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Calculating non-perturbative quantities through the world-line formalism

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Abstract. We present two applications of the world-line formalism to the calculation of non-perturbative quantities in QCD. The first quantity is the free energy of the gluon plasma in the high-temperature limit; the second quantity is the pair-production rate in the chromo-electric field of a flux tube. In the first case, where effects of spatial confinement in the dimensionally-reduced 3D Yang-Mills theory are primarily important, we calculate the free-energy density of a gluon propagating in the stochastic background fields through a suitable parametrization of the area- and the perimeter laws of the Wilson loop, which enters the corresponding one-loop effective action. In this way, we find that the order of the leading correction to the Stefan-Boltzmann free energy changes from $\mathcal{O}(\lambda)$ for $N \sim 1$ to $\mathcal{O}(\lambda^{3/2})$ for $N \gg 1$, where λ is the finite-temperature 't Hooft coupling, and N is the number of colors. In the second case, we find that, in the London limit of the dual superconductor, the Schwinger pair-production rate, $\sim \mathrm{e}^{-\mathrm{const}\cdot m^2}$, goes over to $\mathrm{e}^{-\mathrm{const}\cdot m}$. Given that the flux-tube field is static, we find such a conversion of the Gaussian distribution into an exponential one, remarkable.

1. Free energy of the gluon plasma in the high-temperature limit

In this Section, we address an important issue regarding the leading correction to the Stefan-Boltzmann law for the free-energy density of the gluon plasma at high temperatures. As we will see, this correction has the order[2] $\mathcal{O}(g^2)$ for $N \sim 1$, while this order changes to $\mathcal{O}(\lambda^{3/2})$ for $N \gg 1$, where $\lambda = g^2 N$ is the finite-temperature 't Hooft coupling, and N is the number of colors. The corrections to the Stefan-Boltzmann law stem from the spatial confinement of gluons constituting the plasma, as well as from the Polyakov loop. For our analysis, we will use the method developed in Refs. [1, 2]. We start with representing the partition function of the finite-temperature Euclidean Yang-Mills theory in the form

$$\mathcal{Z}(T) = \left\langle \int \mathcal{D}a_\mu^a \exp\left[-\frac{1}{4g^2}\int_0^\beta dx_4 \int_V d^3x \, (F_{\mu\nu}^a[A])^2\right]\right\rangle, \tag{1}$$

where $\beta \equiv 1/T$, and V is the three-dimensional volume occupied by the system. In Eq. (1), we have modeled spatial confinement of a_μ^a-gluons by means of the stochastic background fields B_μ^a. For this purpose, the full Yang-Mills field A_μ^a has been represented as a sum $A_\mu^a = B_\mu^a + a_\mu^a$, and the stochastic field B_μ^a has been averaged over with some measure $\langle \ldots \rangle$. Clearly, at

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[2] In this Section, we denote for brevity the finite-temperature Yang-Mills coupling g_T simply as g.

finite temperature T, both the a^a_μ- and the B^a_μ-fields obey the periodic boundary conditions $a^a_\mu(\mathbf{x},\beta) = a^a_\mu(\mathbf{x},0)$ and $B^a_\mu(\mathbf{x},\beta) = B^a_\mu(\mathbf{x},0)$. Integrating over the a^a_μ-gluons in the Gaussian approximation, and disregarding for simplicity gluon spin degrees of freedom, one obtains

$$\mathcal{Z}(T) = \langle \left\{\det\left[-(D^a_\mu[B])^2\right]\right\}^{-\frac{1}{2}\cdot 2(N^2-1)}\rangle = \langle \exp\left\{-(N^2-1)\mathrm{Tr}\,\ln\left[-(D^a_\mu[B])^2\right]\right\}\rangle, \qquad (2)$$

with the covariant derivative $(D_\mu[B]f_\nu)^a = \partial_\mu f^a_\nu + f^{abc}B^b_\mu f^c_\nu$. Equation (2) includes the color degrees of freedom of a^a_μ-gluons, and accounts for their $2(N^2-1)$ physical polarizations. In the one-loop approximation for the a^a_μ-field, this equation can be simplified further:

$$\mathcal{Z}(T) \simeq \exp\left\{-(N^2-1)\langle \mathrm{Tr}\,\ln\left[-(D^a_\mu[B])^2\right]\rangle\right\}. \qquad (3)$$

In Eq. (3), "Tr" includes the trace "tr" over color indices and the functional trace over space-time coordinates.

The free-energy density $F(T)$ is defined through the standard formula

$$\beta V F(T) = -\ln \mathcal{Z}(T). \qquad (4)$$

Using further for $\ln\left[-(D^a_\mu[B])^2\right]$ the proper-time representation, one has

$$F(T) = -(N^2-1)\cdot 2\sum_{n=1}^{\infty}\int_0^\infty \frac{ds}{s}\int \mathcal{D}z_\mu\, \mathrm{e}^{-\frac{1}{4}\int_0^s d\tau \dot{z}^2_\mu}\langle W[z_\mu]\rangle. \qquad (5)$$

The integration in Eq. (5) is performed over trajectories $z_\mu(\tau)$, which obey the periodic boundary conditions: $z_4(s) = z_4(0) + \beta n$ and $\mathbf{z}(s) = \mathbf{z}(0)$. The vector-function $z_\mu(\tau)$ describes therefore only the shape of the trajectory, while the factor βV on the left-hand side of Eq. (4) stems from the integration over positions of the trajectories. Furthermore, the summation over the winding number n yields a factor of 2, which accounts for winding modes with $n < 0$. The zero-temperature part of the free-energy density, corresponding to the zeroth winding mode, has been subtracted [1]. Finally, the Wilson loop that enters Eq. (5), reads $W[z_\mu] = \frac{1}{N^2-1}\,\mathrm{tr}\,\mathcal{P}\exp\left(i\oint dz_\mu B_\mu\right)$, where $B_\mu = B^a_\mu t^a$, and $(t^a)^{bc} = -if^{abc}$ is a generator of the adjoint representation of the group SU(N).

According to the lattice data [3], the correlation function $\langle g^2 H_i(x)H_k(x')\rangle$ exceeds by an order of magnitude the correlation function $\langle g^2 E_i(x)H_k(x')\rangle$. This fact allows one to approximately factorize $\langle W[z_\mu]\rangle$ as $\langle W[z_\mu]\rangle \simeq \langle W[\mathbf{z}]\rangle \prod_{n=-\infty}^{+\infty}\langle P^n\rangle$, where $\langle W[\mathbf{z}]\rangle = \left\langle \frac{1}{N^2-1}\,\mathrm{tr}\,\mathcal{P}\,\exp\left(i\oint dz_k B_k\right)\right\rangle$ is the averaged purely spatial Wilson loop, and $\langle P^n\rangle = \left\langle \frac{1}{N^2-1}\,\mathrm{tr}\,\mathcal{T}\exp\left(in\int_0^\beta dz_4 B_4\right)\right\rangle$ is a generalization of the Polyakov loop to the case of n windings. Upon this factorization, the world-line integral over $z_4(\tau)$ in Eq. (5) becomes that of a free particle, which yields

$$F(T) = -2(N^2-1)\sum_{n=1}^{\infty}\int_0^\infty \frac{ds}{s}\,\frac{\mathrm{e}^{-\frac{\beta^2 n^2}{4s}}}{\sqrt{4\pi s}}\,\langle P^n\rangle \oint \mathcal{D}\mathbf{z}\,\mathrm{e}^{-\frac{1}{4}\int_0^s d\tau \dot{\mathbf{z}}^2}\langle W[\mathbf{z}]\rangle. \qquad (6)$$

In order to calculate the world-line integral over $\mathbf{z}(\tau)$, we notice that the Wilson-loop average in the adjoint representation can be written as [4]

$$\langle W[\mathbf{z}]\rangle = \frac{1}{1+\frac{1}{N^2}}\left(\mathrm{e}^{-\sigma\Sigma} + \frac{1}{N^2}\,\mathrm{e}^{-c\cdot g^2\frac{N}{3}T\sqrt{\Sigma}}\right). \qquad (7)$$

Here, Σ is the area of the minimal surface bounded by the contour $\mathbf{z}(\tau)$, and c is some positive dimensionless constant, which will be determined below. Furthermore, Eq. (7) obeys the normalization condition $\langle W[\mathbf{0}]\rangle = 1$. The second exponential on the right-hand side of Eq. (7) represents the perimeter law e^{-mL}, where $L = \int_0^s d\tau|\dot{\mathbf{z}}|$ is the length of the contour $\mathbf{z}(\tau)$, and the constant m has the dimensionality of mass. Here, we have substituted L by $\sqrt{\Sigma}$, and parametrized m through the soft scale g^2NT as $m = c\cdot g^2\frac{N}{3}T$. The spatial string tension σ in the adjoint representation can be expressed in terms of the spatial string tension $\sigma_{\rm f}$ in the fundamental representation by means of Casimir scaling: $\frac{\sigma}{\sigma_{\rm f}} = \frac{2N^2}{N^2-1}$. This ratio is equal to 9/4 for $N = 3$, while going to 2 in the large-N limit. At temperatures $T > T_*$ of interest, where T_* is the temperature of dimensional reduction, one can express $\sigma_{\rm f}$ in terms of the string tension in the 3D Yang-Mills theory, which was calculated analytically in Ref. [5]. The corresponding expression for $\sigma_{\rm f}$ reads[3] $\sigma_{\rm f} = \frac{N^2-1}{8\pi}(g^2T)^2$, which yields the following spatial string tension in the adjoint representation: $\sigma = \frac{1}{4\pi}(g^2NT)^2$.

Hence, the free-energy density (6) can be written in the form $F = F_1 + F_2$, where the term F_1 corresponds to the exponential $\mathrm{e}^{-\sigma\Sigma}$ from Eq. (7), while the term F_2 corresponds to the exponential $\mathrm{e}^{-c\cdot g^2\frac{N}{3}T\sqrt{\Sigma}}$ from the same equation. Clearly, in the large-N limit, $F_1 \gg F_2$ due to the relative factor of $\frac{1}{N^2}$, so that the thermodynamics of the gluon plasma in that limit is fully determined by spatial confinement. Therefore, let us start with calculating the world-line integral $I \equiv \oint \mathcal{D}\mathbf{z}\,\mathrm{e}^{-\frac14\int_0^s d\tau\dot{\mathbf{z}}^2-\sigma\Sigma}$, which enters the term F_1. To this end, we implement for the minimal area Σ the following ansatz: $\Sigma = \frac12\int_0^s d\tau\,|\mathbf{z}\times\dot{\mathbf{z}}|$. It corresponds to a parasol-shaped surface made of thin segments. Furthermore, since $\int_0^s d\tau\,\mathbf{z} = 0$, the point where the segments merge is the origin. Therefore, the chosen ansatz for Σ automatically selects from all cone-shaped surfaces bounded by $\mathbf{z}(\tau)$ the one of the minimal area. We use further the approximation $\Sigma \simeq \sqrt{\mathbf{f}^2}$, where $\mathbf{f} \equiv \frac12\int_0^s d\tau(\mathbf{z}\times\dot{\mathbf{z}})$. In general, $\frac12\int_0^s d\tau\,|\mathbf{z}\times\dot{\mathbf{z}}|$ can be larger than $\sqrt{\mathbf{f}^2}$. This happens if, in the course of its evolution in spatial directions, the gluon performs backward and/or non-planar motions. Once this happens, the vector product $(\mathbf{z}\times\dot{\mathbf{z}})$ changes its direction, and the integral $\int_0^s d\tau(\mathbf{z}\times\dot{\mathbf{z}})$ receives mutually cancelling contributions. This so-called non-backtracking approximation is widely used in order to simplify the parametrizations of minimal surfaces allowing for an analytic calculation of the corresponding world-line integrals [7]. Using this approximation, one can calculate the integral I by representing the exponential $\mathrm{e}^{-\sigma\Sigma}$ as $\mathrm{e}^{-\sigma\Sigma} = \int_0^\infty \frac{d\lambda}{\sqrt{\pi\lambda}}\,\mathrm{e}^{-\lambda-\frac{\sigma^2\mathbf{f}^2}{4\lambda}}$, and introducing further an auxiliary space-independent magnetic field $\mathbf{H}$ according to the formula

$$\mathrm{e}^{-A\mathbf{f}^2} = \frac{1}{(4\pi A)^{3/2}}\int d^3H\,\mathrm{e}^{-\frac{\mathbf{H}^2}{4A}+i\mathbf{H}\mathbf{f}}, \qquad \text{where} \qquad A > 0. \tag{8}$$

The world-line integral gets then reduced to the one for a spinnless particle of an electric charge 1 interacting with the constant magnetic field $\mathbf{H}$, i.e. to the bosonic Euler-Heisenberg-Schwinger Lagrangian, which has the form [8]

$$\oint \mathcal{D}\mathbf{z}\,\mathrm{e}^{-\frac14\int_0^s d\tau\dot{\mathbf{z}}^2+i\mathbf{H}\mathbf{f}} = \frac{1}{(4\pi s)^{3/2}}\,\frac{Hs}{\sinh(Hs)}. \tag{9}$$

Integrating further over λ, we obtain for the world-line integral at issue:

$$I = \frac{\sigma}{2\pi^{5/2}\sqrt{s}}\int_0^\infty dH\,\frac{H^3/\sinh(Hs)}{(H^2+\sigma^2)^2}.$$

[3] Note that, for $N = 3$, the coefficient $\frac{1}{\pi} \simeq 0.32$ in this formula agrees remarkably well with the value of 0.566^2, which was used in Ref. [6] for the parametrization of $\sigma_{\rm f}$ at high temperatures.

In the case of $N = 3$, the corresponding free-energy density reads

$$F_1\big|_{N=3} = -\frac{18\sigma}{5\pi^3}\sum_{n=1}^{\infty}\int_0^\infty \frac{ds}{s^2}\,\mathrm{e}^{-\frac{\beta^2n^2}{4s}}\langle P^n\rangle\int_0^\infty dH\,\frac{H^3/\sinh(Hs)}{(H^2+\sigma^2)^2}.$$

To perform the perturbative expansion of this expression, we introduce a dimensionless integration variable $h = H/\sigma$. Furthermore, we notice that, in the high-temperature limit of interest, $\langle P^n\rangle \simeq \langle P\rangle$, where [9]

$$\langle P\rangle = 1 + \mathcal{O}(g^3). \tag{10}$$

To find the order of the leading g-dependent term of the perturbative expansion, we use the approximation $\sinh(\sigma hs) \simeq \sigma hs\cdot\left(1+\frac{(\sigma hs)^2}{6}\right)$, which yields for $F_1\big|_{N=3}$ the following expression:

$$F_1\big|_{N=3} \simeq -\frac{9\langle P\rangle}{10\pi^2}\sum_{n=1}^{\infty}\int_0^\infty \frac{ds}{s^3}\cdot\frac{\mathrm{e}^{-\frac{\beta^2n^2}{4s}}}{\left(1+\frac{\sigma s}{\sqrt{6}}\right)^2}.$$

Approximating further the sum over winding modes by the first two terms, we obtain

$$F_1\big|_{N=3} \simeq -\frac{9\langle P\rangle T^4}{10\pi^2}\left[17 - \frac{10}{\sqrt{6}}\,\sigma\beta^2 + \mathcal{O}((\sigma\beta^2)^2)\right]. \tag{11}$$

Clearly, since $\sigma = \mathcal{O}(g^4)$, the obtained term $-\frac{10}{\sqrt{6}}\,\sigma\beta^2$ also has the order $\mathcal{O}(g^4)$. Nevertheless, due to Eq. (10), the order of the leading g-dependent term of the perturbative expansion of $F_1\big|_{N=3}$ is 3, rather than 4.

We proceed now to the calculation of the free-energy density F_2 for $N = 3$, which will allow us to find the value of the constant c in Eq. (7). The corresponding world-line integral $\oint \mathcal{D}\mathbf{z}\,\mathrm{e}^{-\frac{1}{4}\int_0^s d\tau\dot{\mathbf{z}}^2 - cg^2T\sqrt{\Sigma}}$ can be calculated by using again the approximation $\Sigma \simeq \sqrt{\mathbf{f}^2}$. The fourth root in the so-emerging exponential, $\mathrm{e}^{-cg^2T\sqrt[4]{\mathbf{f}^2}}$, can be got rid of by using two identical auxiliary integrations as follows:

$$\mathrm{e}^{-cg^2T\sqrt[4]{\mathbf{f}^2}} = \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\sqrt{\lambda}}\int_0^\infty \frac{d\mu}{\sqrt{\mu}}\,\mathrm{e}^{-\lambda-\mu-\frac{(cg^2T)^4\mathbf{f}^2}{64\lambda^2\mu}}.$$

Introducing now once again the auxiliary magnetic field $\mathbf{H}$ according to the formula (8), we obtain for the exponential at issue the following representation:

$$\mathrm{e}^{-cg^2T\sqrt[4]{\mathbf{f}^2}} = \frac{64}{\pi^{5/2}}\frac{1}{(cg^2T)^6}\int_0^\infty d\lambda\lambda^{5/2}\mathrm{e}^{-\lambda}\int_0^\infty d\mu\mu\mathrm{e}^{-\mu}\int d^3H\,\mathrm{e}^{-\frac{16\lambda^2\mu}{(cg^2T)^4}\mathbf{H}^2+i\mathbf{H}\mathbf{f}}.$$

Performing now the functional $\mathbf{z}$-integration as in Eq. (9), and integrating further over μ, which can be done analytically, we obtain the following intermediate expression:

$$F_2\big|_{N=3} = -\frac{256T^4}{\pi^{7/2}}\xi^2\sum_{n=1}^{\infty}\int_0^\infty \frac{ds}{s^2}\mathrm{e}^{-\frac{n^2}{4s}}\langle P^n\rangle\int_0^\infty dh\frac{h^3}{\sinh(\xi^2hs)}\int_0^\infty d\lambda\frac{\lambda^{5/2}\mathrm{e}^{-\lambda}}{(16\lambda^2h^2+1)^2}. \tag{12}$$

Here, we have denoted $\xi \equiv cg^2$, $h \equiv H/(\xi T)^2$, and made s dimensionless by rescaling it as $s_{\mathrm{new}} = T^2 s_{\mathrm{old}}$. By using the approximation $\sinh(\xi^2hs) \simeq \xi^2hs[1 + (\xi^2hs)^2/6]$, we have

$$F_2\big|_{N=3} \simeq -\frac{256T^4}{\pi^{7/2}}\sum_{n=1}^{\infty}\int_0^\infty \frac{ds}{s^3}\mathrm{e}^{-\frac{n^2}{4s}}\int_0^\infty d\lambda\lambda^{5/2}\mathrm{e}^{-\lambda}\int_0^\infty dh\frac{h^2}{(16\lambda^2h^2+1)^2}\cdot\frac{1}{1+(\xi^2hs)^2/6}.$$

The h-integration in this formula can be performed analytically, which yields

$$F_2\big|_{N=3} \simeq -\frac{16T^4}{\pi^{5/2}}\sum_{n=1}^{\infty}\int_0^\infty \frac{ds}{s^3}\mathrm{e}^{-\frac{n^2}{4s}}\int_0^\infty d\lambda\frac{\lambda^{3/2}\mathrm{e}^{-\lambda}}{(4\lambda+\xi^2 s/\sqrt{6})^2}.$$

Approximating again the sum over winding modes by the first two terms, we further have

$$\int_0^\infty \frac{ds}{s^3}\left(\mathrm{e}^{-\frac{1}{4s}}+\mathrm{e}^{-\frac{1}{s}}\right)\int_0^\infty d\lambda\frac{\lambda^{3/2}\mathrm{e}^{-\lambda}}{(4\lambda+\xi^2 s/\sqrt{6})^2} = \frac{17\sqrt{\pi}}{16} - \frac{27\pi^{3/2}}{128\cdot 6^{1/4}}\cdot\xi + \mathcal{O}(\xi^2).$$

This yields the sought free-energy density

$$F_2\big|_{N=3} \simeq -\frac{\langle P\rangle T^4}{10\pi^2}\left(17-\frac{27\pi}{8\cdot 6^{1/4}}\cdot cg^2\right). \tag{13}$$

Once brought together, equations (11) and (13) yield

$$F\big|_{N=3} \simeq -\frac{\langle P\rangle T^4}{\pi^2}\left[17-\frac{27\pi}{80\cdot 6^{1/4}}\cdot cg^2-\frac{9}{\sqrt{6}}\,\sigma\beta^2+\mathcal{O}((\sigma\beta^2)^2)\right]. \tag{14}$$

The two leading terms of this expression can be compared with the known perturbative expansion of the free-energy density [10],

$$F_2\big|_{N=3} = -\frac{8\pi^2T^4}{45}\left[1-\frac{15g^2}{16\pi^2}+\mathcal{O}(g^3)\right]. \tag{15}$$

Comparing the leading term of Eq. (14), $-\frac{17T^4}{\pi^2} \simeq -1.72T^4$, with the Stefan-Boltzmann expression represented by the leading term of Eq. (15), $-\frac{8\pi^2T^4}{45} \simeq -1.75T^4$, we conclude that the above-used approximation of the full sum over winding modes by the $(n=1)$- and the $(n=2)$-terms is very good. Comparing further with each other the $\mathcal{O}(g^2)$-terms of Eqs. (14) and (15), we obtain:

$$c = \frac{80\pi}{27\cdot 6^{3/4}} \simeq 2.4. \tag{16}$$

By using Eq. (7), we proceed now to arbitrary N, which yields

$$F_1 = -\frac{T^4}{8\pi^2}\cdot\frac{N^2(N^2-1)}{N^2+1}\left(1+\frac{\lambda^{3/2}}{8\pi\sqrt{3}}\right)\left(17-\frac{5\lambda^2}{2\pi\sqrt{6}}+\mathcal{O}(\lambda^4)\right)$$

and

$$F_2 = -\frac{T^4}{8\pi^2}\cdot\frac{N^2-1}{N^2+1}\left(1+\frac{\lambda^{3/2}}{8\pi\sqrt{3}}\right)\left(17-\frac{9\pi c}{8\cdot 6^{1/4}}\cdot\lambda+\mathcal{O}(\lambda^2)\right).$$

Here, $\lambda = g^2N$ is the so-called 't Hooft coupling, which stays finite in the large-N limit, and we have used the leading λ-dependent expression for the Polyakov loop (cf. Eq. (10)) [11]: $\langle P\rangle \simeq 1+\frac{\lambda^{3/2}}{8\pi\sqrt{3}}$. Accordingly, we obtain for the full free-energy density $F = F_1+F_2$:

$$F = -\frac{T^4}{8\pi^2}\cdot\frac{N^2(N^2-1)}{N^2+1}\left(1+\frac{\lambda^{3/2}}{8\pi\sqrt{3}}\right)\left[17\left(1+\frac{1}{N^2}\right)-\frac{9\pi c}{8\cdot 6^{1/4}}\cdot\frac{\lambda}{N^2}-\frac{5\lambda^2}{2\pi\sqrt{6}}+\mathcal{O}(\lambda^4)+\mathcal{O}\left(\frac{\lambda^2}{N^2}\right)\right].$$

In the large-N limit of this expression, the c-dependent term, which corresponds to the leading perturbative correction from Eq. (15), gets $\frac{1}{N^2}$-suppressed in comparison with the $\mathcal{O}(\lambda^2)$-term, which corresponds to the leading σ-dependent correction from Eq. (11). This result follows, of course, from the relative factor of $\frac{1}{N^2}$ between the perimeter- and the area-law exponentials in Eq. (7). The large-N limit of the free-energy density thus reads

$$F = -\frac{T^4N^2}{8\pi^2}\left(1+\frac{\lambda^{3/2}}{8\pi\sqrt{3}}\right)\left[17-\frac{5\lambda^2}{2\pi\sqrt{6}}+\mathcal{O}\left(\frac{10}{N^2}\right)+\mathcal{O}\left(\frac{10\lambda}{N^2}\right)+\mathcal{O}(\lambda^4)+\mathcal{O}\left(\frac{\lambda^2}{N^2}\right)\right].$$

Accordingly, $-\frac{F}{T^4}$ could have a maximum corresponding to the most probable configuration of the system, once the relation $\frac{17\lambda^{3/2}}{8\pi\sqrt{3}} = \frac{5\lambda^2}{2\pi\sqrt{6}}$ would hold, i.e. at $\lambda = \frac{289}{8}$. However, since this value of λ is much larger than unity, it lies outside the range of applicability of the λ-expansion, so that such a maximum of $-\frac{F}{T^4}$ is not realized. Thus, the main qualitative result of our study is that the leading correction to the Stefan-Boltzmann expression, while being $\mathcal{O}(\lambda)$ for $N \sim 1$, becomes $\mathcal{O}(\lambda^{3/2})$ for $N \gg 1$, and changes its sign.

2. Pair production in the field of a flux tube

In this Section, we present the calculation of the rate of pair production in the field of a flux tube [12]. Such flux tubes model hadronic strings within the dual-superconductor scenario of confinement [13], and can be viewed as dual Abrikosov-Nielsen-Olesen strings [14]. Here, we are mostly interested in the impact of the dispersion of the (chromo-)electric field of a flux tube on the rate of pair production in this field.

Dual Abrikosov-Nielsen-Olesen strings are the solutions to the classical equations of motion in the 4D dual Abelian Higgs model. The Euclidean Lagrangian of this model has the form

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}^2 + |D_\mu\varphi|^2 + \frac{\lambda}{2}(|\varphi|^2-\eta^2)^2. \tag{17}$$

Here, $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ and $D_\mu\varphi = (\partial_\mu - ig_m B_\mu)\varphi$, where B_μ is the dual gauge field, φ is the complex-valued dual Higgs field, and g_m is the magnetic coupling constant, which is related to the electric coupling constant g via the Dirac quantization condition $g_m = 2\pi/g$. The masses of the dual vector boson and the dual Higgs boson, which stem from the Lagrangian (17), are $m_V = \sqrt{2}g_m\eta$ and $m_H = \sqrt{2\lambda}\eta$, respectively. In what follows, we will consider the model (17) in the so-called London limit. This limit corresponds to the extreme type-II dual superconductor, where not only the Ginzburg-Landau parameter $\frac{m_H}{m_V}$ itself, but also its logarithm $L \equiv \ln\frac{m_H}{m_V}$ is much larger than unity. The electric field of a straight-line dual Abrikosov-Nielsen-Olesen string in the London limit can be calculated analytically, and reads [14]

$$E(r) = \frac{m_V^2}{g_m}K_0(m_V r), \tag{18}$$

where $r = |\mathbf{x}_\perp|$. From now on, $\mathbf{x}_\perp = (x_1, x_2)$ denotes a 2D vector orthogonal to the string, and K_ν's stand for the Macdonald functions. The field averaged over the string cross section, $\langle E\rangle = \frac{1}{S}\int d^2r E(r)$, obeys the relation $g\langle E\rangle = 4\sigma/L$. Here $S = \pi m_V^{-2}$ is the area of the cross section of the string, and $\sigma = 2\pi\eta^2 L$ is the string tension. A correspondence between the London limit of the dual Abelian Higgs model and the genuine Yang-Mills vacuum can then be established through the relation $m_V^2 = \frac{4\pi\sigma}{g^2L}$.

The rate of pair production in the field $E(r)$ can be obtained from the one-loop effective action $\Gamma[A_i]$ through the Schwinger formula (cf. Ref. [8])

$$w = \frac{2}{S}\,\mathrm{Im}\,\Gamma[A_i]. \tag{19}$$

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With the neglection of spin degrees of freedom of the produced quarks, the one-loop effective action has the form

$$\Gamma[A_i] = NN_{\rm f}\int_0^\infty \frac{ds}{s}\,{\rm e}^{-m^2 s}\int \mathcal{D}\mathbf{x}_\perp \mathcal{D}\mathbf{x}_\parallel \exp\left[-\int_0^s d\tau\left(\frac14\dot{\mathbf{x}}_\perp^2+\frac14\dot{\mathbf{x}}_\parallel^2-\frac{ig}{2}E(\mathbf{x}_\perp(\tau))\varepsilon_{ij}\dot{x}_i x_j\right)\right]. \tag{20}$$

Here, $\mathbf{x}_\parallel = (x_3, x_4)$, the indices i and j take the values 3 and 4, and the field of the flux tube reads $A_i = -\frac12\varepsilon_{ij}x_j E(\mathbf{x}_\perp)$. To calculate the world-line integral (20), we impose the condition of largeness of the mass m of the produced pair in comparison with m_V, i.e. $m \gg \frac{2}{g}\sqrt{\frac{\pi\sigma}{L}}$. This condition allows us to treat the field of the flux tube as a nearly constant one. Accordingly, characteristic proper times $s \lesssim \frac{1}{m^2}$ appear sufficiently small, which opens the possibility to calculate the world-line integral semiclassically. Furthermore, we assume that not only the Compton wavelength of a produced pair, $1/m$, is much smaller than the range of the field localization, $1/m_V$, but also that the characteristic pair trajectories are small in comparison with $1/m_V$. As can be seen by solving the corresponding Euler-Lagrange equation, classical Euclidean pair trajectories in a constant electric field $\langle E\rangle$ are circles of the radius $R = \frac{m}{g\langle E\rangle}$. Consequently, the condition of smallness of the pair trajectory, $R \ll \frac{1}{m_V}$, yields $m \ll 2g\sqrt{\frac{\sigma}{\pi L}}$. Both conditions,

$$\frac{2}{g}\sqrt{\frac{\pi\sigma}{L}} \ll m \ll 2g\sqrt{\frac{\sigma}{\pi L}}, \tag{21}$$

are compatible with each other at $g \gg 1$.

Owing to the smallness of the pair trajectory, the field $E(\mathbf{x}_\perp(\tau))$ can be approximated by its value averaged along the trajectory. Namely, one has

$$\int_0^s d\tau E(\mathbf{x}_\perp(\tau))\dot{x}_i x_j \simeq -\Sigma_{ij}\cdot\frac1s\int_0^s d\tau E(\mathbf{x}_\perp(\tau)), \tag{22}$$

where $\Sigma_{ij} \equiv \int_0^s d\tau x_i\dot{x}_j$ is the (i,j)-th component of the tensor area associated with the trajectory. Furthermore, the leading small-s approximation corresponds to the classical limit of the world-line integral $\int\mathcal{D}\mathbf{x}_\perp = \int d^2x_\perp(0)\int\limits_{\mathbf{x}_\perp(0)=\mathbf{x}_\perp(s)}\mathcal{D}\mathbf{x}_\perp(\tau)$ in Eq. (20):

$$\frac{1}{4\pi s}\int d^2x_\perp \exp\left[-\frac{ig}{2}E(\mathbf{x}_\perp)\varepsilon_{ij}\Sigma_{ij}\right]. \tag{23}$$

Accordingly, the effective action in this limit reads

$$\Gamma[A_i] \simeq \frac{NN_{\rm f}}{4\pi}\int_0^\infty\frac{ds}{s^2}\,{\rm e}^{-m^2 s}\int\mathcal{D}\mathbf{x}_\parallel\exp\left(-\frac14\int_0^s d\tau\dot{\mathbf{x}}_\parallel^2\right)\int d^2x_\perp\exp\left[-\frac{ig}{2}E(\mathbf{x}_\perp)\varepsilon_{ij}\Sigma_{ij}\right]. \tag{24}$$

Neglecting for the moment the dispersion of the field $E(\mathbf{x}_\perp)$, we have

$$\int d^2x_\perp\exp\left[-\frac{ig}{2}E(\mathbf{x}_\perp)\varepsilon_{ij}\Sigma_{ij}\right] \simeq S\exp\left[-\frac{ig}{2}\langle E\rangle\varepsilon_{ij}\Sigma_{ij}\right], \tag{25}$$

where we have again used the notation $\langle\cdots\rangle \equiv \frac1S\int d^2x_\perp(\cdots)$. Therefore, within this approximation, we arrive at the Euler-Heisenberg-Schwinger Lagrangian in the constant field $A_i \equiv -\frac12\varepsilon_{ij}x_j\langle E\rangle$:

$$\Gamma[A_i] \simeq S\,\frac{NN_{\rm f}}{(4\pi)^2}\int_0^\infty\frac{ds}{s^2}\,{\rm e}^{-m^2 s}\frac{g\langle E\rangle}{\sin(g\langle E\rangle s)}. \tag{26}$$

Equation (19) yields then for w the result differing from the standard Schwinger formula only by the factor $NN_{\rm f}$, which stems from the non-Abelian nature of quarks. This result reads

$$w \simeq NN_{\rm f}\frac{(g\langle E\rangle)^2}{(2\pi)^3}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}\exp\left(-\frac{\pi k m^2}{g\langle E\rangle}\right). \tag{27}$$

We can further express the inequality $s<\frac{1}{m^2}$ in terms of the parameters of the dual Abelian Higgs model. To this end, we notice that w can only be non-vanishing provided that at least the first pole from the imaginary part of the Euler-Heisenberg-Schwinger Lagrangian, which corresponds to the $(k=1)$-term from the sum (27), yields its contribution to w. For this reason, s may not be arbitrarily small, but it should be bounded from below as $s>\frac{\pi}{g\langle E\rangle}=\frac{\pi L}{4\sigma}$. The inequality $s<\frac{1}{m^2}$ yields then $m<2\sqrt{\frac{\sigma}{\pi L}}$. This new constraint is stronger than the above-obtained one, which is expressed by the right inequality (21), since the large coupling g is now absent. Representing the new constraint in the form

$$L<\frac{4}{\pi}\frac{\sigma}{m^2}\simeq 1.27\frac{\sigma}{m^2}, \tag{28}$$

one can view it as an upper limit for L. Using further the standard value of the string tension, $\sigma=(440\,{\rm MeV})^2$, and a typical value of the hadronic mass, $m=200\,{\rm MeV}$, we get an estimate $L<6.2$, which leaves a sufficient window for having $L\gg 1$. Approximating then the sum (27) by the $(k=1)$-term, we obtain

$$w\simeq\frac{2NN_{\rm f}}{\pi^3}\left(\frac{\sigma}{L}\right)^2\exp\left(-\frac{\pi m^2 L}{4\sigma}\right). \tag{29}$$

We will now calculate w in an alternative way, which allows one to avoid the use of approximation (25) by performing the $d^2x_\perp$-integration of every term in the Taylor expansion of the exponential $\exp\left[-\frac{ig}{2}E(\mathbf{x}_\perp)\varepsilon_{ij}\Sigma_{ij}\right]$. In the London limit, by virtue of the explicit form of $E(\mathbf{x}_\perp)$, the corresponding calculation can be done analytically. Namely, by using Eq. (18), we have

$$\int d^2x_\perp \exp\left[-\frac{ig}{2}E(\mathbf{x}_\perp)\varepsilon_{ij}\Sigma_{ij}\right]=\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{ig}{2}\varepsilon_{ij}\Sigma_{ij}\right)^n\left(\frac{m_V^2}{g_m}\right)^n\int d^2x_\perp (K_0(m_V r))^n.$$

The dominant contribution to the integral on the right-hand side of this expression stems from the distances $r<\frac{1}{m_V}$. By using the leading term in the small-r asymptotic behavior of $K_0(m_V r)$, this contribution can be readily evaluated as $\frac{\pi}{m_V^2}2^{2-n}n!$. Owing to the factor of $n!$, it yields the following closed-form expression for the sum over n:

$$\frac{4\pi}{m_V^2}\sum_{n=0}^{\infty}\left(-\frac{ig}{4}\frac{m_V^2}{g_m}\varepsilon_{ij}\Sigma_{ij}\right)^n=\frac{4\pi/m_V^2}{1+\frac{ig^2m_V^2}{8\pi}\varepsilon_{ij}\Sigma_{ij}}=\frac{4\pi}{m_V^2}\int_0^\infty dt\, {\rm e}^{-t\left(1+\frac{ig^2m_V^2}{8\pi}\varepsilon_{ij}\Sigma_{ij}\right)}. \tag{30}$$

For the n-series in Eq. (30) to be convergent, the condition $\frac{g^2m_V^2}{8\pi}\cdot 2\pi R^2<1$ should hold. This condition yields the following upper limit for L:

$$L<\frac{16}{\pi}\frac{\sigma}{m^2}\simeq 5.09\frac{\sigma}{m^2}. \tag{31}$$

Furthermore, by virtue of the integral representation introduced in the last equality of Eq. (30), we obtain for the effective action (24):

$$\Gamma[A_i]\simeq$$

$$\simeq S\,\frac{NN_{\rm f}}{\pi}\int\limits_0^\infty\frac{ds}{s^2}\,{\rm e}^{-m^2s}\int\mathcal{D}\mathbf{x}_\parallel\int_0^\infty dt\,{\rm e}^{-t\left(1+\frac{ig^2m_V^2}{8\pi}\varepsilon_{ij}\Sigma_{ij}\right)}=S\,\frac{NN_{\rm f}}{4\pi^2}\int_0^\infty\frac{ds}{s^2}\,{\rm e}^{-m^2s}\int_0^\infty dt\,{\rm e}^{-t}\frac{g\mathcal{E}}{\sin(g\mathcal{E}s)}. \tag{32}$$

Here, $\mathcal{E}\equiv\frac{tgm_V^2}{4\pi}$ is a yet another $\mathbf{x}$-independent electric field, which yielded for the integral $\int\mathcal{D}\mathbf{x}_\parallel$ the corresponding Euler-Heisenberg-Schwinger Lagrangian. The pair-production rate stemming from Eq. (32) has the form

$$w\simeq\frac{NN_{\rm f}}{2\pi^3}\left(\frac{\sigma}{L}\right)^2\sum_{k=1}^\infty\frac{(-1)^{k+1}}{k^2}\int_0^\infty dtt^2{\rm e}^{-t-\frac{\pi Lm^2k}{\sigma t}}=NN_{\rm f}\frac{m^3}{\pi^{3/2}}\sqrt{\frac{\sigma}{L}}\sum_{k=1}^\infty\frac{(-1)^{k+1}}{\sqrt{k}}\,K_3\left(2m\sqrt{\frac{\pi Lk}{\sigma}}\right).$$

Due to the exponential fall-off of the Macdonald function K_3 at the large values of its argument, only the terms with $k\lesssim\frac{\sigma}{4\pi Lm^2}$ are relevant in the latter sum. Using again the values $\sigma=(440\,{\rm MeV})^2$ and $m=200\,{\rm MeV}$, we obtain $k<\frac{1}{L}<1$. Therefore, only the first term from the whole sum can be retained, which yields

$$w\simeq NN_{\rm f}\frac{m^3}{\pi^{3/2}}\sqrt{\frac{\sigma}{L}}\,K_3\left(2m\sqrt{\frac{\pi L}{\sigma}}\right). \tag{33}$$

Furthermore, because of the constraint (31), the argument of the Macdonald function in this formula is smaller than 8. Nevertheless, as long as $L>\frac{\sigma}{4\pi m^2}$, this argument is still larger than unity, which results into the formula

$$w\simeq NN_{\rm f}\frac{m^{5/2}\sigma^{3/4}}{2\pi^{5/4}L^{3/4}}\,{\rm e}^{-2m\sqrt{\frac{\pi L}{\sigma}}}. \tag{34}$$

If L additionally respects the inequality (28), the obtained expression (34) can be compared with Eq. (29). This comparison leads us to the conclusion that, averaging the exponential $\exp\left[-\frac{ig}{2}E(\mathbf{x}_\perp)\varepsilon_{ij}\Sigma_{ij}\right]$ without recourse to the cumulant expansion, one obtains a change of the Gaussian m-distribution (29) to the exponential distribution (34). It is remarkable that we have obtained this result for the case of a *static* field E, namely for the field which is produced by the flux tube in the dual-superconductor model of confinement. A similar conversion of the Gaussian mass-distribution of pairs produced in the electric field into an exponential distribution is known to take place for a time-dependent field $E(t)$ which falls off with t as fast as a certain exponential. This is, for example, the case if $E(t)\propto\frac{1}{\cosh^2(\omega t)}$ with sufficiently large ω's [15]. In our case, the obtained exponential m-distribution is a consequence of the logarithmic growth of the flux-tube field (18) towards the core of the string, which takes place in the London limit of the dual superconductor. Therefore, the exponential m-distribution is a specific property of the London limit, which does not hold away from that limit. For instance, in the opposite, so-called Bogomolny, limit of $m_V=m_H$ [16], $E(0)$ was found to be finite [17], rather than growing as $\mathcal{O}(\ln\frac{1}{m_Vr})$. Consequently, the m-distribution in the Bogomolny limit is the standard Gaussian one.

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