New Physics: Sae Mulli,
Vol. 69, No. 12, December 2019, pp. 1308∼1312
http://dx.doi.org/10.3938/NPSM.69.1308

Time-Dependent Harmonic Oscillator and the Wigner Function

Seoktae Koh*
Department of Science Education, Jeju National University, Jeju 63243, Korea

(Received 11 September 2019 : accepted 29 October 2019)

We consider a harmonic oscillator with time-dependent mass and frequency. Using the Lewis-Reisenfeld invariant approach, we calculate the wave function of the time-dependent harmonic oscillator. Then, we calculate the Wigner function of that oscillator. For a specific example, we consider the Caldirola-Kanai oscillator, and we find that the Wigner function of the Caldirola-Kanai oscillator is squeezed.

PACS numbers: 98.80.Cq,04.62.+v, 04.50.Kd, 11.30.−j
Keywords: Harmonic oscillator, Time-dependent, Wigner function, Squeezed state

I. Introduction

The time-dependent harmonic oscillator is ubiquitous in both classical and quantum mechanics. Among them, one example is a scalar field dynamics in an expanding universe [1,2]. Because it provides an exact solvable systems, the time-dependent harmonic oscillator has been extensively studied over the past several decades. One of the techniques to handle these systems is to use the Lewis-Risenfeld invariant method [3–5] which we take in this study. By suitable coordinate transformation [6], the time-dependent harmonic oscillator is transformed into the time-independent harmonic oscillator and the Lewis-Risenfeld invariant corresponds to the conserved energy [7] in this new coordinate system.

The Wigner function [8] is useful to describe the phase space represenation of the quantum state. The Wigner function plays the role of the probability distribution but not exactly same as the classical distribution which is always positive because it could become negative, so called quasi-probability distribution. The Wigner function of the time-dependent harmonic oscillator is studied in this work.

The paper is organized as follows: In Sect. II, we review the classical and quantum mechanical harmonic oscillator with the time-dependent mass and frequency and obtain the wavefunction for the time-dependent harmonic oscillator using the Lewis-Reisenfled invariant approach. In Sect. III, we calculate the Wigner function for the time-dependent harmonic oscillator. Especially, we consider the Caldirola-Kanai oscillator in order to plot the Wigner function. Finally, we summarize in IV.

*E-mail: kundol.koh@jejunu.ac.kr

II. Time-dependent harmonic oscillator

1. Classical harmonic oscillator

The Lagrangian of the time-dependent harmonic oscillator is given by

$$L(q,\dot{q};t) = \frac{1}{2}M(t)(\dot{q}^2 - \omega^2(t)q^2). \tag{1}$$

The classical equation of motion for $q(t)$ is obtained from the Euler-Lagrange equation

$$\ddot{q} + \gamma(t)\dot{q} + \omega^2(t)q = 0, \tag{2}$$

where $\gamma(t) = \frac{\dot{M}(t)}{M(t)}$. Because the term of $\dot{q}$ represents the friction for $\gamma(t) > 0$, the time-dependent mass produces a friction force. It is analogous to the Hubble friction in a time-dependent background or expanding Universe. If $\gamma =$ const. or $M(t) \sim e^{-\gamma t}$, it is the Caldirola-Kanai oscillator [9, 10]. The Caldirola-Kanai oscillator corresponds to the scalar field in de Sitter space.

The canonical momentum of the Lagrangian (1) is given by

$$p = \frac{\partial L}{\partial \dot{q}} = M(t)\dot{q}, \tag{3}$$

and the corresponding Hamiltonian is

$$H(q,p;t) = p\dot{q} - L = \frac{1}{2M(t)}p^2 + \frac{1}{2}M(t)\omega^2(t)q^2. \tag{4}$$

Performing coordinate transformation $q \to \xi/\rho(t)$ [6] where $\rho(t)$ is a time-dependent auxiliary function, the Lagragian (1) in q-coordinate representaion transforms to the Lagrangrain in ξ-coordinate representation which is given by

$$L(\xi,\xi';\tau) = \frac{1}{2}\left[\xi'^2 - M^2(t)\rho^3\{\ddot{\rho}+\gamma(t)\dot{\rho}+\omega^2(t)\rho\}\xi^2\right], \tag{5}$$

where $\xi' = \frac{d\xi}{d\tau}$ and we introduce the new time variable τ defined by

$$\tau(t) = \int^t \frac{1}{M(t')\rho^2(t')}dt'. \tag{6}$$

If the auxiliary function $\rho(t)$ satisfies

$$\ddot{\rho}+\gamma(t)\dot{\rho}+\omega^2(t)\rho = \frac{\Omega^2}{M^2(t)\rho^3}, \tag{7}$$

where Ω^2 is a constant, the Lagrangian (5) becomes

$$L(\xi,\xi';\tau) = \frac{1}{2}\xi'^2 - \frac{1}{2}\Omega^2\xi^2, \tag{8}$$

and the equations of motion for ξ is

$$\frac{d^2\xi}{d\tau^2} + \Omega^2\xi = 0. \tag{9}$$

Eq. (9) is an equation of motion of the time-independent harmonic oscillator with the unit mass and constant frequency Ω.

The Hamiltonian of the Lagrangian (8) is given by

$$H(\xi,\pi;\tau) = \frac{1}{2}\pi^2 + \frac{1}{2}\Omega^2\xi^2, \tag{10}$$

where π is the canonical momentum conjugate to ξ which is given by

$$\pi = \xi' = M(\rho\dot{q} - \dot{\rho}q). \tag{11}$$

If we write the Hamiltonian (10) in terms of q and $\dot{q}$, it becomes

$$H(\xi,\pi;\tau) = \frac{1}{2}\{(\rho p - M\dot{\rho}q)^2 + \Omega^2\left(\frac{q}{\rho}\right)^2\} \equiv I. \tag{12}$$

Eq. (12) turns out to be the Lewis-Reisenfeld invariant satisfying $dI/dt = 0$. This implies that the Lewis-Reisenfeld invariant I is the conserved energy in ξ-coordinate system.

2. Quantum harmonic oscillator

In this section we obtain the exact wave function of the time-dependent harmonic oscillator using the Lewis-Reisenfeld invariant approach. We replace the canonical variables of the classical harmonic oscillator by the quantum operators and then the Hamiltonian of the quantum harmonic oscillator is given by

$$\hat{H}(q,p;t) = \frac{1}{2M(t)}\hat{p}^2 + \frac{1}{2}M(t)\omega^2(t)\hat{q}^2, \tag{13}$$

where $\hat{p}$ and $\hat{q}$ satisfy the commutation relations $[\hat{q},\hat{p}] = i\hbar$.

Through the suitable canonical transformation shown in the previous section, the Hamiltonian (13) transform to the quantum version of the classical harmonic oscillator (10) as

$$\hat{H}(\xi,\pi;\tau) = \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\Omega^2\hat{\xi}^2. \tag{14}$$

Defining the annihilation and creation operator, $\hat{a}$ and $\hat{a}^\dagger$ which satisfy the canonical commutation relation $[\hat{a},\hat{a}^\dagger] = 1$

$$\hat{a} = \sqrt{\frac{\Omega}{2\hbar}}(\hat{\xi} + \frac{i}{\Omega}\hat{\pi}), \tag{15}$$

$$\hat{a}^\dagger = \sqrt{\frac{\Omega}{2\hbar}}(\hat{\xi} - \frac{i}{\Omega}\hat{\pi}), \tag{16}$$

the Hamiltonian (14) yields

$$\hat{H}(\xi,\pi;\tau) = \hbar\Omega(\hat{a}\hat{a}^\dagger + 1). \tag{17}$$

The eigenvalue of (17) is given by $\lambda_n = \hbar\Omega(n+1/2)$ where $n = 0,1,2,\cdots$ and the eigenstate in the ξ-coordinate basis is obtained using $\hat{a}|0\rangle = 0$ and $|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle$ by

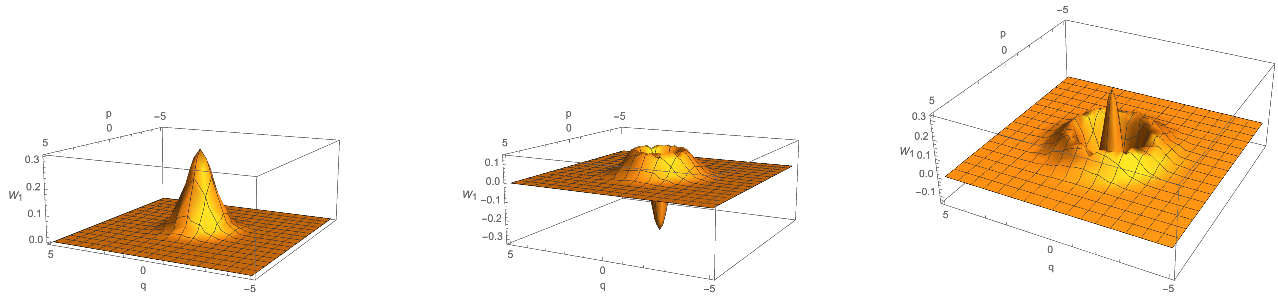
$$\langle\xi|n\rangle = \psi_n(\xi) = \left(\frac{\alpha}{\pi^{1/2}2^n n!}\right)^{1/2} e^{-\alpha^2\xi^2/2}H_n(\alpha\xi), \tag{18}$$

where $\alpha = (\Omega/\hbar)^{1/2}$ and $H_n(x)$ is the Hermite polynomial of order n.

If we change ξ-coordinate representation to q-coordinate representation [6], we have

$$\begin{aligned}\psi(q,t) &= \sum_n c_n e^{-i\lambda_n\tau/\hbar}\frac{1}{\rho^{1/2}}e^{i\frac{M\dot{\rho}}{2\hbar\rho}q^2}\psi_n(\xi)\\ &= \sum_n c_n e^{-i\lambda_n\tau/\hbar}\frac{1}{\rho^{1/2}}e^{i\frac{M\dot{\rho}}{2\hbar\rho}q^2}\left(\frac{\alpha}{\pi^{1/2}2^n n!}\right)^{1/2}\\ &\quad\times e^{-\alpha^2q^2/2\rho^2}H_n\left(\frac{\alpha}{\rho}q\right),\end{aligned} \tag{19}$$

where τ is given by (6) and $\sum_n |c_n|^2 = 1$.



(a) (b) (c)

Fig. 1. The Wigner functions of the harmonic oscillator for (a) $n = 0$, (b) $n = 1$, and (c) $n = 2$ with $\gamma_0 = 0$, $M_0 = 1$, $\omega_0 = 1$, $\hbar = 1$.

III. Wigner function and the Squeezed state

The Wigner function is defined as

$$W(x,p) = \frac{1}{2\pi\hbar}\int e^{-ipy/\hbar}\langle x+\frac{y}{2}|\hat{\rho}|x-\frac{y}{2}\rangle dy \qquad (20)$$

$$= \frac{1}{2\pi\hbar}\int e^{-ipy/\hbar}\psi(x+\frac{y}{2})\psi^*(x-\frac{y}{2})dy \qquad (21)$$

where $\hat{\rho}$ is the density operator and for a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$.

For an excited number state, the Wigner function of the wave function (19) is calculated as

$$W_n(q,p) = \frac{1}{2\pi\hbar}\int e^{-ipy/\hbar}\psi_n(q+\frac{y}{2},t)\psi^*(q-\frac{y}{2},t)dy$$
$$= \frac{(-1)^n}{\pi\hbar}e^{-\frac{2I}{\Omega\hbar}}L_n\left(\frac{4}{\Omega\hbar}I\right), \qquad (22)$$

where $L_n(x)$ is the Laguerre polynomial order of n and I is the Lewis-Reisenfeld invariant which is given in (12). Notice that the Wigner function in q-coordinate representation is expressed in terms of I not $H(q,p;t)$. The Wigner function of the wave function (18) is given by

$$W_n(\xi,\pi) = \frac{(-1)^n}{\pi\hbar}e^{-\frac{2}{\Omega\hbar}H(\xi,\pi;\tau)}L_n\left(\frac{4}{\Omega\hbar}H(\xi,\pi;\tau)\right). \qquad (23)$$

where $H(\xi,\pi;\tau)$ is given in (14)

For a specific example of the time-dependent harmonic oscillator, we consider the Caldirola-Kanai oscillator in which $\gamma(t) = \gamma_0 =$ const. and $\omega(t) = \omega_0 =$ const.. From now on, we consider only for $\Omega = 1$. $\gamma(t) =$ const. implies that $M(t) = M_0 e^{\gamma_0 t}$. The particular solution of (7) for the Caldirola-Kanai oscillator is given by [11,12] with appropriate initial conditions

$$\rho(t) = \frac{e^{-\gamma_0 t/2}}{(M_0(\omega_0^2 - \gamma_0^2/4))^{1/2}}, \qquad (24)$$

where we assume $\omega_0^2 > \gamma_0^2/4$.

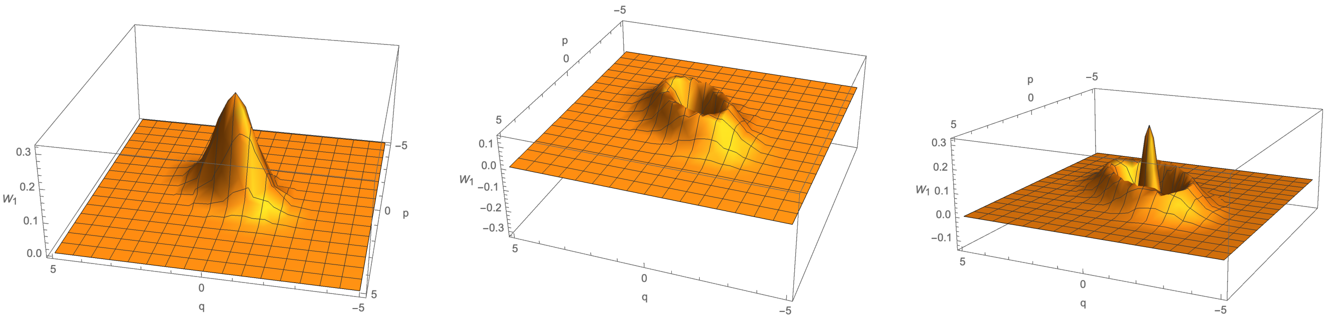
In Fig. 1, we plot the Wigner function (22) for $n = 0$, $n = 1$, and $n = 2$ with $M = M_0 = 1$ and $\omega_0 = 1$ for comparison. This choice of parameters implies the vanishing friction term $\gamma = 0$ such that corresponding to the time-independent harmonic oscillator. The Wigner functions are symmetric centered at the origin $q = 0$, $p = 0$ and give negative values for $n = 1$ and $n = 2$.

We plot the Wigner function for $\gamma = \gamma_0 = 1$ ($M = e^{-\gamma_0 t}$) with $\omega(t) = \omega_0 = 1$ at $t = 0$ in Fig. 2. These figures show that initially the shape is slightly squeezed compared to Fig. 1 and the squeezed axis is oriented to the 45° about the q-axis. In Fig. 3, we plot the Wigner function with the same parameter values of Fig. 2 but at $t = T/4$ where $T = 2\pi/\omega_0$. We find that the Wigner functions is severely squeezed along the q-axis and also the axis is rotated compared to Fig. 2. We briefly explain the squeezing process for the time-dependent harmonic oscillator.

The annihilation and creation operators given in (15) and (16) are expressed in terms of $\hat{q}$ and $\hat{p}$ as

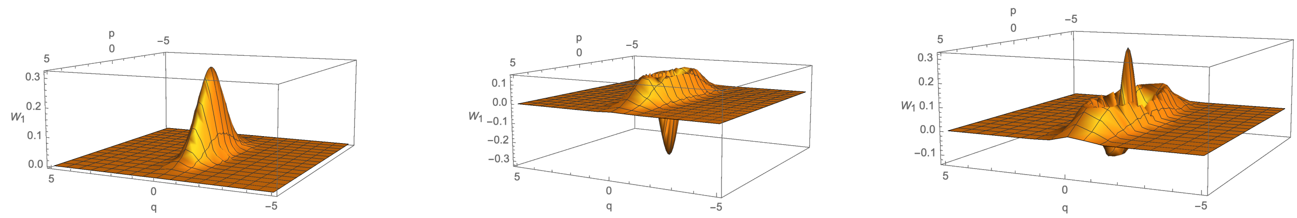
$$\hat{a} = \sqrt{\frac{\Omega}{2\hbar}}\left(\frac{1}{\rho}\hat{q} + i(\rho\hat{p} - M\dot{\rho}\hat{q}\right), \qquad (25)$$

$$\hat{a}^\dagger = \sqrt{\frac{\Omega}{2\hbar}}\left(\frac{1}{\rho}\hat{q} - i(\rho\hat{p} - M\dot{\rho}\hat{q}\right). \qquad (26)$$



(a) (b) (c)

Fig. 2. The Wigner functions of the harmonic oscillator for (a) $n = 0$, (b) $n = 1$, and (c) $n = 2$ at $t = 0$ with $\gamma_0 = 1,\ M_0 = 1,\ \omega_0 = 1,\ \hbar = 1$.



(a) (b) (c)

Fig. 3. The Wigner functions of the harmonic oscillator for (a) $n = 0$, (b) $n = 1$, and (c) $n = 2$ at $t = T/4$ with $\gamma_0 = 1,\ M_0 = 1,\ \omega_0 = 1,\ \hbar = 1$.

Or $\hat{q}$ and $\hat{p}$ are expressed as

$$\hat{q} = \sqrt{\frac{\hbar\rho^2}{2\Omega}}(\hat{a} + \hat{a}^\dagger), \quad (27)$$

$$\hat{p} = -i\sqrt{\frac{\hbar}{2\Omega\rho^2}}[(1 + iM\rho\dot{\rho})\hat{a} - (1 - iM\rho\dot{\rho})\hat{a}^\dagger]. \quad (28)$$

The Hamiltonian of (13) is given in terms of $\hat{a}$ and $\hat{a}^\dagger$ by

$$\hat{H} = \frac{\hbar}{4\Omega}[\zeta\,\hat{a}^2 + \zeta^*\hat{a}^{\dagger 2} + \eta(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})], \quad (29)$$

where

$$\zeta = -\frac{1}{M\rho^2}(1 + iM\rho\dot{\rho})^2 + \rho^2 M\omega^2, \quad (30)$$

$$\eta = \frac{1}{M\rho^2}(1 + M^2\rho^2\dot{\rho}^2) + \rho^2 M\omega^2. \quad (31)$$

This implies that $\hat{a}$ and $\hat{a}^\dagger$ can not factorize the Hamiltonian of (13) into the form of $\hat{a}\hat{a}^\dagger + 1$.

We introduce new annihilation and creation operators, $\hat{b},\ \hat{b}^\dagger$ which are related to $\hat{a},\ \hat{a}^\dagger$ through the Bogoliubov transformation as

$$\hat{b} = \mu\hat{a} + \nu\hat{a}^\dagger,$$
$$\hat{b}^\dagger = \mu^*\hat{a}^\dagger + \nu^*\hat{a}, \quad (32)$$

satisfying $|\mu|^2 - |\nu|^2 = 1$. Then we can parametrize μ and ν as

$$\mu = \cosh(r), \quad \nu = e^{i\varphi}\sinh(r). \quad (33)$$

If the Bogoliubov coefficients μ and ν satisfy the following relations,

$$\mu = \frac{1}{2}\left[\rho\sqrt{\frac{M\omega}{\Omega}} + \frac{1}{\rho}\sqrt{\frac{1}{M\omega\Omega}} + i\dot{\rho}\sqrt{\frac{M}{\omega\Omega}}\right], \quad (34)$$

$$\nu = \frac{1}{2}\left[\rho\sqrt{\frac{M\omega}{\Omega}} - \frac{1}{\rho}\sqrt{\frac{1}{M\omega\Omega}} + i\dot{\rho}\sqrt{\frac{M}{\omega\Omega}}\right], \quad (35)$$

the Hamiltonian of (13) can be expressed as

$$\hat{H} = \hbar\omega(\hat{b}\hat{b}^\dagger + \frac{1}{2}), \quad (36)$$

where

$$\hat{b} = \sqrt{\frac{M\omega}{2\hbar}}(\hat{q} + \frac{i}{M\omega}\hat{p}), \quad (37)$$

$$\hat{b}^\dagger = \sqrt{\frac{M\omega}{2\hbar}}(\hat{q} - \frac{i}{M\omega}\hat{p}). \quad (38)$$

In terms of the squeeze operator defined by

$$\hat{S}(r) = e^{-\frac{1}{2}\zeta(\hat{a}^\dagger)^2 + \frac{1}{2}\zeta^*(\hat{a})^2}, \quad (39)$$

where $\zeta = re^{i\varphi}$, the Bogoliubov transformation (32) can be written as

$$\hat{b} = \hat{S}^{\dagger}(\zeta)\hat{a}\hat{S}(\zeta), \quad \hat{b}^{\dagger} = \hat{S}^{\dagger}(\zeta)\hat{a}^{\dagger}\hat{S}(\zeta). \qquad (40)$$

We can construct squeezed vacuum state by applying the squeeze operator to the vacuum state

$$|0\rangle_{\zeta} = S(\zeta)|0\rangle. \qquad (41)$$

Therefore, when we perfrom the canonical transformation from (17) of (ξ, π) system to (36) of (q, p) system, the squeezed states are generated through the Bogoliubov transformation. This squeezed vacuum cause the Wigner function of the time-dependent harmonic oscillator to be squeezed as time evolves.

IV. Summary

We have studied the harmonic oscillator with the time-dependent mass and frequency. Through the canonical transformation from (q, p) to (ξ, π) we can construct the time-independent harmonic oscillator system and the Hamiltonian or energy in (ξ, π) coordinates corresponds to the Lewis-Reisenfeld invariant which is conserved in time. Quantum mechanically or classically we can obtain the solution of the time-dependent harmonic oscillatro by performing inverse transformation from ξ-coordinates to q-coordinates. With the wave function obtained by the Lewis-Reisenfeld invariant method, we calculate the Wigner function of the time-dependent harmonic oscillator and we have found that the quantum state are squeezed and as time evolves the squeezing is getting strong.

The annihilation and creation operators in ξ-coordinate, defined in (15) and (16), can not make the Hamiltonian in q-coordinate to the diagonalized form. The annihilation and creation operators, defined in (37) and (38), which can make the Hamiltonian in q-coordinates to the diagonalized form are related to $\hat{a}$ and $\hat{a}^{\dagger}$ through the Bogoliubov transformation. If we act the squeeze operator $\hat{S}(\zeta)$ to the vacuum defined in ξ-coordinate, the squeezed vacuum state are generated in q-coordinates. This squeezed vacuum cause the Wigner function of the time-dependent harmonic oscillator to be squeezed as time evolves.

ACKNOWLEDGEMENTS

This work was supported by the 2019 education, research and student guidance grant funded by Jeju National University.

REFERENCES

[1] F. Finelli, G. P. Vacca and G. Venturi, Phys. Rev. D **58**, 103514 (1998).
[2] S. Robles-Perez, Phys. Lett. B **774**, 608 (2017).
[3] H. R. Lewis, Phys. Rev. Lett. **18**, 510 (1967).
[4] H. R. Lewis, J. Math. Phys. **9**, 1976 (1968).
[5] H. R. Lewis and W. B. Riesenfeld, J. Math. Phys. **10**, 1458 (1969).
[6] H. Kanasugi and H. Okada, Prog. Theor. Phys. **93**, 949 (1995).
[7] T. Padmanabhan, Mod. Phys. Lett. A **33**, 1830005 (2018).
[8] M. Hillery, R. F. O'Connell, M. O. Scully and E. P. Wigner, Phys. Rept. **106**, 121 (1984).
[9] P. Caldirola, Nuovo Cim. **18**, 393 (1941).
[10] E. Kanai, Prog. Theor. Phys. **3**, 440 (1948).
[11] I. A. Pedrosa, G. P. Serra and I. Guedes, Phys. Rev. **A56**, 4300 (1997).
[12] C. J. Eliezer and A. Gray, SIAM J. Appl. Math. **30**, 463 (1976).