A symmetric theory of electrons and positrons(*)

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Summary. — It is shown that it is possible to achieve complete formal symmetrization in the electron and proton quantum theory by means of a new quantization process. The meaning of Dirac equations is somewhat modified and there is no longer any reason to speak of negative-energy states nor to assume, for any other types of particles, especially neutral ones, the existence of antiparticles, corresponding to the "holes" of negative energy.

The interpretation of the so-called "negative energy states" proposed by Dirac(1) leads, as is well known, to a substantially symmetric description of electrons and positrons. The substantial symmetry of the formalism consists precisely in that the theory itself gives completely symmetric results, whenever it is possible to apply it while overcoming divergence problems.

The prescriptions needed to cast the theory into a symmetric form, in conformity with its content, are however not entirely satisfactory, because one always starts from an asymmetric form or because symmetric results are obtained only after one applies appropriate procedures, such as the cancellation of divergent constants, that one should

(*) Translated from "Il Nuovo Cimento", vol. 14, 1937, pp. 171-184, by Luciano Maiani in "Soryushiron Kenkyu", *63* (1981) 149-462. (Courtesy of L. Maiani.) The present translation has been revised by the Editor with the addition of the summary which was missing in "Soryushiron Kenkyu".

(1) P. A. M. Dirac, "Proc. Camb. Phil. Soc.", *30*, 150 (1924). See also W. Heinsenberg, "Z. Physik", *90*, 209 (1934).

possibly avoid. For these reasons, we have attempted a new approach, which leads more directly to the desired result.

In the case of electrons and positrons, we may anticipate only a formal progress; but we consider it important, for possible extensions by analogy, that the very notion of negative energy states can be avoided. We shall see, in fact, that it is perfectly, and most naturally, possible to formulate a theory of elementary neutral particles which do not have negative (energy) states.

1. It is well known that quantum electrodynamics can be deduced by quantizing a system of equations which include the Dirac wave equations for the electron and the Maxwell equations. In the latter, the charge density and current are represented by certain expressions containing the electron wave function. The form given to these expressions adds, in reality, something new because it allows to derive the asymmetry with respect to the sign of the electric charge, an asymmetry which is not present in the Dirac equations. These expressions can be derived directly from a variational principle, which yields the Maxwell and the Dirac equations at the same time. Therefore, our first problem will be to examine the foundation of the variational principle itself, and the possibility of replacing it with a more appropriate one.

The Maxwell-Dirac equations contain quantities of two different types. On the one side, we have the electromagnetic potentials, which can be given a classical interpretation, within the limits posed by the correspondence principle. On the other side, there are the matter waves, which represent particles obeying the Fermi Statistic, and which have only a quantum interpretation. In this respect, it seems little satisfactory that the equations as well as the whole quantization procedure have to be derived from a variational principle which can be given only a classical interpretation. It seems more natural to search for a generalization of the variational method, such that the variables appearing in the Lagrange function assume, from the very beginning, their final significance, and, therefore, represent not necessarily commuting quantities.

This is the approach we shall follow. This approach is most important for fields obeying the Fermi statistics; reasons of simplicity may indicate, on the other hand, that nothing has to be added to the old method in the case of the electromagnetic field. In fact, we shall not perform a systematic study of all the logical possibilities offered by the new point of view we are adopting. Rather, we limit ourselves to the description of a quantization procedure for the matter-waves, which is the only important case for applications, at present; this method appears as a natural generalization of the Jordan-Wigner method([2]), and it allows not only to cast the electron-positron theory into a symmetric form, but also to construct an essentially new theory for particles not endowed with an electric charge (neutrons and the hypothetical neutrinos). Even though it is perhaps not yet possible to ask experiments to decide between the new theory and a simple extension of the Dirac equations to neutral particles, one should keep in mind

([2]) P. Jordan and E. Wigner, "Z. Physik", *47*, 631 (1928).

that the new theory introduces a smaller number of hypothetical entities, in this yet unexplored field.

Leaving to the reader the obvious extension of the formulae to the continuous systems, which we shall consider later on, we illustrate in the following the quantization procedure for discrete systems. Let a physical system be described by the real variables $q_1 q_2, \ldots, q_n$ (symmetric, Hermitian matrices). We define a Lagrange function:

$$L = i \sum_{r,s} \big(A_{rs} q_r \dot{q}_s + B_{rs} q_r q_s\big), \tag{1}$$

and set:

$$\delta \int L dt = 0, \tag{2}$$

we understand that A_{rs} and B_{rs} are ordinary real numbers, constant the former and, eventually, time-dependent the latter, which obey the relations:

$$A_{rs} = A_{sr}; \qquad B_{rs} = -B_{sr}, \tag{3}$$

and, furthermore, with $\det\|A_{rs}\| \neq 0$.

If the q's were ordinary, commuting, variables, the variational principle (2) would have no meaning because it would be identically satisfied. In the case of non-commuting variables, eq. (2) implies the vanishing, at any time, of the Hermitian matrix:

$$i \sum_r \left[\delta q_r \left(\sum_s A_{rs} \dot{q}_s + B_{rs} q_s \right) - \sum_s \big(A_{rs} \dot{q}_s + B_{rs} q_s\big) \delta q_r \right] = 0,$$

for arbitrary variations δq_r. This is only possible if the expression $\sum_s (A_{rs}\dot{q}_s + B_{rs} q_s)$ are multiple of the unit matrix so that, after some appropriate modification of the variational principle (2) (*e.g.* by requiring the sum of the diagonal terms in the above expressions to vanish([3])) we may consider the following equations of motion:

$$\sum_s \big(A_{rs}\dot{q}_s + B_{rs} q_s\big) = 0 \qquad r = 1, 2, \ldots, n. \tag{4}$$

We now show that these equations can be derived, following the usual procedure:

$$\dot{q}_r = -\frac{2\pi i}{h}\big(q_r H - H q_r\big)$$

([3]) The physical application which will be illustrated later on suggests the more rigorous restriction that, in any linear combination of q_r and $\dot{q}_r$ to any given eigenvalue there corresponds another one, equal in absolute value and opposite in sign.

from the Hamiltonian:

$$H = -i \sum_{r,s} B_{rs} q_r q_s, \tag{5}$$

(whose exact form will be better justified in the following) provided we assume suitable *anticommutation* relations for the q_r. Substituting in eq. (4) the successive equations, one finds:

$$\begin{aligned}\sum_s B_{rs} q_s &= \frac{2\pi}{h} \sum_{s,l,m} A_{rs} B_{lm} \big(q_s q_l q_m - q_l q_m q_s \big) = \\ &= \frac{2\pi}{h} \sum_{s,l,m} A_{rs} B_{lm} \big[\big(q_s q_l + q_l q_s \big) q_m - q_l \big(q_s q_m + q_m q_s \big) \big] = \\ &= \frac{2\pi}{h} \sum_{lm} B_{lm} \left\{ q_m \left[\sum_s A_{rs} \big(q_s q_l + q_l q_s \big) \right] + \left[\sum_s A_{rs} \big(q_s q_l + q_l q_s \big) \right] q_m \right\},\end{aligned}$$

so that it suffices to set:

$$\sum_s A_{rs} \big(q_s q_l + q_l q_s \big) = \frac{h}{4\pi} \delta_{rl}, \tag{6}$$

for eqs. (4) to be satisfied. Denoting by $\|A^{-1}_{rs}\|$ the inverse matrix of $\|A_{rs}\|$, eq. (6) can be written as:

$$q_r q_s + q_s q_r = \frac{h}{4\pi} A^{-1}_{rs}. \tag{6$'$}$$

In the special case where A is reduced to the diagonal form:

$$A_{rs} = a_r \delta_{rs},$$

we have therefore:

$$q_r q_s + q_s q_r = \frac{h}{4\pi a_r} \delta_{rs}. \tag{7}$$

We shall now apply the present scheme to the Dirac equations.

2. It is well known that one can eliminate the imaginary unit from the Dirac equations with no external field:

$$\left[\frac{W}{c} + (\alpha, p) + \beta mc \right] \psi = 0, \tag{8}$$

with an appropriate choice of the operators α and β (and this can be done in a relativistically invariant fashion). We shall, in fact, refer to a system of intrinsic coordinates such as to make eqs. (8) real, keeping explicitly in mind that the formulae we shall derive are not valid, without suitable modification, in a more general coordinate system. Denoting, as usual, with σ_x, σ_y, σ_z and ρ_1, ρ_2, ρ_3 two independent sets of Pauli matrices, we set:

$$\alpha_x = \rho_1\sigma_x; \qquad \alpha_y = \rho_3; \qquad \alpha_z = \rho_1\sigma_z; \qquad \beta = -\rho_1\sigma_y. \tag{9}$$

Dividing eqs. (9) by $-\frac{h}{2\pi i}$ and defining $\beta' = -i\beta$, $\mu = \frac{2\pi mc}{h}$, we obtain the real equations:

$$\left[\frac{1}{c}\frac{\partial}{\partial t} - (\alpha, \mathrm{grad}) + \beta'\mu\right]\psi = 0. \tag{8$'$}$$

As a consequence, eqs. (8) separate into two independent sets of equations, one for the real and one for the imaginary part of ψ. We set $\psi = U + iV$ and consider the real equations (8′) as acting on U:

$$\left[\frac{1}{c}\frac{\partial}{\partial t} - (\alpha, \mathrm{grad}) + \beta'\mu\right]U = 0. \tag{10}$$

The latter equations, *by themselves*(4), *i.e.* without the similar equations involving V, can be derived from the variational principle previously illustrated and quantized, as indicated above. Nothing similar could be done with elementary methods. Equation (10) can be obtained from the variational principle:

$$\delta\int i\frac{hc}{2\pi}U^*\left[\frac{1}{c}\frac{\partial}{\partial t} - (\alpha, \mathrm{grad}) + \beta'\mu\right]U dq dt = 0. \tag{11}$$

It is easy to verify that the conditions (3), in their natural extension to a continuous system, are obeyed. Following eqs. (7) the anticommutation relations hold:

$$U_i(q)U_k\big(q'\big) + U_k\big(q'\big)U_i(q) = \frac{1}{2}\delta_{ik}\delta\big(q - q'\big), \tag{12}$$

while the energy, according to (5) is:

$$H = \int U^*\big[-c(\alpha, p) - \beta mc^2\big]U dq. \tag{13}$$

The relativistic invariance of (12) and (13) does not require a separate demonstration. If one adds to these equations the analogous ones involving V, as well as the anticommutation relations: $U_r(q)V_s(q') + V_s(q')U_r(q) = 0$, one reobtains the usual Jordan-Wigner scheme, applied to the Dirac equations without external field.

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(4) The behaviour of U under space reflection can be conveniently defined taking into account that a simultaneous change of sign of U_r has no physical significance, as already implied by other reasons. In our scheme: $U'(q) = RU(-q)$ with $R = i\rho_1\sigma_y$ and $R^2 = -1$. Similarly, for a time reflection: $U'(q, t) = i\rho_2 U(q, -t)$.

It is remarkable, however, that the part of the formalism which refers to U (or V) can be considered, in itself, as the theoretical descriptions of some material system, in conformity with the general methods of quantum mechanics. The fact that the reduced formalism cannot be applied to the description of positive and negative electrons may well be attributed to the presence of the electric charge, and it does not invalidate the statement that, at the present level of knowledge, eqs. (12) and (13) constitute the simplest theoretical representation of neutral particles. The advantage, with respect to the elementary interpretation of the Dirac equation, is that there is now no need to assume the existence of antineutrons or antineutrinos (as we shall see shortly). The latter particles are indeed introduced in the theory of positive β-ray emission([5]); the theory, however, can be obviously modified so that the β-emission, both positive and negative, is always accompanied by the emission of a neutrino.

Considering the interest that the above-mentioned hypothesis gives to eqs. (12) and (13), it seems useful to examine their meaning more closely. To this aim, we developed U, inside a cube of side L, over the system of periodical functions:

$$f_\gamma(q) = \frac{1}{L^{3/2}} e^{2\pi i(\gamma,q)}, \tag{14}$$

$$\gamma = \big(\gamma_x, \gamma_y, \gamma_z\big); \qquad \gamma_x = \frac{n_1}{L}, \qquad \gamma_y = \frac{n_2}{L}, \qquad \gamma_z = \frac{n_3}{L};$$

$$n_1, n_2, n_3 = 0, \pm 1, \pm 2, \dots$$

setting:

$$U_r(q) = \sum_\gamma a_r(\gamma) f_\gamma(q). \tag{15}$$

As a consequence of the reality of U, we have:

$$a_r(\gamma) = \bar{a}_r(-\gamma). \tag{16}$$

In the general case, $\gamma \neq 0$, it follows from (12) that:

$$\begin{cases} a_r(\gamma)\bar{a}_s(\gamma) + \bar{a}_s(\gamma)a_r(\gamma) = \dfrac{1}{2}\delta_{rs}, \\ a_r(\gamma)a_s(\gamma) + a_s(\gamma)a_r(\gamma) = 0, \\ \bar{a}_r(\gamma)\bar{a}_s(\gamma) + \bar{a}_s(\gamma)\bar{a}_r(\gamma) = 0. \end{cases} \tag{17}$$

Furthermore, these quantities anticommute with $a(\gamma')$ and $\bar{a}(\gamma')$, when γ' differs both from γ and from $-\gamma$.

([5]) See G. WICK, "Rend. Accad. Lincei", *21*, 170 (1935).

The expression of the energy resulting from (13) is:

$$H = \sum_{\gamma} \sum_{r,s=1}^{4} \big[- hc(\gamma, \alpha^{rs}) - mc^2 \beta^{rs} \big] \bar{a}_r(\gamma) a_s(\gamma). \tag{18}$$

The x component of the linear momentum corresponds to the unit translation along x, up to the factor $\frac{h}{2\pi} i$, as usual:

$$M_x = \int U^* p_x U dq = \sum_{\gamma} \sum_{r=1} h\gamma_x \bar{a}_r(\gamma) a_k(\gamma), \tag{19}$$

and similarly for M_y and M_z.

For any value of γ we have in (18) a Hermitian form which has, notoriously, two positive and two negative eigenvalues, all equal in absolute value to $c\sqrt{m^2c^2 + h^2\gamma^2}$.

We can thus replace (18) by:

$$H = \sum_{\gamma} c\sqrt{m^2c^2 + h^2\gamma^2} \Big[\bar{b}_1(\gamma) b_1(\gamma) + \bar{b}_2(\gamma) b_2(\gamma) - \bar{b}_3(\gamma) b_3(\gamma) - \bar{b}_4(\gamma) b_4(\gamma) \Big] \tag{18$'$}$$

b_r being appropriate linear combinations of the a_r, obtained by a unitary transformation. Furthermore, it follows from (16) that $b_r(\gamma)$ are linearly related to $\bar{b}_r(-\gamma)$.

The Hermitian form (18), for a given value of γ, remains invariant under the exchange of γ with γ, as a consequence of (16) and (17). From this, and keeping again (17) into account, it follows that we can set:

$$b_3(\gamma) = \bar{b}_1(-\gamma); \qquad b_4(\gamma) = \bar{b}_2(-\gamma). \tag{20}$$

We introduce, for simplicity, the new variables:

$$B_1(\gamma) = \sqrt{2} b_1(\gamma); \qquad B_2(\gamma) = \sqrt{2} b_2(\gamma), \tag{21}$$

and we obtain:

$$H = \sum_{\gamma} c\sqrt{m^2c^2 + h^2\gamma^2} \sum_{r=1}^{2} \left[n_r(\gamma) - \frac{1}{2} \right], \tag{22}$$

$$M_x = \sum_{\gamma} h\gamma_x \sum_{r=1}^{2} \left[n_r(\gamma) - \frac{1}{2} \right], \tag{23}$$

where we have set:

$$n_r(\gamma) = \bar{B}_r(\gamma) B_r(\gamma) = \begin{cases} 0 \\ 1 \end{cases}$$

considering, furthermore, that the following relations hold:

$$(24)\qquad \begin{cases} B_r(\gamma)\bar{B}_s(\gamma') + \bar{B}_s(\gamma')B_r(\gamma) = \delta_{\gamma\gamma'}\delta_{rs}, \\ B_r(\gamma)B_s(\gamma') + B_s(\gamma')B_r(\gamma) = 0, \\ \bar{B}_r(\gamma)\bar{B}_s(\gamma') + \bar{B}_s(\gamma')\bar{B}_r(\gamma) = 0, \end{cases}$$

as it would follow formally, in the Jordan-Wigner scheme, for the coefficients in the development of a *two-component* matter-wave.

The preceding formulae are entirely analogous to those obtained in the quantization of the Maxwell equations, except for the different statistic. In the place of massless quanta, we have particles with a finite mass and also for them we have two available polarization states. In the present case, *as in the case of the electromagnetic radiation*, the half-quanta of rest energy and momentum are present, except that they appear with the opposite sign, in apparent connection with the different statistic. They do not constitute a specific difficulty, and they must be considered simply as additive constants, with no physical significance.

Similarly to the case of light quanta, it is not possible to describe with eigenfunctions the states of such particles. In the present case, however, the presence of a rest mass allows one to consider the *non-relativistic approximation*, where all the notions of elementary quantum mechanics apply, obviously. The non-relativistic approximation may be useful primarily in the case of the heavy particles (neutrons).

The simplest way to go to the configuration space representation is to associate the following plane wave to each oscillator:

$$\frac{1}{L^{3/2}} e^{2\pi i(\gamma,q)} \delta_{\sigma\sigma_r}, \qquad (r = 1, 2),$$

corresponding to the same value of the momentum, and with two possible polarization states, to keep into account the multiplicity of oscillators. We can go further, and describe not a simple particle, but a system with an indefinite number of particles with the two-valued, complex eigenfunction $\Phi = (\Phi_1, \Phi_2)$, according to the Jordan-Wigner method. It is sufficient to set:

$$(25)\qquad \begin{cases} \Phi_1(q) = \sum\limits_{\gamma} \dfrac{1}{L^{3/2}} e^{2\pi i(\gamma,q)} B_1(\gamma), \\ \Phi_2(q) = \sum\limits_{\gamma} \dfrac{1}{L^{3/2}} e^{2\pi i(\gamma,q)} B_2(\gamma). \end{cases}$$

In the non-relativistic approximation ($|\gamma| \ll \frac{mc}{h}$) the constants $b_r(\gamma)$ in ($18'$) are linear combinations of $a_r(\gamma)$, *with* γ*-independent coefficients.*

The latter coefficients depend only upon the elements of γ and, according to (9), we have:

$$b_1(\gamma) = \frac{a_3(\gamma) - ia_2(\gamma)}{\sqrt{2}}; \qquad b_3(\gamma) = \frac{a_3(\gamma) + ia_2(\gamma)}{\sqrt{2}},$$
$$b_2(\gamma) = \frac{a_4(\gamma) + ia_1(\gamma)}{\sqrt{2}}; \qquad b_4(\gamma) = \frac{a_4(\gamma) - ia_1(\gamma)}{\sqrt{2}},$$

which satisfy also eq. (20), as a consequence of (16). From eqs. (15) and (25) we have, in the non-relativistic approximation:

$$\begin{cases} \Phi_1(q) = U_3(q) - iU_2(q), \\ \Phi_2(q) = U_4(q) + iU_1(q). \end{cases} \tag{26}$$

On the purely formal side, we note that $\Phi = (\Phi_1, \Phi_2)$ coincides, up to a $\sqrt{2}$ factor, with the pair of *large* eigenfunctions of eqs. (10), when interpreted in the usual way, that is with no reality restriction.

To prove this, it is enough to verify that the transformation $\psi = \frac{1-\rho_2\sigma_y}{\sqrt{2}}U$ allows one to go from the scheme (9) to the usual Dirac scheme ($\alpha = \rho_1\sigma$; $\beta = \rho_3$), so that, effectively:

$$\psi_3 = \frac{1}{\sqrt{2}}\Phi_1, \qquad \psi_4 = \frac{1}{\sqrt{2}}\Phi_2;$$

notoriously, in the latter scheme, ψ_3 and ψ_4 are the large components. This relation clarifies the transformation law of Φ with respect to space rotations, but it has no meaning, obviously, with respect to general Lorentz transformations.

The existence of simple formulae such as (26) could lead one to suspect that, to a certain extent, the passage through plane waves is superfluous. As a matter of fact, such a passage is conceptually needed to obtain the cancellation of the *rest-energy half-quanta*. In fact, after such cancellation, the method of the energy is naturally given by:

$$H = \int \tilde{\Phi} \left(mc^2 + \frac{1}{2m}p^2 \right) \Phi dq, \tag{27}$$

to first approximation, and it differs in an essential way from (13).

3. As we have already said, the scheme (12) is not sufficient to describe charged particles; but, upon the introduction of a further quadruplet of real quantities V_r, analogous to the U_r, one re-obtains the usual electrodynamics, in a form symmetric with respect to the electron and positron. We consider, therefore, two sets of real quantities, representing the matter particles and the electromagnetic field, respectively. Quantities of the first kind are to be interpreted according to the scheme described in Sect. 1. Quantities of the second kind, *i.e.* the electromagnetic potentials φ and $A = (A_x, A_y, A_z)$, can be

interpreted as classical quantities, and have to be quantized according to the Heisenberg rule, based on the correspondence principle. The Maxwell and Dirac equations (with the above-mentioned restriction for the latter) can be obtained from a variational principle:

$$\delta \int L dq dt = 0$$

L being the sum of three terms:

$$L = L' + L'' + L'''.$$

The first term refers to the matter wave:

$$L' = i\frac{hc}{2\pi}\left\{U^*\left[\frac{1}{c}\frac{\partial}{\partial t} - (\alpha, \text{grad}) + \beta'\mu\right]U + \right. \\ \left. + V^*\left[\frac{1}{c}\frac{\partial}{\partial t} - (\alpha, \text{grad}) + \beta'\mu\right]V\right\}, \tag{28}$$

while the second describes the radiation field, which we suppose to be quantized according to the method of FERMI([6]):

$$L'' = \frac{1}{8\pi}(E^2 - H^2) - \frac{1}{8\pi}\left(\frac{1}{c}\dot{\varphi} + \text{div}\, A\right)^2. \tag{29}$$

We must therefore impose the auxiliary condition

$$\frac{1}{c}\dot{\varphi} + \text{div}\, A = 0. \tag{30}$$

The expression given in (29) differs from the one used by Fermi, but for integrable terms only. It leads to a definition of the momentum P_0, conjugate to φ, such as to allow one to eliminate immediately one of the two longitudinal waves, without having to go through the plane-wave development; in this respect, it is completely immaterial whether the second term in the expression (29) for L'' is multiplied by an arbitrary, non-vanishing constant. As for L''' it must be so chosen that $\psi = U + iV$ obeys the Dirac equation (8) completed with the external field, *i.e.* the equation:

$$\left[\frac{W}{c} + \frac{e}{c}\varphi + \left(\alpha, p + \frac{e}{c}A\right) + \beta mc\right]\psi = 0.$$

In practice, this requirement leads to:

$$L''' = ieU^*[\varphi + (\alpha, A)]V - ieV^*[\varphi + (\alpha, A)]U. \tag{31}$$

([6]) E. FERMI, "Rend. Accad. Lincei", *9*, 881 (1929).

Upon variation of the electromagnetic potentials, we obtain the following expressions for the charge and current densities:

$$\text{(32)} \qquad \begin{cases} \rho = -ie(U^*V - V^*U) = -e\dfrac{\tilde{\psi}\psi - \psi^*\bar{\psi}}{2}, \\ I = ie(U^*\alpha V - V^*\alpha U) = e\dfrac{\tilde{\psi}\alpha\psi - \psi^*\alpha\bar{\psi}}{2}. \end{cases}$$

These expressions differ from the usual ones for *infinite constants* only. The cancellation of such infinite constants is required by the symmetry of the theory, which is already implicit in the form chosen for the variational principle; in fact, the exchange of U_r and V_r, which appear symmetrically in L', is equivalent to changing sign to the electric charge.

U and V obey the anticommutation relations:

$$U_r(q)U_s(q') + U_s(q')U_r(q) = \frac{1}{2}\delta(q-q')\delta_{rs},$$
$$V_r(q)V_s(q') + V_s(q')V_r(q) = \frac{1}{2}\delta(q-q')\delta_{rs},$$
$$U_r(q)V_s(q') + V_s(q')U_r(q) = 0,$$

which are equivalent to the usual Jordan-Wigner scheme, if we set $\psi = U + iV$. The electromagnetic potentials φ, A_x, A_y, A_z, on the other side, obey to the usual commutation relations with their conjugate momenta, *e.g.* $P_0(q)\varphi(q') - \varphi(q')P_0(q) = \frac{h}{2\pi i}\delta(q-q')$, with:

$$\text{(33)} \qquad \begin{cases} P_0 = -\dfrac{1}{4\pi c}\left(\dfrac{1}{c}\dot{\varphi} + \operatorname{div} A\right), \\ P_x = -\dfrac{1}{4\pi c}E_x; \qquad P_y = -\dfrac{1}{4\pi c}E_y; \qquad P_z = -\dfrac{1}{4\pi c}E_z. \end{cases}$$

The energy is made up of three terms: $H = H' + H'' + H'''$ is derived from L', according to the rules already illustrated. The second term is obtained from the classical rules: $H'' = \int[P_0\dot{\varphi} + (P, A) - L'']dq$, where $P = (P_x, P_y, P_z)$. As for H''', it can be obtained from L''', following either methods (in our case $H''' = -\int L'''dq$) as it must be, since L''' is a function of both the matter and the electromagnetic field variables. This, by the way, proves the necessity of the ansatz (5). The continuity equation (30), is obeyed at any time, if it holds initially together with the divergence equation $\operatorname{div} E = 4\pi\rho$. It follows from (33) that the kinematics defined by the exchange rules has to be reduced by the use of the equations:

$$\text{(34)} \qquad \begin{cases} P_0(q) = 0, \\ \operatorname{div} P + \dfrac{1}{c}\rho = 0, \end{cases}$$

and therefore by assigning fixed values to two field quantities, with the corresponding indeterminacy in the conjugate variables. The *first* of (34) implies therefore, the elimination of P_0 and φ from the expression of H. The elimination is easily obtained by making use of (33), and one arrives, in this way, at the expression:

$$H = \int \left\{ \tilde{\psi}\left[- c(\alpha, p) - \beta mc^2 \right]\psi - (A, I) + 2\pi c P^2 + \frac{1}{8\pi}\left| \operatorname{rot} A \right|^2 \right\} dq. \tag{35}$$

As for relativistic invariance, we note that $\psi = U + iV$ obeys the Dirac equations, and that the Maxwell equations also hold, with charge and current densities which obey the relativistic transformation law. These two facts guarantee that the complete proof of the invariance of the theory is already implicit in the results of HEISENBERG and PAULI([7]). We turn now to the interpretation of the formalism.

4. Upon developing the U in the basis of the periodical functions considered before, and similarly for the V, we find as the obvious extension of (22), and after cancellation of the rest-energy half-quanta:

$$H' = \sum_{\gamma} c\sqrt{m^2c^2 + h^2\gamma^2} \sum_{r=1}^{2} \left[\bar{B}_r(\gamma)B_r(\gamma) + \bar{B}'_r(\gamma)B'_r(\gamma) \right], \tag{36}$$

where B_r and B'_r refer to the development of U and V, respectively; B_r and B'_r and their conjugate variables obey the usual anticommutation relations. If, for each value of γ, we introduce four appropriate spin functions $\xi_s(\gamma)$ $(s = 1, 2, 3, 4)$ assuming four complex values and forming a unitary system, we can set:

$$\begin{cases} U = \dfrac{1}{\sqrt{2}} \displaystyle\sum_{\gamma} \left\{ B_1(\gamma)\xi_1(\gamma) + B_2(\gamma)\xi_2(\gamma) + \bar{B}_1(-\gamma)\xi_3(\gamma) + \bar{B}_2(-\gamma)\xi_4(\gamma) \right\} f_{\gamma}(q), \\ V = \dfrac{1}{\sqrt{2}} \displaystyle\sum_{\gamma} \left\{ B'_1(\gamma)\xi_1(\gamma) + B'_2(\gamma)\xi_2(\gamma) + \bar{B}'_1(-\gamma)\xi_3(\gamma) + \bar{B}'_2(-\gamma)\xi_4(\gamma) \right\} f_{\gamma}(q), \end{cases} \tag{37}$$

the following relations being, furthermore, satisfied:

$$\begin{cases} \xi_3(\gamma) = \bar{\xi}_1(-\gamma), \\ \xi_4(\gamma) = \bar{\xi}_2(-\gamma). \end{cases} \tag{38}$$

([7]) W. HEISENBERG and W. PAULI, "Z. Physik", *56*, 1 (1929); *59*, 168 (1930).

It follows from the expression (32) for the electric charge density that the total charge is given by:

$$Q = -\frac{ie}{2}\int \left[U^*(q)V(q) - V^*(q)U(q)\right]dq = \tag{39}$$

$$= -\frac{ie}{2}\sum_{\gamma}\sum_{r=1}^{2}\left[B_r(\gamma)\bar{B}'_r(\gamma) + \bar{B}_r(\gamma)B'_r(\gamma) - \bar{B}'_r(\gamma)B_r(\gamma) - B'_r(\gamma)\bar{B}_r(\gamma)\right].$$

If we set:

$$C_r^{\mathrm{el}} = \frac{B_r + iB'_r}{\sqrt{2}}\,; \qquad C_r^{\mathrm{pos}} = \frac{B_r - iB'_r}{\sqrt{2}} \tag{40}$$

we can transform the expressions (36) and (39) for the energy and charge into the form:

$$H' = \sum_{\gamma} c\sqrt{m^2c^2 + h^2\gamma^2}\sum_{r=1}^{2}\left(\bar{C}_r^{\mathrm{el}}C_r^{\mathrm{el}} + \bar{C}_r^{\mathrm{pos}}C_r^{\mathrm{pos}}\right) \tag{41}$$

$$Q = e\sum_{\gamma}\sum_{r=1}^{2}\left[-\left(\bar{C}_r^{\mathrm{el}}C_r^{\mathrm{el}} - \frac{1}{2}\right) + \bar{C}_r^{\mathrm{pos}}C_r^{\mathrm{pos}} - \frac{1}{2}\right] = \tag{42}$$

$$= e\sum_{\gamma}\sum_{r=1}^{2}\left(-\bar{C}_r^{\mathrm{el}}C_r^{\mathrm{el}} + \bar{C}_r^{\mathrm{pos}}C_r^{\mathrm{pos}}\right).$$

The elimination of the *half-quanta of electricity* is, therefore, automatic, provided we perform the internal sum first. Equations (41) ans (42) represent a set of oscillators which are equivalent to a double system of particles obeying the Fermi statistic, with rest mass m and charge $\pm e$; the variables C_r^{pos} refer to positrons and the C_r^{el} to electrons.

The elimination of the longitudinal electric field by the second equation in (34) is somewhat different in a symmetric theory because it is not possible to cast ρ, as it results from (32), in a diagonal form. The result of the elimination is well known in ordinary electrodynamics (though partially illusory because of convergence difficulties) where $\rho = -e\tilde{\psi}\psi$; but it is equally known if one starts from $\rho = e\psi^*\bar{\psi}$ because the latter position is fully equivalent to exchange the role of electron and positron, considering the latter as a real particle and the former as a positron "hole". It seems plausible that those matrix elements which mantain the same form in the two opposite theories remain the same in the symmetric theory.

We thus assume to have already eliminated the irrotational part of A and P. The expression (35) for H is modified in two ways: first by assuming that A and P in this expression represent only the divergence free part of such vectors; secondly by adding a term which represents the electrostatic energy. The latter term takes a different form in the ordinary theory (electron-electron hole) and in the opposite theory. Keeping the interaction of each particle with itself, one has in the first theory:

$$H_{\text{els}} = \frac{e^2}{2} \iint \frac{1}{|q - q'|} \tilde{\psi}(q)\psi(q)\tilde{\psi}(q')\psi(q')\,dq\,dq',$$

while in the second theory:

$$H_{\text{els}} = \frac{e^2}{2} \iint \frac{1}{|q - q'|} \psi^*(q)\bar{\psi}(q)\psi^*(q')\bar{\psi}(q')\,dq\,dq'.$$

Using (37) and (40) one can express the electrostatic energy as a function of the C's. The only terms which have given rise to physical applications are identical in the two theories: they are those which can be interpreted, from the particle viewpoint, as repulsion or attraction between distinct particles of the same or of the opposite type.

For what concerns the interaction with the radiation field, the only difference between the symmetric and the ordinary theory lies in the cancellation of undetermined constants, relative to the single oscillators, in the expression for the current density; again the formulae of interest for the applications remain unchanged.

Comment on the Scientific Paper no. 9 *: "A symmetric theory of electrons and positrons".*

Written in 1937, one year before his tragic disappearence, in a concise and elegant Italian language, this article probably represents the best long-lasting contribution of Ettore Majorana to particle physics.

The article tackles the problem of formulating the Dirac theory without the cumbersome sea of negative-energy states. In the usual formulation one would start from a highly asymmetric situation, to discover only at the end that there is a perfect symmetry between electrons and positrons. The symmetry is so little evident that Dirac himself tried at first to identify the positively charged particles, the holes, with protons!

Referring to the usual formulation, M. notes that:

"*the prescriptions needed to cast the theory into a symmetric form, in conformity to its content, are however not entirely satisfactory because one always starts from an asymmetric form or because symmetric results are obtained after one applies appropriate procedures, such as the cancellation of divergent constants, that one should possibly avoid. For these reasons, we have attempted a new approach, which leads more directly to the desired result.*"

From these premises, M. formulates a field theory based on anticommuting variables, hence without classical interpretation, and derives the Dirac equation from a variational principle.

The electron is represented by a complex field, which can be divided into Hermitian and anti-Hermitian components. However, in the representation where the Dirac matrices are all imaginary (henceforth called the M. representation) each component "*can be considered, in itself, as the theoretical description of some material system, in conformity with the general methods of quantum mechanics.*"

M. promptly recognizes, of course, that we cannot avoid introducing both components for the electron, which admits a conserved charge. But the simplicity of the scheme leads him to speculate that his theory can find application to the case of electrically neutral particles.

"*The advantage... is that there is no reason now to infer the existence of antineutrons or antineutrinos. The latter particles are introduced in the theory of positive β-ray emission; the theory, however, can be obviously modified so that the β-emission, both positive and negative, is always accompanied by the emission of a neutrino.*"

M. refers here to the theory of positive β-rays formulated two years before, in Rome, by Giancarlo Wick.

Unexpectedly, from a reformulation of the Dirac theory, a novel physical possibility emerges, which has since been the object of theoretical and experimental scrutiny. We have not succeeded, yet, to find a definitive answer to M.'s proposal.

The M. neutrino has met with alternating fortunes, somehow superimposing to the "two-component neutrino" theory, formulated by Herman Weyl a few years before, in 1929.

It is a fact that the neutral particles emitted in negative or positive β-decays behave differently: in the interaction with atomic nuclei, the former particles produce invariably positrons, the latter electrons. However, with a $V - A$ interaction, we can associate the different behaviour to the different helicity of the emitted neutral particle.

Since helicity is strictly conserved for massless particles, deviations from this pattern are due to terms in the amplitude of the order of the ratio: (neutrino mass)/(neutrino energy), which is unobservably small in all neutrino-induced reactions.

The Majorana nature of the neutrino can be tested in the so-called *neutrinoless double β-decays*. These are second-order processes in the Fermi theory, whereby a virtual neutrino of positive elicity is emitted, together with an electron, and reabsorbed as if it were a neutrino (negative elicity), with emission of a second electron. The overall process: $N^* \to N + 2e$, violates lepton number conservation and is proportional to the Majorana mass of the neutrino. The observation of neutrinoless double β-decay would be an evident proof that: "*there is no reason to infer... the existence of... antineutrinos.*"

Long considered as an exotic possibility, the M. neutrino has emerged, in our times, as the most natural explanation for the surprisingly small value of neutrino masses.

In addition, the non-conservation of a lepton number, L, leads to speculate that the decay of supermassive M. neutrinos in the primordial Universe may have given rise to an asymmetry in L, transformed in the presently observed baryon number asymmetry by virtue of $B - L$ conservation.

Several laboratories around the world host experiments to detect neutrinoless double β-decay, thus far with no success.

A new-generation experiment, CUORE, is in preparation in the INFN laboratory below the Gran Sasso Mountain in Central Italy. With dimensions never reached before, CUORE should put very stringent limits to the process... or maybe observe it.

I like to think that the answer to the question posed by Majorana more than half a century ago may be found precisely in our country giving, at the same time, a possible explanation to the dominance of matter over antimatter in our Universe, on which our very same existence depends.

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