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PAPER

Minimum detection efficiency for testing a multi-particle Bell inequality

Wenchao Ma[1], **Peng Chen**[1] **and Xiaosong Ma**[1,2,3,*]

[1] National Laboratory of Solid State Microstructures, School of Physics and Collaborative Innovation Center of Advanced Microstructures, Nanjing University, Nanjing 210093, People's Republic of China
[2] Synergetic Innovation Center of Quantum Information and Quantum Physics, University of Science and Technology of China, Hefei, Anhui 230026, People's Republic of China
[3] Hefei National Laboratory, Hefei 230088, People's Republic of China
[*] Author to whom any correspondence should be addressed.

E-mail: xiaosong.ma@nju.edu.cn

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Abstract

Bell's inequality provides a remarkable way to test the consistency between quantum mechanics and classic local realistic theory. However, experimental demonstrations of the loophole-free Bell test are challenging and only recently have been demonstrated with bipartite systems. A central obstacle for the photonic system is that the sampling efficiency, including the collection and detection efficiencies, must be above a certain threshold. We here generalize two-particle Eberhard's inequality to the n-particle systems and derive a Bell-type inequality for multi-particle systems, which significantly relaxes this threshold. Furthermore, an experimental proposal to achieve a multi-partite Bell test without the fair sampling assumption is presented for the case of three particles. For any given value of the sampling efficiency, we give the optimal configurations for actual implementation, the optimal state, the maximum background noise that the system can tolerate, and the lowest fidelity of the quantum state. We believe our work can serve as a recipe for experimentalists planning to violate local realism using a multi-partite quantum state without the sampling loophole.

1. Introduction

In 1935, Einstein, Podolsky, and Rosen argued that quantum mechanics is incomplete when assuming that physical systems satisfy locality and realism [1]. They started their discussion by noting that quantum mechanics predicts perfect correlations between the outcomes of measurements on two distant entangled particles. If one particle of an entangled pair is measured and has a definite outcome, the other particle will be instantly projected onto a well-defined state independent of their spatial separation.

In 1964, Bell proved that the predictions of quantum theory are incompatible with the local realistic theory [2]. If assuming that no physical influences can be faster than the speed of light and that the properties of physical systems are elements of reality, the correlations in measurement outcomes from two distant observers must necessarily obey an inequality. This result showed that there is an upper limit for the observed correlations predicted by local realistic theory. However, quantum mechanics predicts a violation of this limit with certain measurements on entangled particles [3, 4].

Since the first experimental Bell test [5], violations of Bell's inequalities have been observed in a variety of physical systems such as photons [6–11], atoms [12–14], and superconducting qubits [15]. However, due to technological constraints, these experimental tests of Bell's inequality required extra assumptions, and therefore left open loopholes. Performing a loophole-free Bell test faces two main challenges: excluding any possible communication between the observers and guaranteeing efficient measurements [16].

The locality loophole is open if the measurement of one side could be communicated to the other side and hence influence the measurement results remotely. Space-like separation of each local measurement closes the locality loophole. This was firstly explored by Aspect *et al* [6] by employing rapid switching of the measurement settings. Weihes *et al* [7] later improved this with fast random switching. However, these two experiments left open the fair sampling loophole due to inefficient single-photon detection. In 2001, the fair sampling loophole was closed for the first time using entangled ions [17]. The freedom-of-choice loophole refers to the fact that the setting choices are not 'free or random' [18]. This requires that there is no interdependence between the choice of measurement settings and the properties of the system being measured. Because this is difficult to achieve with current experimental techniques, a reasonable assumption is introduced: the hidden variables describing the properties of a system are created with the particles to be measured. Thus, this loophole is closed if the settings are generated independently at the measurement stations and space-like separated from the creation of the particles. In 2010, Scheidl *et al* [19] performed an experiment that violates Bell's inequality while simultaneously closing the freedom-of-choice loophole and the locality loophole, but the fair sampling loophole is open. How to close all loopholes in a single experiment has always been a challenging topic. Until 2015, the first experimental test of Bell's inequality without the above three loopholes was performed based on electron spins of separated nitrogen-vacancy (NV) centers [20]. Independently, three Bell experiments also successfully closed these loopholes. Two of which employed entangled photon pairs [21, 22], and one employed entangled atoms connected with single photons [23].

In the process of performing a loophole-free Bell test, Eberhard's inequality [24], which was proposed by Eberhard in 1993, plays an important role. Since Eberhard's inequality explicitly includes undetected events, its violation implies that the fair-sampling loophole is closed. In addition, Eberhard's inequality is an experiment-friendly inequality for testing local realism without the detection loophole, which has been employed in the landmark experiments in realizing the loophole-free Bell test [9, 21]. It is experiment friendly because it requires a lower threshold efficiency and a rather easy-to-generate non-maximally entangled state for achieving the inequality violation. Therefore, the motivation of this work is to: generalize two-particle Eberhard's inequality to the multi-particle case and see if we can still obtain the similar experimental friendliness as in the two-particle case, i.e., lower threshold efficiency and easy to generate quantum state. Note that there have been some interesting works on sampling efficiency [25–32]. Larsson and Semitecolos extended the two-particle CH inequality to the n-particle systems [25], and Massar and Pironio studied the upper bound of detection efficiency that can be achieved by the local hidden variable model [26]. These are very relevant studies to our present work. Interestingly, although we use a different approach compared to references [25, 26], we obtain the same bounds for sampling efficiency.

2. Eberhard's inequality for two-particle system

Firstly, let us briefly review the two-particle model constructed by Eberhard [24], which contains a source and two observers, Alice and Bob, as shown in figure 1. Each observer can choose two different settings for the measurements, A_1, B_1 for Alice, A_2, B_2 for Bob. For each measurement, Eberhard considers three possible outcomes: 'o' and 'e' indicate two recorded outcomes, and 'u' indicates that no particle is detected.

Since the two observers cannot signal to each other in a Bell test, Alice (Bob) is unaware of the input to Bob (Alice). Thus the no-signaling relationships satisfy,

$$N(a|A_1) = \sum_{b\in\{o,u,e\}} N(ab|A_1A_2) = \sum_{b\in\{o,u,e\}} N(ab|A_1B_2), \tag{1a}$$

$$N(a|B_1) = \sum_{b\in\{o,u,e\}} N(ab|B_1A_2) = \sum_{b\in\{o,u,e\}} N(ab|B_1B_2), \tag{1b}$$

$$N(b|A_2) = \sum_{a\in\{o,u,e\}} N(ab|A_1A_2) = \sum_{a\in\{o,u,e\}} N(ab|B_1A_2), \tag{1c}$$

$$N(b|B_2) = \sum_{a\in\{o,u,e\}} N(ab|A_1B_2) = \sum_{a\in\{o,u,e\}} N(ab|B_1B_2), \tag{1d}$$

which represent that a measurement result at one party is independent of the measurement input at the other party. $N(ab|A_1B_2)$ denote the number of pairs with the outcome a for Alice and b for Bob when measured in settings A_1 and B_2, where $a, b \in \{o, u, e\}$.

In Eberhard's original work [24], he proposed the concept of 'conjugate events': events between different settings that satisfy the no-signaling relationships are 'conjugate events'. Eberhard assumes that classical sample is purely formed by 'conjugate events', and then selects and removes subsets of sample according to the measurement results to obtain the inequality.

With the representation of figure 2, the derivation process of the two-particle Eberhard's inequality is as follows: first, all the $N(oo|A_1A_2)$ events are selected falling into the box marked in red in the setting (A_1A_2),

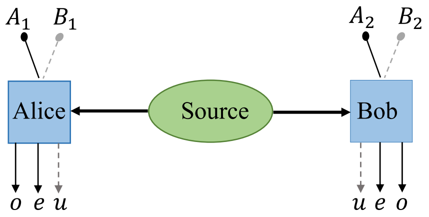

Figure 1. Two-particle experimental schematic. A source produces entangled particle pairs, and two observers, Alice and Bob, perform independent measurements. $\{A_i, B_i\}$ denote the settings selected by the ith observer, $i \in \{1, 2\}$. For each measurement, there are three possible outcomes: 'o' and 'e' for the two recorded outcomes, and 'u' if no particle is detected.

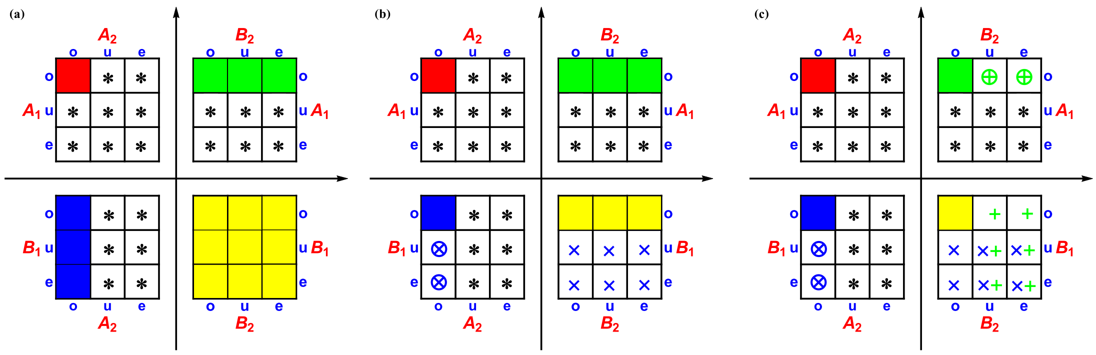

Figure 2. Four boxes located in four quadrants correspond to four combinations of settings, each of which contains nine small boxes corresponding to nine types of events. (a) The box marked in red in the setting (A_1A_2) indicates that all sample events fall into this box. The boxes marked in green indicate that the conjugates of sample events in the setting (A_1B_2) fall into these boxes. The boxes marked in blue and yellow indicate that the conjugates of sample events in the setting (B_1A_2) and setting (B_1B_2) fall into these boxes, respectively. The boxes marked with a * indicate that none of the sample events or their conjugates fall into these boxes. (b) The boxes marked with a $\otimes$ in setting (B_1A_2) indicate that the conjugates of sample falling into these boxes were removed. The boxes marked with a $\times$ in setting (B_1B_2) indicate that when the conjugates in the boxes marked with a $\otimes$ are removed, none of the events in the remaining sample has a conjugate falling into these boxes. (c) The boxes marked with a $\oplus$ in setting (A_1B_2) indicate that the conjugates of sample falling into these boxes were removed. The boxes marked with a $+$ in setting (B_1B_2) indicate that when the conjugates in the boxes marked with a $\oplus$ are removed, none of the events in the remaining sample has a conjugate falling into these boxes.

their conjugates in other settings fall into the corresponding boxes as shown in figure 2(a). Then, the sample is restricted by removing the conjugate events that fall into one of the boxes marked with a $\otimes$ in the setting (B_1A_2) as shown in figure 2(b). The number of events removed is smaller than or equal to the total number $\sum_{b\in\{u,e\}} N(ob|A_1B_2)$ of events of all categories contained in those two boxes. Therefore the restricted sample contains $N(oo|A_1A_2) - \sum_{b\in\{u,e\}} N(ob|A_1B_2)$ events or more; finally, the sample is further restricted by removing the conjugate events falling into the boxes marked with a $\oplus$ in the setting (A_1B_2). Using the same arguments as in the previous procedure, the number of remaining events must be more than or equal to $N(oo|A_1A_2) - \sum_{b\in\{u,e\}} N(ob|A_1B_2) - \sum_{a\in\{u,e\}} N(ao|B_1A_2)$; as shown in figure 2(c), the remaining sample events have conjugates that fall into the only remaining box (oo) in the setting (B_1B_2), i.e., the number of remaining events is smaller than or equal to the total number $N(oo|B_1B_2)$ of events of all categories contained in this box. These processes can be expressed as

$$N(oo|A_1A_2) - \sum_{b\in\{u,e\}} N(ob|A_1B_2) - \sum_{a\in\{u,e\}} N(ao|B_1A_2) \leqslant N(oo|B_1B_2). \tag{2}$$

Introducing the parameter J, equation (2) can be rewritten into the form of equation (13) in reference [24]:

$$J = \sum_{a\in\{u,e\}} N(ao|B_1A_2) + \sum_{b\in\{u,e\}} N(ob|A_1B_2) + N(oo|B_1B_2) - N(oo|A_1A_2) \geqslant 0. \tag{3}$$

When $J \geqslant 0$, the experimental results are consistent with local realistic theory. On the contrary, when $J < 0$, the results are incompatible with local realistic theory, but can be achieved with entangled states.

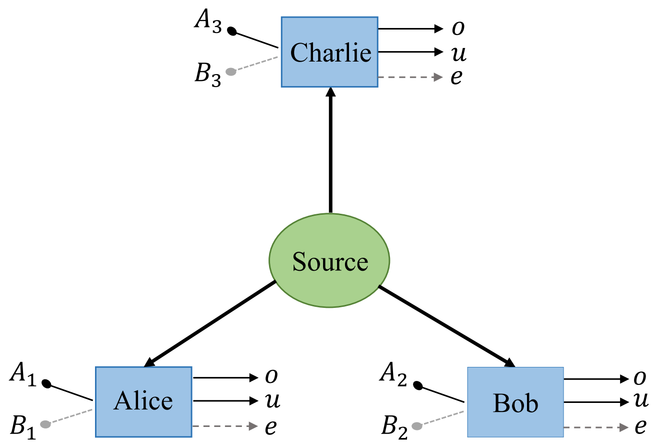

Figure 3. Three-particle experimental schematic. A source that distributes three entangled particles to the observers, and three observers who perform independent measurements. Each observer can choose two different settings, $\{A_i, B_i\}$. For each measurement, there are three possible outputs: 'o', 'u', and 'e'.

3. Eberhard's inequalities for multi-particle systems

Following Eberhard's work, we first derive inequality for the three-particle using the logic of Eberhard and then generalize it to the n-particle case.

3.1. Eberhard's inequality for three-particle system

Three observers, Alice, Bob, and Charlie, satisfy the local realistic theory. They respectively receive entangled particles from the three-particle source and then measure their particles independently. Each observer can choose two different settings for the measurements as shown in figure 3. These settings have eight combinations: $(A_1A_2A_3)$, $(A_1A_2B_3)$, $(A_1B_2A_3)$, $(B_1A_2A_3)$, $(A_1B_2B_3)$, $(B_1A_2B_3)$, $(B_1B_2A_3)$, and $(B_1B_2B_3)$. Because each measurement has three possible outcomes, there are twenty-seven possible events in each combination of settings.

To derive the three-particle Eberhard's inequality, we select all the $N(ooo|A_1A_2A_3)$ events as the initial sample, and their conjugates in other settings fall into the corresponding boxes as shown in figure 4(a). First, we restrict the sample by removing the conjugate events that fall into one of the boxes marked with the green ball in the setting $(A_1A_2B_3)$ as shown in figure 4(b). The number of events subtracted is smaller than or equal to the total number $\sum_{c\in\{u,e\}} N(ooc|A_1A_2B_3)$ of events of the categories contained in those two boxes. Therefore the restricted sample contains $N(ooo|A_1A_2A_3) - \sum_{c\in\{u,e\}} N(ooc|A_1A_2B_3)$ events or more. Then we restrict the sample again by removing the conjugate events falling into the boxes marked with the blue ball in the setting $(A_1B_2A_3)$ as shown in figure 4(c). Using the same argument as the previous process, the restricted sample contains $N(ooo|A_1A_2A_3) - \sum_{c\in\{u,e\}} N(ooc|A_1A_2B_3) - \sum_{b\in\{u,e\}} N(obo|A_1B_2A_3)$ events or more. Finally, we further restrict the sample by removing the conjugate events that fall into the box marked with the cyan ball in the setting $(B_1A_2A_3)$ as shown in figure 4(d). The remaining sample contains $N(ooo|A_1A_2A_3) - \sum_{c\in\{u,e\}} N(ooc|A_1A_2B_3) - \sum_{b\in\{u,e\}} N(obo|A_1B_2A_3) - \sum_{a\in\{u,e\}} N(aoo|B_1A_2A_3)$ events or more. As analyzed in the caption of figure 4, none of the events in the remaining sample has a conjugate that falls in a box marked with green, blue, or cyan balls. Therefore, all events belonging to the remaining sample must have conjugates in the setting $(B_1B_2B_3)$ falling into the only remaining box (ooo), which means that the number of remaining events is smaller than or equal to the total number $N(ooo|B_1B_2B_3)$ of events of all categories contained in the box (ooo). Thus the no-signaling conditions can be satisfied only if the sampling events and their conjugates satisfy,

$$N(ooo|A_1A_2A_3) - \sum_{c\in\{u,e\}} N(ooc|A_1A_2B_3) - \sum_{b\in\{u,e\}} N(obo|A_1B_2A_3) - \sum_{a\in\{u,e\}} N(aoo|B_1A_2A_3) \leqslant N(ooo|B_1B_2B_3). \tag{4}$$

Note that the conjugates of the sample events for the three-particle are more complex than the two-particle, and the conjugate events of the sample are conjugate to each other. As shown in figure 4(a), the conjugates of the sample events may fall into the boxes marked in green in the setting $(A_1A_2B_3)$, or they may fall into the boxes marked in magenta in the setting $(A_1B_2B_3)$. However, for the events in the boxes marked in green, their conjugates also fall into the boxes marked in magenta. Therefore, when we remove

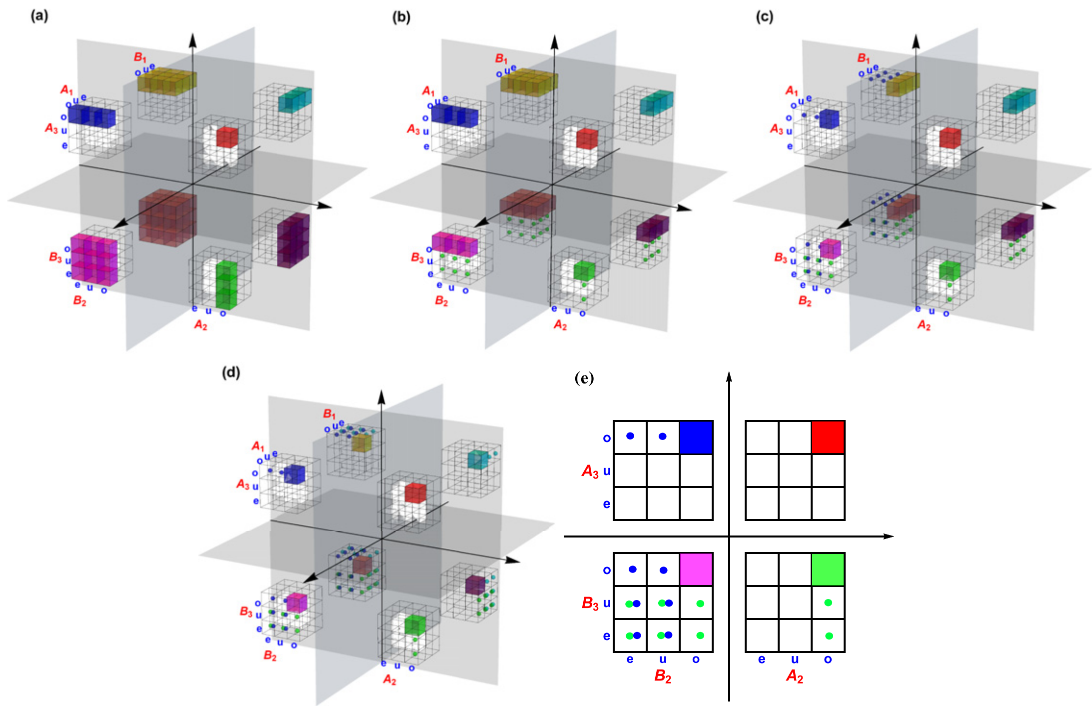

Figure 4. Eight cubes located in the eight quadrants correspond to eight combinations of settings, each of which contains twenty-seven small boxes corresponding to twenty-seven types of events. (a) The box marked in red in the setting $(A_1A_2A_3)$ indicates that all sample events only fall into this box. The boxes marked in green indicate that the conjugates of sample events in the setting $(A_1A_2B_3)$ fall into these boxes. Similarly, the conjugate events fall into the following boxes in other settings: the boxes in the setting $(A_1B_2A_3)$ marked in blue, the boxes in the setting $(B_1A_2A_3)$ marked in cyan, the boxes in the setting $(A_1B_2B_3)$ marked in magenta, the boxes in the setting $(B_1B_2A_3)$ marked in yellow, the boxes in the setting $(B_1A_2B_3)$ marked in purple, and the boxes in the setting $(B_1B_2B_3)$ marked in pink. The remaining unmarked boxes indicate that none of the sample events or their conjugates fall into these boxes. (b) We restrict the sample by removing the conjugate events that fall into the boxes (oou) and (ooe) in the setting $(A_1A_2B_3)$. After this, none of the events in the restricted sample has a conjugate falling into a box where the outputs of B_3 are u or e in the setting $(A_1A_2B_3)$. According to the no-signaling relationships, none of the events in the restricted sample has a conjugate that falls into the box marked with the green ball in the settings $(A_1A_2B_3)$, $(A_1B_2B_3)$, $(B_1A_2B_3)$ and $(B_1B_2B_3)$. (c) We restrict the sample again by removing the conjugate events that fall into the boxes (ouo) and (oeo) in the setting $(A_1B_2A_3)$. Using the same argument as in the (b), none of the events in the restricted sample has a conjugate that falls into a box marked with the blue ball in the settings $(A_1B_2A_3)$, $(B_1B_2A_3)$, $(A_1B_2B_3)$, and $(B_1B_2B_3)$. (d) We further restrict the sample by removing the conjugate events that fall into the box (uoo) and (eoo) in the setting $(B_1A_2A_3)$. Similarly, none of the events in the remaining sample has a conjugate that falls into a box marked with the cyan ball in the settings $(B_1A_2A_3)$, $(B_1B_2A_3)$, $(B_1A_2B_3)$, and $(B_1B_2B_3)$. (e) The conjugates distribution of the remaining and removed events in the settings $(A_1A_2A_3)$, $(A_1A_2B_3)$, $(A_1B_2A_3)$, and $(A_1B_2B_3)$ when the output of A_1 is 'o'. It can be seen that some of the conjugate events that fall into the boxes in the setting $(A_1B_2B_3)$ are removed twice.

sample events whose conjugate events fall into the boxes marked with the green and blue balls, as shown in figure 4(e), some of the events in the setting $(A_1B_2B_3)$ were equivalently removed twice. From figure 4(d), we find that settings $(B_1A_2B_3)$ and $(B_1B_2A_3)$ also have the same situation. To ensure that the sample events are not over-restricted, the number of these conjugate events that are removed multiple times should be added to the inequality. Therefore, equation (4) should be rewritten as

$$N(ooo|A_1A_2A_3) - \sum_{c\in\{u,e\}} N(ooc|A_1A_2B_3) - \sum_{b\in\{u,e\}} N(obo|A_1B_2A_3) - \sum_{a\in\{u,e\}} N(aoo|B_1A_2A_3) + \triangle \leqslant N(ooo|B_1B_2B_3), \tag{5}$$

where $\triangle$ represents the number of the conjugate events that are multi-removed.

Through the previous analysis, we know that the conjugates of the remaining events in setting $(B_1B_2B_3)$ all fall into the box (ooo). Next, we analyze the conjugates of the removed events in the setting $(B_1B_2B_3)$. Figures 5(d) and (e) show that none of the events in the removed sample has a conjugate that falls into the box (ooo), and the conjugates in the boxes (ooc), (obo), and (aoo) are removed only once, where $a, b, c \in \{u, e\}$. Therefore, the conjugate events that are removed multiple times cannot fall into these boxes, and their number must be more than or equal to

$$N(ooo|A_1A_2A_3) - \sum_{a\in\{u,e\}} N(aoo|B_1B_2B_3) - \sum_{b\in\{u,e\}} N(obo|B_1B_2B_3) - \sum_{c\in\{u,e\}} N(ooc|B_1B_2B_3) - N(ooo|B_1B_2B_3), \tag{6}$$

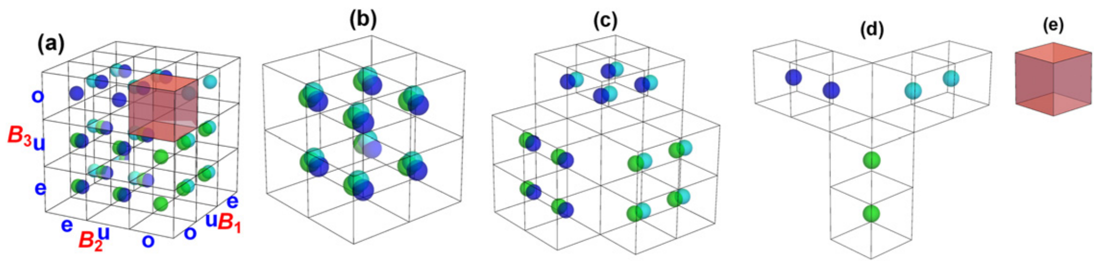

Figure 5. (a) Conjugates distribution of the remaining events and removed events in setting $(B_1B_2B_3)$. The conjugates of the remaining events all fall into the only remaining box marked in pink. On the contrary, the conjugates of the removed events fall into the boxes marked with green, blue, and cyan balls. Different colored balls in the same box indicate that the conjugates falling into this box correspond to different removed events. (b) The conjugates falling into the boxes (uuu), (uue), (ueu), (uee), (euu), (eue), (eeu), and (eee) correspond to three different removed events, respectively. (c) The conjugates falling into the boxes (ouu), (oue), (oeu), (oee), (uou), (uoe), (eou), (eoe), (uuo), (ueo), (euo), and (eeo) correspond to two different removed events, respectively. (d) Only one type of the removed events has conjugates falling into the boxes (oou), (ooe), (ouo), (oeo), (uoo), and (eoo)). (e) None of the removed events has a conjugate that falls into the box (ooo).

that is,

$$\begin{aligned}\triangle \geqslant N(ooo|A_1A_2A_3) &- \sum_{a\in\{u,e\}} N(aoo|B_1B_2B_3) - \sum_{b\in\{u,e\}} N(obo|B_1B_2B_3) - \sum_{c\in\{u,e\}} N(ooc|B_1B_2B_3) \\ &- N(ooo|B_1B_2B_3). \end{aligned} \tag{7}$$

Substituting equation (7) into equation (5),

$$\begin{aligned} 2N(ooo|A_1A_2A_3) &- \sum_{c\in\{u,e\}} N(ooc|A_1A_2B_3) - \sum_{b\in\{u,e\}} N(obo|A_1B_2A_3) - \sum_{a\in\{u,e\}} N(aoo|B_1A_2A_3) \\ &- \sum_{a\in\{u,e\}} N(aoo|B_1B_2B_3) - \sum_{b\in\{u,e\}} N(obo|B_1B_2B_3) - \sum_{c\in\{u,e\}} N(ooc|B_1B_2B_3) \\ &\leqslant 2N(ooo|B_1B_2B_3). \end{aligned} \tag{8}$$

It should be noted that the generalization of the three-particle Eberhard's inequality is not unique, but equation (8) is a better way to obtain a lower efficiency threshold. In appendix A, we present another three-particle inequality different from equation (8), which has a higher threshold efficiency.

3.2. Eberhard's inequalities for *n*-particle systems

Consider a system composed of n observers that are labeled with the index set $I_n = \{1, 2, \ldots, n\}$. For the ith observer, two different measurement settings $\{A_i, B_i\}$ can be selected, and each measurement has three possible outcomes, 'o', 'u', 'e'. Let us denote $A^\tau = \Pi_{i\in\tau}A_i$ for the arbitrary subset $\tau \subseteq I_n$. Moreover, we also denote $\bar{\tau}$ as the complementary set of τ, and $|\tau|$ as the size of the subset τ.

Theorem 1. *For an* n*-particle system, if we select all the* $N(o^n|A^{I_n})$ *events as the classical sample and then remove their conjugates according to the measurement results. These sample events and their conjugates satisfy the following inequalities,*

$$\begin{aligned}(n-1)N(o^n|A^{I_n}) &- \sum_{\bar{\kappa}} \sum_{x\in\{u,e\}} N(o^{|\kappa|}x^{|\bar{\kappa}|}|A^\kappa B^{\bar{\kappa}}) \\ &- \sum_{|\bar{\tau}|=1}^{n-2} (n-1-|\bar{\tau}|) \sum_{\bar{\tau}} \sum_{|\upsilon|=0}^{|\bar{\tau}|} \sum_{\upsilon} N(o^{|\tau|}u^{|\upsilon|}e^{|\bar{\upsilon}|}|B^\tau B^\upsilon B^{\bar{\upsilon}}) - (n-1)N(o^n|B^{I_n}) \leqslant 0, \end{aligned} \tag{9}$$

where, $\kappa \subseteq I_n$, $|\bar{\kappa}| = 1$, $\upsilon \subseteq \bar{\tau}$.

Proof. According to the no-signaling relationships,

$$N(o^n|A^{I_n}) \leqslant N(o^n|A^{I_n}) + \sum_{x\in\{u,e\}} N(o^{|\kappa|}x^{|\bar{\kappa}|}|A^\kappa A^{\bar{\kappa}}) = N(o^n|A^\kappa B^{\bar{\kappa}}) + \sum_{x\in\{u,e\}} N(o^{|\kappa|}x^{|\bar{\kappa}|}|A^\kappa B^{\bar{\kappa}}). \tag{10}$$

Since we only select all the $N(o^n|A^{I_n})$ events as the sample, other events in setting (A^{I_n}) do not contribute to the number of sample events, so

$$\sum_{x\in\{u,e\}} N(o^{|\kappa|}x^{|\bar{\kappa}|}|A^{\kappa}A^{\bar{\kappa}}) = 0. \tag{11}$$

Using equations (11) and (10) is rewritten as,

$$N(o^n|A^{I_n}) = N(o^n|A^{\kappa}B^{\bar{\kappa}}) + \sum_{x\in\{u,e\}} N(o^{|\kappa|}x^{|\bar{\kappa}|}|A^{\kappa}B^{\bar{\kappa}}). \tag{12}$$

Similarly, according to no-signaling relationships, the sampling events and their conjugate events satisfy the following relationships:

$$N(o^n|A^{I_n}) = \sum_{x\in\{u,e\}} N(o^{|\kappa|}x^{|\bar{\kappa}|}|A^{\kappa}B^{\bar{\kappa}}), \tag{13a}$$

$$\sum_{\bar{\kappa}}\sum_{x\in\{u,e\}} N(o^{|\kappa|}x^{|\bar{\kappa}|}|A^{\kappa}B^{\bar{\kappa}}) = \sum_{|\bar{\tau}|=1}^{n}|\bar{\tau}|\sum_{\bar{\tau}}\sum_{|\upsilon|=0}^{|\bar{\tau}|}\sum_{\upsilon} N(o^{|\tau|}u^{|\upsilon|}e^{|\bar{\upsilon}|}|B^{\tau}B^{\upsilon}B^{\bar{\upsilon}}). \tag{13b}$$

Substituting equations (13a) and (13b) into equation (9),

$$\begin{aligned}(n-1)N(o^n|A^{I_n}) &- \sum_{|\bar{\kappa}|=1,\bar{\kappa}}\sum_{x\in\{u,e\}} N(o^{|\kappa|}x^{|\bar{\kappa}|}|A^{\kappa}B^{\bar{\kappa}})\\ &- \sum_{|\bar{\tau}|=1}^{n-2}(n-1-|\bar{\tau}|)\sum_{\bar{\tau}}\sum_{|\upsilon|=0}^{|\bar{\tau}|}\sum_{\upsilon} N(o^{|\tau|}u^{|\upsilon|}e^{|\bar{\upsilon}|}|B^{\tau}B^{\upsilon}B^{\bar{\upsilon}})\\ &- (n-1)N(o^n|B^{I_n}) = -\sum_{|\upsilon|=0}^{n}\sum_{\upsilon} N(u^{|\upsilon|}e^{|\bar{\upsilon}|}|B^{\upsilon}B^{\bar{\upsilon}}) \leqslant 0.\end{aligned} \tag{14}$$

Since the counts of events cannot be negative, equation (14) is true. Thus, if the no-signaling conditions are satisfied, we prove that the sample events and their conjugates always satisfy equation (14).

4. Applications of multi-particle Eberhard's inequalities

Denote the sampling efficiency as η, which is independent of the input. The joint probability under such inefficiency, denoted as P_η, is related to the ideal probability by

$$P_\eta(abc|\alpha_1\beta_2\gamma_3) = \eta^3 P(abc|\alpha_1\beta_2\gamma_3), \tag{15a}$$

$$P_\eta(abu|\alpha_1\beta_2\gamma_3) = \eta^2(1-\eta)P(ab|\alpha_1\beta_2), \tag{15b}$$

$$P_\eta(auc|\alpha_1\beta_2\gamma_3) = \eta^2(1-\eta)P(ac|\alpha_1\gamma_3), \tag{15c}$$

$$P_\eta(ubc|\alpha_1\beta_2\gamma_3) = \eta^2(1-\eta)P(ab|\beta_2\gamma_3), \tag{15d}$$

$$P_\eta(uuu|\alpha_1\beta_2\gamma_3) = (1-\eta)^3, \tag{15e}$$

where, $a, b, c \in \{o, e\}$, $\alpha, \beta, \gamma \in \{A, B\}$. Since η represents the probability that particles emitted by the entangled source are detected, the coefficients in equation (15) represent the probability of various events occurring under inefficiency relative to the ideal situation. If Alice, Bob, and Charlie all successfully detect particles, which happens with probability η^3. If one of them fails, which happens with probability $\eta^2(1-\eta)$. When none of the measurements succeeds, which happens with probability $(1-\eta)^3$.

4.1. Applications of three-particle Eberhard's inequality to the Bell test

Introducing the parameter J^{ideal}, three-particle Eberhard's inequality can be rewritten as

$$
\begin{aligned}
J^{\text{ideal}} = &\sum_{a\in\{u,e\}} N(aoo|B_1A_2A_3) + \sum_{b\in\{u,e\}} N(obo|A_1B_2A_3) + \sum_{c\in\{u,e\}} N(ooc|A_1A_2B_3) \\
&+ \sum_{a\in\{u,e\}} N(aoo|B_1B_2A_3) + \sum_{b\in\{u,e\}} N(obo|B_1B_2A_3) + \sum_{c\in\{u,e\}} N(ooc|B_1B_2B_3) \\
&+ 2N(ooo|B_1B_2B_3) - 2N(ooo|A_1A_2A_3) \geqslant 0.
\end{aligned} \tag{16}
$$

Given an initial state ψ, and a value for η, predictions for the number of events involved in equation (16) can be computed (we restrict measurements to be projective within the x–y plane of the Bloch-sphere. See appendix B for a discussion of the general direction involving the z-axis),

$$N(ooo|\alpha_1\beta_2\gamma_3) = N\frac{\eta^3}{8}\psi^\dagger[I+\sigma(\alpha_1)][I+\tau(\beta_2)][I+\mu(\gamma_3)]\psi, \tag{17a}$$

$$N(ooe|\alpha_1\beta_2\gamma_3) = N\frac{\eta^3}{8}\psi^\dagger[I+\sigma(\alpha_1)][I+\tau(\beta_2)][I-\mu(\gamma_3)]\psi, \tag{17b}$$

$$N(oou|\alpha_1\beta_2\gamma_3) = N\frac{\eta^2(1-\eta)}{4}\psi^\dagger[I+\sigma(\alpha_1)][I+\tau(\beta_2)]\psi, \tag{17c}$$

$$N(oeo|\alpha_1\beta_2\gamma_3) = N\frac{\eta^3}{8}\psi^\dagger[I+\sigma(\alpha_1)][I-\tau(\beta_2)][I+\mu(\gamma_3)]\psi, \tag{17d}$$

$$N(ouo|\alpha_1\beta_2\gamma_3) = N\frac{\eta^2(1-\eta)}{4}\psi^\dagger[I+\sigma(\alpha_1)][I+\mu(\gamma_3)]\psi, \tag{17e}$$

$$N(eoo|\alpha_1\beta_2\gamma_3) = N\frac{\eta^3}{8}\psi^\dagger[I-\sigma(\alpha_1)][I+\tau(\beta_2)][I+\mu(\gamma_3)]\psi, \tag{17f}$$

$$N(uoo|\alpha_1\beta_2\gamma_3) = N\frac{\eta^2(1-\eta)}{4}\psi^\dagger[I+\tau(\beta_2)][I+\mu(\gamma_3)]\psi, \tag{17g}$$

where, $\alpha, \beta, \gamma \in \{A, B\}$,

$$\sigma(\alpha_1) = \begin{pmatrix}
0 & e^{2i(\alpha_1-A_1)} & 0 & 0 & 0 & 0 & 0 & 0 \\
e^{-2i(\alpha_1-A_1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & e^{2i(\alpha_1-A_1)} & 0 & 0 & 0 & 0 \\
0 & 0 & e^{-2i(\alpha_1-A_1)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & e^{2i(\alpha_1-A_1)} & 0 & 0 \\
0 & 0 & 0 & 0 & e^{-2i(\alpha_1-A_1)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{2i(\alpha_1-A_1)} \\
0 & 0 & 0 & 0 & 0 & 0 & e^{-2i(\alpha_1-A_1)} & 0
\end{pmatrix}, \tag{18a}$$

$$\tau(\beta_2) = \begin{pmatrix}
0 & 0 & e^{2i(\beta_2-A_2)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & e^{2i(\beta_2-A_2)} & 0 & 0 & 0 & 0 \\
e^{-2i(\beta_2-A_2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e^{-2i(\beta_2-A_2)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & e^{2i(\beta_2-A_2)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{2i(\beta_2-A_2)} \\
0 & 0 & 0 & 0 & e^{-2i(\beta_2-A_2)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & e^{-2i(\beta_2-A_2)} & 0 & 0
\end{pmatrix}, \tag{18b}$$

$$\mu(\gamma_3) = \begin{pmatrix}
0 & 0 & 0 & 0 & e^{2i(\gamma_3-A_3)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & e^{2i(\gamma_3-A_3)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & e^{2i(\gamma_3-A_3)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{2i(\gamma_3-A_3)} \\
e^{-2i(\gamma_3-A_3)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e^{-2i(\gamma_3-A_3)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & e^{-2i(\gamma_3-A_3)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & e^{-2i(\gamma_3-A_3)} & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{18c}$$

Table 1. Critical conditions for achieving a Bell test without detection loophole in the three-particle system. $\theta = B_1 - A_1 = B_2 - A_2 = B_3 - A_3$.

η (%)	ζ (%)	F (%)	J/N	r	θ (deg)	ω (deg)	ϕ (deg)
100	4.0356	75.59	−0.2421	0.1133	−54.20	-11.59	116.34
95	2.5142	82.49	−0.1508	0.1055	−55.09	-11.02	105.65
90	1.4599	88.41	−0.0876	0.1334	−54.69	-8.44	96.41
85	0.7619	93.17	−0.0457	0.1934	−53.04	-3.66	88.58
80	0.3377	96.60	−0.0203	0.2860	−50.02	3.56	82.35
75	0.1150	98.71	−0.0069	0.4118	−45.38	13.57	78.08
70	0.0242	99.70	−0.0015	0.5710	−38.60	26.97	76.26
65	0.0016	99.98	−0.0001	0.7649	−28.30	45.49	77.72
60	0.0000	100	0.0000	0.9933	−4.69	83.01	87.71

Substituting equations (17a)–(17g) into equation (16),

$$J^{\text{ideal}} = \psi^{\dagger}\mathcal{B}\psi, \tag{19}$$

is obtained. $\mathcal{B}$ is the combination of the projection operators of all events in the equation (16). To achieve a violation, we need a negative value for J^{ideal}. That is possible only if the operator $\mathcal{B}$ has no less than one negative eigenvalue. We compute $\mathcal{B}$ numerically for any given value of the efficiency η, and then find that if and only if the sampling efficiency is greater than $\frac{3}{5}$, there are negative eigenvalues. In table 1, we give the critical conditions to achieve a Bell test without the detection loophole in the three-particle system.

The above computation is under the ideal case where the statistical counting is not affected by the background noise of detection. Considering the influence of the background noise on the actual experiment, equation (19) must be corrected to take into account deviations from that ideal case. Assume that the background noise is independent of settings. Then the events $N(uoo|B_1A_2A_3) + N(eoo|B_1A_2A_3)$, $N(ouo|A_1B_2A_3) + N(oeo|A_1B_2A_3)$, $N(oou|A_1A_2B_3) + N(ooe|A_1A_2B_3)$, $N(uoo|B_1B_2B_3) + N(eoo|B_1B_2B_3)$, $N(ouo|B_1B_2B_3) + N(oeo|B_1B_2B_3)$, and $N(ooe|B_1B_2B_3) + N(ooc|B_1B_2B_3)$ need to add background counts $N\zeta$. In principle, the events of type 'ooo' also introduce additional background counts in settings $(A_1A_2A_3)$ and $(B_1B_2B_3)$. However, since we assume that the background noise is independent of settings, the effect of these two items on equation (19) cancels each other out. After correction for background noise, equation (19) can be rewritten as,

$$J_{\mathcal{B}} = J^{\text{ideal}} + 6N\zeta. \tag{20}$$

The maximum amount of background noise that can be tolerated corresponds to that value of ζ that makes the last negative eigenvalue of $\mathcal{B}$ turn from negative to positive. For $\eta \leqslant \frac{3}{5}$, all eigenvalues of $\mathcal{B}$ are positive. For $\eta > \frac{3}{5}$, there are negative eigenvalues of $\mathcal{B}$ for small values of ζ, increasing from 0 to 0.04 as η increases from $\frac{3}{5}$ to 1. The eigenvector corresponding to the negative eigenvalue has the following form in the circular polarization basis,

$$\begin{aligned}\psi_{LR} = \frac{1}{2\sqrt{6+2r^2}}\big(&-(3-r)(\mathrm{e}^{\mathrm{i}\omega}|LLL\rangle + \mathrm{e}^{-\mathrm{i}\omega}|RRR\rangle) + (1+r)\mathrm{e}^{\mathrm{i}\phi}(|LLR\rangle + |LRL\rangle + |RLL\rangle)\\ &+ (1+r)\mathrm{e}^{-\mathrm{i}\phi}(|LRR\rangle + |RLR\rangle + |RRL\rangle)\big),\end{aligned} \tag{21}$$

where L and R denote left-hand and right-hand circular polarization. The eigenvector ψ_{LR} can be converted from the circular basis to the standard basis,

$$\psi_{HV} = \frac{1}{\sqrt{3+r^2}}\big(r|HHH\rangle + |HVV\rangle + |VHV\rangle + |VVH\rangle\big), \tag{22}$$

where H and V denote horizontal and vertical polarization. Furthermore, the following relationship needs to be satisfied: $A_1 = A_2 = A_3 = -\phi = -\omega/3$.

On the other hand, the infidelity of quantum states is also one of the important factors that determine the success of the Bell test. Here we consider the infidelity of quantum states caused by the white noise, the lowest fidelity of the quantum state that can be tolerated corresponds to the value of F, which also makes the last negative eigenvalue of $\mathcal{B}$ turn from negative to positive. For the three-particle system, F decreases from 1 to 75.59% when the sampling efficiency increases from $\frac{3}{5}$ to 1. Referring to the current experimental conditions [33], we give the threshold of fidelity under the corresponding experimental conditions in figure 6.

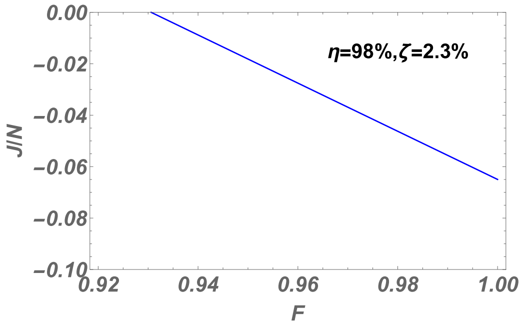

Figure 6. Under the current experimental conditions (the system detection efficiency is 98%, the dark count is 1000 per second), a lower bound on the quantum state fidelity must be satisfied to achieve a violation of the three-particle inequality.

4.2. Threshold efficiency for n-particle Eberhard's inequalities violations

For all events in equation (9), we can replace the measurement counts with quantum probabilities,

$$N(o^n|A^{I_n}) = N\eta^n P(o^n|A^{I_n}), \tag{23a}$$

$$\sum_{x\in\{u,e\}} N(o^{|\kappa|}x^{|\bar{\kappa}|}|A^\kappa B^{\bar{\kappa}}) = N(o^{n-1}|A^\kappa) - N(o^n|A^\kappa B^{\bar{\kappa}})$$
$$= N\eta^{n-1}P(o^{n-1}|A^\kappa) - N\eta^n P(o^n|A^\kappa B^{\bar{\kappa}}), \tag{23b}$$

$$\sum_{\bar{\tau}}\sum_{|\upsilon|=0}^{|\bar{\tau}|}\sum_{\upsilon} N(o^{|\tau|}u^{|\upsilon|}e^{|\bar{\upsilon}|}|B^\tau B^\upsilon B^{\bar{\upsilon}}) = N\sum_{\bar{\tau}}\sum_{|\upsilon|=0}^{|\bar{\tau}|}\sum_{\upsilon} \eta^{|\tau|+|\bar{\upsilon}|}(1-\eta)^{|\upsilon|}P(o^{|\tau|}e^{|\bar{\upsilon}|}|B^\tau B^{\bar{\upsilon}}), \tag{23c}$$

$$N(o^n|B^{I_n}) = N\eta^n P(o^n|B^{I_n}), \tag{23d}$$

where N is the total number of events in an experimental period. Substituting equations (23a)–(23d) into equation (9),

$$\eta \leqslant \frac{\sum_{\kappa,|\kappa|=n-1} P(o^{n-1}|A^\kappa)}{(n-1)P(o^n|A^{I_n}) + \sum_{\bar{\kappa}} P(o^n|A^\kappa B^{\bar{\kappa}}) - \sum_{|\bar{\tau}|=1}^{n-2}(n-1-|\bar{\tau}|)\sum_{\bar{\tau}}\sum_{|\upsilon|=0}^{|\bar{\tau}|}\sum_{\upsilon}\left(\frac{1-\eta}{\eta}\right)^{|\upsilon|}P(o^{|\tau|}e^{|\bar{\upsilon}|}|B^\tau B^{\bar{\upsilon}}) - (n-1)P(o^n|B^{I_n})}, \tag{24}$$

is obtained, in which,

$$P(o^n|A^{I_n}) \leqslant \min_{\kappa\subseteq I_n,|\kappa|=n-1} P(o^{n-1}|A^\kappa), \tag{25a}$$

$$P(o^n|A^\kappa B^{\bar{\kappa}}) \leqslant P(o^{n-1}|A^\kappa), \tag{25b}$$

$$\sum_{|\bar{\tau}|=1}^{n-2}(n-1-|\bar{\tau}|)\sum_{\bar{\tau}}\sum_{|\upsilon|=0}^{|\bar{\tau}|}\sum_{\upsilon}\left(\frac{1-\eta}{\eta}\right)^{|\upsilon|} P(o^{|\tau|}e^{|\bar{\upsilon}|}|B^\tau B^{\bar{\upsilon}}) \geqslant 0, \tag{25c}$$

$$(n-1)P(o^n|B^{I_n}) \geqslant 0. \tag{25d}$$

Because of symmetry, we assume that all $\sum_{\kappa,|\kappa|=n-1} P(o^{n-1}|A^\kappa)$ are equal. The lowest bound in equation (24) would be obtained when equations (25a)–(25d) are all equality. We then would have

$$\eta \leqslant \frac{n}{2n-1}. \tag{26}$$

Because Eberhard's inequalities state what correlations or probabilities are to be expected from a local realistic model, if $\eta \leqslant \frac{n}{2n-1}$, it is impossible to obtain a violation. $\eta > \frac{n}{2n-1}$, implies that quantum correlation cannot be explained by local realistic theory. We note that some works [25, 26] obtained the same bound using different approaches. We believe that further research work on this threshold would be interesting, but this is beyond the goal of our current work.

5. Summary

We have generalized two-particle Eberhard's inequality to the n-particle systems and found that similar experimental friendliness as in the two-particle case can still be obtained in the three-particle case.

Therefore, an experimental proposal to achieve a Bell test without the fair sampling assumption has been presented for the case of three particles. We have also analyzed the influence of the background noise and the quantum state infidelity on the actual experiment. For any given value of the sampling efficiency, we have given the optimal configurations for actual implementation, the optimal state, the maximum background noise that the system can tolerate, and the lowest fidelity of the target state. In addition, we have also demonstrated that it is impossible to yield a violation of the n-particle Eberhard's inequalities if the sampling efficiencies are smaller than or equal to $\frac{n}{2n-1}$.

Finally, let us discuss possible applications of multi-particle Eberahard's inequalities. As we all know, conference key agreement is the task of distributing a secret key among N parties. Often, the security of multi-partite device-independent (DI) protocols, such as DI conference-key agreement, relies on the violation of a multi-partite Bell inequality [34–36]. Fortunately, we have shown that the three-particle Eberhard's inequality is still experimental friendliness, so we believe our work can serve as a recipe for researchers working on DI conference-key agreement.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Another possible inequality

Multi-particle Eberhard's inequality is likely to be written in a simpler form than equation (9). For the three-particle, another possible inequality can be written as

$$N(ooo|A_1A_2A_3) - \sum_{a\in\{u,e\}} N(aoo|B_1A_2A_3) - \sum_{b\in\{u,e\}} N(obo|A_1B_2A_3) - \sum_{c\in\{u,e\}} N(ooc|A_1A_2B_3) - N(ooo|B_1B_2B_3) \leqslant 0. \tag{A.1}$$

Although the form of this inequality seems simpler, the required efficiency threshold for inequality violation is higher. Next, we prove this result in detail. For all events in the above inequality, the number of events can be written as:

$$N(ooo|A_1A_2A_3) = N\eta^3 P(ooo|A_1A_2A_3), \tag{A.2a}$$

$$\sum_{a\in\{u,e\}} N(aoo|B_1A_2A_3) = N\eta^2 P(oo|A_2A_3) - N\eta^3 P(ooo|B_1A_2A_3), \tag{A.2b}$$

$$\sum_{b\in\{u,e\}} N(obo|A_1B_2A_3) = N\eta^2 P(oo|A_1A_3) - N\eta^3 P(ooo|A_1B_2A_3), \tag{A.2c}$$

$$\sum_{c\in\{u,e\}} N(ooc|A_1A_2B_3) = N\eta^2 P(oo|A_1A_2) - N\eta^3 P(ooo|A_1A_2B_3), \tag{A.2d}$$

$$N(ooo|B_1B_2B_3) = N\eta^3 P(ooo|B_1B_2B_3). \tag{A.2e}$$

Substituting equations (A.2a)–(A.2e) into equation (A.1),

$$\eta \leqslant \frac{P(oo|A_2A_3) + P(oo|A_1A_3) + P(oo|A_1A_2)}{P(ooo|A_1A_2A_3) + P(ooo|B_1A_2A_3) + P(ooo|A_1B_2A_3) + P(ooo|A_1A_2B_3) - P(ooo|B_1B_2B_3)}, \tag{A.3}$$

is obtained, in which,

$$P(ooo|A_1A_2A_3) \leqslant P(oo|A_2A_3), \tag{A.4a}$$

$$P(ooo|B_1A_2A_3) \leqslant P(oo|A_2A_3), \tag{A.4b}$$

$$P(ooo|A_1B_2A_3) \leqslant P(oo|A_1A_3), \tag{A.4c}$$

$$P(ooo|A_1A_2B_3) \leqslant P(oo|A_1A_2), \tag{A.4d}$$

$$P(ooo|B_1B_2B_3) \geqslant 0. \tag{A.4e}$$

The lowest bound would be obtained when we have equality in equations (A.4a)–(A.4e), and when $P(oo|A_2A_3) = P(oo|A_1A_3) = P(oo|A_1A_2)$, we then would have

$$\eta \leqslant \frac{3P(oo|A_2A_3)}{4P(oo|A_2A_3)} = \frac{3}{4}. \tag{A.5}$$

This threshold is higher than our result. Therefore, equation (8) is better when we consider the sampling threshold.

Appendix B. Measurements in the *x*–*z* plane of the Bloch sphere

Here, we discuss the case where the measurements are projected in the x–z plane of the Bloch-sphere, i.e., measurements in the form of $\{\Pi(\varphi), 1-\Pi(\varphi)\}$ with $\Pi(\varphi)$ defined by

$$\Pi(\varphi) = \begin{pmatrix} \cos^2(\varphi) & \cos(\varphi)\sin(\varphi) \\ \cos(\varphi)\sin(\varphi) & \sin^2(\varphi) \end{pmatrix}. \tag{2.1}$$

Since each measurement produces one of three possible outcomes, o, e, or u, Alice, Bob, and Charlie's joint probability are given by

$$p(a,b,c|\alpha_1,\beta_2,\gamma_3) = \mathrm{Tr}[\rho(P_{a|\alpha_1} \otimes P_{b|\beta_2} \otimes P_{c|\gamma_3})], \tag{2.2}$$

where, $a, b, c \in \{o, u, e\}$, $\alpha, \beta, \gamma \in \{A, B\}$. $P_{a|\alpha_1}$, $P_{b|\beta_2}$, and $P_{c|\gamma_3}$ can be written as

$$P_{a|\alpha_1} = \delta_{a,o}\Pi(\alpha_1)\eta + \delta_{a,e}(1-\Pi(\alpha_1))\eta + \delta_{a,u}(1-\eta), \tag{2.3a}$$

$$P_{b|\beta_2} = \delta_{b,o}\Pi(\beta_2)\eta + \delta_{b,e}(1-\Pi(\beta_2))\eta + \delta_{b,u}(1-\eta), \tag{2.3b}$$

$$P_{c|\gamma_3} = \delta_{c,o}\Pi(\gamma_3)\eta + \delta_{c,e}(1-\Pi(\gamma_3))\eta + \delta_{c,u}(1-\eta). \tag{2.3c}$$

Given an initial state ψ, and a value for η, predictions for the number of events involved in equation (16) can be computed,

$$N'(ooo|\alpha_1\beta_2\gamma_3) = N\frac{\eta^3}{8}\psi^\dagger[I+\sigma'(\alpha_1)][I+\tau'(\beta_2)][I+\mu'(\gamma_3)]\psi, \tag{2.4a}$$

$$N'(ooe|\alpha_1\beta_2\gamma_3) = N\frac{\eta^3}{8}\psi^\dagger[I+\sigma'(\alpha_1)][I+\tau'(\beta_2)][I-\mu'(\gamma_3)]\psi, \tag{2.4b}$$

$$N'(oou|\alpha_1\beta_2\gamma_3) = N\frac{\eta^2(1-\eta)}{4}\psi^\dagger[I+\sigma'(\alpha_1)][I+\tau'(\beta_2)]\psi, \tag{2.4c}$$

$$N'(oeo|\alpha_1\beta_2\gamma_3) = N\frac{\eta^3}{8}\psi^\dagger[I+\sigma'(\alpha_1)][I-\tau'(\beta_2)][I+\mu'(\gamma_3)]\psi, \tag{2.4d}$$

$$N'(ouo|\alpha_1\beta_2\gamma_3) = N\frac{\eta^2(1-\eta)}{4}\psi^\dagger[I+\sigma'(\alpha_1)][I+\mu'(\gamma_3)]\psi, \tag{2.4e}$$

$$N'(eoo|\alpha_1\beta_2\gamma_3) = N\frac{\eta^3}{8}\psi^\dagger[I-\sigma'(\alpha_1)][I+\tau'(\beta_2)][I+\mu'(\gamma_3)]\psi, \tag{2.4f}$$

$$N'(uoo|\alpha_1\beta_2\gamma_3) = N\frac{\eta^2(1-\eta)}{4}\psi^\dagger[I+\tau'(\beta_2)][I+\mu'(\gamma_3)]\psi, \tag{2.4g}$$

Table B1. Measurements in different planes of the Bloch sphere. $\theta = B_1 - A_1 = B_2 - A_2 = B_3 - A_3$, $\theta' = B_1' - A_1' = B_2' - A_2' = B_3' - A_3'$.

	x–y plane		x–z plane	
η (%)	θ (deg)	J/N	θ' (deg)	J'/N
100	−54.20	−0.2421	−54.17	−0.2421
95	−55.09	−0.1509	−55.07	−0.1509
90	−54.69	−0.0876	−54.66	−0.0876
85	−53.04	−0.0457	−53.01	−0.0457
80	−50.02	−0.0203	−50.00	−0.0203
75	−45.38	−0.0069	−45.36	−0.0069
70	−38.60	−0.0014	−38.58	−0.0015
65	−28.30	−0.0001	−28.34	−0.0001
60	−4.69	−0.0000	−0.00	−0.0000

where,

$$\sigma'(\alpha_1) = \begin{pmatrix} \cos 2\alpha_1 & \sin 2\alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sin 2\alpha_1 & -\cos 2\alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos 2\alpha_1 & \sin 2\alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin 2\alpha_1 & -\cos 2\alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos 2\alpha_1 & \sin 2\alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin 2\alpha_1 & -\cos 2\alpha_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & cos2\alpha_1 & sin2\alpha_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin 2\alpha_1 & -\cos 2\alpha_1 \end{pmatrix}, \tag{2.5a}$$

$$\tau'(\beta_2) = \begin{pmatrix} \cos 2\beta_2 & 0 & \sin 2\beta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos 2\beta_2 & 0 & \sin 2\beta_2 & 0 & 0 & 0 & 0 \\ \sin 2\beta_2 & 0 & -\cos 2\beta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sin 2\beta_2 & 0 & -\cos 2\beta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos 2\beta_2 & 0 & \sin 2\beta_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos 2\beta_2 & 0 & \sin 2\beta_2 \\ 0 & 0 & 0 & 0 & \sin 2\beta_2 & 0 & -\cos 2\beta_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sin 2\beta_2 & 0 & -\cos 2\beta_2 \end{pmatrix}, \tag{2.5b}$$

$$\mu'(\gamma_3) = \begin{pmatrix} \cos 2\gamma_3 & 0 & 0 & 0 & \sin 2\gamma_3 & 0 & 0 & 0 \\ 0 & \cos 2\gamma_3 & 0 & 0 & 0 & \sin 2\gamma_3 & 0 & 0 \\ 0 & 0 & \cos 2\gamma_3 & 0 & 0 & 0 & \sin 2\gamma_3 & 0 \\ 0 & 0 & 0 & \cos 2\gamma_3 & 0 & 0 & 0 & \sin 2\gamma_3 \\ \sin 2\gamma_3 & 0 & 0 & 0 & -\cos 2\gamma_3 & 0 & 0 & 0 \\ 0 & \sin 2\gamma_3 & 0 & 0 & 0 & -\cos 2\gamma_3 & 0 & 0 \\ 0 & 0 & \sin 2\gamma_3 & 0 & 0 & 0 & -\cos 2\gamma_3 & 0 \\ 0 & 0 & 0 & \sin 2\gamma_3 & 0 & 0 & 0 & -\cos 2\gamma_3 \end{pmatrix}. \tag{2.5c}$$

Substituting equations (2.4a)–(2.5c) into equation (16), $J' = \psi^\dagger \mathcal{B}' \psi$ is obtained. We compute $\mathcal{B}'$ numerically for any given value of the efficiency η, and then find that the threshold of efficiency and the violation of inequality coincided with the results in the x–y plane, as shown in table B1.

ORCID iDs

Xiaosong Ma https://orcid.org/0000-0002-0500-5690

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