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Deformation Quasi-Hopf Algebras of Non-semi simple Type from Cochain Twists

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Abstract Given a symmetric decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ of a semisimple Lie algebra $\mathfrak{g}$, we define the notion of a $\mathfrak{p}$-contractible quantized universal enveloping algebra (QUEA): for these QUEAs the contraction $\mathfrak{g} \to \mathfrak{g}_0$ making $\mathfrak{p}$ abelian is nonsingular and yields a QUEA of $\mathfrak{g}_0$. For a certain class of symmetric decompositions, we prove, by refining cohomological arguments due to Drinfel'd, that every QUEA of $\mathfrak{g}_0$ so obtained is isomorphic to a cochain twist of the undeformed envelope $\mathscr{U}(\mathfrak{g}_0)$. To do so we introduce the $\mathfrak{p}$-contractible Chevalley-Eilenberg complex and prove, for this class of symmetric decompositions, a version of Whitehead's lemma for this complex. By virtue of the existence of the cochain twist, there exist triangular quasi-Hopf algebras based on these contracted QUEAs and, in the approach due to Beggs and Majid, the dual quantized coordinate algebras admit quasi-associative differential calculi of classical dimensions. As examples, we consider κ-Poincaré in 3 and 4 spacetime dimensions.

1 Introduction

This paper is concerned with deformation quantizations of the universal enveloping algebras (UEAs) of a certain class of non-semisimple Lie algebras, and more particularly with proving that these deformations are *cochain twists* of their undeformed counterparts. The Lie algebras we consider have the property that they can be obtained by *contraction* of semisimple Lie algebras; among them is the Poincaré algebra, which is the case of clearest physical interest and will be the example we treat in detail.

Let us first recall the situation concerning twists of (semi)*simple* Lie algebras. For any simple Lie algebra $\mathfrak{g}$, the standard Drinfel'd-Jimbo quantization $\mathscr{U}_h(\mathfrak{g})$ comes equipped with a quasitriangular structure $\mathscr{R}$, which provides the isomorphisms required to turn its category of representations into a quasitensor category

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???. As a quasitriangular Hopf algebra, $(\mathscr{U}_h(\mathfrak{g}),\mathscr{R})$ is not twist-equivalent by any cocycle twist to $(\mathscr{U}(\mathfrak{g}),1\otimes 1)$, the undeformed UEA equipped with the usual Hopf algebra structure and trivial triangular structure. It cannot be, because $\mathscr{R}$ is strictly *quasi*triangular (i.e. $\mathscr{R}_{21}\neq\mathscr{R}^{-1}$) and the property of triangularity is preserved by twisting. The celebrated result of Drinfel'd **???** is that $\mathscr{U}_h(\mathfrak{g})$ and $\mathscr{U}(\mathfrak{g})$ *are* twist equivalent in the larger category of quasi-Hopf algebras. Here one drops the requirement that the twist element F obey the cocycle condition. Since this condition is what guarantees the preservation of coassociativity under twisting, a quasi-Hopf algebra may fail to be coassociative; but it does so in a controlled fashion, specified by the *coassociator* Φ. In the special case $\Phi = 1\otimes 1\otimes 1$ one recovers the definition of a Hopf algebra. Drinfel'd showed that $(\mathscr{U}_h(\mathfrak{g}),\mathscr{R},1\otimes 1\otimes 1)$ can be reached by a cochain twist starting from $(\mathscr{U}(\mathfrak{g}),\mathscr{R}_{\mathrm{KZ}},\Phi_{\mathrm{KZ}})$; that is, starting from the quasitriangular quasi-Hopf (qtqH) algebra obtained by equipping $\mathscr{U}(\mathfrak{g})$ with a certain R-matrix $\mathscr{R}_{\mathrm{KZ}}$ and coassociator Φ_{KZ} constructed from the monodromies of a Knizhnik-Zamolodchikov system of equations, which in turn depend on the quadratic Casimir $\mathbf{t}$ of $\mathfrak{g}$. One has $\mathscr{R}_{\mathrm{KZ}} = e^{h\mathbf{t}}$, where the Casimir is split over the tensor product.

An alternative possibility, discussed notably by Beggs and Majid **??**, is to start instead with $(\mathscr{U}(\mathfrak{g}),1^{\otimes 2},1^{\otimes 3})$ and twist by the same cochain F as in Drinfel'd's construction. What results is, necessarily, the same algebra and coproduct as $\mathscr{U}_h(\mathfrak{g})$, but now equipped with non-standard R-matrix $F_{21}F^{-1}$ and coassociator Φ^F (the coboundary of F, closely related to Φ_{KZ}). Φ^F is central in the sense that the coproduct of $\mathscr{U}_h(\mathfrak{g})$ is coassociative, but it is nevertheless non-trivial and thus $(\mathscr{U}_h(\mathfrak{g}),F_{21}F^{-1},\Phi^F)$ is a triangular but strictly *quasi*-Hopf algebra. Dually, the deformed function algebra $C_h(G)$ becomes a co-quasi-Hopf algebra which happens to be associative. The non-triviality of Φ^F is seen at the level of intertwiners of representations (the category of representations is symmetric but non-trivially monoidal). It also appears when one tries to construct a differential calculus on $C_h(G)$, and in fact this was one of the original motivations for considering the set-up: Beggs and Majid showed that, at least for semisimple $\mathfrak{g}$, the standard quantum groups $C_h(G)$ do not admit any bi-covariant associative differential calculus of classical dimensions in deformation theory. But, by the existence of the cochain twist, one can construct a quasi-associative differential calculus $\Omega(C_h(G))$ of classical dimensions **??**.

The results summarized above pertain to semisimple Lie algebras. To the authors' knowledge no systematic extension to quantized universal enveloping algebras (QUEAs) of general Lie algebras is known. The proof of the existence of Drinfel'd's twist element F relies on the vanishing of a certain cohomology class, which holds for semisimple Lie algebras but may fail more generally. Drinfel'd did show **?** that *any* qtqH QUEA is isomorphic to a cochain twist of the undeformed UEA of the underlying Lie algebra $\mathfrak{g}$. So the existence of a qtqH structure is sufficient as well as necessary for the existence of the twist. But when $\mathfrak{g}$ is not semi-simple, one has no general means of knowing whether the given QUEA admits a qtqH algebra structure.

In physics one is also concerned with non-semisimple Lie algebras. In particular, in trying to formulate non-commutative quantum field theory by (paralleling the usual approach in **?**) beginning with particles, regarded as irreducible representations of the algebra of spacetime symmetries, one is certainly interested in the Poincaré algebra $\mathfrak{iso}(1,n)$ and its deformations. Possibly the most well-known deformation of $\mathscr{U}(\mathfrak{iso}(1,n))$ is the θ-deformation, which is dual to the usual non-commutative coordinate algebra $[x_i,x_j] = \theta_{ij}$, with θ_{ij} constant. It is known to be twist-equivalent to $\mathscr{U}(\mathfrak{iso}(1,n))$ by a *cocycle* twist **??**, making it in a sense a rather mild deformation. The results presented here will apply, rather, to what is referred to as the κ-deformation of $\mathscr{U}(\mathfrak{iso}(1,n))$ **????????**. κ-Poincaré can be understood in more than one way. From one perspective, it arises as a certain bicrossproduct **?**, and this viewpoint allows for a nice geometrical interpretation discussed in **?**. Another formulation **???** is as a particular *contraction limit* of the appropriate real form of the standard Drinfel'd-Jimbo QUEA $\mathscr{U}_h(\mathfrak{so}(n+1,\mathbb{C}))$. It is this property which will be relevant in the present work.

The main idea, then, is to consider a class of QUEAs obtained by applying to (*e.g.* the standard) QUEAs $\mathscr{U}_h(\mathfrak{g})$ of semisimple Lie algebras $\mathfrak{g}$ a contraction procedure modelled on that used to obtain κ-Poincaré. As we recall in detail below, to every *symmetric decomposition* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ of a Lie algebra $\mathfrak{g}$ there is associated an Inönu-Wigner contraction, in which $\mathfrak{p}$ is rescaled to become an abelian ideal of the contracted Lie algebra $\mathfrak{g}_0$. Whenever the contraction procedure is non-singular at the level of the $\mathscr{U}_h(\mathfrak{g})$, this yields a quantization $\mathscr{U}_{h'}(\mathfrak{g}_0)$ of $\mathscr{U}(\mathfrak{g}_0)$. One must specify how the formal deformation parameter is rescaled in the contraction limit to produce the limiting parameter h'; obviously there are many possibilities, and we will consider the choice that ensures that κ-Poincaré is captured by our results.

We will show (theorem **??**) that, given a certain restriction on the allowed symmetric decomposition (see Definition **??**), every such QUEA $\mathscr{U}_{h'}(\mathfrak{g}_0)$ is isomorphic to a twist of the undeformed UEA $\mathscr{U}(\mathfrak{g}_0)$ by a cochain F_0. We do this by refining the cohomological arguments of Drinfel'd so as to prove that the twist element F which relates $\mathscr{U}(\mathfrak{g})$ and $\mathscr{U}_h(\mathfrak{g})$ can be chosen to be non-singular in the contraction limit.

Since $\mathscr{U}(\mathfrak{g}_0)$ can always be endowed with the trivial qtqH structure $\mathscr{R} = 1^{\otimes 2}$, $\Phi = 1^{\otimes 3}$, the existence of this twist F_0 means that one can certainly obtain (see Corollary **??** below) a triangular quasi-Hopf algebra $(\mathscr{U}_{h'}(\mathfrak{g}_0), (F_0)_{21}F_0^{-1}, \Phi^{F_0})$. And, from **??**, the deformed coordinate algebra $C_{h'}(G_0)$ dual to $\mathscr{U}_{h'}(\mathfrak{g}_0)$ will admit a quasi-associative bicovariant differential calculus of classical dimensions. It is a separate question whether $\mathscr{U}_{h'}(\mathfrak{g}_0)$ admits a quasitriangular Hopf algebra structure. In Sect. **??**, we give a necessary condition for such a structure to arise by contraction (see corollary **??**). Examples, in the case of κ-Poincaré, are in Sect. **??**.

The paper is organised as follows. In Sect. **??**, we recall the definition of *symmetric* semisimple Lie algebras. The important notion of contractibility is introduced in Sect. **??** after a brief reminder of the definitions of the filtered and graded algebras associated to UEAs. Sect. **??** is dedicated to the cohomology of associative algebras and Lie algebras. After a brief account of Hochschild and Chevalley-Eilenberg cohomology, we introduce the notion of contractible Chevalley-Eilenberg cohomology. We establish, in particular, the vanishing of the first contractible Chevalley-Eilenberg cohomology module for symmetric semisimple Lie algebras possessing the restriction property **??**. This will be crucial in proving the existence of a contractible twist. In Sect. **??**, the usual rigidity theorems for semisimple Lie algebras are then refined, with special regards to the contractibility of the structures. We construct, in particular, a contractible twist from every contractible QUEA of restrictive type to the undeformed UEA of the underlying Lie algebra. The actual contraction is performed in Sect. **??**. In Sect. **??** we comment on the implications of our mathematical results for the particular example of κ-Poincaré. We discuss how they are compatible with previous work and explain certain previous results.

Throughout Sects. **??** through 6, $\mathbb{K}$ denotes a field of characteristic zero.

2 Symmetric Decompositions of Lie Algebras

Let us briefly review some well-known facts concerning symmetric semisimple Lie algebras. Following **??**, we have

A symmetric Lie algebra is a pair $(\mathfrak{g}, \theta)$, where $\mathfrak{g}$ is a Lie algebra and $\theta : \mathfrak{g} \to \mathfrak{g}$ is an involutive (i.e. $\theta \circ \theta = \mathrm{id}$ and $\theta \neq \mathrm{id}$) automorphism of Lie algebras.

As $\theta \circ \theta = \mathrm{id}$, the eigenvalues of θ are $+1$ and -1. Let $\mathfrak{h} = \ker(\theta - \mathrm{id})$ and $\mathfrak{p} = \ker(\theta + \mathrm{id})$ be the corresponding eigenspaces. Every such θ thus defines a *symmetric decomposition* of $\mathfrak{g}$, *i.e.* a triple $(\mathfrak{g}, \mathfrak{h}, \mathfrak{p})$ such that

- $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra;
- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ as $\mathbb{K}$-modules;
- $[\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}$.

Any Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that is the fixed point set of some involutive automorphism will be referred to as a *symmetrizing* subalgebra. If, in addition, $\mathfrak{g}$ is semisimple then $\mathfrak{p}$ must be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the (non-degenerate) Killing form, and thus every given symmetrizing subalgebra $\mathfrak{h}$ uniquely determines $\mathfrak{p}$ and hence θ. In this case, we shall refer to $(\mathfrak{g}, \mathfrak{h})$ as a *symmetric pair*.

A symmetric semisimple Lie algebra $(\mathfrak{g}, \theta)$ is said to be *diagonal* if $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{v}$ for some semisimple Lie algebra $\mathfrak{v}$ and $\theta(x,y) = (y,x)$ for all $(x,y) \in \mathfrak{g}$. A symmetric Lie algebra *splits* into symmetric subalgebras $(\mathfrak{g}_i, \theta_i)_{i \in I}$ if $\mathfrak{g} = \bigoplus_{i \in I} \mathfrak{g}_i$ and the restrictions $\theta_{|\mathfrak{g}_i} = \theta_i$ for all $i \in I$.

Every symmetric semisimple Lie algebra $(\mathfrak{g}, \theta)$ splits into a diagonal symmetric Lie algebra $(\mathfrak{g}_d, \theta_d)$ and a collection of symmetric simple Lie subalgebras $(\mathfrak{g}_i, \theta_i)_{i \in I}$. A proof can be found in Chap. 8 of **?**. Lemma **??** allows for a complete classification of the symmetric semisimple Lie algebras; see **??**. It also follows that we have the following

Let $(\mathfrak{g}, \theta)$ be a symmetric semisimple Lie algebra and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ be the associated symmetric decomposition of $\mathfrak{g}$. Then $\mathfrak{h}$ is linearly generated by $[\mathfrak{p}, \mathfrak{p}]$.

Proof By virtue of Lemma **??**, it suffices to prove this result on symmetric simple Lie algebras and on diagonal symmetric Lie algebras. Let us first assume that $\mathfrak{g}$ is simple. The linear span of $[\mathfrak{p}, \mathfrak{p}]$ defines a non-trivial ideal in $\mathfrak{h}$ and $\mathrm{span}([\mathfrak{p}, \mathfrak{p}]) \oplus \mathfrak{p}$ therefore defines a non-trivial ideal in $\mathfrak{g}$. If we assume that $\mathfrak{g}$ is simple, it immediately follows that $\mathrm{span}([\mathfrak{p}, \mathfrak{p}]) = \mathfrak{h}$. Suppose now that $(\mathfrak{g}, \theta)$ is a diagonal symmetric Lie algebra, *i.e.* that there exists a semisimple Lie algebra $\mathfrak{v}$ such that $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{v}$ and $\theta(x,y) = (y,x)$ for all $(x,y) \in \mathfrak{g}$. In this case, we have a symmetric decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, where $\mathfrak{h}$ is the set of elements of the form (x,x) for all $x \in \mathfrak{v}$, whereas $\mathfrak{p}$ is the set of elements of the form $(x,-x)$ for all $x \in \mathfrak{v}$. We naturally have $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}$. Now, as $\mathfrak{v}$ is semisimple, it follows that for every $x \in \mathfrak{v}$, there exist $y, z \in \mathfrak{v}$ such that $x = [y,z]$. Then for all $(x,x) \in \mathfrak{h}$, we have $(x,x) = ([y,z],[y,z]) = [(y,y),(z,z)] = [(y,-y),(z,-z)]$. But both $(y,-y)$ and $(z,-z)$ are in $\mathfrak{p}$. □

3 Contractible QUEAs

3.1 Filtrations of the Universal Enveloping Algebra

Given a Lie algebra $\mathfrak{g}$ over $\mathbb{K}$, its universal enveloping algebra $\mathscr{U}(\mathfrak{g})$ is defined as the quotient of the graded tensor algebra $T\mathfrak{g} = \bigoplus_{n\geq 0} \mathfrak{g}^{\otimes n}$ by the two-sided ideal $\mathscr{I}(\mathfrak{g})$ generated by the elements of the form $x \otimes y - y \otimes x - [x,y]$, for all $x, y \in \mathfrak{g}$. This quotient constitutes a filtered $\mathbb{K}$-algebra, *i.e.* there exists an increasing sequence

$$\{0\} \subset F_0(\mathscr{U}(\mathfrak{g})) \subset \cdots \subset F_n(\mathscr{U}(\mathfrak{g})) \subset \cdots \subset \mathscr{U}(\mathfrak{g}), \tag{3.1}$$

such that[1]

$$\mathscr{U}(\mathfrak{g}) = \bigcup_{n\geq 0} F_n(\mathscr{U}(\mathfrak{g})) \qquad \text{and} \qquad F_n(\mathscr{U}(\mathfrak{g})) \cdot F_m(\mathscr{U}(\mathfrak{g})) \subset F_{n+m}(\mathscr{U}(\mathfrak{g})) \tag{3.2}$$

The elements of this sequence are, for all $n \in \mathbb{N}_0$,

$$F_n(\mathscr{U}(\mathfrak{g})) = \bigoplus_{m=0}^{n} \mathfrak{g}^{\otimes m} / \mathscr{I}(\mathfrak{g}). \tag{3.3}$$

In particular, $F_0(\mathscr{U}(\mathfrak{g})) = \mathbb{K}$ and $F_1(\mathscr{U}(\mathfrak{g})) = \mathbb{K} \oplus \mathfrak{g}$. Let us identify $\mathfrak{g}$ with its image under the canonical inclusion $\mathfrak{g} \hookrightarrow \mathscr{U}(\mathfrak{g})$, and further write $x_1 \cdots x_n$ for the equivalence class of $x_1 \otimes \cdots \otimes x_n$. In this notation, $F_n(\mathscr{U}(\mathfrak{g}))$ is linearly generated by elements that can be written as words composed of at most n symbols from $\mathfrak{g}$.

We define the left action of $\mathfrak{g}$ on $\mathfrak{g}^{\otimes n}$ by extending the adjoint action $x \triangleright x' = [x,x']$ of $\mathfrak{g}$ on $\mathfrak{g}$ as a derivation:

$$x \triangleright (x_1 \otimes \cdots \otimes x_n) = \sum_{i=1}^{n} x_1 \otimes \cdots \otimes [x,x_i] \otimes \cdots \otimes x_n \in \mathfrak{g}^{\otimes n}, \tag{3.4}$$

for all $x, x_1, \ldots, x_n \in \mathfrak{g}$. In this way we endow $T\mathfrak{g}$ with the structure of a left $\mathfrak{g}$-module. As the ideal $\mathscr{I}(\mathfrak{g})$ is stable under this action, the $F_n(\mathscr{U}(\mathfrak{g}))$ are also left $\mathfrak{g}$-modules. We therefore have a filtration of $\mathscr{U}(\mathfrak{g})$ not only as a $\mathbb{K}$-algebra, but also as a left $\mathfrak{g}$-module.

We will also need such a filtration on $(\mathscr{U}(\mathfrak{g}))^{\otimes 2}$. In fact, for all $m \in \mathbb{N}_0$, there is a $\mathbb{K}$-algebra filtration on the universal envelope $\mathscr{U}(\mathfrak{g}^{\oplus m})$ of the Lie algebra $\mathfrak{g}^{\oplus m}$, as defined above. If we endow $\mathfrak{g}^{\oplus m}$ with the structure of a left $\mathfrak{g}$-module according to

$$x \triangleright (x_1, \ldots, x_m) := ([x,x_1], \ldots, [x,x_m]) \,, \tag{3.5}$$

[1] Although $F_n(\mathscr{U}(\mathfrak{g})) \cdot F_m(\mathscr{U}(\mathfrak{g}))$ is usually strictly contained in $F_{n+m}(\mathscr{U}(\mathfrak{g}))$, it linearly generates the latter.

and extend this action to all of $\mathscr{U}(\mathfrak{g}^{\oplus m})$ as a derivation, then we have a filtration of $\mathscr{U}(\mathfrak{g}^{\oplus m})$ as a left $\mathfrak{g}$-module. But there is a natural isomorphism

$$\rho_m : \mathscr{U}(\mathfrak{g}^{\oplus m}) \xrightarrow{\sim} (\mathscr{U}(\mathfrak{g}))^{\otimes m} \tag{3.6}$$

of $\mathbb{K}$-algebras (see e.g. [**?**, Sect. 2.2]). This induces a left action of $\mathfrak{g}$ on $(\mathscr{U}(\mathfrak{g}))^{\otimes m}$ and a filtration of $(\mathscr{U}(\mathfrak{g}))^{\otimes m}$ as a left $\mathfrak{g}$-module. We write the elements of this filtration as $F_n\left((\mathscr{U}(\mathfrak{g}))^{\otimes m}\right)$.

Given now any symmetric decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}, \tag{3.7}$$

there is an associated bifiltration $(F_{n,m}(\mathscr{U}(\mathfrak{g})))_{n,m\in\mathbb{N}_0}$ of $\mathscr{U}(\mathfrak{g})$, *i.e.* a doubly increasing sequence

$$F_{n,m}(\mathscr{U}(\mathfrak{g})) \subset F_{n+1,m}(\mathscr{U}(\mathfrak{g})) \qquad \text{and} \qquad F_{n,m}(\mathscr{U}(\mathfrak{g})) \subset F_{n,m+1}(\mathscr{U}(\mathfrak{g})), \tag{3.8}$$

such that

$$\mathscr{U}(\mathfrak{g}) = \bigcup_{n,m\geq 0} F_{n,m}(\mathscr{U}(\mathfrak{g})) \qquad \text{and} \qquad F_{n,m}(\mathscr{U}(\mathfrak{g}))\cdot F_{k,l}(\mathscr{U}(\mathfrak{g})) \subset F_{n+k,m+l}(\mathscr{U}(\mathfrak{g})), \tag{3.9}$$

for all $n,m,k,l \in \mathbb{N}_0$. The elements of this sequence are, for all $n,m \in \mathbb{N}_0$,

$$F_{n,m}(\mathscr{U}(\mathfrak{g})) = \bigoplus_{p=0}^{n}\bigoplus_{q=0}^{m} \mathrm{Sym}\left(\mathfrak{h}^{\otimes p} \otimes \mathfrak{p}^{\otimes q}\right) / \mathscr{I}(\mathfrak{g}), \tag{3.10}$$

where, for all $n \in \mathbb{N}_0$ and all $\mathbb{K}$-submodules $X_1, \dots X_n \subset \mathfrak{g}$,

$$\mathrm{Sym}(X_1 \otimes \cdots \otimes X_n) = \bigoplus_{\sigma\in\Sigma_n} X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(n)} \tag{3.11}$$

is the direct sum over all permutations of submodules in the tensor product. Each $F_{n,m}(\mathscr{U}(\mathfrak{g}))$ is therefore the left $\mathfrak{h}$-module linearly generated by elements of $\mathscr{U}(\mathfrak{g})$ that can be written as words containing at most n symbols in $\mathfrak{h}$ and at most m symbols in $\mathfrak{p}$. In particular, $F_{1,0}(\mathscr{U}(\mathfrak{g})) = \mathbb{K} \oplus \mathfrak{h}$ and $F_{0,1}(\mathscr{U}(\mathfrak{g})) = \mathbb{K} \oplus \mathfrak{p}$. We also have, for all $m,n \in \mathbb{N}_0$,

$$F_{n,m}(\mathscr{U}(\mathfrak{g})) \subset F_{n+m}(\mathscr{U}(\mathfrak{g})) \qquad \text{and} \qquad F_n(\mathscr{U}(\mathfrak{g})) = \bigcup_{m=0}^{n} F_{n-m,m}(\mathscr{U}(\mathfrak{g})). \tag{3.12}$$

In complete analogy with the $F_n((\mathscr{U}(\mathfrak{g}))^{\otimes m})$, we can construct bifiltrations $F_{n,p}((\mathscr{U}(\mathfrak{g}))^{\otimes m})$ of all the m-fold tensor products of $\mathscr{U}(\mathfrak{g})$.

3.2 Symmetric tensors

Let $S(\mathfrak{g})$ be the graded algebra associated to the filtration of $\mathscr{U}(\mathfrak{g})$ by setting, for all $n \in \mathbb{N}_0$,

$$S_n(\mathfrak{g}) = F_n(\mathscr{U}(\mathfrak{g}))/F_{n-1}(\mathscr{U}(\mathfrak{g})) \qquad \text{and} \qquad S(\mathfrak{g}) = \bigoplus_{n\geq 0} S_n(\mathfrak{g}). \tag{3.13}$$

Since the $F_n(\mathscr{U}(\mathfrak{g}))$ are left $\mathfrak{g}$-modules, so are the $S_n(\mathfrak{g})$. The *symmetrization map*, $\mathrm{sym}: S(\mathfrak{g}) \to \mathscr{U}(\mathfrak{g})$, defined by

$$\mathrm{sym}(x_1 \cdots x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)} \tag{3.14}$$

for all $n \in \mathbb{N}_0$ and all $x_1, \ldots, x_n \in \mathfrak{g}$, constitutes an isomorphism of left $\mathfrak{g}$-modules[2]. The image of a given $S_n(\mathfrak{g})$ through sym is the $\mathfrak{g}$-module of symmetric tensors in $\mathfrak{g}^{\otimes n}$.

If now $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ is a symmetric decomposition, let

$$S_{m,n}(\mathfrak{g}) = F_{m,n}(\mathscr{U}(\mathfrak{g}))/F_{m+n-1}(\mathscr{U}(\mathfrak{g})), \tag{3.15}$$

for all $m,n \in \mathbb{N}_0$. These obviously constitute left $\mathfrak{h}$-modules. As such, they are isomorphic to the left $\mathfrak{h}$-modules of symmetric tensors in the $\mathrm{Sym}(\mathfrak{h}^{\otimes m} \otimes \mathfrak{p}^{\otimes n})$, which are linearly generated by totally symmetric words with exactly m symbols in $\mathfrak{h}$ and exactly n symbols in $\mathfrak{p}$. Note that these $\mathfrak{h}$-modules are mixed under the left $\mathfrak{p}$-action. Indeed, let $m,n \in \mathbb{N}_0$ be two non-negative integers and let $x \in S_{m,n}(\mathfrak{g})$. We have:

- if $m > 0$ and $n = 0$, then $\mathfrak{p} \triangleright x \in S_{m-1,n+1}(\mathfrak{g})$;
- if $m > 0$ and $n > 0$, then $\mathfrak{p} \triangleright x \in S_{m+1,n-1}(\mathfrak{g}) \oplus S_{m-1,n+1}(\mathfrak{g})$;
- if $m = 0$ and $n > 0$, then $\mathfrak{p} \triangleright x \in S_{m+1,n-1}(\mathfrak{g})$.

[2] Recall that we assume $\mathbb{K}$ has characteristic zero.

This is better represented by the following diagram in $S_{m+n}(\mathfrak{g})$.

$$\cdots @-->[rd]^{\mathfrak{p}\triangleright} \; S_{m+1,n-1}(\mathfrak{g}) @-->[ld]^{\mathfrak{p}\triangleright} @-->[rd]^{\mathfrak{p}\triangleright}[d]^{\mathfrak{h}\triangleright} \; S_{m,n}(\mathfrak{g}) \; S_{m-1,n+1}(\mathfrak{g}) @-->[ld]^{\mathfrak{p}\triangleright} @-->[rd]^{\mathfrak{p}\triangleright}[d]^{\mathfrak{h}\triangleright}$$

$$S_{m-2,n+2}(\mathfrak{g}) @-->[ld]^{\mathfrak{p}\triangleright}[d]^{\mathfrak{h}\triangleright} @-->[rd]^{\mathfrak{p}\triangleright} \quad \cdots @-->[ld]^{\mathfrak{p}\triangleright}$$

$$\cdots \quad S_{m+1,n-1}(\mathfrak{g}) \quad S_{m,n}(\mathfrak{g}) \quad S_{m-1,n+1}(\mathfrak{g})$$

$$S_{m-2,n+2}(\mathfrak{g}) \quad \cdots \tag{3.17}$$

Using the action (**??**) of $\mathfrak{g}$ on $\mathfrak{g}^{\oplus m}$ we have entirely analogous structures for $\mathfrak{g}^{\oplus m}$ with

$$S_{n,p}(\mathfrak{g}^{\oplus m}) = F_{n,p}(\mathscr{U}(\mathfrak{g}^{\oplus m}))/F_{n+p-1}(\mathscr{U}(\mathfrak{g}^{\oplus m})). \tag{3.18}$$

In view of (**??**), it follows that

$$S_{n,p}(\mathfrak{g}^{\oplus m}) \cong F_{n,p}\left((\mathscr{U}(\mathfrak{g}))^{\otimes m})\right)/F_{n+p-1}\left((\mathscr{U}(\mathfrak{g}))^{\otimes m})\right) \tag{3.19}$$

for all $n, p \in \mathbb{N}_0$. We shall therefore identify each $S_{n,p}(\mathfrak{g}^{\oplus m})$ with the left $\mathfrak{h}$-module of symmetric tensors on $(\mathscr{U}(\mathfrak{g}))^{\otimes m}$ containing exactly n factors in $\mathfrak{h}$ and p in $\mathfrak{p}$.

3.3 Symmetric invariants and the restriction property

For all $n, p \in \mathbb{N}_0$, let $S_n(\mathfrak{g}\oplus\mathfrak{g})^{\mathfrak{g}}$ be the set of $\mathfrak{g}$-invariant elements of the left $\mathfrak{g}$-module $S_n(\mathfrak{g}\oplus\mathfrak{g})$ and let $S_{n,p}(\mathfrak{g}\oplus\mathfrak{g})^{\mathfrak{h}}$ denote the set of $\mathfrak{h}$-invariant elements of the left $\mathfrak{h}$-module $S_{n,p}(\mathfrak{g}\oplus\mathfrak{g})$. We have the following two lemmas.

Let n and p be positive integers. Every $x \in S_{n-p,p}(\mathfrak{g}\oplus\mathfrak{g})^{\mathfrak{h}}$ such that $\mathfrak{p}\triangleright x \in S_{n-p+1,p-1}(\mathfrak{g}\oplus\mathfrak{g})$ is in the linear span of $S_{n-p,0}(\mathfrak{g}\oplus\mathfrak{g})^{\mathfrak{g}}\, S_{0,p}(\mathfrak{g}\oplus\mathfrak{g})^{\mathfrak{h}}$.

Proof Let $(h_i)_{i\in I}$ and $(p_j)_{j\in J}$ be ordered bases of $\mathfrak{h}\oplus\mathfrak{h}$ and $\mathfrak{p}\oplus\mathfrak{p}$ respectively. Every element $x \in S_{n-p,p}(\mathfrak{g}\oplus\mathfrak{g})$ can be written as

$$x = \sum_{i_1\le\cdots\le i_{n-p}} \sum_{j_1\le\cdots\le j_p} x_{i_1\ldots i_{n-p} j_1\ldots j_p}\, h_{i_1}\ldots h_{i_{n-p}} p_{j_1}\ldots p_{j_p},$$

where, for all $i_1,\ldots,i_{n-p} \in I$ and $j_1,\ldots,j_p \in J$, $x_{i_1\ldots i_{n-p} j_1\ldots j_p} \in \mathbb{K}$. Then, omitting the ordered sums, we have

$$\mathfrak{p}\triangleright x = x_{i_1\ldots i_{n-p} j_1\ldots j_p}\left[\mathfrak{p}\triangleright\left(h_{i_1}\ldots h_{i_{n-p}}\right) p_{j_1}\ldots p_{j_p} + h_{i_1}\ldots h_{i_{n-p}}\,\mathfrak{p}\triangleright\left(p_{j_1}\ldots p_{j_p}\right)\right].$$

Since $(\mathfrak{p}\triangleright x)\cap S_{n-p-1,p+1}(\mathfrak{g}\oplus\mathfrak{g}) = \{0\}$, we have

$$\mathfrak{p}\triangleright\left(x_{i_1\ldots i_{n-p} j_1\ldots j_p} h_{i_1}\ldots h_{i_{n-p}}\right) = 0,$$

for all $j_1 \le \cdots \le j_p \in J$; it follows that this quantity is also invariant under $[\mathfrak{p},\mathfrak{p}]$ and hence, by Lemma **??**, under $\mathfrak{h}$. Thus it is actually $\mathfrak{g}$-invariant. Introduce a basis $(y_k)_{k\in K}$ of the $\mathbb{K}$-module $S_{n-p,0}(\mathfrak{g}\oplus\mathfrak{g})^{\mathfrak{g}}$, so that we can write

$$x_{i_1\ldots i_{n-p} j_1\ldots j_p} h_{i_1}\ldots h_{i_{n-p}} = \sum_{k\in K} b_{k\, j_1\ldots j_p}\, y_k,$$

with $b_{k\,j_1\ldots j_p} \in \mathbb{K}$, for all $j_1 \leq \cdots \leq j_p \in J$. Now, as x is $\mathfrak{h}$-invariant, we also have

$$\mathfrak{h} \triangleright x = b_{k\,j_1\ldots j_p}\, y_k\, \mathfrak{h} \triangleright \left(p_{j_1} \ldots p_{j_p}\right) = 0.$$

This yields $\mathfrak{h} \triangleright \left(b_{k\,j_1\ldots j_p}\, p_{j_1} \ldots p_{j_p}\right) = 0$, for all $k \in K$. Introduce a basis $(z_l)_{l\in L}$ for the $\mathbb{K}$-module $S_{0,p}(\mathfrak{g}\oplus\mathfrak{g})^{\mathfrak{h}}$, so that we can write, for all $k \in K$,

$$b_{k\,j_1\ldots j_p}\, p_{j_1} \ldots p_{j_p} = \sum_{l\in L} a_{kl} z_l,$$

with $a_{kl} \in \mathbb{K}$ for all $k \in K$ and $l \in L$. Now, x can be rewritten as

$$x = \sum_{k\in K}\sum_{l\in L} a_{kl}\, y_k\, z_l,$$

with $y_k \in S_{n-p,0}(\mathfrak{g}\oplus\mathfrak{g})^{\mathfrak{g}}$ for all $k \in K$ and $z_l \in S_{0,p}(\mathfrak{g}\oplus\mathfrak{g})^{\mathfrak{h}}$ for all $l \in L$. □

Let us now restrict our attention to the class of symmetric Lie algebras encompassed by the following

We say that a symmetric semisimple Lie algebra $(\mathfrak{g},\theta)$ with associated symmetric decomposition $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{p}$ is of **restrictive type** (or has the **restriction property**) if and only if for all $p \in \mathbb{N}_0$, the projection from $\mathfrak{g}$ to $\mathfrak{p}$ maps $S_p(\mathfrak{g}\oplus\mathfrak{g})^{\mathfrak{g}}$ onto $S_{0,p}(\mathfrak{g}\oplus\mathfrak{g})^{\mathfrak{h}}$. This restriction property will be sufficient to allow us to prove a refined version of Whitehead's lemma in the next section. Note that it is similar to the so-called *surjection property* – namely that the restriction from $\mathfrak{g}$ to $\mathfrak{p}$ maps $S(\mathfrak{g})^{\mathfrak{g}}$ onto $S(\mathfrak{p})^{\mathfrak{h}}$ – which is known to hold for all classical symmetric Lie algebras **?** and which has proven useful in a number of contexts **??**. In our case we have, at least,

If a symmetric semisimple Lie algebra splits (as in Lemma **??**), in such a way that its simple factors are drawn only from the following classical families of simple symmetric Lie algebras:

$$\mathrm{AI}_{n>2} : (\mathfrak{su}(n), \mathfrak{so}(n))_{n>2}\,, \quad \mathrm{AII}_n : (\mathfrak{su}(2n), \mathfrak{sp}(2n))_{n\in\mathbb{N}^*}\,,$$
$$\mathrm{BDI}_{n>2,1} : (\mathfrak{so}(n+1), \mathfrak{so}(n))_{n>2},$$

then it is of restrictive type.

Proof See Appendix. □

3.4 Contractible homomorphisms of $\mathbb{K}[[h]]$-modules

Let $\mathbb{K}[[h]]$ denote the $\mathbb{K}$-algebra of formal power series in h with coefficients in the field $\mathbb{K}$ and let $\mathscr{U}(\mathfrak{g})[[h]]$ be the $\mathscr{U}(\mathfrak{g})$-algebra of formal power series in h with coefficients in $\mathscr{U}(\mathfrak{g})$. We have a natural $\mathbb{K}$-algebra monomorphism $i : \mathscr{U}(\mathfrak{g}) \hookrightarrow \mathscr{U}(\mathfrak{g})[[h]]$. There is also an epimorphism of $\mathbb{K}$-algebras $j : \mathscr{U}(\mathfrak{g})[[h]] \twoheadrightarrow \mathscr{U}(\mathfrak{g})$ such that $j\circ i = \mathrm{id}$ on $\mathscr{U}(\mathfrak{g})$. We shall therefore identify $\mathscr{U}(\mathfrak{g})$ with its image $i(\mathscr{U}(\mathfrak{g})) \subset \mathscr{U}(\mathfrak{g})[[h]]$. We shall also consider complete $\mathbb{K}[[h]]$-modules and it is assumed that the tensor products considered from now on are completed in the h-adic topology. In this subsection, we further assume that $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{p}$ is a symmetric decomposition.

Let $p \in \mathbb{Z}$, $m \in \mathbb{N}_0$ be integers. An element x of $(\mathscr{U}(\mathfrak{g}))^{\otimes m}[[h]]$ is $(p,\mathfrak{p})$-contractible if and only if there exists a collection $(x_n)_{n\in\mathbb{N}_0}$ of elements of $(\mathscr{U}(\mathfrak{g}))^{\otimes m}$ such that,

$$x = \sum_{n\geq 0} h^n x_n \tag{3.20}$$

and, for all $n \in \mathbb{N}_0$, there exists $l(n) \in \mathbb{N}_0$ such that $x_n \in F_{l(n),n+p}((\mathscr{U}(\mathfrak{g}))^{\otimes m})$. Similarly, a subset $X \subset (\mathscr{U}(\mathfrak{g}))^{\otimes m}[[h]]$ is $(p,\mathfrak{p})$-contractible if all its elements are, according to the previous definition. Note that for the sake of simplicity, we shall refer to $(0,\mathfrak{p})$-contractible elements or sets as $\mathfrak{p}$-contractible. Let us now define the notion of contractibility for $\mathbb{K}[[h]]$-module homomorphisms in $\mathrm{Hom}(\mathscr{U}(\mathfrak{g})^{\otimes m}[[h]],(\mathscr{U}(\mathfrak{g}))^{\otimes n}[[h]])$.

Let $r,s \in \mathbb{N}_0$ and $p \in \mathbb{Z}$ be integers. A homomorphism of $\mathbb{K}[[h]]$-modules $\phi : (\mathscr{U}(\mathfrak{g}))^{\otimes r}[[h]] \to (\mathscr{U}(\mathfrak{g}))^{\otimes s}[[h]]$ is $\mathfrak{p}$-contractible if and only if, for all $n,m \in \mathbb{N}_0$, $\phi(F_{n,m}(\mathscr{U}(\mathfrak{g})^{\otimes r}))$ is $(m,\mathfrak{p})$-contractible as a subset. Let us emphasize that for every $\mathfrak{p}$-contractible $\mathbb{K}[[h]]$-module homomorphism $\phi : (\mathscr{U}(\mathfrak{g}))^{\otimes r}[[h]] \to (\mathscr{U}(\mathfrak{g}))^{\otimes s}[[h]]$, there exists a collection $(\varphi_n)_{n\in\mathbb{N}_0}$ of $\mathbb{K}[[h]]$-module homomorphisms $\varphi_n : (\mathscr{U}(\mathfrak{g}))^{\otimes r}[[h]] \to (\mathscr{U}(\mathfrak{g}))^{\otimes s}[[h]]$ such that

$$\phi = \sum_{n\geq 0} h^n \varphi_n \tag{3.21}$$

and, for all $n,m,p \in \mathbb{N}_0$, there exists $l(n) \in \mathbb{N}_0$ such that $\varphi_n(F_{m,p}((\mathscr{U}(\mathfrak{g}))^{\otimes r})) \subseteq F_{l(n),n+p}((\mathscr{U}(\mathfrak{g}))^{\otimes s})$. The following two lemmas will be useful in the next sections.

Let ϕ and ψ be two $\mathfrak{p}$-contractible homomorphisms of $\mathbb{K}[[h]]$-modules. Then the $\mathbb{K}[[h]]$-module homomorphism $\phi \circ \psi$ is $\mathfrak{p}$-contractible.

Proof We have

$$\phi = \sum_{n\geq 0} h^n \varphi_n \qquad \text{and} \qquad \psi = \sum_{n\geq 0} h^n \psi_n,$$

with, for all $n,m,p \in \mathbb{N}_0$, $\varphi_n(F_{m,p}) \subseteq F_{*,n+p}$, and $\psi_n(F_{m,p}) \subseteq F_{*,n+p}$. For the sake of simplicity we shall omit the arguments of the bifiltration and denote by $*$ the integer $l(n)$ whose existence is guaranteed by the definition of contractibility. We thus have

$$\phi \circ \psi = \sum_{n\geq 0}\sum_{m\geq 0} h^{n+m} \varphi_n \circ \psi_m = \sum_{n\geq 0} h^n \sum_{m=0}^{n} \varphi_m \circ \psi_{n-m},$$

with, for all $l,m,n,p \in \mathbb{N}_0$, $\varphi_m \circ \psi_{n-m}(F_{l,p}) \subseteq \varphi_m(F_{*,n-m+p}) \subseteq F_{*,n+p}$. □

The following holds for the inverse.

Let ϕ be a $\mathfrak{p}$-contractible homomorphism of $\mathbb{K}[[h]]$-modules, congruent with $\mathrm{id} \mod h$. Then the $K[[h]]$-module homomorphism $\phi^{-1} = \mathrm{id} \mod h$ is $\mathfrak{p}$-contractible.

Proof We shall construct

$$\phi^{-1} = \sum_{n\geq 0} h^n \varphi_n,$$

by recursion on the order in h, by demanding that $\phi \circ \phi^{-1} = \mathrm{id}$. At leading order, we have $\varphi_0 = \mathrm{id}$ and therefore $\varphi_0(F_{m,p}) \subseteq F_{m,p}$, for all $m, p \in \mathbb{N}_0$. Let us assume that we have a polynomial ϕ_n^{-1} of degree $n > 0$ such that

$$\phi \circ \phi_n^{-1} - \mathrm{id} = q \mod h^{n+1}.$$

Assuming that ϕ_n^{-1} is $\mathfrak{p}$-contractible, we have by Lemma **??** that $\phi \circ \phi_n^{-1}$ is $\mathfrak{p}$-contractible, as ϕ is $\mathfrak{p}$-contractible by assumption. Therefore, $q(F_{m,p}) \subseteq F_{*,n+1+p}$. Now, to complete the recursion, we have to find φ_{n+1} such that

$$\phi \circ \left(\phi_n^{-1} + h^{n+1}\varphi_{n+1}\right) - \mathrm{id} = 0 \mod h^{n+2}.$$

This is achieved by taking $\varphi_{n+1} = -q$. We thus have $\varphi_{n+1}(F_{m,p}) \subseteq F_{*,n+1+p}$. □

Finally, when ϕ is not only a $\mathbb{K}[[h]]$-module homomorphism but also a $\mathbb{K}[[h]]$-algebra homomorphism, we have the following useful lemma.

Let $\phi : (\mathscr{U}(\mathfrak{g}))^{\otimes s}[[h]] \to (\mathscr{U}(\mathfrak{g}))^{\otimes t}[[h]]$ be a homomorphism of $\mathbb{K}[[h]]$-algebras. It is $\mathfrak{p}$-contractible if and only if $\phi\left(F_{1,0}((\mathscr{U}(\mathfrak{g}))^{\otimes s})\right)$ is $(0,\mathfrak{p})$-contractible and $\phi\left(F_{0,1}((\mathscr{U}(\mathfrak{g}))^{\otimes s})\right)$ is $(1,\mathfrak{p})$-contractible.

Proof If ϕ is $\mathfrak{p}$-contractible, it follows from the definition that, in particular, $\phi(F_{1,0})$ is $(0,\mathfrak{p})$-contractible and $\phi(F_{0,1})$ is $(1,\mathfrak{p})$-contractible. Now, assuming that $\phi(F_{1,0})$ is $(0,\mathfrak{p})$-contractible and $\phi(F_{0,1})$ is $(1,\mathfrak{p})$-contractible, we want to prove that, for all $m, p \in \mathbb{N}_0$, $\phi(F_{m,p})$ is $(p,\mathfrak{p})$-contractible. We proceed by recursion on m and p. We have assumed the result for $m = 1$ and $p = 0$, as well as for $m = 0$ and $p = 1$. Suppose that, for some $m, p \in \mathbb{N}_0$, we have proven that, for all $m' < m$, $p' < p$ and $n \in \mathbb{N}_0$, there exists $l \in \mathbb{N}_0$ such that $\varphi_n\left(F_{m',p'}\right) \subseteq F_{l,n+p'}$. Then, for all $n \in \mathbb{N}_0$,

$$\begin{aligned}
\varphi_n\left(F_{m,p+1}((\mathscr{U}(\mathfrak{g}))^{\otimes s})\right) &= \varphi_n\left(\bigoplus_{k=0}^{m}\bigoplus_{l=0}^{p} \mathrm{span} \quad F_{k,l}\cdot F_{0,1}\cdot F_{m-k,p-l}\right)\\
&= \bigoplus_{k=0}^{m}\bigoplus_{l=0}^{p} \mathrm{span} \sum_{\sigma\in C_3(n)} \varphi_{\sigma_1}\left(F_{k,l}\right)\cdot\varphi_{\sigma_2}\left(F_{0,1}\right)\cdot\varphi_{\sigma_3}\left(F_{m-k,p-l}\right)\\
&\subseteq \bigoplus_{k=0}^{m}\bigoplus_{l=0}^{p} \mathrm{span}_{\sigma\in C_3(n)} \quad F_{*,\sigma_1+l}\cdot F_{*,\sigma_2+1}\cdot F_{*,\sigma_3+p-l}\\
&= F_{*,n+p+1},
\end{aligned}$$

where, for all $X \subseteq (\mathscr{U}(\mathfrak{g}))^{\otimes s}$, span X denotes the $\mathbb{K}$-module linearly generated by X and $C_3(n)$ is the set $\{\sigma = (\sigma_1,\sigma_2,\sigma_3) \in \mathbb{N}_0^3 : \sum_{i=1}^{3}\sigma_i = n\}$ of weak 3-compositions of n.

Similarly, we have

$$\begin{aligned}
\varphi_n\left(F_{m+1,p}((\mathscr{U}(\mathfrak{g}))^{\otimes s})\right) &= \varphi_n\left(\bigoplus_{k=0}^{m}\bigoplus_{l=0}^{p}\operatorname{span}\quad F_{k,l}\cdot F_{1,0}\cdot F_{m-k,p-l}\right)\\
&= \bigoplus_{k=0}^{m}\bigoplus_{l=0}^{p}\operatorname{span}\sum_{\sigma\in C_3(n)}\varphi_{\sigma_1}\left(F_{k,l}\right)\cdot\varphi_{\sigma_2}\left(F_{1,0}\right)\cdot\varphi_{\sigma_3}\left(F_{m-k,p-l}\right)\\
&\subseteq \bigoplus_{k=0}^{m}\bigoplus_{l=0}^{p}\operatorname{span}_{\sigma\in C_3(n)}\quad F_{*,\sigma_1+l}\cdot F_{*,\sigma_2}\cdot F_{*,\sigma_3+p-l} = F_{*,n+p},
\end{aligned}$$

for all $n\in\mathbb{N}_0$. □

3.5 Contractible deformation Hopf algebras

We recall that $\mathscr{U}(\mathfrak{g})$ possesses a natural cocommutative Hopf algebra structure, whose coproduct is the algebra homomorphism $\Delta_0 : \mathscr{U}(\mathfrak{g}) \to \mathscr{U}(\mathfrak{g})\otimes\mathscr{U}(\mathfrak{g})$ defined by $\Delta_0(x) = x\otimes 1 + 1\otimes x$ for all $x\in\mathfrak{g}$, and whose counit and antipode are specified by $\varepsilon_0(1) = 1$ and $S_0(1) = 1$. We refer to this as the *undeformed* Hopf algebra structure.

Given the notion of contractibility introduced in the preceding subsections, it is natural to specialize the usual notion of a quantization – *i.e.* a deformation – of a universal enveloping algebra, as follows.

Let $(\mathfrak{g},\theta)$ be a symmetric Lie algebra, with symmetric decomposition $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{p}$. A $\mathfrak{p}$-contractible deformation $(\mathscr{U}_h(\mathfrak{g}),\cdot_h,\Delta_h,\varepsilon_h,S_h)$ of the Hopf algebra $(\mathscr{U}(\mathfrak{g}),\cdot,\Delta_0,\varepsilon_0,S_0)$ is a topological Hopf algebra such that

- there exists a $\mathbb{K}[[h]]$-module isomorphism $\eta : \mathscr{U}_h(\mathfrak{g}) \xrightarrow{\sim} \mathscr{U}(\mathfrak{g})[[h]]$;
- $\mu_h := \eta\circ(\cdot_h)\circ\left(\eta^{-1}\otimes\eta^{-1}\right) = \cdot \mod h$ and μ_h is $\mathfrak{p}$-contractible;
- $\tilde{\Delta}_h := (\eta\otimes\eta)\circ\Delta_h\circ\eta^{-1} = \Delta_0 \mod h$ and $\tilde{\Delta}_h$ is $\mathfrak{p}$-contractible;
- $\tilde{S}_h := \eta\circ S_h\circ\eta^{-1} = S_0 \mod h$ and $\tilde{S}_h$ is $\mathfrak{p}$-contractible;
- $\tilde{\varepsilon}_h = \varepsilon_h\circ\eta^{-1} = \varepsilon_0 \mod h$ and $\tilde{\varepsilon}_h$ is $\mathfrak{p}$-contractible.

This definition can be naturally restricted to bialgebras and algebras.

4 On the Cohomology of Associative and Lie Algebras

4.1 The Hochschild cohomology

Let A be a $\mathbb{K}$-algebra. For any (A,A)-bimodule $(M,\triangleright,\triangleleft)$ and all $n\in{\mathbb{N}_0}^*$, we define the (A,A)-bimodule of n-cochains $C^n(A,M) = \mathrm{Hom}(A^{\otimes n},M)$. We also set $C^0(A,M) = M$. To each cochain module $C^n(A,M)$, we associate a coboundary operator, *i.e.* a derivation operator $\delta_n : C^n(A,M) \longrightarrow C^{n+1}(A,M)$, by setting, for all

$f \in C^n(A,M)$,

$$\begin{aligned}\delta_n f(x_1,\dots,x_{n+1}) = {} & x_1 \triangleright f(x_2,\dots,\hat{x}_i,\dots,x_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i f(x_1,\dots,x_i x_{i+1},\dots,x_{n+1}) \\ & + (-1)^{n+1} f(x_1,\dots,x_n) \triangleleft x_{n+1} \end{aligned} \tag{4.1}$$

for all $x_1,\dots,x_{n+1} \in A$. One can check that $\delta_n \circ \delta_{n+1} = 0$ for all n. Therefore, the (C^n, δ_n) thus defined constitute a cochain complex. It is known as the Hochschild or *standard* complex **??** – see also **?** or **?**. An element of the (A,A)-bimodule $Z^n(A,M) = \ker \delta_n \subset C^n(A,M)$ is called an *n-cocycle*, while an element of the (A,A)-bimodule $B^n(A,M) = \operatorname{im} \delta_{n-1} \subset C^n(A,M)$ is called an *n-coboundary*. As usual, the quotient

$$HH^n(A,M) = Z^n(A,M)/B^n(A,M) \tag{4.2}$$

defines the n^{th} cohomology module of A with coefficients in M. In the next section, we shall be particularly interested in the Hochschild cohomology of the universal enveloping algebra of a given Lie algebra $\mathfrak{g}$, *i.e.* $A = \mathscr{U}(\mathfrak{g})$, with coefficients in $M = \mathscr{U}(\mathfrak{g})$. The latter trivially constitutes a $(\mathscr{U}(\mathfrak{g}),\mathscr{U}(\mathfrak{g}))$-bimodule with the multiplication $\cdot$ of $\mathscr{U}(\mathfrak{g})$ as left and right $\mathscr{U}(\mathfrak{g})$-action. Concerning the Hochschild cohomology we will need the following result – see for example Theorem 6.1.8 in **?**.

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{K}$. Then, $HH^2(\mathscr{U}(\mathfrak{g}),\mathscr{U}(\mathfrak{g})) = 0$.

4.2 The Chevalley-Eilenberg cohomology

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and $(M,\triangleright)$ a left $\mathfrak{g}$-module. For all $n \in \mathbb{N}_0{}^*$, we define the left $\mathfrak{g}$-module of n-cochains $C^n(\mathfrak{g},M) = \operatorname{Hom}(\wedge^n \mathfrak{g},M)$, with left $\mathfrak{g}$-action

$$(x \triangleright f)(x_1,\dots,x_n) = x \triangleright (f(x_1,\dots,x_n)) - \sum_{i=1}^{n} f(x_1,\dots,[x,x_i],\dots,x_n), \tag{4.3}$$

for all $f \in C^n(\mathfrak{g},M)$ and all $x,x_1,\dots,x_n \in \mathfrak{g}$. We also set $C^0(\mathfrak{g},M) = M$ with its natural left $\mathfrak{g}$-module structure. To each cochain module $C^n(\mathfrak{g},M)$, we associate a coboundary operator, *i.e.* a derivation operator $d_n : C^n(\mathfrak{g},M) \longrightarrow C^{n+1}(\mathfrak{g},M)$, by setting, for all $f \in C^n(\mathfrak{g},M)$,

$$\begin{aligned} d_n f(x_1,\dots,x_{n+1}) = {} & \sum_{i=1}^{n+1} (-1)^{i+1} x_i \triangleright f(x_1,\dots,\hat{x}_i,\dots,x_{n+1}) \\ & + \sum_{1 \le i \le j \le n+1} (-1)^{i+j} f([x_i,x_j],x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_{n+1}) \end{aligned} \tag{4.4}$$

for all $x_1,\dots,x_{n+1} \in \mathfrak{g}$. In (**??**), hatted quantities are omitted and $\triangleright$ denotes the left $\mathfrak{g}$-action on M. One can check that $d_n \circ d_{n+1} = 0$ for all n. Therefore, the (C^n, d_n) thus defined constitute a cochain complex. It is known as the Chevalley-Eilenberg complex **?**, – see also **?** or **?**. An element of $Z^n(\mathfrak{g},M) = \ker d_n \subset C^n(\mathfrak{g},M)$ is called an *n-cocycle*, while an element of $B^n(\mathfrak{g},M) = \operatorname{im} d_{n-1} \subset C^n(\mathfrak{g},M)$ is called an *n-coboundary*. As usual, the quotient

$$H^n(\mathfrak{g},M) = Z^n(\mathfrak{g},M)/B^n(\mathfrak{g},M) \tag{4.5}$$

defines the n^{th} cohomology module of $\mathfrak{g}$ with coefficients in M. One can check that, for all $n \in \mathbb{N}_0$, $Z^n(\mathfrak{g},M)$, $B^n(\mathfrak{g},M)$ and $H^n(\mathfrak{g},M)$ naturally inherit the left $\mathfrak{g}$-module structure of $C^n(\mathfrak{g},M)$, as for all $n \in \mathbb{N}_0$,

$$d\,(x \triangleright f) = x \triangleright df, \tag{4.6}$$

for all $f \in C^n(\mathfrak{g},M)$ and all $x \in \mathfrak{g}$. An important result about the Chevalley-Eilenberg cohomology of Lie algebras concerns finite dimensional complex semisimple Lie algebras. It is known as Whitehead's Lemma.

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{K}$. If M is any finite-dimensional left $\mathfrak{g}$-module, then $H^1(\mathfrak{g},M) = H^2(\mathfrak{g},M) = 0$. A proof of this result can be found, for instance, in Sect. 7.8 of **?**.

4.3 Contractible Chevalley-Eilenberg cohomology

In the next section, we will be mostly interested in the module $M = \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})$, with the left $\mathfrak{g}$-action induced by (**??**) and (**??**), *i.e.*

$$g \triangleright x = [\Delta_0(g), x], \tag{4.7}$$

for all $g \in \mathfrak{g}$ and all $x \in \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})$. In particular, we shall need a refinement of Whitehead's Lemma, in the case of symmetric semisimple Lie algebras of restrictive type, taking into account the possible $\mathfrak{p}$-contractibility of the generating cocycles of $Z^*(\mathfrak{g}, \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))$. For all $m,n \in \mathbb{N}_0$, we therefore define $C^n_{m,\mathfrak{p}}(\mathfrak{g}, \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))$ as the set of $(m,\mathfrak{p})$-contractible n-cochains, by which we mean the set of n-cochains $f \in C^n(\mathfrak{g}, \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))$, such that, for all $0 \le p \le n$, $f\left((\wedge^{n-p}\mathfrak{h}) \wedge (\wedge^p \mathfrak{p})\right) \subseteq F_{l,m+p}(\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))$, for some $l \in \mathbb{N}_0$. Defining similarly, $Z^n_{m,\mathfrak{p}}(\mathfrak{g}, \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})) = \ker d_n \cap C^n_{m,\mathfrak{p}}(\mathfrak{g}, \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))$ and $B^n_{m,\mathfrak{p}}(\mathfrak{g}, \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})) = d_{n-1} C^{n-1}_{m,\mathfrak{p}}(\mathfrak{g}, \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))$ as the modules of the $(m,\mathfrak{p})$-contractible n-cocycles and of the n-coboundaries of $(m,\mathfrak{p})$-contractible $n-1$-cochains, respectively, we can define the n^{th} $(m,\mathfrak{p})$-contractible cohomology module as

$$H^n_{m,\mathfrak{p}}(\mathfrak{g}, \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})) = Z^n_{m,\mathfrak{p}}(\mathfrak{g}, \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))/B^n_{m,\mathfrak{p}}(\mathfrak{g}, \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})) \tag{4.8}$$

It is worth emphasizing that these cohomology modules generally differ from the usual ones $H^n(\mathfrak{g}, \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))$. Consider for instance a case for which $H^1(\mathfrak{g}, \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})) = 0$. We have that every 1-cocycle in $Z^1(\mathfrak{g}, \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))$, and therefore every cocycle $f \in Z^1_{m,\mathfrak{p}}(\mathfrak{g}, \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))$, is the coboundary of an element $x \in \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})$. However, although the considered f is $(m,\mathfrak{p})$-contractible, it may be that it can only be obtained as the coboundary of an element $x \in \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})$ that does not belong to any $F_{*,m}(\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))$, thus yielding a non-trivial cohomology class in $H^1_{m,\mathfrak{p}}(\mathfrak{g}, \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))$. When $\mathfrak{g}$ is a symmetric semisimple Lie algebra of restrictive type, we nonetheless establish the following lemma concerning the first $(m,\mathfrak{p})$-contractible cohomology module $H^1_{m,\mathfrak{p}}(\mathfrak{g},\mathscr{U}(\mathfrak{g})\otimes\mathscr{U}(\mathfrak{g}))$.

Let $(\mathfrak{g},\theta)$ be a symmetric semisimple Lie algebra of restrictive type over $\mathbb{K}$ and let $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{p}$ be the associated symmetric decomposition of $\mathfrak{g}$. We have $H^1_{m,\mathfrak{p}}(\mathfrak{g},\mathscr{U}(\mathfrak{g})\otimes\mathscr{U}(\mathfrak{g}))=0$, for all $m\in\mathbb{N}_0$.

Proof Let $m\in\mathbb{N}_0$ be a positive integer. We have to prove that every $(m,\mathfrak{p})$-contractible 1-cocycle $f\in Z^1_{m,\mathfrak{p}}(\mathfrak{g},\mathscr{U}(\mathfrak{g})\otimes\mathscr{U}(\mathfrak{g}))$ is the coboundary of an element $\alpha\in F_{l,m}(\mathscr{U}(\mathfrak{g})\otimes\mathscr{U}(\mathfrak{g}))$, for some $l\in\mathbb{N}_0$. From Lemma **??**, there exists an $x\in\mathscr{U}(\mathfrak{g})\otimes\mathscr{U}(\mathfrak{g})$ such that $f=d_0x$. All we have to prove is that we can always find a left $\mathfrak{g}$-invariant $y\in(\mathscr{U}(\mathfrak{g})\otimes\mathscr{U}(\mathfrak{g}))^{\mathfrak{g}}$, such that $x=y$ modulo $F_{l,m}(\mathscr{U}(\mathfrak{g})\otimes\mathscr{U}(\mathfrak{g}))$ for some $l\in\mathbb{N}_0$. Then, we can check that for $\alpha=x-y\in F_{l,m}(\mathscr{U}(\mathfrak{g})\otimes\mathscr{U}(\mathfrak{g}))$, we have

$$d_0\alpha=d_0(x-y)=d_0x=f.$$

In view of (**??**), we can first expand x into its components in the left $\mathfrak{g}$-modules isomorphic to the $S_n(\mathfrak{g}\oplus\mathfrak{g})$, for all $n\in\mathbb{N}_0$. Up to the isomorphism of left $\mathfrak{g}$-modules, which we shall omit here, we have $x=\sum_{n\geq0}x_n$ where, for all $n\in\mathbb{N}_0$, $x_n\in S_n(\mathfrak{g}\oplus\mathfrak{g})$. Similarly, we can further decompose each $S_n(\mathfrak{g}\oplus\mathfrak{g})$ into the left $\mathfrak{h}$-modules $S_{n-p,p}(\mathfrak{g}\oplus\mathfrak{g})$, with $0\leq p\leq n$, and, accordingly, each x_n. We are now going to construct the desired $y\in(\mathscr{U}(\mathfrak{g})\otimes\mathscr{U}(\mathfrak{g}))^{\mathfrak{g}}$ by recursion, submodule by submodule. If $x_n=0$ for all $n>m$, we can set $y=0$ and we are done. So, suppose that there exists an $n>m$ such that $x_n\neq0$ and let $x_{0,n}$ be the component of x_n in $S_{0,n}(\mathfrak{g}\oplus\mathfrak{g})$. If $x_{0,n}$ vanishes, we can skip to the component of x_n in $S_{1,n-1}(\mathfrak{g}\oplus\mathfrak{g})$. Otherwise, we are going to prove that there exists a $\mathfrak{g}$-invariant $y_{n,0}\in S_n(\mathfrak{g}\oplus\mathfrak{g})^{\mathfrak{g}}$, such that the component of $x_n-y_{n,0}$ in $S_{0,n}(\mathfrak{g}\oplus\mathfrak{g})$ vanishes. From f being $(m,\mathfrak{p})$-contractible, we know that

$$f(\mathfrak{h})=d_0x(\mathfrak{h})=\mathfrak{h}\triangleright\left(x_n+\sum_{n'\neq n}x_{n'}\right)\subseteq F_{l,m}(\mathscr{U}(\mathfrak{g})\otimes\mathscr{U}(\mathfrak{g})),\tag{4.9}$$

for some $l\in\mathbb{N}_0$. Therefore, since the $S_{m,p}(\mathfrak{g}\oplus\mathfrak{g})$ are left $\mathfrak{h}$-modules, we have $\mathfrak{h}\triangleright x_{0,n}=0$. Since $\mathfrak{g}$ has the restriction property, Definition **??**, it follows that the $\mathfrak{h}$-invariant tensor $x_{0,n}\in S_{0,n}(\mathfrak{g}\oplus\mathfrak{g})^{\mathfrak{h}}$ is the restriction to $\mathfrak{p}$ of a $\mathfrak{g}$-invariant tensor $y_{n,0}\in S_n(\mathfrak{g}\oplus\mathfrak{g})^{\mathfrak{g}}$. Now consider $x_n-y_{n,0}$. By construction, it has no component in $S_{0,n}(\mathfrak{g}\oplus\mathfrak{g})$. If $n-1\leq m$, we set $y_n=y_{n,0}$ and skip to another $\mathfrak{g}$-module $S_{n'>m}(\mathfrak{g}\oplus\mathfrak{g})$, where x has a non-vanishing component, if any. Otherwise, let $0\leq k<n-m$ and assume that we have found $y_{n,k}\in S_n(\mathfrak{g}\oplus\mathfrak{g})^{\mathfrak{g}}$, such that $x_n-y_{n,k}$ has vanishing component in all the $S_{n-p,p}(\mathfrak{g}\oplus\mathfrak{g})$ with $p\geq n-k>m$. We are going to prove that there exists $y_{n,k+1}\in S_n(\mathfrak{g}\oplus\mathfrak{g})^{\mathfrak{g}}$ such that $x_n-y_{n,k+1}$ has vanishing component in all the $S_{n-p,p}(\mathfrak{g}\oplus\mathfrak{g})$ with $p\geq n-k-1$. To do so, let $x_{k+1,n-k-1}$ be the component of $x_n-y_{n,k}$ in $S_{k+1,n-k-1}(\mathfrak{g}\oplus\mathfrak{g})$. If it is zero, we set $y_{n,k+1}=y_{n,k}$. Otherwise, note that from (**??**), we have $\mathfrak{h}\triangleright x_{k+1,n-k-1}=0$. But the $(m,\mathfrak{p})$-contractibility of f also implies that

$$f(\mathfrak{p})=d_0x(\mathfrak{p})=\mathfrak{p}\triangleright\left(x_n-y_{n,k}+\sum_{n'\neq n}x_{n'}\right)\subseteq F_{l,m+1}(\mathscr{U}(\mathfrak{g})\otimes\mathscr{U}(\mathfrak{g})),$$

from which it follows that $\mathfrak{p} \triangleright x_{k+1,n-k-1} \in S_{k+2,n-k-2}(\mathfrak{g} \oplus \mathfrak{g})$. According to Lemma **??**, we can write $x_{k+1,n-k-1} = \sum_{i,j} a_{ij}\, w_i\, z_j$, with $a_{ij} \in \mathbb{K}$, $w_i \in S_{k+1,0}(\mathfrak{g} \oplus \mathfrak{g})^{\mathfrak{g}}$ and $z_j \in S_{0,n-k-1}(\mathfrak{g} \oplus \mathfrak{g})^{\mathfrak{h}}$. Since $\mathfrak{g}$ has the restriction property, all the z_j are the restrictions to $\mathfrak{p}$ of $\mathfrak{g}$-invariant elements $\zeta_j \in S_{n-k-1}(\mathfrak{g} \oplus \mathfrak{g})^{\mathfrak{g}}$. Now, set $y_{n,k+1} = y_{n,k} + \sum_{i,j} a_{ij}\, w_i\, \zeta_j$. It is obvious that $y_{n,k+1} \in S_n(\mathfrak{g} \oplus \mathfrak{g})^{\mathfrak{g}}$ and, by construction, $x_n - y_{n,k+1}$ has no component in all the $S_{n-p,p}(\mathfrak{g} \oplus \mathfrak{g})$, with $p \geq n-k-1$. The recursion goes on until we have $y_{n,n-m} \in S_n(\mathfrak{g} \oplus \mathfrak{g})^{\mathfrak{g}}$ such that $x_n - y_{n,n-m}$ has vanishing components in all the $S_{n-p,p}(\mathfrak{g} \oplus \mathfrak{g})$, with $p > m$. We therefore set $y_n = y_{n,n-m}$. By repeating this a finite number of times[3], in all the $S_{n'>m}(\mathfrak{g} \oplus \mathfrak{g})$ in which x has non-vanishing components, we obtain the desired $y = \sum_{n \geq 0} y_n$. □

[3] It is rather obvious that x has non-vanishing components in a finite number of submodules $S_n(\mathfrak{g} \oplus \mathfrak{g})$, as there always exists an $l \in \mathbb{N}$ such that $x \in F_l(\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))$.

5 Rigidity Theorems

5.1 Contractible algebra isomorphisms

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{K}$ and let $\mathfrak{h}$ be a symmetrizing Lie subalgebra with orthogonal complement $\mathfrak{p}$ in $\mathfrak{g}$. Then, for every $\mathfrak{p}$-contractible deformation algebra $(\mathscr{U}_h(\mathfrak{g}), \cdot_h)$ of $(\mathscr{U}(\mathfrak{g}), \cdot)$, there exists a $\mathfrak{p}$-contractible isomorphism of $\mathbb{K}[[h]]$-algebras $(\mathscr{U}_h(\mathfrak{g}), \cdot_h) \xrightarrow{\sim} (\mathscr{U}(\mathfrak{g})[[h]], \cdot)$, that is congruent with id $\mod h$.

Proof By definition, there exists a $\mathbb{K}[[h]]$-module isomorphism $\eta : \mathscr{U}_h(\mathfrak{g}) \xrightarrow{\sim} \mathscr{U}(\mathfrak{g})[[h]]$. The latter defines a $\mathbb{K}[[h]]$-algebra isomorphism between $(\mathscr{U}_h(\mathfrak{g}), \cdot_h)$ and $(\mathscr{U}(\mathfrak{g})[[h]], \mu_h)$, where $\mu_h := \eta \circ (\cdot_h) \circ \left(\eta^{-1} \otimes \eta^{-1}\right) = \cdot \mod h$. If we found a $\mathfrak{p}$-contractible $\mathbb{K}[[h]]$-algebra automorphism

$$\phi : (\mathscr{U}(\mathfrak{g})[[h]], \mu_h) \xrightarrow{\sim} (\mathscr{U}(\mathfrak{g})[[h]], \cdot), \tag{5.1}$$

we would prove the proposition as $\phi \circ \eta$ would constitute the desired $\mathbb{K}[[h]]$-algebra isomorphism from $(\mathscr{U}_h(\mathfrak{g}), \cdot_h)$ to $(\mathscr{U}(\mathfrak{g})[[h]], \cdot)$. Let ϕ be a $\mathbb{K}[[h]]$-module automorphism on $\mathscr{U}(\mathfrak{g})[[h]]$. The condition for such an automorphism to be the $\mathbb{K}[[h]]$-algebra automorphism (**??**) is

$$\mu_h = \phi^{-1} \circ (\cdot) \circ (\phi \otimes \phi)\,. \tag{5.2}$$

Let us construct

$$\phi = \sum_{n \geq 0} h^n \varphi_n, \tag{5.3}$$

order by order in h. At leading order, we have $\mu_0 = \cdot$ and we can take $\varphi_0 = \mathrm{id} \in \mathrm{Hom}(\mathscr{U}(\mathfrak{g})[[h]], \mathscr{U}(\mathfrak{g})[[h]])$. We thus have $\varphi_0(F_{m,p}(\mathscr{U}(\mathfrak{g}))) \subseteq F_{m,p}(\mathscr{U}(\mathfrak{g}))$, for all $m, p \in \mathbb{N}_0$. Suppose now that we have found a polynomial of degree $n > 0$,

$$\phi_n = \sum_{m=0}^{n} h^m \varphi_m, \tag{5.4}$$

such that

$$\mu_h - \phi_n^{-1} \circ (\cdot) \circ (\phi_n \otimes \phi_n) = h^{n+1} r \mod h^{n+2}, \tag{5.5}$$

where ϕ_n^{-1} denotes the exact inverse series of ϕ_n defined by $\phi_n \circ \phi_n^{-1} = \mathrm{id}$ and $r \in \mathrm{Hom}(\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})[[h]], \mathscr{U}(\mathfrak{g})[[h]])$. We assume that ϕ_n is $\mathfrak{p}$-contractible. Therefore, $(\cdot) \circ (\phi_n \otimes \phi_n)$ is $\mathfrak{p}$-contractible. By Lemma **??**, ϕ_n^{-1} is $\mathfrak{p}$-contractible and, by Lemma **??**, $\phi_n^{-1} \circ (\cdot) \circ (\phi_n \otimes \phi_n)$ is $\mathfrak{p}$-contractible. By definition of a $\mathfrak{p}$-contractible deformation algebra, we know that μ_h is $\mathfrak{p}$-contractible. It therefore follows from (**??**) at order h^{n+1} that $r(F_{m,p}(\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))) \subseteq F_{*,n+1+p}(\mathscr{U}(\mathfrak{g}))$, for all $m, p \in \mathbb{N}_0$. From the associativity of μ_h, we deduce that r is a 2-cocycle in the Hochschild complex,

$$\delta_2 r = 0. \tag{5.6}$$

As $\mathfrak{g}$ is semisimple, it follows from Lemma **??** that its second Hochschild cohomology module $HH^2(\mathscr{U}(\mathfrak{g}), \mathscr{U}(\mathfrak{g}))$ is empty, so that r is a coboundary. We thus have $r = \delta_1\beta$, for some $\beta \in \mathrm{Hom}(\mathscr{U}(\mathfrak{g})[[h]], \mathscr{U}(\mathfrak{g})[[h]])$. But we know that, in particular, $r(F_{2,0}(\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))) \subseteq F_{*,n+1}(\mathscr{U}(\mathfrak{g}))$ and $r(F_{1,1}(\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))) \subseteq F_{*,n+2}(\mathscr{U}(\mathfrak{g}))$. It follows that β can be consistently chosen so that $\beta(F_{1,0}(\mathscr{U}(\mathfrak{g}))) \subseteq F_{*,n+1}(\mathscr{U}(\mathfrak{g}))$ and $\beta(F_{0,1}(\mathscr{U}(\mathfrak{g}))) \subseteq F_{*,n+2}(\mathscr{U}(\mathfrak{g}))$. To complete the recursion, we have to solve

$$\mu_h = \left(\phi_n^{-1} - h^{n+1}\varphi_{n+1} \mod h^{n+2}\right) \circ \left[\left(\phi_n + h^{n+1}\varphi_{n+1}\right) \cdot \left(\phi_n + h^{n+1}\varphi_{n+1}\right)\right] \mod h^{n+2},$$

that is

$$\delta_1 \varphi_{n+1} = r. \tag{5.7}$$

This equation can be solved by taking $\varphi_{n+1} = -\beta$, which implies that $\varphi_{n+1}(F_{1,0}(\mathscr{U}(\mathfrak{g}))) \subseteq F_{*,n+1}(\mathscr{U}(\mathfrak{g}))$ and $\varphi_{n+1}(F_{0,1}(\mathscr{U}(\mathfrak{g}))) \subseteq F_{*,n+2}(\mathscr{U}(\mathfrak{g}))$. The proposition then follows from Lemma **??**. □

5.2 Contractible twisting for symmetric semisimple Lie algebras

Let $(\mathfrak{g}, \theta)$ be a symmetric semisimple Lie algebra over $\mathbb{K}$ having the restriction property, and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ be the associated symmetric decomposition of $\mathfrak{g}$. Every $\mathfrak{p}$-contractible deformation $(\mathscr{U}_h(\mathfrak{g}), \Delta, \varepsilon, S)$ of the Hopf algebra $(\mathscr{U}(\mathfrak{g}), \Delta_0, \varepsilon_0, S_0)$ is isomorphic, as a Hopf algebra over $\mathbb{K}[[h]]$, to a twist of $(\mathscr{U}(\mathfrak{g}), \Delta_0, \varepsilon_0, S_0)$ by a $\mathfrak{p}$-contractible invertible element $F \in \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})[[h]]$, congruent with $1 \otimes 1 \mod h$.

Proof We consider the composite map

$$\tilde{\Delta} : \mathscr{U}(\mathfrak{g})[[h]] \xrightarrow{\sim} \mathscr{U}_h(\mathfrak{g}) \xrightarrow{\Delta} \mathscr{U}_h(\mathfrak{g}) \otimes \mathscr{U}_h(\mathfrak{g}) \xrightarrow{\sim} \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})[[h]], \tag{5.8}$$

where the existence of a $\mathfrak{p}$-contractible isomorphism of $\mathbb{K}[[h]]$-algebras ϕ follows from Proposition **??**. As ϕ is an algebra isomorphism, the composite map $\tilde{\Delta}$ is an algebra homomorphism. By repeated use of Lemma **??**, one can show that it is $\mathfrak{p}$-contractible. Now, we want to prove that there exists a $\mathfrak{p}$-contractible and invertible element $F \in \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})[[h]]$, such that $F = 1 \otimes 1 \mod h$ and

$$\tilde{\Delta} = F \Delta_0 F^{-1}. \tag{5.9}$$

We shall proceed by recursion on the order in h. To first order, we have, by construction

$$\tilde{\Delta} = \Delta_0 \mod h, \tag{5.10}$$

and we can take $F = 1 \otimes 1 \mod h$. We thus have $F_{|h=0} \in F_{0,0}(\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))$. Suppose now that we have found a polynomial $F_n \in \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})[h]$ of degree n,

$$F_n = \sum_{m=0}^{n} h^m f_m, \tag{5.11}$$

such that

$$\tilde{\Delta} - F_n \Delta_0 F_n^{-1} = h^{n+1} \xi \mod h^{n+2}, \tag{5.12}$$

where $F_n^{-1} \in \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})[[h]]$ is the formal inverse of F in the sense that $F^{-1}F = 1$ and $\xi \in \mathrm{Hom}(\mathscr{U}(\mathfrak{g})[[h]], \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})[[h]])$. We assume that F_n is $\mathfrak{p}$-contractible, *i.e.* for all $n \in \mathbb{N}_0$, $f_n \in F_{*,n}(\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))$. Since $\tilde{\Delta}$ is $\mathfrak{p}$-contractible, we deduce that $\xi(F_{1,0}(\mathscr{U}(\mathfrak{g}))) \subseteq F_{*,n+1}(\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))$ and $\xi(F_{0,1}(\mathscr{U}(\mathfrak{g})) \subseteq F_{*,n+2}(\mathscr{U}(\mathfrak{g}))$. It follows from (**??**) that, for all $X, Y \in \mathfrak{g}$, we have

$$\left(\tilde{\Delta} - F_n \Delta_0 F_n^{-1}\right)([X,Y]) = h^{n+1}\xi([X,Y]) \mod h^{n+2}, \tag{5.13}$$

on one hand and, on the other hand, since $\tilde{\Delta}$ is an algebra homomorphism,

$$\begin{aligned}\left(\tilde{\Delta} - F_n \Delta_0 F_n^{-1}\right)([X,Y]) &= \left[\tilde{\Delta}X, \tilde{\Delta}Y\right] - F_n \Delta_0([X,Y]) F_n^{-1} \\ &= h^{n+1}\left([\Delta_0 X, \xi(Y)] + [\xi(X), \Delta_0 Y]\right) \mod h^{n+2}.\end{aligned} \tag{5.14}$$

Equating (**??**) and (**??**), we finally get

$$d_1 \xi = 0. \tag{5.15}$$

The map ξ is thus a 1-cocycle of $Z^1(\mathfrak{g}, \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))$ in the sense of the Chevalley-
Eilenberg complex[4]. As $\mathfrak{g}$ is semisimple, it follows from Lemma **??** that the cohomology module $H^1(\mathfrak{g}, \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))$ is empty. We therefore conclude that ξ is a coboundary. But we know that $\xi(F_{0,1}(\mathscr{U}(\mathfrak{g}))) \subseteq F_{*,n+2}(\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))$ and $\xi(F_{1,0}(\mathscr{U}(\mathfrak{g}))) \subseteq F_{*,n+1}(\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))$, so that ξ is an $(n+1, \mathfrak{p})$-contractible 1-cocycle in the contractible Chevalley-Eilenberg complex defined in Subsect. **??**. As $\mathfrak{g}$ is of restrictive type, it follows from Lemma **??**, that $H^1_{n+1,\mathfrak{p}}(\mathfrak{g}, \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})) = 0$, so that ξ is the coboundary of an $(n+1, \mathfrak{p})$-contractible element in $\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})$, *i.e.* there exists an $\alpha \in F_{*,n+1}(\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))$ such that $\xi = d_0\alpha = \delta_0\alpha$. In order to complete the recursion, we have to find an $f_{n+1} \in \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})$ such that

$$\tilde{\Delta} - \left(F_n + h^{n+1} f_{(n+1)}\right) \Delta_0 \left(F_n^{-1} - h^{n+1} f_{(n+1)} \mod h^{n+2}\right) = 0 \mod h^{n+2}. \tag{5.16}$$

Expanding the above equation to order h^{n+1} yields

$$\delta_0 f_{n+1} + \xi = 0. \tag{5.17}$$

This equation can then be solved by choosing $f_{n+1} = -\alpha \in F_{*,n+1}(\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}))$. □

[4] By rewriting (**??**–**??**) for the associative product of two arbitrary elements in $\mathscr{U}(\mathfrak{g})$, we also show that ξ is a 1-cocycle in the sense of the Hochschild complex. This indeed provides a unique continuation of ξ from $\mathfrak{g}$ to $\mathscr{U}(\mathfrak{g})$ as a derivation.

5.3 Contractible quasi-Hopf algebras

Generically, cochain twists map quasi-Hopf algebras to quasi-Hopf algebras**???**. Under twisting, the coproduct Δ and coassociator Φ of a given quasi-Hopf algebra transform as

$$\Delta^F X = F\cdot(\Delta X)\cdot F^{-1},\quad \Phi^F = F_{12}\cdot(\Delta\otimes\mathrm{id})(F)\cdot\Phi\cdot(\mathrm{id}\otimes\Delta)(F^{-1})\cdot F_{23}^{-1}, \tag{5.18}$$

and, if the quasi-Hopf algebra is in addition quasitriangular, then the R-matrix $\mathscr{R}$ transforms as

$$\mathscr{R}^F = F_{21}\mathscr{R}F^{-1}. \tag{5.19}$$

In the previous section it happened that both Δ and Δ_0 were coassociative, so that both $\mathscr{U}_h(\mathfrak{g})$ and $\mathscr{U}(\mathfrak{g})$ happened to be Hopf algebras, but the theory applies more generally.

Suppose now that $\mathscr{R}\in(\mathscr{U}_h(\mathfrak{g}))^{\otimes 2}$ and $\Phi\in(\mathscr{U}_h(\mathfrak{g}))^{\otimes 3}$ are any R-matrix and coassociator that make a given QUEA $(\mathscr{U}_h(\mathfrak{g}),\Delta,\varepsilon,S)$ into a (coassociative) qtqH algebra, which we denote, by a slight abbreviation, as $(\mathscr{U}_h(\mathfrak{g}),\Delta,\mathscr{R},\Phi)$. We say that this qtqH algebra is $\mathfrak{p}$-contractible with respect to a symmetric decomposition $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{p}$ if and only if $(\mathscr{U}_h(\mathfrak{g}),\Delta,\varepsilon,S)$ is $\mathfrak{p}$-contractible in the sense of Definition **??** and $\mathscr{R}$ and Φ are $\mathfrak{p}$-contractible as elements of their respective tensor products. It then follows from the definitions above that

For any QUEA $\mathscr{U}_h(\mathfrak{g})$ and any symmetric decomposition $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{p}$, if $(\mathscr{U}_h(\mathfrak{g}),\Delta,\mathscr{R},\Phi)$ is a $\mathfrak{p}$-contractible qtqH algebra and $F\in(\mathscr{U}_h(\mathfrak{g}))^{\otimes 2}$ is a $\mathfrak{p}$-contractible twist then $((\mathscr{U}_h(\mathfrak{g}))^F,\Delta^F,\mathscr{R}^F,\Phi^F)$ is a $\mathfrak{p}$-contractible qtqH algebra. Combining this with Propositions **??** and **??**, we have that every $\mathfrak{p}$-contractible qtqH algebra $(\mathscr{U}_h(\mathfrak{g}),\Delta,\mathscr{R},\Phi)$ can be obtained, via $\mathfrak{p}$-contractible change of basis and twist, from some $\mathfrak{p}$-contractible qtqH algebra $(\mathscr{U}(\mathfrak{g}),\Delta_0,\mathscr{R}',\Phi')$ based on the undeformed UEA. In particular, starting from the trivial triangular quasi-Hopf structure $(\mathscr{R}=1\otimes 1,\Phi=1\otimes 1\otimes 1)$ on $\mathscr{U}(\mathfrak{g})$, which is obviously $\mathfrak{p}$-contractible, we have

For any $\mathfrak{p}$-contractible deformation Hopf algebra $(\mathscr{U}_h(\mathfrak{g}),\Delta,\varepsilon,S)$ based on a symmetric semisimple Lie algebra of restrictive type with symmetric decomposition $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{p}$, there is an R-matrix $\mathscr{R}$ and coassociator Φ such that $(\mathscr{U}_h(\mathfrak{g}),\Delta,\mathscr{R},\Phi)$ is a $\mathfrak{p}$-contractible triangular quasi-Hopf algebra.

Proof Explicitly, by Propositions **??** and **??**, there exists a $\mathfrak{p}$-contractible invertible element $F\in\mathscr{U}(\mathfrak{g})\otimes\mathscr{U}(\mathfrak{g})[[h]]$ and a $\mathfrak{p}$-contractible $\mathbb{K}[[h]]$-algebra isomorphism ϕ, such that

$$\Delta = \left(\phi^{-1}\otimes\phi^{-1}\right)\circ F\Delta_0 F^{-1}\circ\phi.$$

Defining

$$\mathscr{R} := \phi^{-1}\otimes\phi^{-1}\left(F_{21}F^{-1}\right), \tag{5.20}$$

$$\Phi := \phi^{-1}\otimes\phi^{-1}\otimes\phi^{-1}\left(F_{12}\cdot(\Delta_0\otimes\mathrm{id})(F)\cdot(\mathrm{id}\otimes\Delta_0)(F^{-1})\cdot F_{23}^{-1}\right) \tag{5.21}$$

provides the required structure. □

One may also want to know when a given $\mathfrak{p}$-contractible Hopf QUEA $(\mathscr{U}_h(\mathfrak{g}),\Delta,\varepsilon,S)$ admits a $\mathfrak{p}$-contractible quasitriangular structure. When $\mathfrak{g}$ is a semisimple Lie algebra, we can at least give necessary conditions, by adapting the argument surrounding Proposition 3.16 in **?**. Let $\mathbf{t}\in\mathrm{sym}(\mathfrak{g}\otimes\mathfrak{g})^{\mathfrak{g}}$ be a $\mathfrak{g}$-invariant symmetric element. For semisimple $\mathfrak{g}$, $\mathbf{t}$ is a linear combination of invariant symmetric elements of the simple factors of $\mathfrak{g}$. Let $(\mathscr{U}_h(\mathfrak{g}),\Delta,\bar{\mathscr{R}})$ be the corresponding *standard* quasitriangular Hopf QUEA and $(\mathscr{U}(\mathfrak{g})[[h]],\Delta_0,\mathscr{R},\Phi)$ the qtqH algebra with $\mathscr{R}=e^{h\mathbf{t}/2}$, both as defined (simple factor by simple factor) in **?**.

Let $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{p}$ be a symmetric decomposition of restrictive type of a semisimple Lie algebra $\mathfrak{g}$. If $(\mathscr{U}_h(\mathfrak{g}),\Delta,\bar{\mathscr{R}})$ is $\mathfrak{p}$-contractible, then it is isomorphic, via a $\mathfrak{p}$-contractible isomorphism of $\mathbb{K}[[h]]$-algebras, to a $\mathfrak{p}$-contractible twist of $(\mathscr{U}(\mathfrak{g})[[h]],\Delta_0,\mathscr{R}=e^{h\mathbf{t}/2},\Phi)$. Furthermore, $h\mathbf{t}$ is necessarily $\mathfrak{p}$-contractible.

Proof (Outline) One follows the Proof of Proposition 3.16 in **?** to reach the qtqH algebra $(\mathscr{U}(\mathfrak{g})[[h]],\Delta_0,\mathscr{R},\Phi)$, where $\mathscr{R}$ and Φ are $\mathfrak{g}$-invariants but, as above, one knows from Propositions **??** and **??** that the required isomorphism ϕ and twist F can be chosen to be $\mathfrak{p}$-contractible. Indeed, a further $\mathfrak{g}$-invariant twist may be required to ensure that $\mathscr{R}_{21}=\mathscr{R}$, but this twist is $\mathfrak{p}$-contractible as $\mathscr{R}$ is (cf. Prop 3.5 in **?**). Then the rest of the proof is unmodified, and one has that $\mathscr{R}=e^{h\mathbf{t}/2}$ and that Φ is the corresponding coassociator, as defined in **?**. Moreover, since both $\mathscr{R}$ and Φ depend on h and $\mathbf{t}$ solely through $h\mathbf{t}$, their $\mathfrak{p}$-contractibility implies that of $h\mathbf{t}$. □

Knowing, ahead of time, that the standard quasitriangular Hopf QUEAs of semisimple Lie algebras exist allows one to conclude that, to the datum $(\mathfrak{g},\mathbf{t})$, corresponds, via twisting, a quasitriangular Hopf algebra. It does not allow us though to conclude anything about $\mathfrak{p}$-contractibilty. In order to decide whether the existence of a $\mathfrak{p}$-contractible $h\mathbf{t}\in h\,\mathrm{sym}(\mathfrak{g}\otimes\mathfrak{g})^{\mathfrak{g}}$ is also a sufficient condition for the existence of a $\mathfrak{p}$-contractible quasitriangular Hopf algebra $(\mathscr{U}_h(\mathfrak{g}),\Delta,\bar{\mathscr{R}})$ based on $(U_h\mathfrak{g},\Delta,\varepsilon,S)$, it might be helpful to refine the approach of Donin and Shnider, **?**, where it is shown by direct cohomological arguments that there exists a twist from $(\mathscr{U}(\mathfrak{g}),\Delta_0,\mathscr{R},\Phi)$ to the latter, therefore setting the coassociator to unity. In Sect. **??** we will see an example for which a $\mathfrak{p}$-contractible $h\mathbf{t}$ (and a $\mathfrak{p}$-contractible quasitriangular Hopf algebra) does exist, and one for which it does not.

6 Twists and $\mathfrak{p}$-Contractions

We can now finally turn to the objects in which we are really interested in this paper: those deformed enveloping algebras of non-semisimple Lie algebras that are obtained by a certain contraction procedure modelled on that used in **???** to obtain the
κ-deformation of Poincaré. The notion of $\mathfrak{p}$-contractibilty introduced in the previous sections is formulated with this type of contraction in mind, as we now discuss.

Recall first that if $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{p}$ is a symmetric decomposition of a Lie algebra $\mathfrak{g}$, a standard procedure known as *Inönu-Wigner contraction*, **??**, consists in contracting the submodule $\mathfrak{p}$ by means of a one-parameter family of linear automorphisms of the form

$$\Lambda_t = \pi_{\mathfrak{h}} + t\,\pi_{\mathfrak{p}}\,, \tag{6.1}$$

where $\pi_{\mathfrak{h}} : \mathfrak{g} \twoheadrightarrow \mathfrak{h}$ and $\pi_{\mathfrak{p}} : \mathfrak{g} \twoheadrightarrow \mathfrak{p}$ denote the linear projections from $\mathfrak{g}$ to $\mathfrak{h}$ and $\mathfrak{p}$ respectively and $t \in (0,1]$.

For all $t \in (0,1]$, the image of $\mathfrak{g}$ by the automorphism Λ_t^{-1} is the symmetric semisimple Lie algebra $\mathfrak{g}_t$, isomorphic to $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ as a $\mathbb{K}$-module, with Lie bracket

$$[X,Y]_t = \Lambda_t^{-1}\left([\Lambda_t(X), \Lambda_t(Y)]\right) \tag{6.2}$$

for all $X, Y \in \mathfrak{g}$. It has the property that

$$[\mathfrak{h},\mathfrak{h}]_t \subset \mathfrak{h}\,, \qquad [\mathfrak{h},\mathfrak{p}]_t \subset \mathfrak{p}\,, \quad \text{and} \quad [\mathfrak{p},\mathfrak{p}]_t \subset t^2\mathfrak{h}, \tag{6.3}$$

so in the limit $t \to 0$ one obtains a Lie algebra $\mathfrak{g}_0$, isomorphic to $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ as a $\mathbb{K}$-module, whose Lie bracket $[,]_0 = \lim_{t\to 0}[,]_t$ obeys

$$[\mathfrak{h},\mathfrak{h}]_0 \subset \mathfrak{h}\,, \qquad [\mathfrak{h},\mathfrak{p}]_0 \subset \mathfrak{p}\,, \quad \text{and} \quad [\mathfrak{p},\mathfrak{p}]_0 = \{0\}. \tag{6.4}$$

The submodule $\mathfrak{p}$ is therefore an abelian ideal in $\mathfrak{g}_0$. The undeformed Hopf algebra structure defined in Sect. **??** is preserved as t tends to zero. There is thus a natural undeformed Hopf algebra structure on the envelope $\mathscr{U}(\mathfrak{g}_0)$ of the contracted Lie algebra, which we may write as $(\mathscr{U}(\mathfrak{g}_0), \Delta_0, S_0, \varepsilon_0)$ without ambiguity.

We may extend Λ_t over $\mathscr{U}(\mathfrak{g})[[h]]$ as a $\mathbb{K}[[h]]$-algebra homomorphism. Further, by means of the $\mathbb{K}[[h]]$-module isomorphism η of Definition **??**, we can regard Λ_t as a map $\mathscr{U}_h(\mathfrak{g}) \to \mathscr{U}_h(\mathfrak{g})$ on any QUEA $\mathscr{U}_h(\mathfrak{g})$. This specifies how every element of the latter is to be rescaled in the contraction limit.

The relevance of the definition of $\mathfrak{p}$-contractibility from Sect. **??** is then contained in the following

Let $(\mathfrak{g}, \theta)$ be a symmetric semisimple Lie algebra with symmetric decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ and let $(\mathscr{U}_h(\mathfrak{g}), \Delta_h, S_h, \varepsilon_h)$ be a deformation of the Hopf algebra $(\mathscr{U}(\mathfrak{g}), \Delta_0, S_0, \varepsilon_0)$. For all $t \in (0,1]$, set

$$\Delta_{(t)} = (\Lambda_t^{-1} \otimes \Lambda_t^{-1}) \circ \Delta_{th'} \circ \Lambda_t\,, \qquad S_{(t)} = \Lambda_t^{-1} \circ S_{th'} \circ \Lambda_t \quad \text{and} \quad \varepsilon_{(t)} = \varepsilon_{th'} \circ \Lambda_t\,, \tag{6.5}$$

where $h' = h/t$ is the rescaled deformation parameter. Then the limit of $(\mathscr{U}_{th'}(\mathfrak{g}_t), \Delta_{(t)}, S_{(t)}, \varepsilon_{(t)})$ as $t \to 0$ exists if and only if $(\mathscr{U}_h(\mathfrak{g}), \Delta_h, S_h, \varepsilon_h)$ is $\mathfrak{p}$-contractible. If so, one has a deformation of $(\mathscr{U}(\mathfrak{g}_0), \Delta_0, S_0, \varepsilon_0)$ which we denote by $(\mathscr{U}_{h'}(\mathfrak{g}_0), \Delta_{h'}, S_{h'}, \varepsilon_{h'})$, and refer to as the $\mathfrak{p}$-*contraction* of $(\mathscr{U}_h(\mathfrak{g}), \Delta_h, S_h, \varepsilon_h)$.

Proof Let $r, s \in \mathbb{N}$ and let $\phi : (\mathscr{U}(\mathfrak{g}))^{\otimes r}[[h]] \to (\mathscr{U}(\mathfrak{g}))^{\otimes s}[[h]]$ be a homomorphism of $\mathbb{K}[[h]]$-modules. We want to prove that $\phi_t = (\Lambda_t^{-1})^{\otimes s} \circ \phi \circ (\Lambda_t)^{\otimes r}$ has a finite limit when $t \to 0$ if and only if ϕ is $\mathfrak{p}$-contractible. First assume that ϕ is $\mathfrak{p}$-contractible; then from Lemma **??**, there exists a collection $(\varphi_n)_{n\in\mathbb{N}_0}$ of $\mathbb{K}[[h]]$-module homomorphisms $\varphi_n : (\mathscr{U}(\mathfrak{g}))^{\otimes r}[[h]] \to (\mathscr{U}(\mathfrak{g}))^{\otimes s}[[h]]$ such that

$$\phi = \sum_{n\geq 0} h^n\,\varphi_n \tag{6.6}$$

and, for all $n,m,p \in \mathbb{N}_0$, there exists $l \in \mathbb{N}_0$ such that $\varphi_n\left(F_{m,p}((\mathscr{U}(\mathfrak{g}))^{\otimes r})\right) \subseteq F_{l,n+p}\left((\mathscr{U}(\mathfrak{g}))^{\otimes s}\right)$. We thus have, for all $n,m,p \in \mathbb{N}_0$,

$$\begin{aligned} h^n\,(\Lambda_t^{-1})^{\otimes s} \circ \varphi_n \circ (\Lambda_t)^{\otimes r}\left(S_{m,p}(\mathfrak{g}^{\oplus r})\right) &= h'^{-n} t^{n+p}\,(\Lambda_t^{-1})^{\otimes s} \circ \varphi_n\left(S_{m,p}(\mathfrak{g}^{\oplus r})\right) \\ &\subseteq h'^{-n} t^{n+p}\,(\Lambda_t^{-1})^{\otimes s}\left(F_{l,n+p}((\mathscr{U}(\mathfrak{g}))^{\otimes s})\right) \\ &= h'^{-n} t^{n+p}\,O(t^{-(n+p)})\,F_{l,n+p}((\mathscr{U}(\mathfrak{g}))^{\otimes s}) \\ &= h'^{-n} O(1)\,F_{l,n+p}((\mathscr{U}(\mathfrak{g}))^{\otimes s})\,. \end{aligned}$$

This obviously has a finite limit when $t \to 0$ and so does ϕ_t. Conversely, one sees that if ϕ is not $\mathfrak{p}$-contractible, ϕ_t diverges at least as t^{-1}. □

It is worth emphasizing that the notion of $\mathfrak{p}$-contraction defined here is not the only possible contraction that can be performed on a QUEA of $\mathfrak{g}$ with respect to a given symmetric decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$: one could also, for example, consider contractions where the deformation parameter h is not rescaled in the limit.

Finally, we can state our main result concerning twists and $\mathfrak{p}$-contracted QUEAs:

If a deformation Hopf algebra $(\mathscr{U}_{h'}(\mathfrak{g}_0), \Delta_{h'}, S_{h'}, \varepsilon_{h'})$ is the $\mathfrak{p}$-contraction of a QUEA of a symmetric semisimple Lie algebra $(\mathfrak{g}, \theta)$ having the restriction property, then it is isomorphic, as a Hopf algebra over $\mathbb{K}[[h']]$, to a twist of the undeformed Hopf algebra $(\mathscr{U}(\mathfrak{g}_0), \Delta_0, S_0, \varepsilon_0)$ by an invertible element $F_0 \in \mathscr{U}_{h'}(\mathfrak{g}_0) \otimes \mathscr{U}_{h'}(\mathfrak{g}_0)[[h']]$ congruent with $1 \otimes 1$ modulo h'.

Proof By Proposition **??**, Proposition **??** applies. By arguing as in the proof of **??**, we have that if F is the $\mathfrak{p}$-contractible twist element of Proposition **??**, then

$$F_0 = \lim_{t \to 0} (\Lambda_t^{-1} \otimes \Lambda_t^{-1})(F) \tag{6.7}$$

is well-defined. By construction, this is the twist we seek. □

From Corollary **??**, one has similarly that for every such $\mathfrak{p}$-contracted QUEA $\mathscr{U}_{h'}(\mathfrak{g}_0)$ there exists an R-matrix $\mathscr{R}$ and coassociator Φ that make $(\mathscr{U}_{h'}(\mathfrak{g}_0), \mathscr{R}, \Phi)$ into a triangular quasi-Hopf algebra.

7 Examples: κ-Poincaré in 3 and 4 Dimensions

We now turn to explicit examples. Let $\mathbb{K} = \mathbb{C}$, and consider the symmetric decomposition

$$\mathfrak{so}(n+1) = \mathfrak{so}(n) \oplus \mathfrak{p}_n, \qquad n > 2, \tag{7.1}$$

whose Inönu-Wigner contraction of course yields the Lie algebra $\mathfrak{iso}(n)$ of the complexified Euclidean group in n dimensions, $ISO(n, \mathbb{C})$. By Lemma **??**, this decomposition is of restrictive type. Thus, the results above will apply to any $\mathfrak{p}_n$-contractible deformation algebra $\mathscr{U}_h(\mathfrak{so}(n+1))$. Finding such deformations is itself a non-trivial task. In the cases $n = 3, 4$, this was achieved in **??**[5], yielding the κ-deformations $\mathscr{U}_\kappa(\mathfrak{iso}(3))$ and $\mathscr{U}_\kappa(\mathfrak{iso}(4))$. These can be written in terms of the generators

$$M_{ij} = -M_{ji}\,, \quad N_i\,, \quad P_i, \quad P_0 = E, \tag{7.2}$$

for all $1 \leq i, j \leq n-1$ and $n = 3, 4$. The deformation parameter is conventionally denoted as $\kappa = 1/h'$, and the algebra is then given by

$$[M_{ij}, P_k] = \delta_{k[i} P_{j]}\,, \tag{7.3}$$

$$[N_i, P_j] = \delta_{ij}\, \kappa \sinh\left(\frac{E}{\kappa}\right), \qquad [N_i, E] = P_i, \tag{7.4}$$

$$[N_i, N_j] = -M_{ij} \cosh\left(\frac{E}{\kappa}\right) + \frac{1}{4\kappa^2}\left(\vec{P}.\vec{P} M_{ij} + P_k P_{[i} M_{j]k}\right), \tag{7.5}$$

[5] Note that although the κ-Poincaré algebra exists in arbitrary dimension **?**, to the authors' knowledge it has only explicitly been shown to arise as a $\mathfrak{p}$-contraction for $n \leq 4$.

for all $1 \leq i, j, k, l \leq n-1$. The coproduct is given by

$$\Delta_\kappa(E) = E \otimes 1 + 1 \otimes E\,, \tag{7.6}$$

$$\Delta_\kappa(P_i) = P_i \otimes e^{\frac{E}{2\kappa}} + e^{-\frac{E}{2\kappa}} \otimes P_i, \tag{7.7}$$

$$\Delta_\kappa(N_i) = N_i \otimes e^{\frac{E}{2\kappa}} + e^{-\frac{E}{2\kappa}} \otimes N_i + \frac{1}{2\kappa}\left(P_j \otimes e^{\frac{E}{2\kappa}} M_{ij} - e^{-\frac{E}{2\kappa}} M_{ij} \otimes P_j\right), \tag{7.8}$$

$$\Delta_\kappa(M_{ij}) = M_{ij} \otimes 1 + 1 \otimes M_{ij}, \tag{7.9}$$

and the antipode by

$$S_\kappa(P_\mu) = -P_\mu, \quad S_\kappa(M_{ij}) = -M_{ij}, \quad S_\kappa(N_i) = -N_i + \frac{d}{2\kappa}P_i. \tag{7.10}$$

The counit map is undeformed, $\varepsilon(M_{ij}) = \varepsilon(N_i) = \varepsilon(P_\mu) = 0$, for all $0 \leq \mu \leq n-1$.

It follows from the results presented in the previous sections that $\mathscr{U}_\kappa(\mathfrak{iso}(3))$ and $\mathscr{U}_\kappa(\mathfrak{iso}(4))$ are isomorphic to cochain twists of $\mathscr{U}(\mathfrak{iso}(3))$ and $\mathscr{U}(\mathfrak{iso}(4))$ respectively.

Let us comment on the relationship between this statement and various previous results. First, it should not be confused with other statements that exist in the literature, **??**, concerning twists and κ-deformed Minkowski space-time, which involve enlarged algebras that include the dilatation generator.

Next, as we saw above, the existence of the cochain twist means there certainly exist triangular quasi-Hopf algebras $(\mathscr{U}_\kappa(\mathfrak{iso}(n)), \mathscr{R}, \Phi)$, at least for $n = 3, 4$. They are obtained, as in the approach of Beggs and Majid **??**, by twisting $(\mathscr{U}_\kappa(\mathfrak{iso}(n)), 1^{\otimes 2}, 1^{\otimes 3})$. To the first few orders in $h' = 1/\kappa$, the structures $\mathscr{R}$ and Φ were explicitly computed, for any $n \geq 2$, in **?**; see also **??**.

One can also understand the existence of the quasitriangular Hopf algebra structure of $\mathscr{U}_\kappa(\mathfrak{iso}(3))$ exhibited in **?** in the context of the results above. Among the special orthogonal algebras, $\mathfrak{so}(4,\mathbb{C})$ alone is not simple: $\mathfrak{so}(4,\mathbb{C}) = \mathfrak{a}_1 \oplus \mathfrak{a}_1$. There is thus a two-dimensional space of quadratic Casimirs. It is straightforward to verify that a one-dimensional subspace of them are $\mathfrak{p}$-contractible, namely $h\mathbf{t} := h\,\varepsilon_{ijk} M_{ij} P_k$. For $n \neq 3$, it is known that there is no classical r-matrix obeying the classical Yang-Baxter equation **??** and therefore no quasitriangular Hopf algebra structure. This now also follows from Corollary **??**, given that for all $n \neq 3$ the unique quadratic Casimir of $\mathfrak{so}(n+1)$ fails to be $\mathfrak{p}$-contractible.

As for versions of the κ-deformed Poincaré algebra in higher and lower space-time dimensions, a consistent definition was first given in **?**. The main idea is that the four dimensional case is generic enough that the $1+d$-dimensional case can be obtained by simply extending or truncating the range of the spatial indices from $1,\ldots,3$ to $1,\ldots,d$. It is reasonable to think that the twist obtained in the four dimensional case can be similarly extended to arbitrary dimensions, thus extending to all dimensions the existence of a triangular quasi-Hopf algebra structure on the κ-deformation of the Poincaré algebra. In particular, we expect that the κ-deformation of $\mathscr{U}(\mathfrak{sl}(2))$ admits a triangular quasi-Hopf algebra structure **?**, but a proof of this statement would obviously require a refinement of the arguments used here so as to circumvent the obstructions arising in this case – cf. the Appendix. Such a refinement could, for instance, rely on a further symmetry property of the $\mathfrak{p}$-contractible Chevalley-Eilenberg cohomology of $\mathfrak{sl}(2)$.

Finally, we note that it would be interesting to understand the existence of the twist from the point of view of the other, conceptually distinct, construction of κ-Poincaré, namely as a bicrossproduct **????**.

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Appendix: Proof of Lemma ??

In this Appendix, we provide a proof of Lemma **??**. Let $(\mathfrak{g},\theta)$ be a symmetric semisimple Lie algebra obeying the conditions of the lemma. If $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{p}$ is the associated symmetric decomposition of $\mathfrak{g}$, we want to prove that, for all $p\in\mathbb{N}$, the projection from $\mathfrak{g}$ to $\mathfrak{p}$ maps $S_p(\mathfrak{g}\oplus\mathfrak{g})^{\mathfrak{g}}$ onto $S_{0,p}(\mathfrak{g}\oplus\mathfrak{g})^{\mathfrak{h}}$. The isomorphism of left $\mathfrak{g}$-modules (**??**) induces a similar isomorphism $S(\mathfrak{g}\oplus\mathfrak{g})\cong S(\mathfrak{g})\otimes S(\mathfrak{g})$ at the level of the symmetric algebras, from which it follows that

$$S_m(\mathfrak{g}\oplus\mathfrak{g})\cong\bigoplus_{k=0}^{m}S_k(\mathfrak{g})\otimes S_{m-k}(\mathfrak{g}), \tag{7.11}$$

for all $m\in\mathbb{N}$. We thus have a decomposition of $S(\mathfrak{g}\oplus\mathfrak{g})$ into the $\mathfrak{g}$-submodules isomorphic to $S_k(\mathfrak{g})\otimes S_{m-k}(\mathfrak{g})$. There is an analogous decomposition of $S_{0,m}(\mathfrak{g}\oplus\mathfrak{g})$ into $\mathfrak{h}$-submodules isomorphic to $S_{0,k}(\mathfrak{g})\otimes S_{0,m-k}(\mathfrak{g})=S_k(\mathfrak{p})\otimes S_{m-k}(\mathfrak{p})$. It therefore suffices to show that, for all $k,\ell\in\mathbb{N}$, the restriction map induces a surjection

$$(S_k(\mathfrak{g})\otimes S_\ell(\mathfrak{g}))^{\mathfrak{g}}\twoheadrightarrow(S_k(\mathfrak{p})\otimes S_\ell(\mathfrak{p}))^{\mathfrak{h}}\,. \tag{7.12}$$

Identifying $\mathfrak{g}\cong\mathfrak{g}^*$, and in particular $\mathfrak{p}\cong\mathfrak{p}^*$, by means of the Killing form, an element $d\in S_k(\mathfrak{p})\otimes S_\ell(\mathfrak{p})$ can be regarded as a $(k+\ell)$-linear map

$$\mathfrak{p}\times\cdots\times\mathfrak{p}\to\mathbb{K};\quad (X,\dots,Y)\mapsto d(X,\dots,Y) \tag{7.13}$$

that is symmetric in its first k and final ℓ slots. In view of the polarization formulae, such maps are in bijection with polynomials of two variables in $\mathfrak{p}$, according to

$$p_{(d)}(X,Y)=d(\underbrace{X,\dots,X}_{k},\underbrace{Y,\dots,Y}_{\ell}). \tag{7.14}$$

These polynomials are (k,ℓ)-homogeneous, by which we mean that they are homogeneous of degree k with respect to their first argument and of degree ℓ with respect to their second argument. We denote by $\mathbb{K}_{k,\ell}[\mathfrak{p},\mathfrak{p}]$ the left $\mathfrak{h}$-module of (k,ℓ)-homogeneous polynomials on $\mathfrak{p}$. Then for all $k,\ell\in\mathbb{N}$, $(S_k(\mathfrak{p})\otimes S_\ell(\mathfrak{p}))^{\mathfrak{h}}$ is in bijection with the submodule of $\mathfrak{h}$-invariant (k,ℓ)-homogeneous polynomials of $\mathbb{K}_{k,\ell}[\mathfrak{p},\mathfrak{p}]^{\mathfrak{h}}$. Similarly, $(S_k(\mathfrak{g})\otimes S_\ell(\mathfrak{g}))^{\mathfrak{g}}$ is in bijection with $\mathbb{K}_{k,\ell}[\mathfrak{g},\mathfrak{g}]^{\mathfrak{g}}$. Therefore, it suffices to show that the restriction map from $\mathfrak{g}$ to $\mathfrak{p}$ maps $\mathbb{K}_{k,\ell}[\mathfrak{g},\mathfrak{g}]^{\mathfrak{g}}$ onto $\mathbb{K}_{k,\ell}[\mathfrak{p},\mathfrak{p}]^{\mathfrak{h}}$. By virtue of Lemma **??**, it will be sufficient to consider separately the cases of diagonal symmetric Lie algebras and of the symmetric simple Lie algebras listed in **??**.

We recall that a diagonal symmetric Lie algebra is a pair $(\mathfrak{g},\theta)$, where $\mathfrak{g} = \mathfrak{v}\oplus\mathfrak{v}$, for some semisimple Lie algebra $\mathfrak{v}$, and θ is the involutive automorphism of Lie algebras defined by $\theta(x,y) = (y,x)$, for all $(x,y)\in\mathfrak{g}$. We thus have $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{p}$, where $\mathfrak{h}$ is the set of elements of $\mathfrak{g}$ of the form (x,x), whereas $\mathfrak{p}$ is the set of elements of $\mathfrak{g}$ of the form $(x,-x)$, for $x\in\mathfrak{v}$. We are first going to prove that $\mathbb{K}_{k,\ell}[\mathfrak{p},\mathfrak{p}]^{\mathfrak{h}}\cong\mathbb{K}_{k,\ell}[\mathfrak{v},\mathfrak{v}]^{\mathfrak{v}}$. Let $p\in\mathbb{K}_{k,\ell}[\mathfrak{p},\mathfrak{p}]$ be a polynomial. For all $X,Y\in\mathfrak{p}$, we have

$$p(X,Y) = p((x,-x),(y,-y)) = \tilde{p}(x,y), \tag{7.15}$$

for some $x,y\in\mathfrak{v}$. The left $\mathfrak{h}$-action on $\mathfrak{p}$ induces a left $\mathfrak{h}$-action on $\mathfrak{p}\times\mathfrak{p}$, given, for all $h\in\mathfrak{h}$ and all $X,Y\in\mathfrak{p}$, by

$$h\triangleright(X,Y) = (z,z)\triangleright((x,-x),(y,-y)) = ((z\triangleright x,-z\triangleright x),(z\triangleright y,-z\triangleright y)), \tag{7.16}$$

for some $x,y\in\mathfrak{v}$ and some $z\in\mathfrak{v}$; from which it obviously follows that $\tilde{p}$ is $\mathfrak{v}$-invariant if and only if p is $\mathfrak{h}$-invariant. Now, we are going to prove that the restriction map is a surjection from $\mathbb{K}_{k,\ell}[\mathfrak{g},\mathfrak{g}]^{\mathfrak{g}}$ onto $\mathbb{K}_{k,\ell}[\mathfrak{v},\mathfrak{v}]^{\mathfrak{v}}$. Let $p\in\mathbb{K}_{k,\ell}[\mathfrak{g},\mathfrak{g}]^{\mathfrak{g}}$ be a $\mathfrak{g}$-invariant polynomial on $\mathfrak{g}$. The left $\mathfrak{g}$-action on $\mathfrak{g}\oplus\mathfrak{g}$ is given, for all $g\in\mathfrak{g}$ and all $X,Y\in\mathfrak{g}$, by

$$\begin{aligned} g\triangleright(X,Y) &= (g_1,g_2)\triangleright((x_1,x_2),(y_1,y_2)) \\ &= ((g_1\triangleright x_1,g_2\triangleright x_2),(g_1\triangleright y_1,g_2\triangleright y_2)), \end{aligned} \tag{7.17}$$

for some $g_1,g_2\in\mathfrak{v}$ and some $x_1,x_2,y_1,y_2\in\mathfrak{v}$. As one can always choose g_1 and g_2 independently, it follows that in order for p to be $\mathfrak{g}$-invariant, there must be a polynomial $f:\mathbb{K}\times\mathbb{K}\to\mathbb{K}$ and two $\mathfrak{v}$-invariant polynomials $p_1,p_2\in\mathbb{K}_{k,\ell}[\mathfrak{v},\mathfrak{v}]^{\mathfrak{v}}$ such that

$$p((x_1,x_2),(y_1,y_2)) = f(p_1(x_1,y_1),p_2(x_2,y_2)), \tag{7.18}$$

for all $x_1,x_2,y_1,y_2\in\mathfrak{v}$. Now restricting p to $\mathfrak{p}$, we get

$$\begin{aligned} p((x_1,-x_1),(y_1,-y_1)) &= f(p_1(x_1,y_1),p_2(-(x_1,y_1))) \\ &= \tilde{p}(x_1,y_1)\in\mathbb{K}_{k,\ell}[\mathfrak{v},\mathfrak{v}]^{\mathfrak{v}}, \end{aligned} \tag{7.19}$$

for all $x_1,y_1\in\mathfrak{v}$. Now, it is obvious that every polynomial in $\mathbb{K}_{k,\ell}[\mathfrak{v},\mathfrak{v}]^{\mathfrak{v}}$ can be obtained as the restriction to $\mathfrak{p}$ of a polynomial in $\mathbb{K}_{k,\ell}[\mathfrak{g},\mathfrak{g}]^{\mathfrak{g}}$; *e.g.* take $p_2 = 0$, $f = \mathrm{id}$ and $p_1 = \tilde{p}$.

We are now going to consider the different symmetric simple Lie algebras listed in **??**. Let us first consider the symmetric simple Lie algebras of type AI_n for all $n>2$. In this case, we have $\mathfrak{g} = \mathfrak{su}(n)$ endowed with an involutive automorphism θ given by complex conjugation, *i.e.* $\theta(x) = \bar{x}$, for all $x\in\mathfrak{su}(n)$. The fixed points of θ are traceless real antisymmetric matrices which generate an $\mathfrak{so}(n)$ subalgebra. We thus have the symmetric decomposition $\mathfrak{su}(n) = \mathfrak{so}(n)\oplus\mathfrak{p}$, where the orthogonal complement $\mathfrak{p}$ is the left $\mathfrak{so}(n)$-module generated by the traceless imaginary symmetric matrices of $\mathfrak{su}(n)$. It follows from the first fundamental theorem for $\mathfrak{so}(n)$-invariant polynomials on $n\times n$ matrices, **?**, that $\mathbb{K}_{k,\ell}[\mathfrak{p},\mathfrak{p}]^{\mathfrak{so}(n)}$ is generated by the following polynomials:

$$(x,y)\in\mathfrak{p}\times\mathfrak{p}\to\mathrm{tr}\,P(x,y), \tag{7.20}$$

for all (i,j)-homogeneous noncommutative polynomials $P \in \mathbb{K}_{i,j}[X,Y]$, with $i \leq k$ and $j \leq \ell$. The polynomials defined in (**??**) are obviously restrictions to $\mathfrak{p}$ of $\mathfrak{su}(n)$-invariant polynomials on $\mathfrak{su}(n)$ as, for all $P \in \mathbb{K}_{i,j}[X,Y]$ and all $x,y \in \mathfrak{su}(n)$,

$$(x,y) \to \operatorname{tr} P(x,y) \tag{7.21}$$

defines an element in $\mathbb{K}_{m,n}[\mathfrak{su}(n),\mathfrak{su}(n)]^{\mathfrak{su}(n)}$. This proves Lemma **??** for simple symmetric Lie algebras of type $AI_{n>2}$. It is worth noting that in the case of AI_2, there exist obstructions to the above result which are related to the existence of a further $\mathfrak{so}(2)$-invariant with appropriate symmetries, namely the pfaffian $(x,y) \in \mathfrak{p} \times \mathfrak{p} \to \mathrm{Pf}([x,y])$. As the latter is not the restriction to $\mathfrak{p}$ of any $\mathfrak{su}(2)$ invariant on $\mathfrak{su}(2)$, Lemma **??** does not hold in this case.

We now turn to type AII_n. In this case, we have $\mathfrak{g} = \mathfrak{su}(2n)$ endowed with an involutive automorphism θ given by the symplectic transpose, *i.e.*, for all $x \in \mathfrak{su}(2n)$, $\theta(x) = Jx^tJ$, where J is a non-singular skew-symmetric $2n \times 2n$ matrix such that $J^2 = -1$. The fixed point set of θ constitutes an $\mathfrak{sp}(2n)$ subalgebra and we have the following symmetric decomposition $\mathfrak{su}(2n) = \mathfrak{sp}(2n) \oplus \mathfrak{p}$, where $\mathfrak{p} \subset \mathfrak{su}(2n)$ is the left $\mathfrak{sp}(2n)$-module of matrices $x \in \mathfrak{su}(2n)$ such that $\theta(x) = -x$. It follows from the first fundamental theorem for $\mathfrak{sp}(2n)$-invariant polynomials on $2n \times 2n$ matrices, **?**, that $\mathbb{K}_{k,\ell}[\mathfrak{p},\mathfrak{p}]^{\mathfrak{sp}(2n)}$ is generated by the following polynomials:

$$(x,y) \in \mathfrak{p} \times \mathfrak{p} \to \operatorname{tr} P(x,y), \tag{7.22}$$

for all noncommutative (i,j)-homogeneous polynomials $P \in \mathbb{K}_{i,j}[X,Y]$, with $i \leq k$ and $j \leq \ell$. These polynomials are restrictions to $\mathfrak{p}$ of $\mathfrak{su}(2n)$-invariant polynomials on $\mathfrak{su}(2n)$ as, for all $P \in \mathbb{K}_{i,j}[X,Y]$ and all $x,y \in \mathfrak{su}(2n)$,

$$(x,y) \to \operatorname{tr} P(x,y) \tag{7.23}$$

defines an element in $\mathbb{K}_{i,j}[\mathfrak{su}(2n),\mathfrak{su}(2n)]^{\mathfrak{su}(2n)}$. This proves Lemma **??** for simple symmetric Lie algebras of type AII_n.

We finally consider the symmetric simple Lie algebras of type $\mathrm{BDI}_{n,1}$ for all $n > 2$. In this case, we have the symmetric pairs $(\mathfrak{so}(n+1),\mathfrak{so}(n))_{n>2}$. We introduce the usual basis of $\mathfrak{gl}(n+1)$, *i.e.* the $(E_{ij})_{0 \leq i,j \leq n}$ defined as the $(n+1) \times (n+1)$ matrices with a 1 at the intersection of the i^{th} row and j^{th} column and 0 everywhere else. The matrices $M_{ij} = E_{ij} - E_{ji}$, $0 \leq i,j \leq n$, constitute a basis of $\mathfrak{so}(n+1)$, and of these, the M_{ij} with $1 \leq i,j \leq n$ generate an $\mathfrak{so}(n)$ subalgebra. We thus have the symmetric decomposition $\mathfrak{so}(n+1) = \mathfrak{so}(n) \oplus \mathfrak{p}$, where $\mathfrak{p}$ is the n-dimensional $\mathfrak{so}(n)$-module spanned by the $P_i = M_{0,i}$, for all $1 \leq i \leq n$. The P_i transform under the fundamental representation $\mathbf{n}$ of $\mathfrak{so}(n)$, as can be checked from

$$M_{ij} \triangleright P_k = [M_{ij},P_k] = \delta_{jk}P_i - \delta_{ik}P_j, \tag{7.24}$$

for all $1 \leq i,j,k \leq n$. This means that we are looking for $SO(n)$-invariant (k,ℓ)-homogeneous polynomials on $\mathfrak{p} \times \mathfrak{p} = \mathbf{n} \times \mathbf{n}$. For all $n > 2$, it follows from the first fundamental theorem for $\mathfrak{so}(n)$-invariant polynomials on vectors, **??**, that such polynomials only depend on the $SO(n)$ scalars built out of the scalar products of their arguments. Let q be the quadratic form defined on $\mathfrak{p} \times \mathfrak{p}$ by $q(P_i,P_j) = \delta_{ij}$ for all $1 \leq i,j \leq n$. For all $p \in \mathbb{K}_{k,\ell}[\mathfrak{p},\mathfrak{p}]^{\mathfrak{h}}$, there exists a polynomial $f : \mathbb{K}^3 \to \mathbb{K}$ such that, for all $X,Y \in \mathfrak{p}$,

$$p(X,Y) = f(q(X,X),q(X,Y),q(Y,Y)). \tag{7.25}$$

Now, it is obvious that q is the restriction to $\mathfrak{p}$ of the map

$$\mathfrak{so}(n+1) \times \mathfrak{so}(n+1) \to \mathbb{K}; \quad (X,Y) \to -\frac{1}{2}\,\mathrm{tr}(XY),$$

which is $\mathfrak{so}(n+1)$-invariant. This proves the result for symmetric simple Lie algebras of type $\mathrm{BDI}_{n>2,1}$. It is worth noting that in the case of $\mathrm{BDI}_{2,1}$, there exist obstructions to the above result which are related to the existence of a further $SO(2)$ invariant, namely $(X,Y) \in \mathfrak{p} \times \mathfrak{p} \to \det(X,Y)$. As the latter is not the restriction to $\mathfrak{p}$ of any $\mathfrak{so}(3)$ invariant, Lemma **??** does not hold in this case.

By virtue of the special isomorphisms between lower rank simple Lie algebras, the list of summands in Lemma **??** actually includes $\mathrm{CII}_{1,1} = \mathrm{BDI}_{4,1}$ and $\mathrm{BDI}_{3,3} = \mathrm{AI}_4$. The latter respectively correspond to the symmetric decompositions $\mathfrak{sp}(4) = (\mathfrak{sp}(2) \oplus \mathfrak{sp}(2)) \oplus \mathfrak{p}$ and $\mathfrak{so}(6) = (\mathfrak{so}(3) \oplus \mathfrak{so}(3)) \oplus \mathfrak{p}$.

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