symmetry

IMPACT FACTOR 2.2

CITESCORE 5.4

Review

Short-Distance Physics with Rare Kaon Decays

Siavash Neshatpour

Special Issue
Symmetries and Anomalies in Flavour Physics

Edited by
Dr. Stefania Ricciardi, Dr. Thomas Blake and Dr. Farvah Nazila Mahmoudi

MDPI

https://doi.org/10.3390/sym16080946

symmetry

MDPI

Review

Short-Distance Physics with Rare Kaon Decays

Siavash Neshatpour

Institut de Physique des 2 Infinis de Lyon, UMR 5822, Université Claude Bernard Lyon 1, CNRS/IN2P3, F-69622 Villeurbanne, France; s.neshatpour@ip2i.in2p3.fr

Abstract: In this write-up, we provide an overview of the existing theoretical framework concerning rare kaon decays, with a particular emphasis on flavour-changing neutral current processes. These decays offer crucial indirect pathways for investigating short-distance new physics. Our discussion will encompass standard model predictions for relevant observables, alongside an assessment of their capacity to probe new physics through a comparison with experimental data.

Keywords: flavourphysics; kaon physics; rare decays; beyond the standard model; lepton flavour universality

check for updates

Citation: Neshatpour, S. Short-Distance Physics with Rare Kaon Decays. *Symmetry* **2024**, *16*, 946. https://doi.org/10.3390/sym16080946

Academic Editor: Francesco Renga

Received: 30 June 2024
Revised: 17 July 2024
Accepted: 18 July 2024
Published: 24 July 2024

1. Introduction

Kaon decays have historically played a crucial role in understanding and establishing key features of the standard model (SM) of particle physics, such as parity violation [1], the Cabibbo theory of flavour [2], CP violation [3], and the Glashow–Iliopoulos–Maiani (GIM) mechanism [4]. They also remain a powerful tool for probing new physics (NP) beyond the standard model.

Rare kaon decays offer valuable insights into both long-distance and short-distance (SD) physics. These decays complement B-meson physics, presenting unique opportunities to investigate the CKM matrix, CP violation, lepton flavour universality violation, and lepton number violation. The distinct quark content and energy scales of kaon and B-meson decays enhance our ability to explore diverse aspects of fundamental interactions and potential new phenomena, thereby deepening our overall understanding of underlying physics. There are several excellent reviews on rare kaon decays [5–10]; here, we specifically highlight the role of these decays in probing short-distance phenomena (above the electroweak scale $\sim 10^2$ GeV) and uncovering potential contributions from new physics.

The decay rates of rare kaon processes are significantly suppressed due to two factors: the GIM mechanism, which reduces transitions via the heavy mass of gauge bosons, and the small magnitude of the relevant CKM-matrix elements. Consequently, they are even more suppressed than rare B-meson decays. In order to find NP via kaon decays, either the NP effects must be significantly large or highly precise theoretical predictions within the standard model (SM) are required. Only a few decay modes provide such precision, making them especially valuable in the search for NP.

In the study of rare kaon decays, the semi-leptonic decays with neutrinos in the final state, such as $K^+ \to \pi^+ \nu\bar{\nu}$ and $K_L \to \pi^0 \nu\bar{\nu}$, are of particular interest. These decays are primarily generated via Z-penguin and electroweak (EW) box diagrams and are dominated by short-distance physics. Their theoretical cleanliness makes them excellent candidates for detecting NP signals. Other decay modes, like $K^+ \to \pi^+ \ell^+ \ell^-$, $K_S \to \pi^0 \ell^+ \ell^-$, $K_L \to \pi^0 \ell^+ \ell^-$, and $K_{L,S} \to \ell^+ \ell^-$, also provide valuable insights into short-distance physics. However, these modes receive contributions from processes such as $K \to \gamma^{(*)}\gamma^{(*)}$, $K \to \pi\gamma^{(*)}$, and $K \to \pi\gamma^{(*)}\gamma^{(*)}$, resulting in significant long-distance (LD) hadronic contributions, making precise theoretical predictions more challenging.

On the experimental side, ongoing experiments such as NA62 and KOTO are dedicated to studying kaon decays with future measurements expected to further enhance our capabilities and open new avenues for exploration. Furthermore, the LHCb experiment, apart from its main objective of studying beauty and charm physics, has demonstrated its capability for investigating kaon physics, particularly in the decays of K_S.

For a description of rare kaon decays, the $|\Delta S| = 1$ weak effective Hamiltonian can be considered

$$\mathcal{H}_{\text{eff}} = -\frac{4G_F}{\sqrt{2}} V_{td} V_{ts}^* \frac{\alpha_e}{4\pi} \sum_k C_k^\ell O_k^\ell \,, \tag{1}$$

where $V_{qq'}$ are the elements of the CKM matrix, and the semi-leptonic local operators are defined as

$$\begin{gathered} O_L^\ell = (\bar{s}\gamma_\mu P_L d)\,(\bar{\nu}_\ell\,\gamma^\mu (1-\gamma_5)\,\nu_\ell)\,, \\ O_9^\ell = (\bar{s}\gamma_\mu P_L d)\,(\bar{\ell}\gamma^\mu \ell)\,, \qquad O_{10}^\ell = (\bar{s}\gamma_\mu P_L d)\,(\bar{\ell}\gamma^\mu\gamma_5 \ell)\,, \end{gathered} \tag{2}$$

with $P_L = (1-\gamma_5)/2$. The operator O_L applies to decays with neutrinos in the final state, and $O_{9,10}$ are relevant for decays with charged leptons in the final state. The corresponding Wilson coefficients C_k^ℓ encode potential new physics contributions as modifications to the SM Wilson coefficients $C_k^\ell = C_{k,\text{SM}}^\ell + \delta C_k^\ell$. Additionally, NP effects can contribute through scalar, pseudoscalar, and chirality-flipped operators involving right-handed quark currents (e.g., see Refs. [11–13]). Here, we focus solely on the operators given in Equation (2).

This paper is organised as follows: In Section 2, we discuss the golden modes $K^+ \to \pi^+\nu\bar{\nu}$ and $K_L \to \pi^0\nu\bar{\nu}$. Section 3 covers the $K_S \to \pi^0\ell^+\ell^-$ and $K^+ \to \pi^+\ell^+\ell^-$ decays, followed by a discussion of $K_L \to \pi^0\ell^+\ell^-$, and Section 4 examines $K \to \ell^+\ell^-$. We present a global picture of these decays in Section 5 and conclude in Section 6.

2. $K \to \pi\nu\bar{\nu}$

The CP-conserving $K^+ \to \pi^+\nu\bar{\nu}$ and CP-violating $K_L \to \pi^0\nu\bar{\nu}$ decays are predominantly governed by Z-penguin and box diagrams, which ensures that they are theoretically clean. This establishes them as benchmark modes in the study of rare kaon decays.

2.1. $K^+ \to \pi^+\nu\bar{\nu}$

The branching ratio of the $K^+ \to \pi^+\nu\bar{\nu}$ decay, summed over the three neutrino flavours, is given by the following (adapted from Ref. [14] to the notation of Equation (1)):

$$\text{BR}(K^+ \to \pi^+\nu\bar{\nu}) = \frac{\kappa_+(1+\Delta_{\text{EM}})}{\lambda^{10}} \frac{1}{3} s_W^4 \sum_\ell \left\{ \text{Im}^2\Big[\lambda_t C_L^\ell\Big] + \text{Re}^2\Big[-\frac{\lambda_c X_c}{s_W^2} + \lambda_t C_L^\ell\Big] \right\}, \tag{3}$$

where $\lambda = |V_{us}|$, $\lambda_i = V_{id}V_{is}^*$, $s_W = \sin\theta_W$ and $\Delta_{\text{EM}} \approx -0.003$ encodes the electromagnetic radiative correction [15]. The Wilson coefficient corresponding to the top-quark loop contribution in the SM is given by $C_{L,\text{SM}} \equiv -X(x_t)/s_W^2$ with $x_t = m_t^2/M_W^2$. The leading order Inami-Lim loop calculations were first performed in Ref. [16], the next-to-leading order (NLO) QCD corrections were calculated in Ref. [17–19], and the two-loop EW corrections were calculated in Ref. [20,21]. The contribution $X_c \equiv \lambda^4 P_c(X)$ is described by $P_c(X) = P_c^{\text{SD}}(X) + \delta P_{u,c}$, where $P_c^{\text{SD}}(X)$ corresponds to the short-distance charm contribution, known up to and including next-to-next-to-leading order (NNLO) QCD effects [22–24] and NLO EW corrections [25]. The contribution $\delta P_{u,c} = 0.04 \pm 0.02$ corresponds to the long-distance u-quark (from $\Delta S = 1$ four-quark operators) and dimension eight charm-quark contributions as calculated in chiral perturbation theory (ChPT) [26]. The relevant hadronic matrix element of the O_L operator can be related to semi-leptonic decays of kaons encoded in the κ_+ factor. Taking into account isospin breaking effects at NLO in ChPT [15], $\kappa_+ = (5.173 \pm 0.025) \cdot 10^{-11} (\lambda/0.225)^8$.

The SM prediction of the branching ratio of the charged kaon decay is dependent on the CKM inputs. PDG 2022 [27] reports two sets of results, one from the CKM fitter [28,29] and one from UTfit [30,31]

$$\lambda = \begin{cases} 0.22499(67) \\ 0.22500(67) \end{cases} \quad A = \begin{cases} 0.833(11) \\ 0.826^{+0.018}_{-0.015} \end{cases} \quad \bar{\rho} = \begin{cases} 0.159(10) \\ 0.159(10) \end{cases} \quad \bar{\eta} = \begin{cases} 0.348(9) & \text{UTfit} \\ 0.348(10) & \text{CKMfitter} \end{cases}$$

Using these two sets results in slightly different theoretical predictions, as given in Ref. [32]:

$$\mathrm{BR}(K^+ \to \pi^+ \nu\bar{\nu})^{\mathrm{SM}} = \begin{cases} (8.38 \pm 0.17|_{\mathrm{SD}} \pm 0.25|_{\mathrm{LD}} \pm 0.40|_{\mathrm{param.}}) \times 10^{-11} & \text{UTfit input} \\ (8.19 \pm 0.17|_{\mathrm{SD}} \pm 0.25|_{\mathrm{LD}} \pm 0.53|_{\mathrm{param.}}) \times 10^{-11} & \text{CKMfitter input} \end{cases} \quad (4)$$

where the largest uncertainty is parametric and mainly due to CKM inputs (see also Ref. [33] for a further discussion of the uncertainties). An alternative approach through which the CKM inputs are extracted from other observables results in theoretical predictions with slightly smaller parametric uncertainties is given in Ref. [34]:

$$\mathrm{BR}(K^+ \to \pi^+ \nu\bar{\nu})^{\mathrm{SM}} = (8.60 \pm 0.42) \times 10^{-11}\,, \quad (5)$$

with the uncertainty indicating the combined theoretical error.

The most precise measurement for $K^+ \to \pi^+ \nu\bar{\nu}$ comes from NA62 [35] with approximately 40% precision:

$$\mathrm{BR}(K^+ \to \pi^+ \nu\bar{\nu})^{\mathrm{NA62}} = (10.6^{+4.0}_{-3.5} \pm 0.9) \times 10^{-11} \quad (6)$$

where the first and second uncertainties are statistical and systematic, respectively. The above result is in agreement with the SM prediction, placing rather strong constraints on lepton flavour universality-conserving (LFUC) new physics effects. This can be seen in the left plot of Figure 1, where $\delta C_L^e = \delta C_L^\mu = \delta C_L^\tau \equiv \delta C_L$, bounding contributions to the $[0, 17]$ range. Experimentally, the branching ratios are measured as a sum over the three neutrino flavours, making it impossible to distinguish new physics contributions to electrons, muons, and taus. Hence, significant new physics contributions are still possible if there are lepton flavour universality-violating (LFUV) effects. This can be seen on the right plot of Figure 1, where NP effects in the muon and taus are considered different from the electrons, leaving more room for NP effects with $\delta C_L^\mu = \delta C_L^\tau \in [-8, 25]$ and $\delta C_L^e \in [-14, 32]$. A measurement with 15% precision is expected by the conclusion of the NA62 experiment's runtime [36], enabling stronger constraints to be placed on NP contributions.

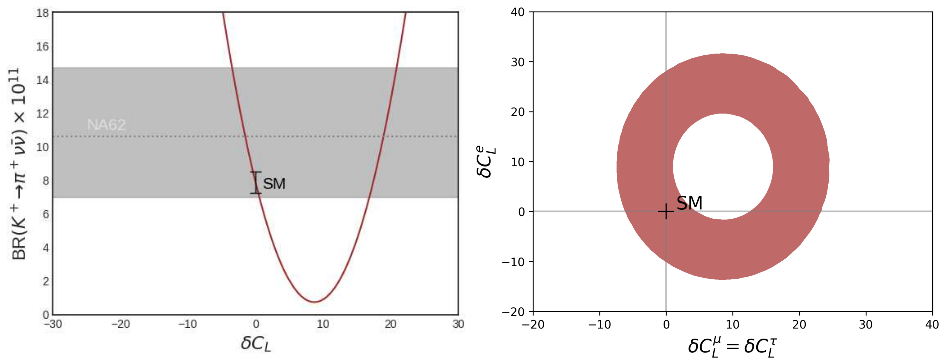

Figure 1. Left plot: The effect of LFUC new physics on $\mathrm{BR}(K^+ \to \pi^+ \nu\bar{\nu})$. The grey dotted line represents the central value measured with NA62, with the grey band indicating the $\pm 1\sigma$ uncertainty. Right plot: The 68% confidence level (CL) region for LFUV new physics, considering the NA62 measurement. Figure adapted from Refs. [37,38].

2.2. $K_L \to \pi^0 \nu\bar{\nu}$

The $K_L \to \pi^0 \nu\bar{\nu}$ decay offers a theoretically clean method to investigate CP violation, as initially highlighted in [39]. This process is particularly useful for studying the CP-violating phase of the CKM matrix, as it involves precise calculations with minimal hadronic uncertainties. The branching ratio is given by [14]

$$\mathrm{BR}(K_L \to \pi^0 \nu\bar{\nu}) = \frac{\kappa_L}{\lambda^{10}} \frac{1}{3} s_W^4 \sum_\ell \mathrm{Im}^2 \left[\lambda_t C_L^\ell\right], \quad (7)$$

where $\kappa_L = (2.231 \pm 0.013) \cdot 10^{-10} (\lambda/0.225)^8$.

The standard model prediction for the branching ratio of the neutral mode is again dependent of the choice of CKM inputs [32]

$$\mathrm{BR}(K_L \to \pi^0 \nu\bar{\nu})^{\mathrm{SM}} = \begin{cases} (2.87 \pm 0.07|_{\mathrm{SD}} \pm 0.02|_{\mathrm{LD}} \pm 0.23|_{\mathrm{param.}}) \times 10^{-11} & \text{UTfit input} \\ (2.78 \pm 0.06|_{\mathrm{SD}} \pm 0.02|_{\mathrm{LD}} \pm 0.29|_{\mathrm{param.}}) \times 10^{-11} & \text{CKMfitter input} \end{cases} \quad (8)$$

Here, the long-distance uncertainties are much smaller than the charged mode, as there are no contributions from the charm-quark loop. The largest source of theoretical error is due to parametric uncertainties. Here also, the prediction of Ref. [34] is slightly different:

$$\mathrm{BR}(K_L \to \pi^0 \nu\bar{\nu})^{\mathrm{SM}} = (2.94 \pm 0.15) \times 10^{-11}, \quad (9)$$

where, again, the smaller uncertainty is due to a different approach in the CKM inputs.

On the experimental side, measuring $K_L \to \pi^0 \nu\bar{\nu}$ presents significant challenges, and current experimental efforts have set upper limits, with the best bounds coming from KOTO [40]

$$\mathrm{BR}(K_L \to \pi^0 \nu\bar{\nu})^{\mathrm{KOTO}} < 3.0 \times 10^{-9} @ 90\%\ \mathrm{CL}, \quad (10)$$

which is two orders of magnitude larger than the SM prediction. This leads to loose bounds on NP contributions, whether LFUC or LFUV, as shown in Figure 2.

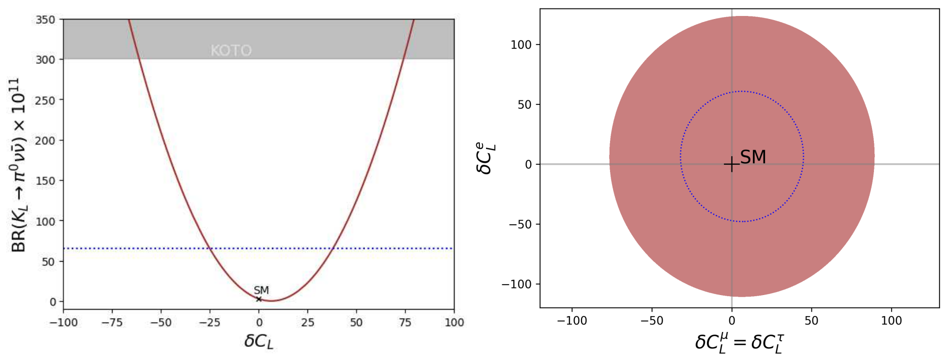

Figure 2. Left plot: The effect of LFUC new physics on $\mathrm{BR}(K_L \to \pi^0 \nu\bar{\nu})$. The grey band shows the KOTO upper limit. Right plot: The 68% confidence level region for LFUV new physics, considering the KOTO upper limit. The blue dotted line corresponds to the Grossman–Nir bound, considering the NA62 measurement for $K^+ \to \pi^+ \nu\bar{\nu}$.

The matrix elements for $K_L \to \pi^0 \nu\bar{\nu}$ and $K^+ \to \pi^+ \nu\bar{\nu}$ transitions are related through isospin, resulting in the Grossman–Nir (GN) bound [41] where $\mathrm{BR}(K_L \to \pi^0 \nu\bar{\nu}) \leq 4.3 \times \mathrm{BR}(K^+ \to \pi^+ \nu\bar{\nu})$. This bound is valid in the presence of most NP models. Considering the NA62 measurement of the charged mode, the GN bound results in stronger constraints on NP contributions, as indicated by the blue dotted contour in Figure 2.

Currently, the GN bound supersedes the upper limit from KOTO. However, in the next 4–5 years, the KOTO experiment will reach a sensitivity below 10^{-10} [32,42], and future plans at KOTO-II aim to measure this decay with an uncertainty of $\sim$25% [32,43,44].

3. $K \to \pi\ell^+\ell^-$

Rare kaon decays with charged leptons in the final state are influenced by long-distance effects involving photon emission in hadronic interactions, making their theoretical descriptions complex. The CP-conserving decays $K^+ \to \pi^+\ell^+\ell^-$ and $K_S \to \pi^0\ell^+\ell^-$ require a different description compared to the $K_L \to \pi^0\ell^+\ell^-$ decay.

3.1. $K^+ \to \pi^+\ell^+\ell^-$ and $K_S \to \pi^0\ell^+\ell^-$

The decays $K^+ \to \pi^+\ell^+\ell^-$ and $K_S \to \pi^0\ell^+\ell^-$ are mainly governed by the long-distance contribution from a single virtual-photon exchange, $K \to \pi\gamma^*$, allowed by CP invariance. This is in contrast to the $K_L \to \pi^0\ell^+\ell^-$ decay, which will be discussed in the next section. In the SM, the amplitude of these decays can be approximated using [45–47]:

$$A_{V_i}^{K\to\pi\gamma^*} = -\frac{G_f\alpha}{4\pi} V_i(z)\bar{u}_l(p_-)(\gamma_\mu k^\mu + \gamma_\mu p^\mu)v_l(p_+)\,, \tag{11}$$

where $z \equiv s/M_K^2$, with s indicating the dilepton-invariant mass squared and $V_i(z)$ being the vector form factor given by

$$V_i(z) = a_i + b_i z + V_i^{\pi\pi}(z), \qquad \text{with } (i = +, S)\,. \tag{12}$$

Theoretically, the one–photon exchange has been studied at $\mathcal{O}(p^4)$ in the chiral expansion in [47], where the calculations include an unknown combination of chiral couplings. This contribution can be described as a linear expansion $(a_i + b_i z)$ where the phenomenological parameters a_i and b_i are extracted from experimental data. Recently, there have been advancements in the theoretical calculation of these parameters (e.g., see [48–50]); nonetheless, they still suffer from rather large uncertainties. Furthermore, beyond the $\mathcal{O}(p^4)$ contributions, there are also unitarity corrections from $K \to \pi\pi\pi$, consistent with the chiral expansion of $\mathcal{O}(p^6)$. The term $V_i^{\pi\pi}(z)$, which describes this two-pion intermediate contribution, has been calculated in Ref. [46] with good precision, employing external parameter fits to $K \to 3\pi$ data [51,52] (see [53] for a recent determination of the external parameters).

The branching ratios of $K^+ \to \pi^+\ell^+\ell^-$ and $K_S \to \pi^0\ell^+\ell^-$ are obtained by integrating the differential decay width with respect to z, as given by [8,46]:

$$\frac{d\Gamma}{dz} = \frac{1}{3}\frac{G_F^2\alpha^2 M_K^5}{(4\pi)^5}\lambda^{3/2}(1,z,r_\pi^2)\sqrt{1-\frac{4r_\ell^2}{z}}\left(1+\frac{2r_\ell^2}{z}\right)|V_i(z)|^2\,, \tag{13}$$

where $\lambda(a,b,c)$ is the Källèn function, $r_\pi = M_\pi/M_K$, and $r_\ell = m_\ell/M_K$.

The decay $K^+ \to \pi^+\ell^+\ell^-$, similar to $B \to K\ell^+\ell^-$, can be influenced by vector and axial short-distance effects represented in the Wilson coefficients C_9 and C_{10}. However, this decay is primarily dominated by long-distance contributions expressed via the a_+ and b_+ parameters, which are not yet theoretically known with high enough precision. Consequently, it is not currently possible to directly extract short-distance information from this decay. However, long-distance effects are purely universal across all lepton flavours, and any deviation from this would indicate LFUV contributions in C_9 [37,54], as given by

$$a_+^{\mu\mu} - a_+^{ee} = -\sqrt{2}\,\mathrm{Re}\left[V_{td}V_{ts}^*(C_9^\mu - C_9^e)\right]. \tag{14}$$

On the experimental side, for the charged mode with final-state electrons, most events have been observed in the BNL-E865 [55] and NA48/2 [56] experiments. The form factor parameters a_+, b_+ of these two datasets agree for most z values except around $z = 0.3$ [48]. A combined determination was done by Ref. [48], rescaling the errors in that region by about 2.5 and obtaining $a_+^{ee} = -0.561 \pm 0.009$ and $b_+^{ee} = -0.694 \pm 0.040$. For final-state muons, the most precise determination of a_+ and b_+ parameters was recently given by NA62 [57] with $a_+^{\mu\mu} = -0.575 \pm 0.013$ and $b_+^{\mu\mu} = -0.722 \pm 0.043$. The measured values result in

$(a_+^{\mu\mu} - a_+^{ee}) = -0.014 \pm 0.016$ [58], consistent with lepton flavour universality within 1σ. With additional statistics on the $K^+ \to \pi^+\mu^+\mu^-$ decay and the expected measurements of the $K^+ \to \pi^+e^+e^-$ decay by the end of the NA62 experiment's runtime, better precision for a_+ and b_+ is anticipated in both the muon and electron channels [59,60]. On the theory side, in the next 5–10 years, lattice calculations are expected to determine these parameters at the 10% uncertainty level [32]. Furthermore, this decay mode can also be used to probe scalar contributions [55,61] (see Ref. [62] for a recent constraint on scalar contribution using the NA62 data [57]).

The branching ratio of the neutral mode, $K_S \to \pi^0\ell^+\ell^-$, is about two orders of magnitude smaller than that of the charged mode, making it even more challenging to directly extract information on short-distance physics. Nonetheless, the experimental determination of form-factor parameter a_S is crucial for the SM prediction of the branching ratio of $K_L \to \pi^0\ell^+\ell^-$, which is sensitive to NP contributions. Unlike the charged mode, the spectra for the neutral mode are not available; only the branching ratio has been measured. The NA48/1 experiment [63,64], considering vector meson dominance with $b_S/a_S = 1/r_V^2$, has determined $|a_S^{ee}| = 1.06^{+0.25}_{-0.21}$ and $|a_S^{\mu\mu}| = 1.54^{+0.40}_{-0.32}$. The LHCb upgrade will be able to reduce the uncertainty in the determination of $a_S^{\mu\mu}$ [65,66].

3.2. $K_L \to \pi^0\ell^+\ell^-$

The decay $K_L \to \pi^0\ell^+\ell^-$ is not predominantly influenced by a single-photon exchange; instead, it is governed by several distinct contributions [47,67–94]. The branching ratio of this decay is given by [69]

$$\mathrm{BR}(K_L \to \pi^0\ell^+\ell^-) = \left(C_{\mathrm{dir}}^\ell \pm C_{\mathrm{int}}^\ell|a_S| + C_{\mathrm{mix}}^\ell|a_S|^2 + C_{\gamma\gamma}^\ell\right) \cdot 10^{-12}\,, \quad (15)$$

where the different terms are

- C_{dir}) a direct CP-violating term: a purely short-distance effect contributing via the vector and axial Wilson coefficients C_9 and C_{10}. It is proportional to the imaginary part of λ_t.
- C_{mix}) an indirect CP-violating term: a long-distance dominated contribution of the single photon exchange via the $K_S \to \pi^0\gamma^*$ vertex through $K^0 - \bar{K}^0$-mixing. It is proportional to ε.
- C_{int}) an interference term from the above two contributions.
- $C_{\gamma\gamma}$) a CP-conserving term: a long-distance-dominated contribution via two virtual photon exchanges.

These components for the electron and muon channel are given by [69]:

	C_{dir}^ℓ	C_{int}^ℓ	C_{mix}^ℓ	$C_{\gamma\gamma}^\ell$
$\ell = e$	$(4.62 \pm 0.24)(w_{7V}^2 + w_{7A}^2)$	$(11.3 \pm 0.3)w_{7V}$	14.5 ± 0.5	≈ 0
$\ell = \mu$	$(1.09 \pm 0.05)(w_{7V}^2 + 2.32w_{7A}^2)$	$(2.63 \pm 0.06)w_{7V}$	3.36 ± 0.20	5.2 ± 1.6

where the C_{dir}^ℓ and C_{int}^ℓ terms are sensitive to short-distance physics via the following (see, e.g., [95]):

$$w_{7V} = \frac{1}{2\pi}\mathrm{Im}\left[\frac{\lambda_t}{1.407 \times 10^{-4}}C_9\right], \quad w_{7A} = \frac{1}{2\pi}\mathrm{Im}\left[\frac{\lambda_t}{1.407 \times 10^{-4}}C_{10}\right]. \quad (16)$$

Considering the combined value of $|a_S| = 1.20 \pm 0.20$ [69] as extracted from the NA48/1 measurements [63,64], the SM predictions are given by [37] the following:

$$\mathrm{BR}^{\mathrm{SM}}(K_L \to \pi^0e^+e^-) = 3.46^{+0.92}_{-0.80}\left(1.55^{+0.60}_{-0.48}\right) \times 10^{-11}\,, \quad (17)$$

$$\mathrm{BR}^{\mathrm{SM}}(K_L \to \pi^0\mu^+\mu^-) = 1.38^{+0.27}_{-0.25}\left(0.94^{+0.21}_{-0.20}\right) \times 10^{-11}\,, \quad (18)$$

where the case of destructive interference between direct and indirect CP-violating contributions, which is theoretically disfavoured [67,96], is shown in parenthesis.

The current experimental limits from KTeV [97,98] are approximately one order of magnitude larger than the SM predictions:

$$\mathrm{BR}^{\mathrm{KTEV}}(K_L \to \pi^0 e^+ e^-) < 28 \times 10^{-11} \quad \text{at } 90\% \text{ CL}, \tag{19}$$

$$\mathrm{BR}^{\mathrm{KTEV}}(K_L \to \pi^0 \mu^+ \mu^-) < 38 \times 10^{-11} \quad \text{at } 90\% \text{ CL}. \tag{20}$$

Nonetheless, the upper limits offer interesting insights into short-distance physics. Figure 3 demonstrates the effects of new physics contributions to $K_L \to \pi^0 \ell^+ \ell^-$, under the assumption that $\delta C_9^\ell = -\delta C_{10}^\ell$. The electron channel is more sensitive to NP contributions than the muon channel. This difference arises primarily from phase space suppression in the muon channel. Specifically, both C_{dir}^μ and C_{int}^μ are approximately 0.4 and 0.2 times their respective values in the electron mode, leading to reduced sensitivity.

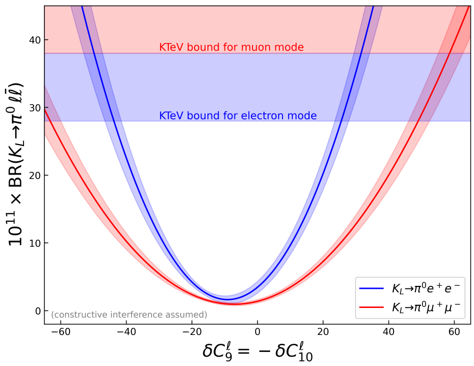

Figure 3. The branching ratio of $K_L \to \pi^0 e^+ e^-$ and $K_L \to \pi^0 \mu^+ \mu^-$ is shown as a function of new physics contributions to $\delta C_9^\ell = -\delta C_{10}^\ell$, with the coloured bands representing the 1σ theoretical uncertainty. Figure adapted from Refs. [37,38].

4. $K \to \ell^+ \ell^-$

The rare leptonic decays $K_{L,S} \to \ell^+ \ell^-$ receive contributions from poorly known long-distance contributions. Nonetheless, they can offer interesting information on short-distance physics. The accurate measurement of $K_L \to \mu^+ \mu^-$, as well as the active effort in the measurement of $K_S \to \mu^+ \mu^-$ via LHCb, justifies the inclusion of these decay modes among the observables of interest to study NP contributions.

The branching fractions for $K_S \to \mu^+ \mu^-$ and $K_L \to \mu^+ \mu^-$ decays, excluding right-handed and (pseudo)scalar operators (adapted to the notation of Equation (1), are given by [13,26]:

$$\mathrm{BR}(K_S \to \mu^+ \mu^-) = \tau_S \frac{f_K^2 m_K^3 \beta_\mu}{16\pi} \left\{ \beta_\mu^2 \left| N_S^{\mathrm{LD}} \right|^2 + \left(\frac{2 m_\mu}{m_K} \frac{G_F \alpha_e}{\sqrt{2}\pi} \right)^2 \mathrm{Im}^2 \left[-\frac{\lambda_c Y_c}{s_W^2} + \lambda_t C_{10}^\ell \right] \right\}, \tag{21}$$

$$\mathrm{BR}(K_L \to \mu^+ \mu^-) = \tau_L \frac{f_K^2 m_K^3 \beta_\mu}{16\pi} \left| N_L^{\mathrm{LD}} - \left(\frac{2 m_\mu}{m_K} \frac{G_F \alpha_e}{\sqrt{2}\pi} \right) \mathrm{Re} \left[-\frac{\lambda_c Y_c}{s_W^2} + \lambda_t C_{10}^\ell \right] \right|^2, \tag{22}$$

where $\beta_\mu = \sqrt{1 - 4 m_\mu^2 / M_K^2}$ and Y_c and $C_{10,\mathrm{SM}}^\ell = -Y(x_t)/s_W^2$ represent the short-distance SM contributions. The LO calculation of the top-quark contribution $Y(x_t)$ was presented in [16], with NLO corrections given in [17–19,99]. The short-distance charm contributions are expressed as $Y_c = \lambda^4 P_c(Y)$, where $P_c(Y)$ has been computed at NNLO in QCD [100]. The long-distance contributions are given in [13] based on [26,69,101,102]:

$$N_S^{\mathrm{LD}} = (-2.65 + 1.14i) \times 10^{-11}\,(\mathrm{GeV})^{-2}, \tag{23}$$

$$N_L^{\mathrm{LD}} = \pm[0.54(77) - 3.95i] \times 10^{-11}\,(\mathrm{GeV})^{-2}. \tag{24}$$

The long-distance contributions to both decays are mainly due to the two-photon exchanges. For the K_L decay, the LD contribution of the 2γ exchange has a dominating absorptive part, which is calculable with good precision [70,103,104], and almost saturates the experimental measurement. While the dispersive part is well established to be smaller [26], it introduces a large theoretical uncertainty (in the real part of $N_L^{\rm LD}$ above). Furthermore, both signs are possible for $N_L^{\rm LD}$, where the ambiguity is due to the unknown sign of the amplitude of the intermediate $K_L \to \gamma\gamma$ decay. The leading $\mathcal{O}(p^4)$ contribution to $\mathcal{A}(K_L \to \gamma\gamma \to \mu^+\mu^-)$ in ChPT is cancelled out due to the Gell–Mann–Okubo formula, and a reliable calculation of higher-order terms is challenging to perform, making it difficult to determine the sign [26,103,105]. For the $K_S \to \mu^+\mu^-$ decay, the LD contribution is cleaner, as the leading $\mathcal{O}(p^4)$ chiral contribution of $K_S \to \pi^+\pi^- \to \gamma\gamma \to \mu^+\mu^-$ is theoretically under better control [101].

The SM predictions for these branching ratios, as given in Ref. [37], are as follows:

$$\mathrm{BR}(K_S \to \mu^+\mu^-)^{\rm SM} = (5.15 \pm 1.50) \times 10^{-12}\,, \tag{25}$$

$$\mathrm{BR}(K_L \to \mu^+\mu^-)^{\rm SM} = \begin{cases} \mathrm{LD}(+):\ \left(6.82^{+0.77}_{-0.24} \pm 0.04\right) \times 10^{-9}\,, \\ \mathrm{LD}(-):\ \left(8.04^{+1.46}_{-0.97} \pm 0.09\right) \times 10^{-9}\,, \end{cases} \tag{26}$$

where the theoretical uncertainties of the K_L decay are asymmetric, especially for LD+. There is ongoing progress on the theoretical calculation of $K_L \to \mu^+\mu^-$, employing dispersion theory and using related leptonic and hadronic decay measurements [106].

Experimentally, the decay $K_L \to \mu^+\mu^-$ has been measured with a precision of less than 2% [27]. Conversely, the theoretical prediction for $K_S \to \mu^+\mu^-$, which is not affected by sign ambiguity, remains challenging to test, as the current upper limit from LHCb [107] is about two orders of magnitude higher than the SM

$$\mathrm{BR}^{\rm LHCb}(K_S \to \mu^+\mu^-) < 2.1(2.4) \times 10^{-10} \quad \text{at } 90(95)\%\ \mathrm{CL}\,, \tag{27}$$

$$\mathrm{BR}^{\rm PDG}(K_L \to \mu^+\mu^-) = (6.84 \pm 0.11) \times 10^{-9}\,. \tag{28}$$

The measured value of BR($K_L \to \mu^+\mu^-$) closely aligns with the theoretical prediction, assuming a positive LD sign, as indicated in Equation (26). However, given the substantial theoretical uncertainty, a negative LD sign cannot be excluded and is within 1.2σ of the experimental measurement. Reducing the theoretical uncertainty in the LD contribution of Equation (24) could provide greater clarity regarding the sign (see Figure 2 of Ref. [108]).

In Figure 4, the impact of NP contributions on the decays $K_L \to \mu^+\mu^-$ and $K_S \to \mu^+\mu^-$ is shown, taking into account both signs of the LD contributions for the former. Despite the great theoretical uncertainty and the unknown LD sign of contributions in the $K_L \to \mu^+\mu^-$ decay, the NP contribution to δC_{10}^μ is constrained within the $[-3, 14]$ range at 1σ. On the other hand, the current upper limit on BR($K_S \to \mu^+\mu^-$) does not significantly constrain NP. Considering the projected LHCb sensitivity with 300 fb^{-1} of data, still, this decay mode alone cannot probe the δC_{10} regions allowed by BR($K_L \to \mu^+\mu^-$). While this is true for vector and axial NP contributions, future measurements of BR($K_S \to \mu^+\mu^-$) at LHCb will be crucial for exploring new physics scenarios involving scalar and pseudoscalar contributions [13]. Moreover, interference effects between $K_L \to \mu^+\mu^-$ and $K_S \to \mu^+\mu^-$, as proposed in [102] (see also [109]), could provide valuable insights into short-distance physics, probing NP contribution $\delta C_{10} \sim \mathcal{O}(1)$ at the high luminosity phase of LHCb [37].

For the electron mode, there is a measurement of the $K_L \to e^+e^-$ decay by BNL-E871 [110], while for $K_S \to e^+e^-$, there is an upper bound by KLOE [111]:

$$\mathrm{BR}^{\rm E871}(K_L \to e^+e^-) = (8.7^{+5.7}_{-4.1}) \times 10^{-12}\,, \tag{29}$$

$$\mathrm{BR}^{\rm KLOE}(K_S \to e^+e^-) < 9 \times 10^{-9} \quad \text{at } 90\%\ \mathrm{CL}\,. \tag{30}$$

Probing short-distance physics with the K_S decay appears to be out of reach for the foreseeable future. However, a percentage-level precision measurement of $\mathrm{BR}(K_L \to e^+e^-)$ could effectively investigate underlying short- and long-distance interactions [58].

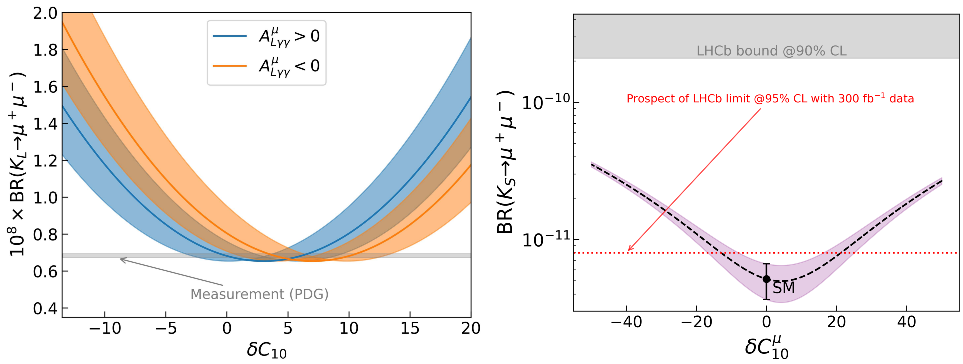

Figure 4. The left panel shows $\mathrm{BR}(K_L \to \mu^+\mu^-)$ as a function of δC^{μ}_{10} for both possible signs of the long-distance contribution from $A^{\mu}_{L\gamma\gamma}$. The right panel depicts $\mathrm{BR}(K_S \to \mu^+\mu^-)$ as a function of NP contributions in δC^{μ}_{10}. The coloured bands denote theoretical uncertainties, and the grey band represents the experimental measurement (left) and upper limit (right). The LHCb projection for $\mathrm{BR}(K_S \to \mu^+\mu^-)$, shown with the dashed red line, is in accordance with Ref. [112]. The figure was adapted from Refs. [37,38].

5. Global Picture

A global study of these rare kaon decays can provide valuable insights beyond what is obtained from individual studies. The global fit integrates data from the various decay modes, allowing for the identification of correlations and interdependencies among different observables, leading to a more accurate and detailed exploration of possible new physics scenarios. Such an analysis is given in Ref. [37] for investigating lepton flavour universality-violating new physics. A fit to LFUV new physics, assuming $\delta C^{\ell}_L \equiv \delta C^{\ell}_9 = -\delta C^{\ell}_{10}$, is given in Figure 5 using the SuperIso public program [113–117]. The 68 and 95% CL fitted regions are shown with the two shades of purple with the best-fit point indicated by the purple cross. The constraining power of each observable is also superimposed on the fit, indicating that, with the current data, $K^+ \to \pi^+\nu\bar{\nu}$, followed by $K_L \to \mu^+\mu^-$, puts the strongest bound on LFUV new-physics contributions. Such global studies, together with improved measurements at future kaon facilities, have the potential to significantly advance the exploration of new physics [37,58].

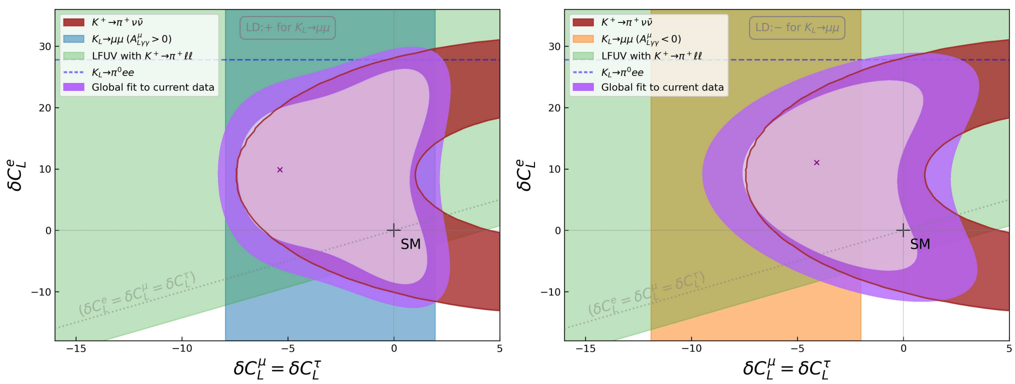

Figure 5. Global fit to rare kaon decays, assuming LFUV new physics effects, together with bounds from individual observables. The two shades of purple correspond to 68 and 95% CL fit regions. The other coloured regions correspond to 68% CL when there is a measurement, and the dashed lines indicate upper limits at 90% CL. On the left (right) plot, the sign of the long-distance contribution to the $K_L \to \mu^+\mu^-$ decay has been assumed to be positive (negative). The figure was adapted from Ref. [37].

6. Conclusions

In this review, we focused on the landscape of rare kaon decays, with particular emphasis on the insights they offer into short-distance physics. These decays offer valuable indirect pathways for probing new physics. The $K \to \pi\nu\bar{\nu}$ decays warrant special attention among rare decays, as they are predicted in the SM with very high precision. Nonetheless, although other rare kaon decays lack this level of precision, they still provide compelling probes for new physics beyond the standard model. The potential of these probes will be further enhanced via advancements in theoretical precision through continuum, data-driven, and lattice calculations. Coupled with increasingly precise experimental measurements, the sensitivity of rare kaon decays to short-distance physics makes them a promising avenue for uncovering signs of new physics.

Funding: This research was funded in part by the National Research Agency (ANR) under project ANR-21-CE31-0002-01.

Data Availability Statement: No new data were created or analysed in this study.

Conflicts of Interest: The author declares no conflicts of interest.

References

1. Lee, T.D.; Yang, C.N. Question of Parity Conservation in Weak Interactions. *Phys. Rev.* **1956**, *104*, 254–258. [CrossRef]
2. Cabibbo, N. Unitary Symmetry and Leptonic Decays. *Phys. Rev. Lett.* **1963**, *10*, 531–533. [CrossRef]
3. Christenson, J.H.; Cronin, J.W.; Fitch, V.L.; Turlay, R. Evidence for the 2π Decay of the K_2^0 Meson. *Phys. Rev. Lett.* **1964**, *13*, 138–140. [CrossRef]
4. Glashow, S.L.; Iliopoulos, J.; Maiani, L. Weak Interactions with Lepton-Hadron Symmetry. *Phys. Rev. D* **1970**, *2*, 1285–1292. [CrossRef]
5. Littenberg, L.; Valencia, G. Rare and radiative kaon decays. *Ann. Rev. Nucl. Part. Sci.* **1993**, *43*, 729–792. [CrossRef]
6. Pich, A. Chiral perturbation theory. *Rept. Prog. Phys.* **1995**, *58*, 563–610. [CrossRef]
7. D'Ambrosio, G.; Isidori, G. CP violation in kaon decays. *Int. J. Mod. Phys. A* **1998**, *13*, 1–94. [CrossRef]
8. Cirigliano, V.; Ecker, G.; Neufeld, H.; Pich, A.; Portoles, J. Kaon Decays in the Standard Model. *Rev. Mod. Phys.* **2012**, *84*, 399. [CrossRef]
9. Bryman, D.; Marciano, W.J.; Tschirhart, R.; Yamanaka, T. Rare kaon and pion decays: Incisive probes for new physics beyond the standard model. *Ann. Rev. Nucl. Part. Sci.* **2011**, *61*, 331–354. [CrossRef]
10. Ceccucci, A. Rare Kaon Decays. *Ann. Rev. Nucl. Part. Sci.* **2021**, *71*, 113–137. [CrossRef]
11. Buras, A.J. New physics patterns in ε'/ε and ε_K with implications for rare kaon decays and ΔM_K. *J. High Energy Phys.* **2016**, *4*, 071. [CrossRef]
12. Bobeth, C.; Buras, A.J. Leptoquarks meet ε'/ε and rare Kaon processes. *J. High Energy Phys.* **2018**, *2*, 101. [CrossRef]
13. Chobanova, V.; D'Ambrosio, G.; Kitahara, T.; Lucio Martinez, M.; Martinez Santos, D.; Fernandez, I.S.; Yamamoto, K. Probing SUSY effects in $K_S^0 \to \mu^+\mu^-$. *J. High Energy Phys.* **2018**, *5*, 24. [CrossRef]
14. Buras, A.J.; Buttazzo, D.; Girrbach-Noe, J.; Knegjens, R. $K^+ \to \pi^+\nu\bar{\nu}$ and $K_L \to \pi^0\nu\bar{\nu}$ in the Standard Model: Status and perspectives. *J. High Energy Phys.* **2015**, *11*, 33. [CrossRef]
15. Mescia, F.; Smith, C. Improved estimates of rare K decay matrix-elements from Kl3 decays. *Phys. Rev. D* **2007**, *76*, 034017 . [CrossRef]
16. Inami, T.; Lim, C.S. Effects of Superheavy Quarks and Leptons in Low-Energy Weak Processes $K_L \to \mu\bar{\mu}$, $K^+ \to \pi^+\nu\bar{\nu}$ and $K^0 \leftrightarrow \bar{K}^0$. *Prog. Theor. Phys.* **1981**, *65*, 297; Erratum in *Prog. Theor. Phys.* **1981**, *65*, 1772. [CrossRef]
17. Buchalla, G.; Buras, A.J. QCD corrections to rare K and B decays for arbitrary top quark mass. *Nucl. Phys. B* **1993**, *400*, 225–239. [CrossRef]
18. Buchalla, G.; Buras, A.J. The rare decays $K \to \pi\nu\bar{\nu}$, $B \to X\nu\bar{\nu}$ and $B \to l^+l^-$: An Update. *Nucl. Phys. B* **1999**, *548*, 309–327. [CrossRef]
19. Misiak, M.; Urban, J. QCD corrections to FCNC decays mediated by Z penguins and W boxes. *Phys. Lett. B* **1999**, *451*, 161–169. [CrossRef]
20. Brod, J.; Gorbahn, M.; Stamou, E. Two-Loop Electroweak Corrections for the $K \to \pi\nu\bar{\nu}$ Decays. *Phys. Rev. D* **2011**, *83*, 034030 . [CrossRef]
21. Buchalla, G.; Buras, A.J. Two loop large m_t electroweak corrections to $K \to \pi\nu\bar{\nu}$ for arbitrary Higgs boson mass. *Phys. Rev. D* **1998**, *57*, 216–223. [CrossRef]
22. Buchalla, G.; Buras, A.J. The rare decays $K^+ \to \pi^+\nu\bar{\nu}$ and $K_L \to \mu^+\mu^-$ beyond leading logarithms. *Nucl. Phys. B* **1994**, *412*, 106–142. [CrossRef]
23. Buras, A.J.; Gorbahn, M.; Haisch, U.; Nierste, U. The Rare decay $K^+ \to \pi^+\nu\bar{\nu}$ at the next-to-next-to-leading order in QCD. *Phys. Rev. Lett.* **2005**, *95*, 261805. [CrossRef]

24. Buras, A.J.; Gorbahn, M.; Haisch, U.; Nierste, U. Charm quark contribution to $K^+ \to \pi^+ \nu\bar{\nu}$ at next-to-next-to-leading order. *J. High Energy Phys.* **2006**, *11*, 002; Erratum in *J. High Energy Phys.* **2012**, *11*, 167. [CrossRef]
25. Brod, J.; Gorbahn, M. Electroweak Corrections to the Charm Quark Contribution to $K^+ \to \pi^+ \nu\bar{\nu}$. *Phys. Rev. D* **2008**, *78*, 034006 . [CrossRef]
26. Isidori, G.; Unterdorfer, R. On the short distance constraints from $K_{L,S} \to \mu^+\mu^-$. *J. High Energy Phys.* **2004**, *1*, 009. [CrossRef]
27. Workman, R.L.; Burkert, V.D.; Crede, V.; Klempt, E.; Thoma, U.; Quadt, A.; de Jong, P.; Demers, S.; Dobrescu, B.A.; D'Onofrio, M.; et al. Review of Particle Physics. *Prog. Theor. Exp. Phys.* **2022**, *2022*, 083C01. [CrossRef]
28. Hocker, A.; Lacker, H.; Laplace, S.; Le Diberder, F. A New approach to a global fit of the CKM matrix. *Eur. Phys. J. C* **2001**, *21*, 225–259. [CrossRef]
29. Charles, J.; Hocker, A.; Lacker, H.; Laplace, S.; Le Diberder, F.R.; Malcles, J.; Ocariz, J.; Pivk, M.; Roos, L. CP violation and the CKM matrix: Assessing the impact of the asymmetric *B* factories. *Eur. Phys. J. C* **2005**, *41*, 1–131. [CrossRef]
30. Bona, M.; Ciuchini, M.; Franco, E.; Lubicz, V.; Martinelli3, G.; Parodi4, F.; Pierini, M.; Roudeau, P.; Schiavi, C.; Silvestrini, L.; et al. The 2004 UTfit collaboration report on the status of the unitarity triangle in the standard model. *J. High Energy Phys.* **2005**, *7*, 28. [CrossRef]
31. Bona, M.; Ciuchini, M.; Franco, E.; Lubicz, V.; Martinelli, G.; Parodi, F.; Pierini, M.; Roudeau, P.; Schiavi, C.; Silvestrini, L.; et al. Model-independent constraints on $\Delta F = 2$ operators and the scale of new physics. *J. High Energy Phys.* **2008**, *3*, 49. [CrossRef]
32. Anzivino, G.; Cuendis, S.A.; Bernard, V.; Bijnens, J.; Bloch-Devaux, B.; Bordone, M.; Brizioli, F.; Brod, J.; Camalich, J.M.; Ceccucci, A.; et al. Workshop summary: Kaons@CERN 2023. *Eur. Phys. J. C* **2024**, *84*, 377. [CrossRef]
33. Brod, J.; Gorbahn, M.; Stamou, E. Updated Standard Model Prediction for $K \to \pi\nu\bar{\nu}$ and ϵ_K. *PoS* **2021**, *BEAUTY2020*, 056. [CrossRef]
34. Buras, A.J.; Venturini, E. Searching for New Physics in Rare *K* and *B* Decays without $|V_{cb}|$ and $|V_{ub}|$ Uncertainties. *Acta Phys. Polon. B* **2021**, *53*, 6-A1. [CrossRef]
35. Cortina Gil, E.; Kleimenova, A.; Minucci, E.; Padolski, S.; Petrov, P.; Shaikhiev, A.; Volpe, R.; Numao, T.; Petrov, Y.; Velghe, B.; et al. Measurement of the very rare $K^+ \to \pi^+ \nu\bar{\nu}$ decay. *J. High Energy Phys.* **2021**, *6*, 93. [CrossRef]
36. NA62 Collaboration. *NA62 Status Report to the CERN SPSC;* Technical report; CERN: Geneva, Switzerland , 2023.
37. D'Ambrosio, G.; Iyer, A.M.; Mahmoudi, F.; Neshatpour, S. Anatomy of kaon decays and prospects for lepton flavour universality violation. *J. High Energy Phys.* **2022**, *9*, 148. [CrossRef]
38. Neshatpour, S.; D'Ambrosio, G.; Iyer, A.; Mahmoudi, F. Kaon rare decays: Theory overview. *PoS* **2024**, *WIFAI2023*, 26. [CrossRef]
39. Littenberg, L.S. The CP Violating Dacay $K_L^0 \to \pi^0 \nu\bar{\nu}$. *Phys. Rev. D* **1989**, *39*, 3322–3324. [CrossRef] [PubMed]
40. Ahn, J.K.; Beckford, B.; Beechert, J.; Bryant, K.; Campbell, M.; Chen, S.H.; Comfort, J.; Dona, K.; Hara, N.; Haraguchi, H.; et al. Search for the $K_L \to \pi^0 \nu\bar{\nu}$ and $K_L \to \pi^0 X^0$ decays at the J-PARC KOTO experiment. *Phys. Rev. Lett.* **2019**, *122*, 21802. [CrossRef]
41. Grossman, Y.; Nir, Y. $K_L \to \pi^0 \nu\bar{\nu}$ beyond the standard model. *Phys. Lett. B* **1997**, *398*, 163–168. [CrossRef]
42. Hsiung, Y.B. KOTO: Search for Direct *CP* Violating $K_L \to \pi^0 \nu\bar{\nu}$ Decays. *PoS* **2023**, *FPCP2023*, 19. [CrossRef]
43. Nanjo, H. KOTO II at J-PARC: Toward measurement of the branching ratio of $K_L \to \pi^0 \nu\bar{\nu}$. *J. Phys. Conf. Ser.* **2023**, *2446*, 012037 . [CrossRef]
44. Aoki , K.; et al. Extension of the J-PARC Hadron Experimental Facility: Third White Paper. *arXiv* **2021**, arXiv:2110.04462.
45. Gilman, F.J.; Wise, M.B. $K \to \pi e^+ e^-$ in the Six Quark Model. *Phys. Rev. D* **1980**, *21*, 3150. [CrossRef]
46. D'Ambrosio, G.; Ecker, G.; Isidori, G.; Portoles, J. The Decays $K \to \pi l^+ l^-$ beyond leading order in the chiral expansion. *J. High Energy Phys.* **1998**, *8*, 4. [CrossRef]
47. Ecker, G.; Pich, A.; de Rafael, E. $K \to \pi l^+ l^-$ Decays in the Effective Chiral Lagrangian of the Standard Model. *Nucl. Phys. B* **1987**, *291*, 692–719. [CrossRef]
48. D'Ambrosio, G.; Greynat, D.; Knecht, M. On the amplitudes for the CP-conserving $K^\pm(K_S) \to \pi^\pm(\pi^0)\ell^+\ell^-$ rare decay modes. *J. High Energy Phys.* **2019**, *2*, 49. [CrossRef]
49. D'Ambrosio, G.; Greynat, D.; Knecht, M. Matching long and short distances at order $\mathcal{O}(\alpha_s)$ in the form factors for $K \to \pi\ell^+\ell^-$. *Phys. Lett. B* **2019**, *797*, 134891. [CrossRef]
50. Boyle, P.A.; Erben, F.; Flynn, J.M.; Gülpers, V.; Hill, R.C.; Hodgson, R.; Jüttner, A.; hÓgáin, F.O.; Portelli, A.; Sachrajda, C.T. Simulating rare kaon decays $K^+ \to \pi^+\ell^+\ell^-$ using domain wall lattice QCD with physical light quark masses. *Phys. Rev. D* **2023**, *107*, L011503. [CrossRef]
51. Kambor, J.; Missimer, J.H.; Wyler, D. $K \to 2\pi$ and $K \to 3\pi$ decays in next-to-leading order chiral perturbation theory. *Phys. Lett. B* **1991**, *261*, 496–503. [CrossRef]
52. Bijnens, J.; Dhonte, P.; Borg, F. $K \to 3\pi$ decays in chiral perturbation theory. *Nucl. Phys. B* **2003**, *648*, 317–344. [CrossRef]
53. D'Ambrosio, G.; Knecht, M.; Neshatpour, S. Determination of the structure of the $K \to \pi\pi\pi$ amplitudes from recent data. *Phys. Lett. B* **2022**, *835*, 137594. [CrossRef]
54. Crivellin, A.; D'Ambrosio, G.; Hoferichter, M.; Tunstall, L.C. Violation of lepton flavor and lepton flavor universality in rare kaon decays. *Phys. Rev. D* **2016**, *93*, 074038. [CrossRef]
55. Appel, R.; Atoyan, G.S.; Bassalleck, B.; Bergman, D.R.; Brown, D.N.; Cheung, N.; Eilerts, S.; Dhawan, S.; Do, H.; Egger, J.; et al. A New measurement of the properties of the rare decay $K^+ \to \pi^+ e^+ e^-$. *Phys. Rev. Lett.* **1999**, *83*, 4482–4485. [CrossRef]
56. Batley, J.R. ; Culling, A.J.; Kalmus, G.; Lazzeroni, C.; Munday, D.J.; Slater, M.W.; Bocquet, G.; Wotton, S.A.; Arcidiacono, R.; Bocquet, G.; et al. Precise measurement of the $K^\pm \to \pi^\pm e^+ e^-$ decay. *Phys. Lett. B* **2009**, *677*, 246–254. [CrossRef]

57. Cortina Gil , E.; Kleimenova, A.; Minucci, E.; Padolski, S.; Petrov, P.; Shaikhiev, A.; Volpe, R.; Numao, T.; Petrov, Y.; Velghe, B.; et al. A measurement of the $K^+ \to \pi^+\mu^+\mu^-$ decay. *J. High Energy Phys.* **2022**, *11*, 11; Addendum in *JHEP* **2023**, *6*, **40**. [CrossRef]
58. D'Ambrosio, G.; Mahmoudi, F.; Neshatpour, S. Beyond the Standard Model prospects for kaon physics at future experiments. *J. High Energy Phys.* **2024**, *2*, 166. [CrossRef]
59. NA62 Collaboration. *ADDENDUM I TO P326 Continuation of the Physics Programme of the NA62 Experiment*; Technical report; CERN: Geneva, Switzerland, 2019.
60. NA62 Collaboration. *NA62 Status Report to the CERN SPSC*; Technical report; CERN: Geneva, Switzerland, 2024.
61. Beder, D.S.; Dass, G.V. Leptonic Coupling Types in $K^+ \to \pi^+\ell^+\ell^-$ Decays and the Question of Strangeness-Changing Semileptonic Neutral Currents. *Phys. Lett. B* **1975**, *59*, 444–450. [CrossRef]
62. D'Ambrosio, G.; Iyer, A.M.; Mahmoudi, F.; Neshatpour, S. Exploring scalar contributions with $K^+ \to \pi^+\ell^+\ell^-$. *Phys. Lett. B* **2024**, *855*, 138824. [CrossRef]
63. Batley, J.R.; Kalmus, G.E.; Lazzeroni, C.; Munday, D.J.; Patel, M.; Slater, M.W.; Wotton, S.A.; Arcidiacono, R.; Bocquet, G.; Ceccucci, A.; et al. Observation of the rare decay $K_S \to \pi^0 e^+e^-$. *Phys. Lett. B* **2003**, *576*, 43–54. [CrossRef]
64. Batley, J.R.; Kalmus, G.E.; Lazzeroni, C.; Munday, D.J.; Patel, M.; Slater, M.W.; Wotton, S.A.; Arcidiacono, R.; Bocquet, G.; Ceccucci, A.; et al. Observation of the rare decay $K_S \to \pi^0\mu^+\mu^-$. *Phys. Lett. B* **2004**, *599*, 197–211. [CrossRef]
65. Chobanova, V.G.; Cid Vidal, X.; Dalseno, J.P.; Lucio Martinez, M.; Martinez Santos, D.; Renaudin, V. Sensitivity of LHCb and its upgrade in the measurement of $\mathcal{B}(K_S^0 \to \pi^0\mu^+\mu^-)$. 2016. Available online: http://cds.cern.ch/record/2195218 (accessed on 1 June 2024).
66. Alves Junior, A.A.; Bettler, M.O.; Rodríguez, A.B.; Vidal, A.C.; Chobanova, V.; Vidal, X.C.; Contu, A.; D'Ambrosio, G.; Dalseno, J.; Dettori, F.; et al. Prospects for Measurements with Strange Hadrons at LHCb. *J. High Energy Phys.* **2019**, *5*, 48. [CrossRef]
67. Buchalla, G.; D'Ambrosio, G.; Isidori, G. Extracting short distance physics from $K_{L,S} \to \pi^0 e^+e^-$ decays. *Nucl. Phys. B* **2003**, *672*, 387–408. [CrossRef]
68. Isidori, G.; Smith, C.; Unterdorfer, R. The Rare decay $K_L \to \pi^0\mu^+\mu^-$ within the SM. *Eur. Phys. J. C* **2004**, *36*, 57–66. [CrossRef]
69. Mescia, F.; Smith, C.; Trine, S. $K_L \to \pi^0 e^+e^-$ and $K_L \to \pi^0\mu^+\mu^-$: A Binary star on the stage of flavor physics. *J. High Energy Phys.* **2006**, *8*, 88. [CrossRef]
70. Martin, B.R.; De Rafael, E.; Smith, J. Neutral kaon decays into lepton pairs. *Phys. Rev. D* **1970**, *2*, 179–200. [CrossRef]
71. Donoghue, J.F.; Holstein, B.R.; Valencia, G. $K_L \to \pi^0 e^+e^-$ as a Probe of CP Violation. *Phys. Rev. D* **1987**, *35*, 2769. [CrossRef]
72. Ecker, G.; Pich, A.; de Rafael, E. Radiative Kaon Decays and CP Violation in Chiral Perturbation Theory. *Nucl. Phys. B* **1988**, *303*, 665–702. [CrossRef]
73. Ecker, G.; Pich, A.; de Rafael, E. $K^0 \to \pi^0\gamma\gamma$ Decays in Chiral Perturbation Theory. *Phys. Lett. B* **1987**, *189*, 363–368. [CrossRef]
74. Sehgal, L.M. CP Violation in $K_L \to \pi^0 e^+e^-$: Interference of One Photon and Two Photon Exchange. *Phys. Rev. D* **1988**, *38*, 808. [CrossRef]
75. Flynn, J.; Randall, L. The CP Conserving Long Distance Contribution to the Decay $K_L \to \pi^0 e^+e^-$. *Phys. Lett. B* **1989**, *216*, 221–226. [CrossRef]
76. Cappiello, L.; D'Ambrosio, G. $K_L \to \pi^0\gamma\gamma$ Decay in the Chiral Effective Lagrangian. *Nuovo Cim. A* **1988**, *99*, 155–161. [CrossRef]
77. Morozumi, T.; Iwasaki, H. The CP Conserving Contribution in the Decays $K_L \to \pi^0\gamma\gamma$ and $K_L \to \pi^0 e^+e^-$. *Prog. Theor. Phys.* **1989**, *82*, 371–379. [CrossRef]
78. Ecker, G.; Pich, A.; de Rafael, E. Vector Meson Exchange in Radiative Kaon Decays and Chiral Perturbation Theory. *Phys. Lett. B* **1990**, *237*, 481–487. [CrossRef]
79. Savage, M.J.; Luke, M.E.; Wise, M.B. The Rare decays $\pi^0 \to e^+e^-$, $\eta \to e^+e^-$ and $\eta \to \mu^+\mu^-$ in chiral perturbation theory. *Phys. Lett. B* **1992**, *291*, 481–483. [CrossRef]
80. Cappiello, L.; D'Ambrosio, G.; Miragliuolo, M. Corrections to $K \to \pi\gamma\gamma$ from $K \to 3\pi$. *Phys. Lett. B* **1993**, *298*, 423–431. [CrossRef]
81. Heiliger, P.; Sehgal, L.M. Analysis of the decay $K_L \to \pi^0\gamma\gamma$ and expectations for the decays $K_L \to \pi^0 e^+e^-$ and $K_L \to \pi^0\mu^+\mu^-$. *Phys. Rev. D* **1993**, *47*, 4920–4938. [CrossRef] [PubMed]
82. Cohen, A.G.; Ecker, G.; Pich, A. Unitarity and $K_L \to \pi^0\gamma\gamma$. *Phys. Lett. B* **1993**, *304*, 347–352. [CrossRef]
83. Buras, A.J.; Lautenbacher, M.E.; Misiak, M.; Munz, M. Direct CP violation in $K_L \to \pi^0 e^+e^-$ beyond leading logarithms. *Nucl. Phys. B* **1994**, *423*, 349–383. [CrossRef]
84. D'Ambrosio, G.; Portoles, J. Vector meson exchange contributions to $K \to \pi\gamma\gamma$ and $K_L \to \gamma l^+l^-$. *Nucl. Phys. B* **1997**, *492*, 417–454. [CrossRef]
85. Donoghue, J.F.; Gabbiani, F. $K_L \to \pi^0\gamma e^+e^-$ and its relation to CP and chiral tests. *Phys. Rev. D* **1997**, *56*, 1605–1611. [CrossRef]
86. Alavi-Harati, A.; Albuquerque, I.F.; Alexopoulos, T.; Arenton, M.; Arisaka, K.; Averitte, S.; Barker, A.R.; Bellantoni, L.; Bellavance, A.; Belz, J.; et al. Measurement of the decay $K_L \to \pi^0\gamma\gamma$. *Phys. Rev. Lett.* **1999**, *83*, 917–921. [CrossRef]
87. Murakami, K.; Hemmi, Y.; Kurashige, H.; Matono, Y.; Nomura, T.; Sakamoto, H.; Sasao, N.; Suehiro, M.; Takeuchi, Y.; Fukushima, Y.; et al. Experimental search for the decay mode $K_L \to \pi^0\gamma e^+e^-$. *Phys. Lett. B* **1999**, *463*, 333–338. [CrossRef]
88. Alavi-Harati, A.; Alexopoulos, T.; Arenton, M.; Arisaka, K.; Averitte, S.; Barker, A.R.; Bellantoni, L.; Bellavance, A.; Belz, J.; Ben-David, R.; et al. First observation of the decay $K_L \to \pi^0 e^+e^-\gamma$. *Phys. Rev. Lett.* **2001**, *87*, 021801. [CrossRef]
89. Diwan, M.V.; Ma, H.; Trueman, T.L. Muon decay asymmetries from $K_L^0 \to \pi^0\mu^+\mu^-$ decays. *Phys. Rev. D* **2002**, *65*, 054020 [CrossRef]

90. Gabbiani, F.; Valencia, G. $K_L \to \pi^0\gamma\gamma$ and the bound on the CP conserving $K_L \to \pi^0 e^+ e^-$. *Phys. Rev. D* **2001**, *64*, 094008 . [CrossRef]
91. Gabbiani, F.; Valencia, G. Vector meson contributions do not explain the rate and spectrum in $K_L \to \pi^0\gamma\gamma$. *Phys. Rev. D* **2002**, *66*, 074006. [CrossRef]
92. Lai, A.; Marras, D.; Bevan, A.; Dosanjh, R.S.; Gershon, T.J.; Hay, B.; Kalmus, G.E.; Lazzeroni, C.; Munday, D.J.; Needham, M.D.; et al. Precise measurement of the decay $K_L \to \pi^0\gamma\gamma$. *Phys. Lett. B* **2002**, *536*, 229–240. [CrossRef]
93. Isidori, G.; Martinelli, G.; Turchetti, P. Rare kaon decays on the lattice. *Phys. Lett. B* **2006**, *633*, 75–83. [CrossRef]
94. Zhao, Y.; Christ, N.H. Calculating $K \to \gamma\gamma$ using lattice QCD. *PoS* **2022**, *LATTICE2021*, 451. [CrossRef]
95. Bobeth, C.; Buras, A.J.; Celis, A.; Jung, M. Patterns of Flavour Violation in Models with Vector-Like Quarks. *J. High Energy Phys.* **2017**, *04*, 079. [CrossRef]
96. Bruno, C.; Prades, J. Rare kaon decays in the 1/N(c) expansion. *Z. Phys. C* **1993**, *57*, 585–594. [CrossRef]
97. Alavi-Harati, A.; Alexopoulos, T.; Arenton, M.; Barbosa, R.F.; Barker, A.R.; Barrio, M.; Bellantoni, L.; Bellavance, A.; Blucher, E.; Bock, G.J.; et al. Search for the rare decay $K_L \to \pi^0 e^+ e^-$. *Phys. Rev. Lett.* **2004**, *93*, 021805. [CrossRef]
98. Alavi-Harati, A.; Albuquerque, I.F.; Alexopoulos, T.; Arenton, M.; Arisaka, K.; Averitte, S.; Barker, A.R.; Bellantoni, L.; Bellavance, A.; Belz, J.; et al. Search for the Decay $K_L \to \pi^0\mu^+\mu^-$. *Phys. Rev. Lett.* **2000**, *84*, 5279–5282. [CrossRef]
99. Buchalla, G.; Buras, A.J. QCD corrections to the $\bar{s}dZ$ vertex for arbitrary top quark mass. *Nucl. Phys. B* **1993**, *398*, 285–300. [CrossRef]
100. Gorbahn, M.; Haisch, U. Charm Quark Contribution to $K_L \to \mu^+\mu^-$ at Next-to-Next-to-Leading. *Phys. Rev. Lett.* **2006**, *97*, 122002. [CrossRef] [PubMed]
101. Ecker, G.; Pich, A. The Longitudinal muon polarization in $K_L \to \mu^+\mu^-$. *Nucl. Phys. B* **1991**, *366*, 189–205. [CrossRef]
102. D'Ambrosio, G.; Kitahara, T. Direct *CP* Violation in $K \to \mu^+\mu^-$. *Phys. Rev. Lett.* **2017**, *119*, 201802. [CrossRef] [PubMed]
103. Gomez Dumm, D.; Pich, A. Long distance contributions to the $K_L \to \mu^+\mu^-$ decay width. *Phys. Rev. Lett.* **1998**, *80*, 4633–4636. [CrossRef]
104. Knecht, M.; Peris, S.; Perrottet, M.; de Rafael, E. Decay of pseudoscalars into lepton pairs and large N(c) QCD. *Phys. Rev. Lett.* **1999**, *83*, 5230–5233. [CrossRef]
105. Gerard, J.M.; Smith, C.; Trine, S. Radiative kaon decays and the penguin contribution to the $\Delta I = 1/2$ rule. *Nucl. Phys. B* **2005**, *730*, 1–36. [CrossRef]
106. Hoferichter, M.; Hoid, B.L.; de Elvira, J.R. Improved Standard-Model prediction for $K_L \to \ell^+\ell^-$. *J. High Energy Phys.* **2024**, *4*, 71. [CrossRef]
107. Aaij, R.; Beteta, C.A.; Ackernley, T.; Adeva, B.; Adinolfi, M.; Afsharnia, H.; Aidala, C.A.; Aiola, S.; Ajaltouni, Z.; Akar, S.; et al. Constraints on the $K_S^0 \to \mu^+\mu^-$ Branching Fraction. *Phys. Rev. Lett.* **2020**, *125*, 231801. [CrossRef]
108. Ashraf, M.U.; Cortina Gil, E.; Jerhot, J.; Lurkin, N.; Numao, T.; Velghe, B.; Wong, V.W.S.; Bryman, D.; Bician, L.; Hives, Z.; et al. *High Intensity Kaon Experiments (HIKE) at the CERN SPS Proposal for Phases 1 and 2*; CERN: Geneva, Switzerland, **2023**.
109. Dery, A.; Ghosh, M.; Grossman, Y.; Schacht, S. $K \to \mu^+\mu^-$ as a clean probe of short-distance physics. *J. High Energy Phys.* **2021**, *07*, 103. [CrossRef]
110. Ambrose, D.; Arroyo, C.; Bachman, M.; de Cecco, P.; Connor, D.; Eckhause, M.; Ecklund, K.M.; Graessle, S.; Hancock, A.D.; Hartman, K.; et al. First observation of the rare decay mode $K_L^0 \to e^+e^-$. *Phys. Rev. Lett.* **1998**, *81*, 4309–4312. [CrossRef]
111. Ambrosino, F.; Antonelli, A.; Antonelli, M.; Archilli, F.; Beltrame, P.; Bencivenni, G.; Bertolucci, S.; Bini, C.; Bloise, C.; Bocchetta, S.; et al. Search for the $K_S \to e^+e^-$ decay with the KLOE detector. *Phys. Lett. B* **2009**, *672*, 203–208. [CrossRef]
112. Aaij, R.; Adeva, B.; Adinolfi, M.; Aidala, C.A.; Ajaltouni, Z.; Akar, S.; Albicocco, P.; Albrecht, J.; Alessio, F.; Alexander, M.; et al. *Physics Case for an LHCb Upgrade II — Opportunities in Flavour Physics, and beyond, in the HL-LHC Era*; Cornell University: Ithaca, NY, USA, 2018.
113. Mahmoudi, F. SuperIso: A Program for calculating the isospin asymmetry of $B \to K^*\gamma$ in the MSSM. *Comput. Phys. Commun.* **2008**, *178*, 745–754. [CrossRef]
114. Mahmoudi, F. SuperIso v2.3: A Program for calculating flavor physics observables in Supersymmetry. *Comput. Phys. Commun.* **2009**, *180*, 1579–1613. [CrossRef]
115. Mahmoudi, F. SuperIso v3.0, flavor physics observables calculations: Extension to NMSSM. *Comput. Phys. Commun.* **2009**, *180*, 1718–1719. [CrossRef]
116. Neshatpour, S.; Mahmoudi, F. Flavour Physics with SuperIso. *PoS* **2021**, *TOOLS2020*, 36. [CrossRef]
117. Neshatpour, S.; Mahmoudi, F. Flavour Physics Phenomenology with SuperIso. *PoS* **2022**, *CompTools2021*, 10. [CrossRef]