Universal nature of quantum supremacy

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Abstract. We disclose the universal nature of computational ♯P-hardness and quantum supremacy of quantum many-body systems. We do so by means of the new powerful technique (the hafnian master theorem) that allows one to address the ♯P-hard problems systematically. We consider a generic example of many-body interacting systems – a trapped BEC-gas of interacting Bose atoms, apply the hafnian master theorem and refer to the Toda's theorem on a ♯P-complete oracle.

1. Introduction

Revealing the nature of the computational ♯P-hardness and quantum supremacy of quantum many-body systems is one of the central problems in modern quantum physics [1, 2, 3]. There is an open fundamental question: What is the nature of the ♯P-hard complexity of quantum many-body systems, or where does the quantum supremacy over classical computers come from? We answer it by means of the new powerful technique (the hafnian master theorem) that allows us to address the ♯P-hard problems on a regular basis. We sketch this technique by considering an example of atomic boson sampling from an interacting Bose-Einstein-condensed gas.

Atomic boson sampling of noncondensed atom numbers in an interacting BEC-gas is a new platform for studying quantum supremacy [4, 5, 6]. It is very different from photonic boson sampling in a linear interferometer widely studied in the last decade [7, 8, 9, 10]. Our analysis is based on the general approach towards unification of nature's complexities via the hafnian and permanent ♯P-complete matrix functions [11], the newly found hafnian master theorem [12] and implementation of the Toda's theorem on a ♯P-complete oracle [13]. We outline an easy-to-follow analytical theory of atomic boson sampling for a BEC-gas in a box trap presented in our recent paper [6]. In this case the sampling probability distribution and its characteristic function can be calculated explicitly. We find that two necessary ingredients of the ♯P-hardness, squeezing and interference, are naturally present in the BEC gas even in equilibrium.

The existence of squeezing in the interacting BEC gas has been known since [14]. Contrary to Gaussian boson sampling of noninteracting photons in a linear interferometer, atomic boson sampling does not require sophisticated external sources of bosons in quantum squeezed states.

We suggest performing proof-of-principle experiments designed to demonstrate manifestations of the ♯P-hard complexity of atomic boson sampling [4, 5, 6]. Extracting the joint probability distribution for the occupations of just two excited atom states [6] (say, two counter-propagating plane waves or their unitary mixed counterparts as in Fig. 1) would be already a remarkable experimental achievement. Such experiments are even easier for implementation than the ones on the statistics of the total noncondensate occupation [15] successfully realized in [16, 17].

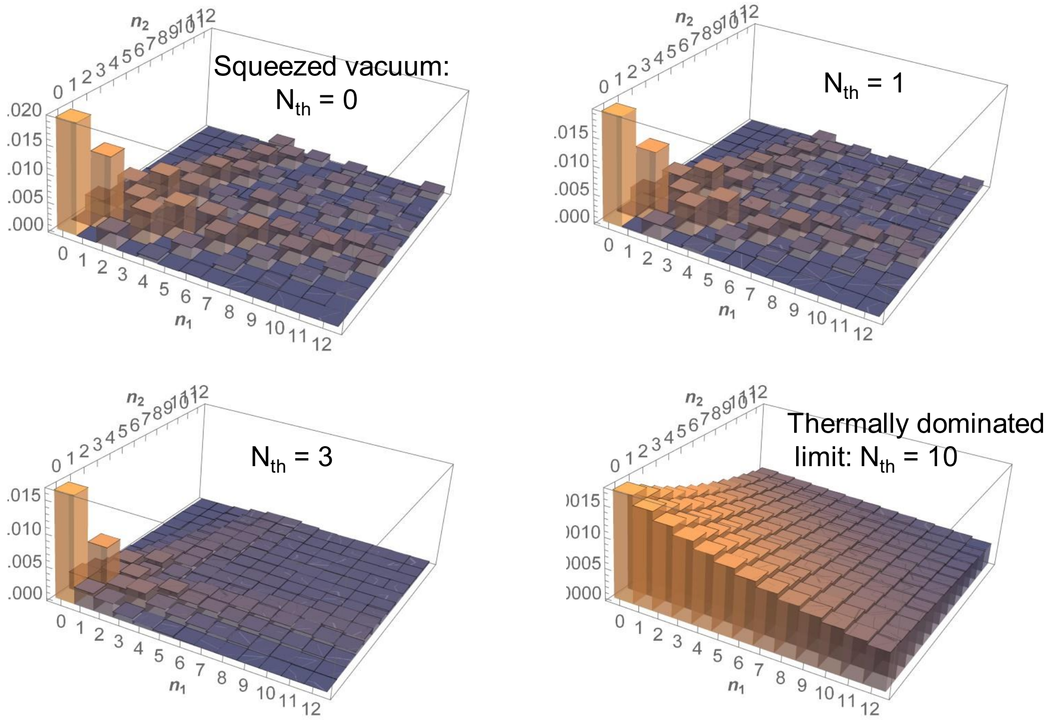

Figure 1. Joint occupation probabilities for two interfering squeezed excited-atom states: Nontrivial quantum statistics, which manifests ♯P-hard complexity of the many-body system, is hiding under thermal fluctuations as the thermal occupation, $N_{\rm th}$, of the relevant quasiparticle state increases from zero to ten quasiparticles.

2. Recent experiments on fluctuations of the atom numbers in the noncondensate of a trapped BEC gas

There are many BEC laboratories worldwide potentially capable of sampling and measuring atom number fluctuations from the noncondensed fraction of a BEC gas. Below we name a few of them. Their recent results are closely related to the proposed atomic boson sampling and leave no doubts for its experimental realization in the near future.

2.1. Fluctuations in the total number of noncondensed atoms (Arlt's group at the Aarhus University, Denmark)

There had been a remarkable experiment on measuring fluctuations in the total number of condensed atoms in a harmonic trap [16, 17]. It successfully resolved the main difficulty in the experiments on the total noncondensate fluctuations – a proper differentiation of the noncondensate from much more populated condensate in the trap with a large number of atoms, e.g., $N \sim 5 \times 10^5$ as in [16, 17].

2.2. Bogoliubov's pair correlations in fluctuations of the atom numbers (Clement's group at the CNRS, France)

The most pertinent to the proposed atomic boson sampling is the experiment done in the Clement's group at the CNRS (France) [18, 19]. They were capable to obtain a full counting statistics of atoms in the momentum space after their release from a trap and subsequent free-fall expansion. They observed strongly correlated pairs of atoms with opposite momenta in the interacting Bose gas. Amazingly, their detectors have a single-atom resolution and a large quantum efficiency.

2.3. Analog of optical boson sampling: A shallow optical-lattice potential as interferometer for an atom beam (Kaufman's group at the JILA, NIST, US)

The ongoing work [20] in the Kaufman's group at JILA and NIST aims at the cold-atom-based experiment implementing a straightforward analogy with the linear-interferometer approach to boson sampling of noninteracting photons. They use a shallow optical-lattice trapping potential as an interferometer for a beam of cold Bose atoms and reproduce an analog of the optical boson sampling experiments, but employing atoms instead of photons. The interference effects, for example, the Hong-Ou-Mandel effect, with cold atoms had been observed before, in particular, in [21]. Yet, such analogs of the linear-interferometer boson sampling with atoms, which are based on the nonequilibrium open system with atomic beams, are very different from the atomic boson sampling in the equilibrium BEC trap originally proposed in [4] and discussed here.

2.4. Analog of optical boson sampling: A boson sampling machine with ultracold atoms in a polarization-synthesized optical lattice (Alberti's group at the University of Bonn, Germany)

Basic building blocks of an analog of optical-boson-sampling interferometer were experimentally demonstrate in [22] by revealing the Hong-Ou-Mandel interference of two bosonic atoms in a 4-mode interferometer. They estimated the sampling rate for a large number of atoms N via a model based on a master equation and showed that quantum supremacy over today's best supercomputers can be reached with $N > 40$.

3. New avenues in the field of the BEC quantum simulation

The proposed atomic boson sampling infers new avenues in the field of the BEC quantum simulation. In particular, it suggests to study possible designs and realizations of the multi-qubit BEC trap [5] most suitable for the experiments on the atomic boson sampling. Another important direction of this research is related to inventing various multi-detector imaging techniques for measuring atom numbers in different groups of excited states and pioneering relevant quantum-statistical experiments.

This field of research promises discovery of a variety of new phenomena related to the effects of interaction, entanglement, and interference in the BEC-gas quantum statistics for the atomic boson sampling, many-body correlations, common or fragmented phase transitions, critical fluctuations, and appearance of the computational $\sharp$P-hard complexity.

Essentially, we suggest the atomic boson sampling in an interacting BEC-condensed gas as an alternative to the photonic boson sampling in a linear interferometer. We show that the process of many-body fluctuations in an interacting equilibrium BEC gas and its output statistics is $\sharp$P-hard for computing and potentially possesses quantum supremacy over classical simulators.

Analysis of the atomic boson sampling clarifies answers to the important general questions. Why do quantum many-body systems possess quantum supremacy over classical computers? What is a specific mechanism behind such supremacy? What is a proper theoretical tool to disclose the mystery of quantum supremacy?

We answer these questions by considering a generic quantum many-body interacting system - Bose-Einstein-condensed atoms confined in a trap. Its physics substantially differs from the physics of boson sampling of massless photons in the interaction-free, nonequilibrium, linear interferometer that has been widely studied and advertised in the last decade in view of its potential quantum supremacy. In the case of the trapped atoms there are the condensate, massive particles, interaction, thermal equilibrium, and no external sources of bosons. Despite these peculiarities, it is possible to solve this problem analytically and address computational ♯P-hardness of atomic boson sampling. In fact, the aforementioned peculiarities turn an atomic, multi-qubit BEC trap into a fruitful platform for testing quantum many-body processes with regard to their quantum supremacy over classical simulators.

4. Quantum statistical physics of the atomic boson sampling

We derive quantum statistics of the atomic boson sampling starting from the Bogoliubov-Popov Hamiltonian for the dilute weakly interacting Bose gas in the equilibrium phase at the temperature much lower than the critical temperature of Bose-Einstein condensation. The new idea is to go beyond and deeper than just calculating quasiparticles and their energy spectrum via the Bogoliubov transformation. This is done by employing the Bloch-Messiah reduction [23] of the Bogoliubov transformation. The analysis involves solving the Gross-Pitaevskii equation for the macroscopic condensate wave function and the Bogoliubov – de Gennes equations for quasiparticle excitations as well as calculating the covariance matrix of the inter-mode normal and anomalous correlations and full joint probability distribution via its characteristic function.

The main novelty of our analysis is finding explicitly the eigen-squeeze modes and their eigenvalues corresponding to the singular value decomposition of the symplectic matrix of the Bogoliubov transformation. As a result, the unique, irreducible Bloch-Messiah representation of the Bogoliubov transformation unambiguously discloses the existence and explicit form of two different fundamental eigen entities of the many-body interacting system – the quasiparticle eigen states and the eigen-squeeze modes. They specify, accordingly, two preferred bases for the creation/annihilation field operators: (a) the basis of the quasiparticles diagonalizing the Hamiltonian and (b) the basis of the eigen-squeeze single-particle excited states diagonalizing the Hermitian factor of the multimode squeeze matrix. The Bloch-Messiah reduction explicitly relates both aforementioned bases to the observational basis of atom excited states which can be selected at will by reconfigurating the atom number detectors. Those detectors make measurements by projecting atom wave functions onto the preselected observational basis wave functions.

The source of the ♯P-hard computational complexity is a combination of squeezing and interference, that is, the interference between the eigen-squeeze modes and both the quasiparticle wave functions, on one hand, and the observational excited atom states, on the other hand. A physical origin of squeezing is a spontaneous creation of two excited atoms from the condensate. Such a process is also the reason for the formation of the noncondensate via quantum depletion of the condensate, even at zero temperature. In the absence of quantum depletion, the squeezing also disappears. A thermal fraction of the noncondensate alone does not lead to squeezing and computational ♯P-hardness of the BEC-gas quantum statistics.

Specifically, we find an analytical solution for the characteristic function and joint probability distribution of excited atom numbers for the atomic boson sampling from a thermal state of a dilute weakly interacting BEC gas, see Fig. 1. The starting point of our analysis is a fact of two-mode squeezing of atom excitations in a trapped BEC gas established in [14] and strongly pronounced in fluctuations of the total number of condensed atoms in the interacting gas which were calculated in [15].

Curiously, two-times decrease, as compared to the case of an ideal, non-interacting BEC gas, in the variance of the total BEC occupation, that occurs due to interaction, had been assigned by Pitaevskii and co-authors in [24] to an accident, not to the effect of squeezing.

The central pillar of our approach is the hafnian master theorem [12] that gives the generating function for the hafnians which determine the joint probability distribution of atom numbers. The point is that it provides the universal, concise and explicit path to expressing the ♯P-hardness of computing quantum statistical processes in finite many-body interacting systems. Note that in general the hafnians and permanents are ♯P-complete, that is, require exponential time, for computing. Besides, the hafnian master theorem establishes a direct connection with the Wick's theorem of the standard quantum field theory and condensed matter physics.

Here we skip discussion of the corresponding equations and formulas which could be found in our detailed papers [4, 5, 6]. The most important fact is that squeezing of atom states naturally appears due to interaction and plays a crucial role in the origin of the computational ♯P-hard complexity of atomic boson sampling. In the absence of squeezing, the joint probability distribution of the excited atom numbers would be a simple function computable via Stockmeyer's approximating algorithm [7, 25] in polynomial time.

5. Conclusions

There is no mystery in the computational ♯P-hardness and quantum supremacy. In fact, the quantum theory of finite quantum systems is simple. It's just a linear algebra and combinatorics of the finite-size matrices, and their ♯P-hardness for computing is fully accounted for by hafnians.

The results [4, 6, 12] discussed above explicitly give the joint probability distribution of atom numbers for atomic boson sampling as a Fourier series of the characteristic function with the coefficients given by the hafnians which are ♯P-hard for computing. Therefore, quantum systems possess quantum supremacy for the problems involving multivariate integral for Fourier-series coefficients since their life naturally consists of performing such Fourier transforms.

Classical simulators, on the contrary, operate with classical functions of continuous variables (inverse Fourier transforms) and cannot perform Fourier-series transform in polynomial time.

Specifically, quantum supremacy and ♯P-hardness encrypted into the Gaussian (equilibrium) states are due to squeezing and interference of the many-body-system modes (see section 4) processing a multivariate Fourier transform of quantum statistics. In other words, the quantum statistics of the atom number fluctuations is determined by an interplay between the two intrinsic entities existing in the interacting BEC gas, the eigen-squeeze modes and the quasipaticle eigen-energy states, and the observational excited atom wave functions. The conventional, textbook approach focuses just on one of those two intrinsic entities, the quasiparticles and their energy spectrum, apparently, because they determine thermodynamic energy-related averaged properties of the many-body system. However, for the quantum statistics of fluctuations the central part is played by the other intrinsic entity – the eigen-squeeze modes of the interacting many-body system. Quantum supremacy, quantum simulations cannot be understood without analysis of the eigen-squeeze modes. This is *the first major conclusion* of our analysis [4, 6, 12].

The second major conclusion is that the nature of the quantum supremacy and ♯P-hard complexity has a universal origin – an intuitively obvious complexity of computing the multivariate integral in Fourier-series coefficients of a sign-indefinite strongly-oscillating function. It is related to the ♯P-hardness of computing matrix hafnians (or permanents) [6, 11, 12] which behave like a lacunary or fractal function with an exponentially wide spectrum.

In fact, any type of the computational ♯P-hardness is equivalent and fully represented by the aforementioned form of the computational ♯P-hardness in view of the Toda's theorem [13] on a ♯P-complete oracle. Computing a ♯P-complete hafnian and using it as an oracle is enough for polynomial-time reduction of every other ♯P-hard problem to an easy, polynomial-time problem. Thus, the multivariate Fourier-series integration fully reveals the general nature of the quantum supremacy and the computational ♯P-hardness of the many-body quantum systems.

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