magnetism so the matter action is simply given by the Maxwell action

$$S_M = \frac{1}{\alpha} \int d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu} \ . \tag{1.4}$$

The constant α depends on conventions, in this chapter we will use $\alpha = 8\pi$. The energy-momentum tensor for the Maxwell field is then

$$T_{\mu\nu} = -\frac{2}{\alpha} \left(F_{\mu\sigma} F_{\nu}^{\sigma} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right) \ . \tag{1.5}$$

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[1] See [134, 203, 57] for some excellent reviews on General Relativity and black holes.

In addition to the Einstein equations one also has the Maxwell equations for the electromagnetic field

$$\nabla^\mu F_{\mu\nu} = 0 \, , \qquad \nabla_{[\mu} F_{\nu\sigma]} = 0 \, . \tag{1.6}$$

A solution to these field equations is given by the following line element

$$ds_4^2 = -\left(1 - \frac{2G_4 M}{r} + \frac{G_4(p^2+q^2)}{r^2}\right) dt^2 + \left(1 - \frac{2G_4 M}{r} + \frac{G_4(p^2+q^2)}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \, , \tag{1.7}$$

and gauge field strength

$$F = -\frac{q}{r^2} dt \wedge dr + p \sin\theta d\theta \wedge d\phi \, . \tag{1.8}$$

This is the four-dimensional Reissner-Nordström solution, by setting $p = q = 0$ one gets the Schwarzschild solution [191]. The metric coefficient of dt^2 will vanish at

$$r_\pm = G_4 M \pm \sqrt{G_4^2 M^2 - G_4(p^2+q^2)} \tag{1.9}$$

and it looks like the metric is singular at these points. However the curvature invariants of the metric are regular at $r = r_\pm$, so these loci are just coordinate singularities. One can explicitly compute the Riemann and Ricci tensors of the Reissner-Nordström metric and find that

$$R = g^{\mu\nu} R_{\mu\nu} = 0 \, , \qquad R_{\mu\nu} R^{\mu\nu} = \frac{4G_4^2(p^2+q^2)^2}{r^8} \, . \tag{1.10}$$

It is clear from these expressions that there is an essential singularity at $r = 0$ where some curvature invariants blow up. A physical observer approaching $r = 0$ will experience infinite tidal forces. This is an example of the singularities that appear in black hole physics. Their existence indicate that Einstein's theory of General Relativity is only an effective low energy theory and should not be valid at very small distances or high energies.

It is clear that there are three distinct regions of parameter space to consider in the Reissner-Nordström solution:

- $G_4^2 M^2 < G_4(p^2 + q^2)$

  In this case g_{tt} never vanishes (since the radial coordinate is real) and the solution is completely regular up to the singularity at $r = 0$. The singularity is not shielded by a horizon and is known as "naked singularity". The cosmic censorship conjecture states that such singularities should not be the product of gravitational collapse and therefore these solutions are generally considered unphysical in General Relativity.[2] One can intuitively understand the condition $G_4^2 M^2 < G_4(p^2 + q^2)$ as stating that the total energy of the solution is less than the energy of the electromagnetic field. This would imply that the mass of the matter that carries the charges is negative which is why such solutions are considered unphysical.

- $G_4^2 M^2 > G_4(p^2 + q^2)$

  This is the more physical situation when the energy of the solution is bigger than the energy of the electromagnetic field. The surfaces $r = r_\pm$ are null and are the two event horizons of the black hole. The singularity at $r = 0$ is time-like and is shielded by the two horizons. This is the non-extremal Reissner-Nordström black hole.

- $G_4^2 M^2 = G_4(p^2 + q^2)$

[2] Naked singularities may be physically relevant in string theory or the gauge/gravity duality but this will not concern us here.

This is the extremal Reissner-Nordström black hole. It has a single horizon at $r = r_+ = r_-$ and a time-like singularity at $r = 0$. The mass of the black hole is exactly balanced by its charge and in fact one can construct an arbitrary number of such black holes, at different points in the space-time, which are stationary and stable. One can show that the extremal Reissner-Nordström black hole is supersymmetric and the near horizon geometry is $AdS_2 \times S^2$. This will be discussed in more detail below.

The extremal RN black hole can serve as a good warm-up example for the solutions that will be the main topic of this thesis so it is worth studying it in more detail. The metric is

$$ds_4^2 = -\left(1 - \frac{G_4 M}{r}\right)^2 dt^2 + \left(1 - \frac{G_4 M}{r}\right)^{-2} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \; . \qquad (1.11)$$

At asymptotic infinity we have $r \to \infty$ and the metric approaches flat Minkowski space, $\mathbb{R}^{1,3}$. The horizon is at $r = G_4 M$ and it is instructive to study the solution near the horizon. For this purpose define a new radial coordinate[3] $\rho = r - G_4 M$, in which the metric is

$$ds_4^2 = -\frac{\rho^2}{(\rho + G_4 M)^2} dt^2 + \frac{(\rho + G_4 M)^2}{\rho^2}(d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)) \; . \qquad (1.12)$$

Now consider the near horizon limit $\rho \to 0$ and define yet another set of new coordinates $\tau = G_4^{-2} M^{-2} t$, $\eta = \rho^{-1}$. The background in this limit is

$$ds_4^2 = G_4^2 M^2 \left(\frac{-d\tau^2 + d\eta^2}{\eta^2} + d\theta^2 + \sin^2\theta d\phi^2\right) \; , \qquad (1.13)$$

[3]These coordinates are called isotropic.

$$F = \frac{q}{\eta^2} d\tau \wedge d\eta + p \sin\theta d\theta \wedge d\phi \ . \tag{1.14}$$

This is the metric on $AdS_2 \times S^2$ with constant electro-magnetic flux on it, indeed it is elementary to show that

$$\int_{S^2} F = 4\pi p \ , \qquad \int_{S^2} \star F = 4\pi q \ . \tag{1.15}$$

This is also known as the Robinson-Bertoti solution [186, 44]. The extremal Reissner-Nordström solution can be thought of as a soliton interpolating between two maximamlly symmetric spaces - four-dimensional Minkowski space-time at asymptotic infinity and $AdS_2 \times S^2$ near the horizon.

There is a simple generalization of the extremal Reissner-Nordström solution. To find it first set the magnetic charge to zero, $p = 0$, and note the g_{tt} and g_{rr} components of the metric as well as the electric gauge potential are expressed in terms of $1 + \frac{G_4 M}{\rho}$, which is a harmonic function on $\mathbb{R}^3$. This observation leads to the following general solution of Einstein-Maxwell theory in four dimensions

$$ds^2 = -\frac{1}{H^2} d\tau^2 + H^2(d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)) \ , \tag{1.16}$$

$$A = -\frac{1}{H} d\tau \ . \tag{1.17}$$

where $H(\vec{x})$ is a solution to the Poisson equation on the $\mathbb{R}^3$ spanned by

$$dx_1^2 + dx_2^2 + dx_3^2 = d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{1.18}$$

The function H for the extremal Reissner-Nordström solution satisfies the Poisson equation with a delta function source

$$\nabla^2 \left(1 + \frac{G_4 M}{\rho} \right) = -4\pi G_4 M \delta(\rho) = -4\pi \sqrt{G_4}\, q\, \delta(\rho)\ . \tag{1.19}$$

In the general solution the function $H(\vec{x})$ will satisfy the Poisson equation with some general charge density $\sigma(\vec{x})$

$$\nabla^2 H(\vec{x}) = \sigma(\vec{x})\ . \tag{1.20}$$

If the charge density is a continuos function one gets solutions of the Einstein-Maxwell equations coupled to extremal dust[4]. These solutions will not have any singularities or horizons and we will not discuss them further here. We will be interested in solutions for which the charge density is a sum of delta functions:

$$H(\vec{x}) = 1 + \sqrt{G_4} \sum_{j=1}^{N} \frac{q_j}{|\vec{x} - \vec{x}_j|}\ , \qquad \nabla^2 H(\vec{x}) = -4\pi \sqrt{G_4} \sum_{j=1}^{N} q_j \delta(\vec{x} - \vec{x}_j)\ . \tag{1.21}$$

This background is a superposition of N extremal Reissner-Nordström black holes and is known as the Majumdar-Papapetrou solution [160, 176]. In the isotropic coordinates used here the black hole horizons are located at the poles, $\vec{x} = \vec{x}_j$, of the harmonic function $H(\vec{x})$.

Extremality is quite important for the simple form of this class of solutions. It is only in the extremal limit that the repulsion induced by the electric charge can "cancel" the gravitational attraction so that the solution can remain static. If one adds any additional energy to the solution, the non-linearities of gravity become more directly manifest.

[4]Extremal dust has the property that when two grains of dust are at rest, their electrostatic repulsion is exactly sufficient to balance their gravitational attraction and they remain at rest.

Note that the source in (1.19) lies at the origin of the coordinates (1.18); i.e., at the horizon of the black hole (remember that we are in isotropic coordinates). However, since the horizon of the black hole is in fact not just a single point in space, $\vec{x} = \vec{x}_k$ is clearly a coordinate singularity. This means that although the support of the delta function lies at $\vec{x} = \vec{x}_k$, this should not be interpreted as the location of the black hole charge. Rather, the role of this delta function is to enforce a boundary condition on the electric flux emerging from the black hole so that the hole does indeed carry the proper charge.

Of course, in four dimensions, we can also have magnetically charged black holes. In fact, as for the single-center Reissner-Nordström black hole, we can have dyons, carrying both electric and magnetic charge. The corresponding extremal solutions are given directly by electro-magnetic duality rotations of the electric Majumdar-Papapetrou solutions presented above.

As we have already noted, there is a coordinate singularity at the black hole horizon. Thus, the isotropic coordinates does not allow us to see to what extent the black hole, or even the horizon, is non-singular. However, if the black hole is to have a smooth horizon, then the horizon should have non-zero (and finite) area. It is clear that this is the case for the Majumdar-Papapetrou metrics since as $\rho \to \rho_j$ we have a finite size S^2 of radius $\sqrt{G_4} q_j$. In fact as in the single center case we have an $AdS_2 \times S^2$ throat near each of the extremal black-hole horizons.

The Reissner-Nordström black hole, and its multi-center generalization, is a solution to Einstein-Maxwell theory in any space-time dimension[5] $D > 3$. There are also very similar solutions of supergravity theories in various dimensions. These solutions are not point-like but are sourced by extended objects known as D-branes (or M-branes) and may look significantly more complicated. We will discuss them in detail below but it

[5]Of course, the dyonic Reissner-Nordström solution exists only in $D = 4$.

will be useful to always keep in mind the intuition from the simple Reissner-Nordström solution in four dimensions.

1.1.2 Black-hole thermodynamics

The is a well-known theorem for black holes which states that in any physically allowed process, the total area of all black holes in the universe cannot decrease [134, 203]. This law strongly resembles the second law of thermodynamics, which states that in any physically allowed process the total entropy in the universe cannot decrease. This relation is not accidental and one can show that there is a correspondence between the laws of black hole mechanics and the laws of thermodynamics [14]. By studying quantum fields near the horizon of a black hole Hawking made this analogy precise and showed that black holes are thermal objects and posses entropy and temperature [136]. Here we will review the laws of black hole thermodynamics and will briefly discuss their physical implications.

The zeroth law of black hole mechanics states that the surface gravity, κ, on the horizon of an arbitrary stationary black hole is constant. This resembles the zeroth law of thermodynamics which states that a physical system in thermal equilibrium has constant temperature. This suggest that the surface gravity of the black hole horizon should be identified with the temperature of the black hole

The first law of black hole mechanics for a charged, rotating black hole is

$$dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ + \Phi_H dQ \; , \tag{1.22}$$

where M, Q and J are the mass, charge and angular momentum of the black hole, κ is the surface gravity of the horizon, A is the area of the horizon, Ω_H is the angular velocity of the horizon and Φ_H is the electrostaic potential at the horizon. It is clear that this relation bears a similarity with the first law of thermodynamics

$$dE = TdS + \text{work terms} \ . \tag{1.23}$$

As already stated above the second law of black hole mechanics suggests that the entropy of the black hole is proportional to its area

$$S_{BH} = \frac{A}{4G_4} \ . \tag{1.24}$$

This is the Bekenstein-Hawking formula for black-hole entropy [18, 136].

It can be shown that it is impossible to make the surface gravity of a black hole to vanish by any physical process. This is the the third law of black hole mechanics, which again indicates that the surface gravity of the black hole horizon should be related to the temperature of the black hole.

To summarize, the relations between thermodynamic and black-hole quantities are [203]:

$$E \leftrightarrow M \ , \qquad T \leftrightarrow \frac{\kappa}{2\pi} \ , \qquad S \leftrightarrow \frac{A}{4G_4} \ . \tag{1.25}$$

As an illustration we will present here some of the thermodynamic quantities of the Reissner-Nordström black hole. For more details on the calculation see [203, 57]. The temperature of the Reissner-Nordström black hole is given by

$$T_{RN} = \frac{r_+ - r_-}{4\pi r_+^2} = \frac{\sqrt{G_4^2 M^2 - G_4(q^2+p^2)}}{2\pi\left(G_4 M + \sqrt{G_4^2 M^2 - G_4(q^2+p^2)}\right)^2} \ . \tag{1.26}$$

The extremal black hole has $r_+ = r_-$ and therefore $T = 0$. The entropy of the Reissner-Nordström black hole is (A is the area of the outer horizon)

$$S_{RN} = \frac{A}{4G_4} = \frac{\pi r_+^2}{G_4} = \frac{\pi \left(G_4 M + \sqrt{G_4^2 M^2 - G_4(q^2+p^2)} \right)^2}{G_4} . \tag{1.27}$$

It is clear that the extremal Reissner-Nordström black hole has finite entropy

$$S_{ERN} = \pi(q^2 + p^2) , \tag{1.28}$$

despite it being at zero temperature. For $p = q = 0$ we have the Schwarzschild black hole with temperature and entropy given by

$$T_{Sch} = \frac{1}{8\pi G_4 M} , \qquad S_{Sch} = 4\pi G_4 M^2 . \tag{1.29}$$

The underlying physical reason why the entropy of a black hole is proportional to its area is unclear in general relativity. The entropy of a conventional physical system is proportional to the logarithm of the number of microscopic states with a given set of macroscopic parameters. Therefore a statistical interpretation of the black hole entropy will imply that in a full quantum theory of gravity the number of microscopic states with the same conserved charges as a given black hole should be order

$$N_{micro} \sim e^{A_{BH}} . \tag{1.30}$$

The black hole uniqueness theorems in general relativity are at odds with this relation, which is one more clue that one needs a quantum theory of gravity to understand the entropy of black holes. String theory is a theory of quantum gravity and as such should contain a microscopic explanation of the entropy of black holes. In the last fifteen years we have gained some detailed understanding of what the microstates of a large class of black holes in string theory are and how to reproduce the Bekenstein-Hawking entropy by counting them. However there are still many unsolved problems and this thesis is devoted to some of them.

1.2 Supergravity, String Theory and D-branes

String theory grew out of attempts to construct a fundamental theory of the strong nuclear interactions. It turned out that non-Abelian gauge theories are better suited for this task but nevertheless string theory has had a life of its own since then (and recently via the gauge/gravity duality has taught us some important lessons about nuclear physics). It was realized that there are massless spin two excitations in the spectrum of the closed string which can be interpreted as gravitons. Thus string theory is a theory of quantum gravity. There are also low energy excitations in string theory which resemble the standard model of particle physics. This has lead to a great research effort to construct a unified theory of all fundamental interaction based on string theory, see [125, 126, 179, 180, 16] for comprehensive reviews on string theory. As a quantum theory of gravity string theory has also one more challenging task - to explain and resolve the puzzles of black hole physics.

By now it is well established that there are five consistent string theories in ten dimensions, all of which involve supersymmetry - type I string theory with gauge group $SO(32)$, two heterotic string theories with gauge groups $E_8 \times E_8$ and $SO(32)$ and type IIA and type IIB string theories. These theories were first constructed perturbatively but it was later revealed that there are various string dualities which relate the five consistent string theories. There is also a conjectured non-perturbative theory in eleven-dimensions known as M-theory which is believed to be related to the five string theories by a web of dualities [141, 206].

All five string theories have a low energy excitation spectrum which yields supergravity theories - i.e. supersymmetric extensions of general relativity in ten-dimensions. There are two supergravity theories with maximal supersymmetry in ten dimensions - type IIA and type IIB supergravity. As the names suggest they can be obtained by restricting to the low energy modes of the type IIA and IIB superstring theory. There is also a unique supergravity theory in eleven dimensions which is conjectured to be the low energy limit of M-theory. We will use these supergravity theories and their compactifications to lower dimensions throughout this thesis and we will review them briefly in Sections 1.2.2 and 1.2.3.

In type IIA and IIB string theory as well as in M-theory there are massive objects called branes which can form black holes in the supergravity limit. When supersymmetry is preserved these black holes can be thought of as ground states of the theory and can use different tools to study them. This has lead to a successful counting of the microstates of a large class of black holes in string theory which we will review in Section 1.2.4.

1.2.1 BPS states

Most of the supergravity solutions studied in this thesis are supersymmetric and saturate a Bogomol'nyi-Prasad-Sommerfield (BPS) bound, so it is useful to briefly review this here. The concept of a BPS bound and its saturation can be illustrated by massive particles in four dimensions [182, 47, 207]. The $\mathcal{N}$ extended supersymmetry algebra restricted to particles of mass $M > 0$ at rest in four dimensions is

$$\{Q^I_\alpha, Q^{\dagger J}_\beta\} = 2M\delta^{IJ}\delta_{\alpha\beta} + 2iZ^{IJ}\Gamma^0_{\alpha\beta}\,. \tag{1.31}$$

Here Q^I_α are the fermionic supercharges, $I, J = 1, \ldots \mathcal{N}$ label the supersymmetries and $\alpha, \beta = 1, \ldots, 4$ are Majorana spinor indices of the supercharges. The matrix Z^{IJ} is antisymmetric and is called the central charge matrix, clearly it is non-vanishing only for $\mathcal{N} \geq 2$. The central charges are conserved quantities that commute with all other generators of the algebra. They can be also thought of as electric and magnetic charges that couple to the gauge fields in the supergravity multiplet. For simplicity we will concentrate on the case $\mathcal{N} = 2$. It is clear from (1.31) that the matrix

$$\begin{pmatrix} M & Z \\ Z^\dagger & M \end{pmatrix} , \tag{1.32}$$

should be positive definite, which implies that

$$M \geq |Z| \, . \tag{1.33}$$

States that have $M = |Z|$ are said to saturate the BPS bound. The similarity with the extremal Reissner-Nordström black hole is clear. There we had, after setting $G_4 = 1$, $M = q$. Indeed the similarity can be made precise and one can show that the Reissner-Nordström black hole is a BPS state of $\mathcal{N} = 2$ minimal supergravity in four dimensions [111].

In the case of maximal, $\mathcal{N} = 8$, supersymmetry in four dimensions there are four eigenvalues of the matrix Z^{IJ} and one has four different possible BPS states:

- $M = |Z_1| = |Z_2| = |Z_3| = |Z_4|$

  This state is called half-BPS and preserves 16 of the 32 real supercharges.

- $M = |Z_1| = |Z_2| = |Z_3| > |Z_4|$

  This state is called quarter-BPS and preserves 8 of the 32 real supercharges.

- $M = |Z_1| = |Z_2| > |Z_3| > |Z_4|$

  This state is called eighth-BPS and preserves 4 of the 32 real supercharges.

- $M = |Z_1| > |Z_2| > |Z_3| > |Z_4|$

  This state is called sixteenth-BPS and preserves 2 of the 32 real supercharges.

For a large portion of this thesis we will discuss eighth-BPS states in four, five, ten and eleven-dimensional supergravity. There are black holes with macroscopic horizons which are eighth-BPS and can be thought of as generalizations of the extremal Reissner-Nordström black hole discussed in some detail in Section 1.1. Black holes with more than four supercharges usually do not have macroscopic horizons and look like naked singularities which should be resolved by higher order supergravity or string theory effects.

1.2.2 Eleven-dimensional supergravity

Eleven-dimensional supergravity is the unique supergravity in eleven dimensions and is therefore of fundamental nature [67]. The field content of eleven-dimensional supergravity is relatively simple. There is the graviton, which is a symmetric traceless tensor of the little group in eleven dimensions - $SO(9)$. It has 44 independent physical degrees of freedom which are encoded in the metric $g_{\mu\nu}$ (or the vielbein e^a_μ). There is also a fermionic partner of the graviton - the gravitino. This is a 32-component Majorana spinor which also transforms as a space-time vector, ψ^α_μ. The gravitino transforms as a vector of $SO(9)$ and as a 16-component real spinor of $Spin(9)$. Naively there are $9 \times 16 = 144$ physical degrees of freedom in the gravitino, however there is a local gauge symmetry $\psi^\alpha_\mu \to \psi^\alpha_\mu + \partial_\mu \chi^\alpha$, where χ^α is an arbitrary spinor. This reduces the number of physical degrees of freedom in the gravitino to $144 - 16 = 128$. In order for the eleven-dimensional theory to be supersymmetric there should be an equal number of fermionic and bosonic degrees of freedom. The extra bosonic degrees of freedom come from a 3-form field $A_{(3)}$. The theory has the usual gauge invariance $A_{(3)} \to A_{(3)} + d\Lambda_{(2)}$ for an arbitrary 2-form $\Lambda_{(2)}$. The 3-form is a massless completely antisymmetric field and thus has 84 degrees of freedom, which adds up to a total of 128 bosonic degrees of freedom as required by supersymmetry.

The bosonic part of the action of eleven-dimensional supergravity is [16]

$$S_{11} = \frac{1}{2\kappa_{11}^2} \int d^{11}x \left(R - \frac{1}{48} F_{\mu_1\mu_2\mu_3\mu_4} F^{\mu_1\mu_2\mu_3\mu_4} \right) - \frac{1}{12\kappa_{11}^2} \int A_{(3)} \wedge F_{(4)} \wedge F_{(4)} \ , \tag{1.34}$$

where

$$2\kappa_{11}^2 = 16\pi G_{11} = \frac{(2\pi l_p)^9}{2\pi} \ , \qquad F_{(4)} = dA_{(3)} \ , \tag{1.35}$$

and G_{11} and l_p are the eleven-dimensional Newton constant and Planck length. The full action of eleven-dimensional supergravity is invariant under local supersymmetry transformations. These transformations are parametrized by a 32-component Majorana spinor, ϵ, and the infinitesimal transformations of the fields are[6]

$$\begin{aligned} \delta e_\mu^a &= \bar{\epsilon}\Gamma^a \psi_\mu \ , \\ \delta A_{\mu\nu\rho} &= -3\bar{\epsilon}\Gamma_{[\mu\nu}\psi_{\rho]} \ , \\ \delta\psi_\mu &= \nabla_\mu \epsilon + \frac{1}{12}\left(\frac{1}{4!}\Gamma_\mu F_{\nu\rho\sigma\lambda}\Gamma^{\nu\rho\sigma\lambda} - \frac{1}{2} F_{\mu\rho\sigma\lambda}\Gamma^{\rho\sigma\lambda} \right)\epsilon \ . \end{aligned} \tag{1.36}$$

Here Greek letters denote space-time (“curved”) indices and Latin letters denote tangent space (“flat”) indices. We have also used the standard notation

$$\Gamma_\mu = e_\mu^a \Gamma_a \ , \qquad \Gamma^{\mu_1\mu_2\ldots\mu_n} = \Gamma^{[\mu_1}\Gamma^{\mu_2}\ldots\Gamma^{\mu_n]} \ , \tag{1.37}$$

[6]The spinor index of fermions will be suppressed from now on.

and the covariant derivative on spinors is given by

$$\nabla_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4}\omega_{\mu ab}\Gamma^{ab}\epsilon \ . \tag{1.38}$$

Given a supersymmetric solution, there exist spinors, called Kiling spinors, that characterize the supersymmetries of the solutions. These are similar to the Killing vectors which characterize bosonic symmetries of a gravitational background. Killing spinors are spinors that parametrize infinitesimal supersymmetry transformations under which the fields are invariant for a specific field configuration. Since the supersymmetry variations of the bosonic fields always contain one or more fermionic fields, which vanish classically, these variations are guaranteed to vanish. Thus in exploring supersymmetry of bosonic solutions one has to look only at the variations of the fermionic fields. In other words to have a supersymmetric bosonic background we need a solution to the following equations

$$\delta\psi_\mu = \nabla_\mu \epsilon + \frac{1}{12}\left(\frac{1}{4!}\Gamma_\mu F_{\nu\rho\sigma\lambda}\Gamma^{\nu\rho\sigma\lambda} - \frac{1}{2}F_{\mu\rho\sigma\lambda}\Gamma^{\rho\sigma\lambda}\right)\epsilon = 0 \tag{1.39}$$

Solutions to this equations, $\epsilon(x)$, are the Killing spinors.

The presence of a three-form gauge potential, $A_{(3)}$ in eleven-dimensional supergrvaity suggests the existence of a fundamental two-dimensional extended object present in the theory. Indeed this turns out to be the case and this object is called M2 brane [42]. The M2 brane has a magnetic dual M5 brane which sources the six-form gauge potential Hodge dual to $A_{(3)}$. Unlike the situation in string theory, the fundamental theory living on the world-volume of these membranes is still somewhat mysterious[7]. Nevertheless it is known that the M2 and M5 branes are massive and can backreact on the eleven-dimensional space-time to produce solutions of supergravity. We will discuss

[7] See [8, 3] for some recent progress in this direction.

extensively a large class of such solution in Chapter 2. For illustration here we will just present the supergravity solution sourced by N M2 branes.

The M2 brane solution is [81]

$$\begin{aligned} ds^2_{M2} &= H^{-2/3}(r)(-dt^2 + dx_1^2 + dx_2^2) + H^{1/3}(r)(dr^2 + r^2 d\Omega_7^2) \ , \\ A_{(3)} &= -H^{-1}(r) dt \wedge dx_1 \wedge dx_2 \ , \end{aligned} \tag{1.40}$$

where $d\Omega_7^2$ is the unit radius S^7 and $H(r)$ is a harmonic function on the $\mathbb{R}^8$ transverse to the worldvolume of the M2 brane

$$H(r) = 1 + \frac{32\pi^2 l_p^6 N}{r^6} \ . \tag{1.41}$$

The similarity between this solution and the extremal Reissner-Nordström solution in four dimensions is obvious. One can easily study the near horizon limit of the M2 brane and show that the resulting space is $AdS_4 \times S^7$ with N units of flux. This is again similar to the extremal Reissner-Nordström solution where the near horizon limit is $AdS_2 \times S^2$.

One can also similarly construct the solution for a stack of M5 branes [133]. It is again determined by a single harmonic function on the space transverse to the branes. Not surprisingly its near horizon limit is $AdS_7 \times S^4$.

1.2.3 Type IIA and Type IIB supergravity

Eleven-dimensional supergravity is related to the various ten-dimensioal supergravities which are the low-energy limits of superstring theories. The most direct connection is between eleven-dimension supergravity and type IIA supergravity [108, 53]. Type IIA supergravity can be constructed by a dimensional reduction of eleven-dimensional supergravity on a circle. This is the low-energy version of the statement that M-theory compactified on a circle of radius R corresponds to type IIA superstring theory in ten dimensions with string coupling $g_s = R\alpha'^{-1/2}$, where $\alpha' = \frac{1}{2\pi T}$ and T is the string tension. The details of how this reduction is done (at least for the bosonic sector of the theory) are discussed in Appendix A. Here we will only describe the field content of the theory and will refrain from presenting its action and supersymmetry transformations.

By reducing along a circle to ten dimensions the eleven-dimensional metric decomposes into a ten-dimenisonal metric $g^{(10)}_{\mu\nu}$, a $U(1)$ gauge field described by the 1-form $C_{(1)}$ and a scalar Φ called the dilaton. The three-form in eleven dimensions reduces to a 2-form, $B_{(2)}$, and a 3-form, $C_{(3)}$, in ten dimensions. There are again 128 bosonic degrees of freedom - 35 in $g^{(10)}_{\mu\nu}$, 8 in $C_{(1)}$, 1 in Φ, 28 in $B_{(2)}$ and 56 in $C_{(3)}$. The first ten components of the eleven-dimensional gravitino become two ten-dimensional Majorana-Weyl gravitini of opposite chirality, $\psi^{(1)}_{\mu}$ and $\psi^{(2)}_{\mu}$. Each of the two ten-dimensional gravitini has 56 physical degrees of freedom. There are also two ten-dimensional Majorana-Weyl spinors, the dialtini, which are the fermionic partners of the dilaton and each of them has 8 degrees of freedom. It is clear that by reducing on a circle we have kept the same number of degrees of freedom in the ten dimensional theory as we had in eleven dimensions. This is a general feature of dimensional reduction on flat manifolds.

One can keep on doing the same exercise and reduce type IIA supergravity further to lower dimensions by compactifying on circles or torii and keeping only the massless degrees of freedom. This is a well established procedure which leads to maximally supersymmetric supergravity theories in various dimensions. For example, if we reduce eleven-dimensional supergravity on T^7 and keep all massless degrees of freedom we will get the maximally supersymmetric, $\mathcal{N} = 8$ supergravity in four dimensions. In general when one reduces eleven dimensional supergravity on T^D there is a global $E_{D(D)}$ internal symmetry which is called the U-duality group[8].

There is another supergravity theory in ten dimensions with maximal supersymmetry - type IIB supergravity [190, 140]. Its field content can be deduced from the massless spectrum of type IIB string theory. It can also be constructed as a ten dimensional supergravity theory using guidance from the supersymmetric representation theory, as well as gauge and diffeomorphism invariance. Unlike type IIA supergravity it cannot be obtained from a reduction of eleven-dimensional supergravity. There is however a string theory duality, T-duality, which relates type IIA and Type IIB superstring theory and thus IIA and IIB supergravity. Some details of how this duality works will be discussed in Appendix A. The field content of the theory is - the metric $g_{\mu\nu}$, the 2-form $B_{(2)}$, the dilaton Φ, the axion $C_{(0)}$, a 2-form $C_{(2)}$ and a 4-form $C_{(4)}$. The field strength of $C_{(4)}$ should be self-dual under Hodge duality. It is notoriously difficult to construct a ten-dimensional supergravity action that incorporates this self-duality and it is usually imposed by hand as an addition to the equations of motion [190, 140]. Type IIB supergravity has an $SL(2,\mathbb{R})$ global symmetry[9] under which $C_{(4)}$ and the metric are invariant. The dilaton and axion $(\Phi, C_{(0)})$ and the 2-forms $(B_{(2)}, C_{(2)})$ transform as doublets. The theory has two ten-dimensional Majorana-Weyl gravitini of the same chirality as well as two dilatini. There are again 128 bosonic and 128 fermionic degrees of freedom.

The bosonic fields $(g_{\mu\nu}, \Phi, B_{(2)})$ are common to type IIA and IIB supergravity. These are actually the massless bosonic excitations in the NS-NS sector of the closed

[8]Strictly speaking string/M- theory will break this symmetry to a discrete subgroup of $E_{D(D)}$ and thus the U-duality group is discrete.

[9]This is broken down to $SL(2,\mathbb{Z})$ in the full string theory.

type IIA and IIB string theory. The R-R sector of the two theories is different which explains the different dimensions of the forms, $C_{(n)}$, in the two ten-dimensional supergravities.

As shown in [71] type II open superstrings can have Dirichlet boundary conditions and can end on extdended objects in the ten-dimensional space time. These objects are called D-branes. It was later shown that in fact these D-branes are sources for the various R-R gauge potentials, $C_{(n)}$, appearing in IIA and IIB string theory and the corresponding supergravities [178]. See [145] for a review of the physics of D-branes.

In type IIB theory we have D1, D3, D5, D7 and D9 branes. The D1 and D5 branes couple to the $C_{(2)}$ and it dual $C_{(6)}$ gauge potentials and are related by electric-magnetic duality. The D3 branes couple to $C_{(4)}$ and are self-dual. The D7 and the space-time filling D9 branes are somewhat exotic and we will not discuss them here, see [179, 180, 145] for more details.

In type IIA theory we have D0, D2, D4, D6 and D8 branes. The D0 and D6 branes couple to the $C_{(1)}$ and it dual $C_{(7)}$ gauge potentials and are related by electric-magnetic duality. The D2 and D4 branes couple to the $C_{(3)}$ and its dual $C_{(5)}$ gauge potentials and are also related by electric-magnetic duality. The D8 branes couple to $C_{(9)}$ which has a ten-form field strength, dual to a non-dynamical scalar known as the Romans mass [187, 178].

There are two more types of extended objects in string theory - the fundamental string, F1, and the NS5 brane which is the magnetic dual object to the fundamental string. The fundamental string couples electrically to the two-form potential $B_{(2)}$ and the NS5 brane couple magnetically to $B_{(2)}$.

All D-branes and the NS5 brane are massive non-perturbative objects in string theory[10]. They source non-trivial supergravity solutions, akin to the Reissner-Nordström black hole and the M2 brane solution of eleven-dimensional supergravity [100, 138]. Many of these D-brane supergravity solutions are related by a combination of dualities and dimensional reduction to the M2 and M5 brane solutions in eleven-dimensional supergravity. We will use these relations throughout this thesis and will explain them along the way.

1.2.4 Black hole entropy from string theory

One of the great success of string theory, which came out of studying the physics of the non-perturbative D-branes, is the microscopic counting of the entropy of a certain class of black holes. Strominger and Vafa counted the ground state degeneracy of a certain bound state of D1-D5 branes and momentum [198]. At strong gravitational coupling this configuration is a five-dimensional BPS black hole. The calculation was done by considering type IIB string theory on $K3 \times S^1$, where the D5-branes wrap the whole compact manifold[11] and the D1 branes are wrapped along the S^1. The original counting of Strominger-Vafa uses some advanced technology involving a non-linear sigma model and the cohomology of the target space manifold of this sigma model. We will not review the calculation here but rather present a short overview of a more intuitive (and less rigorous) approach followed by Callan and Maldacena [52], see also [161].

The configuration studied in [52] is a bound state of D5 and D1 branes in IIB string theory. The D5 branes wrap a T^5, the D1 branes are bound to the world-volume of the D5 branes and wrap an S^1 in T^5. The other five space-time dimensions are simply

[10] See Appendix D for our conventions about the tensions of the extended objects in string and M-theory.

[11] $K3$ is the unique non-trivial compact Calabi-Yau manifold in two complex dimensions.

five-dimensional Minkowski space. In addition to the D-branes there is also an ensemble of open strings with one end attached to the D1 and the other attached to the D5 branes[12], these strings move along the S^1 and the solution carries a large momentum in this internal direction.

The gravitational backreaction of this configuration leads to the following metric:

$$ds^2_{IIB} = \frac{1}{\sqrt{H_1 H_5}}(-dt^2+dx_5^2+(H_p-1)(dt-dx_5)^2)+\sqrt{H_1 H_5}(dx_1^2+dx_2^2+dx_3^2+dx_4^2) + \sqrt{\frac{H_1}{H_5}}(dx_6^2+dx_7^2+dx_8^2+dx_9^2) \; , \quad (1.42)$$

where the $T^4\times S^1$ is spanned by $(x_5, x_6, x_7, x_8, x_9)$, the functions (H_1, H_5, H_p) are three harmonic functions on the transverse space to the branes spanned by (x_1, x_2, x_3, x_4)

$$H_1 = 1 + \frac{Q_1}{r^2} \; , \qquad H_5 = 1 + \frac{Q_5}{r^2} \; , \qquad H_p = 1 + \frac{Q_p}{r^2} \; , \quad (1.43)$$

and $r^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$. The solution has also a non-trivial dilaton and R-R two-form field but we will not present them here[13]. Upon dimensional reduction on the $T^4 \times S^1$ one obtains a BPS black hole with three charges and four supercharges (i.e. an eighth-BPS state) which is very similar to the extremal Reissner-Nordström black hole:

$$ds_5^2 = -\frac{1}{(H_1 H_5 H_p)^{2/3}}dt^2 + (H_1 H_5 H_p)^{1/3}(dx_1^2+dx_2^2+dx_3^2+dx_4^2) \; . \quad (1.44)$$

[12]Open strings stretching between a Dm and a Dn brane are called (m,n) strings.

[13]See Chapter 3 and Appendix B for the full D1-D5-P type IIB supergravity solution. The solution presented there is more general and is in slightly different notation.

The three charges of the black hole correspond to the D1 and D5 brane charges, Q_1 and Q_5, and the momentum charge Q_p. One can easily calculate the horizon area of this black hole and find its Bekenstein-Hawking entropy

$$S_{BH} = \frac{A}{4G_5} = 2\pi\sqrt{Q_1 Q_5 Q_p}\,. \tag{1.45}$$

Now, following [52], we will reproduce this entropy by counting the ground state degeneracy of the D1-D5-P system. We will assume that the string coupling (and therefore the gravitational coupling) is small so that one can use perturbative string theory. One can show that the massless supersymmetric excitations of the systems are described by the $(1,5)$ and $(5,1)$ strings. One can describe these excitations effectively as a two-dimensional conformal field theory (CFT) living on the cylinder (t, x_5). One can also argue that for each string momentum mode one has $4Q_1Q_5$ bosons and $4Q_1Q_5$ fermions in the CFT. The total central charge of this system is

$$c = 1 \times 4Q_1Q_5 + \frac{1}{2} \times 4Q_1Q_5 = 6Q_1Q_5\,. \tag{1.46}$$

The degeneracy for level $Q_p >> 1$ states in a CFT of central charge, c, is given by the Cardy formula [55]

$$d(Q_p) \sim e^{2\pi\sqrt{\frac{cQ_p}{6}}}\,. \tag{1.47}$$

Taking the logarithm of this and using (1.46) we find the entropy of the system to be

$$S_{D1-D5-P} = \log(d(Q_p)) = 2\pi\sqrt{Q_1Q_5Q_p}\,, \tag{1.48}$$

which precisely matches the Bekenstein-Hawking entropy of the black hole (1.45). This counting was also done for the near extremal D1-D5-P black hole [52] as well as for more general black holes [199].

It is remarkable that one can reproduce the Bekenstein-Hawking entropy of a black hole by counting bound states of strings and branes. However one should remember that this is just counting of the microstates for large charges of the black hole and it was done at weak string (and gravitational) coupling. When one varies the string coupling one will interpolate between a configuration of strings and branes described by field theory and a black hole described by supergravity. There is a general argument that during this process the number of microstates should stay the same [139] (up to coefficient of order one), this argument applies to supersymmetric and non-supersymmetric black holes in string theory and is one of the explanataions of the success of the calculations in [198, 52, 144, 199]. Another reason for the correct entropy counting is probably the universality of the Cardy formula and the presence of an AdS near horizon region in the class of black hole constructed in string theory.

Despite the fact that we can count the microstates of the black hole in string theory we still lack information about their structure at strong gravitational coupling, the details of the resolution of black hole singularities and the information paradox. In the next Chapters we will present some recent advances in our understanding of these issues in black hole physics.

Chapter 2

Three-charge solutions and microstate geometries

The counting of black hole microstates which matches with the Bekenstein-Hawking entropy is a remarkable success of string theory as a theory of quantum gravity. However we still do not understand what is the structure of the black hole microstates at strong gravitational coupling and how this relates to the possible resolution of black hole singularities and the formation of horizons. It is usually thought that quantum gravity will become important at distance of the order of the Planck length away from the black hole singularity. Recent studies, however, suggest that quantum gravity effects do not stay confined to the region near the black hole singularity and can indeed extend to macroscopic distances all the way to the black hole horizon. This implies some important consequences for the physics of black holes in string theory and this Chapter is devoted to a review of some of the recent advances in the subject.

The core question is given a set of boundary conditions at asymptotic infinity for a supergravity black hole what are the possible solutions of string or M-theory that will fit these boundary conditions. We will call such solutions, with some abuse of terminology, microstate geometries. One of the surprising advances in the last few years is that there is a vast number of such solutions within supergravity which are completely regular and without horizons. It still remains to be seen whether these solutions can provide a semiclassical accounting of the entropy.

Independent of the microscopic counting issue, the study of microstate geometries is interesting for other reasons. It is important to understand the failure of black hole uniqueness in string theory and to classify all possible solutions with the same asymptotics as a given black hole. Given that there are vast numbers of regular solutions without horizons that are within the validity of supergravity and have the same structure at infinity as a black hole one should understand their implication for black-hole physics. It seems that within string theory the black hole singularity is an artifact of the symmetry and the microstate geometries discussed here will resolve the singularity.

It is important to emphasize that the program of constructing all microstate geometries for a black hole with finite entropy and temperature is not completed. One can separate the solution of the big problem in three smaller problems:

There should be solutions of supergravity (or string theory) which resolve the black hole singularity and are completely smooth and regular. The solutions discussed in this Chapter achieve precisely this for a large class of supersymmetric black holes and black rings. A crucial ingredient in the singularity resolution is the geometric transition mechanism. This mechanism is common to many systems in string theory and is intuitively simple to understand. There are certain branes wrapping a topologically trivial cycle, there is also a dual, Gaussian, cycle that "measures" the charge of the branes. The branes have tension and they collapse the cycle which they wrap and the geometry looks singular. The resolution that string theory provides is a new geometry with two topologically non-trivial cycles - one corresponding to the cycle that the branes wrapp and the other to the Gaussian cycle. We explain the geometric transition mechanism in more detail in Section 2.2.9, but it is important to note that for the black-hole singularity resolution the geometric transition happens on the non-compact space-time. This is crucial for our construction and is a novel feature, different from all other brane systems which undergo geometric transition [124, 202, 51].

Another problem is to show that the set of microstate geometries which resolve the black hole singularity account for a macroscopic part of the entropy of the black hole. In other words one needs to show that the moduli space of regular supergravity solutions is large enough and upon quantization it can store the vast entropy of a black hole. For the one- and two-charge BPS systems in string theory this has been already done and is reviewed in Section 2.1. However these black holes have Planck scale horizons and do not have finite entropy in supergravity. The simplest system with macorscopic horizon for which one can construct microstate geometries is the three-charge system, this system is supersymmetric which somewhat simplifies the construction of supergravity solutions. There are BPS black holes and black rings with three charges which have finite entropy and one can construct vast numbers of regular solutions with the same charges and supersymmetry which we review in Section 2.2. It is still unclear whether these solutions are enough to account for the entropy of the black hole and black ring but in Chapters 3 and 4 we present some strong arguments in favor of this proposal.

One should remember that BPS black holes have vanishing temperature and thus do not emit Hawking radiation. Ideally one would like to construct many non-BPS regular supergravity solutions with the same charges and asymptotics as a non-BPS black hole (or black ring). Then it should be possible to study the details of the Hawking radiation of the black hole (or black ring) using these microstate geometries. One possible mechanism is that the regular microstate geometries are unstable and their decay rate matches with the rate of Hawking emission from the corresponding black hole. Very little is known about regular non-BPS solutions so it is still unclear whether this mechanism will work, see [63, 64, 65, 6, 7] for some recent work on this problem. In Chapter 6 we will describe a way to overcome the technical difficulties associated with the construction of non-BPS microstate geometries and will present one of the first examples of such a regular non-BPS solution. Our results suggest that the large number of microstate geometries and geometric transition mechanism for singularity resolution established for the BPS black holes and black rings may as well be features of their non-BPS black counterparts.

2.1 Systems with microscopic horizons

To address some of the questions in black hole physics Mathur and collaborators considered a simple two-charge system in string theory - a bound state of D1 and D5 branes [155, 156, 157]. Two-charge black holes in string theory appear singular and do not have macroscopic horizons, thus their entropy vanishes in the classical (supergravity) approximation. However one can consider higher derivative corrections to the Einstein-Hilbert action and show that these systems in fact develop effective horizons with radius of the order of the Planck scale (called also "stretched horizons") [192]. The entropy of the two-charge system is

$$S = 2\pi\sqrt{2}\sqrt{Q_1 Q_2 - J} \tag{2.1}$$

where Q_1 and Q_2 are the two conserved charges and J is the angular momentum. As shown by Mathur and collaborators (see [168] for a review) this entropy can be accounted for by counting regular supergravity solutions specified by an arbitrary closed curve in $\mathbb{R}^4$. These solutions can be thought of as U-duals to another well-known two charge system in string theory - supertubes. Supertubes are tubular D2 branes with worldvolume D0 and F1 fluxes which are responsible for the two electric charges [166]. Although the two-charge result is quite appealing it applies only to black holes with effective, Planck scale, horizons.

It is natural to conjecture that a similar picture will arise for the three-charge case, namely that there are enough regular, classical supergravity solutions which will account for (a significant part of) the black hole entropy. This is often referred to as the "fuzzball proposal." If this conjecture is true one will have a resolution to the information paradox - the horizon is just a coarse grained description of individual regular microstate geometries and thus all scattering processes are unitary; and will have an explanation of the entropy of black holes as statistical average over microstate geometries. To prove (or disprove) this conjecture one has to find large classes of regular, three charge, BPS solutions in five dimensions, which have the same supersymmetries, asymptotic structure and macroscopic charges as BPS black obejcts[1].

Another system for which all BPS supergravity solutions with the same asymptotic charges as a given black hole are constructed is the BPS D3 brane in IIB supergrvaity [153]. The Lin-Lunin-Maldacena solutions are half-BPS (i.e. preserving 16 of the 32 supercharges) have the same asymptotic structure as the D3 brane background and have non-trivial topology in the bulk. The system does not have a macroscopic horizon due the the large number of supersymmetry preserved by the solution. Nevertheless one can map each regular geometry to a half-BPS state of the dual $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) and on has a complete description of the half-BPS states of $\mathcal{N} = 4$ SYM in terms of supergravity solutions. It is still unclear whether one can have a similar description of all eighth-BPS states of the D1-D5-P CFT, dual to the three-charge black hole, in terms of regular supergravity solutions but there is clear evidence that this is the case for a large part of these states. An interesting technical point is that the Lin-Lunin-Maldacena solutions are determined by solving a Laplace equation on a certain subspace of the ten-dimensional space-time. We will see a similar structure emerging in the three-charge, eighth-BPS solutions discussed below.

It is remarkable that one can construct all supergravity solutions with the same charges and asymptotic structure as in the half-BPS and quarter-BPS systems discussed

[1]In five dimensions apart from black holes, with S^3 horizon topology, there are black rings with $S^1 \times S^2$ topology of the horizon [92, 93].

above. The crucial difference between the half- and quarter-BPS systems and the eighth-BPS system is that the latter generically has a macroscopic horizon, whereas the former have no horizon or only have an effective horizon at the Planck or string scale. Indeed, historically, the link between microstate counting and Bekenstein-Hawking entropy (at vanishing string coupling) was first investigated by Sen [192] for the two-charge system. While this work was extremely interesting and suggestive, the result became compelling only when the problem was later solved for the three-charge system by Strominger and Vafa [198]. Similarly, the work on the microstate geometries of the one- and two-charge systems is extremely interesting and suggestive, but to be absolutely compelling, it must be extended to the three-charge problem. This would amount to establishing that the boundary D1-D5-P CFT microstates are dual to bulk microstates – configurations that have no horizons or singularities, and which look like a black hole from a large distance, but start differing significantly from the black hole solution at the location of the would-be horizon. String theory would then indicate that a black hole solution should not be viewed as a fundamental object in quantum gravity, but rather as an effective thermodynamic description of an ensemble of horizonless configurations with the same macroscopic charges and asymptotic properties [168]. The black hole horizon would be the place where these configurations start differing from each other, and the classical thermodynamic description of the physics via the black hole geometry stops making sense.

2.2 The three-charge system

Two-charge supertubes in the probe limit are described by the Dirac-Born-Infeld (DBI) action. In a similar way one can study supertubes with three charges and one or two dipole charges using the DBI action [19]. This provides useful intuition about the three-charge system and the exact supergravity solutions one can expect to find.

The radial size of a three-charge supertube with angular momentum J and all three charges of the same order, $Q_1 \approx Q_2 \approx Q_3 \approx Q$ is

$$r_{\rm ST}^2 \ \sim \ g_s \, \frac{J^2}{Q^2} \,, \tag{2.2}$$

where g_s is the string coupling constant[2]. One can also compute the proper length of the circumference of the horizon of the three-charge spinning black hole (sometimes called the BMPV black hole[49]) to be

$$r_{\rm BH}^2 \ \sim \ g_s \, \frac{Q^3 - J^2}{Q^2} \ . \tag{2.3}$$

The most important aspect of the equations (2.2) and (2.3) is that for comparable charges and angular momenta, the black hole and the three-charge supertube have comparable sizes. Moreover, these sizes grow with g_s in the same way. This is a somewhat counter-intuitive behavior, most massive physical objects one can think about tend to become smaller when gravity is made stronger and this is consistent with the fact that gravity is an attractive force. The only well-known object that becomes larger with stronger gravity is a black hole. Nevertheless, three-charge supertubes also become larger as gravity becomes stronger. The size of a tube is determined by a balance between the angular momentum of the system and the tension of the tubular brane. As the string coupling is increased, the D-brane tension decreases, and thus the size of the tube grows, at exactly the same rate as the Schwarzschild radius of the black hole[3]. This is the distinguishing

[2]See Appendix D for more details, and [30] for a review of these calculations.

[3]Note that this is a feature only of three-charge supertubes; ordinary (two-charge) supertubes have a growth that is duality-frame dependent.

feature that makes the three-charge supertubes, as well as the smooth geometries that we will obtain from their geometric transitions, unlike any other configuration that one counts in studying black hole entropy.

To be more precise, let us consider the Strominger-Vafa counting of states that leads to the correct black hole entropy. One counts microscopic configurations of branes and strings at weak coupling where the system's size is of string scale, and its Schwarzschild radius is even smaller. One then imagines increasing the gravitational coupling; the Schwarzschild radius grows, becoming comparable to the size of the brane configuration at the "correspondence point" [139], and larger thereafter. When the Schwarzschild radius is much larger than the Planck scale, the system can be described as a black hole. There are thus two very different descriptions of the system: as a microscopic string theory configuration for small g_s, and as a black hole for large g_s. One then compares the entropy in the two regimes and as we summarized in Section 1.2.4 one finds agreement between the two calculations.

Three-charge supertubes behave differently. Their size grows at the same rate as the Schwarzschild radius, and thus they have no "correspondence point." Their supergravity description is valid in the same regime as the description of the black hole. If by counting such configurations one could reproduce the entropy of the black hole, then one should think about the supertubes as the large g_s continuation of the microstates counted at small g_s in the string/brane picture, and therefore as the microstates of the corresponding black hole.

It is interesting to note that if the supertubes did not grow with exactly the same rate with g_s as the size of the black hole horizon, they would not be good candidates for being black hole microstates, and Mathur's conjecture would not seem very plausible. The fact that there exists a huge number of configurations that do have the same growth with g_s as the black hole is a non-trivial confirmation that these configurations may well represent black-hole microstates for the three-charge system.

The existence of supersymmetric three-charge supertubes with arbitrary shape suggests the existence of a large number of regular supergravity solutions build out of the same branes that source the supertube. In addition to that, three-charge supertubes point to the existence of supersymmetric black rings. As we will review in this Chapter there are indeed supersymmetric black rings with three electric and threep dipole charges as well as a large class of regular three-charge supergravity solutions.

In the following Sections we will summarize the construction of asymptotically flat supersymmetric solutions of M-theory on T^6 and the physics of these solutions. Most of the material presented in this Chapter is well known, see [168, 193, 13, 66] for some reviews . The presentation here is based on [30].

2.2.1 Supersymmetric configurations

We will consider brane configurations that preserve the same supersymmetries and have the same asymptotic structure as the five-dimensional, asymptotically flat, three-charge black hole. In M-theory, these solutions can be constructed by compactifying on a six-torus, T^6, and wrapping three sets of M2 branes on three orthogonal two-tori. One can also add three sets of M5 branes while preserving the same supersymmetries. Each set of M5 branes can be thought of as magnetically dual to a set of M2 branes in that the M5 branes wrap the four-torus, T^4, orthogonal to the T^2 wrapped by the M2 branes. The remaining spatial direction of the M5 brane's worldvolume wraps a non-intersecting closed curve, $y^\mu(\sigma)$, in the five-dimensional space-time. We will take this curve to be the same for all three sets of M5 branes, but in principle one can choose three different curves for the three sets of M5 branes and one can construct such supergravity solutions by following similar procedure as the one outlined here. In [21] it was argued that this was the most general three-charge brane configuration consistent with the supersymmetries of the three-charge black-hole.

One can view these backgrounds as solutions to the STU model of five-dimensional ungauged $\mathcal{N} = 2$ supergravity. This is the minimal ungauged supergravity theory in five-dimensions coupled to two vector multiplets. The bosonic degrees of freedom are the metric, three vector fields (one from the gravity multiplet and two from the vector multiplets) and two independent real scalars. The STU model is a consistent truncation to a subset of the fields of the maximal, $\mathcal{N} = 8$, ungauged supergravity in five dimensions [129]. This five-dimensional description of supersymmetric black holes is often used in the literature. Here we will emphasize the eleven-dimensional origin of the solutions and their interpretation as intersecting branes. This will provide some intuition about their physics and will allow for comparison to solutions of type IIA and IIB supergravity.

The metric corresponding to this brane configuration can be written as

$$\begin{aligned} ds_{11}^2 = ds_5^2 &+ \left(Z_2 Z_3 Z_1^{-2}\right)^{\frac{1}{3}} (dx_5^2 + dx_6^2) + \left(Z_1 Z_3 Z_2^{-2}\right)^{\frac{1}{3}} (dx_7^2 + dx_8^2) \\ &+ \left(Z_1 Z_2 Z_3^{-2}\right)^{\frac{1}{3}} (dx_9^2 + dx_{10}^2) \,, \end{aligned} \tag{2.4}$$

where the five-dimensional space-time metric has the form:

$$ds_5^2 \equiv -\left(Z_1 Z_2 Z_3\right)^{-\frac{2}{3}} (dt + k)^2 + \left(Z_1 Z_2 Z_3\right)^{\frac{1}{3}} ds_4^2 \,, \tag{2.5}$$

for some one-form, k, defined on the spatial section of this metric. Since we want the metric to be asymptotic to flat $\mathbb{R}^{4,1} \times T^6$, we require

$$ds_4^2 \equiv h_{\mu\nu} dy^\mu dy^\nu \,, \tag{2.6}$$

to limit to the flat, Euclidean metric on $\mathbb{R}^4$ at spatial infinity and we require the warp factors, Z_I, to limit to constants at infinity. To fix the normalization of the corresponding Kaluza-Klein $U(1)$ gauge fields, we will take $Z_I \to 1$ at infinity.

The supersymmetry, ϵ, consistent with the brane configuration discussed above, must satisfy:

$$\big(\mathbb{1} \,-\, \Gamma^{056}\big)\,\epsilon \;=\; \big(\mathbb{1} \,-\, \Gamma^{078}\big)\,\epsilon \;=\; \big(\mathbb{1} \,-\, \Gamma^{09\,10}\big)\,\epsilon \;=\; 0\,, \tag{2.7}$$

where Γ^a are eleven-dimensional, tangent space gamma matrices and we use the standard notation $\Gamma^{a_1 \ldots a_n} = \Gamma^{[a_1} \ldots \Gamma^{a_n]}$. Since we have

$$\Gamma^{012345678910} \;=\; 1\,, \tag{2.8}$$

this implies

$$\big(\mathbb{1} \,-\, \Gamma^{1234}\big)\,\epsilon \;=\; 0\,, \tag{2.9}$$

which means that one of the four-dimensional helicity components of the supersymmetry must vanish identically. The holonomy of the metric, (2.6), acting on the spinors is determined by

$$[\,\nabla_\mu\,,\,\nabla_\nu\,]\,\epsilon \;=\; \tfrac{1}{4}\,R^{(4)}_{\mu\nu cd}\,\Gamma^{cd}\,\epsilon\,, \tag{2.10}$$

where $R^{(4)}_{\mu\nu cd}$ is the Riemann tensor of (2.6). Observe that (2.10) vanishes identically as a consequence of (2.9) if the Riemann tensor is self-dual:

$$R^{(4)}_{abcd} \;=\; \tfrac{1}{2}\,\varepsilon_{cd}{}^{ef}\,R^{(4)}_{abef}\,. \tag{2.11}$$

Such four-metrics are called "half-flat" [86]. The holonomy of these four-dimensional metrics is $SU(2)$, which means that the base space has to be hyper-Kähler[4].

Thus we can preserve the supersymmetry if and only if we take the four-metric to be hyper-Kähler. However, there is a theorem, [110], that states that any metric that is (i) Riemannian (signature $+4$) and regular, (ii) hyper-Kähler and (iii) asymptotic to the flat metric on $\mathbb{R}^4$, must be globally the flat metric on $\mathbb{R}^4$. The obvious conclusion, which we will follow for the moment, is that we simply take (2.6) to be the flat metric on $\mathbb{R}^4$. However, there are very important exceptions. First, we require the four-metric to be asymptotic to flat $\mathbb{R}^4$ because we want to interpret the object in asymptotically flat, five-dimensional space-time. If we want something that can be interpreted in terms of asymptotically flat, four-dimensional space-time then we want the four-metric to be asymptotic to the flat metric on $\mathbb{R}^3 \times S^1$. This allows for a lot more possibilities, and includes the multi-Taub-NUT metrics [135]. Using such Taub-NUT metrics provides a straightforward technique for reducing the five-dimensional solutions to four dimensions [24, 96, 97, 90, 25].

The other exception will be discussed in the following Sections: The requirement that the four-metric be globally Riemannian is too strong. The four-metric can be allowed to change overall sign since this can be compensated by a sign change in the warp factors of (2.5). For the moment, however, we will suppose that the four-metric is simply that of flat $\mathbb{R}^4$.

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[4]Hyper-Kähler manifolds are complex $4n$-dimensional manifolds ($n \in \mathbb{N}$) with $Sp(n)$ holonomy and three integrable complex structures, which satisfy the quaternionic algebra. Note that in four (real) dimensions Calabi-Yau manifolds have $SU(2) \cong Sp(1)$ holonomy and are thus hyper-Kähler.

2.2.2 The BPS equations

For the class of solutions of interest here the Maxwell three-form potential in eleven dimensions is given by

$$C_{(3)} = A^{(1)} \wedge dx_5 \wedge dx_6 \; + \; A^{(2)} \wedge dx_7 \wedge dx_8 \; + \; A^{(3)} \wedge dx_9 \wedge dx_{10} \,, \tag{2.12}$$

where the six coordinates, x_i, parameterize the compactification torus, T^6, and $A^{(I)}$, $I = 1, 2, 3$, are one-form Maxwell potentials in the five-dimensional space-time and depend only upon the coordinates, y^μ, that parameterize the spatial directions. It is convenient to introduce the Maxwell dipole field strengths, $\Theta^{(I)}$, obtained by removing the contributions of the electrostatic potentials

$$\Theta^{(I)} \;\equiv\; dA^{(I)} + d\big(Z_I^{-1}\,(dt + k)\big)\,. \tag{2.13}$$

With this Ansatz in hand one can try to solve the first order differential equations coming from imposing the vanishing of the gravitino variation in eleven-dimensional supergravity (1.39). It turns out that these first order equations are not enough to determine all functions in the solution and one should also use the equations of motion for the three-form potential [101, 21]. We will not present the details of the calculation here but one can show that the most general supersymmetric configuration is obtained by solving the following *BPS equations*:

$$\Theta^{(I)} \;=\; \star_4\, \Theta^{(I)}\,, \tag{2.14}$$

$$\nabla^2 Z_I \;=\; \frac{1}{2} C_{IJK} \star_4 (\Theta^{(J)} \wedge \Theta^{(K)})\,, \tag{2.15}$$

$$dk \;+\; \star_4\, dk \;=\; Z_I\, \Theta^{(I)}\,, \tag{2.16}$$

where $\star_4$ is the Hodge dual and ∇^2 is the Laplacian on the four-dimensional base metric $h_{\mu\nu}$, and $C_{IJK} \equiv |\epsilon_{IJK}|$. It is important to note that if these equations are solved in the order presented above, then one is solving a linear system.

At each step in the solution-generating process one has the freedom to add homogeneous solutions of the equations. Since we are requiring that the fields fall off at infinity, this means that these homogeneous solutions must have sources in the base space and since there is no topology in the $\mathbb{R}^4$ base, these sources must be singular. One begins by choosing the profiles, in $\mathbb{R}^4$, of the three types of M5 branes that source the $\Theta^{(I)}$. These fluxes then give rise to the explicit sources on the right-hand side of (2.15), but one also has the freedom to choose singular sources for (2.15) corresponding to the densities, $\rho_I(\sigma)$, of the three types of M2 branes. The M2 branes can be distributed at the same location as the M5 profile, and can also be distributed away from this profile. The functions, Z_I, then appear in the final solution as warp factors and as the electrostatic potentials. There are thus two contributions to the total electric charge of the solution: The localized M2 brane sources described by $\rho_I(\sigma)$ and the induced charge from the fields, $\Theta^{(I)}$, generated by the M5 branes. It is in this sense that the solution contains electric charges that are dissolved in the fluxes generated by M5 branes. This is much like in the well-known Klebanov-Strassler or Klebanov-Tseytlin solutions of type IIB supergravity [149, 150].

The final step is to solve the last BPS equation, (2.16), which is sourced by a cross term between the magnetic and electric fields. Again there are homogeneous solutions that may need to be added, however they need to be adjusted so as to ensure that (2.5) has no closed time-like curves (CTC's). Roughly one must make sure that the angular momentum at each point does not exceed what can be supported by local energy density.

These solutions can be generalized further by allowing the compact manifold to be an arbitrary six-dimensional Calabi-Yau manifold, CY_3. Since Calabi-Yau manifolds are Kähler and Ricci-flat the solutions will preserve the same amount of supersymmetry. The constants C_{IJK} are then the triple intersection numbers of the CY_3. The number of Maxwell fields is $h_{1,1}$, where $h_{1,1}$ is one of the Hodge numbers of the CY_3. This leads to $h_{1,1}$ conserved charges at inifity. Such solutions were discussed in [60].

It is worth making some comments about the asymptotic charges of the BPS solutions. Even though the generic solution is build from six sets of branes, there are only three conserved electric charges that can be measured at infinity. These are obtained from the three vector potentials, $A^{(I)}$, defined in (2.12), by integrating $\star_5 dA^{(I)}$ over the three-sphere at spatial infinity. Since the M5 branes wrap a closed curve, they do not directly contribute to the electric charges. The electric charges are determined by electric fields at infinity, and hence by the functions Z_I

$$Z_I \sim 1 + \frac{Q_I}{\rho^2} \,, \qquad \qquad \rho \to \infty \,. \tag{2.17}$$

Note that while the M5 branes do not directly contribute to the electric charges, they do contribute indirectly via charges dissolved in fluxes, that is, through the source terms on the right-hand side of (2.15).

There are two commuting angular momenta, J_1 and J_2, corresponding to the components of rotation in the two orthogonal planes in $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$. One can read off the angular momenta of the solution by making an expansion at infinity of the angular momentum one-fomrm, k, in (2.5):

$$k \ \sim \ \left(J_1 \, \frac{u^2 d\varphi_1}{(u^2+v^2)^2} \ + \ J_2 \, \frac{v^2 d\varphi_2}{(u^2+v^2)^2} \right) \ + \ \ldots \ , \qquad u, v \to \infty \,, \tag{2.18}$$

where the metric on $\mathbb{R}^4$ is

$$ds_4^2 = d\rho^2 + \rho^2 (d\vartheta^2 + \sin^2\vartheta d\varphi_1^2 + \cos^2\vartheta d\varphi_2^2) = du^2 + u^2 d\varphi_1^2 + dv^2 + v^2 d\varphi_2^2 \,. \tag{2.19}$$

The charges, Q_I, and the angular momenta, J_1, J_2, need to be correctly normalized in order to express them in terms of the quantized charges. The normalization depends upon the eleven-dimensional Planck length, ℓ_p, and the volume of the compactifying torus, T^6. See Appendix D for more details on the units and conventions used in this thesis.

2.2.3 BMPV black hole

The simplest example of a three-charge supergravity solution in the class discussed above is the BMPV black hole [49]. This is a three-charge, black hole with two equal angular momenta, $J_1 = J_2 = J$. When written in the Ansatz of Section 2.2.1 the BMPV black hole is given by

$$\Theta^{(I)} = 0 \,, \qquad Z_I = 1 + \frac{Q_I}{\rho^2} \,, \qquad k = \frac{J}{\rho^2}(\sin^2\vartheta d\varphi_1 + \cos^2\vartheta d\varphi_2) \,, \tag{2.20}$$

where the base is $\mathbb{R}^4$ with the metric (2.19). Another interesting supersymmetric solution with a horizon is the three-charge black ring with three dipole charges [87, 21]. The Ansatz of Section 2.2.1 also easily incorporates an arbitrary superposition of BMPV black holes as well as concentric black rings with a BMPV black hole in the center [21, 103, 104]. One can also use the methods above to study processes in which black holes and black rings are brought together and ultimately merge [27]. Such processes are interesting in their own right, but they can also be very useful in the study of microstate geometries [28, 29].

Our main interest in this thesis is the construction of regular supergravity solutions with no horizons, therefore in the next Sections we will study a large class of such solutions.

2.2.4 Gibbons-Hawking Metrics

In Section 2.2.1 we noted that supersymmetry allows us to take the base-space metric to be any hyper-Kähler metric. There is a well-known class of interesting four-dimensional hyper-Kähler metrics - the multi-centered Gibbons-Hawking metrics [109]. These provide examples of asymptotically locally Euclidean (ALE) and asymptotically locally flat (ALF) spaces, which are asymptotic to $\mathbb{R}^4/\mathbb{Z}_n$ and $\mathbb{R}^3 \times S^1$ respectively [85]. Using ALF metrics provides a smooth way to transition between a five-dimensional and a four-dimensional interpretation of a certain configurations. Indeed, the size of the S^1 is usually a modulus of a solution, and thus is freely adjustable. When this size is large compared to the size of the source configuration, this configuration is essentially five-dimensional; if the radius of the S^1 is small, then the configuration has a four-dimensional description.

We noted earlier that a regular, Riemannian, hyper-Kähler metric that is asymptotic to flat $\mathbb{R}^4$ is necessarily flat $\mathbb{R}^4$ globally. The non-trivial ALE metrics get around this by having a discrete identification at infinity but, as a result, do not have an asymptotic structure that lends itself to a space-time interpretation. However, there is an important loophole in this line of reasoning. One should remember that only the five-dimensional metric (2.5) should be regular and Lorentzian and this might be achievable if singularities of the four-dimensional base space were canceled by the warp factors. More specifically, we are going to consider base-space metrics (2.6) whose overall sign is allowed to change in interior regions. That is, we are going to allow the signature to flip from $+4$ to -4. Such metrics were dubbed ambipolar [26].

The potentially singular regions could actually be regular if the warp factors, Z_I, all flip sign whenever the four-metric signature flips. We will show below that this can be done for ambipolar Gibbons-Hawking metrics. One of the results of this thesis presented in Chapter 5 is that one can use more general ambipolar hyper-Kähler metrics and still produce regular five-dimensional supergravity solutions.

Below we give a review of Gibbons-Hawking geometries [135, 109] and their ambipolar generalization. Gibbons-Hawking metrics have the form of a $U(1)$ fibration over a flat $\mathbb{R}^3$ base:

$$h_{\mu\nu}dx^\mu dx^\nu \;=\; V^{-1}\left(d\psi + \vec{A}\cdot d\vec{y}\right)^2 \;+\; V\left(dy_1^2 + dy_2^2 + dy_3^2\right), \tag{2.21}$$

where we write $\vec{y} = (y_1, y_2, y_3)$. The function, V, is harmonic on the flat $\mathbb{R}^3$ while the connection, $A = \vec{A}\cdot d\vec{y}$, is related to V via

$$\vec{\nabla}\times\vec{A} \;=\; \vec{\nabla}V\,. \tag{2.22}$$

This family of metrics is the unique set of hyper-Kähler metrics with a tri-holomorphic $U(1)$ isometry[5]. Moreover, four-dimensional hyper-Kähler manifolds with $U(1)\times U(1)$ symmetry must, at least locally, be Gibbons-Hawking metrics with an extra $U(1)$ symmetry around an axis in the $\mathbb{R}^3$ [113].

Before we continue the general discussion of Gibbons-Hawking spaces we will present two simple examples of well known four-dimensional metrics in this class. Euclidean $\mathbb{R}^4$ is given by $V = \dfrac{1}{r}$ and

$$ds^2_{\mathbb{R}^4} = r(d\psi + (1+\cos\theta)d\phi)^2 + \frac{1}{r}(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2)\,. \tag{2.23}$$

[5]Tri-holomorphic means that the $U(1)$ isometry preserves all three complex structures of the hyper-Kähler metric.

To go from the Gibbons-Hawking coordinates to the more familiar forms of the metric on $\mathbb{R}^4$ (2.19), one has to perform the following change of coordinates

$$r = \frac{u^2+v^2}{4} \equiv \frac{\rho^2}{4}\,, \quad \theta = 2\arctan(u/v) = 2\vartheta\,, \quad \psi = 2\varphi_1\,, \quad \phi = -\varphi_1 - \varphi_2 \tag{2.24}$$

Another well-known example of a GH metric is the Taub-NUT metric [201, 171]. This is a solution of the Euclidean vacuum Einstein equations and is given by $V = \varepsilon_0 + \dfrac{Q_{TN}}{r}$ and

$$\begin{aligned} ds^2_{TN} = \left(\varepsilon_0 + \frac{Q_{TN}}{r}\right)^{-1} (d\psi + (1+Q_{TN}\cos\theta)d\phi)^2 \\ + \left(\varepsilon_0 + \frac{Q_{TN}}{r}\right)(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2)\,. \end{aligned} \tag{2.25}$$

The metric is asymptotic to $\mathbb{R}^3 \times S^1$ for $r \to \infty$ and Q_{TN} is called the NUT charge [110].

After this short digression let us contiune with the general Gibbons-Hawking metrics. In the standard form of the Gibbons-Hawking metrics one takes V to have a finite set of isolated sources. That is, let $\vec{y}^{(j)}$ be the positions of the source points in the $\mathbb{R}^3$ and let $r_j \equiv |\vec{y} - \vec{y}^{(j)}|$. Then one takes:

$$V = \varepsilon_0 \; + \; \sum_{j=1}^{N} \frac{q_j}{r_j}\,, \tag{2.26}$$

where one usually takes $q_j \geq 0$ to ensure that the metric is Riemannian (positive definite). We will relax this restriction later. There appear to be singularities in the metric at $r_j = 0$, however, if one changes to polar coordinates centered at $r_j = 0$ with radial coordinate $\rho = 2\sqrt{|\vec{y} - \vec{y}^{(j)}|}$, then the metric is locally of the form:

$$ds^2_4 \;\sim\; d\rho^2 \;+\; \rho^2\, d\Omega^2_3\,, \tag{2.27}$$

where $d\Omega_3^2$ is the standard metric on $S^3/\mathbb{Z}_{|q_j|}$. In particular, this means that one must have $q_j \in \mathbb{Z}$ and if $|q_j| = 1$ then the space looks locally like $\mathbb{R}^4$. If $|q_j| \neq 1$ then there is an orbifold singularity. Orbifold singularities are allowed in string theory so we will view such backgrounds as regular.[6]

If $\varepsilon_0 \neq 0$, then $V \to \varepsilon_0$ at infinity and so the metric (2.21) is asymptotic to flat $\mathbb{R}^3 \times S^1$, that is, the base is asymptotically locally flat (ALF). The five-dimensional space-time is thus asymptotically compactified to a four-dimensional space-time. This a standard Kaluza-Klein reduction and the vector field, $\vec{A}$, yields a non-trivial, four-dimensional Maxwell field whose sources, from the point of view of type IIA supergravity, are simply D6 branes. Later on we will use the fact that introducing a constant term into V yields a further compactification and we can relate five-dimensional to four-dimensional physics.

For the moment suppose that one has $\varepsilon_0 = 0$. At infinity in $\mathbb{R}^3$ one has $V \sim q_0/r$, where $r \equiv |\vec{y}|$ and

$$q_0 \equiv \sum_{j=1}^{N} q_j \, . \tag{2.28}$$

Hence spatial infinity in the Gibbons-Hawking metric also has the form (2.27), where $r = \frac{1}{4}\rho^2$ and $d\Omega_3^2$ is the standard metric on $S^3/\mathbb{Z}_{|q_0|}$. For the Gibbons-Hawking metric to be asymptotic to the positive definite, flat metric on $\mathbb{R}^4$ one must have $q_0 = 1$. Note that for the Gibbons-Hawking metrics to be globally positive definite one would also have to take $q_j \geq 0$ and thus the only such metric would have to have $V \equiv \frac{1}{r}$. The metric (2.21) is then the flat metric on $\mathbb{R}^4$ globally, as can be seen by using the change of variables (2.24). The only way to get non-trivial metrics that are asymptotic to flat $\mathbb{R}^4$ is by taking some of the $q_j \in \mathbb{Z}$ to be negative. We will see how this works later in this Chapter.

[6]We will see later that for the essential physical points it will be sufficient to take $|q_j| = 1$ so one can avoid the orbifold singularities.

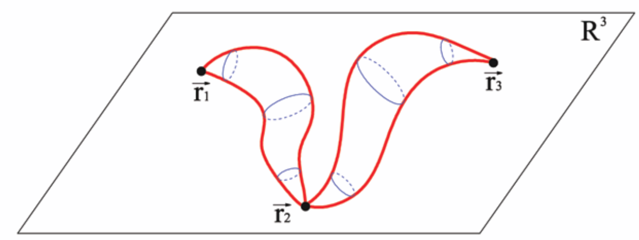

Figure 2.1: This figure depicts some non-trivial cycles of the Gibbons-Hawking geometry. The behaviour of the $U(1)$ fiber is shown along curves between the sources of the potential, V. Here the fibers sweep out a pair of intersecting homology spheres.

It is instructive to discuss the topology, homology and cohomology of the Gibbons-Hawking metrics. They contain $\frac{1}{2}N(N-1)$ topologically non-trivial two-cycles, Δ_{ij}, that run between the GH centers. These two-cycles can be defined by taking any curve, γ_{ij}, between $\vec{y}^{(i)}$ and $\vec{y}^{(j)}$ and considering the $U(1)$ fiber of (2.21) along the curve. This fiber collapses to zero size at the GH centers, and so the curve and the fiber sweep out a 2-sphere (up to $\mathbb{Z}_{|q_j|}$ orbifolds), see Fig. 2.1. These spheres intersect one another at the common points $\vec{y}^{(j)}$. There are $(N-1)$ linearly independent homology two-spheres, and the set $\Delta_{i\,(i+1)}$ represents a basis.

It is also convenient to introduce a set of frames

$$\hat{e}^1 \;=\; V^{-\frac{1}{2}}\,(d\psi \;+\; A)\,, \qquad \hat{e}^{a+1} \;=\; V^{\frac{1}{2}}\,dy^a\,, \quad a = 1,2,3\,. \tag{2.29}$$

and two sets of two-forms:

$$\Omega_{\pm}^{(a)} \;\equiv\; \hat{e}^1 \wedge \hat{e}^{a+1} \;\pm\; \tfrac{1}{2}\,\epsilon_{abc}\,\hat{e}^{b+1} \wedge \hat{e}^{c+1}\,, \qquad a = 1,2,3\,. \tag{2.30}$$

The two-forms, $\Omega_-^{(a)}$, are anti-self-dual, harmonic and non-normalizable and they are the three complex structures of the hyper-Kähler base. They also close the quaternionic algebra

$$(\Omega_-^{(a)})_{\mu\rho}(\Omega_-^{(b)})_\nu^\rho = -\delta^{ab}\delta_{\mu\nu} + \epsilon_{abc}(\Omega_-^{(c)})_{\mu\nu} \,, \tag{2.31}$$

as required for any hyper-Kähler manifold.

The forms, $\Omega_+^{(a)}$, are self-dual and can be used to construct harmonic fluxes that are dual to the two-cycles. Consider the self-dual two-form:

$$\Theta \;\equiv\; -\sum_{a=1}^{3} \big(\partial_a\big(V^{-1}\, H\big)\big)\, \Omega_+^{(a)} \,. \tag{2.32}$$

Then Θ is closed (and hence co-closed and harmonic) if and only if H is harmonic in $\mathbb{R}^3$, *i.e.* $\nabla^2 H = 0$. We now have the choice of how to distribute sources of H throughout the $\mathbb{R}^3$ base of the GH space; such a distribution may correspond to having multiple black rings and black holes in this space. Nevertheless, if we want to obtain a geometry that has no singularities and no horizons, Θ has to be regular, and this happens if and only if H/V is regular, which in turn happens only if H has the form:

$$H \;=\; h_0 \;+\; \sum_{j=1}^{N} \frac{h_j}{r_j} \,. \tag{2.33}$$

Also note that the "gauge transformation:"

$$H \;\rightarrow\; H \;+\; c\,V \,, \tag{2.34}$$

for some constant, c, leaves Θ unchanged, and so there are only N independent parameters in H. In addition, if $\varepsilon = 0$ then one must take $h_0 = 0$ for Θ to remain finite at infinity. The remaining $(N-1)$ parameters then describe harmonic forms that are dual to the non-trivial two-cycles. If $\varepsilon \neq 0$ then the extra parameter is that of a Maxwell field whose gauge potential gives the Wilson line around the S^1 at infinity.

It is straightforward to find a local potential such that $\Theta = dB$:

$$B \;\equiv\; V^{-1}\, H\,(d\psi \;+\; A) \;+\; \vec{\xi} \cdot d\vec{y}\,, \tag{2.35}$$

where

$$\vec{\nabla} \times \vec{\xi} \;=\; -\,\vec{\nabla} H\,. \tag{2.36}$$

Hence, $\vec{\xi}$ is a vector potential for magnetic monopoles located at the singular points of H.

To determine how these fluxes thread the two-cycles we need the explicit forms for the vector potential, B, and to find this we first need the vector fields, $\vec{v}_i$, that satisfy:

$$\vec{\nabla} \times \vec{v}_i \;=\; \vec{\nabla} \left(\frac{1}{r_i} \right). \tag{2.37}$$

One then has:

$$\vec{A} \;=\; \sum_{j=1}^{N} q_j\, \vec{v}_j\,, \qquad \vec{\xi} \;=\; \sum_{j=1}^{N} h_j\, \vec{v}_j\,. \tag{2.38}$$

If we choose coordinates so that $\vec{y}^{(i)} \;=\; (0,0,a)$ and let ϕ denote the polar angle in the (y_1, y_2)-plane, then:

$$\vec{v}_i \cdot d\vec{y} \;=\; \left(\frac{(y_3 - a)}{r_i} \;+\; c_i \right) d\phi\,, \tag{2.39}$$

where c_i are constants. The vector field, $\vec{v}_i$, is regular away from the y_3-axis, but has a Dirac string along the y_3-axis. By choosing c_i appropriately we can cancel the string along the positive or negative y_3-axis, and by moving the axis we can arrange these strings to run in any direction we choose, but they must start or finish at some $\vec{y}^{(i)}$, or run out to infinity.

Now consider what happens to B in the neighborhood of $\vec{y}^{(i)}$. Since the circles swept out by ψ and ϕ are shrinking to zero size, the string singularities near $\vec{y}^{(i)}$ are of the form:

$$B \ \sim \ \frac{h_i}{q_i}\Big(d\psi + q_i\Big(\frac{(y_3-a)}{r_i}+c_i\Big)\,d\phi\Big) \ - \ h_i\Big(\frac{(y_3-a)}{r_i}+c_i\Big)\,d\phi \ \sim \ \frac{h_i}{q_i}\,d\psi\,. \quad (2.40)$$

This shows that the vector, $\vec{\xi}$, in (5.16) cancels the string singularities in the $\mathbb{R}^3$. The singular components of B thus point along the $U(1)$ fiber of the GH metric.

Choose any curve, γ_{ij}, between $\vec{y}^{(i)}$ and $\vec{y}^{(j)}$ and define the two-cycle, Δ_{ij}, as in Fig. 2.1. If one has $V > 0$ then the vector field, B, is regular over the whole of Δ_{ij} except at the end-points, $\vec{y}^{(i)}$ and $\vec{y}^{(j)}$. Let $\widehat{\Delta}_{ij}$ be the cycle Δ_{ij} with the poles excised. Since Θ is regular at the poles, then the expression for the flux, Π_{ij}, through Δ_{ij} can be obtained as follows:

$$\begin{aligned}\Pi_{ij} \quad &\equiv \quad \frac{1}{4\,\pi}\int_{\Delta_{ij}}\Theta \ = \ \frac{1}{4\,\pi}\int_{\widehat{\Delta}_{ij}}\Theta \ = \ \frac{1}{4\,\pi}\int_{\partial\widehat{\Delta}_{ij}} B \\ &= \quad \frac{1}{4\,\pi}\int_0^{4\pi} d\psi\,\big(B|_{y^{(j)}} \ - \ B|_{y^{(i)}}\big) \ = \ \Big(\frac{h_j}{q_j} \ - \ \frac{h_i}{q_i}\Big)\,. \end{aligned} \quad (2.41)$$

We have normalized these periods for later convenience.

On an ambipolar GH space where the cycle runs between positive and negative GH points, the flux, Θ, and the potential B are both singular when $V = 0$ and so this integral is a rather formal object. However, we will see in Section 2.2.11 that when we extend to the five-dimensional metric, the physical flux of the complete Maxwell field combines Θ with another term so that the result is completely regular. Moreover, the physical flux through the cycle is still given by (2.41). We will therefore refer to (2.41) as the magnetic flux even in ambipolar metrics and we will see that such fluxes are directly responsible for holding up the cycles

2.2.5 Solving the BPS equations on a Gibbons-Hawking base

Our task now is to solve the BPS equations (2.14)–(2.16) but now with a Gibbons-Hawking base metric. Such solutions have been derived before for positive-definite Gibbons-Hawking metrics [101], and it is trivial to generalize them to the ambipolar form. For the present we will not impose any conditions on the sources of the BPS equations.

In the previous Section we saw that there was a simple way to obtain self-dual two-forms, $\Theta^{(I)}$, that satisfy (2.14). That is, we introduce three harmonic functions, K^I, on $\mathbb{R}^3$ that satisfy $\nabla^2 K^I = 0$, and define $\Theta^{(I)}$ as in (2.32) by replacing H with K^I. We will not, as yet, assume any specific form for K^I.

Using the two-forms Θ^I one can show that the second BPS equation is solved by (2.15):

$$Z_I \;=\; \tfrac{1}{2} C_{IJK}\, V^{-1}\, K^J K^K \;+\; L_I\,, \tag{2.42}$$

where L_I are three more independent harmonic functions on $\mathbb{R}^3$.

We now write the one-form, k, as:

$$k \;=\; \mu\,(d\psi + A) \;+\; \omega\;, \tag{2.43}$$

with $\omega = \vec{\omega}\cdot d\vec{y}$, then (2.16) becomes:

$$\vec{\nabla}\times\vec{\omega} \;=\; (V\vec{\nabla}\mu \;-\; \mu\vec{\nabla}V) \;-\; V\sum_{I=1}^{3} Z_I\,\vec{\nabla}\bigg(\frac{K^I}{V}\bigg)\,. \tag{2.44}$$

Taking the divergence yields the following equation for μ:

$$\nabla^2\mu \;=\; V^{-1}\,\vec{\nabla}\cdot\bigg(V\sum_{I=1}^{3} Z_I\,\vec{\nabla}\frac{K^I}{V}\bigg)\,, \tag{2.45}$$

which is solved by:

$$\mu \;=\; \tfrac{1}{6}\, C_{IJK}\, \frac{K^I K^J K^K}{V^2} \;+\; \frac{1}{2\,V}\, K^I L_I \;+\; M\,, \tag{2.46}$$

where M is yet another harmonic function on $\mathbb{R}^3$. The function M determines the anti-self-dual part of dk that cancels out of (2.16). Substituting this result for μ into (2.44) we find that ω satisfies:

$$\vec{\nabla} \times \vec{\omega} \;=\; V \vec{\nabla} M \;-\; M \vec{\nabla} V \;+\; \tfrac{1}{2}\, (K^I \vec{\nabla} L_I - L_I \vec{\nabla} K^I)\,. \tag{2.47}$$

The integrability condition for this equation is simply the fact that the divergence of both sides vanish, which is true because K^I, L_I, M and V are harmonic.

2.2.6 Properties of the solution

The solution is thus characterized by the harmonic functions K^I, L_I, V and M. The gauge invariance, (2.34), extends in a straightforward manner to the complete solution:

$$\begin{aligned} K^I &\;\to\; K^I \;+\; c^I\, V\,, \\ L_I &\;\to\; L_I \;-\; C_{IJK}\, c^J\, K^K \;-\; \tfrac{1}{2}\, C_{IJK}\, c^J\, c^K\, V\,, \\ M &\;\to\; M - \tfrac{1}{2}\, c^I\, L_I + \frac{1}{12}\, C_{IJK} \left(V\, c^I\, c^J\, c^K + 3\, c^I\, c^J\, K^K \right)\,, \end{aligned} \tag{2.48}$$

where the c^I are three arbitrary constants[7].

[7] Note that this gauge invariance exists for any C_{IJK} coming from a compactifaction of M-theory on a Calabi-Yau three-fold. Of course the number of constants c^I is then $h^{1,1}$.

The eight functions that give the solution may also be identified with the eight independent parameters in the **56** of the $E_{7(7)}$ duality group of four-dimensional $\mathcal{N} = 8$ ungauged supergravity [22]:

$$
\begin{aligned}
x_{12} &= L_1\,, & x_{34} &= L_2\,, & x_{56} &= L_3\,, & x_{78} &= -V\,, \\
y_{12} &= K^1\,, & y_{34} &= K^2\,, & y_{56} &= K^3\,, & y_{78} &= 2\,M\,.
\end{aligned}
\tag{2.49}
$$

With these identifications, the right-hand side of (2.47) is the symplectic invariant of the **56** of $E_{7(7)}$:

$$
\vec{\nabla} \times \vec{\omega} \;=\; \tfrac{1}{4} \sum_{A,B=1}^{8} \big(y_{AB} \vec{\nabla} x_{AB} \;-\; x_{AB} \vec{\nabla} y_{AB}\big)\,. \tag{2.50}
$$

We also note that the quartic invariant of the **56** of $E_{7(7)}$ is determined by:

$$
\begin{aligned}
J_4 \quad = \quad & -\frac{1}{4}(x_{12}y^{12} + x_{34}y^{34} + x_{56}y^{56} + x_{78}y^{78})^2 - x_{12}x_{34}x_{56}x_{78} \\
& -y^{12}y^{34}y^{56}y^{78} + x_{12}x_{34}y^{12}y^{34} + x_{12}x_{56}y^{12}y^{56} + x_{34}x_{56}y^{34}y^{56} \\
& +x_{12}x_{78}y^{12}y^{78} + x_{34}x_{78}y^{34}y^{78} + x_{56}x_{78}y^{56}y^{78}\,,
\end{aligned}
\tag{2.51}
$$

and we will see that this plays a direct role in the expression for the scale of the $U(1)$ fibration. It also plays a central role in the expression for the horizon area of the four-dimensional supersymmetric black hole [146] and the five-dimensional supersymmetric black ring [22].

In principle we can choose the harmonic functions K^I, L_I and M to have sources that are localized anywhere on the base. These solutions then have localized brane sources, and include, for example, supertubes and black rings in Taub-NUT [24, 97, 90, 25]. Such solutions also include more general multi-center black hole configurations in four dimensions, of the type considered by Denef and collaborators [77, 15, 78].

Nevertheless, our focus for the moment is on obtaining smooth horizonless solutions, which correspond to microstates of black holes and black rings and we choose the harmonic functions so that there are no brane charges anywhere, and all the charges come from the smooth cohomological fluxes that thread the non-trivial cycles.

2.2.7 Closed time-like curves

To make sure that we are dealing with physically reasonable space-times we have to ensure that causality is preserved, i.e. there should be no closed time-like curves in the solutions. To look for the presence of closed time-like curves in the metric one considers the space-space components of the metric given by (2.4), (2.5) and (2.21). That is, one goes to the space-like slices obtained by taking t to be a constant. The T^6 directions immediately yield the requirement that $Z_I Z_J > 0$ while the metric on the four-dimensional base reduces to:

$$\begin{aligned} ds_4^2 &= -W^{-4}\big(\mu(d\psi + A) + \omega\big)^2 \\ &\qquad + W^2 V^{-1}\big(d\psi + A\big)^2 + W^2 V\big(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\big)\,, \end{aligned} \tag{2.52}$$

where we have chosen to write the metric on $\mathbb{R}^3$ in terms of a generic set of spherical polar coordinates, (r, θ, ϕ) and where we have defined the warp-factor, W, by:

$$W \equiv (Z_1\, Z_2\, Z_3)^{1/6}\,. \tag{2.53}$$

There is some potentially singular behavior arising from the fact that the Z_I, and hence W, diverge on the locus, $V = 0$ (see (2.42)). However, one can show that if one expands the metric (2.52) and uses the expression, (2.46), then all the dangerous divergent terms cancel and the metric is regular. We will discuss this further below and in Section 2.2.8.

Expanding (2.52) leads to:

$$
\begin{aligned}
ds_4^2 &= W^{-4}\,(W^6V^{-1}-\mu^2)\Big(d\psi + A - \frac{\mu\,\omega}{W^6V^{-1}-\mu^2}\Big)^2 - \frac{W^2\,V^{-1}}{W^6V^{-1}-\mu^2}\,\omega^2 \\
&\quad +\; W^2V\big(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\big) \\
&= \frac{\mathcal{Q}}{W^4V^2}\Big(d\psi + A - \frac{\mu\, V^2}{\mathcal{Q}}\,\omega\Big)^2 \;+\; W^2V\Big(r^2\sin^2\theta\, d\phi^2 - \frac{\omega^2}{\mathcal{Q}}\Big) \\
&\quad +\; W^2V(dr^2 + r^2 d\theta^2)\,, \qquad (2.54)
\end{aligned}
$$

where we have introduced the quantity:

$$
\mathcal{Q} \;\equiv\; W^6\,V \;-\; \mu^2\,V^2 \;=\; Z_1Z_2Z_3V \;-\; \mu^2\,V^2\,. \qquad (2.55)
$$

Upon evaluating $\mathcal{Q}$ as a function of the harmonic functions that determine the solution one obtains the following result:

$$
\begin{aligned}
\mathcal{Q} &= -M^2\,V^2 - \tfrac{1}{3}\,M\,C_{IJK}K^I\,K^J\,K^k - M\,V\,K^I\,L_I - \tfrac{1}{4}\,(K^I L_I)^2 \\
&\quad +\tfrac{1}{6}\,VC^{IJK}L_IL_JL_K + \tfrac{1}{4}\,C^{IJK}C_{IMN}L_JL_KK^MK^N \qquad (2.56)
\end{aligned}
$$

with $C^{IJK} \equiv C_{IJK}$. We can straightforwardly see that when we consider M-theory compactified on T^6, then $C^{IJK} = |\epsilon^{IJK}|$, and $\mathcal{Q}$ is nothing other than the $E_{7(7)}$ quartic invariant (2.51) where the x's and y's are identified as in (2.49). This is expected from the fact that the solutions on a GH base have an extra $U(1)$ invariance, and hence can be thought of as four-dimensional. The four-dimensional supergravity obtained by compactifying M-theory on T^7 is $N = 8$ supergravity, which has an $E_{7(7)}$ symmetry group. Of course, the analysis above and in particular equation (2.56) are valid for solutions of arbitrary five-dimensional $U(1)^N$ ungauged supergravities on a GH base. More details on the explicit relation for general theories can be found in [17].

Note that $\mathcal{Q}$ is invariant under the gauge transformation (2.48). Observe that the metric coefficients in (2.54) only involve V in the combinations W^2V and $\mathcal{Q}$ and both of these are regular as $V \to 0$. Thus, at least the spatial metric is regular at $V = 0$. In Section 2.2.8 we will show that the complete solution is regular as one passes across the surface $V = 0$.

From (2.54) and (2.4) we see that to avoid CTC's, the following inequalities must be true everywhere:

$$\mathcal{Q} \, \geq \, 0 \,, \quad W^2 \, V \, \geq \, 0 \,, \quad \big(Z_J \, Z_K \, Z_I^{-2} \big)^{\frac{1}{3}} \, = \, W^2 Z_I^{-1} \, \geq \, 0 \,, \;\; I = 1, 2, 3 \,. \tag{2.57}$$

The last two conditions can be subsumed into:

$$V \, Z_I \, = \, \tfrac{1}{2} \, C_{IJK} \, K^J \, K^K \, + \, L_I \, V \, \geq \, 0 \,, \qquad I = 1, 2, 3 \,. \tag{2.58}$$

The obvious danger arises when V is negative. We will show in the next sub-section that all these quantities remain finite and positive in a neighborhood of $V = 0$, despite the fact that W blows up. Nevertheless, these quantities could possibly be negative away from the $V = 0$ surface, we will comment about this further below. One should also note that $\mathcal{Q} \geq 0$ requires $\prod_I (V Z_I) \geq \mu^2 V^4$, and so, given (2.58), the constraint $\mathcal{Q} \geq 0$ is still somewhat stronger.

Also note that there is a danger of CTC's arising from Dirac-Misner strings in ω. That is, near $\theta = 0, \pi$ the $-\omega^2$ term could be dominant unless ω vanishes on the polar axis. We will analyze this issue completely when we consider bubbled geometries in Section 2.2.10.

Finally, one can also try to argue [41] that the complete metric is stably causal and that the t coordinate provides a global time function [134]. In particular, t will then be monotonic increasing on future-directed non-space-like curves and hence there can be no CTC's. The coordinate t is a time function if and only if

$$- g^{\mu\nu}\partial_\mu t\, \partial_\nu t = -g^{tt} = (W^2 V)^{-1}(\mathcal{Q} - \omega^2) > 0\,, \tag{2.59}$$

where ω is squared using the $\mathbb{R}^3$ metric. This is obviously a slightly stronger condition than $\mathcal{Q} \geq 0$ in (2.57).

2.2.8 Regularity of the solution and critical surfaces

As we have seen, the general solutions we will consider have functions, V, that change sign on the $\mathbb{R}^3$ base of the GH metric. Our purpose here is to show that such solutions are completely regular, with positive definite metrics, in the regions where V changes sign. As we will see the "critical surfaces," where V vanishes are simply a set of completely harmless, regular hypersurfaces in the full five-dimensional geometry.

The most obvious issue is that if V changes sign, then the overall sign of the metric (2.21) changes and there might be whole regions of closed time-like curves when $V < 0$. However, we remarked above that the warp factors, in the form of W, prevent this from happening. Specifically, the expanded form of the complete, eleven-dimensional metric when projected onto the GH base yields (2.54). Moreover

$$W^2\, V \;=\; (Z_1\, Z_2\, Z_3\, V^3)^{\frac{1}{3}} \;\sim\; ((K_1\, K_2\, K_3)^2)^{\frac{1}{3}} \tag{2.60}$$

on the surface $V = 0$. Hence $W^2 V$ is regular and positive on this surface, and therefore the space-space part (2.54) of the full eleven-dimensional metric is regular.

There is still the danger of singularities at $V = 0$ for the other background fields. We first note that there is no danger of such singularities being hidden implicitly in the $\vec{\omega}$ terms. Even though (2.44) suggests that the source of $\vec{\omega}$ is singular at $V = 0$, we see from (2.47) that the source is regular at $V = 0$ and thus there is nothing hidden in $\vec{\omega}$. We therefore need to focus on the explicit inverse powers of V in the solution.

The factors of V cancel in the torus warp factors, which are of the form $(Z_I Z_J Z_K^{-2})^{\frac{1}{3}}$. The coefficient of $(dt+k)^2$ is W^{-4}, which vanishes as V^2. The singular part of the cross term, $dt\, k$, is $\mu\, dt\, (d\psi + A)$, which, from (2.46), diverges as V^{-2}, and so the overall cross term, $W^{-4} dt\, k$, remains finite at $V = 0$.

So the metric is regular at critical surfaces. The inverse metric is also regular at $V = 0$ because the $dt\, d\psi$ part of the metric remains finite and so the determinant is non-vanishing.

This surface is therefore not an event horizon even though the time-like Killing vector defined by translations in t becomes null when $V = 0$. Indeed, when a metric is stationary but not static, the fact that g_{tt} vanishes on a surface does not make it an event horizon (the best known example of this is the boundary of the ergosphere of the Kerr metric). The necessary condition for a surface to be a horizon is rather to have $g^{rr} = 0$, where r is the coordinate transverse to this surface. This is clearly not the case here.

Hence, the surface given by $V = 0$ is like a boundary of an ergosphere, except that the solution has no ergosphere because this Killing vector is time-like on both sides and does not change character across the critical surface. In the Kerr metric the time-like Killing vector becomes space-like and this enables energy extraction by the Penrose process. Here there is no ergosphere and so energy extraction is not possible, as is to be expected from a BPS geometry.

At first sight, it does appear that the Maxwell fields are singular on the surface $V = 0$. Certainly the "magnetic components," $\Theta^{(I)}$, (see (2.32)) are singular when $V = 0$. However, one must remember that the complete Maxwell fields are the $A^{(I)}$, and these are indeed non-singular at $V = 0$. One finds that the singularities in the "magnetic terms" of $A^{(I)}$ are canceled by singularities in the "electric terms" of $A^{(I)}$, and this is possible at $V = 0$ precisely because g_{tt} goes to zero, and so the magnetic and electric terms can communicate. Specifically, one has, from (2.13) and (2.35):

$$dA^{(I)} = d\left(B^{(I)} - \frac{(dt+k)}{Z_I}\right). \tag{2.61}$$

Near $V = 0$ the singular parts of this behave as:

$$\begin{aligned} dA^{(I)} &\sim d\left(\frac{K^I}{V} - \frac{\mu}{Z_I}\right)(d\psi + A) \\ &\sim d\left(\frac{K^I}{V} - \frac{K^1\,K^2\,K^3}{\frac{1}{2}\,V\,C_{IJK}\,K^J\,K^K}\right)(d\psi + A) \sim 0\,. \end{aligned} \tag{2.62}$$

The cancellations of the V^{-1} terms here occur for much the same reason that they do in the metric (2.54).

Therefore, even if V vanishes and changes sign and the base metric becomes negative definite, the complete eleven-dimensional solution is regular and well-behaved around the $V = 0$ surfaces. It is this fact that gets us around the uniqueness theorems for asymptotically Euclidean hyper-Kähler metrics in four dimensions, and as we will see, there are now a vast number of candidates for the base metric.

2.2.9 The geometric transition

It is natural to try to obtain microstates by starting with brane configurations that do not develop a horizon at large effective coupling, or alternatively to consider a black ring solution in the limit where its entropy decreases and becomes zero. However, the geometry of a zero-entropy black ring (or three charge supertube) is singular [21]. This singularity is not a curvature singularity, since the curvature is bounded above by the inverse of the dipole charges. Rather, the singularity is caused by the fact that the size of the S^1 of the horizon shrinks to zero size and the result is a “null orbifold.” One can also

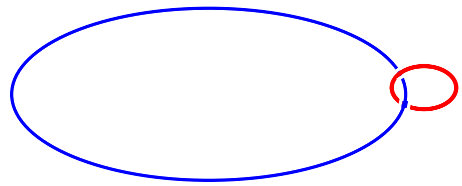

Figure 2.2: Geometric transitions: The branes wrap the large (blue) cycle; the flux through the Gaussian (small, red) cycle measures the brane charge. In the open-string picture the small (red) cycle has non-zero size, and the large (blue) cycle is contractible. After the geometric transition the size of the large (blue) cycle becomes zero, while the small (red) cycle becomes topologically non-trivial.

think about this singularity as caused by the gravitational back-reaction of the branes that form the three-charge supertube, which causes the S^1 wrapped by these branes to shrink to zero size.

String theory is very good at resolving this kind of singularities, and the mechanism by which it does is that of "geometric transition." To understand what a geometric transition is, consider a collection of branes wrapped on a certain cycle. At weak effective coupling one can describe these branes by studying the open strings that live on them. One can also find the number of branes by integrating the corresponding flux over a Gaussian cycle dual to that wrapped by the branes. However, when one increases the coupling, the branes back-react on the geometry, and shrink the cycle they wrap to zero size. At the same time, the Gaussian cycle becomes large and topologically non-trivial. (See Fig. 2.2.) The resulting geometry has a different topology, and no brane sources. The only information about the branes is now in the integral of the flux over the blown-up dual Gaussian cycle. Hence, even if in the open-string (weakly coupled) description we had a configuration of branes, in the closed-string (large effective coupling) description these branes have disappeared and have been replaced by a non-trivial topology with flux.

Geometric transitions appear in many systems [124, 150, 202, 153]. A classic example of such system are the brane models that break an $\mathcal{N} = 2$ superconformal field

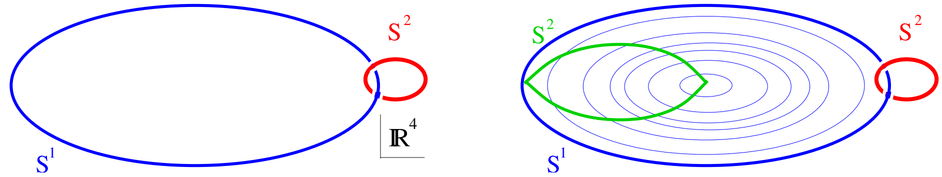

Figure 2.3: The geometric transition of the black ring: Before the transition the branes wrap the large (blue) S^1; the flux through the Gaussian S^2 (small, red) cycle measures the brane charge. After the transition the Gaussian S^2 (small, red) cycle is topologically non-trivial and of finite size and a new (green) S^2 appears, coming from the fact that the blue S^1 shrinks to zero so that the disk spanning the S^1 becomes an S^2. The resulting geometry has two non-trivial S^2's and no brane sources.

theory down to an $\mathcal{N} = 1$ supersymmetric field theory [150, 51]. Typically, the $\mathcal{N} = 2$ superconformal field theory is realized on a stack of D3 branes on some Calabi-Yau three-fold. One can then break the supersymmetry to $\mathcal{N} = 1$ by introducing extra D5 branes that wrap a two-cycle. When one investigates the closed-string picture, the two-cycle collapses and the dual three-cycle blows up (this is also known as a conifold transition). The D5 branes disappear and are replaced by non-trivial fluxes on the three-cycle. The resulting geometry has no more brane sources, and has a different topology than the one we started with.

Our purpose here is to argue that geometric transitions resolve the singularity of the zero-entropy black ring. Here the ring wraps a curve $y^\mu(\sigma)$, that is topologically an S^1 inside $\mathbb{R}^4$. (In Fig. 2.3 this S^1 is depicted as a large, blue cycle.) The Gaussian cycle for this S^1 is a two-sphere around the ring (illustrated by the red small cycle in Fig. 2.3). If one integrates the field strengths $\Theta^{(I)}$ on the red Gaussian two-cycle one obtains the M5 brane dipole charges of the ring, q^I.

After the geometric transition the large (blue) S^1 becomes of zero length, and the red S^2 becomes topologically non-trivial. Moreover, because the original topology is trivial, the curve $y^\mu(\sigma)$ was the boundary of a disk. When after the transition this boundary curve collapses, the disk becomes a (topologically non-trivial) two-sphere. Alternatively, one can think about this two-sphere (shown in Fig. 2.3 in green) as coming from having an S^1 that has zero size both at the origin of the space $r = 0$ and at the location of the ring. Hence, before the transition we had a ring wrapping a curve of arbitrary shape inside $\mathbb{R}^4$, and after the transition we have a manifold that is asymptotically $\mathbb{R}^4$, and has two non-trivial two-spheres, and no brane sources.

We can now try to determine the geometry of this manifold. If the curve has an arbitrary shape the only information about this manifold is that it is asymptotically $\mathbb{R}^4$ and that it is hyper-Kähler, as required by supersymmetry. If the curve wrapped by the supertube has arbitrary shape, this is not enough to determine the space that will come out after the geometric transition. However, if one considers a circular supertube, the solution before the transition has a $U(1) \times U(1)$ invariance, and so one naturally expects the solution resulting from the transition should also have this invariance.

With such a high level of symmetry we do have enough information to determine what the result of the geometric transition is. By a theorem of Gibbons and Ruback [113], a hyper-Kähler manifold that has a $U(1) \times U(1)$ invariance must have a translational $U(1)$ invariance and hence, must be Gibbons-Hawking. We also know that this manifold should have two non-trivial two-cycles, and hence, as we have discussed in Section 2.2.4 it should have three centers. Each of these centers must have integer GH charge. The sum of the three charges must be 1, in order for the manifold to be asymptotically $\mathbb{R}^4$. Moreover, we expect the geometric transition to be something that happens locally near the ring, and so we expect the region near the center of the ring (which is also the origin of our coordinate system) to remain the same. Hence, the GH center at the origin of the space must have charge + 1.

The conclusion of this argument is that the space that results from the geometric transition of a $U(1) \times U(1)$ invariant three-charge supertube must be a GH space with three

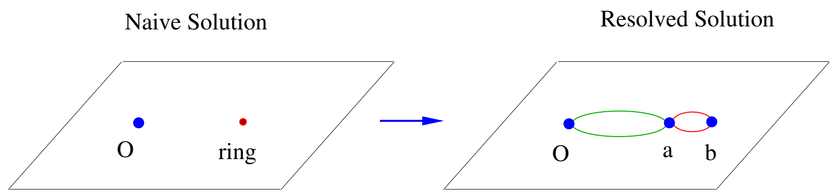

Figure 2.4: Geometric transition of supertube: The first diagram shows the geometry before the transition. The second shows the resolved geometry, where a pair of GH charges has nucleated at positions a and b.

centers, that have charges $1, \ +Q, \ -Q$, where Q is any integer. In the "transitioned" solution, the singularity of the zero-entropy black ring is resolved by the nucleation, or "pair creation," of two equal and oppositely charged GH points.

This process is depicted in Fig. 2.4. The nucleation of a GH pair of oppositely-charged centers blows up a pair of two-cycles. In the resolved geometry there are no more brane sources, only fluxes through the two-cycles. The charge of the solution does not come from any brane sources, but from having non-trivial fluxes over intersecting two-cycles (or "bubbles").

Similarly, if one considers the geometric transition of multiple concentric black rings, one will nucleate one pair of GH points for each ring, resulting in a geometry with no brane sources, and with a very large number of positive and negative GH centers. As we will see, these centers are not restricted to be on a line, but can have arbitrary positions in the $\mathbb{R}^3$ base of the GH space, as long as certain algebraic equations (discussed in Section 2.2.11) are satisfied.

There is one further piece of physical intuition that is extremely useful in understanding these bubbled geometries. As we have already remarked, GH points can be interpreted, from a ten-dimensional type IIA perspective, as D6 branes. Since these branes are mutually BPS, there should be no force between them. On the other hand, D6 branes of opposite charge attract one another, both gravitationally and electromagnetically. If one simply compactifies M-theory to an ambipolar GH space, one can only hold in equilibrium GH points of opposite charge at the cost of having large regions where the metric has the wrong signature and CTC's. To eliminate these singular regions, one must hold the GH points apart by some other mechanism. In the geometries we seek, this is done by having fluxes threading the bubbles: Collapsing a bubble concentrates the energy density of the flux and increases the energy in the flux sector. Thus a flux tends to blow up a cycle. The regular, ambipolar BPS configurations that we construct come about when these two competing effects - the tendency of oppositely charged GH points to attract each other and the tendency of the fluxes to make the bubbles large - are in balance. We will see precisely how this happens in Section 2.2.11.

Hence, we have arrived at the following conclusion: The singularity of the zero-entropy black ring is resolved by the nucleation of GH centers of opposite charge. The solutions that result, as well as other three-charge microstate solutions, are topologically non-trivial, have no brane sources, and are smooth despite the fact that they are constructed using an ambipolar GH metric with regions where the metric is negative-definite.

2.2.10 The bubbling solutions

We now proceed to construct the general form of bubbling solutions constructed using an ambipolar Gibbons-Hawking base [26, 41, 120]. Remember that the harmonic function that determines the geometry of the GH base is

$$V = \varepsilon_0 + \sum_{j=1}^{N} \frac{q_j}{r_j} \,. \tag{2.63}$$

In Section 2.2.4 we saw that the two-forms, $\Theta^{(I)}$, will be regular, self-dual, harmonic two-forms, and thus representatives of the cohomology dual to the two-cycles, provided that the K^I have the form:

$$K^I \;=\; k_0^I \;+\; \sum_{j=1}^{N} \frac{k_j^I}{r_j}\,. \tag{2.64}$$

Moreover, from (2.41), the flux of the two-form, $\Theta^{(I)}$, through the two-cycle Δ_{ij} is given by

$$\Pi_{ij}^{(I)} \;=\; \left(\frac{k_j^I}{q_j} \;-\; \frac{k_i^I}{q_i}\right)\,, \qquad 1 \le i,j \le N\,. \tag{2.65}$$

The functions, L_I and M, must similarly be chosen to ensure that the warp factors, Z_I, and the function, μ, are regular as $r_j \to 0$. This means that we must take:

$$L^I \;=\; \ell_0^I \;+\; \sum_{j=1}^{N} \frac{\ell_j^I}{r_j}\,, \quad M \;=\; m_0 \;+\; \sum_{j=1}^{N} \frac{m_j}{r_j}\,, \tag{2.66}$$

with

$$\ell_j^I \quad = \quad -\tfrac{1}{2}\, C_{IJK}\, \frac{k_j^J\, k_j^K}{q_j}\,, \qquad j = 1, \dots, N\,; \tag{2.67}$$

$$m_j \quad = \quad \tfrac{1}{12}\, C_{IJK} \frac{k_j^I\, k_j^J\, k_j^K}{q_j^2} \;=\; \tfrac{1}{2}\, \frac{k_j^1\, k_j^2\, k_j^3}{q_j^2}\,, \qquad j = 1, \dots, N\,. \tag{2.68}$$

Since we have now fixed the eight harmonic functions, all that remains is to solve for ω in equation (2.47). The right-hand side of (2.47) has two kinds of terms:

$$\frac{1}{r_i}\,\vec{\nabla}\,\frac{1}{r_j} \;-\; \frac{1}{r_j}\,\vec{\nabla}\,\frac{1}{r_i} \qquad \text{and} \qquad \vec{\nabla}\,\frac{1}{r_i}\,. \tag{2.69}$$

Hence ω will be built from the vectors $\vec{v}_i$ of (2.37) and some new vectors, $\vec{w}_{ij}$, defined by:

$$\vec{\nabla} \times \vec{w}_{ij} \;=\; \frac{1}{r_i}\,\vec{\nabla}\,\frac{1}{r_j} \;-\; \frac{1}{r_j}\,\vec{\nabla}\,\frac{1}{r_i}\,. \tag{2.70}$$

To find a simple expression for $\vec{w}_{ij}$ it is convenient to use the coordinates outlined earlier with the z-axis running through $\vec{y}^{(i)}$ and $\vec{y}^{(j)}$. Indeed, choose coordinates so that $\vec{y}^{(i)} = (0,0,a)$ and $\vec{y}^{(j)} = (0,0,b)$ and one may take $a > b$. Then the explicit solutions may be written very simply:

$$w_{ij} \;=\; -\,\frac{(y_1^2 + y_2^2 + (y_3 - a)(y_3 - b))}{(a-b)\, r_i\, r_j}\, d\phi\,. \tag{2.71}$$

This is then easy to convert to a more general system of coordinates. One can then add up all the contributions to ω from all the pairs of points.

There is, however, a more convenient basis of vector fields that may be used instead of the w_{ij}. Define:

$$\omega_{ij} \;\equiv\; w_{ij} + \tfrac{1}{(a-b)}\big(v_i - v_j + d\phi\big) \;=\; -\,\frac{(y_1^2 + y_2^2 + (y_3 - a + r_i)(y_3 - b - r_j))}{(a-b)\, r_i\, r_j}\, d\phi\,, \tag{2.72}$$

These vector fields then satisfy:

$$\vec{\nabla} \times \vec{\omega}_{ij} \;=\; \frac{1}{r_i}\,\vec{\nabla}\,\frac{1}{r_j} \;-\; \frac{1}{r_j}\,\vec{\nabla}\,\frac{1}{r_i} \;+\; \frac{1}{r_{ij}}\left(\vec{\nabla}\,\frac{1}{r_i} \;-\; \vec{\nabla}\,\frac{1}{r_j}\right), \tag{2.73}$$

where

$$r_{ij} \;\equiv\; |\vec{y}^{(i)} \;-\; \vec{y}^{(j)}| \tag{2.74}$$

is the distance between the $i^{\rm th}$ and $j^{\rm th}$ center in the Gibbons-Hawking metric.

We then see that the general solution for $\vec{\omega}$ may be written as:

$$\vec{\omega} \;=\; \sum_{i,j}^{N} a_{ij}\,\vec{\omega}_{ij} \;+\; \sum_{i}^{N} b_i\,\vec{v}_i\,, \tag{2.75}$$

for some constants a_{ij}, b_i.

The important point about the ω_{ij} is that they have no string singularities whatsoever. They can be used to solve (2.47) with the first set of source terms in (2.69), without introducing Dirac-Misner strings, but at the cost of adding new source terms of the form of the second term in (2.69). If there are N source points, $\vec{y}^{(j)}$, then using the w_{ij} suggests that there are $\frac{1}{2}N(N-1)$ possible string singularities associated with the axes between every pair of points $\vec{y}^{(i)}$ and $\vec{y}^{(j)}$. However, using the ω_{ij} makes it far more transparent that all the string singularities can be reduced to those associated with the second set of terms in (2.69) and so there are at most N possible string singularities and these can be arranged to run in any direction from each of the points $\vec{y}^{(j)}$.

Finally, we note that the constant terms in (2.26), (2.64) and (2.66) determine the behavior of the solution at infinity. If the asymptotic geometry is Taub-NUT, all these constants can be non-zero, and they correspond to combinations of the moduli. However, in order to obtain solutions that are asymptotic to five-dimensional Minkowski space, $\mathbb{R}^{4,1}$, one must take $\varepsilon_0 = 0$ in (2.26), and $k_0^I = 0$ in (2.64). Moreover, μ must vanish at infinity, and this fixes m_0. For simplicity we also fix the asymptotic values of the moduli that give the size of the three T^2's, and take $Z_I \to 1$ as $r \to \infty$. Hence, the solutions that are asymptotic to five-dimensional Minkowski space have:

$$\varepsilon_0 = 0\,, \qquad k_0^I = 0\,, \qquad l_0^I = 1\,, \qquad m_0 = -\tfrac{1}{2}\, q_0^{-1} \sum_{j=1}^{N} \sum_{I=1}^{3} k_j^I\,. \tag{2.76}$$

It is straightforward to generalize these results to solutions with different asymptotics, and in particular to Taub-NUT. We will do this in Chapter 3.

2.2.11 The bubble equations

In Section 2.2.7 we examined the conditions for the absence of CTC's and in general the following must be true globally:

$$\mathcal{Q} \ \geq\ 0\,, \qquad V\, Z_I \ =\ \tfrac{1}{2}\, C_{IJK}\, K^J\, K^K \ +\ L_I\, V \ \geq\ 0\,, \qquad I = 1, 2, 3\,. \tag{2.77}$$

As yet, we do not know how to verify these conditions in general, but one can learn a great deal by studying the limits in which one approaches a Gibbons-Hawking point, *i.e.* $r_j \rightarrow 0$. From this one can derive some simple, physical conditions (the bubble equations) that in some examples ensure that (2.77) are satisfied globally.

To study the limit in which $r_j \rightarrow 0$, it is simpler to use (2.52) than (2.54). In particular, as $r_j \rightarrow 0$, the functions, Z_I, μ and W limit to finite values while V^{-1} vanishes. This means that the circle defined by ψ will be a CTC unless we impose the additional condition:

$$\mu(\vec{y} = \vec{y}^{(j)}) \ =\ 0\,, \quad j = 1, \dots, N\,. \tag{2.78}$$

There is also potentially another problem: The small circles in ϕ near $\theta = 0$ or $\theta = \pi$ will be CTC's if ω has a finite $d\phi$ component near $\theta = 0$ or $\theta = \pi$. Such a finite $d\phi$ component corresponds precisely to a Dirac-Misner string in the solution and so we must make sure that ω has no such string singularities.

It turns out that these two sets of constraints are exactly the same. One can check this explicitly, but it is also rather easy to see from (2.44). The string singularities in $\vec{\omega}$ potentially arise from the $\vec{\nabla}(r_j^{-1})$ terms on the right-hand side of (2.44). We have already arranged that the Z_I and μ go to finite limits at $r_j = 0$, and the same is automatically true of $K^I V^{-1}$. This means that the only term on the right hand side of (2.44) that could, and indeed will, source a string is the $\mu \vec{\nabla} V$ term. Thus removing the string singularities is equivalent to (2.78).

One should note that by arranging that μ, ω and Z_I are regular one has also guaranteed that the physical Maxwell fields, $dA^{(I)}$, in (2.61) are regular. Furthermore, by removing the Dirac strings in ω and by requiring μ to vanish at GH points one has guaranteed that the physical flux of $dA^{(I)}$ through the cycle Δ_{ij} is still given by (2.41) (and (2.65)). This is because the extra terms, $d(Z_I^{-1}k)$, in (2.61), while canceling the singular behaviour when $V = 0$, as in (2.62), give no further contribution in (2.41). Thus the fluxes, $\Pi_{ij}^{(I)}$, are well-defined and represent the true physical, magnetic flux in the five-dimensional extension of the ambipolar GH metrics.

Performing the expansion of μ using (2.46), (2.64), (2.66) and (2.68) around each Gibbons-Hawking point one finds that (2.78) becomes the bubble equations:

$$\frac{1}{6} C_{IJK} \sum_{j=1,\, j\neq i}^{N} \Pi_{ij}^{(I)} \Pi_{ij}^{(J)} \Pi_{ij}^{(K)} \frac{q_i q_j}{r_{ij}} \;=\; 2(\varepsilon_0 m_i - m_0 q_i) + \sum_{I=1}^{3} (k_0^I l_i^I - l_0^I k_i^I) \tag{2.79}$$

for $i = 1, \dots, N$, and where $r_{ij} \equiv |\vec{y}^{(i)} - \vec{y}^{(j)}|$. Summing both sides of this equation and using the skew-symmetry of $\Pi_{ij}^{(I)}$ leads to:

$$m_0 \;=\; q_0^{-1} \Big(\varepsilon_0\, m_i - \frac{1}{2} \sum_{j=1}^{N} \sum_{I} \big(l_0^I\, k_j^I - k_0^I\, \ell_j^I \big) \Big) \,. \tag{2.80}$$

This is the generalization of (2.76) for general values of ε_0 and k_0^I, which is simply the condition $\mu \to 0$ as $r \to \infty$ and means that there is no Dirac-Misner string running out to infinity. Thus there are only $(N-1)$ independent bubble equations.

We refer to (2.79) as the bubble equations because they relate the flux through each bubble to the physical size of the bubble, represented by r_{ij}. Note that for a generic configuration, a bubble size can only be non-zero if and only if all three of the fluxes are non-zero. Thus the bubbling transition will only be generically possible for the three-charge system. We should also note that from a four-dimensional perspective these equations describe a collection of BPS stacks of branes, and are thus particular case of a collection of BPS black holes. Such configurations have been constructed in [77, 15, 78], and the equations that must be satisfied by the positions of the black holes are called "integrability equations" and reduce to the bubble equations when the charges are such that the five-dimensional solution is smooth.

While the bubble equations are necessary to avoid CTC's near the Gibbons-Hawking points, they are not sufficient to guarantee the absence of CTC's globally. It has been shown numerically in some non-trivial examples that the bubble equations do indeed ensure the global absence of CTC's. It is an open question as to how and when a bubbled configuration that satisfies (2.79) is globally free of CTC's, see [30] for a more detailed discussion on this issue.

2.2.12 Asymptotic charges

One can obtain the electric charges and angular momenta of bubbled geometries by expanding Z_I and k at infinity. It is, however, more convenient to translate the asymptotics into the standard coordinates of the Gibbons-Hawking spaces. Thus, remembering that $r = \frac{1}{4}\rho^2$, one has

$$Z_I \sim 1 + \frac{Q_I}{4\,r} + \ldots\,, \qquad \rho \to \infty\,, \tag{2.81}$$

and from (2.42) one easily obtains

$$Q_I = -\,2\,C_{IJK} \sum_{j=1}^{N} q_j^{-1}\,\hat{k}_j^J\,\hat{k}_j^K\,, \tag{2.82}$$

where

$$\hat{k}_j^I \equiv k_j^I - q_j\,N\,k_0^I\,, \qquad \text{and} \qquad k_0^I \equiv \frac{1}{N}\sum_{j=1}^{N} k_j^I\,. \tag{2.83}$$

Note that $\hat{k}^I_j$ is gauge invariant under (2.34).

One can read off the angular momenta using an expansion like that of (2.18). However, it is easiest to re-cast this in terms of the Gibbons-Hawking coordinates. The flat GH metric (near infinity) has $V = \frac{1}{r}$ and using the change of variables (2.24) in (2.18) one finds that

$$k \ \sim \ \frac{1}{4\,\rho^2}\Big((J_1 + J_2) \ + \ (J_1 - J_2)\,\cos\theta\Big)\,d\psi \ + \ \dots\,. \tag{2.84}$$

Thus, one can get the angular momenta from the asymptotic expansion of $g_{t\psi}$, which is given by the coefficient of $d\psi$ in the expansion of k, which is proportional to μ. There are two types of such terms, the simple $\frac{1}{r}$ terms and the dipole terms arising from the expansion of $V^{-1}K^I$. Following [41], define the dipoles

$$\vec{D}_j \ \equiv \ \sum_I \hat{k}^I_j\,\vec{y}^{(j)}\,, \qquad \vec{D} \ \equiv \ \sum_{j=1}^N \vec{D}_j\,, \tag{2.85}$$

and then the expansion of k takes the form (2.84) if one takes $\vec{D}$ to define the polar axis from which θ is measured. One then finds the "left" and "right" angular momenta of the five-dimensional solutions

$$J_R \ \equiv \ J_1 + J_2 \ = \ \tfrac{4}{3}\,C_{IJK}\sum_{j=1}^N q_j^{-2}\,\hat{k}^I_j\,\hat{k}^J_j\,\hat{k}^K_j\,, \tag{2.86}$$

$$J_L \ \equiv \ J_1 - J_2 \ = \ 8\,\big|\vec{D}\big|\,. \tag{2.87}$$

One can use the bubble equations to obtain another, rather more intuitive expression for $J_1 - J_2$. One should first note that the right-hand side of the bubble equation, (2.79), may be written as $-\sum_I \hat{k}_i^I$. Multiplying this by $\vec{y}^{(i)}$ and summing over i yields:

$$\begin{aligned} \vec{J}_L &\equiv 8\,\vec{D} \;=\; -\,\tfrac{4}{3}\,C_{IJK} \sum_{\substack{i,j=1 \\ j\neq i}}^{N} \Pi_{ij}^{(I)}\,\Pi_{ij}^{(J)}\,\Pi_{ij}^{(K)}\;\frac{q_i\,q_j\,\vec{y}^{(i)}}{r_{ij}} \\ &= -\tfrac{2}{3}\,C_{IJK} \sum_{\substack{i,j=1 \\ j\neq i}}^{N} q_i\,q_j\,\Pi_{ij}^{(I)}\,\Pi_{ij}^{(J)}\,\Pi_{ij}^{(K)}\;\frac{(\vec{y}^{(i)} - \vec{y}^{(j)})}{\big|\vec{y}^{(i)} - \vec{y}^{(j)}\big|}\,, \end{aligned} \tag{2.88}$$

where we have used the skew symmetry $\Pi_{ij} = -\Pi_{ji}$ to obtain the second identity. This result suggests that one should define an angular momentum flux vector associated with the $ij^{\rm th}$ bubble:

$$\vec{J}_{L\,ij} \;\equiv\; -\,\tfrac{4}{3}\,q_i\,q_j\,C_{IJK}\,\Pi_{ij}^{(I)}\,\Pi_{ij}^{(J)}\,\Pi_{ij}^{(K)}\;\hat{y}_{ij}\,, \tag{2.89}$$

where $\hat{y}_{ij}$ are *unit* vectors,

$$\hat{y}_{ij} \;\equiv\; \frac{(\vec{y}^{(i)} - \vec{y}^{(j)})}{\big|\vec{y}^{(i)} - \vec{y}^{(j)}\big|}\,. \tag{2.90}$$

This means that the flux terms on the left-hand side of the bubble equation actually have a natural spatial direction, and once this is incorporated, it yields the contribution of the bubble to J_L.

2.3 Summary and open problems

In this Chapter we have reviewed the construction of a large class of regular asymptotically flat supergravtiy solutions with four supercharges. The solutions are determined by fixing a GH base with non-trivial two-cycles and distributing cohomological fluxes on these two-cycles. The solution is then completely fixed by imposing regularity and causality. It was shown in [28, 29] that a generic distribution of fluxes on the two-cycles will result in a regular solution with charges that are the same as those of a black hole or black ring with vanishing entropy. To get microstate geometries corresponding to black holes and rings with non-zero entropy one needs “scaling solutions” [29, 32]. These are regular solutions with a GH base in which the fluxes are distributed in such a way that a subset of the GH points can come arbitrarily close together on the four dimensional base and still obey the bubble equations (2.79). In the five-dimensional solution the GH points are still finite metric distance apart and the topological cycles are of finite size. Such scaling solutions develop long AdS throats and look very much like a black hole or black ring. One can also show that the lowest energy excitations in the sclaing solution have energy of the same order as the mass gap in the dual CFT [29]. This suggests that scaling solutions correspond to the typical microstates of the three-charge supersymmetric black hole (or black ring). In [79] it was also shown, following a completely different approach, that the scaling solutions will play an important role in accounting for the entropy of the three-charge black hole. The authors of [79] studied at weak (but non-zero) coupling the quiver quantum field theory on the world-volume of the D-branes that form the black hole and found that only quiver theories which corresponds to a scaling solutions in supergravity have large enough ground state degeneracies to account for the black hole entropy.

It is clear from the results presented in this Chapter that black-hole uniqueness is violated in string theory and M-theory. Microstate geometries provide large families of interesting smooth solutions with no horizons that have the same asymptotics at innity as a supersymmetric black hole or black ring. While this is interesting in its own right, it is also possible that there might be enough microstate geometries to account for the classical black-hole entropy. It is clear that it will not be possible to use the supergravity approximation to describe every black hole microstate but it is yet not clear whether supergravity is able to sample enough of the states in the Hilbert space to account for a macroscopic of the black hole entropy. To establish this one needs a proper way to quantize the moduli space of regular supergravity solutions which will in turn provide an effective way to compute how much entropy they can account for. There were some recent attempts to do this [74, 75, 13, 76] using techniques from geometric quantization [68]. However the results of this counting are not conclusive because the authors studied only a restrictive class of regular solutions.

The regular solutions with a GH base represent only a limited subset of all possible three-charge BPS solutions. The GH metrics are hyper-Kähler metrics with a very special $U(1)$ isometry and a general black hole microstate geometry will not have this isometry. In Chapter 5 we will dsicuss how to construct supergravity solutions with a $U(1)$ isometry that is less restrictive. More generally one would expect a large number of BPS solutions with no isometry, this expectation is based on the fact that there are three-charge supertubes of arbitrary shape in the probe approximation, which after backreacting on the geometry, should yield supergravity solutions with no isometry. To construct these more general BPS solutions one has to use hyper-Kähler metrics with no isometries. To the best of our knowledge there are no such explicit metrics known in the literature and this present a formidable technical difficulty. However one can use a combination of string dualities and the physics of supertubes to make some progress in constructing new microstate geometries and counting them, this will be discussed in Chapters 3 and 4.

An important open problem in the study of microstate geometries is how to construct large number of regular non-supersymmetric solutions with the same charges and asymptotic structure as non-supersymmetric black holes and black rings. After all the black holes that are observed in Nature are not supersymmetric and ultimately one would like to understand their structure. After one breaks supersymmetry one loses a lot of technical simplifications. This usually means that it is very hard to construct explicit gravitational solutions due to the non-linearities of the gravitaional equations of motion. Nevertheless, there has been some recent progress in the construction of regular non-supersymmetric solutions and we will discuss this in Chapter 6.

Chapter 3

Spectral flow from supergravity

In the quest to understand black hole entropy in terms of microstate geometries, two problems appear to be most difficult to overcome. The first is to determine which of the microstate solutions are more "typical" than others. The second is to construct very large classes of microstate solutions whose counting can give the black hole entropy.

Spectral flow has proven to be a useful tool in addressing these kinds of questions. In the dual conformal field theory the spectral flow operation is initiated by redefining the R-charge current by mixing it with some other conserved $U(1)$ current. This then requires a modification of the Hamiltonian in order to preserve the supersymmetry. In the bulk gravity theory, the $U(1)$ R-current and the other conserved $U(1)$ current are dual to isometries of the background and spectral flow can be achieved simply by a change of coordinates that mixes these two $U(1)$ directions. One can then add an asymptotically flat region to this new geometry to obtain a geometry that has different charges from the original. This is an effective method of obtaining some five-dimensional three-charge and four-dimensional four-charge microstate geometries from two-charge geometries [159, 118, 119, 120, 24, 143]. In addition, spectral flow can be used to determine exactly the CFT state dual to the black hole microstate one constructs, and hence is a useful tool in determining how typical a certain microstate geometry is.

Despite its usefulness, spectral flow appears to be a rather cumbersome operation on asymptotically flat five-dimensional geometries: One must first strip the geometry of its asymptotically-flat region, then perform the spectral flow, and then add back the aysmptotically-flat geometry. The last step can be quite non-trivial, especially for geometries that do not have a large number of isometries (see, for example [95]).

In this Chapter, based on [33] , we explore a simpler way to use spectral flow to generate asymptotically four-dimensional geometries starting from other asmptotically four-dimensional geometries, without stripping away the asymptotically flat region. This method has two immediate applications which we believe are quite useful in the program of constructing microstate geometries and finding their CFT dual. First, it allows us to use a known microstate solution to generate a huge number of other smooth microstate solutions. Secondly, it gives us new insights into which microstate geometries represent bound states in the CFT. Since a configuration that consists purely of concentric, two-charge supertubes is unbound, any spectral flow of this will give unbound states. In particular, we expect such solutions will not correspond to CFT states in the sector that is primarily responsible for the entropy. We will use this observation to examine the status of some of the microstate geometries that have been studied in the past.

The fact that one can relate bubbling solutions with a Gibbons-Hawking, multi-centered base to solutions with a supertube in a bubbling background also indicates that in the vicinity of the black hole microstates with a GH base there exists a very large family of other, less symmetric microstate solutions with the same macroscopic charges. Indeed, we know from the Dirac-Born-Infeld (DBI) action that two-charge supertubes can have arbitrary shapes [166], and that these arbitrary shapes correspond (upon dualizing to the D1-D5-P duality frame) to smooth geometries [156, 158]. Hence, one can use spectral flow to transform a GH center into a supertube, wiggle the supertube, and undo the spectral flow, to obtain bubbling three-charge solutions that depend classically on several arbitrary continuous functions. Hence the dimension of the moduli space of smooth black hole microstate solutions is classically infinite. If, upon counting these solutions, one finds a black-hole-like entropy, this will be, in our opinion, compelling evidence that the microstates of black holes are given by horizonless configurations. In Chapter 4 we will indeed argue that for the deep, smooth microstate solutions of [29, 32] one can obtain an entropy with the correct charge dependence using the methods outlined here.

To clarify the relationship of the solutions discussed here with some earlier results, we note that it was shown in [101, 21] that general BPS configurations with the same supersymmetries as a black hole or black ring require that the four-dimensional spatial base of the solution be hyper-Kähler. It should be remembered that in establishing this result it was assumed that the solution was independent of the internal directions of the compactification tori. The solutions that we discuss here, which come from the spectral flow of supertubes of arbitrary shape, necessarily depend upon one of these internal directions. Hence, they are more general than those considered in [21], corresponding to solutions of ungauged supergravity in six dimensions [131], and their base space is not hyper-Kähler but almost hyper-Kähler.

3.1 A chain of dualities

Three-charge solutions with four supercharges are most simply written in the M-theory duality frame in which the three charges are treated most symmetrically and correspond to three types of M2 branes wrapping three T^2's inside T^6 [21]. These solutions were presented in some detail in the previous Chapter. Here we will use string theory dualities to transform the solutions to solutions of type IIA and IIB supergravity.

In order to study spectral flow as well as two-charge supertubes in three-charge backgrounds, it is useful to dualize to a frame in which the two-charge supertube action is simple. One such frame is where the three electric charges of the background correspond to D0 branes, D4 branes and F1 strings and the supertube carries D0 and F1 electric charges and D2 dipole charge [166]. On the other hand, in order to study the supergravity solutions describing supertubes in black-ring or bubbling backgrounds, it is useful to work in a duality frame in which the supergravity solution for the supertubes is smooth. In this frame the electric charges of the background correspond to D1 branes, D5 branes, and momentum P, and the supertube carries D1 and D5 charges, with KKM dipole charge. We therefore dualize the foregoing M-theory solution to these frames and give all the details of the solutions explicitly. More details about this procedure can be found in Appendix A and B.

3.1.1 Three-charge solutions in the D0-D4-F1 duality frame

Here we will present the three-charge solutions in the duality frame in which they have electric charges corresponding to D0 branes, D4 branes, and F1 strings, and dipole charges corresponding to D6, D2 and NS5 branes. We use the T-duality rules (given in Appendix A) to transform field-strengths. It should be emphasized that our results are correct for any three-charge solution (including those without a tri-holomorphic $U(1)$ [31]), however, finding the explicit form of the RR and NS-NS potentials (which is crucial if we want to investigate this solution using probe supertubes) is straightforward only when the solution can be written in Gibbons-Hawking form.

Label the coordinates by $(x^0, \dots, x^8, z)$[1]. The electric charges N_1, N_2 and N_3 of the solution then correspond to:

$$N_1 : D0 \qquad N_2 : D4\,(5678) \qquad N_3 : F1\,(z) \tag{3.1}$$

[1]See Appendix A for more details about the brane configuration that we use.

where the numbers in the parentheses refer to spatial directions wrapped by the branes and $z \equiv x^{10}$. The magnetic dipole moments of the solutions correspond to:

$$n_1 : \; D6\,(y5678z) \qquad n_2 : \; D2\,(yz) \qquad n_3 : \; NS5\,(y5678)\,, \tag{3.2}$$

where y denotes the brane profile in the spatial base, $(x^1, \ldots, x^4)$. The metric of the solution is:

$$ds^2_{IIA} = -\frac{1}{Z_3\sqrt{Z_1 Z_2}}(dt+k)^2 + \sqrt{Z_1 Z_2}\,ds_4^2 + \frac{\sqrt{Z_1 Z_2}}{Z_3}dz^2 + \sqrt{\frac{Z_1}{Z_2}}(dx_5^2 + dx_6^2 + dx_7^2 + dx_8^2)\,. \tag{3.3}$$

The dilaton and the Kalb-Ramond fields are:

$$\Phi = \frac{1}{4}\log\left(\frac{Z_1^3}{Z_2 Z_3^2}\right), \qquad B = -dt \wedge dz - A^{(3)} \wedge dz\,. \tag{3.4}$$

The RR field strengths are

$$F^{(2)} = -\mathcal{F}^{(1)}\,, \qquad \widetilde{F}^{(4)} = -\left(\frac{Z_2^5}{Z_1^3 Z_3^2}\right)^{1/4} \star_5 (\mathcal{F}^{(2)}) \wedge dz\,, \tag{3.5}$$

where we define $\mathcal{F}^{(I)} \equiv dA^{(I)}$ and $\star_5$ is the Hodge dual with respect to the five dimensional metric:

$$ds_5^2 = -\frac{1}{Z_3\sqrt{Z_1 Z_2}}(dt+k)^2 + \sqrt{Z_1 Z_2}\,ds_4^2\,. \tag{3.6}$$

The foregoing results are valid for any three-charge solution with an arbitrary hyper-Kähler base. As we show in Appendix B, when the base has a Gibbons-Hawking metric one can easily find the RR 3-form potential:

$$C^{(3)} \;=\; \left(\zeta_a \,+\, V^{-1}K^3\xi_a^{(1)}\right)\Omega_-^{(a)} \wedge dz - \left(Z_3^{-1}(dt+k)\wedge B^{(1)} \,+\, dt \wedge A^{(3)}\right)\wedge dz\,, \tag{3.7}$$

where $\xi_a^{(1)}$ and ζ_a are defined by equations (2.35) and (B.29). Thus we have the full three-charge supergravity solution in the D0-D4-F1 duality frame.

3.1.2 Three-charge solutions in the D1-D5-P duality frame

One can T-dualize the solution above along z to obtain a solution with D1, D5 and momentum charges:

$$N_1 : \; D1\,(z) \qquad N_2 : \; D5\,(5678z) \qquad N_3 : \; P\,(z) \tag{3.8}$$

and dipole moments corresponding to wrapped D1 branes, D5 branes and Kaluza Klein Monopoles (kkm) [194]:

$$n_1 : \; D5\,(y5678) \qquad n_2 : \; D1\,(y) \qquad n_3 : \; kkm\,(y5678z)\,. \tag{3.9}$$

The metric is

$$\begin{aligned} ds^2_{IIB} = -\frac{1}{Z_3\sqrt{Z_1 Z_2}}\,(dt+k)^2 \quad &+ \quad \sqrt{Z_1 Z_2}\,ds_4^2 + \frac{Z_3}{\sqrt{Z_1 Z_2}}\,(dz + A^{(3)})^2 \\ &+ \quad \sqrt{\frac{Z_1}{Z_2}}\,(dx_5^2 + dx_6^2 + dx_7^2 + dx_8^2) \end{aligned} \tag{3.10}$$

and the dilaton and the Kalb-Ramond field are:

$$\Phi = \frac{1}{2}\log\left(\frac{Z_1}{Z_2}\right), \qquad\qquad B = 0\,. \tag{3.11}$$

The only non-zero RR three-form field strength is:

$$F^{(3)} = -\left(\frac{Z_2^5}{Z_1^3 Z_3^2}\right)^{1/4} \star_5 (\mathcal{F}^{(2)}) - \mathcal{F}^{(1)} \wedge (dz - A^{(3)})\,. \tag{3.12}$$

If we specialize our general result to the supersymmetric black ring solution in the D1-D5-P frame then it agrees (up to conventions) with [88]. It is also elementary to find the RR two-form potential for a general BPS solution with GH base in D1-D5-P frame. This can be done by T-dualizing the IIA D0-D4-F1 result (3.7), to obtain:

$$C^{(2)} = \big(\zeta_a + V^{-1}K^3\xi^{(1)}_a\big)\,\Omega^{(a)}_- - \big(Z_3^{-1}(dt+k)\wedge B^{(1)} + dt\wedge A^{(3)}\big) + A^{(1)}\wedge(A^{(3)} - dz - dt) + dt\wedge(A^3 - dz)\,, \qquad (3.13)$$

where again $\xi^{(1)}_a$ and ζ_a are defined in equations (2.35) and (B.29). This is the full three-charge supergravity solution in the D1-D5-P duality frame. As shown in [158], two-charge supertubes in flat space are regular only in this duality frame, so our general result can be used to analyze the regularity of two charge supertubes in a general three-charge solution. This will be the subject of the next section.

3.2 Spectral flow

In this Section we discuss the three-charge BPS solutions in IIB duality frame, their relation to solutions of six-dimensional supergravity, and the way in which the spectral flow transformation acts on solutions with a $U(1)$ isometry on the base. In Section 3.2.3 we specialize this to a translational $U(1)$ isometry where the solution has a multi-centered Gibbons-Hawking base. We show that spectral flow acts by interchanging the harmonic functions underlying these solutions, while keeping the solutions smooth. The explicit transformation is given in equations (3.26) and (3.27). We also show that spectral flow is part of a larger $SL(2,\mathbb{Z})^3$ subgroup of the four-dimensional $E_{7(7)}$ U-duality group, and this particular subgroup of $E_{7(7)}$ is distinguished because, for generic parameters, it generates orbits of smooth solutions.

In Section 3.2.6 we show that spectral flow can transform a configuration containing one or several supertubes in Taub-NUT into a multi-center bubbling solution; conversely, it can transform such a solution into a solution where at least one of the centers is replaced by a two-charge supertube. This demonstrates that the black hole microstates with a GH base constructed so far in the literature are part of an infinite-dimensional moduli space of smooth supersymmetric solutions. In Section 3.2.7 we explore the action of generalized spectral flow on multi-center D6-D4-D2-D0 configurations and use the physics of supertubes to argue that some multi-center configurations that appear bound from a four-dimensional perspective are in fact not bound when seen as full ten-dimensional solutions.

3.2.1 From six to five dimensions

As shown in Section 3.1.2 one can dualize the three-charge BPS solutions in a IIB frame in which the three fundamental charges are those of the D1-D5-P system. In this form, the D5-brane wraps a four-torus, T^4, while the D1-brane, the remaining spatial part of the D5-brane and the momentum follow a common S^1. The metric thus naturally decomposes into a six-dimensional part and the T^4-part (3.10). To facilitate comparison with some previous result in the literature [131] it is useful to rewrite the six-dimensional part of the metric as

$$ds_6^2 = -\frac{2}{H}(dv+\beta)\left(du + k + \frac{1}{2}F(dv+\beta)\right) + H\, h_{\mu\nu}dx^\mu dx^\nu \ , \tag{3.14}$$

where

$$H \ = \ \sqrt{Z_1 Z_2}\,, \qquad F \ = \ -\,Z_3\,, \qquad d\beta = \Theta^{(3)}\,. \tag{3.15}$$

In this formulation there is obviously no longer a symmetry between the three fundamental charges and, with the foregoing choices, Z_1 corresponds to the D1-charge, Z_2 to the D5-charge and Z_3 to the KK-momentum charge.

We have cast the six-dimensional metric in the form (3.14) because it affords the easiest comparison with previous work on the classification of all supersymmetric solutions of six-dimensional minimal supergravity, obtained in [131]. In the minimal theory, two of the $U(1)$ Maxwell fields, $\Theta^{(I)}$, are set equal and they appear in a three-form field strength:

$$G^{(3)} = d(H^{-1}(dv+\beta)\wedge(du+k)) + (dv+\beta)\wedge\mathcal{G}^{+} + \star_4 dH \tag{3.16}$$

with

$$\mathcal{G}^{+} \;=\; \Theta^{(1)} \;=\; \Theta^{(2)} \qquad d\beta = \Theta^{(3)}\,. \tag{3.17}$$

One also has $Z_1 = Z_2$. We will, however, not make this restriction here[2] but this earlier work is of relevance here because it allowed more general backgrounds that could depend upon the extra background coordinate, v. The spectral flow operations that we wish to consider could generate such v-dependent solutions. See, for example, [95].

The important point in this type IIB or six-dimensional form of the three-charge solutions is that one of the $U(1)$ gauge fields has been converted to a six-dimensional Kaluza-Klein field, β. This then puts it on the same footing as a $U(1)$ isometry on the four-dimensional base. In particular, one can then mix these two directions with a coordinate transformation and, as we will see, this generates a spectral flow transformation. One should also note that one has the freedom to choose which of the three $U(1)$ Maxwell fields in five dimensions will become the six-dimensional Kaluza-Klein field

[2]This corresponds to solutions of six-dimensional, ungauged supergravity with one tensor multiplet, and was studied in [56].

and so there are three independent ways of generating the spectral flow. We now discuss this in detail.

3.2.2 Spectral flow

From the six-dimensional perspective the operation of spectral flow is simply a coordinate change that mixes periodic coordinates on the base with the extra Kaluza-Klein coordinate, v (see, for example [11, 164]). When the base is asymptotic to $\mathbb{R}^4$, the size of the circles that are mixed with the Kaluza-Klein circle becomes infinite, and the spectral flow operation changes the asymptotics of the solution. We will bypass this problem by focusing on solutions that are asymptotically $\mathbb{R}^3 \times S^1$.

If the base metric has an isometry then one can adapt the coordinate system to that isometry and take the metric to be invariant under translations of a coordinate, τ. In particular, the base metric can be written in the form:

$$ds_4^2 \;=\; h_{\mu\nu} dx^\mu dx^\nu \;=\; V^{-1}(d\tau + A)^2 \;+\; V \gamma_{ij} dx^i dx^j \,, \tag{3.18}$$

where $i, j = 1, 2, 3$ and every component of the metric is independent of τ. The one-form, A, and the three-metric, γ_{ij} are, *a priori*, arbitrary[3].

We will also assume that the complete six-dimensional solution is invariant under τ-translations and for simplicity we will also assume that the six-dimensional solution is independent of v but neither of these assumptions is essential to the spectral flow transformations. It is convenient to decompose the one-forms, k and β, according to:

$$k \;=\; \mu(d\tau + A) \;+\; \omega \,, \qquad \beta \;=\; \nu(d\tau + A) \;+\; \sigma \,, \tag{3.19}$$

[3]However, the condition that the base metric be hyper-Kähler means that this metric can be completely determined by solving the $SU(\infty)$ Toda equation [48, 72, 10, 31], see also Chapter 5. This fact will not be needed here.

where ω and σ are one-forms in the three-dimensional space.

A spectral flow is then generated by the change of coordinate:

$$\tau \;\to\; \tau \,+\, \gamma\, v\,, \tag{3.20}$$

for some real parameter, γ. For this to be a well-defined coordinate transformation on the two circles, γ must be properly quantized[4]. More generally we could consider any global diffeomorphism in the $SL(2,\mathbb{Z})$ that acts on the two-torus defined by these $U(1)$'s. We will return to this later. The important point is that because these mappings are diffeomorphisms, they map regular solutions without closed time-like curves (CTC's) onto regular solutions without closed time-like curves.

Inserting (3.20) into (3.14), one can collect terms and restore the entire metric back to its canonical form, (3.14). One finds that this coordinate transformation is equivalent to:

$$ds_6^2 \;\to\; d\tilde{s}_6^2 \;\equiv\; -\,2\widetilde{H}^{-1}\,(dv+\tilde{\beta})\,\big(du+\tilde{k}+\tfrac{1}{2}\,\widetilde{\mathcal{F}}\,(dv+\tilde{\beta})\big) \;+\; \widetilde{H}\,d\tilde{s}_4\,, \tag{3.21}$$

where

$$\begin{aligned}
\widetilde{V} \quad &= \quad (1+\gamma\,\nu)\,V\,, \qquad \widetilde{A} \;=\; A-\gamma\,\sigma\,, \qquad \widetilde{H} \;=\; (1+\gamma\,\nu)^{-1}H\,,\\
\tilde{\beta} \quad &= \quad (1+\gamma\,\nu)^{-1}\beta\,, \quad \widetilde{F} \;=\; (1+\gamma\,\nu)\,F+2\gamma\mu+(1+\gamma\,\nu)^{-1}V^{-1}\gamma^2H^2,\\
\tilde{k} \quad &= \quad k \;-\; \frac{\gamma\mu}{(1+\gamma\,\nu)}\,\beta+\frac{\gamma^2H^2}{V\,(1+\gamma\,\nu)^2}\,\beta-\frac{\gamma H^2}{V\,(1+\gamma\,\nu)}\,(d\tau+A)\,.
\end{aligned} \tag{3.22}$$

For a general hyper-Kähler metric with a rotational $U(1)$ isometry, two of the three complex structures depend explicitly upon τ [10, 31] and so, after the shift (3.20), these

[4]For a Gibbons-Hawking base τ has a period of 4π and v has a period of 2π so γ has to be an even integer.

two complex structures depend upon v. As a result, the metric, $d\tilde{s}_4^2$ is almost-hyper-Kähler [131] but not hyper-Kähler. On the other hand, if the $U(1)$ isometry is translational then the hyper-Kähler metric may be put into Gibbons-Hawking form [109] and all three complex structures are independent of τ and so $d\tilde{s}_4^2$ will also be hyper-Kähler with a translational $U(1)$ isometry and hence must have Gibbons-Hawking form [113]. We now investigate this in more detail.

3.2.3 Spectral flow in Gibbons-Hawking metrics

We will now present the explicit spectral flow transformation on the three-charge solutions with a GH base discussed in Chapter 2. It is useful to rewrite the six-dimensional supergravity solution with a GH base (3.14) as [131]:

$$ds_6^2 = -\frac{F}{H}\left[dv + \beta + \frac{1}{F}(du + k)\right]^2 + \frac{1}{HF}(du + k)^2 + H\left[\frac{1}{V}(d\tau + A)^2 + V(dx^2 + dy^2 + dz^2)\right], \quad (3.23)$$

where one should recall that $H = \sqrt{Z_1 Z_2}$ and $F = -Z_3$. As before we define:

$$k = \mu(d\tau + A) + \omega\,, \qquad \beta = \nu(d\tau + A) + \sigma\,. \quad (3.24)$$

Starting from M-theory on T^6, one can choose to dualize to six dimensions so that any one of the K^I becomes the Kaluza-Klein potential, and if we take this to be K^3 then one has:

$$\nu \;=\; V^{-1}\, K^3\,, \qquad \vec{\nabla} K^3 \;=\; -\,\vec{\nabla} \times \vec{\sigma}\,. \quad (3.25)$$

The spectral flow transformation (3.22) then corresponds to:

$$\begin{aligned}
\widetilde{L}_3 &= L_3 - 2\gamma M\,, \qquad \widetilde{L}_2 = L_2\,, \qquad \widetilde{L}_1 = L_1\,, \\
\widetilde{K}^1 &= K^1 - \gamma L_2\,, \qquad \widetilde{K}^2 = K^2 - \gamma L_1\,, \qquad \widetilde{K}^3 = K^3\,, \\
\widetilde{V} &= V + \gamma K^3\,, \qquad \widetilde{M} = M\,, \qquad \vec{\widetilde{\omega}} = \vec{\omega}\,.
\end{aligned} \tag{3.26}$$

We can also consider a more general process in which each of the K^I's is successively chosen to be the special one, and a spectral flow, with parameter γ_I, is made. The result is:

$$\begin{aligned}
\widetilde{L}_I &= L_I - 2\gamma_I M\,, \qquad \widetilde{M} = M\,, \qquad \vec{\widetilde{\omega}} = \vec{\omega}\,, \\
\widetilde{K}^I &= K^I - C^{IJK}\gamma_J L_K + C^{IJK}\gamma_J\gamma_K M\,, \\
\widetilde{V} &= V + \gamma_I K^I - \tfrac{1}{2}C^{IJK}\gamma_I\gamma_J L_K + \tfrac{1}{3}C^{IJK}\gamma_I\gamma_J\gamma_K M\,,
\end{aligned} \tag{3.27}$$

where $C^{IJK} \equiv C_{IJK} \equiv |\epsilon_{IJK}|$. We will refer to the transformations (3.27) as "generalized spectral flow." Unlike the transformation (3.26), which transforms smooth six-dimensional solutions into other smooth six-dimensional solutions, the generalized spectral flow may, in some instances, transform a smooth solution to a duality frame in which it is no longer smooth. We will discuss this further later, in the remainder of this section we examine the $SL(2,\mathbb{Z})$ actions in more detail and explicitly verify how $SL(2,\mathbb{Z})$ transformations preserve regularity.

3.2.4 $SL(2,\mathbb{Z})$ transformations of bubbling solutions

The spatial part of the metric (3.23) may be thought of as a T^2 fibration over $\mathbb{R}^3$, where τ and v define the T^2 fiber. As we have seen, spectral flows are generated by the coordinate transformation (3.20). Similarly, it follows directly from (3.23) and (3.25) that the gauge transformations (2.48) with $c^1 = c^2 = 0, c^3 = c$ can be obtained from the coordinate transformation:

$$v \;\to\; v \,+\, c\,\tau\,. \tag{3.28}$$

More generally, one can make any $SL(2,\mathbb{Z})$ transformation in the global diffeomorphisms of the T^2 defined by (τ, v):

$$\begin{pmatrix} \tilde\tau \\ 2\tilde v \end{pmatrix} = \mathcal{M} \begin{pmatrix} \tau \\ 2v \end{pmatrix} = \begin{pmatrix} m & n \\ p & q \end{pmatrix} \begin{pmatrix} \tau \\ 2v \end{pmatrix}\,, \tag{3.29}$$

Here $\mathcal{M} \in SL(2,\mathbb{Z})$ and the factors of 2 insure the correct periodicities for the $\tilde\tau$ and $\tilde v$ coordinates. Since it is a diffeomorphism, any such transformation will take smooth (CTC-free) solutions to smooth (CTC-free) solutions.

If one uses this transformation in (3.23) one can easily recast the metric back into the same form:

$$\begin{aligned} d\tilde s_6^2 = -\frac{\widetilde F}{\widetilde H}\left[d\tilde v + \tilde\beta + \frac{1}{\widetilde F}(du + \tilde k)\right]^2 + \frac{1}{\widetilde H \widetilde F}(du+\tilde k)^2 \\ + \widetilde H\left[\frac{1}{\widetilde V}(d\tilde\tau + \widetilde A)^2 + \widetilde V(dx^2+dy^2+dz^2)\right]\,, \end{aligned} \tag{3.30}$$

where

$$\tilde k \;\equiv\; \tilde\mu(d\tilde\tau + \widetilde A) + \tilde\omega\,, \qquad \tilde\beta \;\equiv\; \tilde\nu(d\tilde\tau + \widetilde A) + \tilde\sigma\,, \tag{3.31}$$

and

$$\begin{aligned} &\widetilde V = (m-2n\nu)V, \qquad \widetilde H = \frac{H}{m-2n\nu}, \\ &\widetilde F = (m-2n\nu)F - 4n\mu - 4n^2\frac{H^2}{(m-2n\nu)V}\,, \\ &\tilde\nu = -\frac{\frac{p}{2} - q\nu}{m-2n\nu}, \qquad \tilde\mu = \frac{1}{m-2n\nu}\left(\mu + 2n\frac{H^2}{(m-2n\nu)V}\right)\,, \\ &\widetilde A = mA + 2n\sigma, \qquad \tilde\sigma = q\sigma + \tfrac{p}{2}A, \qquad \tilde\omega = \omega. \end{aligned} \tag{3.32}$$

The effect of this $SL(2,\mathbb{Z})$ transformation on the functions determining the underlying five-dimensional solutions is:

$$
\begin{aligned}
&\widetilde{V} = (m-2n\nu)V, \qquad \tilde{\mu} = \frac{V}{\widetilde{V}}\left(\mu + 2n\frac{Z_1 Z_2}{\widetilde{V}}\right) , \\
&\widetilde{Z}_1 = \frac{V}{\widetilde{V}} Z_1, \qquad \widetilde{Z}_2 = \frac{V}{\widetilde{V}} Z_2, \qquad \widetilde{Z}_3 = \frac{\widetilde{V}}{V} Z_3 + 4n\mu + 4n^2\frac{Z_1 Z_2}{\widetilde{V}} .
\end{aligned} \tag{3.33}
$$

Note that because the functions Z_I are gauge invariant, their transformations only depend upon the spectral flow parameter, $\gamma = -2n$.

Upon identifying the harmonic functions V, K^I, L^I and M that give the solution with the eight $E_{7(7)}$ parameters x and y (2.49), the $SL(2,\mathbb{Z})$ transformation becomes simply

$$
\begin{aligned}
&\begin{pmatrix} \tilde{y}_{12} \\ 2\tilde{x}_{34} \end{pmatrix} = \mathcal{M}\begin{pmatrix} y_{12} \\ 2x_{34} \end{pmatrix}, \qquad \begin{pmatrix} \tilde{y}_{34} \\ 2\tilde{x}_{12} \end{pmatrix} = \mathcal{M}\begin{pmatrix} y_{34} \\ 2x_{12} \end{pmatrix} \\
&\begin{pmatrix} \tilde{x}_{56} \\ 2\tilde{y}_{78} \end{pmatrix} = \mathcal{M}\begin{pmatrix} x_{56} \\ 2y_{78} \end{pmatrix}, \qquad \begin{pmatrix} \tilde{x}_{78} \\ 2\tilde{y}_{56} \end{pmatrix} = \mathcal{M}\begin{pmatrix} x_{78} \\ 2y_{56} \end{pmatrix}
\end{aligned} \tag{3.34}
$$

From the point of view of the five-dimensional solution, the transformation (3.34) is simply a subgroup of the $E_{7(7)}(\mathbb{Z})$ duality group that takes solutions into solutions. Nevertheless, the important feature of this transformation is that it takes smooth solutions into smooth solutions. As we will discuss below, for generic parameters, (3.34) transforms bubbling solutions into bubbling solutions, while for specific parameters it can transform them into bubbling solutions that contain one or several two-charge supertubes, with charges corresponding to x_{12} and x_{34}. As we will see, these solutions are smooth in the six-dimensional duality frame (3.14), but not in five-dimensions.

In order to arrive at the foregoing transformation we chose to dualize using the function K^3 to get the six-dimensional background. One can obviously use the other two functions, K^1 and K^2 and obtain two other $SL(2,\mathbb{Z})$ subgroups of $E_{7(7)}(\mathbb{Z})$. Indeed, these three $SL(2,\mathbb{Z})$'s commute with one another and thus form an $SL(2,\mathbb{Z})^3$ subgroup of $E_{7(7)}(\mathbb{Z})$. As could be expected this general $SL(2,\mathbb{Z})^3$ transformation leaves the quartic invariant J_4 unchanged. This is discussed further in [33].

3.2.5 Regularity and the bubble equations

Suppose that the harmonic functions take their usual form for an ambi-polar Gibbons-Hawking base

$$V = \varepsilon_0 + \sum_{j=1}^{N} \frac{q_j}{r_j}, \qquad K^I = k_0^I + \sum_{j=1}^{N} \frac{k_j^I}{r_j}, \tag{3.35}$$

$$L^I = l_0^I + \sum_{j=1}^{N} \frac{l_j^I}{r_j}, \qquad M = m_0 + \sum_{j=1}^{N} \frac{m_j}{r_j}, \tag{3.36}$$

where $r_j \equiv |\vec{y} - \vec{y}^{(j)}|$ and ε_0, q_j, k_a^I, l_a^I, m_a ($a = 0, 1, \ldots, N$) are, as yet, arbitrary constants. As usual define:

$$q_0 \equiv \sum_{j=1}^{N} q_j, \qquad \tilde{k}_j^I \equiv k_j^I - q_0^{-1} q_j \sum_{j=1}^{N} k_j^I, \qquad \Pi_{ij}^{(I)} \equiv \left(\frac{k_j^I}{q_j} - \frac{k_i^I}{q_i} \right). \tag{3.37}$$

Recall that for the functions Z_I and μ to be regular as $r_j \to 0$, one must take:

$$l_j^I = -\frac{1}{2} C_{IJK} \frac{k_j^J k_j^K}{q_j}, \qquad m_j = \frac{1}{12} C_{IJK} \frac{k_j^I k_j^J k_j^K}{q_j^2}, \qquad j = 1, \ldots, N. \tag{3.38}$$

The constant terms, ε_0, k_0^I, l_0^I and m_0, determine the asymptotic behaviour of the solution. The original M-theory geometry is generically asymptotic to $\mathbb{R}^{1,3} \times S^1 \times (T^2)^3$

and the constant terms determine the scales of the S^1 and T^2 factors and fix the $U(1)$ Wilson lines around the S^1 [25]. If one tunes the constants appropriately (*e.g.* if one sets $\varepsilon_0 = 0$) then various circles in the five-dimensional or six-dimensional metrics will decompactify.

To remove closed time-like curves in the neighborhood of the points where $r_j \to 0$ one must impose that $\mu \to 0$ as $r_j \to 0$. Explicitly this yields the bubble equations:

$$\frac{1}{6} C_{IJK} \sum_{j=1,\, j\neq i}^{N} \Pi_{ij}^{(I)} \Pi_{ij}^{(J)} \Pi_{ij}^{(K)} \frac{q_i q_j}{r_{ij}} \;=\; 2(\varepsilon_0 m_i - m_0 q_i) + \sum_{I=1}^{3} (k_0^I l_i^I - l_0^I k_i^I) \qquad (3.39)$$

for $i = 1, \ldots, N$, and where $r_{ij} \equiv |\vec{y}^{(i)} - \vec{y}^{(j)}|$. Summing both sides of this equation and using the skew-symmetry of $\Pi_{ij}^{(I)}$ leads to:

$$m_0 \;=\; q_0^{-1} \left(\varepsilon_0 \, m_i - \frac{1}{2} \sum_{j=1}^{N} \sum_{I} \left(l_0^I \, k_j^I - k_0^I \, \ell_j^I \right) \right), \qquad (3.40)$$

where q_0 is given by (3.37).

We expect that both the regularity of the six-dimensional solution and the bubble equations are preserved under the simple spectral flow (3.26) and, more generally, under the global diffeomorphisms (3.29) precisely because they are diffeomorphisms of the torus. Moreover, these diffeomorphisms only involve the space-like sections of the metric and hence they should not introduce new CTC's. One can see this explicitly from (3.33). Suppose that n is generic so that $\widetilde{V}$ and V have *exactly the same* singular points. Then $V^{\pm 1} \widetilde{V}^{\mp 1}$ is regular and so if one starts with regular Z_I and μ then one will end up with regular $\widetilde{Z}_I$ and $\tilde{\mu}$. Moreover, if the bubble equations are satisfied then $\mu \to 0$ as $r_j \to 0$ and hence $\tilde{\mu} \to 0$ as $r_j \to 0$. Thus the bubble equations are satisfied in the new solution.

This argument obviously generalizes to any combination of transformations in $SL(2,\mathbb{Z})^3$ that do not change the singular structure of V. Therefore such transformations clearly map smooth bubbling solutions into smooth bubbling solutions and preserve the bubble equations.

If the spectral flow parameter, n is not generic, then V and $\widetilde{V}$ can have different sets of singular points, but the solution generated by the simple spectral flow will still be smooth in six dimensions, and its physics is the subject of the next section. It turns out that this feature does not generalize to non-generic many-parameter spectral flow transformations (3.27). These flows will take multi-center black hole solutions into other multi-center solutions, by preserving the bubble equations and not introducing closed timelike curves. However, they may transform microstate solutions that are smooth in supergravity into solutions that do not appear smooth in supergravity. This will be the subject of Section 3.2.7.

3.2.6 Supertubes, bubbling geometries and spectral flow

Perhaps the most physically interesting spectral flow transformation occurs when V and $\widetilde{V}$ have different sets of singular points. Suppose that we start with a regular, bubbled solution and that we use the simple spectral flow (3.26) so that $\widetilde{V}$ has (at least) one less singularity than V. It follows that $\widetilde{Z}_1$, $\widetilde{Z}_2$ and $\tilde{\mu}$ now develop singularities, but these singularities have a very special form. As we will show, these singularities correspond exactly to having a two-charge supertube at the location of the old pole (or poles) of V. Going in the opposite direction, one can start from a geometry containing one or several two-charge supertubes and obtain a bubbling solution by doing the inverse spectral flow[5].

[5]This is exactly the way in which the first three-charge microstates were obtained by Lunin, [159] and independently by Giusto, Mathur, and Saxena [118, 119].

It is well known that two-charge supertubes give smooth supergravity solutions when in the duality frame in which they have D1 and D5 charges and KKM dipole charge, both in flat space [156, 158] and in Taub-NUT [24]. Since the standard regularity conditions only involve the local geometry around the supertube, one would expect two-charge supertubes to be regular in more generic three-charge backgrounds [35]. Hence, the fact that the spectral flow transformation (3.27) takes smooth solutions into smooth solutions is not surprising; after all, from a six-dimensional perspective, the flow (3.27) is nothing but a coordinate transformation.

The effect of the spectral flow transformation may, at first, appear surprising from the geometric perspective of the four-dimensional base: GH-based solutions are bubbling geometries with fluxes threading topologically non-trivial cycles while supertubes are thought of as rotating supersymmetric ensembles of branes that do not involve topology. The spectral flow maps one picture into the other and, once again, from the six-dimensional perspective it is easy to see how this happens. Consider the (spatial) $U(1)$ fiber parametrized by v in (3.14) over any disk that spans the closed loop of the supertube. At the supertube the function H in (3.14) becomes singular and pinches-off the $U(1)$ fiber. The result is a topologically non-trivial 3-sphere and the three-form, (3.16), has a non-zero flux through this 3-cycle. In the metric with a GH base, this 3-cycle simply appears as a non-trivial $U(1)$ fibration (parametrized by v) over a non-trivial 2-cycle in the base. The spectral flow merely "undoes" the topology in the base at the cost of introducing an apparent singularity but both perspectives are equivalent, and describe the same, completely regular, six-dimensional solution.

It is useful to begin by illustrating the spectral-flow procedure on the solution corresponding to one supertube in Taub-NUT [24]. The smooth six-dimensional solution describing two-charge supertubes can be written as a solution with a GH base using the following harmonic functions [25]:

$$\begin{aligned} V &= \epsilon_0 + \frac{1}{r}\,, \quad L_1 = 1 + \frac{Q_1}{4|\vec{r}-\vec{R}|}\,, \quad L_2 = 1 + \frac{Q_2}{4|\vec{r}-\vec{R}|}\,, \quad L_3 = 1\,, \\ K_1 &= 0\,, \quad K_2 = 0\,, \quad K_3 = -\frac{q_3}{2|\vec{r}-\vec{R}|}\,, \quad M = \frac{J_T}{16}\left(\frac{1}{R} - \frac{1}{|\vec{r}-\vec{R}|}\right). \end{aligned} \tag{3.41}$$

where $\vec{R}$ defines the position of a round supertube that is wrapping the fiber of the Taub-NUT metric. Not all the constant parts in the harmonic functions are independent. The absence of closed timelike curves requires that

$$J_T\left(\epsilon_0 + \frac{1}{R}\right) = 4q_3 \tag{3.42}$$

Moreover, in six dimensions the metric constructed using (3.41) is smooth (up to harmless $\mathbb{Z}_{q_3}$ orbifold singularities) if [24]:

$$q_3 J_T = Q_1 Q_2\,. \tag{3.43}$$

This condition comes from the requirement that ω in (5.26) has no Dirac-Misner strings.

Before performing the spectral flow, we should observe that the harmonic functions above can be shifted using a subset of the gauge transformation (2.48) that preserves $K_1 = K_2 = 0$ and that sets the sum of the coefficients of the poles in K_3 to be zero:

$$\begin{aligned} V &= \epsilon_0 + \frac{1}{r}\,, \quad L_1 = 1 + \frac{Q_1}{4|\vec{r}-\vec{R}|}\,, \quad L_2 = 1 + \frac{Q_2}{4|\vec{r}-\vec{R}|}\,, \quad L_3 = 1\,, \quad K_1 = 0\,, \\ K_2 &= 0\,, \quad K_3 = \frac{q_3\epsilon_0}{2} + \frac{q_3}{2r} - \frac{q_3}{2|\vec{r}-\vec{R}|}\,, \quad M = \frac{J_T}{16}\left(\frac{1}{R} - \frac{1}{|\vec{r}-\vec{R}|}\right) - \frac{q_3}{4}. \end{aligned} \tag{3.44}$$

Under a spectral flow with parameter γ_3 one obtains a new solution with the harmonic functions:

$$\begin{aligned}
V &= \epsilon_0 \left(1 + \frac{\gamma_3 q_3}{2}\right) + \frac{1}{r}\left(1 + \frac{\gamma_3 q_3}{2}\right) - \frac{q_3 \gamma_3}{2|\vec{r} - \vec{R}|} \,, \quad K_1 = -\gamma_3 - \frac{\gamma_3 Q_2}{4|\vec{r} - \vec{R}|} \,, \\
K_2 &= -\gamma_3 - \frac{\gamma_3 Q_1}{4|\vec{r} - \vec{R}|} \,, \quad K_3 = \frac{q_3 \epsilon_0}{2} + \frac{q_3}{2r} - \frac{q_3}{2|\vec{r} - \vec{R}|} \,, \\
L_1 &= 1 + \frac{Q_1}{4|\vec{r} - \vec{R}|} \,, \quad L_2 = 1 + \frac{Q_2}{4|\vec{r} - \vec{R}|} \,, \\
L_3 &= 1 + \frac{\gamma_3 q_3}{2} - \frac{\gamma_3 J_T}{8}\left(\frac{1}{R} - \frac{1}{|\vec{r} - \vec{R}|}\right) \,, \quad M = \frac{J_T}{16}\left(\frac{1}{R} - \frac{1}{|\vec{r} - \vec{R}|}\right) - \frac{q_3}{4} \,.
\end{aligned} \tag{3.45}$$

It is not hard to check that the harmonic functions above satisfy the condition (3.38), and hence they give a smooth three-charge two-centered bubbling solution. Moreover, the equation that gives the radius of the supertube in Taub-NUT (3.42) becomes exactly the "bubble equation" (3.39) governing the two-center bubbling solution. Hence, a spectral flow transformation can be used to change a smooth two-charge supertube in six dimensions into a smooth three-charge bubbling solution. This solution has the same singular parts as the four-dimensional microstate solution obtained in [189], but has different constant parts in the harmonic functions.

Of course, to obtain asymptotically five-dimensional solutions from other asymptotically flat solutions using spectral flow is a little more complicated. These solutions must not have any constant term in the K_I [26, 41]. Nevertheless, the solution before the spectral flow necessarily has all the Z_I (and hence L_I) limiting to constant values. Hence, a spectral flow will necessarily introduce a constant term in at least one of the K_I. The way this problem is usually remedied [159, 118, 119, 24, 95] is to strip off the asymptotically-flat region of the solution to obtain an asymptotically AdS_3 geometry, spectral flow this geometry, and then add back by hand the asymptotically-flat part of the solution.

On the other hand, by looking at the solutions that have four-dimensional asymptotics, there is no need to eliminate the constant terms in the K_I harmonic function. A spectral flow will simply match two solutions with different values of the moduli at infinity.

We can generalize the foregoing example by starting with a solution describing N two-charge supertubes in Taub-NUT. The solution is specified by eight harmonic functions which have the form

$$\begin{aligned} V &= \epsilon_0 + \frac{1}{r}\,, \qquad K^1 = K^2 = 0\,, \qquad K^3 = k_0^3 - \sum_{i=1}^{N} \frac{q_3^i}{2r_i}\,, \\ L_1 &= l_0^1 + \sum_{i=1}^{N} \frac{Q_1^i}{4\,r_i}\,, \qquad L_2 = l_0^2 + \sum_{i=1}^{N} \frac{Q_2^i}{4\,r_i}\,, \qquad L_3 = l_0^3 \\ M &= m_0 - \sum_{i=1}^{N} \frac{J_i}{16\,r_i}\,, \end{aligned} \tag{3.46}$$

where $r_i = |\vec{r} - \vec{r}_i|$ and $\vec{r}_i$ are the locations of the supertubes in the base space. We will also define $R_i \equiv |\vec{r}_i|$.

If we choose all $\vec{r}_i$ to lie on the negative z axis (in GH coordinates) this will correspond to a configuration of N concentric supertubes of "radius[6]," R_i. It is clear that the straightforward generalization of the analysis in [24] will imply that, in the duality frame where the two charges of the supertubes are D1 and D5 charges, the type IIB supergravity solution will be smooth if (3.43) is satisfied for each center:

$$Q_1^i Q_2^i = q_3^i J_i\,. \tag{3.47}$$

[6] This is the distance from the Taub-NUT center to the supertube as measured in the three-dimensional base, and not the physical radius of the supertube.

These N conditions guarantee that the full metric is completely regular (again up to $\mathbb{Z}_{q_3^i}$ orbifold singularities). The solution should be free of CTC's and imposing this condition at the locations of the supertubes and at the origin of the four-dimensional base yields $N+1$ equations: N expressions that give the radius of each supertube and generalize (3.42), as well as a relation that fixes the parameter m_0:

$$\left(\epsilon_0 + \frac{1}{R_i}\right) J_i = 4\, l_0^3\, q_3^i\,, \qquad\qquad m_0 = \frac{1}{16} \sum_{i=1}^{N} \frac{J_i}{R_i}\,. \tag{3.48}$$

We can use the gauge freedom (2.48) to fix a gauge in which $\sum_{i=1}^{N+1} q_3^i = 0$:

$$\begin{aligned} V &\to V, \qquad K^1 \to K^1\,, \qquad K^2 \to K^2\,, \qquad K^3 \to K^3 + c\,V\,, \\ L_1 &\to L_1 - c\,K^2 = L_1\,, \qquad L_2 \to L_2 - c\,K^1 = L_2\,, \qquad L_3 \to L_3\,, \\ M &\to M - \frac{c}{2}\,L_3\,, \end{aligned} \tag{3.49}$$

where

$$c = \sum_{i=1}^{N} \frac{q_3^i}{2}\,. \tag{3.50}$$

This will ensure that the sum of the GH charges of the solution will remain the same after the spectral flow. After the gauge transformation, the harmonic functions take the following form:

$$\begin{aligned} V &= \epsilon_0 + \frac{1}{r}\,, \qquad K^1 = K^2 = 0\,, \qquad K^3 = k_0^3 + c\,\epsilon_0 + \frac{c}{r} - \sum_{i=1}^{N} \frac{q_3^i}{2r_i}\,, \\ L_1 &= l_0^1 + \sum_{i=1}^{N} \frac{Q_1^i}{4r_i}\,, \qquad L_2 = l_0^2 + \sum_{i=1}^{N} \frac{Q_2^i}{4r_i}\,, \qquad L_3 = l_0^3 \\ M &= m_0 - \frac{c\,l_0^3}{2} + \sum_{i=1}^{N} \frac{J_i}{16r_i}\,. \end{aligned} \tag{3.51}$$

To transform the solution corresponding to many supertubes to a bubbling solution with an ambipolar Gibbons-Hawking base, we perform a spectral flow transformation (3.26) with parameter γ to obtain.

$$\begin{aligned}\widetilde{V} &= V + \gamma K_3\,, \qquad \widetilde{K}^1 = K^1 - \gamma L_2\,, \qquad \widetilde{K}^2 = K^2 - \gamma L_1\,, \qquad \widetilde{K}^3 = K^3\,, \\ \widetilde{L}_1 &= L_1\,, \qquad \widetilde{L}_2 = L_2\,, \qquad \widetilde{L}_3 = L_3 - 2\gamma M\,, \qquad \widetilde{M} = M\,. \end{aligned} \tag{3.52}$$

The GH base space of the transformed solution has $N+1$ centers. The new harmonic functions:

$$\begin{aligned}\widetilde{V} &= \tilde{\epsilon}_0 + \sum_{j=1}^{N+1} \frac{\tilde{q}_j}{r_j}\,, \qquad \widetilde{K}^I = \tilde{k}_0^I + \sum_{j=1}^{N+1} \frac{\tilde{k}_j^I}{r_j}\,, \\ \widetilde{L}^I &= \tilde{l}_0^I + \sum_{j=1}^{N+1} \frac{\tilde{l}_j^I}{r_j}\,, \qquad \widetilde{M} = \tilde{m}_0 + \sum_{j=1}^{N+1} \frac{\tilde{m}_j}{r_j}\,,\end{aligned} \tag{3.53}$$

can be found straightforwardly from (3.51) and (3.52). It is also straightforward to check that (3.47), which insures the regularity of the supertubes, implies that the constants in these harmonic functions satisfy (3.38) for any value of γ. Moreover, the bubble equations (3.39) are equivalent to the $N+1$ equations (3.48) that give the radii of the N supertubes and the value of the m_0 parameter. This establishes explicitly that for any even integer γ the spectral flow transformation (3.52) maps smooth solutions containing supertubes to smooth multi-center GH bubbling solutions.

Having shown that a solution corresponding to many concentric supertubes can be transformed into a GH bubbling solutions, it is interesting to investigate the opposite transformation - that of a bubbling solution into a solution containing supertubes.

It is not hard to see that given a generic smooth bubbling solution, whose parameters respect (3.38) and (3.39), one can perform a spectral flow (3.26) with parameter $\gamma = -\frac{q_i}{k_i^3}$ to obtain a solution in which there is no GH charge at the $i^{\rm th}$ point. Equations

(3.38) then insure that the functions K^1, K^2 and L_3 will also not have a pole at the position of the i^{th} point. The poles of the other harmonic functions are

$$K_3 \sim \frac{k_i^3}{r_i} \,, \quad L_1 \sim \frac{-k_i^2 k_i^3}{r_i} \,, \quad L_2 \sim \frac{-k_i^2 k_i^3}{r_i} \,, \quad M \sim \frac{k_i^1 k_i^2 k_i^3}{2 r_i} \,. \tag{3.54}$$

This solution corresponds to an object with two charges, one dipole charge, and angular momentum, and it is simply a circular[7] two-charge supertube at position $\vec{r}_i$.

It is clear that this solution will be smooth from a six-dimensional perspective, simply because spectral flow takes smooth solutions into smooth solutions. Moreover, the coefficients of the singular parts of L_1, L_2, K_3 and M satisfy the same relation, (3.43), as do the coefficients in the smooth two-charge supertube solutions in $\mathbb{R}^4$ or Taub-NUT. In [35] we showed that the smoothness conditions coming from the supergravity analysis coincide with the equations of motion for a two-charge supertube in a GH background that one obtains using the Dirac-Born-Infeld action of the supertube. Hence, a spectral flow transformation with a well-chosen parameter can transform any multi-center Gibbons-Hawking solution to a solution where one (or several) of the centers has been replaced by a two-charge supertube.

3.2.7 Generalized spectral flow

It is also interesting to consider generalized spectral flow transformations that can take a GH center into an even simpler configuration. We begin by exploring the orbit of generalized spectral flow. We then use the physics of supertubes to argue that many multi-center configurations that appear to be bound states from a four-dimensional perspective do so only because of the limited supergravity Ansatz used to study their stability. When

[7] The circle is along the $U(1)$ fiber of the ambi-polar Taub-NUT base.

one explores them using a more complete supergravity Ansatz, based on the underlying holographic dual, they are in fact unbound.

In order to describe generalized spectral flow on multi-center solutions it is convenient to work in the five-dimensional duality frame in which the electric charges of the solution are those of three sets of M2 branes (2.4). When the base space of these solutions is ambi-polar, multi-center Taub-NUT, they can be reduced to four-dimensional multi-center solutions. The M2 charges correspond to D2 charges, the M5 dipole charges correspond to D4 charges, the Kaluza-Klein momentum along the Taub-NUT fiber becomes the D0 charge and the geometric GH charges correspond to D6 branes. The sources that appear in the eight harmonic functions that determine the solutions thus correspond exactly to the four-dimensional D6, D4, D2 and D0 charges.

A smooth multi-center, five-dimensional solution corresponds, in four dimensions, to a multi-center solution where each center is a "primitive" D6 brane, that is, a D6 brane that has non-trivial world-volume flux and locally preserves sixteen supercharges[8]. From the perspective of the D-brane world-volume, primitivity places non-trivial constraints upon the fluxes. In the supergravity background these constraints amount to imposing smoothness, which fixes the flux parameters as in (3.38) [26, 41]. In the same manner, a two-charge supertube, which is also smooth in the D1-D5-P duality frame, has D4, D2 and D0 charges that satisfy (3.43). Thus it corresponds to a "primitive" D4 brane - a D4 brane with non-trivial world-volume flux that locally preserves sixteen supercharges.

In Section 3.2.5 we have established that spectral flow generically takes multi-center, primitive D6 configurations into other such configurations. Moreover, in Section 3.2.6 we have seen that for some specially-chosen parameters it can transform a primitive

[8]Only four of those supercharges are common to all the D6 branes, and thus common to the complete multi-center solution.

D6 center into a primitive D4 center. One can take this further, and consider two- and three-parameter generalized spectral flow. A two-parameter spectral flow, with

$$\gamma_1 = -\frac{q}{k^1}\,, \quad \gamma_2 = -\frac{q}{k^2} \tag{3.55}$$

can take a GH center with GH charge q into a center with only two singular harmonic functions, K_3 and M. This corresponds to a set of primitive D2 branes that have a non-trivial D0 brane charge. Furthermore, one can perform another spectral flow to take this center into a center that only has a non-zero M, and hence corresponds to a collection of D0 branes. The parameters of this flow are:

$$\gamma_1 = -\frac{q}{k^1}\,, \quad \gamma_2 = -\frac{q}{k^2}\,, \quad \gamma_3 = -\frac{q}{k^3}\,. \tag{3.56}$$

Since this last configuration consists of only a single species of D-brane, primitivity (the local preservation of sixteen supercharges) is now manifest. One should note that each successive spectral flow decreases the number of types of D-brane charge possessed by the brane and that this reduction critically depends upon the selection of parameters, (3.38), that made the original fluxed D6-brane smooth. By reversing these multiple spectral flows one thus obtains another way to understand the primitivity of the original D6-brane configuration.

Unlike the primitive D6 branes, which give smooth five-dimensional solutions in all duality frames, or the primitive D4 branes, which are smooth in the D1-D5-P frame, the primitive D2 and D0 branes are not smooth in supergravity in any duality frame. This is not unexpected, because the U-duality group in four dimensions can take smooth solutions into singular ones, and generalized spectral flow is nothing but a three-parameter family of this group. This is also not in conflict with the fact that each spectral flow can be realized by a six-dimensional coordinate change and therefore will preserve the regularity of six-dimensional solutions. The point is that one cannot realize all three independent spectral flows as coordinate changes of a single regular metric and so concatenating spectral flows can generate singular solutions.

One can also extend spectral flow to $U(1)^N$ five-dimensional ungauged supergravities compactified on a GH space (or, after dimensional reduction, $\mathcal{N} = 2$ supergravities in four dimensions), that come from M-theory compactified on a CY manifold. The equation that gives generalized spectral flow (3.27) is written in a way that trivially expands to such supergravities[9]. For such solutions a six-dimensional lift of the solution, and the smooth supertube interpretation of the primitive D4 centers are not straightforward (unless the CY is $K^3 \times T^2$). Nevertheless, for generic parameters the generalized spectral flow still takes smooth solutions into smooth solutions, while for special choices of parameters it can interpolate between solutions with primitive D6, D4, D2 and D0 centers.

To recapitulate, from a five-dimensional perspective a spectral flow with one parameter can take a smooth GH solution into a smooth solution that contains a two-charge supertube in a GH background. Furthermore, two-parameter and three-parameter generalized spectral flows can transform a GH center into a singular configuration, that from a four-dimensional perspective has D2-D0 and pure D0 charges respectively.

In studying the microstates of a black-hole one obviously wants to ensure that one is studying a single black hole and not merely an ensemble, or gas, of unbound BPS black holes and black rings. That is, one should only consider a system as being a single black hole if there are no "separation moduli" that can be used to physically deform the system into widely separated components without changing the energy or other asymptotic charges.

[9]For a general CY compactification C_{IJK} are the triple intersection numbers of the CY three-fold.

Establishing whether a solution is bound or unbound can be rather subtle. Consider for example an asymptotically-flat five-dimensional solution containing two two-charge supertubes. These do not interact with each other, and one can move them arbitrary far apart at no energy cost (without affecting the asymptotic charges and angular momenta). Hence, this configuration has flat directions, and is unbound. However, when considered as a four-dimensional multi-center solution, this solution has three centers that have a nontrivial four-dimensional $\vec{E}\times\vec{B}$ interaction, and appears bound. Of course, the answer to this puzzle is that the separation moduli of the five-dimensional solution break its tri-holomorphic $U(1)$ isometry, and hence are not visible in four dimensions.

If one's purpose is to describe microstates of five-dimensional three-charge black holes or black rings, the ultimate arbiter of whether a multi-center solution is bound is to dualize it to the D1-D5-P duality frame, and take the limit in which it becomes asymptotic to $AdS_3 \times S^3$, see Appendix C and [35, 22]. If the six-dimensional solution has separation moduli, the solution is not bound. As we will see below, these separation modes are often not visible if one constructs and analyzes the solution using a more limited four- or five-dimensional Ansatz.

It is also possible that a certain multi-center solution, which is unbound when embedded in an asymptotically $AdS_3 \times S^3$ spacetime, can become bound when embedded in an asymptotically $\mathbb{R}^{3,1} \times S^1 \times S^1$ spacetime. The simplest example is again that of two concentric two-charge supertubes in Taub-NUT. These supertubes have no zero-modes [195] because of the constraining nature of the Taub-NUT geometry. However, when taking the limit in which their solution becomes asymptotically $AdS_3 \times S^3$, the base of the solution becomes $\mathbb{R}^4$, and the two supertubes become indistinguishable from two supertubes in an asymptotically-flat five-dimensional space, which are unbound. The same analysis extends trivially to more concentric supertubes in Taub-NUT.

Hence, two or more supertubes in Taub-NUT do not form a true bound state. Rather they are geometrically bound: their lack of separation moduli is a result of the compactification geometry rather than of binding interactions. Intuitively, one should think of such geometrically-bound configurations as being the analogue of an ideal gas in a box: there is no binding energy between the atoms, but the system cannot be deformed into widely separated components because of the walls of the box.

As we have seen in Section 3.2.6, using spectral flow one can transform a solution that contains concentric two-charge supertubes to a bubbling multi-center solution. The analysis above implies that such a bubbling multi-center solution is not a bound state. Indeed, upon spectral flow, a solution where the supertubes are not concentric anymore becomes a v-dependent six-dimensional solution (3.14) of the type described in Section 3.2.2, [131, 56]. Hence, as explained above, multi-center bubbling solutions that can be obtained by spectral flow from a concentric-supertube configuration only appear bound from a four- and five-dimensional perspective because of the limited supergravity Ansatz used to describe them. Upon embedding it in the correct holographic background this configuration has at least one zero-mode and this involves making the six-dimensional lift of the solution (3.14) v-dependent.

It is also important to realize that starting from concentric supertubes, a spectral flow transformation only generates a very specific type of bubbling geometries. Indeed, spectral flow leaves the bubble equations invariant, and hence the bubble equations governing the unbound bubbling solutions do not contain any terms that depend on the distance between any two of the GH centers that come from the supertubes. Hence, from a four-dimensional perspective these GH points are free to move on two-spheres of radius, R_i (given by (3.48)) around a central GH point. In the quiver language, used to describe multi-center four-dimensional solutions [77, 15, 78], these bubbling solutions

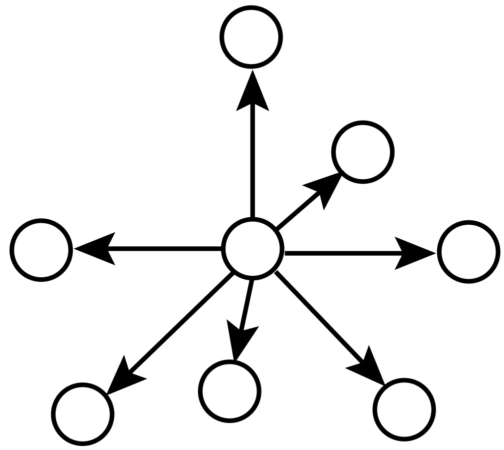

Figure 3.1: The "hedgehog" quiver corresponding to the unbound multi-center solution generated from the spectral flow of many concentric supertubes.

can be depicted as "hedgehog" quivers, with all the arrows originating from one of the nodes, and joining it to all the other nodes.

It is also possible to argue that, at least for a large enough number of centers, spectral flow can be used to generate all the hedgehog multi-center GH solutions in Figure (3.1) from two-charge round supertubes in flat space. Indeed, by simple parameter counting, a solution with $N+1$ GH centers has $4N+1$ parameters (three k_i^I's and one GH charge q_i for each point minus three gauge transformations). Requiring the vanishing of $\Pi_{ij}^{(1)}\Pi_{ij}^{(2)}\Pi_{ij}^{(3)}$ between any two of N centers naively imposes $N(N-1)$ constraints, which, in general, cannot be satisfied. Nevertheless, since $\Pi_{ij}^{(I)}$ are given by (3.37), it is not hard to see that one can also have all of them zero if for one of the I, the value of k_i^I/q_i is the same for all the N points. Choosing, for example, $I=2$, this implies

$$k_i = (k_i^1, q_i\kappa^2, k_i^3)\ , \tag{3.57}$$

and imposes $N-1$ conditions, leaving $3N+2$ independent parameters. This is exactly the number of parameters that describe all possible spectral flows of N round supertubes of arbitrary charge in a Taub-NUT space: three independent parameters (Q_1, Q_2, d_3) for each supertube, one for the Taub-NUT center and one spectral flow parameter, γ. It is not hard to see that a spectral flow with parameter $\gamma_2 = -1/\kappa^2$ transforms the foregoing set of N GH centers into concentric two-charge supertubes.

There exists another way to make all the fluxes between the N GH points vanish: one can divide them in three sets, A, B, C, that have fluxes

$$\begin{aligned} k_i &= (k_i^1, q_i\kappa^2, q_i\kappa^3)\ , \quad \text{for } i \in A \\ k_i &= (q_i\kappa^1, k_i^2, q_i\kappa^3)\ , \quad \text{for } i \in B \\ k_i &= (q_i\kappa^1, q_i\kappa^2, k_i^3)\ , \quad \text{for } i \in C \end{aligned} \tag{3.58}$$

where $\kappa^1, \kappa^2, \kappa^3$ are constants, and k_i^1, k_i^2 and k_i^3 can be arbitrary for the GH centers in the A, B and C set respectively. A two-parameter spectral flow with parameters $\gamma_1 = -1/\kappa^1$ and $\gamma_2 = -1/\kappa^2$ transforms the GH centers in the A and B sets into two-charge supertubes of different type, and transforms the GH centers in the C set into singular D2-D0 centers. Normally, different kinds of two-charge supertubes have an $\vec{E} \times \vec{B}$-type electric-magnetic interaction, and cannot go arbitrarily far away from each other without changing the asymptotic charges of the solution. Therefore such a solution is generically a bound state. However, for the particular supertubes that are created from the spectral flow of (3.58) the $\vec{E} \times \vec{B}$-type interaction vanishes, and hence they can move freely away from each other. Hence this type of configuration also corresponds to an unbound state. It is quite clear that for N sufficiently large all hedgehog quivers can be only of the type (3.57) or (3.58), and hence they are all unbound from the point of view of six-dimensional supergravity.

Another intuitive way to think of our formulation of bound state classification is as follows. Bound states generically emerge through $\vec{E} \times \vec{B}$ interactions. In four dimensions there are four independent $U(1)$ Maxwell fields and thus many ways in which to generate the interaction. In five dimensions there are only three $U(1)$ Maxwell fields and thus some of the four-dimensional $\vec{E} \times \vec{B}$ interactions become trivial upon oxidation to five dimensions. Indeed, this is precisely what happens with the hedgehog quiver: One can map the sources of the four-dimensional $\vec{E} \times \vec{B}$ interaction to a single D6 at the center of the quiver and D0 charges on the nodes. Upon lifting to five dimensions, the D6 brane disappears (becoming the "center of space") and all the nodes become free.

One should note that our analysis here indicates that the hedgehog quiver describes unbound states only when the center of the quiver is primitive. Indeed, one can consider quivers in which the exterior nodes have charges corresponding to black rings, and the center is a primitive (fluxed) D6 brane. This solution can be lifted to one or many concentric black rings on an $\mathbb{R}^4$ base. As with supertubes, the absence of arrows between the exterior nodes of the quiver is equivalent to the absence of $\vec{E} \times \vec{B}$-like interactions between the black rings. Hence, in the asymptotically five-dimensional solution, these rings can slide away from each other, and the configuration has zero modes and is unbound. However, if the center node is not a fluxed D6 brane, but a BMPV black hole, the sliding away of the rings becomes impossible. Indeed, as shown in [27], one cannot take a BMPV black hole away from the center of a black ring without modifying the asymptotic angular momenta of the solution. In that case, the black ring and the black hole interact via $\vec{E} \times \vec{B}$-type interactions, that render the sliding-away mode massive.

To summarize, our arguments indicate that all asymptotically five-dimensional solutions given by hedgehog quivers are unbound when their centers are primitive branes. Although a more detailed analysis is needed, this also seems to be true for hedgehog quivers whose central node is primitive and the outside nodes are not. However, the system may well be a bound state when the central node is not primitive.

A few examples of quivers describing unbound states include, for example the "Hall halo" configurations with a primitive center discussed in [77, 15, 78], the three-center configuration discussed in Section 6 of [41], and possibly also the "foaming quiver" (with charges equal to those of a maximally-spinning BMPV black hole) considered in [28]. As we stressed earlier, the unbound status of such geometrically bound systems can only be seen when considering asymptotically-flat five-dimensional solutions, or asymptotically $AdS_3 \times S^3$ solutions in six dimensions.

This analysis of bound and unbound systems is also in agreement with the recent findings that only quivers with closed loops can give solutions that have the charges of black holes and black rings with classically large horizon radius [29, 32], and that at weak coupling only these quivers give a macroscopic (black-hole-like) entropy [79]. Based on this, one expects that the closed, deep or scaling quivers describe bound states, which they indeed do [29]. Intuitively, one can think about the microstates that come from hedgehog quivers (and are necessarily "shallow"), and possibly about other "shallow" microstates as unbound or very weakly bound; conversely, the deep AdS throat, which is the hallmark of the scaling solution, is a direct manifestation of the binding of the geometric components of the microstate geometry.

3.3 Conclusions

We have investigated spectral flow - a transformation that takes multi-center solutions into other multi-center solutions, by shifting the underlying harmonic functions. For generic parameters, this transformation takes bubbling solutions that have multiple GH centers into other bubbling solutions with GH centers. However, for specially-chosen parameters, a spectral flow transformation can take a bubbling solution into a smooth solution that contains one or several supertubes in a GH multi-center background.

The fact that spectral flow can be used to interchange solutions containing two-charge supertubes and bubbling solutions is a very powerful fact, which we have studied in this Chapter, and will continue to use in Chapter 4.

The first problem we have addressed using this tool is to understand which of the three-charge bubbling solutions constructed in the literature are bound states, and which are not. We have found that the solutions that correspond to quivers without loops or bifurcations can be transformed into solutions that in the five-dimensional lift can be taken apart. Therefore they should not correspond to bound states in the dual CFT.

The second use of spectral flow has been to point to the existence of three-charge smooth BPS solutions that depend on arbitrary functions in the vicinity of any multi-center GH solution. Indeed, any GH center can be related to a round supertube via spectral flow. Furthermore, supertubes can have arbitrary shape while still remaining regular and supersymmetric. Hence the inverse spectral flow of a wiggly supertube gives a new smooth black-hole microstate solution, that does not have a GH base, and that can depend on arbitrary functions.

Even without knowing the explicit form of these BPS solutions, it is still possible to investigate their physics (at least in the vicinity of GH solutions), analyze their moduli space, and count their entropy by using supertube counting techniques [175, 9, 188]. The fact that GH solutions can be deformed to BPS solutions that depend on arbitrary functions establishes the existence of families of black hole microstates that depend on an infinite number of continuous parameters. In Chapter 4 we will explore these microstates, and argue that they can have a macroscopically large (black-hole-like) entropy.

We have also explored a larger class of spectral flow transformations (called generalized spectral flow) that for generic parameters transform multi-center bubbling solutions into other multi-center bubbling solutions, but for special parameters can transform one or several of the centers of a bubbling solution into a two-charge supertube, D2-D0 or D0 center.

By taking the limit of parameters in which the D2-D0, or the D0 branes do not back-react on the geometry, one can study them using their (non-abelian) Born-Infeld action, and count their entropy. In fact, such a counting has been performed in several circumstances. For example, in [80, 115] it was found that the entropy coming from D0 branes in a D6-$\overline{\mathrm{D6}}$ background (which lifts to a multi-center GH solution in five dimensions) is of the same order as the would-be black hole entropy. One could then use generalized spectral flow to transform the D0-D6-$\overline{\mathrm{D6}}$ system considered there into a multi-center D6-$\overline{\mathrm{D6}}$ configuration, which (unlike the system with D0 branes) is well-described by supergravity in the regime of parameters where the classical black hole exists. It would be very interesting to follow the spectral flow, and find the description of the D0 configurations that give the black-hole-like entropy [80, 115] in the multi-center D6 frame, and to find whether these configurations are still smooth in supergravity.

Chapter 4

Entropy enhancement

Supertubes [166] can have arbitrary shapes and give smooth supergravity solutions in the duality frame in which they have D1 and D5 charges [156, 158]. This has been very useful in matching the entropy of two-charge smooth supergravity solutions to that of the dual CFT and served as one of the motivations of the formulation of the fuzzball proposal. However, even if supertubes can have arbitrary shapes, and hence a lot of entropy, their naive quantization cannot account for the entropy of a black hole with a non-trivial, macroscopic horizon

$$S_{BH} \sim \sqrt{Q_1 Q_2 Q_3}\,. \tag{4.1}$$

Indeed, as found in [175, 9, 188], since supertubes only carry two charges, their entropy scales like:

$$S_{ST} \;\sim\; \sqrt{Q_1\, Q_2}\,. \tag{4.2}$$

The new insight here, based on [34], comes from considering supertubes in the background of a scaling bubbling solution with large magnetic fluxes. We generalize the analysis of [175], and use the supertube DBI–WZ action to count states of quantized supertubes in non-trivial background geometries. We find that, for the purposes of entropy counting, the supertube charges Q_I that appear in (4.2) are replaced by the local effective charges of the supertube, Q_I^{eff}, which are a combination of the supertube charges and terms coming from the interaction between the supertube magnetic dipole moment and the background magnetic dipole fields.

If there are strong magnetic fluxes in the background, as there are in deep, bubbled microstate geometries, these effective charges can be much larger than the asymptotic charges of the configuration, and can thus lead to a very large entropy enhancement! Indeed, one finds that if the supertube is put in certain deep scaling solutions, the effective charges can diverge if the supertube is suitably localized or if the length of the throat goes to infinity. Of course, this divergence is merely the result of not considering the back-reaction of the wiggly supertube on its background. Once this back-reaction is taken into account, the supertube will delocalize and the fine balance needed to create extremely deep scaling solutions might be destroyed if the tube wiggles too much.

Hence, we expect a huge range of possibilities in the semi-classical configuration space, from very shallow solutions to very deep solutions. In very shallow solutions, the supertubes can oscillate a lot, but they will not have their entropy enhanced and for very deep solutions the supertube will have vastly enhanced charges but, if the solution is to remain deep, the supertube will be very limited in its oscillations. One can thus imagine that the solutions with most of the entropy will be intermediate, neither too shallow (so as to obtain effective charge enhancement), nor too deep (to alow the supertube to fluctuate significantly). To fully support this intuition one will need to construct the full back-reacted supergravity solution for wiggly supertubes in bubbling three-charge backgrounds. Even though we do not yet have such solutions, it is possible to use the AdS/CFT correspondence to estimate the depth of the bulk microstate solutions dual to states in the typical sector of the dual CFT [29]. We will use this result to determine the depth of the typical throat and then argue that the effective supertube charges corresponding to this throat depth yields an enhanced supertube entropy that is macroscopic (4.1).

It is interesting to note that entropy enhancement is not just a red-shift effect. There is no entropy enhancement unless there are strong background magnetic fluxes. A three-charge BPS black hole will not enhance the entropy of supertubes: it is only solutions that have dipole charges, like bubbled black holes or black rings that can generate supertube entropy enhancement.

The last ingredient that we use is the generalized spectral flow transformation of Chapter 3, [33], that enables us to start from a simple, bubbled black hole microstate geometry [26, 41] and generate a bubbled geometry in which one or several of the Gibbons-Hawking (GH) centers are transformed into smooth two-charge supertubes. One can then study the particular class of fluctuating microstate geometries that result from allowing the supertube component to oscillate in the deep bubbled geometries. The naive expectation is that one would recover an entropy of the form (4.2) but, as we indicated, the Q_I are replaced by the enhanced Q_I^{eff}, and the entropy of these supertubes can become "macroscopic" in that it corresponds to the entropy of a black hole with a macroscopic horizon. One can then "undo" the spectral flow to argue that this entropy is present in the BPS fluctuations of three-charge bubbling solutions in any duality frame. In fact, spectrally flowing configurations with oscillating supertubes into other duality frames is not strictly speaking necessary for the purpose of illustrating entropy enhancement and arguing that smooth solutions can give macroscopically large entropy. After all, one could do the full analysis in the D1-D5-P duality frame and consider smooth black hole microstates containing both GH centers and supertubes. Nevertheless, since such solutions have not been studied in the past in great detail, it is easiest to construct them by spectrally flowing multi-center GH solutions, which have been studied much more and are better understood.

We start this Chapter by studying probe supertubes in various backgrounds, then we move on to calculate the entropy stored in the oscillations of the supertubes and uncover the entropy enhancement mechanism.

4.1 Supertube probes in BPS solutions

We begin with a review of supertubes in the background of a three-charge rotating BPS (BMPV) black hole and then extend this to more general three-charge backgrounds.

4.1.1 Supertubes in a three-charge black hole background

As a warm up exercise, we first consider a probe supertube with two charges and one dipole charge in the background of a three-charge (BMPV) black hole. This example was considered in [19, 165] and was generalized to a probe supertube with three charges and two dipole charges in [94]. The full supergravity solution describing a BMPV black hole on the symmetry axis of a black ring with three charges and three dipole charges was found in [21, 104], and a more general solution in which the black hole is not at the center of the ring was found in [27].

First, we need the BMPV black hole solution in the D0-D4-F1 duality frame. The metric (in the string frame) is:

$$ds_{10}^2 = -\frac{1}{\sqrt{Z_1 Z_2} Z_3}(dt+k)^2 \quad + \quad \sqrt{Z_1 Z_2}\,(d\rho^2 + \rho^2(d\vartheta^2 + \sin^2\vartheta d\varphi_1^2 + \cos^2\vartheta d\varphi_2^2)) \\ + \quad \frac{\sqrt{Z_1 Z_2}}{Z_3}\,dz^2 \;+\; \sqrt{\frac{Z_1}{Z_2}}\,ds_{T^4}^2 \tag{4.3}$$

and the dilaton and the Kalb-Ramond field are given by:

$$\Phi = \frac{1}{4}\log\left(\frac{Z_1^3}{Z_2 Z_3^2}\right)\,, \qquad B = (Z_3^{-1}-1)dt\wedge dz + Z_3^{-1}k\wedge dz\,. \tag{4.4}$$

The non-trivial RR potentials are:

$$C^{(1)} = (Z_1^{-1}-1)dt+Z_1^{-1}k\,, \qquad C^{(3)} = -(Z_2-1)\rho^2\cos^2\vartheta d\varphi_1\wedge d\varphi_2\wedge dz+Z_3^{-1}dt\wedge k\wedge dz\,. \tag{4.5}$$

The one-form k and the functions Z_I are given by

$$k = k_1 d\varphi_1 + k_2 d\varphi_2 = \frac{J}{\rho^2}(\sin^2\vartheta d\varphi_1 - \cos^2\vartheta d\varphi_2)\,, \qquad Z_I = 1 + \frac{Q_I}{\rho^2}\,, \tag{4.6}$$

where J is the angular momentum of the black hole. The charges, Q_1, Q_2 and Q_3 correspond to the respective D0 brane, D4 brane and F1 string charges of the black hole.

This solution is indeed a BPS, five-dimensional, rotating black hole [49] with an event horizon at $r = 0$, whose area is proportional to $\sqrt{Q_1Q_2Q_3 - J^2}$. For $J^2 > Q_1Q_2Q_3$ the solution has closed time-like curves and is unphysical.

We will denote the world-volume coordinates on the supertube by ξ^0, ξ^1 and $\xi^2 \equiv \theta$. To make the supertube wrap z we take $\xi^1 = z$ and we will fix a gauge in which $\xi^0 = t$. Note that $z \in (0, 2\pi L_z)$. The profile of the tube, parameterized by θ, lies in the four-dimensional non-compact $\mathbb{R}^4$ parameterized by $(\rho, \vartheta, \varphi_1, \varphi_2)$ and for a generic profile all four of these coordinates will depend on θ. It is convenient to use polar coordinate (u, φ_1) and (v, φ_2) in $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$, where the $\mathbb{R}^4$ metric takes the form (2.19).

There is also a gauge field, $\mathcal{F}$, on the world-volume of the supertube. Supersymmetry requires that $\mathcal{F}$ essentially has constant components and we can then boost the frames so that $\mathcal{F}_{t\theta} = 0$.

In this frame supersymmetry also requires $\mathcal{F}_{tz} = 1$ [166]. For the present we take

$$2\pi\alpha' F \equiv \mathcal{F} = \mathcal{F}_{tz} dt \wedge dz + \mathcal{F}_{z\theta} dz \wedge d\theta\,, \tag{4.7}$$

where the components are constant. Keeping $\mathcal{F}_{tz}$ as a variable will enable us to extract the charges below.

The supertube action is a sum of the DBI and Wess-Zumino (WZ)actions:

$$S = -T_{D2}\int d^3\xi e^{-\Phi}\sqrt{-\det\left(\widetilde{G}_{ab}+\widetilde{B}_{ab}+\mathcal{F}_{ab}\right)}+T_{D2}\int d^3\xi[\widetilde{C}^{(3)}+\widetilde{C}^{(1)}\wedge(\mathcal{F}+\widetilde{B})]\,, \tag{4.8}$$

where, as usual, $\widetilde{G}_{ab}$ and $\widetilde{B}_{ab}$ are the induced metric and Kalb-Ramond field on the world-volume of the supertube. We have also chosen the orientation such that $\epsilon_{tz\theta}=1$. It is also convenient to define the following induced quantities on the world-volume:

$$\Delta_{\mu\nu}=\partial_\mu u\partial_\nu u+u^2\partial_\mu\varphi_1\partial_\nu\varphi_1+\partial_\mu v\partial_\nu v+v^2\partial_\mu\varphi_2\partial_\nu\varphi_2\,,\quad \gamma_\mu=k_1\partial_\mu\varphi_1+k_2\partial_\mu\varphi_2\,, \tag{4.9}$$

where $\partial_\mu\equiv\frac{\partial}{\partial\xi^\mu}$.

After some algebra, the DBI part of the action simplifies to:

$$S_{DBI}=-T_{D2}\int dtdzd\theta\{Z_1^{-2}\left(\mathcal{F}_{z\theta}-\gamma_\theta(\mathcal{F}_{tz}-1)\right)^2 + Z_2Z_1^{-1}\Delta_{\theta\theta}[2(1-\mathcal{F}_{tz})-Z_3(\mathcal{F}_{tz}-1)^2]\}^{1/2}\,, \tag{4.10}$$

while the WZ piece of the action takes the form

$$S_{WZ}=T_{D2}\int dtdzd\theta\left[(1-\mathcal{F}_{tz})\gamma_\theta Z_1^{-1}+\mathcal{F}_{z\theta}\left(Z_1^{-1}-1\right)\right]\,. \tag{4.11}$$

For a supersymmetric configuration ($\mathcal{F}_{tz}=1$) we have

$$S_{\mathcal{F}_{tz}=1}=S_{DBI}+S_{WZ}=-T_{D2}\int dtdzd\theta\,\mathcal{F}_{z\theta} \tag{4.12}$$

The foregoing supertube carries D0 and F1 "electric" charges, given by

$$N_1^{ST} = \frac{T_{D2}}{T_{D0}} \int dz d\theta \, \mathcal{F}_{z\theta}, \qquad N_3^{ST} = \frac{1}{T_{F1}} \int d\theta \, \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{tz}}\bigg|_{\mathcal{F}_{tz}=1}. \tag{4.13}$$

The Hamiltonian density is:

$$\mathcal{H}|_{\mathcal{F}_{tz}=1} = \left[\frac{\partial \mathcal{L}}{\partial \mathcal{F}_{tz}} \mathcal{F}_{tz} - \mathcal{L}\right]_{\mathcal{F}_{tz}=1} = T_{D2}\mathcal{F}_{z\theta} + \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{tz}}\bigg|_{\mathcal{F}_{tz}=1}. \tag{4.14}$$

One can easily integrate this to get the total Hamiltonian of the supertube[1] (we assume constant charge density $\mathcal{F}_{z\theta}$)

$$\int dz d\theta \, \mathcal{H}|_{\mathcal{F}_{tz}=1} = N_1^{ST} + N_3^{ST}\,. \tag{4.15}$$

Thus the energy of the supertube is the sum of its conserved charges which shows that the supertube is indeed a BPS object.

Now choose a static round supertube profile $u' = v' = \varphi_2' = 0$, $\varphi_1 = \theta$. One then has:

$$\gamma_\theta = k_1 = J \, \frac{u^2}{(u^2+v^2)^2} \,, \qquad \Delta_{\theta\theta} = u^2 \tag{4.16}$$

and the supertube "electric" charges are:

$$N_1^{ST} = n_2^{ST} \mathcal{F}_{z\theta} \,, \qquad N_3^{ST} = n_2^{ST} \frac{Z_2 u^2}{\mathcal{F}_{z\theta}} \,, \tag{4.17}$$

where n_2^{ST} is the number of times the supertube wraps the angular variable θ, which in our conventions is also the supertube dipole charge given by the number of "tubular" D2 branes. So we find

$$N_1^{ST} N_3^{ST} = (n_2^{ST})^2 u^2 \, Z_2 \,. \tag{4.18}$$

[1]See Appendix D for details about our units and conventions.

This is an important relation in that it fixes the location of the supertube in terms of its intrinsic charges.

This computation was used in [19] to study the merger of a supertube and a black hole. In particular, a supertube can merge with a black hole if and only if $N_1^{ST} N_3^{ST} \le (n_2^{ST})^2 N_2$, where N_2 is the number of D4 branes in the black hole. Moreover, the supertube will "crown" the horizon of the black hole at "latitude", $\vartheta = \alpha$, given by:

$$\sin\alpha = \sqrt{\frac{N_1^{ST} N_3^{ST}}{(n_2^{ST})^2 N_2}}\,. \tag{4.19}$$

One can also show that one cannot violate chronology protection by throwing a supertube into the black hole, that is, one cannot over-spin the black hole and that the bound $J^2 \le N_1 N_2 N_3$ is preserved after the merger.

In [35] we also studied various two- and three-charge supertubes in the background of supersymetric black rings. We have also studied mergers of black rings and supertubes and showed that chronology is protected in such mergers. We will not present the details of the calculation here but it is worth noting that there are new features due to the dipole charges of the black ring which interact non-trivially with the dipole charge of the supertube.

4.1.2 Supertubes in solutions with a Gibbons-Hawking base

We now consider two-charge supertubes probing a general three-charge BPS solution with a Gibbons-Hawking base and we will again work in the D0-D4-F1 duality frame. The general BPS solution with three charges and three dipole charges and a GH base is given in Section 3.1.1 and we proceed as we did for the black-hole background in the previous Section. We denote the supertube coordinates as ξ^0, ξ^1 and $\xi^2 \equiv \theta$ and consider the simplified case of a circular supertube along the $U(1)$ fiber of the GH base:

$$\xi^0 \;=\; t\,, \qquad \xi^1 \;=\; z\,, \qquad \theta = \psi\,. \tag{4.20}$$

The supertube action takes the explicit form

$$S = T_{D2} \int d^3\xi \left\{ \left[\left(\frac{1}{Z_1} - 1 \right) \mathcal{F}_{z\theta} + \frac{K^3}{Z_1 V} + \left(\frac{\mu}{Z_1} - \frac{K^1}{V} \right) (\mathcal{F}_{tz} - 1) \right] \right.$$
$$\left. - \left[\frac{1}{V^2 Z_1^2} \left[(K^3 - V(\mu(1-\mathcal{F}_{tz}) - \mathcal{F}_{z\theta}))^2 + V Z_1 Z_2 (1-\mathcal{F}_{tz})(2 - Z_3(1-\mathcal{F}_{tz})) \right] \right]^{1/2} \right\}. \tag{4.21}$$

For $\mathcal{F}_{tz} = 1$ the tube is supersymmetric and, as before, the Hamiltonian density is the sum of the charge densities (4.14). Due to the supersymmetry there is a constraint similar to (4.19), which determines the location of the supertube in terms of its charges

$$\left[N_1^{ST} + n_2^{ST} \frac{K^3}{V} \right] \left[N_3^{ST} + \frac{K^1}{V} \right] = (n_2^{ST})^2 \frac{Z_2}{V}\,, \tag{4.22}$$

where the charges are still defined by (4.13). There is an important new feature here in that there is a contribution from the interactions of the dipole charges of the supertube and the background. This appears through the pull-back of the magnetic gauge potentials, $B^{(I)}$ (given by the harmonic functions K^I), to the world-volume of the supertube and it gives an added contribution to the basic supertube charges to yield what we will refer to as the local effective charges of the supertube. As we will see later in this Chapter this will be essential when one computes the entropy of a fluctuating supertube in backgrounds with non-trivial dipole charges.

Here we presented a brief review of probe supertubes in a general GH bubbling solution. We have studied this configuration in greater detail in [35] and have shown that 2-charge supertubes are regular in any 3-charge bubbling solution. This was done by studying both the exact supergravity solution and the probe action of the supertube.

4.2 Fluctuating supertubes and entropy enhancement

Our goal is to quantize the small oscillations about round two-charge supertubes in flat space, black hole and generic three-charge backgrounds, and to examine the entropy coming from these fluctuations. We find it convenient to work in the D0-D4-F1 duality frame, and our approach follows that of [175, 34] (see also [9]).

As first reported in [34], in a generic three-charge background we find a non-trivial enhancement of the entropy of a supertube when the dipole magnetic fields are large. This enhancement arises because the entropy that can be stored in a supertube is governed not by the electric charges of the supertube (as in flat space or in a black hole background) but by its locally-defined *effective charges*, that can get large contributions from the interactions of the dipole moment of the supertube with the magnetic fluxes of the background.

As discussed in [35] it is expected that supertube fluctuations will give rise to smooth horizonless solutions. Hence, our analysis strongly supports the existence of smooth horizonless three-charge solutions that depend on arbitrary continuous functions, and whose entropy is much larger than their typical charge, and might even be as large as the square root of the cube of their charge. That is, it might be black-hole-like.

4.2.1 Flat space

In the absence of background fluxes, the WZ action of the supertube is zero, and the DBI action (4.8) reduces to

$$S = -T_{D2} \int dt dz d\theta \sqrt{R^2(1 - \mathcal{F}_{tz}^2) + \mathcal{F}_{z\theta}^2} \,, \tag{4.23}$$

where R is the radius of the supertube and its embedding is

$$t = \xi^0 \,, \qquad z = \xi^1 \,, \qquad \varphi_1 = \theta \,. \tag{4.24}$$

The charges of the tube are given by (4.13):

$$N_1^{ST} \;=\; n_2^{ST} \mathcal{F}_{z\theta} \,, \qquad N_3^{ST} \;=\; n_2^{ST} \frac{R^2}{\mathcal{F}_{z\theta}} \,, \tag{4.25}$$

where the factors of n_2^{ST} come from multiple windings in θ. Similarly the radius relation (4.22) reduces to:

$$N_1^{ST} N_3^{ST} \;=\; \left(n_2^{ST}\right)^2 R^2 \,. \tag{4.26}$$

The angular momentum of the supertube is[2]:

$$J^{ST} \;=\; \frac{N_1^{ST} N_3^{ST}}{n_2^{ST}} \;=\; n_2^{ST} R^2 \,. \tag{4.27}$$

The foregoing results apply to round (maximally spinning) supertubes. Supertubes of arbitrary shape will have more complicated expressions for their conserved quantities and will generically have smaller angular momentum.

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[2]See [35] for more details on how to compute this angular momentum.

Now we will perform a simplified version of the analysis in [175], which will be enough to give us the correct dependence of the entropy on the supertube charges. We consider small fluctuations of the supertube in the six directions transverse to its world-volume:

$$x_i \to x_i + \eta_i(t,\theta)\,, \qquad\qquad i = 1,\ldots,6\,, \tag{4.28}$$

where four of these fluctuations take place on the compact T^4 and the other two are radial coordinates in the non-compact space. In general there are eight independent fluctuation modes for the supertube, consisting of seven transverse coordinate motions and a charge density fluctuation (which also affects the shape). To keep the computations simple here, we have restricted to a representative sample of oscillations in both the compactification space and in the space-time. Since we are only interested in BPS fluctuations we will also restrict η_i to depend only upon t and θ [175][3].

The effective Lagrangian for the fluctuations is obtained by expanding the DBI Lagrangian of the supertube

$$L_\eta = -T_{D2}\left[(1-\mathcal{F}_{tz}^2-\dot\eta_i\dot\eta_i)(R^2+\eta_i'\eta_i')-2\mathcal{F}_{tz}\mathcal{F}_{z\theta}\dot\eta_i\eta_i'+\mathcal{F}_{z\theta}^2(1-\dot\eta_i\dot\eta_i)+(\dot\eta_i\eta_i')^2\right]^{1/2}\,, \tag{4.29}$$

where the repeated index i is summed over. The canonical momenta conjugate to η_i are:

$$\Pi_i = \int_0^{2\pi L_z} dz\,\frac{\partial L_\eta}{\partial\dot\eta_i}\bigg|_{\dot\eta_i=0\,,\,\mathcal{F}_{tz}=1} = \frac{1}{2\pi}\eta_i'\,, \tag{4.30}$$

and the canonical commutation relations are:

$$[\eta_j(t,\theta),\Pi_k(t,\theta')] = i\delta_{jk}\delta(\theta-\theta')\,. \tag{4.31}$$

[3]The time dependent modes will break supersymmetry. Hence, we will retain the time dependence of η_i to compute momenta and quantize the system but then we will set $\partial_t\eta_i \equiv \dot\eta_i = 0$.

The BPS modes η_i then can be expanded as:

$$\eta_i = \frac{1}{\sqrt{2}}\left[\sum_{k>0} e^{ik\theta/n_2^{ST}} \frac{(a_k^i)^\dagger}{\sqrt{|k|}} \ + \ \text{h.c.}\right] , \tag{4.32}$$

where $(a_k^i)^\dagger$ and a_k^i are creation and annihilation operators for the k^{th} harmonic. The normalization has been chosen such that[4]:

$$[(a_k^i)^\dagger, a_{k'}^j] = \delta^{ij}\delta_{k,k'} \tag{4.33}$$

It is not hard to see that the fluctuations do not change N_1^{ST} and the angular momentum J^{ST}. The charge N_3^{ST} becomes:

$$N_3^{ST} \ = \ \frac{1}{T_{F1}}\int_0^{2\pi n_2^{ST}} d\theta \, \frac{\partial\mathcal{L}}{\partial\mathcal{F}_{tz}}\bigg|_{\mathcal{F}_{tz}=1} \ = \ \frac{T_{D2}}{T_{F1}}\int_0^{2\pi n_2^{ST}} d\theta \, \frac{(R^2+\eta_i'\eta_i')}{\mathcal{F}_{z\theta}} \ , \tag{4.34}$$

from which one finds

$$\sum_{i=1}^{6}\sum_{k>0} k(a_k^i)^\dagger a_k^i \ = \ L_z T_{D2}\int_0^{2\pi n_2^{ST}} d\theta \int_0^{2\pi n_2^{ST}} d\theta' \, \sum_{i=1}^{6}\eta_i'\eta_i' \tag{4.35}$$

$$= \ N_1^{ST}N_3^{ST} - (n_2^{ST})^2 R^2 \ = \ N_1^{ST}N_3^{ST} - n_2^{ST}J^{ST} \ . \tag{4.36}$$

The left hand side of this expression can be thought of as the energy of a system of six massless bosons in $(1+1)$ dimensions (we have effectively $(1+1)$ dimensional dynamics in (t,θ) space). Due to supersymmetry there will also be six corresponding

[4]Technically, to get this normalization correct we need to include the mode expansion of the non-BPS modes in (4.32). Ignoring the non-BPS modes gives an incorrect factor of $\sqrt{2}$ in the normalization of the η_i. Here we have given the correctly normalized expressions that one would obtain if one included the non-BPS modes.

fermionic degrees of freedom. The total central charge of the system is thus $c = 9$, and so the entropy of this system is given by the Cardy formula[5] [55]:

$$S = 2\pi\sqrt{\frac{c}{6}}\sqrt{N_1^{ST}N_3^{ST} - n_2^{ST}J^{ST}} = 2\pi\sqrt{\frac{3}{2}}\sqrt{N_1^{ST}N_3^{ST} - n_2^{ST}J^{ST}}\,. \qquad (4.37)$$

If we had included all eight bosonic fluctuation modes then we would have had eight bosons and eight fermions and hence a theory with $c = 12$ and with the entropy:

$$S_{ST} = 2\pi\sqrt{2}\sqrt{N_1^{ST}N_3^{ST} - n_2^{ST}J}\,. \qquad (4.38)$$

This is the correct central charge and it yields the correct supertube entropy [175]. By restricting our analysis to six of the shape modes and ignoring the other supersymmetric modes we have obtained a finite, but well understood, fraction of the supertube entropy. Since our purpose here is to analyze when entropy enhancement happens, and when it does not, we will only be interested on the dependence of the supertube entropy on the macroscopic charges, and not pay particular attention to numerical coefficients. Restricting our analysis in more general backgrounds to transverse BPS fluctuations and counting the entropy coming from these modes will therefore be enough to illustrate the physics of entropy enhancement.

4.2.2 The three-charge black hole

A two-charge round supertube in the background of a three-charge BPS rotating (BMPV) black hole was discussed in section 4.1.1. Here we will use the metric and background fields presented in section 4.1.1 and consider small shape fluctuations in the

[5]We are assuming that the electric and dipole charges of the supertube are large enough so that the use of the Cardy formula is justified.

directions transverse to the world-volume of the supertube. We are again interested only in BPS excitations, which have the following form

$$x_i \to x_i + \eta_i(t,\theta)\,, \;\; i=1,2,3,4\,, \qquad u \to u + \eta_5(t,\theta)\,, \qquad v \to v + \eta_6(t,\theta)\,, \tag{4.39}$$

where we have defined the metric on the four-torus to be

$$ds^2_{T^4} = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2\,. \tag{4.40}$$

and the supertube embedding is the same as (4.24). One can use the sum of the DBI and WZ actions, find an effective action for the supertube fluctuations and compute the momenta conjugate to η_5, η_6 and η_i:

$$\begin{aligned}
\Pi_{\eta_5} &= \int dz \left(\frac{\partial \mathcal{L}}{\partial \dot{\eta}_5}\right)\bigg|_{BPS} = \frac{Z_2}{2\pi}\eta_5'\,, \\
\Pi_{\eta_6} &= \int dz \left(\frac{\partial \mathcal{L}}{\partial \dot{\eta}_6}\right)\bigg|_{BPS} = \frac{Z_2}{2\pi}\eta_6'\,, \\
\Pi_{\eta_i} &= \int dz \left(\frac{\partial \mathcal{L}}{\partial \dot{\eta}_i}\right)\bigg|_{BPS} = \frac{1}{2\pi}\eta_i'\,,
\end{aligned} \tag{4.41}$$

where the subscript “BPS” means that we have evaluated everything “on shell,” which means we have imposed the BPS conditions of no time dependence and $\mathcal{F}_{tz} = 1$.

The BPS modes η_i, η_5 and η_6 then can be expanded as

$$\eta_i = \frac{1}{\sqrt{2}}\Bigg[\sum_{k>0} e^{ik\theta/n_2^{ST}} \frac{(a_k^i)^\dagger}{\sqrt{|k|}} \;+\; \text{h.c.}\Bigg],$$

$$\eta_5 = \frac{1}{\sqrt{2}}\Bigg[\sum_{k>0} e^{ik\theta/n_2^{ST}} \frac{(a_k^5)^\dagger}{\sqrt{|k|}} \;+\; \text{h.c.}\Bigg], \tag{4.42}$$

$$\eta_6 = \frac{1}{\sqrt{2}}\Bigg[\sum_{k>0} e^{ik\theta/n_2^{ST}} \frac{(a_k^6)^\dagger}{\sqrt{|k|}} \;+\; \text{h.c.}\Bigg].$$

At first glance, the physics of the η_i fluctuations along the torus appears very different from that of the fluctuations in the spacetime direction, η_5 and η_6; indeed the latter have a factor of Z_2 in the denominator, and this factor becomes arbitrarily large when the supertube is near the horizon of a black hole.

The charge N_1^{ST} is the same as that of the round supertube in flat space, but the charge N_3^{ST} is modified to:

$$N_3^{ST} = \frac{1}{T_{F1}} \int d\theta \, \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{tz}}\bigg|_{BPS} = \frac{T_{D2}}{T_{F1}\mathcal{F}_{z\theta}} \int d\theta \left(Z_2 u^2 + Z_2[(\eta_5')^2 + (\eta_6')^2] + \sum_{i=1}^{4} (\eta_i')^2 \right). \tag{4.43}$$

Using similar arguments to those given for the flat space background one finds the entropy of the BPS shape modes to be:

$$S = 2\pi \sqrt{\frac{3}{2}} \sqrt{N_1^{ST} N_3^{ST} - (n_2^{ST})^2 Z_2 u^2}\,. \tag{4.44}$$

Hence, despite the presence of the warp factor Z_2 in the radius relation and in the mode expansions (4.42), the entropy of the supertube depends on its electric charges in exactly the same way as in flat space, and hence there is no entropy enhancement.

4.2.3 General GH background

Now we consider the general situation of a probe supertube with D0 and F1 charges and D2 dipole charge in the three-charge BPS solution with a Gibbons-Hawking base. We choose the supertube world-volume coordinates ξ to be $(t, \theta = \psi, z = x^5)$, where ψ is the $U(1)$ fiber of the GH base.

The DBI–WZ action of the supertube is:

$$S = T_{D2} \int d^3\xi \Bigg\{ \left[\left(\frac{1}{Z_1} - 1 \right) \mathcal{F}_{z\theta} + \frac{K^3}{Z_1 V} + (\mathcal{F}_{tz} - 1) \left(\frac{\mu}{Z_1} - \frac{K^1}{V} \right) \right]$$

$$- \left[\frac{1}{V^2 Z_1^2} \left[(K^3 - V(\mu(1-\mathcal{F}_{tz}) - \mathcal{F}_{z\theta}))^2 + V Z_1 Z_2 (1-\mathcal{F}_{tz})(2 - Z_3(1-\mathcal{F}_{tz})) \right] \right]^{1/2} \Bigg\}, \tag{4.45}$$

where $2\pi\alpha' F \equiv \mathcal{F} = \mathcal{F}_{tz} dt \wedge dz + \mathcal{F}_{z\theta} dz \wedge d\theta$ is the world-volume gauge field of the D2 brane. Our goal is to semi-classically quantize BPS fluctuations around certain supertube configurations, and compute their entropy. Supersymmetry requires that these fluctuations be independent of t and z, and that $\mathcal{F}_{tz} = 1$.

All the fluctuations of the supertube lead to similar values for the entropy, but for the purpose of illustrating entropy enhancement it is best to focus on the fluctuations in the four torus directions:

$$x_i \to x_i + \eta_i(t, \theta) \qquad i = 1 \dots 4. \tag{4.46}$$

Since the BPS modes are independent of z, it is convenient to work with a Lagrangian density that has already been integrated over the z direction, which gives the conjugate momenta for the excitations η_i:

$$\Pi_i = \left(\frac{\partial}{\partial \dot{\eta}_i} \int_0^{2\pi L_z} dz [\mathcal{L}_{WZ} + \mathcal{L}_{DBI}] \right)_{\dot{\eta}_{i=0},\ \mathcal{F}_{tz}=1} = 2\pi L_z T_{D2}\, \eta_i' \,, \tag{4.47}$$

where $\dot{\eta}_i \equiv \frac{\partial \eta_i}{\partial t}$ and $\eta_i' \equiv \frac{\partial \eta_i}{\partial \theta}$. To semi-classically quantize the BPS oscillations we impose the canonical commutation relations:

$$[\eta_j(t,\theta),\ \Pi_k(t,\theta')] = i\delta_{jk}\delta(\theta-\theta')\,. \tag{4.48}$$

To work with proper normalization one should remember that s supertube with dipole charge n_2^{ST} can be thought of as wrapped n_2^{ST} times around the θ circle and that both the BPS and non-BPS modes contribute to the delta-function in (4.48) and the inclusion of both contributions is essential to the proper normalization of the modes. The result is simply an extra factor of $\sqrt{2}$ in the coefficient of the BPS mode expansion compared to the naive expansion that neglects non-BPS modes:

$$\eta_i = {\eta_i}^{\rm BPS} + {\eta_i}^{\rm nonBPS} = \frac{1}{\sqrt{8\pi^2 T_{D2} L_z}} \sum_{k>0} \left[e^{ik\theta/n_2} \frac{(a_k^i)^\dagger}{\sqrt{|k|}} + \text{h.c.} \right] + {\eta_i}^{\rm nonBPS}\,. \tag{4.49}$$

The creation and annihilation operators, $(a_k^i)^\dagger$ and a_k^i, for the modes in the $k^{\rm th}$ harmonic satisfy canonical commutation relations:

$$[a_k^i, (a_{k'}^j)^\dagger] = \delta^{ij}\delta_{k,k'}\,. \tag{4.50}$$

The D0 and F1 quantized charges of the supertube are:

$$N_1^{ST} = \frac{T_{D2}}{T_{D0}} \int_0^{2\pi L_z} dz \int_0^{2\pi n_2} d\theta\, \mathcal{F}_{z\theta} = 4\pi^2 \frac{T_{D2}}{T_{D0}} L_z n_2^{ST} \mathcal{F}_{z\theta} \tag{4.51}$$

$$N_3^{ST} = \frac{T_{D2}}{T_{F1}} \int_0^{2\pi n_2} d\theta \left[-\frac{K^1}{V} + \frac{1}{\mathcal{F}_{z\theta} + V^{-1}K^3} \left(\frac{Z_2}{V} + (\eta')^2 \right) \right] \tag{4.52}$$

Substituting (4.49) into (4.52) and rearranging using (4.51) leads to:

$$\sum_{i=1}^{4}\sum_{k>0} k(a_k^i)^\dagger a_k^i = L_z T_{D2}\int_0^{2\pi n_2} d\theta \int_0^{2\pi n_2} d\theta' \sum_{i=1}^{4} \eta_i' \eta_i'$$
$$= \left[N_1^{ST} + 2\pi T_{F1} L_z n_2^{ST}\frac{K^3}{V}\right]\left[N_3^{ST} + \frac{2\pi T_{D2}}{T_{F1}} n_2^{ST}\frac{K^1}{V}\right] - 4\pi^2 T_{D2} L_z (n_2^{ST})^2 \frac{Z_2}{V}\,, \tag{4.53}$$

where the integrals over θ and θ' are precisely those appearing in each of (4.51) and (4.52). This is the result we have been seeking. The left hand side of (4.53) can be thought of as the total energy L_0 of a set of four harmonic oscillators in $1+1$ dimensions. For large L_0, the entropy coming from the different ways of distributing this energy to various modes of these oscillators is given by the Cardy formula:

$$S = 2\pi\sqrt{\frac{cL_0}{6}}. \tag{4.54}$$

Since we count BPS excitations, there will be also 4 fermionic degrees of freedom, and the central charge associated to the torus oscillations will be $c = 4 + 2 = 6$, giving the entropy:

$$S = 2\pi\sqrt{\left[N_1^{ST} + n_2^{ST}\frac{K^3}{V}\right]\left[N_3^{ST} + n_2^{ST}\frac{K^1}{V}\right] - (n_2^{ST})^2\frac{Z_2}{V}}$$
$$= 2\pi\sqrt{Q_1^{eff}\,Q_3^{eff} \;-\; (n_2^{ST})^2\frac{Z_2}{V}}\,, \tag{4.55}$$

where we have used the conventions presented in Appendix D and

$$Q_1^{eff} \equiv N_1^{ST} + n_2^{ST}\frac{K^3}{V}\,, \qquad Q_3^{eff} \equiv N_3^{ST} + n_2^{ST}\frac{K^1}{V} \tag{4.56}$$

Equation (4.53) has two important consequences. First, for a supertube with a given set of BPS modes, this equation is nothing but a "radius formula" that determines its size by fixing, in the spatial base, the location of the $U(1)$ fiber that it wraps. When the supertube is maximally spinning, and has no BPS modes, this equation simply becomes the radius formula of the maximally spinning supertube [35]. The second result is that this formula also determines the capacity of the supertube to store entropy: In flat space, this capacity is determined by the asymptotic charges, $Q_1^{ST} \equiv N_1^{ST}$ and $Q_3^{ST} \equiv N_3^{ST}$, whereas, in a more general background, the capacity to store entropy is determined by Q_1^{eff} and Q_3^{eff}. In certain backgrounds, the latter can be made much larger than the former and so a supertube of given asymptotic charges can have a lot more modes and thus store a lot more entropy by the simple expedient of migrating to a location where the effective charges are very large. We will discuss this further below.

Clearly, for bubbling backgrounds, and even for black ring backgrounds, the right hand side of (4.53) can diverge, and one naively gets an infinite value for the entropy. Nevertheless, as we mentioned in the introduction, this calculation is done in the approximation that the supertube does not back-react on the background, and taking this back-reaction into account will modify this naive conclusion.

We have also explicitly calculated the supertube entropy in a general three-charge black-ring background, where the supertube oscillates both in the T^4, and in two of the transverse $\mathbb{R}^4$ directions [35]. The result is identical to (4.55), except that now there are six possible bosonic modes (and thus after we include the corresponding fermions the central charge of the system is $c = 9$). Based on this result, we expect that upon including the four bosonic shape modes in the transverse space, as well as the fermionic counterparts of all the eight bosonic modes, the central charge c should jump from 6 to 12, and equation (4.55) to be modified accordingly. We have also explicitly computed the entropy coming from arbitrary shape modes, and the formulas do display entropy enhancement. Our calculation agrees with the entropy of supertubes in flat space-time, computed using similar methods in [175, 9], and using different methods in [188].

It is also possible to compute the angular momentum of the supertube along the GH fiber

$$J^{ST} = \frac{N_1^{ST} N_3^{ST}}{n_2^{ST}} - \frac{Q_1^{eff}\, Q_3^{eff}}{n_2^{ST}} + n_2^{ST} \frac{Z_2}{V}. \tag{4.57}$$

From this identity we may simply re-write (4.55) as

$$S \;=\; 2\pi \sqrt{Q_1^{eff}\, Q_3^{eff} \;-\; (n_2^{ST})^2 \frac{Z_2}{V}} \;=\; 2\pi \sqrt{N_1^{ST} N_3^{ST} - n_2^{ST} J^{ST}}\,. \tag{4.58}$$

Hence, in a certain sense, (4.55) is the same as the entropy formula for a supertube in empty space and it naively appears that entropy enhancement has gone away. It has not. The important point is that (4.57) implies that it is possible for J^{ST} to become extremely large and negative as the number of BPS modes on the tube increases[6]. In flat space, $|J^{ST}|$ is limited by $|N_1^{ST} N_3^{ST}|$ but in a general background our Born-Infeld analysis (equations (4.53) and (4.57)) imply that the upper bound is the same but there is no lower bound.

From the supergravity perspective, the limits on J^{ST} usually emerge from requiring that there are no CTC's near the supertube. This is a local condition set by the local behavior of the metric, and particularly by the Z_I, near the supertube. Although we do not have the explicit solution, our analysis suggests that the lower limit of the angular momentum of the supertube is controlled by Q_1^{eff} and Q_3^{eff} as opposed to N_1^{ST} and N_3^{ST}. Thus entropy enhancement can occur if the supertube moves to a region where Q_1^{eff} and Q_3^{eff} are extremely large and then a vast number of modes can be supported on a supertube (of fixed N_1^{ST} and N_3^{ST}) by making J^{ST} large and negative. We

[6]This is not unexpected: As in flat space, every BPS mode on the supertube takes away one quantum of angular momentum of the tube.

therefore expect the corresponding supergravity solution to be CTC-free provided that $|n_2^{ST} J^{ST}| < Q_1^{eff} Q_3^{eff}$.

One should thus think of a supertube of given n_2^{ST}, N_1^{ST} and N_3^{ST} as being able to store a certain number of modes before it over-spins. The "storage capacity" of the supertube is determined by the local conditions around the supertube and, specifically, by n_2^{ST}, Q_1^{eff} and Q_3^{eff}. Magnetic dipole interactions, like those evident in bubbling backgrounds, can thus greatly modify the capacity of a given supertube to store entropy.

4.2.4 Entropy Enhancement - the Proposal

As we have seen, the entropy of a supertube, and hence the entropy of a fluctuating geometry, depends upon the *local* effective charges and not upon the asymptotic charges measured at infinity. In the derivation of (4.53) we started with a maximally spinning, round supertube with zero entropy and perturbed around it. For the maximally spinning tube, the equilibrium position is determined by the vanishing of the right-hand side. Upon adding wiggles to the tube, the right hand side no longer vanishes and the imperfect cancelation is responsible for the entropy.

It is interesting to ask how much entropy can equation (4.53) accommodate. The answer is not so simple. At first glance one might say that the both terms in the right hand side of (4.53) can be divergent, and hence the entropy of the fluctuating tube is infinite. Nevertheless, one can see that the leading order divergent terms in $Q_1^{eff} Q_3^{eff}$ and in $(n_2^{ST})^2 Z_2/V$ come entirely from bulk supergravity fields, and exactly cancel for the supertube in a GH background.

It is likely that this partial cancelation is an artefact of the extremely symmetric form of the solution, and that in a more general solution such cancellation may not take place.

In particular, both Q_1^{eff} and Q_3^{eff} are integrals of "effective charge" densities on the supertube world-volume, and the right hand side of equation (4.53) should be written as

$$Q_1^{eff}\,Q_3^{eff} - n_2^2\frac{Z_2}{V} = \int \rho_1^{eff} d\theta \int \rho_3^{eff} d\theta - \int \rho_1^{eff}\,\rho_3^{eff} d\theta \tag{4.59}$$

If this generalized formula is correct, certain density and shape modes will disturb the balance between the product of integrals and the integral of the product, and the leading behavior of the entropy will still be of the order

$$S \sim \sqrt{Q_1^{eff}\,Q_2^{eff}}\,. \tag{4.60}$$

Regardless of this, the next-to-leading divergent terms in (4.55) are a combination of supertube world-volume terms and bulk supergravity fields. In a scaling solution, or when the tube is close to a black ring, these terms can diverge, giving naively an infinite entropy. We expect the back-reaction of the supertubes to render this entropy finite.

The idea of entropy enhancement is that one can find backgrounds in which the effective charges of a two-charge supertube can be made far larger than the asymptotic charges of the solution, and that, in the right circumstances, the oscillations of this humble supertube could give rise to an entropy that grows with the asymptotic charges much faster than $\sqrt{Q^2}$ (as typical for supertubes), and might even grow as fast as $\sqrt{Q^3}$, as typical for black holes in five dimensions.

To achieve such a vast enhancement requires a very strong magnetic dipole-dipole interaction and this means that multiple magnetic fluxes must be present in the solution. It is *not* sufficient to have a large red-shift: BMPV black holes have infinitely long throats and arbitrarily large red-shifts but have no magnetic dipole moments to enhance the effective charges and thus increase the entropy that may be stored on a given supertube.

Hence, the obvious places to obtain entropy enhancement are solutions with large dipole magnetic fields, such as black ring or bubbling microstate solutions. Since we are focussing on trying to obtain the entropy of black holes from horizonless configurations, we will focus on the latter. These bubbling solutions are constructed using an ambi-polar base GH metric, and near the "critical surfaces," where V vanishes, the term $\frac{K^I}{V}$ in the effective charge diverges. It is therefore natural to expect entropy enhancement for supertubes that localize near the critical ($V = 0$) surfaces.

We also believe that placing supertubes in deep scaling solutions [29, 32, 79] will prove to be an equally crucial ingredient. Indeed, as we will see in the next section, in a deep microstate geometry the K^I at the location of the tube can also become large, and hence there will be a double enhancement of the effective charge, both because of the vanishing V in the denominator and because of the very large K^I in the numerator. There is another obvious reason for this: It is only the scaling microstate geometries that have the same quantum numbers as black holes with macroscopic horizons [29] .

This must mean that the simple entropy enhancement one gets from the presence of critical surfaces is not sufficient for matching the black hole entropy. The fundamental reason for this may well be the following: Even if the round supertube can be brought very close to the $V = 0$ surface, once the supertube starts oscillating it will necessarily sample the region around this surface, and the charge enhancement will correspond to the average Q_I^{eff} in that region. For this to be very large the entire region where the supertube oscillates must have a very significant charge enhancement. The only such region in a horizonless solution is the bottom of a deep or scaling throat, where the average of the K^I is indeed very large.

All the issues we have raised here have to do with the details of the entropy enhancement mechanism, and involve some very long and complex calculations that we intend to pursue in future work. We believe their clarification is very important, as it will shed light on how the entropy of black holes can be realized at the level of horizonless configurations.

Our goals here are rather more modest. We have shown via a Dirac-Born-Infeld probe calculation that the entropy of supertubes is given by their effective charges, and not by their brane charges, and that these effective charges can be very large. However, because the supertube has been treated as a probe in our calculations, it is logically possible that, once we take into account its back-reaction, the bubble equations may forbid the supertube to get suitably close to the $V = 0$ surfaces, and to have a suitable entropy enhancement.

In principle this is rather unlikely, as we know that in all the examples studied to date, the solutions of the Born-Infeld action of supertubes always correspond to configurations that are smooth and regular in supergravity [35]. However, settling the issue completely is not possible before constructing the full supergravity solutions corresponding to wiggly supertubes. We recently made some progress in this direction [40]. In the remainder of this chapter we will show that at least for the maximally-spinning supertubes, their effective charges in deep scaling solutions can lead to a black-hole-like enhanced entropy.

4.2.5 Supertubes in scaling microstate geometries

As explained in Chapter 3 and [33], one can perform a spectral flow transformation on a GH background with N GH points and obtaina bacground with $N-1$ GH points and one two-charge supertube. The dipole charge and "bare" electric charges of the supertube are given by the coefficients of the divergent terms in $\widetilde{K}^2$, $\widetilde{L}_1$ and $\widetilde{L}_3$[7]. One can furthermore

[7]The harmonic functions after the spectral flow transformation are denoted with tilde. See Chapter 3 for more detials on the spectral flow transformation for GH bubbling solutions

show that "effective" charges of the supertube are given by the divergence of the electric potentials, Z_I, near the supertube:

$$Q_1^{eff} \equiv 4 \lim_{r_N \to 0} r_N \widetilde{Z}_1 = 4 \, q_N \left(\widetilde{V}^{-1} Z_1\right)\Big|_{r_N=0} = 4\tilde{\ell}_N^1 + 4\tilde{k}_N^2 (\widetilde{V}^{-1} \widetilde{K}^3)|_{r_N=0} \,, \quad (4.61)$$

and similarly for Q_3^{eff}.

To find bubbling solutions that contain supertubes with enhanced charges one could look for solutions of the bubble equations (2.79) that describe scaling solutions where some of the centers are GH points, and the other centers are supertubes. However, it is more convenient to construct such solutions by spectrally flowing multi-center GH solutions, which have been studied much more. The parameters of the equations are the same as in Section 2.2.10. One obtains a scaling solution when a subset, $\mathcal{S}$, of the GH points approach one another arbitrarily closely, that is, $r_{ij} \to 0$ for $i, j \in \mathcal{S}$. In terms of the physical geometry, these points are remaining at a fixed distance from each other, but are descending a long AdS throat that, in the intermediate region, looks almost identical to the throat of a black hole or black ring (depending upon the total GH charge in $\mathcal{S}$). In particular, in the intermediate regime, one has $Z_I \sim \frac{\hat{Q}_I}{4\,r}$, where we have taken $\mathcal{S}$ to be centered at $r = 0$ and the $\hat{Q}_I$ are the electric charges associated with $\mathcal{S}$. Similarly, if $\mathcal{S}$ has a non-zero total GH charge of $\hat{q}_0$, then one has $V \sim \frac{\hat{q}_0}{r}$. More precisely:

$$Z_I \, V = l_0^I V + \varepsilon_0 \, (L_I - \ell_0^I) - \tfrac{1}{4} \, C_{IJK} \sum_{i,j=1}^{N} \Pi_{ij}^{(J)} \, \Pi_{ij}^{(K)} \, \frac{q_i \, q_j}{r_i \, r_j} \,. \quad (4.62)$$

Suppose that we perform a spectral flow so that some point, $p \in \mathcal{S}$, becomes a supertube. Let $\widetilde{V}_p$ be the value of $\widetilde{V}$ at p. Then, from (4.61), the effective charges of this supertube are dominated by terms from interactions with the magnetic fluxes in the throat:

$$Q_I^{eff} \ \sim \ -2\, q_p\, \widetilde{V}_p^{-1}\, C_{IJK} \sum_{j \in \mathcal{S}\,,\, j \neq p} \Pi_{jp}^{(J)}\, \Pi_{jp}^{(K)}\, \frac{q_j}{r_{jp}}\,. \tag{4.63}$$

However, observe that $\tilde{q}_j = (k_p^2)^{-1} q_p q_j \Pi_{jp}^{(2)}$ and so

$$q_p^{-1} \widetilde{V}_p \ \sim \ (k_p^2)^{-1} \sum_{j \in \mathcal{S}\,,\, j \neq p}^{N} \frac{q_j \Pi_{jp}^{(2)}}{r_{jp}}\,. \tag{4.64}$$

Therefore the numerator and denominator of (4.63) have the same naive scaling behavior as $r_{jp} \to 0$ and so, in general, Q_I^{eff} will attain a finite limit that only depends upon the q_j, k_j^I for $j \in \mathcal{S}$. Indeed, the finite limit of Q_I^{eff} scales as the square of the k's for large k_j^I parameters. This is no different from the typical values of asymptotic electric charges in bubbled geometries.

However, since we are in a bubbled microstate geometry, V and $\widetilde{V}$ change sign throughout the bubbled region. In particular, there are surfaces at the bottom of the throat where $\widetilde{V}$ vanishes and there are regions around them where $\widetilde{V}$ remains finite and bounded as $r_{ij} \to 0$. Suppose that we can arrange for the supertube point p to be in such a region of a scaling throat and at the same time we can arrange that Z_I still diverges as $\frac{1}{r}$. Then, in principle, the effective charges, of the supertube Q_I^{eff}, could become arbitrarily large.

As mentioned above, we expect the entropy of the system to come from wiggly supertubes in throats that are neither very deep (to allow the tubes to wiggle), nor very shallow (to give enhancement). We do not, as yet, know how to take the back-reaction of the wiggly supertubes into account, and hence we do not have any supergravity argument about the length of these throats. However, we can use the AdS/CFT correspondence and the fact that we know what the typical CFT microstates are, to argue [29] that the typical bulk microstates are scaling solutions that have GH size r_T given by

$$r_T \sim \overline{Q}^{\,-1/2} \sim \frac{1}{\bar{k}}, \tag{4.65}$$

where $\overline{Q}$ is the charge and $\bar{k}$ is the typical flux parameter.

If one takes this AdS/CFT result as given, and moreover assumes that the wiggling supertube remains in a region of finite $\widetilde{V}$ in the vicinity of the $\widetilde{V} = 0$ surface, one then has:

$$Q_I^{eff} \sim (\bar{k})^3 \sim \overline{Q}^{\,3/2} \tag{4.66}$$

because $\Pi^{(K)}_{jp} \sim \bar{k}$, and hence the entropy of the fluctuating supertube (4.60) would depend upon the asymptotic charges as:

$$S \sim \sqrt{Q_1^{eff}\, Q_2^{eff}} \sim \overline{Q}^{\,3/2}. \tag{4.67}$$

which is precisely the correct behavior for the entropy of a classical black hole!

These simple arguments indicate that fluctuating supertubes at the bottom of deep scaling microstate geometries can give rise to a black-hole-like macroscopic entropy, provided that they oscillate in a region of bounded $\widetilde{V}$.

Obviously there is a great deal to be checked in this argument, particularly about the effect of the back-reaction of the supertube on its localization near the $\widetilde{V} = 0$ surface. We conclude this section by demonstrating that at least maximally spinning tubes, for which we can construct the supergravity solution, have no problem localizing in a region of finite $\widetilde{V}$. As the solution scales, the effective charges diverge, as is needed for entropy enhancement.

4.2.6 An example

One can construct a very simple deep scaling solution using three GH centers with charges q_1, q_2 and q_3, and fluxes arranged so that $|\Gamma_{ij}| \equiv |q_i\, q_j \Pi^{(1)}_{ij} \Pi^{(2)}_{ij} \Pi^{(3)}_{ij}|$, $i, j = 1, 2, 3$, satisfy the triangle inequalities. The GH points then arrange themselves asymptotically as a scaled version of this triangle:

$$r_{ij} \;\to\; \lambda \left|\Gamma_{ij}\right| , \qquad \lambda \to 0 \,. \tag{4.68}$$

One can then take a spectral-flow of this solution so that the second GH point becomes a two-charge supertube. For simplicity, we will choose $q_1\, \Pi^{(2)}_{12} \;=\; q_3\, \Pi^{(2)}_{23}$ so that after the flow the GH charges of the remaining two GH points will be equal and opposite:

$$\tilde q_1 \;=\; -\, \tilde q_3 \,. \tag{4.69}$$

For $\widetilde V_p$ to remain finite in the scaling limit, the supertube must approach the plane equidistant from the remaining GH points.

We have performed a detailed analysis of such solutions and used the absence of CTC's close to the GH points, in the intermediate throat and in the asymptotic region to constrain the possible fluxes. We have found a number of such solutions that have the desired scaling properties for Q_I^{eff} and we have performed extensive numerical analysis to check that there are no regions with CTC's. In particular, we checked numerically that the inverse metric component, g^{tt}, is globally negative and thus the metric is stably causal. We will simply present one example here.

Consider the asymptotically Taub-NUT solution with:

$$q_1 \;=\; 16 \,, \quad q_2 \;=\; 96 \,, \quad q_3 \;=\; -40 \,, \quad \epsilon_0 \;=\; 1 \,, \quad Q_0 \;\equiv\; q_1 + q_2 + q_3 \;=\; 72 \,, \tag{4.70}$$

and

$$k_1^I \;=\; (8,-88,8)\,, \qquad k_2^I \;=\; (0,96,0)\,, \qquad k_3^I \;=\; (20,64,20)\,, \tag{4.71}$$

where Q_0 is the KK monopole charge of the solution. With these parameters one has the following fluxes:

$$\Pi_{12}^{(I)} \;=\; (-\tfrac{1}{2},\tfrac{13}{2},-\tfrac{1}{2})\,, \qquad \Pi_{23}^{(I)} \;=\; (-\tfrac{1}{2},-\tfrac{13}{5},-\tfrac{1}{2})\,, \qquad \Pi_{13}^{(I)} \;=\; (-1,\tfrac{39}{10},-1)\,, \tag{4.72}$$

and

$$\Gamma_{12} \;=\; \Gamma_{23} \;=\; \Gamma_{31} \;=\; 2496\,. \tag{4.73}$$

In this scaling solution the GH points form an equilateral triangle and thus, after the spectral flow, the supertube will tend to be equidistant from the two GH points of equal and opposite charges (4.69), and therefore will approach the surface where $\widetilde{V} = 0$.

The solution to the bubble equations yields

$$r_{12} \;=\; \frac{11232\, r_{13}}{11232 + 359\, r_{13}}\,, \qquad r_{23} \;=\; \frac{11232\, r_{13}}{11232 + 731\, r_{13}}\,, \tag{4.74}$$

which satisfies the triangle inequalities for $r_{13} \le \frac{11232}{\sqrt{262429}} \approx 21.9$. After spectral flow the value of $\widetilde{V}$ at the location of the supertube (point 2) is

$$\widetilde{V}_2 \;=\; 1 \;+\; \frac{104}{r_{12}} \;-\; \frac{104}{r_{23}} \;=\; -\,\frac{22}{9}\,, \tag{4.75}$$

independent of r_{13}. In particular, it remains finite and bounded as the three points scale and the distances between them go to zero. The effective charges of the supertube are given by

$$Q_1^{eff} \;=\; Q_3^{eff} \;=\; 384\, \widetilde{V}_2^{-1}\Big(1 \;+\; \frac{52}{r_{12}} \;+\; \frac{52}{r_{23}}\Big)\,, \tag{4.76}$$

and scale as λ^{-1} as $\lambda \to 0$ in (4.68). We thus have effective charges that naively scale to arbitrarily large values. As described earlier, we expect this scaling to stop as the supertubes become more and more wiggly, and we expect the entropy to come from configurations of intermediate throat depth.

4.3 Conclusions

The most important result presented in this Chapter is that the entropy of a supertube in a given background is not determined by its charges, but rather by its "effective charges," which receive a contribution from the interaction of the magnetic dipole moment of the tube with the magnetic fluxes in the background. As a result, one can get very dramatic entropy enhancement if a supertube is placed in a suitable background. We have argued that this enhancement can give rise to a macroscopic (black-hole-like) entropy, coming entirely from smooth horizonless configurations.

Three ingredients are needed for this dramatic entropy enhancement:

- Deep or scaling solutions
- Ambi-polar base metrics
- BPS fluctuations that localize near the critical ($V = 0$) surfaces of the ambi-polar metrics

These are also precisely the ingredients that have emerged from recent developments in the study of finite-sized black-hole microstates in the regime of parameters where the gravitational back-reaction of some of the branes is negligible. Indeed, deep scaling ambi-polar configurations are needed both to get a macroscopic entropy in the quiver quantum field theory regime' [79], and to get smooth microstates of black holes with macroscopic horizons [29]. Furthermore, the D0 branes that can give a black-hole-like entropy in a D6-$\overline{\text{D6}}$ background [80] must localize near the critical surface of the ambi-polar base, much like the supertubes in our analysis. It would be fascinating to find a link between the microscopic configurations constructed in these papers, and those we consider here.

We have referred to the entropy enhancement mechanism as a "proposal" because a number of the details need to be checked by careful computation. Most importantly, we have performed a classical calculation using a brane probe near a critical surface. It is important to study the fluctuating supertubes in the full supergravity theory and determine how the back-reaction of the fluctuations modifies the picture presented here. One important issue is whether fluctuating supertubes can still remain in the region close to the critical surface with V finite and bounded. Another is to understand the interplay between how much a supertube wiggles and how long its throat can get or how much the supergravity solution it sources can scale.

In [40] we found exact supergravity solutions corresponding to wiggling supertubes. We were able to find the exact solution that corresponds to the backreaction of the charge density mode along the supertube profile. The solution that we find is valid for a circular supertube in a general GH background . Taking into account the backreaction of the shape modes of the supertube seems to be technically challenging since one breaks a lot of the symmetries of the background. The results of [40] however support the general results of the entropy enhancement mechanism described in this Chapter for probe supertubes. In particular we have shown in [40] that (4.59) is valid for the charge density modes.

While some of the details need to be explored very carefully, we believe that the mechanism and the approach of this Chapter may well provide the key to understanding how fluctuating microstate geometries can provide a semi-classical description of black-hole entropy in the regime of parameters where the black hole exists.

Chapter 5

Going beyond triholomorphy

Despite the remarkable results that have been obtained using Gibbons-Hawking geometries, such metrics represent a major restriction. In particular, they all have a translational (tri-holomorphic) $U(1)$ isometry [113], which is a combination of the two $U(1)$'s in the $\mathbb{R}^2$ planes that make up the $\mathbb{R}^4$ in the asymptotic region[1]. Thus, bubbling solutions with a GH base cannot capture quite a host of interesting physical processes that do not respect this symmetry, like the merger of two BMPV black holes, or the geometric transition of a three-charge supertube of arbitrary shape. In [26] it was argued that this geometric transition results in bubbling solutions that have an ambi-polar hyper-Kähler base, and that depend on a very large number of arbitrary continuous functions. It is of great interest to construct and understand such solutions since they will provide important information about the structure of the microstates of supersymmetric black holes.

In this Chapter, based on [31], we will make a step in this direction by considering metrics that have a general $U(1)$ isometry which are much less restrictive than the GH metrics. Moreover, they could also arise from the geometric transition of supertubes that preserve a rotational $U(1)$, and hence could also depend on an arbitrarily large number of continuous functions. Even if the entropy in these symmetric configurations will be smaller than the entropy of the black hole, it might give some insight into the structure and charge dependence of the most general, non-symmetric configuration.

[1]Alternatively, one can see that the tri-holomorphic $U(1)$ necessarily lies in one of the $SU(2)$ factors of $SO(4) \equiv (SU(2) \times SU(2))/\mathbb{Z}_2$.

An important feature of all bubbled solutions is that the ambi-polar base space and the fluxes dual to the homology are singular on the critical surfaces where the metric changes sign. As we discussed in Chapter 2, for ambi-polar GH spaces it was possible to use the explicit solutions to show that all these singularities were canceled and the final result was a regular, five-dimensional compactification of M-theory. Our analysis here will illustrate how this happens for the general $U(1)$-invariant BPS bubbled background, and this work suggests that the most general bubbling geometries will have also this property.

Before beginning, we would like to stress that constructing solutions that only have a rotational $U(1)$ is a rather tedious and challenging task. For classical black holes and black rings, only two such solutions exist: one describing a black ring with an arbitrary charge density [23], and one describing a black ring with a black hole away from the center of the ring [27]. In [31] we were able to construct the first explicit bubbling solution in this class, using a base that is a generalization of the Atiyah-Hitchin metric[2]. Nevertheless, the most general bubbling solutions that only have a rotational $U(1)$ invariance will be much more complicated, and perhaps even impossible to write down explicitly.

In this Chapter we will follow [31] and present a general analysis of supergravity solutions on hyper-Kähler manifolds with no tri-holomorphic $U(1)$ symmetry. The solutions cannot be constructed as explicitly as for case of GH metrics but nevertheless we are able to show that the general structure for the construction of bubbling solutions is still present. We also present the explicit five-dimensional supergravity solution with a four-dimensional Atiyah-Hitchin base and show how one can construct a bubbling

[2]Our solutions can also be used to construct black holes or black rings in the Atiyah-Hitchin space. See [197, 105, 106, 107] for recent work in this direction.

solution on it. We present also a curious construction of a five-dimensional supergr-vaity solution based on the irregular Eguchi-Hanson metric, the result is the well-known $AdS_2 \times S^3$ solution in global coordinates with constant electro-magnetic flux.

5.1 Hyper-Kähler base with a general $U(1)$

5.1.1 Prelude

It has been known for a long time that hyper-Kähler metrics with a generic (rotational) $U(1)$ isometry can be obtained by solving the $SU(\infty)$ Toda equation [48, 72, 10]. The coordinates can be chosen so that the metric takes the form:

$$ds_4^2 \;=\; V^{-1}\,(d\tau + A_i dx^i)^2 \;+\; V\gamma_{ij}\,dx^i dx^j\,, \tag{5.1}$$

with $\gamma_{ij} = 0$ for $i \neq j$ and

$$\gamma_{11} \;=\; \gamma_{22} \;=\; e^{\nu}\,, \qquad \gamma_{33} \;=\; 1\,, \tag{5.2}$$

for some function, ν. The function, V, and the vector field, A, are given by

$$V \;=\; \partial_z\nu\,, \qquad A_1 \;=\; \partial_y\nu\,, \qquad A_2 \;=\; -\,\partial_x\nu\,, \qquad A_3 \;=\; 0\,, \tag{5.3}$$

and the function ν must satisfy:

$$\partial_x^2\,\nu \;+\; \partial_y^2\,\nu \;+\; \partial_z^2\,(e^{\nu}) \;=\; 0\,. \tag{5.4}$$

This equation is called the $SU(\infty)$ Toda equation, and may be viewed as a continuum limit of the $SU(N)$ Toda equation. Even though the $SU(\infty)$ Toda equation is integrable, surprisingly little is know about its solutions, and there appears to be no known analog of the known soliton solutions of the $SU(N)$ Toda equation. On the other hand, the metric is determined in terms of a single function and (5.1) is a relatively mild generalization of the Gibbons-Hawking metrics.

Our purpose here is to construct three-charge solutions based upon ambi-polar hyper-Kähler metrics with generic (non-tri-holomorphic) $U(1)$ isometries. We will do this in two different ways, first by building such solutions using a general metric of the form (5.1) on the base space. We will then consider the Atiyah-Hitchin metric: This metric has an $SO(3)$ isometry, but none of the $U(1)$ subgroups is tri-holomorphic. Just as with the GH metric, the metric (5.1), is ambi-polar if we allow $V = \partial_z \nu$ to change sign. Thus the primary issue of regularity in the five-dimensional metric arises on the critical surfaces where $V = 0$. While we will not be able to construct general solutions as explicitly as can be done for GH metrics, we will show that the five-dimensional metric is regular and Lorentzian in the neighborhood of these critical surfaces.

The standard Atiyah-Hitchin metric [5] arises as the solution of a first order, non-linear Darboux-Halphen system for the three metric coefficient functions. This system is analytically solvable in terms of the solution of a single, second order linear differential equation. Indeed, the solutions of the latter equation are expressible in terms of elliptic functions. The standard practice is to choose the solution of this linear equation so that the metric functions are regular, and the result is a smooth geometry that closes off at a non-trivial "bolt," or two-cycle in the center. We will show that if one selects the most general solution of the linear differential equation, then one obtains an ambi-polar generalization of the Atiyah-Hitchin metric. Moreover, one can set up regular, cohomological fluxes on the two-cycle and the resulting warp factors render the five-dimensional metric perfectly smooth and regular across the critical surface.

The ambi-polar Atiyah-Hitchin metric actually continues through the bolt and initially appears to have two regions, one on each side of the bolt, that are asymptotic to $\mathbb{R}^3 \times S^1$. It thus looks like a wormhole. Unfortunately, the solution cannot be made regular on *both* its asymptotic regions. Indeed, upon imposing asymtotic flatness on one side of the wormhole, one finds that the warp factors change sign twice, once on the critical surface and again as one enters one of the asymptotic regions. Thus the critical surface is regular, but there is another potentially singular region elsewhere. However, we find that if we tune the flux through the bubble to exactly the proper value, one can pinch off the metric just as the warp factors change sign for the second time. The result is a Lorentzian metric, that extends smoothly through the critical surface ($V = 0$). The pinching off of the metric does however result in a curvature singularity that is very similar to the one encountered in the Klebanov-Tseytlin solution [149]. We will argue that the singularity of our new, non-trivial BPS solution is also a consequence of the very high level of symmetry, and that it should be resolved via a mechanism similar to that in [150].

We also consider solutions based upon an ambi-polar generalization of the Eguchi-Hanson metric, obtained by making an analytic continuation of the standard Eguchi-Hanson metric, and extending the range of one of the coordinates[3]. The singular structure of this metric is precisely what is needed to render it ambi-polar. Hence, upon adding fluxes and warp factors this metric gives us regular five-dimensional solutions that have similar features to the bubling Atiyah-Hitchin solution. There is also one surprise: One of the Eguchi-Hanson "wormhole" solutions is completely regular everywhere and is nothing other than the global $AdS_2 \times S^3$ Robinson-Bertotti solution [44].

[3] The unextended version of this metric was also discussed in the original Eguchi-Hanson paper [83], but was discarded because it is singular.

We will start by discussing the general construction of supergravity solutions on hyper-Kähler bases with a general $U(1)$ symmetry and their ambipolar generalizations. In the next Section we will present two explicit solutions on the Atiyah-Hitchin and the singular Eguchi-Hanson spaces.

5.1.2 The BPS equations

We will use the same Ansatz for the metric and the 3-form gauge potential of eleven-dimensional supergravity as in Chapter 2 and will study the system of BPS equations (2.14), (2.15), (2.16) on a general hyper-Kähler manifold with a $U(1)$ isometry.

The first step in solving the system of BPS equations is to identify the self-dual, harmonic two-forms, $\Theta^{(I)}$. In a Kähler manifold this is, at least theoretically, straightforward because such two-forms are related to the moduli of the metric. For a hyper-Kähler metric there are three complex structures, $J_{(i)}$, $i = 1, 2, 3$, and given a harmonic two-form, ω, one can define three symmetric tensors via:

$$h^{(i)}_{\mu\nu} \;\equiv\; J_{(i)\,\mu}{}^{\rho}\,\omega_{\rho\nu} \;-\; J_{(i)\,\nu}{}^{\rho}\,\omega_{\mu\rho}\,. \tag{5.5}$$

These tensors may be viewed as metric perturbations and as such they represent perturbations that preserve the hyper-Kähler structure. In particular, they are zero modes of the Lichnerowicz operator.

For the metric (5.1) it is convenient to introduce vierbeins:

$$\hat{e}^1 = V^{-1/2}(d\tau + A_i dx^i)\,, \qquad \hat{e}^2 = V^{1/2}e^{\nu/2}dx\,, \qquad \hat{e}^3 = V^{1/2}e^{\nu/2}dy\,, \qquad \hat{e}^4 = V^{1/2}dz\,, \tag{5.6}$$

and introduce a basis for the self-dual and anti-self dual two forms:

$$\Omega_\pm^{(1)} = (d\tau + A_2 dy)\wedge dx \pm V dy \wedge dz = e^{-\nu/2}(\hat{e}^1\wedge\hat{e}^2 \pm \hat{e}^3\wedge\hat{e}^4)\,, \tag{5.7}$$

$$\Omega_\pm^{(2)} = (d\tau + A_1 dx)\wedge dy \pm V dz \wedge dx = e^{-\nu/2}(\hat{e}^1\wedge\hat{e}^3 \pm \hat{e}^4\wedge\hat{e}^2)\,, \tag{5.8}$$

$$\Omega_\pm^{(3)} = (d\tau + A_1 dx + A_2 dy)\wedge dz \pm e^{\nu} V dx \wedge dy = (\hat{e}^1\wedge\hat{e}^4 \pm \hat{e}^2\wedge\hat{e}^3)\,. \tag{5.9}$$

The three Kähler forms are then given by [10]:

$$J_{(1)} = e^{\nu/2}\cos\left(\frac{\tau}{2}\right)\Omega_-^{(1)} + e^{\nu/2}\sin\left(\frac{\tau}{2}\right)\Omega_-^{(2)}, \tag{5.10}$$

$$J_{(2)} = e^{\nu/2}\sin\left(\frac{\tau}{2}\right)\Omega_-^{(1)} - e^{\nu/2}\cos\left(\frac{\tau}{2}\right)\Omega_-^{(2)}\,, \tag{5.11}$$

$$J_{(3)} = \Omega_-^{(3)}\,, \tag{5.12}$$

and one can check that they satisfy the proper quaternionic algebra:

$$J_{(i)\,\mu}{}^{\rho}\, J_{(j)\,\rho}{}^{\nu} = \delta_{ij}\,\delta_\mu^\nu - \varepsilon_{ijk}\, J_{(k)\,\mu}{}^{\nu}\,. \tag{5.13}$$

Following [26], we make an Ansatz for the harmonic, self-dual field strengths, $\Theta^{(I)}$:

$$\Theta^{(I)} = \sum_{a=1}^{3} \partial_a(\dot{\nu}^{-1}K^I)\,\Omega_+^{(a)}\,, \tag{5.14}$$

where the dot represents derivative with respect to z. We then find that the K^I must satisfy the linearized Toda equation (it follows from (5.4) that $\dot{\nu}$ also solves this equation):

$$\mathcal{L}_T\, K^I \equiv \partial_x^2 K^I + \partial_y^2 K^I + \partial_z^2(e^{\nu}K^I) = 0\,. \tag{5.15}$$

For later convenience, we note that there are relatively simple vector potentials such that $\Theta^{(I)} = dB^{(I)}$:

$$B^{(I)} \equiv \dot\nu^{-1} K^I (d\tau + A) + \vec\xi^{(I)} \cdot d\vec x \,, \tag{5.16}$$

where

$$(\vec\nabla \times \vec\xi^{(I)})^j = -\partial_i\big(\gamma^{ij} e^\nu K^I\big) \,. \tag{5.17}$$

Hence, $\vec\xi^{(I)}$ is a vector potential for magnetic monopoles located at the singular points of K^I.

Since the K^I satisfy the linearized Toda equation, we see the direct relationship between the harmonic forms and linearized fluctuations of the metric. In practice, (5.5) and (5.14) do not yield exactly the same result as the direct substitution of fluctuations in ν into (5.1) but they are equivalent up to infinitesimal diffeomorphisms. For example, the metric fluctuation obtained from using (5.14) and $J_{(3)}$ in (5.5) is identical with the metric fluctuation, $\nu \to \nu + \epsilon K^I$, combined with the infinitesimal diffeomorphism, $z \to z - \epsilon \dot\nu^{-1} K^I$.

The second BPS equation (2.15) reduces to:

$$\mathcal{L} Z_I = \dot\nu \, e^\nu \, C_{IJK} \, \gamma^{ij} \partial_i\Big(\frac{K^J}{\dot\nu}\Big) \, \partial_j\Big(\frac{K^K}{\dot\nu}\Big) \,, \tag{5.18}$$

where γ_{ij} is the three-metric in (5.2) and $\mathcal{L}$ is given by:

$$\mathcal{L}\, F \equiv \dot\nu \, e^\nu \, \nabla^2_\gamma F = \partial_x^2 F + \partial_y^2 F + \partial_z(e^\nu \partial_z F) \,. \tag{5.19}$$

The operator, ∇^2_γ, denotes the Laplacian in the metric γ_{ij}.

The natural guess for the solution is to follow, once again, [26] and try:

$$Z_I \equiv \tfrac{1}{2} C_{IJK} \, \dot\nu^{-1} \, K^J K^K + Z_I^{(0)} \,. \tag{5.20}$$

One then finds that $Z_I^{(0)}$ is not a solution of the homogeneous equation, but

$$\mathcal{L}\, Z_I^{(0)} \;=\; -\,\partial_z\big(\tfrac{1}{2}\, e^{\nu}\, C_{IJK}\, K^J K^K\big)\,. \tag{5.21}$$

Intriguingly, one can also check that:

$$\mathcal{L}_T\big(\tfrac{1}{2}\, C_{IJK}\, \dot\nu^{-1}\, K^J K^K\big) \;=\; \dot\nu\, e^{\nu}\, C_{IJK}\, \gamma^{ij}\partial_i\Big(\frac{K^J}{\dot\nu}\Big)\,\partial_j\Big(\frac{K^K}{\dot\nu}\Big)\,, \tag{5.22}$$

where $\mathcal{L}_T$ is the linearized Toda operator (5.15) and so one has the explicit solution but to the wrong equation.

The important point, however, is that the source on the right-hand side of (5.21) is regular as $\dot\nu \to 0$, and so $Z_I^{(0)}$ is regular on any critical surface where one has $\dot\nu = 0$.

To solve the last BPS equation (2.16) for the angular momentum vector, k, we make the Ansatz:

$$k \;=\; \mu\,(d\tau + A) \;+\; \omega\,, \tag{5.23}$$

where ω is a one form in the three-dimensional space defined by (x, y, z). Define yet another linear operator:

$$\widetilde{\mathcal{L}}\, F \;\equiv\; e^{\nu}\gamma^{ij}\,\partial_i\,\partial_j F \;=\; \partial_x^2 F +\; \partial_y^2 F \;+\; e^{\nu}\,\partial_z^2 F\,, \tag{5.24}$$

and then one finds that μ and ω must satisfy:

$$\widetilde{\mathcal{L}}\,\mu \;=\; \dot\nu^{-1}\,\partial_i\Big(\dot\nu\, e^{\nu}\,\gamma^{ij}\,\sum_{I=1}^{3} Z_I\,\partial_j\Big(\frac{K^I}{\dot\nu}\Big)\Big)\,, \tag{5.25}$$

and

$$(\vec\nabla\times\vec\omega)^i \;=\; \dot\nu\, e^{\nu}\gamma^{ij}\partial_j\mu \;-\; \mu\,\partial_j(e^{\nu}\gamma^{ij}\dot\nu) \;-\; \dot\nu\, e^{\nu}\, Z_I\,\gamma^{ij}\,\partial_j\Big(\frac{K^I}{\dot\nu}\Big)\,. \tag{5.26}$$

Note that the integrability of the equation for ω is precisely the equation (5.25) for μ, provided that one also uses the fact that ν satisfies (5.4). The structure of these equations also closely parallels those encountered for a GH base metric [26, 41].

Once again one can try a form of the solution based upon the results for GH spaces. Define μ_0 by:

$$\begin{aligned} \mu &= \tfrac{1}{2}\,\dot\nu^{-1}\,Z_I\,K^I \;-\; \tfrac{1}{12}\,\dot\nu^{-2}\,C_{IJK}\,K^IK^JK^K \;+\; \mu_0 \\ &= \tfrac{1}{2}\,\dot\nu^{-1}\,Z_I^{(0)}\,K^I \;+\; \tfrac{1}{6}\,\dot\nu^{-2}\,C_{IJK}\,K^IK^JK^K \;+\; \mu_0\,, \end{aligned} \tag{5.27}$$

and one then finds that μ_0 must satisfy:

$$\widetilde{\mathcal{L}}\,\mu_0 \;=\; -\,\tfrac{1}{2}\,e^{\nu}\,K^I\,\partial_z Z_I^{(0)} \;+\; \tfrac{1}{12}\,e^{\nu}\,C_{IJK}\,K^IK^JK^K\,. \tag{5.28}$$

Again note that the source is regular as $\dot\nu \to 0$ and so μ_0 will be similarly regular as $\dot\nu \to 0$.

Finally, if one substitutes these expressions for Z_I and μ into (5.26), one obtains:

$$\begin{aligned} (\vec\nabla\times\vec\omega)^i \;=\;& \dot\nu\,e^{\nu}\gamma^{ij}\partial_j\mu_0 \;-\; \mu_0\,\partial_j(e^{\nu}\gamma^{ij}\dot\nu) \;+\; \tfrac{1}{2}\,K^I\partial_j\big(e^{\nu}\gamma^{ij}Z_I^{(0)}\big) \;-\; \tfrac{1}{2}\,e^{\nu}\gamma^{ij}Z_I^{(0)}\partial_jK^I \\ &-\; \tfrac{1}{6}\,\delta^i_3\,e^{\nu}\,C_{IJK}\,K^IK^JK^K\,, \end{aligned} \tag{5.29}$$

where the δ^i_3 means that the last term only appears for $i=3$. Note that $\vec\omega$ has sources that are regular as $\dot\nu \to 0$ and so $\vec\omega$ will be regular on critical surfaces.

Therefore, in this more general class of metrics, we cannot find the solutions to the BPS equations as explicitly as one can for GH base metrics. However, one can completely and explicitly characterize the singular parts of the solutions as one approaches critical surfaces where $\dot\nu \to 0$.

5.1.3 Regularity on the critical surfaces

Consider the behavior of the metric (2.5) as $\dot{\nu} \to 0$. The warp factors, Z_I diverge as $\dot{\nu}^{-1}$, μ diverges as $\dot{\nu}^{-2}$ and so the only potentially divergent part of the metric is:

$$-\big(Z_1 Z_2 Z_3\big)^{-\frac{2}{3}}\mu^2(d\tau{+}A)^2{+}\big(Z_1 Z_2 Z_3\big)^{\frac{1}{3}}\,\dot{\nu}^{-1}\,(d\tau{+}A)^2 \;=\; \big(Z_1 Z_2 Z_3\,\dot{\nu}^3\big)^{-\frac{2}{3}}\,\mathcal{Q}\,(d\tau{+}A)^2\,, \tag{5.30}$$

where

$$\mathcal{Q} \;\equiv\; Z_1\,Z_2\,Z_3\,\dot{\nu} \;-\; \mu^2\,\dot{\nu}^2\,. \tag{5.31}$$

Every other part of the metric has a finite limit as $\dot{\nu} \to 0$. Since $(Z_1 Z_2 Z_3\,\dot{\nu}^3)$ is finite as $\dot{\nu} \to 0$, we need to show that $\mathcal{Q}$ is finite. Using (5.20) and (5.27) one has

$$\begin{aligned}
\mathcal{Q} \;&=\; \dot{\nu}^{-2}\Big[\big(K^2K^3 \,+\, \dot{\nu}\,Z_1^{(0)}\big)\big(K^1K^3 \,+\, \dot{\nu}\,Z_2^{(0)}\big)\big(K^1K^2 \,+\, \dot{\nu}\,Z_3^{(0)}\big) \\
&\qquad\qquad -\; \big(K^1K^2K^3 \,+\, \tfrac{1}{2}\,\dot{\nu}\,Z_I^{(0)}K^I \,+\, \dot{\nu}^2\mu_0\big)^2\Big] \\
&\to\; \big(Z_1^{(0)}Z_2^{(0)}\,K^1K^2 + Z_1^{(0)}Z_3^{(0)}\,K^1K^3 + Z_2^{(0)}Z_3^{(0)}\,K^2K^3\big) \\
&\qquad\qquad -\; \tfrac{1}{4}\,\big(\dot{\nu}\,Z_I^{(0)}K^I\big)^2 \;-\; 2\,(K^1K^2K^3)\,\mu_0\,,
\end{aligned} \tag{5.32}$$

as $\dot{\nu} \to 0$. Thus the metric is finite on the critical surfaces. To avoid CTC's, $\mathcal{Q}$ must also be positive everywhere and, as with solutions on GH base metrics, this will depend upon the details of particular solutions.

The Maxwell fields are also regular on the critical surfaces. From (5.14) we see that the $\Theta^{(I)}$ are, in fact, singular on the critical surfaces, however from (2.13) and (5.16) we see that the complete Maxwell fields are given by:

$$A^{(I)} \;=\; -\,Z_I^{-1}\,\big(dt + \mu\,(d\tau + A) + \omega\big) \;+\; \dot{\nu}^{-1}\,K^I\,(d\tau + A) \;+\; \vec{\xi}^{(I)}\cdot d\vec{x}\,. \tag{5.33}$$

As we remarked earlier, ω is regular on the critical surfaces and the vectors, $\vec{\xi}^{(I)}$, defined by (5.17) are similarly regular. The only possible singular terms are thus

$$\begin{aligned} A^{(I)} &\sim \big(\dot\nu^{-1}\, K^I \,-\, Z_I^{-1}\mu\big)\,(d\tau + A) \\ &\sim \dot\nu^{-1}\,\big(K^I \,-\, \big(\tfrac{1}{2}\, C_{IJK}\, K^J\, K^K\big)^{-1}\, K^1\, K^2\, K^3\big)\,(d\tau + A) \;=\; 0\,. \end{aligned} \tag{5.34}$$

Thus the $A^{(I)}$ are regular on the critical surfaces.

5.1.4 Asymptotia

We would like the four dimensional base metric to be asymptotic to $\mathbb{R}^4$ and there are several ways to arrange this, depending upon how the $U(1)$ defined by τ-translations acts in $\mathbb{R}^4$. The simplest is to take $\nu \sim \log(z)$ and then:

$$ds_4^2 \;\sim\; z\, d\tau^2 \;+\; z^{-1} dz^2 \;+\; dx^2 \;+\; dy^2 \;=\; dr^2 \;+\; r^2\, d\phi^2 \;+\; dx^2 \;+\; dy^2\,, \tag{5.35}$$

where $z = \frac{1}{4} r^2$ and $\tau = 2\phi$. This metric is that of $\mathbb{R}^2 \times \mathbb{R}^2$ provided that τ has period 4π so that ϕ has period 2π. The $U(1)$ acts in one of the $\mathbb{R}^2$ planes and so this is the natural boundary condition appropriate to a system with this symmetry.

Another possible boundary condition at infinity is is to require:

$$\nu \;\sim\; \log\left(\frac{z^2}{\big(1 + \frac{1}{8}\,(x^2 + y^2)\big)^2}\right)\,, \tag{5.36}$$

and then

$$ds_4^2 \;\sim\; 2\, z^{-1} dz^2 \;+\; \tfrac{1}{2}\, z\,(d\tau + A_0)^2 \;+\; z\, \frac{dx^2 \;+\; dy^2}{\big(1 + \frac{1}{8}\,(x^2 + y^2)\big)^2}\,, \tag{5.37}$$

where

$$A_0 \;=\; \frac{1}{2}\,\frac{(x dy \;-\; y dx)}{\left(1+\frac{1}{8}\,(x^2+y^2)\right)}\,. \tag{5.38}$$

Now set $x = \tan\frac{\theta}{2}\cos\phi$ and $y = \tan\frac{\theta}{2}\sin\phi$ and one arrives at the metric:

$$\begin{aligned} ds_4^2 \;&\sim\; 2\,z^{-1}dz^2 \;+\; \tfrac{1}{2}\,z\,(d\tau + 2\,(1-\cos\theta)\,d\phi)^2 \;+\; 2\,z\left(d\theta^2 \;+\; \sin^2\theta d\phi^2\right) \\ &\sim\; dr^2 \;+\; \tfrac{1}{4}\,r^2\left(\sigma_1^2 \;+\; \sigma_2^2 \;+\; \sigma_3^2\right), \end{aligned} \tag{5.39}$$

where $z = \frac{1}{8}r^2$, the σ_i are the $SU(2)$ left invariant one-forms:

$$\begin{aligned} \sigma_1 \;&\equiv\; \cos\psi\, d\theta \;+\; \sin\psi\,\sin\theta\, d\phi\,, \\ \sigma_2 \;&\equiv\; \sin\psi\, d\theta \;-\; \cos\psi\,\sin\theta\, d\phi\,, \\ \sigma_3 \;&\equiv\; d\psi \;+\; \cos\theta\, d\phi\,, \end{aligned} \tag{5.40}$$

and $\tau = -2(\psi+\phi)$. Once again, the $U(1)$ generated by τ acts in one of the $\mathbb{R}^2$ planes in $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$.

With either of these asymptotic behaviors, the integral:

$$\int \sqrt{\gamma}\,\gamma^{ij}\,\partial_i\nu\,\partial_j\nu\,d^3x\,. \tag{5.41}$$

converges at infinity. The integrand is manifestly non-negative and if ν is regular everywhere then we may integrate by parts. This generates the Toda equation, (5.4), and so the integral vanishes. We therefore conclude that the only solution that is regular on $\mathbb{R}^3$ is a constant. Hence, ν must have singularities on $\mathbb{R}^3$.

While general Toda metrics may have complicated singularities, we are interested in metrics that, upon adding fluxes, give rise to smooth bubbling solutions. For Gibbons-Hawking base metrics, one has positive and negative sources (GH points) for the metric function, V, and pairs of these GH points then define the homology cycles. If one moves sufficiently close to one of these singular points of V in a GH metric, then the metric is, in fact, regular and caps off into a piece of $\mathbb{R}^4$ (perhaps divided out by a discrete group) with $SO(4)$ rotation symmetry. Guided by this, it is natural to consider singularities in ν that lead to local geometry that looks like $\mathbb{R}^4/\mathbb{Z}_q$ for some integer, q, and which locally has an $SO(4)$ invariance about the singular point.

Equivalently, near the singularities of ν, the Toda metric has a $U(1)\times U(1) \subset SO(4)$ symmetry and so can be mapped into a Gibbons-Hawking form. Thus the interesting class of metrics for bubbling should be those that can be put into Gibbons-Hawking form in the immediate vicinity of each singular point of ν. The non-trivial part of the Toda solution then relates to the transitions between these special regions. One can thus think of the Toda function as quilting together a collection of GH pieces.

It is elementary to see from the foregoing that, in the neighborhood of a singular point of charge ± 1, one must have:

$$\nu \;\sim\; \log|z-\alpha| \,, \qquad \pm(z-\alpha) > 0 \,. \tag{5.42}$$

With these choices the metric becomes precisely that of $\mathbb{R}^4$ and is positive or negative definite depending on the sign of the charge. By taking the $z \to 0$ limit in (5.36) one can also see that for a point of charge $+2$ one has $\nu \;\sim\; 2\,\log|z-\alpha|$. One can continue to higher charges via a series expansion in z but the geometry gets more complicated. This is because a charge q leads to a local geometry that is $\mathbb{R}^4/\mathbb{Z}_q$. In GH spaces this discrete identification was factored out of the $U(1)$ fiber, but in a general Toda geometry it will be factored out of the base and so the geometry near the singular points of ν will involve orbifold points in $\mathbb{R}^3$. It is therefore simpler to restrict to geometric charges of

± 1 and take the view that other geometric charges can be obtained via mergers of the more fundamental unit charges.

While we do not yet know how to progress beyond these simple observations, we believe that similar considerations will apply to bubbled geometries constructed from completely general ambi-polar, hyper-Kähler metrics. In the neighborhood of singular points of the Kähler potential they will locally be of GH form and so one might at least construct an approximate description as a quilt of GH patches with transition functions. Indeed, with such an approximating metric one might be able to establish existence theorems and perhaps even count moduli in the same manner that Yau established the existence of Calabi-Yau metrics [209].

5.2 Some examples

5.2.1 The Atiyah-Hitchin metric

The Atiyah-Hitchin metric has the form [5, 10]:

$$ds^2 \;=\; \tfrac{1}{4}\, a^2 b^2 c^2 \, d\eta^2 \;+\; \tfrac{1}{4}\, a^2 \, \sigma_1^2 \;+\; \tfrac{1}{4}\, b^2 \, \sigma_2^2 \;+\; \tfrac{1}{4}\, c^2 \, \sigma_3^2 \,, \tag{5.43}$$

where the σ_i are defined in (5.40) and satisfy $d\sigma_i = \frac{1}{2}\epsilon_{ijk}\sigma_j \wedge \sigma_k$. For (5.43) to be hyper-Kähler, the functions $a(\eta)$, $b(\eta)$ and $c(\eta)$ must satisfy:

$$\frac{\dot a}{a} \;=\; \tfrac{1}{2}\left((b-c)^2 \;-\; a^2\right) \tag{5.44}$$

$$\frac{\dot b}{b} \;=\; \tfrac{1}{2}\left((c-a)^2 \;-\; b^2\right) \tag{5.45}$$

$$\frac{\dot c}{c} \;=\; \tfrac{1}{2}\left((a-b)^2 \;-\; c^2\right) , \tag{5.46}$$

where the dot denotes $\frac{d}{d\eta}$.

5.2.2 The standard solution

This system of equations may be mapped onto a Darboux-Halphen system by introducing $w_1 = bc$, $w_2 = ac$ and $w_3 = ab$. One then finds

$$\frac{d}{d\eta}(w_1+w_2) = -2w_1w_2, \qquad \frac{d}{d\eta}(w_2+w_3) = -2w_2w_3, \qquad \frac{d}{d\eta}(w_1+w_3) = -2w_1w_3 . \tag{5.47}$$

To solve this system one first defines a new coordinate, θ, via

$$d\eta = \frac{d\theta}{u^2(\theta)} , \tag{5.48}$$

where u is defined to be the solution of

$$\frac{d^2u}{d\theta^2} + \frac{u}{4\sin^2\theta} = 0 . \tag{5.49}$$

One then finds that the solutions are given by [5]:

$$\begin{aligned} w_1 &= -uu' - \tfrac{1}{2}u^2 \csc\theta , \\ w_2 &= -uu' + \tfrac{1}{2}u^2 \cot\theta , \\ w_3 &= -uu' + \tfrac{1}{2}u^2 \csc\theta , \end{aligned} \tag{5.50}$$

where the prime denotes derivative with respect to θ.

One can find the explicit solution for u in terms of elliptic functions:

$$u(\theta) = \frac{c_1}{\pi}\sqrt{\sin\theta}\, K\big(\sin^2\tfrac{\theta}{2}\big) + \frac{c_2}{\pi}\sqrt{\sin\theta}\, K\big(\cos^2\tfrac{\theta}{2}\big) , \tag{5.51}$$

where c_1 and c_2 are constants and

$$K(x^2) \;\equiv\; \int_0^{\pi/2} (1 - x^2 \sin^2 \varphi)^{-1/2}\, d\varphi\,. \tag{5.52}$$

A first order system for three functions like (5.47) should involve three constants of integration. These are represented by c_1, c_2 and the trivial freedom to shift η by a constant. However, in order to get a regular, positive definite metric one must choose only one of the non-trivial solutions, which is then canonically normalized to:

$$u(\theta) = \frac{1}{\pi}\sqrt{\sin\theta}\, K\big(\sin^2 \tfrac{\theta}{2}\big)\,. \tag{5.53}$$

With this choice, the function $u(\theta)$ is non-vanishing on $(0, \pi)$ and so the change of variables (5.48) is well-defined. Moreover one has $w_1 < 0$, $w_2 < 0$ and $w_3 > 0$ on $(0, \pi)$ and so the metric coefficients:

$$a^2 \;=\; \frac{w_2\, w_3}{w_1}\,, \qquad b^2 \;=\; \frac{w_1\, w_3}{w_2}\,, \qquad c^2 \;=\; \frac{w_1\, w_2}{w_3}\,, \tag{5.54}$$

are all positive.

5.2.3 The geometry of the Atiyah-Hitchin metric

The standard Atiyah-Hitchin geometry is asymptotic to $\mathbb{R}^3 \times S^1$ and has a non-trivial two-cycle, or "bolt" in the center. To see this we first look at the structure at infinity, which corresponds to $\theta \to \pi$. In this limit one has:

$$\begin{aligned} u(\theta) &\sim -\tfrac{1}{\pi}\sqrt{2\cos\tfrac{\theta}{2}}\, \log(\cos\tfrac{\theta}{2})\,, \qquad & d\eta &\sim \frac{\pi^2\, d\theta}{2\cos\frac{\theta}{2}(\log(\cos\frac{\theta}{2}))^2}\,, \\ w_1(\theta) &\sim \tfrac{1}{\pi^2}\, \log(\cos\tfrac{\theta}{2})\,, & w_2(\theta) &\sim \tfrac{1}{\pi^2}\, \log(\cos\tfrac{\theta}{2})\,, \\ w_3(\theta) &\sim \tfrac{1}{\pi^2}\, (\log(\cos\tfrac{\theta}{2}))^2\,, \end{aligned} \tag{5.55}$$

which implies

$$a(\theta) \ \sim \ \tfrac{1}{\pi}\, \log(\cos\tfrac{\theta}{2})\,, \qquad b(\theta) \ \sim \ \tfrac{1}{\pi}\, \log(\cos\tfrac{\theta}{2})\,, \qquad c(\theta) \ \sim \ \tfrac{1}{\pi}\,. \tag{5.56}$$

Define $r = -\log(\cos\frac{\theta}{2})$ and then the asymptotic form of the metric becomes:

$$ds^2 \ \sim \ \tfrac{1}{4\,\pi^2}\,\big(dr^2 \ + \ r^2(\sigma_1^2 \ + \ \sigma_2^2) \ + \ \sigma_3^2\big)\,, \tag{5.57}$$

which indeed has the structure of a $U(1)$ fibration over $\mathbb{R}^3$.

At the other end of the interval, $\theta \to 0$, one finds:

$$\begin{aligned} u(\theta) \ &\sim \ \tfrac{1}{2}\,\theta^{\frac{1}{2}} \ - \ \tfrac{1}{96}\,\theta^{\frac{5}{2}} \ + \ \mathcal{O}(\theta^{\frac{7}{2}})\,, & d\eta \ &\sim \ 4\theta^{-1}d\theta \\ w_1(\theta) \ &\sim \ -\tfrac{1}{4} \ - \ \tfrac{1}{2048}\,\theta^4 \ + \ \mathcal{O}(\theta^6)\,, & w_2(\theta) \ &\sim \ -\tfrac{1}{32}\,\theta^2 \ - \ \tfrac{1}{3072}\,\theta^4 \ + \ \mathcal{O}(\theta^6)\,, \\ w_3(\theta) \ &\sim \ \tfrac{1}{32}\,\theta^2 \ + \ \tfrac{7}{3072}\,\theta^4 \ + \ \mathcal{O}(\theta^6)\,, & a(\theta) \ &\sim \ \tfrac{1}{16}\,\theta^2 \ + \ \tfrac{1}{384}\,\theta^4 \ + \ \mathcal{O}(\theta^6), \\ b(\theta) \ &\sim \ \tfrac{1}{2} \ + \ \tfrac{1}{64}\,\theta^2 \ + \ \mathcal{O}(\theta^4)\,, & c(\theta) \ &\sim \ \tfrac{1}{2} \ - \ \tfrac{1}{64}\,\theta^2 \ + \ \mathcal{O}(\theta^4) \end{aligned} \tag{5.58}$$

Define $\rho = \frac{1}{64}\theta^2$ and the metric near $\theta = 0$ has the form:

$$ds^2 \ \sim \ d\rho^2 \ + \ 4\,\rho^2\,\sigma_1^2 \ + \ \tfrac{1}{16}\,\big(\sigma_2^2 \ + \ \sigma_3^2\big) \tag{5.59}$$

Thus we see the "bolt" at the origin. Note that the scale of the metric has been fixed and the radius of the bolt has been set to $\frac{1}{4}$. The fact that the coefficient of σ_1 vanishes as $\sim 4\rho^2$ also has important implications for the global geometry. There is a very nice discussion of this in the appendices of [70].

For future reference, we will chose the constant of integration (5.48) so that $\eta \to 0$ at infinity ($\theta = \pi$) and take:

$$\eta \ \equiv \ -\int_\theta^\pi \frac{d\theta}{u^2}\,. \tag{5.60}$$

With this choice, η has the following asymptotic behavior:

$$\eta \;\sim\; 4\,\log(\theta) \;\;\text{as}\;\; \theta \to 0\,; \qquad\qquad \eta \;\sim\; -\,\frac{\pi^2}{r} \;\;\text{as}\;\; \theta \to \pi\,, \tag{5.61}$$

where $r = -\log(\cos\frac{\theta}{2})$.

Since there is a non-trivial two-cycle, there must be a non-trivial, dual element of cohomology. That is, there must be precisely one square-integrable, harmonic two-form. In particular, this means the two-form must be a singlet under $SO(3)$. To determine this two form, it is convenient to introduce the vierbeins:

$$e^1 \;=\; -\,\tfrac{1}{2}\,abc\,d\eta\,, \qquad e^2 \;=\; \tfrac{1}{2}\,a\,\sigma_1\,, \qquad e^3 \;=\; \tfrac{1}{2}\,b\,\sigma_2\,, \qquad e^4 \;=\; \tfrac{1}{2}\,c\,\sigma_3\,, \tag{5.62}$$

and define some manifestly $SO(3)$-invariant, self-dual two-forms via:

$$\begin{aligned}
\Omega_1 \;&\equiv\; h_1\left(a^2\,d\eta\wedge\sigma_1 \;-\; \sigma_2\wedge\sigma_3\right),\\
\Omega_2 \;&\equiv\; h_2\left(b^2\,d\eta\wedge\sigma_2 \;+\; \sigma_1\wedge\sigma_3\right),\\
\Omega_3 \;&\equiv\; h_3\left(c^2\,d\eta\wedge\sigma_3 \;-\; \sigma_1\wedge\sigma_2\right),
\end{aligned} \tag{5.63}$$

for some functions, $h_j(\eta)$. The condition that Ω_j be closed, and hence harmonic is:

$$\frac{d}{d\eta}\,\log(h_j) \;=\; -\,a_i^2 \quad\Leftrightarrow\quad \frac{d}{d\theta}\,\log(h_j) \;=\; -\,\frac{a_i^2}{u^2}\,, \tag{5.64}$$

where

$$(a_1, a_2, a_3) \equiv (a, b, c)\,. \tag{5.65}$$

These equations imply that there are obvious local potentials, B_j, for Ω_j:

$$\Omega_j \;=\; d\,B_j\,, \quad \text{where} \quad B_j \;\equiv\; -\,h_j\,\sigma_j\,. \tag{5.66}$$

Remarkably enough, the equations for the h_j are integrable in terms of $u(\theta)$ and we find:

$$h_1 \;=\; \tfrac{1}{4}\,\alpha_1\,\frac{u^2}{w_1\,\sin(\frac{\theta}{2})}\,, \qquad h_2 \;=\; \tfrac{1}{4}\,\alpha_2\,\frac{u^2}{w_2}\,, \qquad h_3 \;=\; \tfrac{1}{4}\,\alpha_3\,\frac{u^2}{w_3\,\cos(\frac{\theta}{2})}\,, \tag{5.67}$$

where the α_j are constants of integration. One should note that these solutions follow from (5.49) and (5.50) and do not depend upon the specific choice in (5.53). However here we focus on the solutions that arise from (5.53). To determine which, if any, of the h_j gives the desired harmonic form, we look at the regularity of these two-forms and examine their behavior as $\theta \to 0$ and $\theta \to \pi$.

As $\theta \to 0$ we have:

$$h_1 \;\sim\; -\tfrac{1}{2}\,\alpha_1 \,+\, \mathcal{O}(\theta^4)\,, \qquad h_2 \;\sim\; -2\,\alpha_2\,\theta^{-1} \,+\, \mathcal{O}(\theta)\,, \qquad h_3 \;\sim\; 2\,\alpha_3\,\theta^{-1} \,+\, \mathcal{O}(\theta)\,, \tag{5.68}$$

and as $\theta \to \pi$ we have:

$$h_1 \;\sim\; \tfrac{1}{4}\,\alpha_1\,r\,e^{-r} + \mathcal{O}(e^{-r})\,, \qquad h_2 \;\sim\; \tfrac{1}{4}\,\alpha_2\,r\,e^{-r} + \mathcal{O}(e^{-r})\,, \qquad h_3 \;\sim\; \tfrac{1}{2}\,\alpha_3 + O(r^{-1})\,, \tag{5.69}$$

where $r = -\log(\cos\frac{\theta}{2})$. It follows that h_1 is regular at $\theta = 0$ and falls off very fast at infinity. The corresponding two-form, Ω_1, is globally regular and square-integrable and is thus the harmonic form we seek. Indeed, at $\theta = 0$ one has $\Omega_1 = \frac{1}{2}\alpha_1\sigma_2 \wedge \sigma_3$ and $\sigma_2 \wedge \sigma_3$ is the volume form on the bolt of unit radius, which means the period integral is given by:

$$\int_{Bolt} \Omega_1 \;=\; \tfrac{1}{2}\,\alpha_1 \int_{Bolt} \sigma_2 \wedge \sigma_3 \;=\; 2\,\pi\,\alpha_1\,. \tag{5.70}$$

5.2.4 Ambi-polar Atiyah-Hitchin metrics

The most general $SO(3)$ invariant metric governed by (5.47) requires one to use the most general function, $u(\theta)$, in (5.51). As we will see, this possibility is usually ignored because it leads to ambi-polar metrics, and we will show, in the next section, how such solutions can be used to make new Lorentzian BPS solutions in five dimensions.

To understand how the inclusion of the extra function changes the Atiyah-Hitchin metric, define $\tilde{u}(\theta) \equiv u(\pi - \theta)$ and let $\tilde{w}_j(\theta)$ be defined by (5.50) with u replaced by $\tilde{u}$. It is evident that $\tilde{u}(\theta)$ also solves (5.49), indeed, it simply interchanges c_1 and c_2 in (5.51). Therefore the functions $\tilde{w}_j$ also solve the system (5.47). On the other hand, from (5.50) one can easily see that:

$$w_1(\pi-\theta) \;=\; -\tilde{w}_3(\theta)\,, \qquad w_2(\pi-\theta) \;=\; -\tilde{w}_2(\theta)\,, \qquad w_3(\pi-\theta) \;=\; -\tilde{w}_1(\theta)\,. \tag{5.71}$$

Thus allowing a non-zero value for c_1 and c_2 means that asymptotic behavior of the w_j at $\theta = 0$ is related to the asymptotic behavior at $\theta = \pi$. In particular, because we now have

$$\begin{aligned} u(\theta) &\sim -\tfrac{c_1}{\pi}\sqrt{2\cos\tfrac{\theta}{2}}\,\log(\cos\tfrac{\theta}{2})\,, \qquad \theta \to \pi\,, \\ u(\theta) &\sim -\tfrac{c_2}{\pi}\sqrt{2\sin\tfrac{\theta}{2}}\,\log(\sin\tfrac{\theta}{2})\,, \qquad \theta \to 0\,, \end{aligned} \tag{5.72}$$

we therefore have, as $\theta \to \pi$:

$$w_1(\theta) \;\sim\; \tfrac{c_1^2}{\pi^2}\,\log(\cos\tfrac{\theta}{2})\,, \quad w_2(\theta) \;\sim\; \tfrac{c_1^2}{\pi^2}\,\log(\cos\tfrac{\theta}{2})\,, \quad w_3(\theta) \;\sim\; \tfrac{c_1^2}{\pi^2}\,(\log(\cos\tfrac{\theta}{2}))^2\,, \tag{5.73}$$

and, as $\theta \to 0$:

$$w_1(\theta) \;\sim\; -\tfrac{c_2^2}{\pi^2}\,(\log(\sin\tfrac{\theta}{2}))^2\,, \quad w_2(\theta) \;\sim\; -\tfrac{c_2^2}{\pi^2}\,\log(\sin\tfrac{\theta}{2})\,, \quad w_3(\theta) \;\sim\; -\tfrac{c_2^2}{\pi^2}\,\log(\sin\tfrac{\theta}{2})\,. \tag{5.74}$$

This means that the metric now has two regions that are asymptotic to $\mathbb{R}^3 \times S^1$ with $a \sim r$ and $b \sim r$ as $\theta \to \pi$ and with $c \sim r$ and $b \sim r$ as $\theta \to 0$. It therefore, naively looks like a "wormhole" geometry. The asymptotics also imply that if the metric is positive definite in one asymptotic region then it is negative definite in the other: a^2, b^2 and c^2 all change sign as one goes from $\theta = 0$ to $\theta = \pi$. One also sees from the asymptotics of w_2 that w_2 must have at least one zero in $(0, \pi)$ and so the metric is singular at such a point. It is for all these reasons that the generalization of the Atiyah-Hitchin metric is usually ignored. However, this metric is ambi-polar and, as we will show, all the pathologies itemized here are not present in the five-dimensional solution that can be constructed from this metric.

For simplicity, we will restrict our attention to ambi-polar metrics based upon:

$$u(\theta) = \frac{1}{\pi} \sqrt{\sin\theta} \, \Big(K\big(\sin^2 \tfrac{\theta}{2}\big) \ + \ K\big(\cos^2 \tfrac{\theta}{2}\big)\Big) \,, \tag{5.75}$$

then one has

$$\begin{aligned} w_1(\pi - \theta) &= -w_3(\theta) \,, \qquad w_2(\pi - \theta) = - \, w_2(\theta) \,, \qquad w_3(\pi - \theta) = - \, w_1(\theta) \,, \\ a^2(\pi - \theta) &= -c^2(\theta) \,, \qquad b^2(\pi - \theta) = - \, b^2(\theta) \,, \qquad c^2(\pi - \theta) = - \, a^2(\theta) \end{aligned} \tag{5.76}$$

With this choice one has $u > 0$, $w_1 < 0$ and $w_3 > 0$ for $\theta \in [0, \pi]$ and w_2 has a simple zero at $\theta = \pi/2$. See Fig. 5.1. This means that the metric coefficients, a_j^2, simultaneously change sign at $\theta = \pi/2$ and this is the only point at which this happens. Moreover, a^2 and c^2 have simple zeroes while b^2 has a simple pole at $\theta = \pi/2$. This behavior of the metric coefficients precisely mimics that of the ambi-polar GH metrics.

We note that the forms given by (5.63) and (5.67) are still "harmonic" in that they are self-dual and closed. Moreover, Ω_1 and Ω_3 are non-singular in the wormhole geometry, except that Ω_1 remains finite as $\theta \to 0$ while Ω_3 remains finite as $\theta \to \pi$. This means that neither is square-integrable on the complete geometry. On the other hand, Ω_2 falls off exponentially at both $\theta = 0$ and $\theta = \pi$ but is singular at $\theta = \pi/2$, where the metric

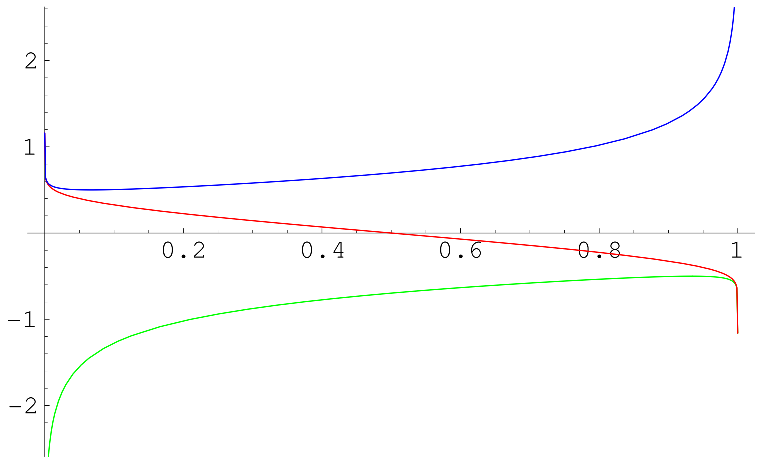

Figure 5.1: The three functions, w_j, as a function of $x = \sin^2 \frac{\theta}{2}$ when u is given by (5.75). One has $w_1 < 0$, $w_3 > 0$ and w_2 has a simple zero at $\theta = \pi/2$. All three functions diverge at both ends of the interval.

changes sign. Once again this last flux has a behavior precisely analogous to the two-form fields that were essential building blocks for the regular five-dimensional solutions that can be built from ambi-polar GH metrics.

Finally, we should comment that more general choices of $u(\theta)$, such as taking $c_1 = -c_2 = 1$ in (5.51), can result in solutions with zeroes for w_1, w_2 and w_3. We have not studied these in detail.

5.2.5 The BPS solutions

Solving the BPS equations

Since there is only one independent harmonic form in the Atiyah-Hitchin metric, this means that the two-forms, $\Theta^{(I)}$, in (2.14) must all be proportional to one another for $I = 1, 2, 3$. For simplicity, we will, in fact, take them all to be equal. We will also take the three warp factor functions to be equal, $Z_I = Z$, $I = 1, 2, 3$. Ignoring, for the present, issues of regularity, the $SO(3)$ invariant solutions of (2.14) are given by the Ω_i of (5.63) and so we will take

$$\Theta^{(I)} \;=\; \Theta \;=\; \Omega_1 \;+\; \Omega_2 \;+\; \Omega_3 \,. \tag{5.77}$$

The functions, h_j, in (5.67) contain integration constants, α_j, that make this an arbitrary linear combination. Note: One should not confuse the index, $I = 1, 2, 3$ on $\Theta^{(I)}$ with the index, $i = 1, 2, 3$ on Ω_i. The former indexes the $U(1)$ gauge groups of three-charge system while the latter labels the three distinct type of two-form in (5.63) that satisfy (2.14).

With this choice, the second BPS equation becomes:

$$\frac{d^2 Z}{d\eta^2} \;=\; 8 \sum_{j=1}^{3} h_j^2 \, a_j^2 \,. \tag{5.78}$$

Given the form of Θ, there is a unique Ansatz for the angular momentum vector, k:

$$k \;=\; \sum_{j=1}^{3} \mu_j \, \sigma_j \,, \tag{5.79}$$

which means that the third BPS equation yields three equations:

$$\frac{d\mu_j}{d\eta} \;-\; a_j^2 \, \mu_j \;=\; 3 \, h_j \, a_j^2 \, Z \,, \qquad j = 1, 2, 3 \,. \tag{5.80}$$

The factor of three comes from the sum over the $U(1)$ label, I, in (2.16) and the choices: $\Theta^{(I)} = \Theta$, $Z_I = Z$.

These equations can, once again, be integrated explicitly in terms of the the elliptic function, u. First, from (5.64) we have:

$$\frac{dZ}{d\eta} \;=\; \gamma_0 \;-\; 4\sum_{j=1}^{3} h_j^2\,, \tag{5.81}$$

for some constant, γ_0. Using (5.49) and (5.50) one can easily show that

$$\frac{d}{d\eta}\,\frac{\alpha_j^2}{w_j} \;=\; u^2\,\frac{d}{d\theta}\,\frac{\alpha_j^2}{w_j} \;=\; \alpha_j^2 \;+\; 4\,(-1)^j\,h_j^2\,, \tag{5.82}$$

and hence:

$$Z \;=\; \delta \;+\; \gamma\,\eta \;-\; \sum_{j=1}^{3}(-1)^j\,\frac{\alpha_j^2}{w_j}\,, \tag{5.83}$$

where $\gamma = \gamma_0 + \sum_{j=1}^{3}(-1)^j\alpha_j^2$.

The last BPS equation, (5.80), can be integrated to yield:

$$\mu_j \;=\; \frac{3}{h_j}\int h_j^2\,a_j^2\,Z\,d\eta \;=\; \frac{3}{h_j}\int\Big(-\frac{1}{2}\,\frac{d}{d\eta}h_j^2\Big)\,Z\,d\eta\,, \qquad j=1,2,3\,. \tag{5.84}$$

It is easy to integrate this explicitly. First, by integrating by parts one can show:

$$\frac{3}{h_j}\int h_j^2\,a_j^2\,(\delta \;+\; \gamma\,\eta)\,d\eta \;=\; -\tfrac{3}{2}\,\delta\,h_j \;-\; \tfrac{3}{2}\,\gamma\,\Big[\,h_j\eta \;-\; (-1)^j\,\frac{\alpha_j^2}{4\,h_j}\,\Big(\frac{1}{w_j}-\eta\Big)\Big] \;+\; \frac{\beta_j}{h_j}\,, \tag{5.85}$$

where the β_j are constants of integration. The other parts of the integrals for μ_j can be obtained from:

$$\begin{aligned}\frac{3}{h_j}\int\frac{h_j^2\,a_j^2}{w_j}\,d\eta \;&=\; (-1)^j\,\frac{\alpha_j^2}{8\,h_j}\,\Big[\,\frac{2\,w_i\,w_k}{w_j^3} \;-\; \frac{w_i+w_k}{w_j^2}\,\Big]\,,\\ \frac{3}{h_j}\int\frac{h_j^2\,a_j^2}{w_i}\,d\eta \;&=\; (-1)^{j+1}\,\frac{3\,\alpha_j^2}{8\,h_j}\,\frac{(w_j-w_k)}{w_j^2}\,,\end{aligned} \tag{5.86}$$

where $i, j, k \in \{1, 2, 3\}$ are all distinct.

Thus, rather surprisingly, we can obtain the complete solution analytically in terms of elliptic functions.

The bubbled solution on the standard Atiyah-Hitchin base

The physical intuition underlying BPS solutions is that all charges have to be of the same sign so that the electromagnetic repulsion balances the gravitational attraction. Bubbled geometries generically have geometric charges of all signs and then the attractive forces are balanced by threading cycles with fluxes that then resist the collapse of the bubbles. The result is then a stable configuration where the sizes of some of the bubbles are fixed in terms of the fluxes that thread them. Such relationships are typically embodied in a system of "Bubble Equations" [26, 41]. If one insists that a solution is a BPS configuration but one does not have the forces properly balanced the result is the appearance of CTC's. Thus, when investigating BPS geometries one typically encounters the constraints of bubble equations through the process of eliminating CTC's.

The standard Atiyah-Hitchin base metric is, in its own right, a well-behaved BPS solution with no additional fluxes. Indeed, the addition of a flux through the non-trivial two-cycle should drive the configuration out of equilibrium and expand the bubble. We should therefore find irremovable CTC's if we attempt to include a non-trivial flux. We now show that this is precisely what happens.

As we remarked earlier, the only non-trivial, harmonic flux on the standard Atiyah-Hitchin base is given by Ω_1 and so we set $\alpha_2 = \alpha_3 = 0$ in the results of the previous sub-section[4]. We then find:

$$Z \;=\; \delta \,+\, \gamma\,\eta \,+\, \frac{\alpha_1^2}{w_1} \tag{5.87}$$

[4] If one is interested in solutions that are asymptotically $AdS \times S^2$, one could also investigate solutions that contain the Ω_3 component of the 2-form field strength Θ, which corresponds to constant flux on the S^2. Nevertheless, in our investigations this did not give any sensible solutions.

and $k = \mu\sigma_1$, where

$$\mu = -\tfrac{3}{2}\,\delta\,h_1 - \tfrac{3}{2}\,\gamma\left[h_1\eta + \frac{\alpha_1^2}{4\,h_1}\Big(\frac{1}{w_1} - \eta\Big)\right] - \frac{\alpha_1^4}{8\,h_1}\left[\frac{2\,w_2\,w_3}{w_1^3} - \frac{w_2 + w_3}{w_1^2}\right] + \frac{\beta_1}{h_1}\,. \tag{5.88}$$

It is interesting to note that the part of Z corresponding to the flux sources in (5.87) (*i.e.* the w_1^{-1} term) is always negative, and therefore at infinity this warp factor looks like it is coming from an object of negative mass and charge. This is however not surprising, considering that the Atiyah-Hitchin space also looks asymptotically as a negative-mass Taub-NUT space.

The value of β_1 is fixed by requiring that μ does not diverge, and indeed falls off at infinity. We find that if we set:

$$\beta_1 = \frac{\pi^2\,\alpha_1^4}{8}\,, \tag{5.89}$$

then this removes all the terms that diverge at infinity and leaves only terms that fall off. Indeed, there are two types of such terms: Those proportional to γ, which fall off as $\frac{1}{r}$, and the remainder that fall off as re^{-r}.

Near $\theta = 0$ the function η is logarithmically divergent and so Z is logarithmically divergent unless $\gamma = 0$. Physically, a non-zero value of γ corresponds to a uniform distribution of M2 branes smeared over the bolt at $\theta = 0$, with negative values of γ corresponding to positive charge densities. If $\gamma = 0$ then $Z = \delta - 4\alpha_1^2$ at $\theta = 0$.

For constant time slices, the five-dimensional metric (2.5) becomes

$$ds^2 = \left(\tfrac{1}{4}\,a^2\,Z - \mu^2\,Z^{-2}\right)\sigma_1^2 + \tfrac{1}{4}\,Z\,a^2\,b^2\,c^2\,d\eta^2 + \tfrac{1}{4}\,Z\,b^2\sigma_2^2 + \tfrac{1}{4}\,Z\,c^2\sigma_3^2\,, \tag{5.90}$$

and so to avoid CTC's, one must have $Z \geq 0$ and the quantity:

$$\mathcal{Q} \;=\; \tfrac{1}{4}\, a^2\, Z^3 \;-\; \mu^2 \tag{5.91}$$

must be non-negative. The function $a(\theta) \sim \frac{1}{16}\theta^2$ as $\theta \to 0$ and Z diverges, at worst, logarithmically. Thus we must have $\mu \to 0$ as $\theta \to 0$ in order to avoid CTC's on the bolt[5]. This means that we must take

$$\gamma \;=\; \tfrac{1}{4}\,\delta \;-\; \tfrac{2}{3}\,\alpha_1^{-2}\,\beta_1 \;=\; \tfrac{1}{4}\,\delta \;-\; \tfrac{1}{12}\,\pi^2\,\alpha_1^2\,. \tag{5.92}$$

For pure-flux solutions, which have no singular sources, one must take $\gamma = 0$ and the CTC condition (5.92) reduces to $\delta = \frac{1}{3}\,\pi^2\,\alpha_1^2$. Then one finds

$$\tfrac{1}{4}\, a^2\, Z \;-\; \mu^2\, Z^{-2} \;\sim\; -\,\tfrac{1}{3072}\,(12 - \pi^2)\,\alpha_1^2\,\theta^4 \;<\; 0\,, \tag{5.93}$$

and so one necessarily has CTC's in the immediate neighborhood of the bolt. This is a signal that there is no physical BPS solution based upon the standard Atiyah-Hitchin metric with pure flux: The flux will blow up the cycle and there is no gravitational attraction holding the bubble back.

One might hope that one could stabilize the solution with a distribution of M2 branes on the bolt. While this might be possible in general, it does not seem to be possible with a uniform, $SO(3)$ invariant distribution. For this, one must have $\gamma < 0$ for Z to remain positive near $\theta = 0$ and then (5.92) means that $\alpha_1^2 > \frac{3}{\pi^2}\delta$. In addition, we must have $\delta \geq 0$ for $Z > 0$ at infinity. From (5.81) one has

$$\frac{dZ}{d\eta} \;=\; \gamma \;+\; \alpha_1^2 \;-\; 4\,h_1^2 \;=\; \tfrac{1}{4}\,\delta \;+\; (1 - \tfrac{1}{12}\,\pi^2)\,\alpha_1^2 \;-\; 4\,h_1^2\,, \tag{5.94}$$

[5]This is how the bubble equations arise on GH spaces.

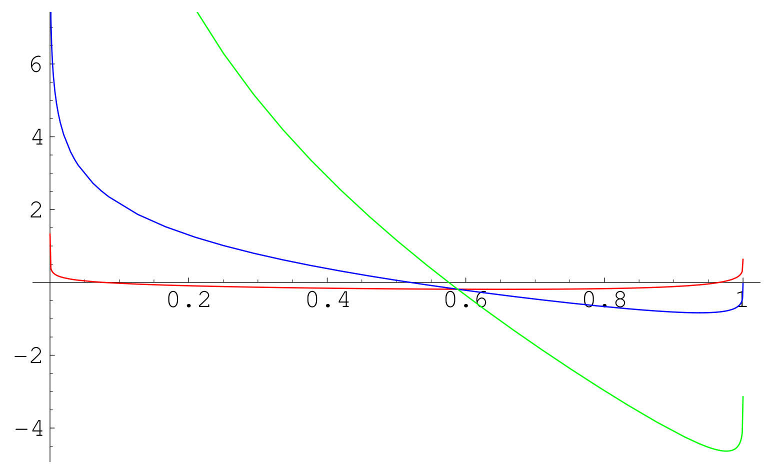

Figure 5.2: Plots of Z as a function of $x = \sin^2 \frac{\theta}{2}$. We have taken $\delta = 1$, fixed γ in terms of α_1 using (5.92) and then we have chosen three values of α_1 that ensure that γ is negative: $\alpha_1 = 0.6$, $\alpha_1 = 1.0$ and $\alpha_1 = 2.0$. The steeper graphs at $x = 0.5$ correspond to larger values of α_1. Note that $Z \to 1$ as $x \to 1$, but that Z is generically negative for $x > 0.6$.

and since $h_1 = -\frac{1}{2}\alpha_1$ at $\theta = 0$ and $h_1 \to 0$ at infinity ($\theta = \pi$) we see that $\frac{dZ}{d\eta}$ is negative at $\theta = 0$ and positive at $\theta = \pi$. Therefore, Z has a minimum for $\theta \in (0, \pi)$. While we have not done an exhaustive analysis, we generally find that Z is negative at this minimum value. Some examples are shown in Fig. 5.2. Obviously, the complete five-dimensional metric is singular when $Z < 0$.

Adding the singular M2 brane sources does render $\mathcal{Q}$ positive in a region around the bolt but, as one can see from (5.91), $\mathcal{Q}$ also goes negative shortly before Z goes negative. Thus adding M2 branes sources moves CTC's away from the bolt but at the cost of more extensive singular behavior elsewhere in the solution.

Bubbling the ambi-polar Atiyah-Hitchin base

We now consider adding flux to one of the ambi-polar Atiyah-Hitchin metrics discussed in Section 5.2.4. That is, we will start with the ambi-polar "wormhole" geometry that arises from the choice (5.75), which therefore has the reflection symmetry given by

(5.76). The solutions to the BPS equations have exactly the same functional form as those given in Section 5.2.5 for the standard Atiyah-Hitchin background. However, the underlying functions now have very different asymptotic behavior and this affects all of the choices based upon regularity and square integrability.

Let $r = -\log(\pi - \theta)$ and $\hat{r} = -\log(\theta)$, then as $\theta \to 0$ one has

$$\begin{aligned} u(\theta) &\sim \tfrac{1}{\pi}\,\hat{r}\,e^{-\hat{r}/2}\,, \qquad \eta \sim -\eta_0 + \frac{\pi^2}{\hat{r}}\,, \\ w_1(\theta) &\sim -\tfrac{1}{\pi^2}\,\hat{r}^2\,, \quad w_2(\theta) \sim \tfrac{1}{\pi^2}\,\hat{r}\,, \quad w_3(\theta) \sim \tfrac{1}{\pi^2}\,\hat{r}\,, \end{aligned} \tag{5.95}$$

which implies

$$a^2(\theta) \sim -\tfrac{1}{\pi^2}\,, \qquad b^2(\theta) \sim -\tfrac{1}{\pi^2}\,\hat{r}^2\,, \qquad c^2(\theta) \sim -\tfrac{1}{\pi^2}\,\hat{r}^2\,. \tag{5.96}$$

The constant, η_0, is defined by[6]:

$$\eta_0 \equiv \int_0^\pi \frac{1}{u(\theta)^2} = 2\pi\,. \tag{5.97}$$

As $\theta \to \pi$ one has:

$$\begin{aligned} u(\theta) &\sim \tfrac{1}{\pi}\,r\,e^{-r/2}\,, \qquad \eta \sim -\frac{\pi^2}{r}\,, \\ w_1(\theta) &\sim -\tfrac{1}{\pi^2}\,r\,, \quad w_2(\theta) \sim -\tfrac{1}{\pi^2}\,r\,, \quad w_3(\theta) \sim \tfrac{1}{\pi^2}\,r^2\,, \end{aligned} \tag{5.98}$$

which implies

$$a^2(\theta) \sim \tfrac{1}{\pi^2}\,r^2\,, \qquad b^2(\theta) \sim \tfrac{1}{\pi^2}\,r^2\,, \qquad c^2(\theta) \sim \tfrac{1}{\pi}\,. \tag{5.99}$$

[6]While we haven't proven that $\eta_0 = 2\pi$ analytically, we have checked numerically to over 100 significant figures.

The metric in each of these asymptotic regions becomes:

$$
\begin{aligned}
ds^2 &\sim -\tfrac{1}{4\,\pi^2}\left(d\hat{r}^2 \,+\, \hat{r}^2(\sigma_2^2 \,+\, \sigma_3^2) \,+\, \sigma_1^2\right), \qquad \theta \to 0\,, \\
ds^2 &\sim \tfrac{1}{4\,\pi^2}\left(dr^2 \,+\, r^2(\sigma_1^2 \,+\, \sigma_2^2) \,+\, \sigma_3^2\right), \qquad \theta \to \pi\,.
\end{aligned}
\tag{5.100}
$$

We thus have an ambi-polar metric with two regions that are asymptotic to different $U(1)$ fibrations over different $\mathbb{R}^3$ bases. The metric changes sign precisely at $\theta = \frac{\pi}{2}$ at which point the metric function $b^2(\theta)$ has a simple pole, while $a^2(\theta)$ and $c^2(\theta)$ have simple zeroes.

This time the appropriate "harmonic" form is Ω_2 because we have :

$$
\begin{aligned}
h_1(\theta) &\sim -\tfrac{1}{2}\,\alpha_1\,, \quad h_2(\theta) \sim \tfrac{1}{4}\,\alpha_2\,\hat{r}\,e^{-\hat{r}}\,, \quad h_3(\theta) \sim \tfrac{1}{4}\,\alpha_3\,\hat{r}\,e^{-\hat{r}}\,, \quad \theta \to 0\,; \\
h_1(\theta) &\sim -\tfrac{1}{4}\,\alpha_1\,r\,e^{-r}\,, \; h_2(\theta) \sim -\tfrac{1}{4}\,\alpha_2\,r\,e^{-r}\,, \; h_3(\theta) \sim \tfrac{1}{2}\,\alpha_3\,, \quad \theta \to \pi;
\end{aligned}
\tag{5.101}
$$

and so Ω_2 is the only solution that falls off in both asymptotic regions. It is, however, not really harmonic in that it is singular precisely on the critical surface where $w_2 = 0$. This is, however, the standard behavior for the flux that goes into making the complete, five-dimensional solution and, as was noted in (5.34), the complete flux, $C^{(3)}$, is smooth on the critical surface.

One now has

$$
Z \;=\; \delta \,+\, \gamma\,\eta \,-\, \frac{\alpha_2^2}{w_2} \tag{5.102}
$$

and $k = \mu\sigma_2$, where

$$
\begin{aligned}
\mu \;=\; & -\tfrac{3}{2}\,\delta\,h_2 \,-\, \tfrac{3}{2}\,\gamma\left[h_2\eta \,-\, \frac{\alpha_2^2}{4\,h_2}\left(\frac{1}{w_2} - \eta\right)\right] \\
& -\, \frac{\alpha_2^4}{8\,h_2}\left[\frac{2\,w_1\,w_3}{w_2^3} \,-\, \frac{w_1 + w_3}{w_2^2}\right] \,+\, \frac{\beta_2}{h_2}\,.
\end{aligned}
\tag{5.103}
$$

Recall that the vector potential for Ω_2 is given in (5.66) and so the potential for the complete Maxwell field is:

$$A \;=\; Z^{-1}\left(dt + \mu\,\sigma_2\right) \;-\; h_2\,\sigma_2\,. \tag{5.104}$$

and so the only potentially singular term is:

$$Z^{-1}\,\mu \;-\; h_2 \;\sim\; -\,\frac{\alpha_2}{4\,u^2\,w_2}\,(4\,w_1\,w_3 \;+\; u^4)\,, \tag{5.105}$$

as $w_2 \to 0$. However, from (5.50) one has

$$w_1\,w_3 \;+\; \tfrac{1}{4}\,u^4 \;=\; w_2\,(w_1 + w_3) - w_2^2\,, \tag{5.106}$$

and so the complete Maxwell field is regular.

The spatial sections of the complete five-dimensional metric are:

$$ds^2 \;=\; \left(\tfrac{1}{4}\,b^2\,Z \;-\; \mu^2\,Z^{-2}\right)\sigma_2^2 \;+\; \tfrac{1}{4}\,Z\,a^2\,b^2\,c^2\,d\eta^2 \;+\; \tfrac{1}{4}\,Z\,a^2\sigma_1^2 \;+\; \tfrac{1}{4}\,Z\,c^2\sigma_3^2\,. \tag{5.107}$$

First note that:

$$\begin{aligned} Z\,a^2 \;&=\; \frac{w_3}{w_1}\,((\delta \;+\; \gamma\,\eta)\,w_2 \;-\; \alpha_2^2)\,, \qquad Z\,c^2 \;=\; \frac{w_1}{w_3}\,((\delta \;+\; \gamma\,\eta)\,w_2 \;-\; \alpha_2^2)\,, \\ Z\,a^2\,b^2\,c^2 \;&=\; w_1\,w_3((\delta \;+\; \gamma\,\eta)\,w_2 \;-\; \alpha_2^2)\,. \end{aligned} \tag{5.108}$$

Since one has $w_1 < 0$ and $w_3 > 0$ everywhere (see Fig.5.1) it follows that these three metric coefficients are regular and positive near $w_2 = 0$.

More generally, observe that $\delta + \gamma\eta \to \delta$ and $w_2 \to -\infty$ as $\theta \to \pi$ and $\delta + \gamma\eta \to \delta - 2\pi\gamma$ and $w_2 \to +\infty$ as $\theta \to 0$. This means that for the metric coefficients in (5.108) to remain positive at infinity one must have:

$$\gamma \;\geq\; \frac{\delta}{2\pi} \;\geq\; 0\,. \tag{5.109}$$

Indeed observe that the function, $\eta + \frac{1}{2}\eta_0$, is odd under $\theta \to \pi - \theta$ and so, for $\gamma > 0$, the function

$$\gamma(\eta + \tfrac{1}{2}\eta_0)\, w_2 \;=\; \gamma(\eta + \pi)\, w_2 \tag{5.110}$$

is globally negative with a double zero at $\theta = \frac{\pi}{2}$. Thus the metric coefficients (5.108) are globally positive when δ is the middle of the range specified by (5.109).

Now consider the remaining coefficient, $Z^{-2}\mathcal{Q}$, where

$$\mathcal{Q} \;\equiv\; \tfrac{1}{4}\, b^2\, Z^3 \;-\; \mu^2\,. \tag{5.111}$$

Near $w_2 = 0$ one has $Z^{-2} \sim \alpha_2^{-4} w_2^2$ and

$$\mathcal{Q} \;\sim\; \frac{\alpha_2^6\, w_1\, w_3}{u^4\, w_2^4}\,\big(w_2\,(w_1 + w_3) \;-\; (w_1\, w_3 + \tfrac{1}{4}\, u^4)\big) \;+\; \mathcal{O}(w_2^{-2})\,. \tag{5.112}$$

However, it follows from (5.106) that, in fact, $\mathcal{Q} \sim \mathcal{O}(w_2^{-2})$ and so the metric coefficient $Z^{-2}\mathcal{Q}$ is regular around $w_2 = 0$.

The regularity of the solution near the critical surface was, of course, guaranteed by our general analysis of the Toda metrics in Section 5.1.3, but it is still useful to see how it comes about here.

Finally there is the angular momentum vector and the issue of global positivity of $\mathcal{Q}$. For this it is most convenient to consider the combination $h_2\,\mu$:

$$h_2\,\mu \;\sim\; \tfrac{3}{8}\,\gamma\,\eta_0\,\alpha_2^2 \,+\, \beta_2 \,+\, \tfrac{1}{8}\,\pi^2\,\alpha_2^4 \,+\, \mathcal{O}(\theta^2)\,, \qquad \theta\to 0\,, \tag{5.113}$$

$$h_2\,\mu \;\sim\; \beta_2 \,-\, \tfrac{1}{8}\,\pi^2\,\alpha_2^4 \,+\, \mathcal{O}((\pi-\theta)^2)\,, \qquad \theta\to \pi\,. \tag{5.114}$$

Since h_2 vanishes exponentially fast in r and $\hat{r}$ in the two asymptotic regions (see (5.101)), this means that μ will diverge exponentially in r and $\hat{r}$ unless

$$\beta_2 \;=\; \tfrac{1}{8}\,\pi^2\,\alpha_2^4\,, \qquad \gamma \;=\; -\,\tfrac{2}{3}\,\pi^2\,\eta_0^{-1}\,\alpha_2^2 \;=\; -\,\tfrac{1}{3}\,\pi\,\alpha_2^2\,. \tag{5.115}$$

If these two conditions are met then μ also vanishes exponentially in r and $\hat{r}$ in both of the asymptotic regions.

Unfortunately this value of γ is inconsistent with (5.109). If one allows μ to diverge exponentially in one of the asymptotic regions then $\mathcal{Q}$ will become negative in the asymptotic regions. This is because Z limits to a finite value and b^2 diverges as a power of r or $\hat{r}$. Therefore there is no way to arrange the metric to be positive definite in the asymptotic regions on both sides of the wormhole: Either one has (5.109) and arranges that three coefficients in (5.108) to be globally positive, or one arranges that $\mathcal{Q} > 0$ only to have the three coefficients in (5.108) to change sign in one of the asymptotic regions.

Thus we have a beautifully regular metric across the critical surface, but it fails to be globally well-behaved as a "wormhole" metric. We suspect that the problem is due to the high level of symmetry. With more bubbles and thus more parameters we believe that one could simultaneously control behavior in both asymptotic regions. Even with the very high level of symmetry, there is another way to remove the regions of CTC's.

Pinching off the wormhole

One way to remove the region of CTC's is to pinch off the wormhole before one encounters the region where CTC's occur. Here we will consider the ambi-polar metric described exactly as above with the asymptotic regions as $\theta \to \pi$ arranged to be regular and asymptotic to the $U(1)$ fibration over $\mathbb{R}^3$ as in (5.100). This requires one to take:

$$\beta_2 \;=\; \tfrac{1}{8}\,\pi^2\,\alpha_2^4\,, \qquad \delta \;>\; 0\,. \tag{5.116}$$

The metric coefficients, a_i^2, are non-vanishing away from the critical surface, and so to pinch off the complete metric away from the critical surface we must arrange that the function Z vanish at some point. To avoid CTC's one must also ensure that $\mathcal{Q}$ is non-negative near the pinch-off and so one must arrange that μ vanishes simultaneously with Z. Thus we are looking for a point, θ_0, such that

$$Z\big|_{\theta=\theta_0} \;=\; 0\,, \qquad \mu|_{\theta=\theta_0} \;=\; 0\,. \tag{5.117}$$

Given these conditions, the equation of motion, (5.80), for μ then implies that $\frac{d}{d\theta}\mu$ must also vanish at θ_0. Therefore, near the pinching-off point we have:

$$Z \;\sim\; z_0\,(\theta-\theta_0)\,, \qquad \mu \;\sim\; \mu_0\,(\theta-\theta_0)^2\,, \qquad \big(\tfrac{1}{4}\,b^2\,Z \;-\; \mu^2\,Z^{-2}\big) \;\sim\; \tfrac{1}{4}\,b_0^2\,z_0(\theta-\theta_0)\,. \tag{5.118}$$

This means that the spatial part of the complete metric (5.107) is indeed pinching off in every direction with surfaces of constant θ being a set of collapsing, squashed three-spheres. The metric is not smooth at θ_0: There is a curvature singularity in the spatial part of the metric and the coefficient of dt^2 is diverging as $(\theta-\theta_0)^{-2}$. This reflects a similar divergence in the electric component of the Maxwell fields, $A^{(I)}$, (see (2.13)) at

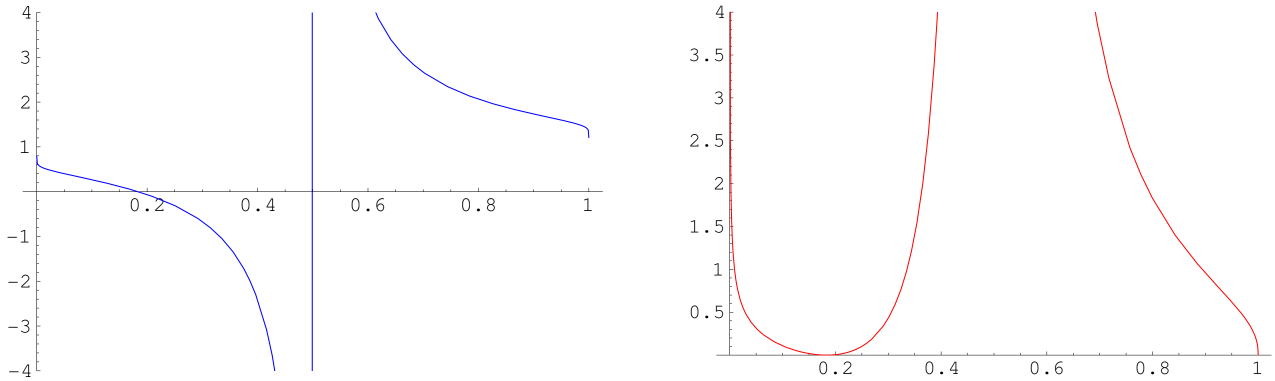

Figure 5.3: Plots of Z (on the left) and μ (on the right) as functions of $x = \sin^2 \frac{\theta}{2}$ for $\gamma = 0$, $\delta = 1$ and $\alpha_2 \approx 0.4890$. Both functions are singular at $x = 0.5$ and both vanish, μ with a double root, at $x \approx 0.1837$.

$Z = 0$. One should also note that the flux, Θ, is also singular at $Z = 0$ in that it remains constant while the cycle that supports it is collapsing.

Define

$$\hat{\gamma} \equiv \alpha_2^{-2} \gamma \, , \qquad \hat{\delta} \equiv \alpha_2^{-2} \delta \, , \tag{5.119}$$

then the conditions (5.117) relate $\hat{\gamma}$ and $\hat{\delta}$ to θ_0. Thus we can, in principle, choose the pinching-off point and then (5.117) yields the corresponding values of $\hat{\gamma}$ and $\hat{\delta}$. In practice, there is the constraint that $\delta > 0$. We know from the analysis above that we cannot arrange for Z and μ to vanish simultaneously at $\theta = 0$. Numerical analysis shows that one cannot have Z and μ vanish simultaneously unless $\theta_0 \gtrsim 0.6158$. Since we are interested in solutions that contain the critical surface ($w_2 = 0$), we have found a number of solutions that pinch off for $0.6158 \lesssim \theta_0 < \frac{\pi}{2}$. We also checked numerically that it does not appear to be possible to have all three of μ, Z and $\frac{dZ}{d\theta}$ vanish simultaneously for $\theta \in (0, \frac{\pi}{2})$. Thus (5.118) appears to be the general behavior at a pinch-off: Z does not appear to be able to have a double root.

We have verified in several numerical examples that the spatial metric is indeed globally positive definite in the region at and to the right of the pinch. These solutions

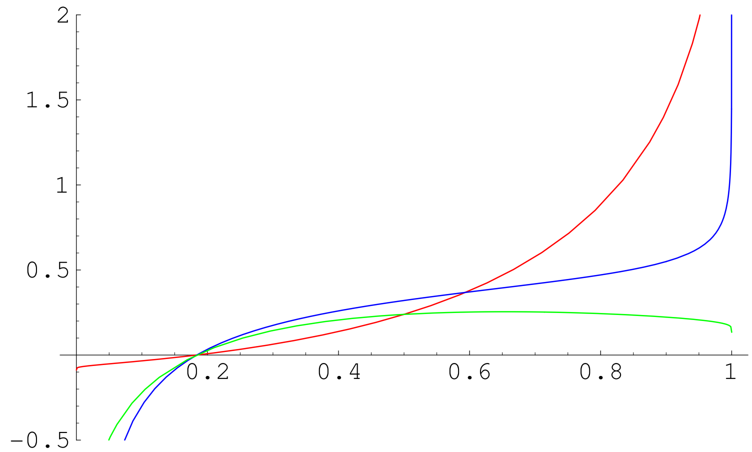

Figure 5.4: This graph shows the three metric coefficients in the angular directions, Za^2, $Z^{-2}\mathcal{Q}$ and Zc^2 (in this order from top to bottom on the right-hand side of the graph), as functions of $x = \sin^2\frac{\theta}{2}$ for $\gamma = 0$, $\delta = 1$ and $\alpha_2 \approx 0.4890$. All of these functions vanish at the pinching-off point, $x \approx 0.1837$, and are positive-definite to the right of it.

still contain the critical surface where the a_i^2 and Z simultaneously change sign and these solutions are perfectly regular across the critical surface. The cost of ensuring the global absence of CTC's is to include a non-standard, singular point-source at the center of the solution.

To present an example, we considered the solution with $\gamma = 0$, $\delta = 1$. Solving (5.117) leads to $\alpha_2 \approx 0.4890$ and the pinch-off at $x = \sin^2(\frac{\theta}{2}) \approx 0.1837$. In Fig. 5.3 we show plots of the functions Z and μ for these parameter values. Note that both are singular at $x = 0.5$ and that both vanish, μ with a double root, at $x \approx 0.1837$. In Fig. 5.4 we have shown the three metric coefficients in the angular directions, Za^2, Zc^2 and $Z^{-2}\mathcal{Q}$. All of them are positive and vanish exactly at the pinching-off point.

Before ending this section we should make a few more comments about the metric that is pinching off. The singularity at the pinch-off point is caused by the fact that the warp factors Z_I go to zero. This causes the size of the two-cycles wrapped by fluxes to shrink to zero size, and hence the energy density coming from these fluxes to be infinite. A well-known solution with a similar type of singularity is the one obtained by Klebanov and Tseytlin [149]. However, for this solution it is well understood that the singularity comes about because of the high level of symmetry in the Ansatz, and that upon considering a less-symmetric base space the singularity is resolved [150]. Since the base space considered here also has a high level of symmetry, it is tempting to conjecture that, in analogy to the Klebanov-Strassler solution [150], the pinching off will be resolved by the blowing up of a two-cycle on the base, which will only be possible in a less-symmetric, non-singular background.

We should also remark that in our discussion we have taken all three warp factors to be equal, but generically we can also imagine pinching off the metric using only one of the warp factors, and keeping the others finite. This will change the structure of the metric near the singularity (some of the two-tori will blow up and some others will shrink), but the singularity will also come from shrinking cycles on the base, and will probably be resolved also by considering a less-symmetric base with a blown-up two-cycle

5.2.6 Variations on the Eguchi-Hanson metric

Given the foregoing results, particularly those involving wormholes, it is interesting to look at the corresponding story for the Eguchi-Hanson metric [84]. This metric has an $SO(3) \times U(1)$ invariance and the diagonal $U(1)$ action is triholomorphic. The metric is equivalent to a GH metric with two GH points of equal charge [183]. The manifestly $SO(3) \times U(1)$ invariant form of this metric is:

$$ds^2 = \left(1 - \frac{a^4}{\rho^4}\right)^{-1} d\rho^2 \; + \; \frac{\rho^2}{4}\left(1 - \frac{a^4}{\rho^4}\right)\sigma_3^2 \; + \; \frac{\rho^2}{4}(\sigma_1^2 + \sigma_2^2)\,. \tag{5.120}$$

The space contains an S^2 (bolt) at $\rho = a$ and so the range of the radial coordinate is $a \leq \rho < \infty$. At infinity this space is asymptotic to $\mathbb{R}^4/\mathbb{Z}_2$.

To avoid closing off of the space at the bolt, we analytically continue by taking $a^2 = ib$, with b real, and introduce a new radial coordinate $\eta = \rho^2$. One thereby obtains:

$$ds^2 = \left(1 + \frac{b^2}{\eta^2}\right)^{-1} \frac{d\eta^2}{4\eta} + \frac{\eta}{4}(\sigma_1^2 + \sigma_2^2) + \frac{\eta}{4}\left(1 + \frac{b^2}{\eta^2}\right)\sigma_3^2 \,. \tag{5.121}$$

This metric was also considered by Eguchi and Hanson in [83], where it was called "type I," and was given in the form:

$$ds^2 = \left(1 + \left(1 - \frac{a^4}{r^4}\right)^{-1/2}\right)^2 \frac{dr^2}{4} + \frac{r^2}{8}\left(1 + \left(1 - \frac{a^4}{r^4}\right)^{1/2}\right)(\sigma_1^2 + \sigma_2^2) + \frac{r^2}{4}\sigma_3^2 \,. \tag{5.122}$$

This may be mapped to (5.121) via the coordinate change

$$\eta \;=\; r^2 \left(1 + \sqrt{1 - \frac{a^4}{r^4}}\right) \,. \tag{5.123}$$

In terms of the Toda frame, (5.1)–(5.4), this metric was found in [72] and is given by

$$\nu = \log\left(z^2 + \frac{a^4}{16}\right) - \log(2) - 2\log\left(1 + \frac{x^2 + y^2}{8}\right) \,. \tag{5.124}$$

The reason why this metric was never studied in the past is that it is not geodesically complete, and there is a singularity at $\eta = 0$. Nevertheless, we can extend the coordinate $\eta = \rho^2$ to negative values, and the resulting space (5.121) has two regions, one where the signature is $(+,+,+,+)$ and one where the signature is $(-,-,-,-)$. This makes (5.121) into precisely an ambi-polar metric of the type that can give a good five-dimensional BPS solution: The overall sign of the metric changes as one passes through $\eta = 0$, with the coefficient of a $U(1)$ fiber becoming singular at this critical surface.

The BPS solutions

In order to solve the BPS equations, (2.14)–(2.16), it is convenient to introduce the basis of frames given by:

$$\hat{e}_1 = -\left(1+\frac{b^2}{\eta^2}\right)^{-1/2}\frac{d\eta}{2\sqrt{\eta}}\,, \quad \hat{e}_2 = \frac{\sqrt{\eta}}{2}\sigma_1\,, \quad \hat{e}_3 = \frac{\sqrt{\eta}}{2}\sigma_2\,, \quad \hat{e}_4 = \frac{\sqrt{\eta}}{2}\left(1+\frac{b^2}{\eta^2}\right)^{1/2}\sigma_3 \tag{5.125}$$

One can then show that:

$$\Theta = \frac{\alpha}{\eta^2}(\hat{e}_1\wedge\hat{e}_4 + \hat{e}_2\wedge\hat{e}_3) \tag{5.126}$$

defines a harmonic, self-dual, "normalizable" two form for constant α. One also has $\Theta = dB$ with $B = \frac{\alpha}{4\eta}\sigma_3$. As before, we take all three flux forms $\Theta^{(I)}$ to be equal to Θ and set $Z_I = Z$. Then the equation for $Z(\eta)$ becomes

$$\frac{d}{d\eta}\left((\eta^2+b^2)\frac{dZ}{d\eta}\right) = \frac{\alpha^2}{2\eta^3} \tag{5.127}$$

which is solved by

$$Z(\eta) = \gamma + \frac{\alpha^2}{4b^2\eta} + \left(\frac{\alpha^2}{4b^3}+\frac{\beta}{b}\right)\arctan\left(\frac{\eta}{b}\right)\,, \tag{5.128}$$

where β and γ are integration constants. The angular momentum vector, k, has a solution of the form $k = \mu(\eta)\sigma_3$ where the function $\mu(\eta)$ satisfies

$$\eta^3\frac{d\mu}{d\eta} - \eta^2\mu + \frac{3\alpha\gamma}{4}\eta + \frac{3\alpha}{4}\left(\frac{\alpha^2}{4b^3}+\frac{\beta}{b}\right)\arctan\left(\frac{\eta}{b}\right) + \frac{3\alpha^3}{16b^2} = 0\,. \tag{5.129}$$

The solution to this equation is

$$\mu(\eta) = \delta\eta + \frac{3\alpha}{8b^2}\left(\frac{\alpha^2}{4b^2}+\beta\right) + \frac{3\alpha\gamma}{8}\frac{1}{\eta} + \frac{\alpha^3}{16b^2}\frac{1}{\eta^2} + \frac{3\alpha}{8b^3}\left(\frac{\alpha^2}{4b^2}+\beta\right)\eta\arctan\left(\frac{\eta}{b}\right) + \frac{3\alpha}{8b}\left(\frac{\alpha^2}{4b^2}+\beta\right)\frac{1}{\eta}\arctan\left(\frac{\eta}{b}\right) . \quad (5.130)$$

To complete the solution we have to impose boundary condition on the functions Z and μ.

A regular "wormhole"

If the solution is to have two asymptotic regions corresponding to $\eta \to \pm\infty$ then we must require that the angular momentum vector falls off in these regions or there will generically be CTC's. This implies:

$$\delta = 0\,, \qquad \beta = -\frac{\alpha^2}{4b^2}\,, \quad (5.131)$$

and then the functions Z and μ simplify to:

$$Z(\eta) = \gamma + \frac{\alpha^2}{4b^2\eta}\,, \qquad \mu(\eta) = \frac{3\alpha\gamma}{8}\frac{1}{\eta} + \frac{\alpha^3}{16b^2}\frac{1}{\eta^2}\,. \quad (5.132)$$

If $\gamma \neq 0$, Z will have a zero at $\eta = -\frac{\alpha^2}{4b^2\gamma}$ and thus we will inevitably have CTC's unless we pinch off the solution before, or at, this point.

We consider $\gamma = 0$ first, for which we have:

$$Z = \frac{\alpha^2}{4b^2}\frac{1}{\eta}\,, \qquad \mu = \frac{\alpha^3}{16b^2}\frac{1}{\eta^2}\,. \quad (5.133)$$

Note that the angular momentum function μ is always positive and is diverging on the critical surface $\eta = 0$. The Z also diverges and changes sign on the critical surface.

This behavior ensures that the five-dimensional metric is regular and Lorentzian. The explicit form of the space-time metric is:

$$ds_5^2 = -\frac{16b^4}{\alpha^4}\eta^2 dt^2 - \frac{2b^2}{\alpha}dt\,\sigma_3 + \frac{\alpha^2}{16b^2}\frac{d\eta^2}{(\eta^2+b^2)} + \frac{\alpha^2}{16b^2}(\sigma_1^2+\sigma_2^2+\sigma_3^2)\,. \qquad (5.134)$$

This metric can be cast into a more familiar form by first diagonalizing the metric by shifting the ψ-coordinate in (5.40) so that $\sigma_3 \to \sigma_3 + \frac{16b^4}{\alpha^3}dt$:

$$ds_5^2 = -\frac{16b^4}{\alpha^4}(\eta^2+b^2)dt^2 + \frac{\alpha^2}{16b^2}\frac{d\eta^2}{\eta^2+b^2} + \frac{\alpha^2}{16b^2}(\sigma_1^2+\sigma_2^2+\sigma_3^2)\,. \qquad (5.135)$$

Change variables via $\eta = b\sinh\chi$, $\tilde{t} = \frac{16b^4}{\alpha^3}t$ and then the metric becomes

$$ds_5^2 = \frac{\alpha^2}{16b^2}\left(-\cosh^2\chi\,d\tilde{t}^2 + d\chi^2 + \sigma_1^2+\sigma_2^2+\sigma_3^2\right), \qquad (5.136)$$

which is the well known metric for global $AdS_2 \times S^3$. The complete Maxwell field on this space is given by

$$dA \;=\; d\Theta - d(Z^{-1}(dt+k))\,, \qquad (5.137)$$

and, using $\Theta = dB$ with $B = \frac{\alpha}{4\eta}\sigma_3$, we find

$$A \;=\; -\frac{4b^2}{\alpha^2}\,\eta\,dt\,, \qquad F \;=\; \frac{4b^2}{\alpha^2}\,dt\wedge d\eta\,. \qquad (5.138)$$

The Maxwell field is thus proportional to the volume form on AdS_2 and we have obtained the global form of a Robinson-Bertotti solution [44]. The wormhole thus reduces to the usual global AdS solution.

It is interesting to try to understand the reason for which we could find an Eguchi-Hanson “wormhole” but not an Atiyah-Hitchin one. At an algebraic level, the problem comes from the form of μ, which in the Eguchi-Hanson background goes to zero on both asymptotic regions (5.132), while in the Atiyah-Hitchin background μ, (5.88), diverges in one region or in the other. If one relaxes the requirement that the Atiyah-Hitchin solutions be asymptotically flat, one can choose a more generic Θ, containing all three Ω_i. However, this still does not give a μ that decays properly at the two asymptotic regions.

A "pinch-off" solution

The other way to remove CTC's is to allow $\gamma \neq 0$ in (5.128) and pinch-off the asymptotic region with $\eta \to -\infty$ at the point, η_0, where Z vanishes. This means that we only have to require that μ vanishes as $\eta \to \infty$ and this implies

$$\delta = -\frac{3\pi\alpha}{16b^3}\left(\beta + \frac{\alpha^2}{4b^2}\right) \tag{5.139}$$

in (5.130).

As with the Atiyah-Hitchin solution, the solution will have CTC's near the pinching off point unless we also require that μ vanishes at the same point. Specifically, the constant time slices of the metric have the form:

$$ds^2 = -\frac{\mu^2}{Z^2}\,\sigma_3^2 + Z\left(\left(1+\frac{b^2}{\eta^2}\right)^{-1}\frac{d\eta^2}{4\eta} + \frac{\eta}{4}(\sigma_1^2+\sigma_2^2) + \frac{\eta}{4}\left(1+\frac{b^2}{\eta^2}\right)\sigma_3^2\right) \tag{5.140}$$

and to avoid CTC's we must have

$$\mathcal{Q} \;\equiv\; Z^3\left(\frac{\eta^2+b^2}{4\eta}\right) - \mu^2 \;\geq\; 0\,. \tag{5.141}$$

If Z vanishes then μ must vanish and this imposes a relationship, akin to the bubble equations, on β, α and b. Unlike the corresponding solution in the Atiyah-Hitchin background, there is still a free parameter in the final result, and if one choses these parameters in the proper ranges one can arrange that the pinch-off occurs at $\eta_0 < 0$ and that there are no CTC's in the region $\eta > \eta_0$. There is still, however, a curvature singularity in the metric at $\eta = \eta_0$, similar to the one in the pinched-off Atiyah-Hitchin solution, and probably caused also by the fact that the ansatz used is very symmetric. It is quite likely that this singularity will also be resolved in the same manner as the Klebanov-Tseylin/Klebanov-Strassler solutions [149, 150]

It is easy to find numerical examples that exhibit a "pinch off." For example, one can take the following values of the parameters:

$$\alpha \approx 4.2619 \qquad \beta \approx -4.5358 \qquad \gamma = b = 1 \qquad \text{and} \qquad \delta = -\frac{3\pi\alpha}{16b^3}\left(\beta + \frac{\alpha^2}{4b^2}\right), \tag{5.142}$$

and the pinch off point is $\eta_0 \approx -4.5721$. Since η_0 is negative this represents a solution based upon a non-trivial ambi-polar base metric.

5.3 Concluding Remarks

We have investigated the construction of three-charge solutions that do not have a tri-holomorphic $U(1)$ isometry. We have found that the most general form of these solutions, can be expressed in terms of several scalar functions. One of these functions satisfies the (non-linear) $SU(\infty)$ Toda equation, while the other functions satisfy linear equations that can be thought of as various linearizations of the $SU(\infty)$ Toda equation.

We have also shown generically that in the region where the signature of the four-dimensional base space changes from $(+,+,+,+)$ to $(-,-,-,-,)$, the fluxes, warp factors, and the rotation vector diverge as well, but the overall five-dimensional (or eleven-dimensional) solution is smooth. This is similar to what happens when the base-space is Gibbons-Hawking, and strongly suggests that this phenomenon is generic: Any ambi-polar, four-dimensional, hyper-Kähler metric with at least one non-trivial two-cycle can be used to construct a regular supersymmetric five-dimensional three-charge solution upon adding fluxes, warp factors and rotation according to the BPS equations (2.14), (2.15) and (2.16).

Since the most general form of hyper-Kähler, four-dimensional spaces with a rotational $U(1)$ isometry is not known explicitly, one cannot explicitly construct the most general three-charge bubbling solution with this isometry. Nevertheless, we have been able to construct the first explicit bubbling solution with a rotational $U(1)$ starting from an ambi-polar generalization of the Atiyah-Hitchin metric. For both the standard Atiyah-Hitchin and Eguchi-Hanson metrics, it is not possible to construct regular three-charge bubbling solutions. This reflects the fact that fluxes tend to stabilize cycles that would shrink by themselves, and hence only "pathological" generalizations to ambi-polar metrics can be used as base-spaces to create bubbling solutions. We have obtained the ambi-polar generalizations of both the Atiyah-Hitchin and the Eguchi-Hanson spaces, and have constructed the full three-charge solutions based on these spaces.

As expected from our general analysis, the full solutions are completely regular at the critical surface where the metric on the base space changes sign. Moreover, for the ambi-polar Eguchi-Hanson space, one can construct the full solution, which, interestingly enough, turns out to be global $AdS_2 \times S^3$. We could also obtain solutions that pinch off, and have a curvature singularity. We argued that this singularity has the same structure as the one in the Klebanov-Tseytlin solution [149] and we believe the presence of this singularity is a consequence of the high level of symmetry of the base space, and that the singularity will similarly be resolved by considering a less-symmetric base space.

This work opens several interesting directions of research. First, having shown that singular, $U(1)$-invariant, ambi-polar, four-dimensional, hyper-Kähler metrics can give smooth five-dimensional solutions upon adding fluxes, it is important to go back to the $SU(\infty)$ Toda equation and to construct more general solutions. A first step in this investigation would be to find the solutions of the Toda equations that give the $U(1) \times U(1)$ invariant ambi-polar Gibbons-Hawking metrics, following perhaps the techniques of [10]. One could then find other solutions in the vicinity of the latter, and count them using the techniques of [175, 188].

Finally, we have seen that the ambi-polar generalization of the Eguchi-Hanson space yields a geometry that is $AdS_2 \times S^3$. Moreover, unlike in the case of usual bubbling BPS solutions, the AdS_2 solution is not the Poincaré patch, but the full global AdS solution. While the distinction between global and Poincaré AdS_2 is relatively trivial, the appearance of something like a regular wormhole suggests that bubbling geometries might be even richer and more interesting than was originally anticipated.

Chapter 6

Leaving supersymmetry behind

Finding and understanding supersymmetric solutions of supergravity theories is a very important task, and significant advances have been achieved in this direction. To find and classify supersymmetric solutions one is typically utilizing the supersymmetry variations of the fermionic fields, which lead to first order differential equations that are more tractable than the second order equations of motion. Undoubtedly, supersymmetric gravity solutions have very interesting physics, some intriguing mathematical structure and provide a good laboratory for testing new ideas on tractable examples. However, one would ultimately like to construct and understand non-supersymmetric and non-extremal solutions and it is important to have as much exact solutions as possible to gain intuition about their structure and properties.

Of separate, albeit related, interest are asymptotically flat supergravity solutions with no horizons and singularities. As we discussed in Chapter 2, such regular solutions may represent possible microstates for black holes (or black rings) having the same charges and asymptotic structure. In the supersymmetric case, large classes of two and three charge BPS solutions with the same asymptotic structure as five-dimensional black holes and black rings have been found. The solutions are typically constructed by first choosing a four-dimensional hyper-Kähler base space with non-trivial topology. One then constructs a five-dimensional supergravity solution by turning on magnetic fluxes on the non-trivial cycles of the base. These fluxes stabilize the two-cycles and are ultimately responsible for the non-trivial asymptotic charges. The homological two-cycles on the base ensure that there are no singular sources and the solutions can be made regular and causal. To argue in favor of the validity of Mathur's conjecture for non-supersymmetric, and non-extremal, black holes, one needs to construct a large number of similar smooth, horizonless gravity solutions which break supersymmetry and have the same charges and asymptotics as the black holes. There are very few solutions of this kind found so far, notable examples are the solutions of [143, 38]. Certainly it is of great interest to find more examples of such solutions and understand the possible implications for the resolution of black hole singularities and the information paradox. In this Chapter we discuss how to construct certain classes of non-BPS solutions of eleven-dimensional supergravity compactified on CY_3.

6.1 Breaking supersymmetry

6.1.1 Almost BPS solutions

Implicit in the construction of the supersymmetric solutions of Chapter 2 is the choice of an orientation for the hyper-Kähler four-dimensional base: The curvature tensor can be arranged to be either self-dual or anti-self dual. For supersymmetry it is crucial that the Riemann curvature of this base has the same duality as the three magnetic two-forms: They must all be self-dual or anti-self-dual. The difference in choice merely amounts to an overall reversal of orientation and is usually neglected. However, there has been a very nice recent observation [123] that one can obtain extremal non-supersymmetric solutions of the supergravity equations of motion by flipping the relative dualities of the hyper-Kähler base and the magnetic two-forms[1]. This means that supersymmetries are

[1]We will consistently fix our hyper-Kähler base to be self-dual (*i.e.* with self-dual curvature) and so this new prescription amounts to starting with anti-self-dual magnetic two-forms and solving the supersymmetric BPS equations with flipped dualities.

"locally preserved" by the sources but globally broken by the incompatible holonomy of the background metric on the base.

The basic technique is also easily understood in terms of the underlying brane construction. For example, an asymptotically five-dimensional black ring solution (with a flat $\mathbb{R}^4$ base) preserves the four supersymmetries respected by its three constituent electric M2 branes. When one replaces the $\mathbb{R}^4$ base by a Taub-NUT space and considers the solution from the IIA perspective, the M2 branes descend to D2 branes while the tip of Taub-NUT descends to a D6 brane. In the BPS embedding, the four Killing spinors preserved by the three sets of D2 branes are the same as those of the D6 brane, and thus the solution is supersymmetric. In the non-BPS embedding the D6 brane has opposite orientation, and hence it does not preserve any of the four Killing spinors of the D2 branes.

Note that if there are only two sets of D2 branes present, the D6 brane will be mutually BPS with them irrespective of its orientation. Hence, a two-charge supertube embedded in Taub-NUT in the "duality-matched" embedding [24] or in the "duality-flipped" embedding [123] will still be supersymmetric.

Let us discuss in some detail how one can go about and construct these simple almost BPS solutions. BPS solutions of eleven-dimensional supergravity with M2 and M5 branes wrapped on two- and four-cycles of a CY_3 take the form discussed in Chapter 2[2]. It is useful to remember that the dipole field strengths are

$$\Theta^{(I)} = dB^{(I)} \,, \qquad\qquad I = 1, 2, 3 \,, \tag{6.1}$$

and the equations following from the supersymmetry variations for a self-dual hyper-Kähler base metric are given in (2.14)-(2.16)

[2]For simplicity we will assume in this Chapter that $CY_3 = T^6$, most of the result are trivially extended for arbitrary compact Calabi-Yau three-fold as long as one knows its triple interesection numbers C_{IJK}.

It was observed in [123] that a class of extremal solutions of the equations of motion can be obtained by reversing the duality of the Θ_I and of k relative to the duality of the curvature of the four-dimensional base. That is, one preserves the metric, ds_4^2, and the duality of its Riemann tensor but flips $\star_4 \to -\star_4$ in (2.14)-(2.16):

$$\begin{aligned} &\Theta^{(I)} = -\star_4 \Theta^{(I)} \, , \\ &\nabla^2 Z_I = \frac{|\epsilon_{IJK}|}{2} \star_4 (\Theta^{(J)} \wedge \Theta^{(K)}) \, , \\ &dk - \star_4 dk = Z_I \Theta^{(I)} \, . \end{aligned} \tag{6.2}$$

When the base metric ds_4^2 is flat $\mathbb{R}^4$, the flip of orientation can be re-written as a change of coordinates, and solutions to equations(6.2) are still BPS. When ds_4^2 is not flat, as in Taub-NUT space, equations (6.2) determine, in general, non-BPS solutions, which were named "almost BPS" in [123].

It is intuitively clear why this simple flip of orientation leads to new supergravity solutions that break supersymmetry. The equations of motion are second order and the flip of orientation does not affect them, the only thing affected are some of the supersymmetry variations. There are now a number of examples of such solutions in the literature - almost BPS black rings, black holes as well as multi-center configurations of such objects [36, 37, 99, 116, 117, 59]. They are however always a superposition of black holes, black rings and supertubes, i.e. they have singularities and/or horizons. There are no analogs of the regular BPS bubbling geometries. To find such solutions one has to completely relax the condition that the four-dimensional base is hyper-Kähler. This will be the subject of the next Section.

6.1.2 Non-BPS solutions and the floating brane Ansatz

Recently, there has been important progress in overcoming the difficulties of constructing exact non-BPS solutions of $\mathcal{N} = 2$ five-dimensional supergravity (or M-theory compactified on CY_3) [38, 123, 36, 39]. The underlying idea is to find a linear system of differential equations yielding non-supersymmetic solutions. Motivated by these advances the authors of [39] revisited the Ansatz and assumptions in the construction of BPS solutions to five-dimensional $\mathcal{N} = 2$ supergravity coupled to vector multiplets [101, 21]. In this paper, the authors rederived the equations of motion, imposing a simple relation between the warp factor in the metric and the gauge fields, dubbed the "floating brane" Ansatz. This Ansatz greatly simplifies the equations of motion and allows one not only to recover almost all known, BPS and non-BPS, classes of solutions, but also to find a new *linear* system of equations. Using this result, new regular, horizonless and non-supersymmetric solutions were found in [38]. These solutions were constructed by solving the same linear system of equations as for BPS solutions, but on a Ricci-flat (instead of hyper-Kähler) four-dimensional base. The particular examples discussed in [38] were based on the Euclidean Schwarzschild and Kerr-Taub-Bolt black holes.

In [45] we found a five-parameter family of smooth, horizonless solutions with a dyonic Euclidean Reissner-Nordström base. The solutions have general fluxes with no definite self-duality and are asymptotic to $\mathbb{R}^{1,3} \times S^1$. We then generalized these solutions by including rotation and a NUT charge on the four-dimensional base, i.e. we use the Kerr-Newman-NUT background as a base. This more general family of solutions, still regular and horizonless, has six independent parameters, however their range is constrained by imposing regularity and causality of the five-dimensional background. Our solutions are not supersymmetric and have the same asymptotic structure as non-extremal black holes. They are therefore of interest, not only by themselves as new non-supersymmetric solutions, but also as candidates for microstates of non-extremal black holes. These solutions can be viewed as a generalization of the ones discussed in [38] since we consider four-dimensional base spaces which are electrovac and are not not Ricci-flat. A general feature of the solutions is that the mass is linearly dependent on the electric charges. This property is due to the "floating brane" Ansatz of [39], which relates the warp factors in the five-dimensional metric to the electric gauge potentials. We also showed that some of the solutions based on the Euclidean four-dimensional Kerr-Newman-NUT background exhibit ambipolar behavior: the four-dimensional base is allowed to have regions of positive and negative signature while the five-dimensional solution is everywhere completely regular and of definite Lorentzian signature. This provides some evidence that non-supersymmetric ambipolar solutions may also be ubiquitous like their BPS cousins [26, 41].

It should be emphasized that our solutions will have no singular M2 and M5 brane sources. Because of the non-trivial topology of the four dimensional base the asymptotic charges of the solution are due to "charges dissolved in fluxes". This is essentially the same geometric transition mechanism as the one discussed in Chapter 2 and in [26] for BPS solutions. By using the results of Chapter 3 one can recast our solutions as six-dimensional solutions of IIB supergravity compactified on T^4 [35]. This duality frame may be useful for understanding the holographic dual field theory description of the solutions.

The five-dimensional Ansatz

We will work with $\mathcal{N}=2$, five-dimensional ungauged supergravity with two $U(1)$ vector multiplets and we use the conventions of [39]. This supergrvaity theory (known also as the STU model) comes from compactifying the M-theory configuration of Chapter 2 on T^6. The bosonic action is

$$S = \frac{1}{2\kappa_5}\int \sqrt{-g}\, d^5x \Big(R - \tfrac{1}{2} Q_{IJ} F^I_{\mu\nu} F^{J\mu\nu} - Q_{IJ}\partial_\mu X^I \partial^\mu X^J - \tfrac{1}{24} C_{IJK} F^I_{\mu\nu} F^J_{\rho\sigma} A^K_\lambda \bar{\epsilon}^{\mu\nu\rho\sigma\lambda}\Big)\,,$$

with $I, J = 1, 2, 3$. The scalars X^I satisfy the constraint

$$X^1 X^2 X^3 = 1\,, \tag{6.3}$$

and there are therefore only two independent scalars. This is explained by the fact that one of the vectors is in the gravity multiplet, and thus there are only two vector multiplets. For convenience, we introduce three other scalar fields, Z_I

$$X^1 = \left(\frac{Z_2\, Z_3}{Z_1^2}\right)^{1/3}, \quad X^2 = \left(\frac{Z_1\, Z_3}{Z_2^2}\right)^{1/3}, \quad X^3 = \left(\frac{Z_1\, Z_2}{Z_3^2}\right)^{1/3}. \tag{6.4}$$

This automatically solves the constraint (6.3). The scalar kinetic term can be written as

$$Q_{IJ} \;=\; \frac{1}{2}\,\mathrm{diag}\left((X^1)^{-2}, (X^2)^{-2}, (X^3)^{-2}\right). \tag{6.5}$$

It is useful to introduce the scalar

$$Z \;\equiv\; \left(Z_1\, Z_2\, Z_3\right)^{1/3}. \tag{6.6}$$

If one reduces the theory to four dimensions this will be a third independent scalar field. Having defined this new scalar, we will work with the following metric Ansatz

$$ds_5^2 \;=\; -\,Z^{-2}\,(dt+k)^2 \;+\; Z\, ds_4^2\,, \tag{6.7}$$

We will denote the frames for (6.7) by e^A, $A = 0, \ldots, 4$ and let $\hat{e}^a$, $a = 1, \ldots, 4$ denote frames for ds_4^2. Explicitly,

$$e^0 \;\equiv\; Z^{-1}\,(dt + k)\,, \qquad\qquad e^a \;\equiv\; Z^{1/2}\,\hat{e}^a\,. \tag{6.8}$$

We will assume also the "floating brane" Ansatz of [39], which means that we take the metric coefficients to be related to the electrostatic potentials. The Maxwell field is thus

$$A^{(I)} \;=\; -\,Z_I^{-1}\,(dt + k) + B^{(I)}\,, \tag{6.9}$$

where $B^{(I)}$ is a one-form on the base ds_4^2. Upon uplifting this solutions to eleven-dimensional supergravity, this Ansatz implies that M2 brane probes that have the same charges as the M2 branes sourcing the solution will have equal and opposite Wess-Zumino and Dirac-Born-Infeld terms and hence will not feel any force. Such brane probes may be placed anywhere in the base and may thus be viewed as "floating."

Equations of motion

The general equations of motion following from the above Ansatz were derived in [39] and we will use their results and conventions. We introduce the magnetic two-from field strengths

$$\Theta^{(I)} = dB^{(I)}\,, \tag{6.10}$$

and it will also be convenient to introduce the two-forms $\omega_-^{(I)}$ defined by

$$\frac{1}{2}\left(\Theta^{(I)} - *_4\Theta^{(I)}\right) \equiv C_{IJK} Z_J \omega_-^{(K)}\,, \tag{6.11}$$

where the $*_4$ is the Hodge dual with respect to the four-dimensional metric ds_4^2 in (6.7). Note that in contrast to the discussion in Chapters 2-5 the magnetic fluxes do not have a definite self-duality. Following [39] we will simplify the equations of motion by assuming

$$dk + *_4 dk = \frac{1}{2}\sum_I Z_I \left(\Theta^{(I)} + *_4\Theta^{(I)}\right) , \qquad \text{and} \qquad \omega_-^{(1)} = \omega_-^{(2)} = 0\,. \tag{6.12}$$

The four-dimensional base space has to be a solution of Euclidean Einstein-Maxwell theory[3] with (symbols with a hat live on the four-dimensional base)

$$\hat{R}_{\mu\nu} = \frac{1}{2}\left(F_{\mu\rho}F_\nu^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right) , \tag{6.13}$$

and

$$F = \Theta^{(3)} - \omega_-^{(3)}\,. \tag{6.14}$$

The rest of the equations of motion reduce to[4]

$$\hat{\nabla}^2 Z_1 = *_4(\Theta^{(2)} \wedge \Theta^{(3)})\,, \qquad (\Theta^{(2)} - *_4\Theta^{(2)}) = 2Z_1\,\omega_-^{(3)}\,, \tag{6.15}$$

$$\hat{\nabla}^2 Z_2 = *_4(\Theta^{(1)} \wedge \Theta^{(3)})\,, \qquad (\Theta^{(1)} - *_4\Theta^{(1)}) = 2Z_2\,\omega_-^{(3)}\,, \tag{6.16}$$

$$\hat{\nabla}^2 Z_3 = *_4[\Theta^{(1)} \wedge \Theta^{(2)} - \omega_-^{(3)} \wedge (dk - *_4 dk)]\,, \tag{6.17}$$

---

[3] The normalization of the flux in this equation is different from the one used in Chapter 1 and most standard sources on general relativity, this choice is made so that the conventions in this Chapter agree with the four-dimensional conventions in [39].

[4] It is important to note that we have fixed the constant ϵ used in [39] to be $\epsilon = 1$, which amounts to a particular self-duality convention for the fluxes. This choice is not restrictive and it is straightforward to repeat all our calculations for $\epsilon = -1$.

$$dk + *_4 dk = \frac{1}{2}\sum_{I=1}^{3} Z_I(\Theta^{(I)} + *_4\Theta^I)\,. \qquad (6.18)$$

An important point about this system of equations is that it can be solved in a linear fashion. In order to do that, one has to solve the equations in the right order. The starting point is to choose a four-dimensional metric and its associated two-form field strength that solve (6.13). Then using (6.14) one can read off $\Theta^{(3)}$ and $\omega_-^{(3)}$ from the field strength. Knowing these fields, (6.15) and (6.16) become systems of two linear coupled equations for Z_1 and $\Theta^{(2)}$ and Z_2 and $\Theta^{(1)}$ respectively. Finally, k and Z_3 are solutions to the system of linear equations (6.17) and (6.18). We will show in the next section how to solve these equations starting from the Euclidean Reisner-Nordström backgrounds [45].

6.2 Examples of non-BPS solutions

Here we will discuss non-supersymmetric solutions of five-dimensional supergravity with an Euclidean Reissner-Nordström base.

6.2.1 The four-dimensional background

Our starting point in this section will be the Euclidean dyonic Reissner-Nordström background [185, 173]

$$ds_4^2 = \left(1 - \frac{2m}{r} + \frac{p^2 - q^2}{r^2}\right) d\tau^2 + \left(1 - \frac{2m}{r} + \frac{p^2 - q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \qquad (6.19)$$

$$F = \frac{2q}{r^2}\, d\tau \wedge dr + 2p \sin\theta\, d\theta \wedge d\phi\,. \qquad (6.20)$$

Where m corresponds to the mass, q to the electric charge and p to the magnetic monopole charge of the solution. This background solves the four-dimensional Einstein equations (6.13). It is useful to rewrite the metric as

$$ds_4^2 = \frac{(r-r_+)(r-r_-)}{r^2}d\tau^2 + \frac{r^2}{(r-r_+)(r-r_-)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \quad (6.21)$$

The constants $r_\pm$ are the Euclidean analogs of the inner and outer horizon of the Reissner-Nordström black hole

$$r_\pm = m \pm \sqrt{m^2 - p^2 + q^2}\,. \quad (6.22)$$

To render $r_\pm$ real we restrict to the range of parameters[5] $m^2 > p^2 - q^2$. Near the outer horizon one can set

$$r = r_+ + \frac{r_+ - r_-}{4r_+^2}\rho^2\,, \qquad \chi = \frac{r_+ - r_-}{2r_+^2}\tau\,, \quad (6.23)$$

and rewrite the metric as

$$ds_{NH}^2 = d\rho^2 + \rho^2 d\chi^2 + r_+^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \quad (6.24)$$

which means that for a regular solution we should restrict to $r \geq r_+$ and the coordinate τ should be made periodic

$$\tau \sim \tau + \frac{4\pi r_+^2}{r_+ - r_-}\,. \quad (6.25)$$

With this identification the metric is asymptotic to $\mathbb{R}^2 \times S^2$ for $r \to r_+$ (i.e. we have a bolt of radius r_+ [110]) and to $\mathbb{R}^3 \times S^1$ for $r \to \infty$. The angles θ and ϕ are the

[5] The case $m^2 = p^2 - q^2$ corresponds to the extremal Euclidean Reissner-Nordström black hole. We discuss this case in Appendix E.

coordinates on S^2. In the next section we will solve the equations of motion of $\mathcal{N} = 2$ five-dimensional supergravity with this Euclidean metric as a base space.

6.2.2 The five-dimensional supergravity solution

A convenient set of frames on the four-dimensional base is given by

$$\begin{aligned} \hat{e}^1 &= \left(1 - \frac{2m}{r} + \frac{p^2 - q^2}{r^2}\right)^{1/2} d\tau\,, \quad \hat{e}^2 = \left(1 - \frac{2m}{r} + \frac{p^2 - q^2}{r^2}\right)^{-1/2} dr\,, \\ \hat{e}^3 &= r\,d\theta\,, \qquad\qquad \hat{e}^4 = r\sin\theta\,d\phi\,, \end{aligned} \tag{6.26}$$

and the usual self-dual and anti-self-dual two-forms are

$$\Omega_\pm = \hat{e}^1 \wedge \hat{e}^2 \pm \hat{e}^3 \wedge \hat{e}^4\,. \tag{6.27}$$

With this in hand it is easy to show that

$$\Theta^{(3)} = \frac{p+q}{r^2}\,\Omega_+\,, \qquad\qquad \omega_-^{(3)} = \frac{p-q}{r^2}\,\Omega_-\,. \tag{6.28}$$

It will be useful to have the explicit expression for the potential $B^{(3)}$ satisfying $\Theta^{(3)} = dB^{(3)}$

$$B^{(3)} = \frac{(p+q)}{r}\,d\tau - (p+q)\cos\theta\,d\phi\,. \tag{6.29}$$

The solution to equations (6.15) and (6.16) is

$$Z_1 = 1 - \frac{2q_2(p+q)}{m}\,\frac{1}{r}\,, \qquad\qquad Z_2 = 1 - \frac{2q_1(p+q)}{m}\,\frac{1}{r}\,, \tag{6.30}$$

$$\Theta^{(1)} = f_1(r)\Omega_+ + g_1(r)\Omega_-\,, \qquad\qquad \Theta^{(2)} = f_2(r)\Omega_+ + g_2(r)\Omega_-\,, \tag{6.31}$$

where

$$f_1 = \frac{2q_1}{r^2} - \frac{2q_1(p^2-q^2)}{m\,r^3}\,, \qquad f_2 = \frac{2q_2}{r^2} - \frac{2q_2(p^2-q^2)}{m\,r^3}\,, \tag{6.32}$$

$$g_1 = \frac{(p-q)}{r^2} - \frac{2q_1(p^2-q^2)}{m\,r^3}\,, \qquad g_2 = \frac{(p-q)}{r^2} - \frac{2q_2(p^2-q^2)}{m\,r^3}\,. \tag{6.33}$$

Note that with these functions $f_I(r)$ and $g_I(r)$ one can show that $d\Theta^{(I)} = 0$, which means that locally one can express $\Theta^{(1)}$ and $\Theta^{(2)}$ in terms of potential one-forms, $\Theta^{(I)} = dB^{(I)}$. Explicitly, these one-forms are

$$B^{(I)} = K_I\,d\tau + b_I\,d\phi\,, \tag{6.34}$$

with

$$K_1 = \frac{2q_1+p-q}{r} - \frac{2q_1(p^2-q^2)}{m\,r^2}\,, \qquad b_1 = (-2q_1+p-q)\cos\theta\,, \tag{6.35}$$

$$K_2 = \frac{2q_2+p-q}{r} - \frac{2q_2(p^2-q^2)}{m\,r^2}\,, \qquad b_2 = (-2q_2+p-q)\cos\theta\,. \tag{6.36}$$

To solve (6.17) and (6.18), we will use the Ansatz

$$k = \mu(r)d\tau + \nu(\theta)d\phi\,. \tag{6.37}$$

One can then show that

$$\nu(\theta) = \nu_0 + \xi\cos\theta\,, \tag{6.38}$$

with ν_0 and ξ constants. Then the problem reduces to a system of two coupled *linear* ordinary differential equatons for $\mu(r)$ and $Z_3(r)$

$$\frac{d\mu}{dr} = -\left(\frac{\xi}{r^2} + Z_1 f_1 + Z_2 f_2 + \frac{p+q}{r^2} Z_3\right), \tag{6.39}$$

$$\hat{\nabla}^2 Z_3 = 2\left(f_1 f_2 - g_1 g_2 + \frac{\xi(p-q)}{r^4} - \frac{(p-q)}{r^2}\frac{d\mu}{dr}\right). \tag{6.40}$$

A solution to these equations is given by

$$\begin{aligned} Z_3 &= 1 - \left(\frac{4q_1 q_2(m^2 - p^2 + q^2)}{m^3} + \frac{2(p-q)(q+q_1+q_2)}{m}\right)\frac{1}{r} \\ &\quad + \frac{4q_1 q_2(p^2-q^2)}{m^2}\frac{1}{r^2}, \end{aligned} \tag{6.41}$$

$$\begin{aligned} \mu &= (p+q+2(q_1+q_2))\left(\frac{1}{r} - \frac{1}{r_+}\right) \\ &\quad - \left(\frac{2q_1 q_2(p+q)(3m^2-p^2+q^2)}{m^3} + \frac{(p^2-q^2)(q+2q_1+2q_2)}{m}\right)\left(\frac{1}{r^2} - \frac{1}{r_+^2}\right) \\ &\quad + \frac{4q_1 q_2(p^2-q^2)(p+q)}{m^2}\left(\frac{1}{r^3} - \frac{1}{r_+^3}\right). \end{aligned} \tag{6.42}$$

To arrive at this particular solution we have chosen

$$\nu_0 = \xi = 0\,, \qquad \to \qquad \nu = 0\,, \tag{6.43}$$

which ensures that there are no closed time-like curves (CTCs) coming from the $d\phi^2$ term in the five-dimensional metric, at $\theta = 0, \pi$. We have also chosen the additive constant in the solution for μ such that $\mu(r_+) = 0$, which ensures the absence of CTCs near the bolt. This implies that μ has a non vanishing value γ at infinity,

$$\begin{aligned} \lim_{r\to\infty} \mu = \gamma \equiv \quad & -\frac{1}{r_+}(p+q+2(q_1+q_2)) \\ & +\frac{1}{r_+^2}\left(\frac{2q_1q_2(p+q)(3m^2-p^2+q^2)}{m^3} + \frac{(p^2-q^2)(q+2q_1+2q_2)}{m}\right) \\ & -\frac{1}{r_+^3}\frac{4q_1q_2(p^2-q^2)(p+q)}{m^2}\,, \end{aligned} \tag{6.44}$$

this will be important in the calculation of the asymptotic charges of the five-dimensional solution. Note also that we have set the constants terms in Z_I to 1 by which we fix the asymptotic values of the scalar fields[6].

An important difference between this solution and the magnetized Euclidean Schwarzschild solution in [38] is that the fluxes here are not self-dual. It is clear that if we set

$$q = p = \frac{\tilde{q}_3}{2}\,, \qquad q_1 = \frac{\tilde{q}_1}{2}\,, \qquad q_2 = \frac{\tilde{q}_2}{2}\,, \tag{6.45}$$

we will recover the five-dimensional solution based on the Euclidean Schwarzschild black hole found in [38]. Note that all q_I in [38] should be identified with $\tilde{q}_I$, this is due to the different conventions in the normalization of the fluxes.

An important step in the analysis of the five-dimensional solution constructed above is to ensure the global absence of CTCs. This means that for constant time slices one should make sure that the coefficient of $d\tau^2$ in the five-dimensional metric is non-negative and all Z_I are positive definite. To analyze this condition in an explicit example we will take

$$q = q_1 = q_2 = Q > 0\,, \qquad p = \frac{Q}{2}\,. \tag{6.46}$$

[6] In an eleven-dimensional uplift of our solution this choice will fix the asymptotic volumes of the two-cycles of T^6.

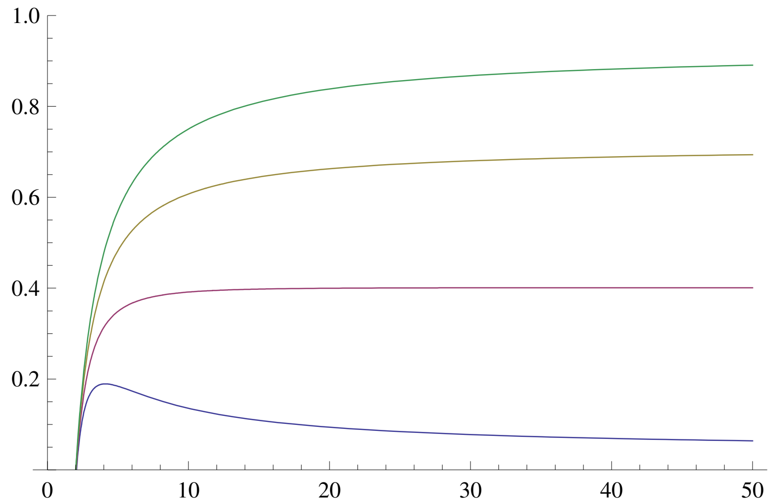

Figure 6.1: $\mathcal{M}$ as a function of $\rho = r/r_+$ for four different values of Q/m. The curves correspond to $Q/m = (0.1, 0.2, 0.3, 0.4)$ from top to bottom.

Then we have

$$r_\pm = m \pm \sqrt{m^2 + \frac{3Q^2}{4}}\,, \tag{6.47}$$

and the condition that Z_1 and Z_2 are positive for $r \geq r_+$ imposes

$$0 < \frac{Q}{m} < \frac{\sqrt{3}}{2} \approx 0.8660\,. \tag{6.48}$$

Requiring that Z_3 is positive for $r > r_+$ leads to

$$0 < \frac{Q}{m} \lessapprox 0.7783\,, \tag{6.49}$$

which is clearly a stronger constraint. Finally we have to make sure that the coefficient of $d\tau^2$ is non-negative

$$\mathcal{M} \equiv \frac{1}{r^2(Z_1Z_2Z_3)^{2/3}}[Z_1Z_2Z_3(r-r_+)(r-r_-) - \mu^2r^2] \geq 0\,. \tag{6.50}$$

Expanding this expression for $r \to \infty$ we find a sextic algebraic inequality in Q/m, which can be solved numerically. The allowed range of parameters coming from this constraint is

$$0 < \frac{Q}{m} \lessapprox 0.4118\,, \qquad\qquad 0.8811 \lessapprox \frac{Q}{m} \lessapprox 1.2587\,. \tag{6.51}$$

The bottom line is that for the choice of parameters (6.46) the five-dimensional solution is completely regular and there are no CTCs (globally) if

$$0 < \frac{Q}{m} \lessapprox 0.4118\,. \tag{6.52}$$

Some plots of $\mathcal{M}$ for different values of Q/m are presented in Fig. 6.1. We have performed a detailed numerical analysis for a number of other choices for the parameters (p, q, q_1, q_2) and the conclusions are qualitatively the same. Namely, there is a region in parameter space in which the five-dimensional solution is regular and has no global CTCs.

6.2.3 The asymptotic charges

Having found a regular five-dimensional solution of $\mathcal{N} = 2$ ungauged supergravity (or alternatively M-theory on T^6), asymptotic to $\mathbb{R}^{1,3} \times S^1$, it is instructive to compute its asymptotic charges. The dipole charges , d_I, of the solution are directly encoded in the magnetic part of the gauge field, $B^{(I)}$. We thus have from (6.29), (6.35) and (6.36)

$$\begin{aligned} d_1 &= 2q_1 - p + q\,, \\ d_2 &= 2q_2 - p + q\,, \\ d_3 &= p + q\,. \end{aligned} \tag{6.53}$$

If the solution is viewed as a compactification of eleven-dimensional supergravity on T^6 these will correspond to the M5 brane charges. The electric charges of the solution are given by

$$Q_I = \int_{S^1\times S^2}\Big[(X^I)^{-2} *_5 dA^I - \tfrac{1}{2}C_{IJK}A^J\wedge dA^K\Big]\,, \tag{6.54}$$

where the integral is computed over the $S^1 \times S^2$ at spatial infinity, parameterized by (τ,θ,ϕ). The Chern-Simons term gives a non-vanishing contribution to the charge, due to the fact that the one-form k goes to a constant non-zero value at infinity. A straightforward calculation yields

$$\begin{aligned}
Q_1 &= -\frac{16\pi^2 r_+^2}{r_+-r_-}\left(\frac{2(p+q)q_2}{m}+\gamma(q+q_2)\right)\,,\\
Q_2 &= -\frac{16\pi^2 r_+^2}{r_+-r_-}\left(\frac{2(p+q)q_1}{m}+\gamma(q+q_1)\right)\,,\\
Q_3 &= -\frac{16\pi^2 r_+^2}{r_+-r_-}\left(\frac{4q_1q_2}{m}+\gamma(q_1+q_2+p-q)+\frac{2(p-q)(q+q_1+q_2)}{m}\right.\\
&\qquad\left.-\frac{4q_1q_2(p^2-q^2)}{m^3}\right)\,.
\end{aligned} \tag{6.55}$$

To compute the mass and the Kaluza-Klein (KK) electric charge of the solution one has to analyze the asymptotic form of the metric. The fact that the one-form, k, does not vanish at infinity implies that the coordinates (τ, t) define a frame which is not asymptotically at rest. One can go to an asymptotically static frame by casting the large r limit of the metric in the form

$$ds^2 \approx (1-\gamma^2)\Big(d\tau - \frac{\gamma}{1-\gamma^2}dt\Big)^2 - \frac{1}{1-\gamma^2}dt^2 + dr^2 + r^2(d\theta^2+\sin^2\theta d\phi^2)\,, \tag{6.56}$$

and redefining the coordinates as

$$\hat\tau = (1-\gamma^2)^{1/2}\Big(\tau - \frac{\gamma}{1-\gamma^2}t\Big)\,, \qquad \hat t = (1-\gamma^2)^{-1/2}t\,. \tag{6.57}$$

To compute the mass and KK charge, one needs to reduce our solution along the the $\hat{\tau}$ coordinate. The metric takes the form

$$ds_5^2 = \frac{g^2}{Z^2}\hat{I}_4\Big[d\hat{\tau} + \Big(\gamma - \frac{\mu}{g^2\hat{I}_4}\Big)\,d\hat{t}\Big]^2 + \frac{Z}{g\hat{I}_4^{1/2}}ds_E^2\,, \tag{6.58}$$

where we have defined,

$$g = 1 - \frac{2m}{r} + \frac{p^2 - q^2}{r^2}\,, \qquad \hat{I}_4 = \frac{1}{1-\gamma^2}\left(g^{-1}Z^3 - g^{-2}\mu^2\right)\,, \tag{6.59}$$

and

$$ds_E^2 = -\hat{I}_4^{-1/2}d\hat{t}^2 + \hat{I}_4^{1/2}\Big[dr^2 + gr^2(d\theta^2 + \sin^2\theta d\phi^2)\Big] \tag{6.60}$$

is the four-dimensional Einstein metric. From the asymptotic behavior of the $d\hat{t}^2$ coefficient in the Einstein frame metric one can read off the mass of the solution

$$\begin{aligned} M &= \frac{1}{G_4(1-\gamma^2)}\Big[\frac{m}{2}(1-2\gamma^2) - \frac{q_1q_2 + pq_1 + pq_2 + \frac{q(p-q)}{2}}{m} \\ &\qquad -\gamma(q_1 + q_2 + \frac{p+q}{2}) + \frac{q_1q_2(p^2-q^2)}{m^3}\Big]\,. \end{aligned} \tag{6.61}$$

Here G_4 is the four-dimensional Newton's constant, whose relation to the five-dimensional Newton's constant G_5 is

$$G_4 = \frac{G_5}{\mathrm{vol}(\tau)} = \frac{G_5}{(1-\gamma^2)^{1/2}}\frac{(r_+ - r_-)}{4\pi r_+^2}\,, \tag{6.62}$$

and $\text{vol}(\tau)$ is the length of the S^1 parametrized by τ. The KK electric charge, Q_e, is encoded in the KK gauge field[7]

$$A_{KK} = \left(\gamma - \frac{\mu}{g^2 \hat{I}_4}\right) d\hat{t} \,, \tag{6.63}$$

and is given by

$$\begin{aligned} Q_e &= -\frac{1}{G_4(1-\gamma^2)}\Big[\gamma\frac{m}{2} + \gamma\frac{q_1 q_2 + p q_1 + p q_2 + \frac{q(p-q)}{2}}{m} \\ &\quad + \frac{1+\gamma^2}{2}(q_1 + q_2 + \frac{p+q}{2}) - \gamma\frac{q_1 q_2 (p^2 - q^2)}{m^3}\Big] \,. \end{aligned} \tag{6.64}$$

Finally it is instructive to compute the rest-mass, M_0, of the solution, i.e. the mass with respect to the (t, τ) frame

$$M_0 \equiv (1-\gamma^2)^{-1/2}(M - \gamma Q_e) = \frac{1}{16\pi G_5}\left(\frac{32\pi^2 r_+^2 m}{r_+ - r_-} + Q_1 + Q_2 + Q_3\right) \,. \tag{6.65}$$

It is clear from this expression that if we set the mass of the four-dimensional Reissner-Nordström black hole to zero we will recover the usual relation between the mass and the charges of a BPS black hole solution. Note also that despite the fact that we start our construction from a four-dimensionnal black hole with a magnetic charge p, A_{KK} has a component only along $d\hat{t}$, which implies that the final solution does not carry any global magnetic charge.

6.3 Outlook

Starting from a four-dimensional Euclidean background that solves Einstein-Maxwell equations, we found a five-parameter family of solutions to five-dimensional $\mathcal{N} = 2$

[7] We use the conventions of [90].

ungauged supergravity coupled to two vector multiplets. Our solutions are regular, horizonless, do not preserve any supersymmetries and have the same charges at infinity as a non-extremal black hole. They generalize substantially the solutions found in [38] which were based on a Ricci-flat four-dimensional base and had only self-dual (or anti-self-dual) fluxes. The key point of the construction, in both [38] and our work, is the existence of a bolt (a topological S^2) in the four-dimensional base [110], on which one can put magnetic fluxes. These fluxes provide non-singular sources for the warp factors of the solution, ensure its regularity and are ultimately responsible for the charges at spatial infinity. It would be interesting to construct other non-supersymmetric five-dimensional supergravity solutions with a four-dimensional electrovac base. If this base space has interesting topology one should be able to find regular solutions by putting fluxes on it. In [45] we found a more general six-parametr family of solutions based on the Kerr-Newman-NUT gravitational instanton. This is a generalization of the Euclidean Reissner-Nordström solution which includes rotation and NUT charge. An important aspect of this regular non-supersymmetric solution is that there is a range of parameters for which it can be made ambipolar in much the same way as the BPS solutions of Chapter 2 and the non-supersymmetric solution with a Kerr-Taub-bolt base found in [38]. There are some other well-known backgrounds that could be used for constructing non-supersymmetric supergrvity solutions. The ten-parameter family of solutions constructed by Carter [58] is a notable example, which includes the Kerr-Newman solution. Another interesting example is the Euclidean Melvin solution [169]. This solution is not asymptotically flat and may lead to non-supersymmetric solutions with interesting asymptotic structure. Trying to build five-dimensional solutions on these spaces may be challenging, but the presence of two commuting Killing vectors on the four-dimensional base should render the problem tractable.

In the supergravity action (6.3), the three gauge fields have symmetric roles. This symmetry is explicitly broken by our assumptions (6.12), which leads to a linear system of differential equations. A very natural question is whether one can put all three $U(1)$ gauge fields on the same footing, and find solutions which are symmetric under the interchange of the three gauge fields. While the "floating brane" Ansatz presumably allows for such solution, it seems to be a rather difficult task to find completely general solutions in this Ansatz. Indeed, turning on $\omega_-^{(1)}$ and $\omega_-^{(2)}$ modifies the equations of motion and they can no longer be solved in a linear way.

As we discussed above the solutions presented in this Chapter can be obtained by compactifying eleven-dimensional supergravity on T^6 with three sets of M2 and M5 branes wrapping two- and four-cycles on the torus. It should be in principle straightforward to construct analogous compactifications replacing the T^6 by an arbitrary Calabi-Yau threefold. These would correspond to solutions of five-dimensional $\mathcal{N} = 2$ ungauged supergravity coupled to $h_{1,1} - 1$ vector multiplets, where $h_{1,1}$ is one of the Hodge numbers of the Calabi-Yau. In the BPS case such solutions were discussed in [60].

Rather than finding new solutions by solving the equations of motion, a very fruitful approach is the use of solution generating techniques. In this context, it is useful to note that the solutions discussed in this Chapter have at least two commuting space-like Killing vectors. This symmetry can be utilized to generate an even more general class of non-extremal solutions by using spectral flow [33]. This may proceed in the following way - first one has to use the results of Chapter 3 to dualize the eleven-dimensional solution to IIB supergravity and then perform the spectral flow transformation of [33]. The action of spectral flow on non-BPS supergravity solutions has already shown its efficiency [39], [4], and it is natural to expect that it will be useful for generating new interesting solutions.

The construction of our solutions relies on the "floating brane" Ansatz of [39], which states that the metric warp factors and the electric potentials are related. All the solutions found so far within this Ansatz have a mass that is linear in the sum of the electric charges. It should be expected that for a generic non-supersymmetric supergravity solution this linear dependence should not be present. Very few such more general non-BPS solutions are known [143, 89, 184, 151] and it would be quite interesting to find more of them. It is also worth exploring the limitations on the types of solutions that can be constructed via the "floating brane" Ansatz and to find new more general techniques for constructing non-BPS solutions.

An interesting open question is whether the solutions presented in this Chapter are stable. Since the solutions have the same asymptotics as a non-exrtremal black hole, one can expect that they will be unstable, it will be very interesting to understand the details of this putative instability. We have not performed the stability analysis of our solutions and we expect this to be a non-trivial task, see [127] for a discussion of the instability of the Schwarzschild instanton. It is known that the regular non-BPS solutions found in [143] are unstable [54]. It was later shown that this instability has a natural interpretation in terms of Hawking radiation [63, 64, 65, 6, 7]. It is tempting to speculate that if the non-BPS solutions presented here are unstable their instability should also be interpreted as Hawking radiation for the corresponding non-extremal black hole with the same asymptotic charges.

Conclusion

In this thesis we have studied in detail supersymmetric and non-supersymetric solutions of supergravity. The solutions are regular, horizonless and with the same charges and asymptotic structure as supersymmetric and non-supersymmetric black holes and black rings. The existence of a large number of such solutions provides insights into the structure of black hole microstates, the information paradox and the resolution of black hole singularities in string theory.

We used spectral flow as a solution generating technique which also provides an efficient way to related two and three charge solutions with a GH base. Using the physics of supertubes and spectral flow we have also discussed which of the two and three charge solutions with GH base are bound states. We also studied in detail probe supertubes in the background of various supersymmetric three-charge solutions. Using the DBI action we analyzed the fluctuations of the supertubes in a general three-charge solution with large magnetic dipole fluxes and uncovered a novel mechanism of entropy enhancement. We also argued how via this mechanism one may account for a large portion of the entropy of the three-charge black hole. In Chapter 5 we studied supergravity solutions with a four-dimensional base with a generic $U(1)$ isometry. We presented the first explicit example of such more general solution and showed that the use of ambipolar four-dimensional bases may be a general feature of the black hole microstate geometries. Finally we studied a large class of non-supersymmetric and non-extremal supergravity solutions by using a four-dimensional base which is a solution to the Euclidean Einstein-Maxwell theory. We presented an explicit five-parameter family of regular non-BPS solutions. Our construction suggests that there may be a large number of regular non-supersymmetric solutions with the same charges and asymptotics as non-supersymmetric black holes. These solutions will play an important role in understanding the microstates and singularity resolution of non-supersymmetric black holes.

Without trying to be comprehensive we will outline some interesting directions for future work:

In addition to spectral flow there are more sophisticated solution-generating techniques based on the U-duality groups of supergravities in four and five dimensions. One can generalize spectral flow to the full U-duality group, for the STU model this is $SO(4,4)$ [98, 130]. The general form of the BPS solutions with GH base is known and it is unlikley that these more general solution generating techniques will teach us anything new about the supersymmetric solutions. It is also not guaranteed that the orbit of the $SO(4,4)$ U-duality group will preserve the regularity of the microstate geometries., which was one of the features of spectral flow. Nevertheless it will be very interesting to apply the $SO(4,4)$ U-duality group to generate new non-BPS solutions starting from the examples discussed in Chapter 6. Since the construction of new non-supersymmetric solutions is technically challenging it will be beneficial to construct as many new solutions as possible no matter if they are regular or not.

As pointed out in Chapter 4 it will be very interesting to find exact supergravity solutions that take into account the back-reaction of the wiggling supertubes which we studied only in the probe limit. Such a solution is hard to find for all oscillatory modes since we break a lot of the symmetry. However one can find the full backreacted solution for a supertube with fluctuating charge density. There is already some work in this direction with encouraging results and support for the entropy enhancement proposal [40].

One can also try to construct more general non-BPS solutions. One can use the Euclidean Melvin universe as a four-dimensional Einstein-Maxwell base. It has the same number of commuting Killing vectors as the Reissner-Nordström solution so finding the five-dimensional background should not be harder. It is interesting also to see whether there are more general linear equations of motion in the non-BPS floating brane Ansatz. Can one use four-dimensional Einstein-Maxwell-Dilaton solutions or four-dimensional solutions with cosmological constant as a base for construction five-dimensional non-supersymmetric solutions? There is a large class of metrics found by LeBrun [152] which can be turned into four-dimensional Euclidean Einstein-Maxwell solutions. These solutions are Kähler but not hyper-Kähler and have non-trivial two-cycles akin to the GH solutions. It will be quite interesting if one can construct a large number of non-BPS regular five-dimensional solutions with non-trivial topology based on the LeBrun metrics. There is some recent work in progress which suggests that this may indeed be possible [46].

It is well known that higher derivative corrections to the gravitational action introduce corrections to the Bekenstein-Hawking entropy of a black hole [204]. It will be very interesting to study supergravity actions with higher derivative terms and analyze the effect of these corrections on the regular microstate geometries. Of course it will be also desirable to find a way to count such geometries and reproduce the Wald formula which incorporates all corrections to the Bekenstein-Hawking entropy.

It will be very interesting to clarify the implications for the OSV conjecture [174] posed by the existence of the regular microstate geometries. It is established by now that there should be some modifications to the original conjecture due to the presence of the scaling solutions discussed in Chapter 2, see [79].

It may be fruitful to apply techniques from the theory of crystal melting, toric geometry and Young diagrams [142] to count the solutions with a $U(1) \times U(1)$ invariant hyper-Kähler base. It will be then be possible to compare the results of this approach to the counting performed by geometric quantization in [74, 75, 13, 76].

It will be interesting to apply the ideas that lead to the construction of the regular microstates geometries to gauged supergravity in five dimensions and look for regular asymptotically AdS solutions. These will be of interest for black holes in AdS and for AdS/CFT. The BPS solutions of five-dimensional gauged supergravity were classified in [102] where it was shown that they have a four-dimensional Kähler base. There is also a known BPS black hole with finite entropy in gauged supergravity [132], which has a simple Käler base. It remains to be seen whether one can construct ambipolar solutions with a Kähler base with non-trivial topology which can be interpreted as microstates of this black hole. Of course it will also be interesting to study non-BPS solutions of gauged supergravity along the lines of Chapter 6.

It is clear that studying the puzzles of black hole physics has lead to a lot of advances in theoretical physics and in our understanding of the nature of space-time and quantum gravity. There is no doubt that pondering about black holes will uncover new surprises and exciting physics. We hope that the results summarized in this thesis will be an important step along the way.

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Appendix A: T-duality transformations

In this Appendix we summarize the T-duality transformation rules for type II theories with non-zero R-R fields. These rules are derived in [43] and can be considered a generalization of the Buscher rules [50]. In the expressions below we will adopt the conventions and notation of [170], the different R-R forms are denoted with $C^{(n)}$ and the fields obtained after the T-duality transformations are denoted with a tilde, $w = x_9$ is the M-theory compactification direction and x is the T-duality direction.

The set of bosonic fields in the low energy limit of M-theory, *i.e.* eleven-dimensional supergravity, are:

$$G_{\mu\nu} \qquad \text{and} \qquad A_{\mu\nu\rho}\,. \tag{A.1}$$

After the compactification along $w = x_9$ we are left with type IIA supergravity with the fields

$$g_{\mu\nu}, \qquad C^{(3)}_{\mu\nu\rho}, \qquad B_{\mu\nu}, \qquad C^{(1)}_{\mu}, \qquad \Phi\,, \tag{A.2}$$

which are related to the eleven-dimensional fields as follows (note that we are working in string frame):

$$\begin{aligned} &g_{\mu\nu} = \sqrt{G_{ww}}\left(G_{\mu\nu} + \frac{G_{\mu w}G_{\nu w}}{G_{ww}}\right), \qquad\qquad C^{(1)}_{\mu} = \frac{G_{\mu w}}{G_{ww}}\,, \\ &C^{(3)}_{\mu\nu\rho} = A_{\mu\nu\rho}\,, \qquad B_{\mu\nu} = A_{\mu\nu w}\,, \qquad \Phi = \frac{3}{4}\log(G_{ww})\,. \end{aligned} \tag{A.3}$$

The type IIB fields are:

$$g_{\mu\nu}, \qquad B_{\mu\nu}, \qquad \Phi, \qquad C^{(0)}, \qquad C^{(2)}_{\mu\nu}, \qquad C^{(4)}_{\mu\nu\rho\sigma}\,. \tag{A.4}$$

The T-duality rules for the metric and the NS-NS fields are:

$$\tilde{g}_{xx} = \frac{1}{g_{xx}}, \qquad \tilde{g}_{\mu x} = \frac{B_{\mu x}}{g_{xx}}, \qquad \tilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{g_{\mu x}g_{\nu x} - B_{\mu x}B_{\nu x}}{g_{xx}}\,,$$
$$\tilde{B}_{\mu x} = \frac{g_{\mu x}}{g_{xx}}, \qquad \tilde{B}_{\mu\nu} = B_{\mu\nu} - \frac{B_{\mu x}g_{\nu x} - g_{\mu x}B_{\nu x}}{g_{xx}}\,, \qquad \tilde{\Phi} = \Phi - \frac{1}{2}\log g_{xx}\,. \tag{A.5}$$

The R-R forms transform under T-duality as:

$$\tilde{C}^{(n)}_{\mu\ldots\nu\alpha x} = C^{(n-1)}_{\mu\ldots\nu\alpha} - (n-1)\frac{C^{(n-1)}_{[\mu\ldots\nu|x}g_{|\alpha]x}}{g_{xx}}\,,$$
$$\tilde{C}^{(n)}_{\mu\ldots\nu\alpha\beta} = C^{(n+1)}_{\mu\ldots\nu\alpha\beta x} + nC^{(n-1)}_{[\mu\ldots\nu\alpha}B_{\beta]x} + n(n-1)\frac{C^{(n-1)}_{[\mu\ldots\nu|x}B_{|\alpha|x}g_{|\beta]x}}{g_{xx}}\,. \tag{A.6}$$

Alternatively one can transform the RR field strengths as follows (for a detailed derivation of these rules see Appendix A of [147])

$$\tilde{F}^{(n)}_{\mu_1\ldots\mu_{n-1}x} = F^{(n-1)}_{\mu_1\ldots\mu_{n-1}} + (n-1)(-1)^n\frac{g_{x[\mu_1}F^{(n-1)}_{\mu_2\ldots\mu_{n-1}]x}}{g_{xx}}\,,$$
$$\tilde{F}^{(n)}_{\mu_1\ldots\mu_n} = F^{(n+1)}_{\mu_1\ldots\mu_n x} - n(-1)^n B_{x[\mu_1}F^{(n-1)}_{\mu_2\ldots\mu_n]} - n(n-1)\frac{B_{x[\mu_1}g_{\mu_2|x|}F^{(n-1)}_{\mu_3..\mu_n]x}}{g_{xx}}\,. \tag{A.7}$$

In the next Appendix we give the explicit transformations that take us from the M-theory duality frame used in Chapter 2, to solutions in other useful duality frames.

Appendix B: Three charge solutions in different duality frames

Here we will present the three-charge BPS solutions, discussed throughout this thesis, in different string theory duality frames. This is needed for the calculations presented in Chapter 3 and 4. The results in this Appendix were presented in [35].

Compactification along x_9

The first step is to compactify the eleven-dimensional solution, presented in Section 2, along x_9, in this way we obtain the following combination of "electric"[8]

$$N_1 : \; D2\,(56) \qquad N_2 : \; D2\,(78) \qquad N_3 : \; F1\,(z) \tag{B.1}$$

and "dipole" branes

$$n_1 : \; D4\,(y78z) \qquad n_2 : \; D4\,(y56z) \qquad n_3 : \; NS5\,(y5678) \tag{B.2}$$

in Type IIA. The numbers in the parentheses refer to spatial directions wrapped by the branes, N_i and n_i denote the electric charges and dipole charges, respectively. The label

[8]We are choosing x_9 to be the M-theory circle in order to match the conventions in the literature for the global signs of the B-field and the RR potentials for the BMPV black hole [49] and the supersymmetric black ring solutions [88].

y refers to the brane profile on the spatial base and from now on we will denote $x_{10} = z$. The ten-dimensional *string frame* metric is

$$ds_{10}^2 = -\frac{1}{Z_3\sqrt{Z_1 Z_2}}(dt+k)^2 + \sqrt{Z_1 Z_2}ds_4^2 + \frac{\sqrt{Z_1 Z_2}}{Z_3}dz^2 + \sqrt{\frac{Z_2}{Z_1}}(dx_5^2 + dx_6^2) + \sqrt{\frac{Z_1}{Z_2}}(dx_7^2 + dx_8^2) \quad \text{(B.3)}$$

The dilaton and the Kalb-Ramond field are

$$\Phi = \frac{1}{4}\log\left(\frac{Z_1 Z_2}{Z_3^2}\right), \qquad B = -A^{(3)} \wedge dz\,. \quad \text{(B.4)}$$

The RR ("electric") forms are

$$C^{(1)} = 0\,, \qquad C^{(3)} = A^{(1)} \wedge dx_5 \wedge dx_6 + A^{(2)} \wedge dx_7 \wedge dx_8\,, \quad \text{(B.5)}$$

and the four-form field strength is[9]

$$\widetilde{F}^{(4)} = dC^{(3)} + dB \wedge C^{(1)} = A^{(1)} \wedge dx_5 \wedge dx_6 + dA^{(2)} \wedge dx_7 \wedge dx_8 \quad \text{(B.6)}$$

$$= d\mathcal{F}^{(1)} \wedge dx_5 \wedge dx_6 + \mathcal{F}^{(2)} \wedge dx_7 \wedge dx_8\,, \quad \text{(B.7)}$$

where we have used the notation $\mathcal{F}^{(I)} = dA^{(I)}$. Now we will perform a chain of T-dualities in order to arrive at the desired frame.

T-duality along x_5

A T-duality along the x_5 direction brings us to Type IIB with the following sets of "electric"

$$N_1 :\ D1\,(6) \qquad N_2 :\ D3\,(578) \qquad N_3 :\ F1\,(z) \quad \text{(B.8)}$$

[9]Note that we are using the notation of [180] $\widetilde{F}^{(4)} = dC^{(3)} + dB \wedge C^{(1)}$.

and "dipole" branes

$$n_1 : \, D5\,(y578z) \qquad n_2 : \, D3\,(y6z) \qquad n_3 : \, NS5\,(y5678)\,. \tag{B.9}$$

The metric is

$$ds_{10}^2 = -\frac{1}{Z_3\sqrt{Z_1 Z_2}}(dt+k)^2 + \sqrt{Z_1 Z_2}ds_4^2 + \frac{\sqrt{Z_1 Z_2}}{Z_3}dz^2 + \sqrt{\frac{Z_2}{Z_1}}dx_6^2 + \sqrt{\frac{Z_1}{Z_2}}(dx_5^2 + dx_7^2 + dx_8^2)\,. \tag{B.10}$$

The other NS-NS fields are

$$\Phi = \frac{1}{4}\log\left(\frac{Z_1^2}{Z_3^2}\right), \qquad B = -A^{(3)} \wedge dz\,. \tag{B.11}$$

The RR field strengths are

$$\begin{aligned} F^{(3)} &= -\mathcal{F}^{(1)} \wedge dx_6\,, \\ \widetilde{F}^{(5)} &= \mathcal{F}^{(2)} \wedge dx_5 \wedge dx_7 \wedge dx_8 + \star_{10}(\mathcal{F}^{(2)} \wedge dx_5 \wedge dx_7 \wedge dx_8)\,, \end{aligned} \tag{B.12}$$

where in the expression for $\widetilde{F}^{(5)}$ we have added the Hodge dual piece by hand to ensure self-duality [190]. Note that if one is working in the "democratic formalism" (*i.e.* with both electric and magnetic field strengths) $\widetilde{F}^{(5)}$ will be automatically self-dual, however since we have chosen to T-dualize explicitly only the electric field strengths we have to add the self-dual piece by hand whenever we encounter a five-form field strength after T-dualizing a four-form field strength.

Using the form of the ten-dimensional metric (B.10) one can show that

$$\star_{10}\left(dA^{(2)}\wedge dx_5\wedge dx_7\wedge dx_8\right) = -\left(\frac{Z_2^5}{Z_1^3 Z_3^2}\right)^{1/4}\star_5\left(dA^{(2)}\wedge dz\wedge dx_6\right), \qquad \text{(B.13)}$$

where $\star_5$ is the Hodge dual on the five-dimensional subspace given by the metric

$$ds_5^2 = -\frac{1}{Z_3\sqrt{Z_1 Z_2}}(dt+k)^2 + \sqrt{Z_1 Z_2}\,ds_4^2\,. \qquad \text{(B.14)}$$

T-duality along x_6

Now perform T-duality along x_6 to get

$$N_1:\ D0 \qquad N_2:\ D4\,(5678) \qquad N_3:\ F1\,(z) \qquad \text{(B.15)}$$

"electric"

$$n_1:\ D6\,(y5678z) \qquad n_2:\ D2\,(yz) \qquad n_3:\ NS5\,(y5678) \qquad \text{(B.16)}$$

and "dipole" branes in Type IIA. The metric is

$$ds_{10}^2 = -\frac{1}{Z_3\sqrt{Z_1 Z_2}}(dt+k)^2 + \sqrt{Z_1 Z_2}\,ds_4^2 + \frac{\sqrt{Z_1 Z_2}}{Z_3}dz^2 + \sqrt{\frac{Z_1}{Z_2}}(dx_5^2+dx_6^2+dx_7^2+dx_8^2)\,. \qquad \text{(B.17)}$$

The dilaton and the Kalb-Ramond fields are

$$\Phi = \frac{1}{4}\log\left(\frac{Z_1^3}{Z_2 Z_3^2}\right), \qquad B = -A^{(3)}\wedge dz\,. \qquad \text{(B.18)}$$

The RR field strengths are

$$F^{(2)} = -\mathcal{F}^{(1)}\,, \qquad \widetilde{F}^{(4)} = -\left(\frac{Z_2^5}{Z_1^3 Z_3^2}\right)^{1/4} \star_5 \left(\mathcal{F}^{(2)}\right) \wedge dz\,. \tag{B.19}$$

Since we are interested in studying probe two charge supertubes in this background, we will also need the RR potentials since they enter the Wess-Zumino action of the supertube.

Finding the RR and NS-NS potentials in the D0-D4-F1 frame

If everything is consistent, then the Bianchi identities for the field strengths should be satisfied. For the solution given by (3.3)–(3.5), the non-trivial Bianchi identity is:[10]

$$d\widetilde{F}^{(4)} = -F^{(2)} \wedge dB\,. \tag{B.20}$$

Indeed we can use the BPS equations to show that

$$\begin{aligned} d\widetilde{F}^{(4)} &= -d\left(\left(\frac{Z_2^5}{Z_1^3 Z_3^2}\right)^{1/4} \star_5 \left(\mathcal{F}^{(2)}\right)\right) \wedge dz \\ &= -\left[d\left(\frac{1}{Z_1 Z_3}\right) \wedge dk \wedge (dt+k) - d\left(\frac{(dt+k)}{Z_1}\right) \wedge \Theta^3 \right. \\ &\qquad \left. - d\left(\frac{(dt+k)}{Z_3}\right) \wedge \Theta^1 + \Theta^3 \wedge \Theta^1\right] \wedge dz\,. \end{aligned} \tag{B.21}$$

On the other hand

$$\begin{aligned} F^{(2)} \wedge dB &= dA^{(1)} \wedge dA^{(3)} \wedge dz \\ &= \left[d\left(\frac{1}{Z_1 Z_3}\right) \wedge dk \wedge (dt+k) - d\left(\frac{(dt+k)}{Z_1}\right) \wedge \Theta^3 \right. \\ &\qquad \left. - d\left(\frac{(dt+k)}{Z_3}\right) \wedge \Theta^1 + \Theta^3 \wedge \Theta^1\right] \wedge dz\,. \end{aligned} \tag{B.22}$$

[10] See [180] p. 86.

So the Bianchi identity is obeyed and it can be checked in a similar manner that the equations of motion of type IIA supergravity are obeyed. This confirms the consistency of our calculations.

We will now find the RR three-form potential $C^{(3)}$ in the same duality frame. It satisfies the following differential equation

$$dC^{(3)} \equiv \tilde{F}^{(4)} + C^{(1)} \wedge H^{(3)} . \tag{B.23}$$

Note that this depends upon a gauge choice for $C^{(1)}$, we choose a gauge in which $C^{(1)}$ is vanishing at asymptotic infinity, namely[11]

$$C^{(1)} = -A^{(1)} - dt . \tag{B.24}$$

Computing explicitly one finds

$$dC^{(3)} = \big[\big(-\star_4 dZ_2 + B^{(1)}\wedge\Theta^{(3)}\big) - d\big(Z_3^{-1}(dt+k)\wedge B^{(1)} + dt\wedge A^{(3)}\big)\big]\wedge dx_5 , \tag{B.25}$$

and hence

$$C^{(3)} = -\big(\gamma + Z_3^{-1}(dt+k)\wedge B^{(1)} + dt\wedge A^{(3)}\big)\wedge dx_5 , \tag{B.26}$$

where

$$d\gamma = \big(\star_4 dZ_2 - B^{(1)}\wedge\Theta^{(3)}\big) . \tag{B.27}$$

So the calculation boils down to integrating for the 2-form γ. Up to this stage we have not assumed any particular form of the four-dimensional base space. If this space is

[11] We have fixed $Z_I \sim 1 + \mathcal{O}(r^{-1})$.

Gibbons-Hawking then the equation for γ can be integrated explicitly. Using the BPS supergravity solutions presented in Chapter 2 it is not hard to show that

$$
\begin{aligned}
\star_4 dZ_2 \;-\; B^{(1)} \wedge \Theta^{(3)} \;=\;& \big(-\partial_a Z_2 + K^1 \partial_a (V^{-1} K^3)\big)\, \frac{1}{2} \epsilon_{abc} (d\psi + A) \wedge dy^b \wedge dy^c \\
&-\; \xi^{(1)}_a \,\big(\partial_b (V^{-1} K^3)\big) (d\psi + A) \wedge dy^a \wedge dy^b \\
&+\; V \big(\vec{\xi}^{(1)} \cdot \vec{\nabla} (V^{-1} K^3)\big)\, dy^1 \wedge dy^2 \wedge dy^3 \,.
\end{aligned}
\tag{B.28}
$$

Recall that $Z_2 = L_2 + V^{-1} K^1 K^3$ and define $\vec{\zeta}$ by:

$$
\vec{\nabla} \times \vec{\zeta} \;\equiv\; -\vec{\nabla} L_2 \,,
\tag{B.29}
$$

then using

$$
\Omega^{(a)}_{\pm} = \hat{e}^1 \wedge \hat{e}^{a+1} \pm \frac{1}{2} \epsilon_{abc} \hat{e}^{b+1} \wedge \hat{e}^{c+1} \,,
\tag{B.30}
$$

one can show that:

$$
\begin{aligned}
\star_4 dZ_2 \;-\; B^{(1)} \wedge \Theta^{(3)} \;=\;& d\big[\big(- \zeta_a \;-\; V^{-1} K^3 \xi^{(1)}_a \big) \Omega^{(a)}_{-} \big] \\
&-\; \big(V\, \vec{\nabla} \cdot \vec{\zeta} \;+\; K^3\, \vec{\nabla} \cdot \vec{\xi}^{(1)} \big)\, dy^1 \wedge dy^2 \wedge dy^3 \,.
\end{aligned}
\tag{B.31}
$$

The last term is a multiple of the volume form on $\mathbb{R}^3$ and so is necessarily exact, however, it can be simplified if we chose a gauge for $\vec{\xi}^{(1)}$ and $\vec{\zeta}$:

$$
\vec{\nabla} \cdot \vec{\zeta} \;=\; \vec{\nabla} \cdot \vec{\xi}^{(1)} \;=\; 0 \,.
\tag{B.32}
$$

Then one has:

$$
\gamma \;=\; - \big[\big(\zeta_a \;+\; V^{-1} K^3 \xi^{(1)}_a \big)\, \Omega^{(a)}_{-} \big] \,.
\tag{B.33}
$$

Finally, let $\vec{r}_i = (y_1 - a_i, y_2 - b_i, y_3 - c_i)$ and let $F \equiv \frac{1}{r_i}$ and then define $\vec{w}$ by $\vec{\nabla} \times \vec{w} \equiv -\vec{\nabla} F$, then the standard solution for $\vec{w}$ is:

$$w = -\frac{y_3 - c_i}{r_i} \frac{(y_1 - a_i)\, dy_2 - (y_2 - b_i)\, dy_1}{((y_1 - a_i)^2 + (y_2 - b_i)^2)}\,. \tag{B.34}$$

It is elementary to verify that $\vec{\nabla} \cdot \vec{w} = 0$ and so this is the requisite gauge. Finally the explicit form of the RR three-form potential for a solution with GH base in the D0-D4-F1 frame is

$$C^{(3)} = \left(\zeta_a + V^{-1} K^3 \xi_a^{(1)}\right) \Omega_-^{(a)} \wedge dz - \left(Z_3^{-1}(dt + k) \wedge B^{(1)} + dt \wedge A^{(3)}\right) \wedge dz\,. \tag{B.35}$$

T-duality along z

Another T-duality along z transforms the system into D1-D5-P frame with

$$N_1 :\ D1\,(z) \qquad N_2 :\ D5\,(5678z) \qquad N_3 :\ P\,(z) \tag{B.36}$$

“electric”

$$n_1 :\ D5\,(y5678) \qquad n_2 :\ D1\,(y) \qquad n_3 :\ kkm\,(y5678z) \tag{B.37}$$

and “dipole” branes. The metric is

$$ds^2_{IIB} = -\frac{1}{Z_3\sqrt{Z_1 Z_2}}(dt + k)^2 + \sqrt{Z_1 Z_2}\, ds_4^2 + \frac{Z_3}{\sqrt{Z_1 Z_2}}(dz + A^3)^2 + \sqrt{\frac{Z_1}{Z_2}}(dx_5^2 + dx_6^2 + dx_7^2 + dx_8^2)\,. \tag{B.38}$$

The dilaton and the Kalb-Ramond field are:

$$\Phi = \frac{1}{2}\log\left(\frac{Z_1}{Z_2}\right), \qquad B = 0\,. \tag{B.39}$$

The RR three-form field strength (it is the only non-zero field strength) is:

$$F^{(3)} = -\left(\frac{Z_2^5}{Z_1^3 Z_3^2}\right)^{1/4} \star_5 (\mathcal{F}^{(2)}) - \mathcal{F}^{(1)} \wedge (dz - A^{(3)})\,. \tag{B.40}$$

We can also easily find the RR 2-form potential by T-dualizing (B.35)

$$\begin{aligned} C^{(2)} \;=\; & \big(\zeta_a \;+\; V^{-1} K^3 \xi_a^{(1)}\big)\, \Omega_-^{(a)} - \big(Z_3^{-1}(dt + k) \wedge B^{(1)} \;+\; dt \wedge A^{(3)}\big) \\ & + A^{(1)} \wedge (A^{(3)} - dz - dt) \;+\; dt \wedge (A^3 - dz)\,. \end{aligned} \tag{B.41}$$

Appendix C: Solutions in D1-D5-P frame and a decoupling limit

In this Appendix we consider the decoupling limit of the three-charge metric in the D1-D5-P duality frame (B.38). As shown in [22, 88], for a supersymmetric black ring, such a limit takes an asymptotically-flat solution into a solution that is asymptotically $AdS_3 \times S^3 \times T^4$, and is thus dual to a state or an ensemble of states in the D1-D5 CFT.

Like for three-charge black holes and black rings, one can take a similar limit for any of the regular three-charge solutions with GH base discussed in this thesis. This is achieved by sending $\alpha' \to 0$ and scaling the coordinates and the parameters of the solution in such a way that the type IIB metric scales as α'. At this point it is useful to give the form of the "electric" charges Q_I in terms of the parameters of the harmonic functions specifying the solution:

$$Q_I = -2C_{IJK} \sum_{j=1}^{N} \frac{\tilde{k}_j^J \tilde{k}_j^K}{q_j} \qquad \text{where} \qquad \tilde{k}_j^I = k_j^I - q_j \sum_{i=1}^{N} k_i^I \,. \tag{C.1}$$

The angular momenta are obtained by expanding the one-form k at infinity and one finds:

$$J_R \equiv J_1 + J_2 = C_{IJK} \sum_{j=1}^{N} \frac{\tilde{k}_j^I \tilde{k}_j^J \tilde{k}_j^K}{q_j^2}, \qquad J_L = J_1 - J_2 = 8 \Big| \sum_{j=1}^{N} \sum_{I=1}^{3} \tilde{k}_j^I \vec{y}^{(j)} \Big| \,, \tag{C.2}$$

where the $\vec{y}^{(j)}$ are the positions of the GH centers. The scaling with α' of the coordinates is the same as for the black hole solution

$$y_1 \sim y_2 \sim y_3 \sim (\alpha')^2 \,, \quad x_a \sim (\alpha')^{1/2}, \quad a = 5,6,7,8 \,, \quad t \sim z \sim \psi \sim (\alpha')^0 \,, \tag{C.3}$$

where we have written the four-dimensional base as a GH space (2.21).

The electric charges have also the same scaling as for the black hole:

$$Q_1 \sim Q_2 \sim \alpha', \qquad Q_3 \sim (\alpha')^2 \,. \tag{C.4}$$

Hence, to preserve the fact that the charges of bubbling solutions come entirely from magnetic fluxes, the latter need to scale as

$$k_j^1 \sim k_j^2 \sim \alpha', \qquad k_j^3 \sim (\alpha')^0 \tag{C.5}$$

In particular, we have $r^2 = y_1^2 + y_2^2 + y_3^2$, so $r \sim (\alpha')^2$. At infinity in the M-theory solution the functions Z_I behave like

$$Z_I \sim 1 + \frac{Q_I}{4r} + ... \tag{C.6}$$

and so

$$Z_1 \sim \frac{1}{\alpha'} \qquad Z_2 \sim \frac{1}{\alpha'} \qquad Z_3 \sim \text{const} \,. \tag{C.7}$$

So in the limit $\alpha' \to 0$ we can ignore the constant in Z_1 and Z_2 but we should keep it in Z_3. It can be shown that $k \sim A^3 \sim (\alpha')^0$ which finally leads to the desired scaling

$$ds^2_{IIB} \sim \alpha' \,. \tag{C.8}$$

After we have taken the $\alpha' \to 0$ limit we can take the large $r = \frac{\rho^2}{4}$ limit and switch to four-dimensional spherical polar coordinates (2.19), with radial coordinate ρ, in which we have:

$$ds^2_{IIB} \sim \frac{\rho^2}{\sqrt{Q_1 Q_2}}(-dt^2 + dz^2) + \sqrt{Q_1 Q_2}\frac{d\rho^2}{\rho^2} + \sqrt{Q_1 Q_2}(d\vartheta^2 + \sin^2\vartheta d\varphi_1^2 + \cos^2\vartheta d\varphi_2^2) + \sqrt{\frac{Q_1}{Q_2}} ds^2_{T^4} \quad \text{(C.9)}$$

where we have used the freedom to change $A^{(3)}$ by pure gauge transformations. This metric is indeed that of the product space $AdS_3 \times S^3 \times T^4$, where the radii of the AdS_3 and the S^3 are the same and are equal to $(Q_1 Q_2)^{1/4}$. Therefore, the bubbling solutions in the decoupling limit are asymptotic to $AdS_3 \times S^3 \times T^4$ and thus should be described by the D1-D5 CFT as expected[12].

Note that the asymptotic metric in the decoupling limit of any of the BPS solutions of Chapter 2 is the same as the metric of the three-charge BPS black hole in the decoupling limit. This implies that the ge=ometries we are analyzing have a field theory description in the same D1-D5 CFT as the three-charge black hole with identical electric charge. The same result was found for supersymmetric black rings [22, 88].

We should also emphasize that in the decoupling limit only the three-charge black holes and the two-charge supertubes have metrics that are everywhere locally $AdS_3 \times S^3 \times T^4$. A general BPS solution like a black ring or a horizonless bubbling solution will have non-trivial geometry and topology.

[12] See [74] for a discussion of a different decoupling limit in which some of these bubbling solutions become dual to microstates of the MSW CFT [162]

Appendix D: Units and conventions

Here we summarize some of the conventions used in the thesis (see [180, 177] for more details).

The tensions of the extended objects in string and M-theory are:

$$T_{F1} = \frac{1}{2\pi\alpha'} , \qquad T_{Dp} = \frac{1}{g_s(2\pi)^p(l_s)^{p+1}} , \qquad T_{NS5} = \frac{1}{g_s^2(2\pi)^5(l_s)^6} , \tag{D.1}$$

$$T_{M2} = \frac{1}{(2\pi)^2(l_{11})^3} , \qquad T_{M5} = \frac{1}{(2\pi)^5(l_{11})^6} \tag{D.2}$$

where $\alpha' = l_s^2$, l_s is the string length, g_s is the string coupling constant (in the particular duality frame in which one works) and l_D is the D-dimensional Planck length. The Newton's constant in different dimensions is

$$16\pi G_{11} = (2\pi)^8(l_{11})^9 , \quad 16\pi G_{10} = (2\pi)^7(g_s)^2(l_s)^8 , \quad 16\pi G_D = (2\pi)^{D-3}(l_D)^{D-2} . \tag{D.3}$$

One can show that

$$l_{11} = g_s^{1/3} l_s = g_s^{1/3}(\alpha')^{1/2} . \tag{D.4}$$

T-duality along a circle of radius R changes the coupling constants to:

$$\widetilde{R} = \frac{\alpha'}{R} , \qquad \tilde{g}_s = \frac{l_s}{R} g_s , \qquad \tilde{l}_s = l_s . \tag{D.5}$$

where $\widetilde{R}$ is the radius after T-duality.

When one compactifies M-theory on a circle of radius L_9, the coupling constants of the resulting type IIA string theory satisfy:

$$L_9 = g_s l_s . \tag{D.6}$$

If one compactifies M-theory on a T^6 (along the directions $5, 6, 7, 8, 9, 10$) and the radius of each circle is L_i ($i = \{5, 6, 7, 8, 9, 10\}$), the five-dimensional Newton's constant is

$$G_5 = \frac{G_{11}}{vol(T^6)} = \frac{G_{11}}{(2\pi)^6 L_5 L_6 L_7 L_8 L_9 L_{10}} = \frac{\pi}{4} \frac{(l_{11})^9}{L_5 L_6 L_7 L_8 L_9 L_{10}} \,. \tag{D.7}$$

The relations between the number of M2 and M5 branes, N_I and n_I, and the physical charges of the five-dimensional solution obtained by compactifying M-theory on a T^6, Q_I and q_I, are

$$Q_1 = \frac{(l_{11})^6}{L_7 L_8 L_9 L_{10}}\, N_1 \,, \qquad Q_2 = \frac{(l_{11})^6}{L_5 L_6 L_9 L_{10}}\, N_2 \,, \qquad Q_3 = \frac{(l_{11})^6}{L_5 L_6 L_7 L_8}\, N_3 \,, \tag{D.8}$$

$$q_1 = \frac{(l_{11})^3}{L_5 L_6}\, n_1 \,, \qquad q_2 = \frac{(l_{11})^3}{L_7 L_8}\, n_2 \,, \qquad q_3 = \frac{(l_{11})^3}{L_9 L_{10}}\, n_3 \,. \tag{D.9}$$

We will choose a system of units in which all three T^2 are of equal volume

$$L_5 L_6 = L_7 L_8 = L_9 L_{10} = (l_{11})^3 \equiv g_s l_s^3 \,, \tag{D.10}$$

note that this is a numerical identity and is not dimensionally correct since g_s is dimensionless. With this choice we will have

$$G_5 = \frac{\pi}{4} \,, \qquad Q_I = N_I \,, \qquad q_I = n_I \,. \tag{D.11}$$

and these identities hold in every duality frame we use in the paper. Furthermore we will choose

$$g_s l_s = 1 \,. \tag{D.12}$$

Since we are compactifying M-theory on L_9 we will have $L_9 = g_s l_s = 1$ and $L_{10} = l_s^2$, this implies (note that throughout the paper we put $L_{10} \equiv L_z$)

$$T_{D0} = 1 \, , \qquad 2\pi T_{F1} L_{10} = 1 \, , \qquad \text{and} \qquad \frac{2\pi T_{D2}}{T_{F1}} = 1 \, . \tag{D.13}$$

We have fixed $l_s = g_s^{-1}$ so that a lot of the various brane tension factors, appearing in the probe supertube calculations throughout the thesis, cancel. Note that with our choices g_s is still a free parameter but we have fixed the volume of the compactification torii.

Appendix E: Extremal Reissner-Nordström

An interesting limiting case of the solution presented in Section 6.2 is when the two horizons of the four-dimensional base coincide. This is the extremal Euclidean dyonic Reissner-Nordström background

$$ds_4^2 = \left(1 - \frac{m}{r}\right)^2 d\tau^2 + \left(1 - \frac{m}{r}\right)^{-2} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \,, \tag{E.1}$$

$$F = \frac{2q}{r^2}\, d\tau \wedge dr + 2p \sin\theta\, d\theta \wedge d\phi \,. \tag{E.2}$$

This background is a limit of the dyonic Reissner-Nordström black hole which is obtained by taking $m^2 = p^2 - q^2$. The two horizons degenerate and we have

$$r_+ = r_- = m \,. \tag{E.3}$$

The near horizon limit of the Lorentzian extremal Reissner-Nordström black hole is the Bertoti-Robinson solution which is $AdS_2 \times S^2$ with electric and magnetic flux [44]. In the Euclidean solution of interest the horizon has become a bolt of radius m and near the bolt we can set

$$r = m + \frac{m^2}{\rho^2} \,, \tag{E.4}$$

and rewrite the metric as

$$ds_{NH}^2 = m^2 \left(\frac{d\rho^2 + d\tau^2}{\rho^2} + d\theta^2 + \sin^2\theta d\phi^2\right) \,. \tag{E.5}$$

This is the metric on $H_2^+ \times S^2$, where H_2^+ is the Poincaré half plane and we have the following range of coordinates $\tau \in (-\infty, \infty)$ and $\rho \in (0, \infty)$. Note that we still have a finite size bolt (S^2) at $r = m$ on which we can put flux[13]. At asymptotic infinity the metric approaches the flat metric on $\mathbb{R}^4$. This should be contrasted with the case of the non-extremal Euclidean Reissner-Nordström black hole of Section 6.2, where we had to periodically identify the coordinate τ to get a regular metric near the outer horizon. The five-dimensional supegravity solution based on this four-dimensional base has the same warp factors and fluxes as the solution in Section 6.2, however one should remember to set $m^2 = p^2 - q^2$. The coordinate τ is non-compact but it is still an isometry of the five-dimensional solution. This means that we have the electric charges corresponding to the three $U(1)$ gauge fields smeared along τ. What happens effectively is that in the extremal limit the coordinate τ decompactifies and the five-dimensional solution is asymptotic to $\mathbb{R}^{1,4}$ and corresponds to a smeared distribution of charges along τ. With this in mind one can proceed in the same way as in Section 6.2 and compute the asymptotic charges and mass densities of the five-dimensional solution[14]

$$\begin{aligned}
Q_1 &= -4\pi \left(\frac{2(p+q)q_2}{m} + \gamma(q+q_2) \right) , \\
Q_2 &= -4\pi \left(\frac{2(p+q)q_1}{m} + \gamma(q+q_1) \right) , \\
Q_3 &= -4\pi \left(\frac{4q_1 q_2}{m} + \gamma(q_1+q_2+p-q) + \frac{2(p-q)(q+q_1+q_2)}{m} - \frac{4q_1q_2(p^2-q^2)}{m^3} \right) , \\
M_0 &= \frac{1}{16\pi G_5} \left(8\pi m + Q_1 + Q_2 + Q_3 \right) .
\end{aligned} \tag{E.6}$$

[13] Five dimensional solutions with an $H_2^+ \times S^2$ base are discussed in [82]. Note that since the construction in [82] is based on a four-dimensional Euclidean Israel-Wilson base the five-dimensional solutions they find preserve some supersysmmetry. The four-dimensional solutions constructed here do not have an Israel-Wilson base and have fluxes with no definite self-duality, therefore they are not BPS.

[14] Note that, since the τ coordinate is not compact anymore, we are now computing charge and mass densities.

It is clear from the dependence of the mass on the charges that we again have a non-BPS five-dimensional solution that has the same asymptotic charges as a non-extremal black hole. This may seem somewhat strange because we have started with an extremal four-dimensional solution, which is also known to be BPS[15]. There is nothing puzzling going on here, to get the five-dimensional solution we have added fluxes to the four-dimensional base which break the supersymmetry completely. In addition the difference between the mass and the sum of the electric charges corresponds to the "solitonic" contribution of the bolt, and therefore one should not expect to have a solution with the same charges as an extremal black hole.

[15]The Lorentzian extremal Reissner-Nordström solution is a BPS background interpolating between $AdS_2 \times S^2$ and $\mathbb{R}^{1,3}$. Going to the Euclidean regime does not spoil the supersymmetry of the solution which now interpolates between $H_2^+ \times S^2$ and $\mathbb{R}^4$, see for example [154].