From fields to paths, or there and back again

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De champs vers chemins, histoire d'un aller et retour

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Mémoire d'Habilitation à Diriger des Recherches

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Abstract

The Gaussian free field (GFF) is one of the most fundamental objects of Statistical Physics and Quantum Field Theory. There is a strong connection between this GFF and the random walks (in discrete) and the Brownian motion (in continuum). These are the so called "isomorphism theorems", which originate from Euclidean Quantum Field Theory. This manuscript presents an overview of the results obtained by myself and my coauthors on this topic. An important aspect of my work was to relate in dimension 2 the isomorphism theorems to the theory of Schramm-Loewner Evolutions (SLE) and to describe the intrinsic geometry of the 2D continuum GFF in terms of clusters of Brownian trajectories. A method commonly used throughout this manuscript is that of metric graphs, which I introduced earlier during my PhD. Other aspects of the isomorphism theorems covered by this manuscript include their relation to the 2D Gaussian multiplicative chaos, their relation to the topological expansion, and the inversion of the isomorphism theorems and how it involves self-repelling processes.

Résumé

Le champ libre gaussien (CLG) est un des objets le plus fondamentaux de la Physique Statistique et de la Théorie Quantique des Champs. Il y a une forte relation entre ce CLG et les marches aléatoires (dans le discret) et le mouvement brownien (dans le continu). Ce sont les théorèmes dit "d'isomorphisme", qui proviennent de la Théorie Quantique des Champs Euclidienne. Ce manuscrit présente un panorama des résultats obtenus par moi-même et mes coauteurs sur ce sujet. Un aspect important de mon travail a été de relier en dimension 2 les théorèmes d'isomorphisme à la théorie des Évolutions de Schramm-Loewner (SLE) et de décrire la géométrie intrinsèque du CLG sur espace continu 2D en termes d'amas de trajectoires browniennes. Une méthode beaucoup utilisée dans ce manuscrit est celle des graphes métriques, que j'ai introduite pendant mon Doctorat. Les autres aspects des théorèmes d'isomorphisme traités dans ce manuscrit incluent leurs relations avec le chaos multiplicatif gaussien en dimension 2, leurs relations avec l'expansion topologique, ainsi que l'inversion des théorèmes d'isomorphisme et comment cela fait intervenir des processus auto-répulsifs.

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Contents

Introduction

There are two main characters in this memoir. The first one is the Gaussian free field (GFF), also known as Euclidean bosonic free field. It will appear both on continuous domains and on discrete electrical networks, and importantly, on so called metric graphs (or cable systems), but more on them later. The GFF is ubiquitous in Statistical Physics, but it traces its origins back to the earliest works on Quantum Field Theory (QFT) in the 1920s. One can turn the QFT into essentially a subdomain of Statistical Physics and Probability Theory in generally by a procedure known as the Wick rotation: in the functional integrals associated to quantum fields one sees the time coordinate as a pure imaginary additional space coordinate. The Wick rotation relates a dynamical quantum picture in space dimension $d-1$ to a static statistical picture in space dimension d. It is a major ingredient in the mathematically rigorous constructive QFT [GJ87, Sim74]. The GFF is obtained by Wick-rotating the (quantum) bosonic free field, which is a gas of non-interacting spin-0 bosons. This memoir will mostly deal with the free field, but it is worth mentioning that by adding an interaction potential to the free field one gets an interacting quantum field, such as the φ^4 (Phi 4), and then the major questions are the renormalization of the field in continuum, the asymptotic triviality and the symmetry breaking.

The second main character of this memoir is the random walk in discrete or the Brownian motion in continuum (these two count for one). The two characters, GFF and random walk/Brownian motion are related through the so called *isomorphism theorems* which tell that the square of the former is of the same nature as the occupation times of the latter. The term "isomorphism" is not completely well suited here. It is rather a correspondence, a family of exact identities, rather than an isomorphism between objects of the same category. These isomorphisms emerged from the Euclidean QFT, i.e. the QFT which deals with the Wick-rotated quantum fields. The precursor of these isomorphisms was a work of the physicist Kurt Symanzik [Sym66] in the 1960s. He expressed the correlations of the Euclidean φ^4 field in terms of Brownian paths, closed Brownian loops, and their intersection local times. His expression was purely formal and did not deal with the renormalization or the triviality of the φ^4 in higher dimensions. Symanzik's identities are related, but different from the expansion of the correlations of the φ^4 into Feynman diagrams, and unlike the latter do not require a small value of the coupling constant in the interaction potential. Then, a mathematically cleaner identity of the same nature was given separately by Brydges, Fröhlich and Spencer [BFS82] and by Dynkin [Dyn84a] in the 1980s. It will be referred to in this memoir as the BFS-Dynkin isomorphism. Other identities of the same nature were discovered or rediscovered over time by different probabilists. These isomorphism identities are more general than that of Symanzik [Sym66] in the following sense. It is no longer just about the correlations of the φ^4 and the self-intersection local times, it is more generally about the square of the GFF and the occupation times. Then indeed, if an interaction potential depends on the square of the field (and so does the φ^4), and does not take into account its signs, then the isomorphism identities give an expression of the corresponding correlations in terms of random walks in discrete, and after suitable renormalization in terms of Brownian motions in continuum. A major achievement in this domain was the proof by Fröhlich of the asymptotic triviality of the φ^4 field in dimensions $d \geq 5$ [Frö82]. He related through isomorphisms this asymptotic triviality to the asymptotic non-intersection of two independent random walks.

More recently, a new perspective emerged in dimension 2 both on the GFF and on the Brownian motion with the introduction of Schramm-Loewner Evolutions (SLE) [Sch07]. This is related to the conformal invariance, since both the 2D continuum GFF and the 2D Brownian motion are conformally invariant in law. The SLEs are random fractal curves in 2D domains that satisfy the conformal invariance in law and a domain Markov property. The SLE curves appear, or are conjectured so, as scaling limits of interfaces in different models of Statistical Physics at criticality in dimension 2. The SLE curves are primarily classified by a parameter $\kappa > 0$ (SLE_κ). The SLE_κ curves join two boundary points of a 2D domain. There are also versions of these curves that are loops (i.e. closed curves) inside the domain. These are the conformal loop ensembles CLE_κ [SW12].

Schramm and Sheffield introduced the notion of level lines of the 2D continuum GFF [SS09, SS13]. The GFF on a continuous domain in dimensions $d \geq 2$ is not a pointwise defined function, rather a random generalized function living in a Sobolev space. Nevertheless one can make sense of the level lines of the GFF in dimension $d = 2$. These are SLE_4 type curves in the case of level lines touching the boundary of the domain, and CLE_4 type loops in case in the case of level lines staying inside the domain.

Also in this context of conformally invariant stochastic processes, Lawler and Werner rediscovered in [LW04] the Symanzik's measure on Brownian loops that Symanzik used to express the correlations of the φ^4 field [Sym66]. Lawler and Werner also introduced Poisson point processses of Brownian loops with intensity proportional to these measure, which they called Brownian loop soups. So a Brownian loop soup comes with an intensity parameter $\alpha > 0$. Actually the Symanzik's loop measure has an ultraviolet divergence, therefore a Brownian loop soup contains countably infinitely many loops, with small loops of each scale. In dimension 2 the Brownian loop soups satisfy a conformal invariance in law if one considers only the range of the loops and forgets about their time parametrization. Because of this conformal invariance, the 2D Brownian loop soups are related to the SLE and CLE processes. Sheffield and Werner studied in [SW12] the clusters of Brownian loops in a Brownian loop soup on a simply connected 2D domain. They showed that there is a phase transition. For $\alpha > 1/2$ there is a.s. only one cluster that is everywhere dense in the domain, and for $\alpha \in (0, 1/2]$ there are infinitely many clusters. Moreover, in the latter case, the outermost boundaries of clusters are distributed as a CLE_κ conformal loop ensemble, with a correspondence between α and κ. Actually, if one adds on top of the Brownian loops an additional independent Poisson point process of Brownian boundary excursions, then one can construct an interface distributed as an SLE_κ joining two boundary points.

So in the picture above the intensity parameter $\alpha = 1/2$ plays a particular role, as it is the critical one for the Brownian loop clusters. The corresponding value of κ is $\kappa = 4$. Thus, the Brownian loop soup with $\alpha = 1/2$ is related to SLE_4, CLE_4, and therefore indirectly related to the 2D continuum GFF, because of the level lines story. There is also an other relation between the Brownian loop soup with $\alpha = 1/2$ and the continuum GFF, the one that comes from the isomorphism theorems that were discussed at the beginning. Actually in discrete, the occupation times of the random walk loop soup with $\alpha = 1/2$, the discrete equivalent of the Brownian loop soup, are distributed as the square of the discrete GFF. This is the Le Jan's isomorphism [LJ10, LJ11]. In dimension 2 one can renormalize this relation in the continuum limit by considering the Wick's square of the GFF. So one has a relation between the occupation times of the Brownian loop soup and the square of the GFF which originates from the Euclidean

QFT, and an other relation between the boundaries of clusters in a Brownian loop soup and the level lines of GFF, which is specific to the dimension 2 and is related to the conformal invariance.

A major part of my research was to unify these two different relations between the Brownian loop soup with $\alpha = 1/2$ and the GFF. The first step was to answer the following question. The isomorphism theorems related the square of a GFF to occupation times of random walk/Brownian paths. But what about the sign of the GFF? How is it related to these random walk/Brownian paths? I answered this question during my PhD under the supervision of Prof. Yves Le Jan at Université Paris-Sud (now Université Paris-Saclay). There is one case when it is simple to pin down the signs. It is the continuous 1D setting. The Brownian loops in 1D do not have much geometry. Their range is just a line segment, however it carries a non-trivial 1D Brownian local time process, distributed as a square Bessel 4 bridge. As for the GFF, in dimension 1 is is just a Brownian motion. It is easy to see that the excursions of this GFF away from 0 are exactly the clusters of the Brownian loop soup with $\alpha = 1/2$. Further, the idea is to force this continuous 1D picture into frameworks with more geometry. For this I extended the discrete GFFs to metric graphs. A metric graph (or cable system) is obtained from a discrete undirected graph by replacing each edge by a continuous line segment, so as to obtain a continuous topological object. The discrete GFF on the vertices of the graph has a natural interpolation to the metric graph by adding conditionally independent Brownian bridges inside the edges. Similarly, the nearest neighbor random walk on the discrete graph can be interpolated to a continuous Markovian diffusion on the metric graph, the metric graph Brownian motion. On the metric graph the sign clusters of the GFF again coincide with the clusters of the Brownian loop soup with $\alpha = 1/2$. This is explained in my article [16]. By relying on this correspondence, I proved in [14] the convergence in dimension 2 of boundaries of clusters in both random walk loop soup and metric graph loop soup to the CLE_κ-s in the scaling limit.

My research after my PhD relied heavily on the results I obtained and methods I introduced during my PhD. In particular I worked on the relation between the Brownian loop soup with $\alpha = 1/2$ and the continuum GFF in dimension 2. For instance, with Juhan Aru (now at EPFL) and Avelio Sepúlveda (now at Universidad de Chile) we constructed the so called first passage sets (FPS) of the 2D continuum GFF [9, 8]. We gave two constructions of this object. The first one is by iterating an infinite sequence of level lines of the GFF. The second one is by taking a Brownian loop soup with $\alpha = 1/2$ plus an independent Poisson point process of Brownian boundary excursions, and by considering the clusters connected to the boundary in this soup of Brownian paths. The equivalence of these two constructions is achieved by using the isomorphism theorems on metric graphs and taking the scaling limit. An FPS essentially carries a chunk of the GFF. The restriction of the GFF to an FPS is a positive Radon measure, actually a Minkowski content measure of the FPS in the gauge $|\log r|^{1/2} r^2$. One can see it as a renormalized uniform measure on the FPS, since the Lebesgue measure of the FPS is 0.

Further, one can combine these results on the FPS with the results of Qian and Werner [QW19] on the decomposition of Brownian loop soup clusters (which in turn relied on my convergence result from [14]) to obtain a decomposition of the 2D continuum GFF in terms of clusters of the Brownian loop soup. Each cluster has a sign, -1 or $+1$, chosen independently and uniformly. Each cluster also carries a Radon measure, the Minkowski content measure of the FPS in the gauge $|\log r|^{1/2} r^2$. By summing over the clusters these measures multiplied by the respective signs one gets the 2D continuum GFF. Note however that this does not mean that the GFF is a signed measure. Without the compensations induced by the signs, the absolute value of the GFF diverges in every open subset. At the end of the day, given a Brownian loop soup with $\alpha = 1/2$, plus an additional randomness given by the signs -1 or $+1$ on clusters, one can simultaneously obtain the GFF, its renormalized square and the CLE_4 associated to the GFF.

Recently, in a collaboration with Elie Aïdekon (Sorbonne Université), Nathanaël Berestycki (University of Vienna) and Antoine Jégo (now at EPFL), we obtained a description of the Gaussian multiplicative chaos (GMC) associated to the 2D GFF in terms of the Brownian loop soup [1]. One can renormalize the exponential of a 2D GFF. This is its GMC. There is a general theory of the multiplicative chaos of log-correlated Gaussian fields, introduced by Kahane [Kah85]. The multiplicative chaos of the 2D GFF in particular is a very popular topic now because it appears in the Liouville quantum gravity. See [RV14] for a review. In [1] we essentially showed that this GMC of the 2D GFF is a measure supported on points that are an intersection of infinitely many Brownian loops in a Brownian loop soup with $\alpha = 1/2$, and such that for each of the loops it is a point of infinite multiplicity. We also constructed similar measures for every value of the intensity parameter α of the loop soup. These measures have many features in common with the GMC, such as the values of the carrying dimensions and the conformal covariance. For $\alpha \neq 1/2$, these measures are a sort of non-Gaussian multiplicative chaoses.

To conclude this discussion, one can say that the 2D continuum GFF, despite being only a generalized function, has an intrinsic geometry. This geometry can be accessed either through SLE and CLE processes, or through the Brownian loop soup and other soups of Brownian trajectories, and there is to some extent a correspondence between the two descriptions. However, the description through SLE and CLE is related to the conformal invariance and is specific to the dimension 2. But the isomorphism theorems between the GFF and the random walks in discrete are not dimension specific. This suggests that it should be possible to describe the intrinsic geometry of the continuum GFF through Brownian loops, in particular through clusters of Brownian loops, in some higher dimensions too. Wendelin Werner formulated some conjectures in this direction in [Wer21]. His conjectures were inspired by my works on metric graphs and in 2D continuum.

My other works presented in this memoir are all either related to the isomorphism theorems or at least to the GFF. These works include the law of the extremal distance of a CLE_4 loop, the Lévy type transformation for the metric graph GFF, the combination of the isomorphism theorems for the matrix-valued fields with the topological expansion, the isomorphism theorems for the β-Dyson's Brownian motions, and the inversion of the isomorphism theorems in relation with the self-repelling random walks and self-repelling diffusions.

Organization of the manuscript

My works in this memoir are grouped thematically rather than chronologically. The memoir is made of 6 parts. In Part I is introduced the background on the isomorphism theorems and on SLE, CLE and level lines of the 2D continuum GFF. It is to be considered as a more detailed introduction into my research topics. The results I obtained during my PhD are also presented there. Parts II to V are an overview of my research after the PhD. Part VI presents some further directions of research. These are either works in progress, or problems on my to-do list, or longer term open problems. At the end there is a bibliography. It is split into two. First appears the list of the articles that I authored or coauthored. These publications and preprints are numbered from 1 to 18. Then comes the list of all other bibliographical references. These are cited by the authors' initials and the year. All my publications can be downloaded

- on HAL `https://hal.archives-ouvertes.fr/search/index/q/*/authIdHal_s/titus-lupu`,

- on arXiv `https://arxiv.org/search/math?query=Lupu%2C+Titus&searchtype=author&abstracts=show&order=-announced_date_first&size=50`.

Next follows a more detailed presentation of the content of Parts II to V.

Part II is dedicated to the construction of random conformally invariant or covariant fields in dimension 2 out of Brownian loop soups. In Chapter 4 is presented the joint work with Aru and Sepúlveda [9, 8] where we construct the first passage sets of the GFF in continuum and describe them as clusters of Brownian loops and excursions. In Chapter 5 is presented a joint work with Hao Wu (YMSC, Tsinghua University) [2]. By relying on the results presented in Chapter 4, we showed that the notion of level lines of the continuum GFF can be extended to measure-valued boundary conditions. Previously more regularity was assumed on the boundary conditions. In Chapter 6 is presented the joint work with Aïdekon, Berestycki and Jégo [1] on the multiplicative chaoses of the Brownian loop soups.

Part III is an overview of my others works related to the 2D GFF, but not necessarily to the isomorphism theorems. In Chapter 7 is presented a joint work with Aru and Sepúlveda [4]. We showed that certain geometrical quantities related to a CLE_4 loop can be read on a one-dimensional Brownian trajectory. More precisely, we give the joint law of the conformal radius and extremal distance (or extremal length) of a CLE_4 loop in terms of a first exit time and last passage time of a 1D Brownian motion. The law of the conformal radius was known previously, but not that of the extremal distance. Our proof relies on the coupling of the CLE_4 with the continuum GFF. In Chapter 8 is presented an article coauthored with Wendelin Werner [11]. There we show that the classical Lévy transformation for the 1D Brownian motion can be generalized to GFFs on any metric graph. Further, we conjecture that in dimension 2 this Lévy transformation has a fine mesh limit in continuum, and that this limit is related to a growth process for the CLE_4 loops.

In Part IV are presented my works on the relations between the isomorphism theorems and the topological expansion. The topological expansion appears in random matrix theory and expresses the moments of random matrices as sums over maps on 2D surfaces, with weights depending on the topology of the surface and the number of faces of the map. In Chapter 9 is presented my article [6]. There I considered Gaussian fields of real symmetric, complex Hermitian or quaternionic Hermitian matrices over an electrical network, and described how the isomorphisms between these fields and random walks give rise to topological expansions encoded by ribbon graphs. I further considered matrix-valued Gaussian fields twisted by an orthogonal, unitary or symplectic (quaternionic unitary) connection. In this case the isomorphisms involve traces of holonomies of the connection along random walk loops parametrized by boundary cycles of ribbon graphs. In Chapter 10 is presented my article [3]. There I show that the β-Dyson's Brownian motions for general values of β satisfy both a Le Jan type and BFS-Dynkin type isomorphism with the local times of one-dimensional Brownian trajectories. The Le Jan type isomorphism involves a whole range of intensity parameters α for the 1D Brownian loop soup, not just α half-integer. I further ask the question whether the β-Dyson's Brownian motions have natural generalizations in a multi-dimensional setting. This is motivated by the study of random walk and Brownian loop soups with a non half-integer intensity parameter α.

Part V is dedicated to the inversion of isomorphism theorems. The general question is the following. Rather than starting from random walks and constructing the square of discrete GFF, or the whole discrete GFF, from these random walks, one starts from the GFF or its square and asks for the conditional law of the random walks. This is a topic initiated by Christophe Sabot (Université Lyon 1 Claude Bernard) and Pierre Tarrès (NYU Shanghai) in [ST15a]. They showed in particular that the conditional law is related to a natural model for a self-repelling random walk, the Vertex Diminished Jump Process (VDJP). Part V begins with Chapter 11 where are recalled some elements on the combinatorics of the Ising model, in relation with the FK-Ising random cluster model and the random currents. This is because the inversion of the isomorphisms is related to these FK-Ising random clusters and to the random currents. In Chapter 11 is also presented a short note by Wendelin Werner and myself [12] where we gave a probabilistic coupling between the random currents and the FK-Ising random clusters. In Chapter 12 is presented my joint work with Sabot and Tarrès [10]. There we do the inversion in discrete when one conditions by the GFF with its signs, rather than just the square of the GFF as in [ST15a]. What we obtain is a VDJP on FK-Ising type random clusters, where the clusters themselves evolve over time by being progressively eroded. In Chapter 13 are presented two other articles coauthored with Sabot and Tarrès [7, 5]. There we consider the VDJP and its reinforced dual, the Vertex Reinforced Jump Process (VRJP). We show that in dimension one the VRJP and the VDJP admit continuous fine mesh limits that can be expressed through stochastic flows introduced by Bass and Burdzy in [BB99]. These limits can be seen a continuous 1D reinforced, respectively self-repelling diffusion processes. The self-repelling diffusion in particular inverts the isomorphism theorems in 1D continuum.

Part I

Random walk representations of scalar bosonic fields: old and new

Chapter 1

From occupation times of random walks to the square of bosonic fields

In this Chapter we first introduce the Gaussian free field (GFF) and explain how it is related to scalar bosonic fields in Quantum Field Theory (Section 1.1). We then present the description of the square of the discrete, respectively continuum space GFF, in terms of occupation times of random walks, respectively Brownian motions (Section 1.2). These random walk/Brownian motion representations of the GFF, sometimes called *isomorphism theorems*, originated in Mathematical Physics, motivated by the study of interacting bosonic fields, such as the φ^4 field [BFS82, Frö82], following the seminal work of Symanzik who formally expanded the correlations of the continuum φ^4 field in terms of Brownian loops [Sym65, Sym66, Sym69]. These isomorphism identities were one of the starting points of my research.

1.1 Gaussian free field and interacting bosonic fields: origins and motivations

1.1.1 Discrete GFF

A central object in this manuscript is the *Gaussian free field* [She07]. We first present it in the discrete setting. Let $\mathcal{G} = (V, E)$ be a connected undirected graph. For simplicity, we assume it finite, without multiple edges or self-loops. The set of vertices V is partitioned $V = V_{\text{int}} \amalg V_\partial$, where V_{int} is considered as the subset of interior edges and V_∂ as the boundary, and both are non-empty. Each edge $\{x, y\} \in E$ is endowed with a conductance $C(x, y) = C(y, x) > 0$, thus making $\mathcal{G}$ into an *electrical network*. We will denote by $x \sim y$ the fact that $\{x, y\} \in E$. For an example of electrical network, one could think $\mathcal{G}$ to be a finite box in $\mathbb{Z}^d$, with $V = [-n, n]^d \cap \mathbb{Z}^d$, $V_{\text{int}} = [-(n-1), (n-1)]^d \cap \mathbb{Z}^d$, and each edge having a unit conductance.

Definition 1.1. The massless *Gaussian free field* (GFF) on $\mathcal{G}$ with boundary condition f on V_∂ is the random Gaussian field $(\phi(x))_{x \in V}$ which coincides with f on V_∂, and for the values on V_{int} with density

$$\frac{1}{Z_f} \exp\Big(-\frac{1}{2} \sum_{\{x,y\} \in E} C(x, y)(\varphi(y) - \varphi(x))^2 \Big).$$

The massive GFF with boundary condition f on V_∂ and *square-mass* $K > 0$ is the random Gaussian field $(\phi_K(x))_{x \in V}$ which coincides with f on V_∂, and for the values on V_{int} with density

$$\frac{1}{Z_{f,K}} \exp\Big(-\frac{1}{2} \sum_{\{x,y\} \in E} C(x, y)(\varphi(y) - \varphi(x))^2 - \frac{K}{2} \sum_{x \in V_{\text{int}}} \varphi(x)^2 \Big).$$

The constant K above is proportional to the square of a mass in Quantum Field Theory; see Section 1.1.3.

Let $\Delta_{\mathcal{G}}$ be the discrete Laplacian on $\mathcal{G}$:

$$(\Delta_{\mathcal{G}} u)(x) = \sum_{y \sim x} C(x,y)(u(y) - u(x)). \tag{1.1}$$

The covariance structure of the massless GFF is given by the Green's function $(G(x,y))_{x,y\in V}$ of $-\Delta_{\mathcal{G}}$, with 0 boundary conditions. For the massive GFF the covariance structure is given by $(G_K(x,y))_{x,y\in V}$, the Green's function of $-\Delta_{\mathcal{G}}+K$, with 0 boundary conditions. The expectation of the massless GFF is given by the discrete harmonic extension of its boundary conditions.

The discrete GFF, both massless and massive, satisfies an obvious *Markov property*. Let $\widehat{V} \subset V$, such that both $\widehat{V}$ and $V \setminus \widehat{V}$ are non-empty. Then, conditionally of $(\phi(x))_{x\in\widehat{V}}$, resp. $(\phi_K(x))_{x\in\widehat{V}}$, the distribution of $(\phi(x))_{x\in V\setminus\widehat{V}}$, resp. $(\phi_K(x))_{x\in V\setminus\widehat{V}}$, is that of a massless, resp. massive, GFF on $V \setminus \widehat{V}$ with boundary conditions given by$(\phi(x))_{x\in\widehat{V}}$, resp. $(\phi_K(x))_{x\in\widehat{V}}$. Note that for these boundary conditions, only matter the values on the vertices of $\widehat{V}$ that are adjacent to $V \setminus \widehat{V}$. This *weak Markov property* can be upgraded to a *strong Markov property*, where $\widehat{V}$ is random and a *stopping set*, i.e. for every deterministic $U \subset V$, the event $\{\widehat{V} = U\}$ is measurable with respect to $(\phi(x))_{x\in U}$, resp. $(\phi_K(x))_{x\in U}$.

1.1.2 Continuum GFF

Let $d \geq 1$. Let be an open bounded connected subset $D \subset \mathbb{R}^d$. For simplicity, we assume that D has a Lipschitz boundary, although this is not strictly necessary. For $N \geq 1$, let $D_N = D \cap \frac{1}{N}\mathbb{Z}^d$. We see D_N as a subgraph of $\mathbb{Z}^d$. Let ϕ_N be the massless discrete GFF on D_N with 0 boundary conditions, where the conductances on D_N are given by $C_N(x,y) = N^{-(d-2)}$ for $\|y - x\| = \frac{1}{N}$. We are interested in the continuum limit of ϕ_N as $N \to +\infty$.

In dimension $d = 1$, this is just a Brownian bridge. However, in dimension $d \geq 2$, it cannot be defined as a random function, but only as a random generalized function, which we will denote by ϕ_D. To properly construct ϕ_D, one can use its expansion in the eigenbasis of the Dirichlet Laplacian on D. Let $(\lambda_i)_{i\geq 1}$ be the sequence of eigenvalues of $-\Delta$ with 0 boundary condition on ∂D, $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$. Let $(u_i)_{i\geq 1}$ be the corresponding orthonormal basis of $L^2(D)$. Let $(Z_i)_{i\geq 1}$ be an i.i.d. sequence of $\mathcal{N}(0,1)$ r.v.s. One can decompose

$$\phi_D = \sum_{i\geq 1} \lambda_i^{-1/2} Z_i u_i. \tag{1.2}$$

According to the Weyl law, λ_i is of order $i^{2/d}$. Thus, it is easy to see that the sum (1.2) converges a.s. in the Sobolev space $H^{-\alpha}(D)$ for the Sobolev norm

$$\|u\|^2_{H^{-\alpha}(D)} = \sum_{i\geq 1} \lambda_i^{-\alpha} \langle u, u_i\rangle^2,$$

provided that $\alpha > d/2 - 1$. For details, we refer to [She07, Section 2.3]. This leads to the following definition.

Definition 1.2. Given $f : \partial D \to \mathbb{R}$ a bounded measurable function, the continuum massless GFF on D with boundary condition f is given by

$$\phi_D = h(f) + \sum_{i\geq 1} \lambda_i^{-1/2} Z_i u_i,$$

where $h(f)$ is the harmonic extension of f and $(Z_i)_{i\geq 1}$ is an i.i.d. sequence of $\mathcal{N}(0,1)$ r.v.s. For $K > 0$, the continuum massive GFF on D with boundary condition f and square-mass K is given by

$$\phi_{D,K} = h_K(f) + \sum_{i\geq 1} (\lambda_i + K)^{-1/2} Z_i u_i,$$

where $h_K(f)$ is the solution to $(-\Delta + K)u = 0$ with boundary condition f.

The covariance kernel of ϕ_D, resp. $\phi_{D,K}$, is given by G_D, resp. $G_{D,K}$, the Green's function of $-\Delta$, resp. $-\Delta + K$, with 0 boundary conditions of ∂D. The divergence on the diagonal is as follows:

$$d = 2: \quad G_D(x,y) \sim \frac{1}{2\pi} \log(1/\|y-x\|), \qquad d \geq 3: \quad G_D(x,y) \asymp \|y-x\|^{-(d-2)}.$$

The same for $G_{D,K}$.

Just as the discrete GFF, the continuum GFF, both massless and massive, satisfies a Markov property. For more on this, we refer to Section 4.1.

One particular feature of the massless GFF in dimension 2, and which is at the heart of many results presented in this manuscript, is its conformal invariance. Given D and D' two conformally equivalent domains, ψ a conformal map from D to D' and ϕ_D a massless GFF on D, then $\phi_D \circ \psi^{-1}$ is distributed as a massless GFF on D'. This can be seen with the conformal invariance of Green's functions $G_{D'}(x,y) = G_D(\psi^{-1}(x), \psi^{-1}(y))$. Ultimately this comes from the conformal covariance of the Laplacian: $\Delta(u \circ \psi) = |\psi'|^2 (\Delta u) \circ \psi$.

1.1.3 Relation to the quantum bosonic free field

Here we present the physicists point of view on the continuum GFF, which is mathematically non-rigorous. They see the distribution of the massless GFF as

$$\frac{1}{Z} \exp\Big(-\frac{1}{2} \int_D \|\nabla \varphi\|^2 \Big) \mathcal{D}\varphi, \tag{1.3}$$

where $\mathcal{D}\varphi$ is a uniform measure on fields on D, with some fixed boundary conditions on ∂D. Such a measure actually does not exist. Also the partition function Z is not well defined, and $\int_D \|\nabla \varphi\|^2$ does not make sense for φ a GFF. This corresponds to the Feynman's functional integral. For the massive free field, one has

$$\frac{1}{Z_m} \exp\Big(-\frac{1}{2} \int_D \|\nabla \varphi\|^2 + m^2 \varphi^2 \Big) \mathcal{D}\varphi, \tag{1.4}$$

with $m^2 = K$.

Now we explain briefly the relation to the Quantum Field Theory (QFT). Consider the Euclidean space $\mathbb{R}^d$ and let $(x_1, x_2, \ldots, x_d)$ be the coordinates. Then formally perform the change of variables

$$t = -i x_d, \tag{1.5}$$

with $i = \sqrt{-1}$. This is the *Wick rotation.* The variable t represents a physical coordinate time. The Wick rotations transforms the Euclidean inner product $x_1^2 + \cdots + x_{d-1}^2 + x_d^2$ into the Lorentzian inner product $x_1^2 + \cdots + x_{d-1}^2 - t^2$, which is an invariant in Einstein's special relativity. We will denote $\mathbb{R}^d$ endowed with this inner product by $\mathbb{R}^{d-1,1}$. This is the Minkowski space-time. The functional integrals (1.3), resp.(1.4), become after Wick rotation (also with $\mathbb{R}^{d-1,1}$ instead of D)

$$\frac{1}{\widetilde{Z}} \exp\Big(\frac{i}{2} \int_{\mathbb{R}^{d-1,1}} -\|\nabla_{\mathbb{R}^{d-1}} \varphi\|^2 + \Big(\frac{\partial \varphi}{\partial t}\Big)^2 \Big) \mathcal{D}\varphi, \tag{1.6}$$

resp.

$$\frac{1}{\widetilde{Z}_m} \exp\Big(\frac{i}{2}\int_{\mathbb{R}^{d-1,1}} -\|\nabla_{\mathbb{R}^{d-1}}\varphi\|^2 + \Big(\frac{\partial\varphi}{\partial t}\Big)^2 - m^2\varphi^2\Big)\mathcal{D}\varphi, \tag{1.7}$$

where $\nabla_{\mathbb{R}^{d-1}}$ denotes the gradient with respect to the variables $(x_1, \dots, x_{d-1})$. These are the functional integral representations for a gas of non-interacting bosons (several particles can occupy the same quantum state) with spin zero. The mass of a particle is the m in (1.7). This is the *bosonic free field*, and it has been introduced in the earliest works on QFT [BJ25, BHJ26]. The wavefunction of an individual particle obeys the *free Klein-Gordon equation*

$$\frac{\partial^2}{\partial t^2}\psi = \Delta_{\mathbb{R}^{d-1}}\psi - m^2\psi. \tag{1.8}$$

For more details, we refer to [Fol08]. As for the probabilistic GFF, it is often called *Euclidean bosonic free field*, since the Wick rotation transforms the Minkowski space-time into the Euclidean space.

We would like to point out that although the physicists often manipulate the functional integrals (1.6) and (1.7) as complex-valued measures on real scalar fields, they cannot be defined mathematically as such, not even as complex-valued measures on generalized fields (Schwartz distributions). This is because the corresponding total variation measures would not be sigma-finite. This is a difference between the functional integrals (1.3) and (1.4) on one hand and (1.6) and (1.7) on the other. However, one can perform analytic continuations from the Euclidean to the Minkowski setting.

1.1.4 Interacting bosonic fields

Both from the Quantum Field Theory and from Statistical Physics point of view, it is often interesting to add interactions to the bosonic fields. We will not develop this in our manuscript, which mostly focuses on the free theory. However, this is something worth mentioning, since the random walk representations of the GFF (Section 1.2) originated from the study of interacting fields; see Section 1.2.4.

One way to achieve interactions is to consider a potential $\mathcal{V}$ and the functional integral (in the Euclidean setting)

$$\frac{1}{Z_{\mathcal{V}}} \exp\Big(-\frac{1}{2}\int_D \|\nabla\varphi\|^2 - \int_D \mathcal{V}(\varphi)\Big)\mathcal{D}\varphi. \tag{1.9}$$

The most commonly studied is the φ^4 interaction with $\mathcal{V}(\varphi) = \dfrac{\lambda}{4!}\varphi^4$, for $\lambda > 0$. The definition of the φ^4 (Phi 4) field is straightforward on a finite graph or in continuum in dimension $d = 1$. However, in dimensions $d \geq 2$ in continuum it requires a renormalization. In dimension $d = 2$, instead of the fourth power of the continuum GFF ϕ^4, one uses the Wick-renormalized fourth power $:\phi^4:$

$$:\phi^4 := \lim_{\varepsilon\to 0} \phi_\varepsilon^4 - \mathbb{E}[\phi_\varepsilon^2]\phi_\varepsilon^2 + 3\mathbb{E}[\phi_\varepsilon^2]^2,$$

where ϕ_ε is a mollification of ϕ. In this way, on a bounded domain D, the renormalized φ^4 field is absolutely continuous with respect to the GFF; see [GJ87, Section 8.6]. In dimension $d = 3$ the renormalization of the φ^4 field is more complicated and in particular it is not absolutely continuous with respect to the GFF. See [GJ87, Section 23.1] and the references therein. In dimensions $d \geq 4$, the φ^4 field is asymptotically trivial, that is to say a formal lattice approximation would in fact converge just to the free field [Frö82, ADC21]. This is also related to the fact that in dimension $d \geq 4$, two independent Brownian motions with different starting points do not intersect; see Section 1.2.4.

In dimension 2 in continuum, one can also take for $\mathcal{V}$ any polynomial P of even degree with positive leading term and apply to the corresponding $P(\varphi)$ field the Wick renormalization; see [GJ87, Section 8.6] and [Sim74]. One can also take $\mathcal{V}(\varphi) = e^{\sqrt{2\pi}\gamma\varphi}$ for a suitable range of γ ($\gamma \in (0, 2)$) [HK71]. This also corresponds to the Liouville field theory [RV14, Section 5.2]. See also Chapter 6.

Also note that the Standard Model of Particle Physics contains primarily different types of fields, that is to say fermions and gauge bosons. The latter correspond in the Euclidean setting to random holonomy fields; see [Fol08, Chapter 9] and [GJ87, Chapter 22].

1.2 Bosonic fields and gases of random walk loops and excursions

1.2.1 Interior excursions and the BFS-Dynkin isomorphism

Here we return to the discrete setting of Section 1.1.1 and consider a finite electrical network $\mathcal{G} = (V, E)$ endowed with the conductances $C(x, y) = C(y, x) > 0$ for $x \sim y$. Let $(X_t)_{t\geq 0}$ be the nearest-neighbor Markovian jump process on $\mathcal{G}$ with the jump rate from a vertex $x \in V$ to a neighbor y given by the conductance $C(x, y)$. The infinitesimal generator of $(X_t)_{t\geq 0}$ is the discrete Laplacian $\Delta_{\mathcal{G}}$ (1.1). Let T_{V_∂} be the following first hitting time:

$$T_{V_\partial} = \inf\{t \geq 0 | X_t \in V_\partial\}.$$

Then $(X_t)_{0\leq t<T_{V_\partial}}$ is the Markov jump process killed upon hitting V_∂. Let $p(t, x, y)$ be the corresponding transition densities, where

$$\forall x \in V_{\text{int}}, \forall t \geq 0, \sum_{y\in V_{\text{int}}} p(t, x, y) = \mathbb{P}_x(T_{V_\partial} > t).$$

For every $x, y \in V_{\text{int}}$ and $K > 0$, we have that

$$\int_0^{+\infty} p(t, x, y)dt = G(x, y), \qquad \int_0^{+\infty} e^{-Kt}p(t, x, y)dt = G_K(x, y). \tag{1.10}$$

Given $x, y \in V_{\text{int}}$ and $t > 0$, let $\mathbb{P}_t^{x,y}$ be the law of $(X_s)_{0\leq s\leq t}$ with $X_0 = x$, conditioned on $X_t = y$ and $T_{V_\partial} > t$.

Definition 1.3. Given $x, y \in V_{\text{int}}$, the *interior-to-interior* excursion measure from x to y is

$$\mu^{x,y} = \int_0^{+\infty} dt\, p(t, x, y)\, \mathbb{P}_t^{x,y}.$$

Given (1.10), we have that the total mass of a measure $\mu^{x,y}$ is given by the Green's function $G(x, y)$. The image of $\mu^{x,y}$ by time reversal in $\mu^{y,x}$.

Given a generic path $\wp$ parametrized by continuous time, we will denote by $T(\wp)$ its total duration. We will be interested in the *occupation times* on paths on $\mathcal{G}$. Given $(\wp(t))_{0\leq t\leq T(\wp)}$ such a path and $x \in V$, we will denote

$$\ell^x(\wp) = \int_0^{T(\wp)} \mathbf{1}_{\wp(t)=x} dt.$$

Next we present a result relating the local times of paths under mesures $\mu^{x,y}$ and the square of the Gaussian free field. It is often referred to as *Dynkin's isomorphism* [Dyn84a, Dyn84b], however it appeared first in the work of Brydges, Fröhlich and Spencer [BFS82, Theorem 2.2]. Therefore we will refer to it as the *BFS-Dynkin isomorphism.*

Theorem 1.4 (BFS-Dynkin)**.** *Let ϕ be the massless GFF on $\mathcal{G}$ with* 0 *boundary condition on V_∂ (Definition 1.1). Let $x_1, x_2, \dots, x_{2k} \in V_{\text{int}}$. Then for any F bounded measurable function on $\mathbb{R}^V$,*

$$\mathbb{E}\Big[\Big(\prod_{i=1}^{2k} \phi(x_i)\Big) F\Big(\Big(\frac{1}{2}\phi(x)^2\Big)_{x\in V}\Big)\Big] = \sum_{\substack{(\{a_j,b_j\})_{1\leq j\leq k} \\ \text{partition in pairs} \\ \text{of } \{1,2,\dots,2k\}}} \int \prod_{j=1}^{k} \mu^{x_{a_j},x_{b_j}}(d\wp_j) \mathbb{E}\Big[F\Big(\Big(\frac{1}{2}\phi(x)^2 + \sum_{j=1}^{k} \ell^x(\wp_j)\Big)_{x\in V}\Big)\Big], \quad (1.11)$$

where the sum runs over the $(2k)!/(2^k k!)$ partitions in pairs as in the Wick's rule for Gaussians.

Note that on the right-hand side of (1.11) appear the occupation times of k continuous-time random walk trajectories and they are homogeneous to the square of the GFF.

A standard way to prove the BFS-Dynkin isomorphism is to consider $F(f) = e^{-\sum_{x\in V}\lambda_x f(x)}$. Then one gets a Laplace transform on both sides of (1.11), which is explicit in both cases. For a different and recent approach we refer to [BHS21].

The BFS-Dynkin isomorphism can be extended to the case of general **non-negative** boundary conditions for the GFF. It will then involve not only interior-to-interior excursions, but also interior-to-boundary excursions, which we introduce next.

Definition 1.5. Given $x \in V_{\text{int}}$, and $y \in V_\partial$, the *interior-to-boundary* excursion measure from x to y is

$$\mu^{x,y} = \sum_{\substack{z\in V_{\text{int}} \\ z\sim y}} C(z,y)\mu^{x,z},$$

where the $\mu^{x,z}$ above are given by Definition 1.3. Thus defined, $\mu^{x,y}$ is a measure on paths from x to a neighbor z of y in V_{int}. By convention, we add to such a path a terminal jump to y, without adding holding time at y, thus turning it into an actual path from x to y which spends zero time in y.

The total mass of an interior-to-boundary excursion measure $\mu^{x,y}$ is

$$H(x,y) = \sum_{\substack{z\in V_{\text{int}} \\ z\sim y}} C(z,y)G(x,z) = \mathbb{P}_x(X_{T_{V_\partial}} = y).$$

The function $H(x,y)$ is the *discrete Poisson kernel*. Moreover, the probability measure $\mu^{x,y}/H(x,y)$ is the law of $(X_t)_{0\leq t\leq T_{V_\partial}}$ with $X_0 = x$, conditioned on $X_{T_{V_\partial}} = y$.

We do not know if the identity below appeared previously as such in the literature, but it is closely related to the *Eisenbaum's isomorphism* [Eis95].

Proposition 1.6. *Take $f : V_\partial \to \mathbb{R}_+$. Let ϕ be the massless GFF on $\mathcal{G}$ with boundary condition*

f on V_∂. Let $x_1, \dots, x_n \in V_{\text{int}}$. Then for any F bounded measurable function on $\mathbb{R}^V$,

$$\mathbb{E}\Big[\Big(\prod_{i=1}^{n} \phi(x_i)\Big) F\Big(\Big(\frac{1}{2}\phi(x)^2\Big)_{x\in V}\Big)\Big] =$$

$$\sum_{\substack{I \subset \{1,\dots,n\} \\ n-|I| \text{ even}}} \sum_{(y_i)_{i\in I} \in V_\partial^I} \prod_{i\in I} f(y_i) \sum_{\substack{(\{a_j,b_j\})_{1\leq j\leq (n-|I|)/2} \\ \text{partition in pairs} \\ \text{of } \{1,\dots,n\}\setminus I}}$$

$$\int \prod_{i\in I} \mu^{x_i,y_i}(d\tilde{\wp}_i) \prod_{j=1}^{(n-|I|)/2} \mu^{x_{a_j},x_{b_j}}(d\wp_j) \mathbb{E}\Big[F\Big(\Big(\frac{1}{2}\phi(x)^2 + \sum_{i\in I} \ell^x(\tilde{\wp}_i) + \sum_{j=1}^{(n-|I|)/2} \ell^x(\wp_j)\Big)_{x\in V}\Big)\Big].$$

In particular, for $n = 1$,

$$\mathbb{E}\Big[\phi(x_1) F\Big(\Big(\frac{1}{2}\phi(x)^2\Big)_{x\in V}\Big)\Big] = \mathbb{E}\Big[f(X_{T_{V_\partial}}) F\Big(\Big(\frac{1}{2}\phi(x)^2 + \ell^x((X_t)_{0\leq t\leq T_{V_\partial}})\Big)_{x\in V}\Big)\Big],$$

where on the right-hand side ϕ and $(X_t)_{t\geq 0}$ are independent and $X_0 = x_1$.

1.2.2 Boundary excursions and the generalized Ray-Knight theorem

Next we define the boundary-to-boundary excursion measures.

Definition 1.7. Given $x, y \in V_\partial$, the *boundary-to-boundary* excursion measure from x to y is

$$\mu^{x,y} = \sum_{\substack{z\in V_{\text{int}} \\ z\sim x}} \sum_{\substack{w\in V_{\text{int}} \\ w\sim y}} C(x,z)C(w,y)\mu^{z,w},$$

where the $\mu^{z,w}$ above are given by Definition 1.3. Thus defined, $\mu^{x,y}$ is a measure on paths from a neighbor z of x in V_{int} to a neighbor w of y in V_{int}. By convention, we add to such a path an initial jump from x to z and a terminal jump from w to y, without adding holding time at x or y, thus turning it into an actual path from x to y which spends zero time in x and y.

The total mass of a boundary-to-boundary excursion measure $\mu^{x,y}$ is

$$H(x,y) = \sum_{\substack{z\in V_{\text{int}} \\ z\sim x}} \sum_{\substack{w\in V_{\text{int}} \\ w\sim y}} C(x,z)C(w,y)G(z,w). \tag{1.12}$$

Here, $H(x,y)$ is the *boundary Poisson kernel.* The image of $\mu^{x,y}$ by time reversal is $\mu^{y,x}$.

Given a **non-negative** boundary condition $f : V_\partial \to \mathbb{R}_+$, let Ξ^f denote the Poisson point process (PPP) of boundary-to-boundary excursions with intensity measure equal to

$$\frac{1}{2} \sum_{(x,y)\in V_\partial^2} f(x)f(y)\mu^{x,y}. \tag{1.13}$$

We see Ξ^f as a rendom collection of excursions. Given $x \in V$, $\ell^x(\Xi^f)$ will denote its occupation time in x:

$$\ell^x(\Xi^f) = \sum_{\wp\in\Xi^f} \ell^x(\wp).$$

The following identity in law relates $\ell^x(\Xi^f)$ to the square of a GFF with boundary condition f. This is an extension of the *generalized Ray-Knight theorem* [EKM+00]. The details of the proof can be found in [8, Appendix]. It is also closely related to Sznitman's isomorphism for random interlacements, who rather considered an infinite volume setting [Szn12a]. Note that the original Ray-Knight theorem [Ray63, Kni63] dealt only with the local times of a one-dimensional Brownian motion, which is actually the GFF in the continuum one-dimensional setting; see Theorem 13.7.

Theorem 1.8 (Generalized Ray-Knight). *Take $f : V_\partial \to \mathbb{R}_+$. Let ϕ_f be the massless GFF on $\mathcal{G}$ with boundary condition f on V_∂. Let ϕ_0 be the massless GFF on $\mathcal{G}$ with 0 boundary condition on V_∂. Let Ξ^f be a PPP of boundary-to-boundary excursions independent from ϕ_0. Then the following identity in law holds:*

$$\Big(\frac{1}{2}\phi_0(x)^2 + \ell^x(\Xi^f)\Big)_{x\in V} \overset{(law)}{=} \Big(\frac{1}{2}\phi_f(x)^2\Big)_{x\in V}.$$

1.2.3 Loops and Le Jan's isomorphism

Next we introduce a measure on nearest-neighbor loops associated with the Markov jump process $(X_t)_{t\geq 0}$. Its Brownian analogue has been first introduced by Symanzik [Sym65, Sym66, Sym69]. Therefore we will call this measure *Symanzik's loop measure*. The discrete-space continuous-time loops were studied by Le Jan [LJ10, LJ11]. For discrete-time random walk loops, see [LTF07] and [LL10, Chapter 9]. Also, we will consider *rooted* loops. It is often useful to consider *unrooted* loops, however, we will not emphasize the distinction between the two.

Definition 1.9. The *Symanzik's loop measure* is

$$\mu^{\text{loop}}(d\wp) = \sum_{x\in V_{\text{int}}} \int_0^{+\infty} \frac{dt}{t}\, p(t,x,x)\, \mathbb{P}_t^{x,x}(d\wp) = \frac{1}{T(\wp)} \sum_{x\in V_{\text{int}}} \mu^{x,x}(d\wp).$$

The measure μ^{loop} has an infinite total mass. Actually for every $x \in V_{\text{int}}$, it puts an infinite mass on trivial "loops" that stay only in x and do not jump. The induced measure on the duration of such loops is

$$\mathbf{1}_{t>0} e^{-t/G(x,x)} \frac{dt}{t}.$$

However, μ^{loop} puts a finite weight on loops that visit at least two different vertices:

$$\mu^{\text{loop}}(\{\text{Loops with skeleton } x_1 \to x_2 \to \cdots \to x_n \to x_1\}) = \frac{1}{n} \frac{C(x_1,x_2)\dots C(x_{n-1},x_n)C(x_n,x_1)}{W(x_1)W(x_2)\dots W(x_n)},$$

where

$$W(x) = \sum_{\substack{y\in V \\ y\sim x}} C(x,y).$$

So the measure induced on the discrete skeletons is the same appearing in [LTF07] and [LL10, Chapter 9].

Definition 1.10. A continuous-time *random walk loop soup* on $\mathcal{G}$ is a Poisson point process (PPP) $\mathcal{L}^\alpha$ with intensity measure $\alpha\mu^{\text{loop}}$ for an intensity parameter $\alpha > 0$. It is a random countable collection of nearest neighbor loops in $\mathcal{G}$ that do not touch V_∂, parametrized by continuous time. The *occupation field* of $\mathcal{L}^\alpha$ is given by the occupation times

$$\ell^x(\mathcal{L}^\alpha) = \sum_{\wp\in\mathcal{L}^\alpha} \ell^x(\wp).$$

Definition 1.11. Given $\alpha \in \mathbb{C}$ and a matrix $M = (M_{ij})_{1\leq i,j\leq n}$, the *$\alpha$-permanent* of M is

$$\mathrm{Perm}_\alpha(M) = \sum_{\substack{\sigma \text{ permutation} \\ \text{of } \{1,\dots,n\}}} \alpha^{\#\text{ cycles of } \sigma} \prod_{i=1}^{n} M_{i\sigma(i)}.$$

Theorem 1.12 (Le Jan)**.** *Fix $\alpha > 0$. The occupation field $(\ell^x(\mathcal{L}^\alpha))_{x\in V}$ is the α-permanental field on V with kernel $(G(x,y))_{x,y\in V}$ characterized by its moments:*

$$\mathbb{E}\big[\ell^{x_1}(\mathcal{L}^\alpha)\ell^{x_2}(\mathcal{L}^\alpha)\dots\ell^{x_n}(\mathcal{L}^\alpha)\big] = \mathrm{Perm}_\alpha((G(x_i,x_j))_{1\leq i,j\leq n}).$$

For every $\lambda = (\lambda_x)_{x\in V_{\mathrm{int}}}$,

$$\mathbb{E}\Big[\exp\Big(\sum_{x\in V_{\mathrm{int}}} \lambda_x \ell^x(\mathcal{L}^\alpha)\Big)\Big] = \Big(\frac{\det(-\Delta_{\mathcal{G}})}{\det(-\Delta_{\mathcal{G}}-\lambda)}\Big)^\alpha, \tag{1.14}$$

where $\Delta_{\mathcal{G}}$ is considered to be the Dirichlet Laplacian with 0 boundary condition on V_∂. In particular, the Laplace transform is finite if and only if $-\Delta_{\mathcal{G}}-\lambda$ is positive definite.

Fix $x \in V_{\mathrm{int}}$. Then $\ell^x(\mathcal{L}^\alpha)$ follows a $\Gamma(\alpha, G(x,x))$ distribution with density

$$\mathbf{1}_{t>0}\frac{1}{\Gamma(\alpha)G(x,x)^\alpha} t^{\alpha-1} e^{-t/G(x,x)} dt.$$

As a process in α, $\alpha \mapsto \ell^x(\mathcal{L}^\alpha)$ is a Gamma subordinator with Lévy measure

$$\mathbf{1}_{t>0} e^{-t/G(x,x)} \frac{dt}{t}.$$

In particular, conditionally on $\ell^x(\mathcal{L}^\alpha)$, the family $(\ell^x(\wp))_{\wp\in\mathcal{L}^\alpha}$ is a Poisson-Dirichlet partition $PD(\alpha)$ of the interval $(0, \ell^x(\mathcal{L}^\alpha))$.

For the intensity parameter $\alpha = 1/2$, (1.14) is the Laplace transform of a square Gaussian. This is the *Le Jan's isomorphism* [LJ10, LJ11].

Theorem 1.13 (Le Jan)**.** *For $\alpha = 1/2$, the following identity in law holds:*

$$(\ell^x(\mathcal{L}^{1/2}))_{x\in V} \overset{(law)}{=} \Big(\frac{1}{2}\phi(x)^2\Big)_{x\in V},$$

where ϕ is the massless GFF on $\mathcal{G}$ with 0 boundary condition on V_∂.

1.2.4 All together

The results of Proposition 1.6, Theorem 1.8 and Theorem 1.13 can all be regrouped into a single statement.

Proposition 1.14. *Fix $f : V_\partial \to \mathbb{R}_+$. Let ϕ be the massless GFF on $\mathcal{G}$ with boundary condition f on V_∂. take Ξ^f and $\mathcal{L}^{1/2}$ PPPs of boundary-to-boundary excursions and loops respectively, with Ξ^f independent from $\mathcal{L}^{1/2}$. Let $x_1, \dots, x_n \in V_{\mathrm{int}}$. Then for any F bounded measurable function on $\mathbb{R}^V$, the following holds:*

$$\mathbb{E}\Big[F\Big(\Big(\frac{1}{2}\phi(x)^2\Big)_{x\in V}\Big)\Big] = \mathbb{E}[F((\ell^x(\mathcal{L}^{1/2}) + \ell^x(\Xi^f))_{x\in V})]; \tag{1.15}$$

$$\mathbb{E}\Big[\Big(\prod_{i=1}^{n}\phi(x_i)\Big)F\Big(\Big(\frac{1}{2}\phi(x)^2\Big)_{x\in V}\Big)\Big] = \sum_{\substack{I\subset\{1,\dots,n\}\\ n-|I|\ \text{even}}}\ \sum_{(y_i)_{i\in I}\in V_\partial^I}\ \prod_{i\in I}f(y_i) \sum_{\substack{(\{a_j,b_j\})_{1\leq j\leq(n-|I|)/2}\\ \text{partition in pairs}\\ \text{of } \{1,\dots,n\}\setminus I}}$$

$$\int\prod_{i\in I}\mu^{x_i,y_i}(d\tilde{\wp}_i)\prod_{j=1}^{(n-|I|)/2}\mu^{x_{a_j},x_{b_j}}(d\wp_j)\mathbb{E}\Big[F\Big(\Big(\ell^x(\mathcal{L}^{1/2})+\ell^x(\Xi^f)+\sum_{i\in I}\ell^x(\tilde{\wp}_i)+\sum_{j=1}^{(n-|I|)/2}\ell^x(\wp_j)\Big)_{x\in V}\Big)\Big]. \tag{1.16}$$

Note that one the right-hand side of (1.15) and (1.16) the GFF does not appear at all. Everything is expressed through the occupation field of a family of continuous-time random walk like trajectories: loops $\mathcal{L}^{1/2}$, always present, boundary-to-boundary excursions Ξ^f accounting for a non-zero boundary condition, excursions $\tilde{\wp}_i$ from some of the x_i to the boundary and excursions $\wp_j$ joining x_{a_j} and x_{b_j}. See Figure 1.1. See also the discussion in [Szn12b, Section 4.3]. One can also replace the massless GFF by a massive GFF with square mass K. In this case one only needs to weight each measure on trajectories $\mu^{z,w}$ and μ^{loop} by $e^{-KT(\wp)}$ with $T(\wp)$ the duration of the path.

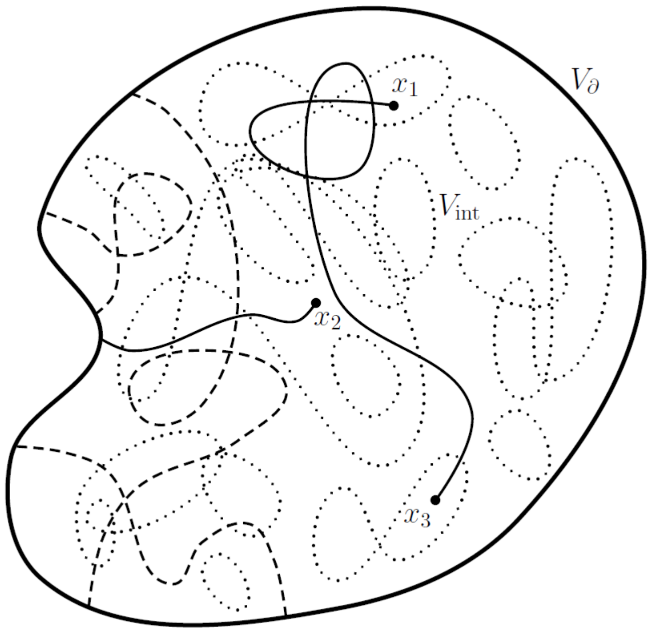

Figure 1.1: Conceptual illustration of Proposition 1.14. The loops are in dotted lines, the boundary-to-boundary excursions in dashed lines, and the excursions from the x_i in full lines. Picture inspired by Figure 0.1 in [Szn12b].

One of the original motivations for the development of the above isomorphism identities, in particular in Symanzik [Sym65, Sym66, Sym69], Brydges-Fröhlich-Spencer [BFS82], Fröhlich [Frö82], is that it provides a method to deal with interacting bosonic fields (1.9). Indeed, one can take in (1.15) and (1.16) for F:

$$F\Big(\Big(\frac{1}{2}\varphi(x)^2\Big)_{x\in V}\Big) = \exp\Big(-\sum_{x\in V}\mathcal{V}(|\varphi(x)|)\Big),$$

where $\mathcal{V}(|\varphi|)$ is some interaction potential depending only on $|\varphi|$ and not on the sign of φ. Then Proposition 1.14 provides and expression for the moments of such fields purely in terms of random walks. For instance, for the discrete φ^4 field, $\mathcal{V}(|\varphi|) = \dfrac{\lambda}{4!}\varphi^4$, this would involve exponentials of *intersection local times* $\ell^x(\wp)\ell^x(\tilde{\wp})$, for $\wp$ and $\tilde{\wp}$ two different paths, and *self-intersection local times* $\ell^x(\wp)^2$, since

$$\ell^x(\wp)^2 = 2\int_{0<t_1<t_2<T(\wp)} \mathbf{1}_{\wp(t_1)=\wp(t_2)=x} dt_1 dt_2.$$

Proposition 1.14 leaves open the question how the signs of the field are related to these random walk trajectories. I answered this question during my PhD Thesis [13, 16], and it is all about clusters of paths. See Chapter 3 for details.

1.2.5 The continuum setting

The identities of Proposition 1.14 also extend to the continuum setting in dimensions $d \leq 3$. Let us first see the continuum analogues of the measures on paths.

Assume $d \geq 2$. Let $D \subset \mathbb{R}^d$ be an open bounded subset with nice enough boundary. Let $(B_t)_{t\geq 0}$ be the Brownian motion on $\mathbb{R}^d$. Actually we will use the Brownian motion with infinitesimal generator Δ rather than the standard Brownian motion, which has for infinitesimal generator $\frac{1}{2}\Delta$. This choice will simplify the tracking of different normalization constants. Let $T_{\partial D}$ denote the first exit time

$$T_{\partial D} = \inf\{t \geq 0 | B_t \notin D\}.$$

Let $p_D(t, x, y)$ denote the transition densities of the Brownian motion $(B_t)_{0\leq t<T_{\partial D}}$ killed upon exiting D, so that for every $x \in D$ and $t > 0$,

$$\int_D p_D(t, x, y)dy = \mathbb{P}_x(T_{\partial D} > t).$$

Recall that G_D denotes the Green's function of $-\Delta$ on D with 0 boundary conditions on ∂D. For every $x, y \in D$, we have that

$$G_D(x, y) = \int_0^{+\infty} p_D(t, x, y)dt.$$

For $x \in D$ and $y \in \partial D$, with ∂D $\mathcal{C}^1$ around y, let $H_D(x, y)$ denote the *Poisson kernel*

$$H_D(x, y) = \partial_{\vec{n}} G_D(x, z)|_{z=y},$$

where $\partial_{\vec{n}}$ is the normal derivative pointing inwards. The kernel $H_D(x, y)$ is the density of the harmonic measure seen from x. For $x \neq y \in \partial D$, with ∂D $\mathcal{C}^1$ around x and around y, let $H_D(x, y)$ denote the *boundary Poisson kernel*

$$H_D(x, y) = \partial_{\vec{n}}\partial_{\vec{n}} G_D(z, w)|_{\substack{z=x\\ w=y}}.$$

Note that

$$H_D(x, x) = \lim_{y\to x} H_D(x, y) = +\infty.$$

For $x, y \in D$, let $\mathbb{P}^{x,y}_{D,t}$ denote the Brownian bridge probability measure conditioned on the bridge staying in D.

Definition 1.15. 1. Given $x, y \in D$, the Brownian measure on interior-to-interior excursions from x to y in D is

$$\mu_D^{x,y} = \int_0^{+\infty} dt \, p_D(t,x,y) \, \mathbb{P}_{D,t}^{x,y}. \tag{1.17}$$

Its total mass in $G_D(x,y)$.

2. Given $x \in D$ and $y \in \partial D$, the Brownian measure on interior-to-boundary excursions from x to y in D is

$$\mu_D^{x,y} = \lim_{\varepsilon \to 0} \varepsilon^{-1} \mu_D^{x, y+\varepsilon \overrightarrow{n}_y},$$

where $\overrightarrow{n}_y$ is the unit normal vector at y pointing inwards. The total mass of $\mu_D^{x,y}$ is given by the Poisson kernel $H_D(x,y)$. The probability measure $\mu_D^{x,y}/H_D(x,y)$ is the law of $(B_t)_{0 \leq t \leq T_{\partial D}}$ with $B_0 = x$, conditioned on $B_{T_{\partial D}} = y$.

3. Given $x, y \in \partial D$, the Brownian measure on boundary-to-boundary excursions from x to y in D is

$$\mu_D^{x,y} = \lim_{\varepsilon \to 0} \varepsilon^{-2} \mu_D^{x+\varepsilon \overrightarrow{n}_y, y+\varepsilon \overrightarrow{n}_y},$$

The total mass of $\mu_D^{x,y}$ is given by the boundary Poisson kernel $H_D(x,y)$.

4. The Symanzik's Brownian loop measure on D is

$$\mu_D^{\text{loop}} = \int_{x \in D} dx \int_0^{+\infty} \frac{dt}{t} \, p_D(t,x,x) \, \mathbb{P}_{D,t}^{x,x}.$$

It has an infinite total mass because of the ultraviolet divergence.

For details on these Brownian measures, we refer to [Law05, Chapter 5] and [LW04]. In dimension $d = 1$ in continuum, there are also the analogues of these measures; see [18].

Definition 1.16. 1. Given $f : \partial D \to \mathbb{R}_+$ a bounded measurable function, the soup of Brownian boundary-to-boundary excursions in D induced by f is the Poisson point process Ξ_D^f with intensity measure

$$\frac{1}{2} \iint_{\partial D \times \partial D} dx \, dy \, f(x) f(y) \mu_D^{x,y}, \tag{1.18}$$

where the integral is with respect to the hypersurface measure on ∂D (counting measure if $d = 1$).

2. Given $\alpha > 0$, the *Brownian loop soup* in D of intensity parameter α is the Poisson point process $\mathcal{L}_D^\alpha$ with intensity measure $\alpha \mu_D^{\text{loop}}$.

A Brownian loop soup $\mathcal{L}_D^\alpha$ contains a.s. infinitely many small loops. The same is true for an excursion soup Ξ_D^f provided f is not 0 almost everywhere on ∂D. However, in both cases the number of paths of diameter larger than a fixed scale $\varepsilon > 0$ is finite a.s.

Now we have the measures and the random collections of Brownian paths involved in the continuum version of Proposition 1.14. It remains to see what the square of the continuum GFF and the occupation fields of paths mean. In dimension $d = 1$ this is straightforward. The continuum GFF is actually a Brownian bridge if D is an interval. It is defined pointwise, and so is its square. As for the quantities $\ell^x(\wp)$ for $\wp$ Brownian like paths, these are to be understood as the Brownian local times

$$\ell^x(\wp) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^{T(\wp)} \mathbf{1}_{|\wp(t) - x| < \varepsilon} dt \tag{1.19}$$

which in dimension 1 are defined pointwise; see [RY99, Chapter VI].

In dimensions $d \geq 2$ one faces a difficulty. The continuum GFF ϕ_D is not defined pointwise, so one has to make sense of its square first. In dimensions 2 and 3 on can use the Wick's square of the GFF:

$$: \phi_D^2 := \lim_{\varepsilon \to 0} (\phi_{D,\varepsilon}^2 - \mathbb{E}[\phi_{D,\varepsilon}^2])$$

for $\phi_{D,\varepsilon}$ a mollification of ϕ_D. $: \phi_D^2 :$ is a centered random generalized function (not a positive measure) measurable w.r.t. ϕ_D. To ensure the convergence, one can use a second moment method, and in the limit

$$\begin{aligned} \mathbb{E}\big[\langle : \phi_D^2 :, u \rangle^2\big] \quad = \quad & 2 \iint_{D\times D} u(x) G_D(x,y)^2 u(y) dx dy \\ & + 4 \iint_{D\times D} u(x)(h(f))(x) G_D(x,y) u(y)(h(f))(y) dx dy < +\infty, \end{aligned} \tag{1.20}$$

where $h(f)$ is the harmonic extension of f. See [LJ11, Section 10.1] for details. Note that the right-hand side of (1.20) is infinite in dimensions $d \geq 4$. In dimension 2 there is a more general theory of the Wick renormalization of powers of the GFF; see [Sim74] and [Jan97, Chapter 3].

As for the occupation times of Brownian paths, these can be defined as random Radon measures on D:

$$\ell(\wp, A) = \int_0^{T(\wp)} \mathbf{1}_{\wp(t)\in A} dt. \tag{1.21}$$

Each Brownian path involved will have a finite *occupation measure* given by (1.21). However the PPPs Ξ_D^f and $\mathcal{L}_D^{1/2}$ contain each infinitely many paths. It is easy to see that the occupation measure of Ξ_D^f is still a.s. finite and

$$\mathbb{E}[\ell(\Xi_D^f)] = \frac{1}{2} h(f)^2.$$

To the contrary, the occupation measure of $\mathcal{L}_D^{1/2}$ diverges in every open subset of D because of the accumulation of small loops. Yet, in dimensions 2 and 3 one can define a renormalized, centered occupation field of $\mathcal{L}_D^{1/2}$, denoted $: \ell(\mathcal{L}_D^{1/2}) :$. We have that

$$: \ell(\mathcal{L}_D^{1/2}) := \lim_{\varepsilon \to 0} \Big(\sum_{\wp \in \mathcal{L}_D^{1/2}, \mathrm{diam}(\wp) > \varepsilon} \ell(\wp) - \mathbb{E}\Big[\sum_{\wp \in \mathcal{L}_D^{1/2}, \mathrm{diam}(\wp) > \varepsilon} \ell(\wp) \Big] \Big).$$

For details see [LJ11, Section 10.2].

So in dimensions 2 and 3 in continuum, a renormalized version of the identities (1.15) and (1.16) holds. One has to subtract the expectation of the fields involved on both sides of the equalities. In particular this involves the Wick's square of the GFF $: \phi_D^2 :$ and the centered occupation field of $\mathcal{L}_D^{1/2}$, $: \ell(\mathcal{L}_D^{1/2}) :$. See [LJ11, Section 10.2]. It is not known whether it is possible to get a meaningful renormalization of the identities (1.15) and (1.16) in dimensions 4 and higher.

To end this Chapter, I would like to point out that none of the proofs of isomorphism theorems is technically complicated, but all may appear surprising and relying on miraculous coincidences. So one question I personally get often asked is how people came up with these identities in the first place. So here is what I think. One should keep in mind the relation of the continuum GFF to the QFT through the Wick rotation (1.5), as mentioned in Section 1.1.3. The same Wick rotation transforms an expectation w.r.t. a Brownian motion into a Feynman path integral describing the propagation of a relativistic spin-0 particle, whose wave-function satisfies the free Klein-Gordon equation (1.8). This space-time trajectory of the particle is actually virtual, as it is not observed through a measurement. Now, the correlations of fields in perturbative QFT are expanded into infinite sums on Feynman diagrams. An edge in a Feynman diagrams corresponds to the propagation of a particle, and the (complex) weight of the edge is given by the path integral describing the propagation of this particle in space-time. So the isomorphism theorems appear to be probabilistic reinterpretations of relations in QFT between fields and particles, virtual particles in particular. Symanzik points to it, as he sees his Brownian loop expansion of the φ^4 field [Sym66] to be analogous to expansions in Quantum Electrodynamics [Fey50]. Note however that the isomorphism identities, unlike the expansions into Feynman diagrams for the interacting fields, are not perturbative and do not require a small value of the coupling constant in the interaction potential.

Chapter 2

Dimension two: clusters of Brownian paths and Schramm-Loewner Evolutions

In this Chapter is presented a second source of inspiration for my research, alongside the isomorphism identities described in Chapter 1. This second source is the theory of Schramm-Loewner Evolutions (SLE) in dimension 2, in particular in relation to the continuum GFF (Section 2.2) and to clusters in a Brownian loop soup (Section 2.3).

2.1 SLE and CLE: a brief overview

The *Schramm-Loewner evolutions* (SLE) are random growth processes for curves in a 2D domain that satisfy the conformal invariance in law and a domain Markov property. They were introduced by Schramm [Sch00] and appear, or are conjectured so, as scaling limits of interfaces in different models of Statistical Physics at criticality in dimension 2 [Sch07]. Let us consider the chordal SLE from 0 to ∞ in the upper half-plane

$$\mathbb{H} = \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}.$$

By conformal invariance this is the same as considering any other simply connected domain (no holes) with two marked points. $(\eta(t))_{t\geq 0}$ is a continuous curve in $\overline{\mathbb{H}}$ from 0 to ∞. Let H_t denote the unbounded connected component of $\mathbb{H} \setminus \eta([0,t])$. Let g_t be the unique conformal from H_t to $\mathbb{H}$ normalized at ∞ as

$$g_t(z) = z + \frac{a}{z} + O(|z|^{-2}),$$

where the quantity a above is *the half-plane capacity* of $\eta([0,t])$, $\operatorname{hcap}(\eta([0,t]))$. One also chooses the time-parametrization such that $\operatorname{hcap}(\eta([0,t])) = 2t$. Then g_t satisfies the *Loewner differential equation*

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \xi_t}, \qquad z \in H_t,$$

where $(\xi_t)_{t\geq 0}$ is the *driving function* of $(\eta(t))_{t\geq 0}$. In the case of an SLE_κ, one take $\xi_t = \sqrt{\kappa} W_t$, where $(W_t)_{t\geq 0}$ is a standard Brownian motion on $\mathbb{R}$ starting at 0. The SLE_κ is defined for all $\kappa > 0$. It is a random fractal continuous curve, with Hausdorff dimension $(1 + \kappa/8) \wedge 2$. There are three phases:

- For $\kappa \in (0, 4]$, the SLE_κ is a simple (i.e. non self-intersecting) curve, and hits the boundary $\mathbb{R} = \partial\mathbb{H}$ only at the origin, for $t = 0$.

- For $\kappa \in (4, 8)$, the curve SLE_κ has self-intersections and also hits the boundary $\mathbb{R} = \partial\mathbb{H}$ for $t > 0$. However, the range on η has an empty interior
- For $\kappa \geq 8$, the SLE_κ is a space-filling curve.

For more on SLE, we refer to [Law05].

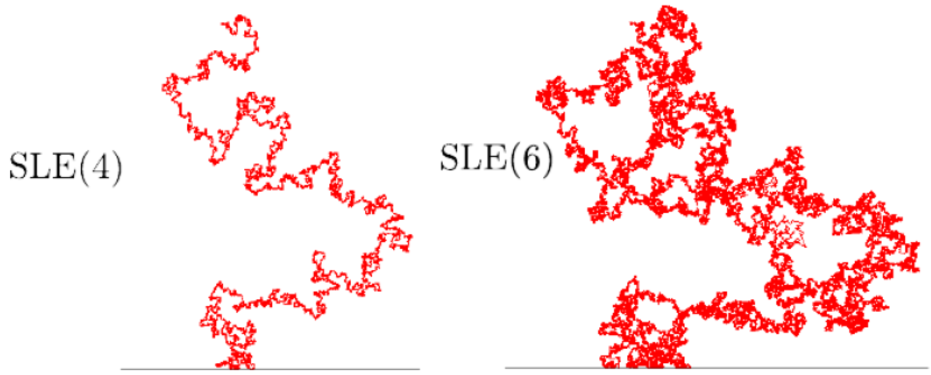

Figure 2.1: SLE_4 on the left and SLE_6 on the right. Pictures kindly provided by Hao Wu.

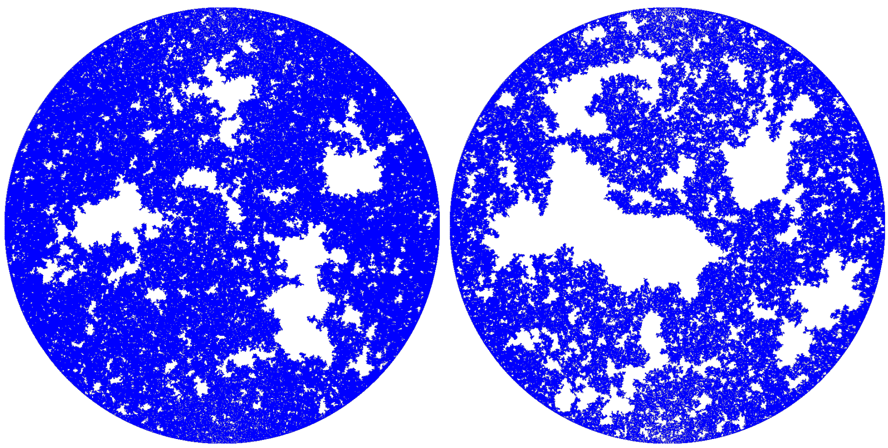

Figure 2.2: CLE_3 on the left and CLE_4 on the right. The CLE loops are the boundaries between the blue and the white. Computer simulation by David Bruce Wilson available on his webpage http://dbwilson.com/.

Next are the most remarkable values of κ for SLE.

- The SLE_2 is the scaling limit of the loop-erased random walk [Sch00].
- The SLE_8 is the scaling limit of contours of the uniform spanning tree [Sch00].
- The $SLE_{8/3}$ appears as the outer boundary of a 2D Brownian trajectory. It also satisfies a conformal restriction property [LSW03]. It is conjectured to be the scaling limit of the self-avoiding walk [Sch07].
- The SLE_6 satisfies the locality property. It is conjectured to be the scaling limit of interfaces in the critical percolation. So far this has been proved for the triangular lattice [Smi09].

- The SLE_3 is the scaling limit of interfaces in the critical Ising model [CS12].
- The $\mathrm{SLE}_{16/3}$ is the scaling limit of boundaries of clusters in the critical FK-Ising model (random cluster model with $q = 2$) [CS12].
- The SLE_4 is a *level line* of the continuum GFF; see Section 2.2. It is also conjectured to appear in the scaling limit of the double dimer model [Ken14, Dub19, BC18].

The chordal SLE curves join two points on the boundary of a simply connected domain. There is also a version of SLE-type loops in the interior of the domain, that describe the scaling limits of inner interfaces. These are the *Conformal Loop Ensembles* CLE_κ, for $\kappa \in (8/3, 4]$, introduced in [She09, SW12]. A CLE_κ is a random infinite countable collection of loops in a simply connected domain D, and satisfies the conformal invariance in law and a domain Markov property. Each loop is a closed Jordan curve and does not touch the boundary ∂D. Two different loops do not intersect and do not surround each other. Each loop looks locally as an SLE_κ curve. See Figure 2.2. For a construction of the CLE through Brownian loop soups, see the forthcoming Theorem 2.3.

2.2 SLE_4, CLE_4 and the continuum GFF

In dimension 2, the continuum massless GFF is invariant in law under conformal transformations. Despite not being defined pointwise, it has intrinsic interfaces that are SLE-type curves. These interfaces fully encode the GFF and provide a new understanding of it, different and complementary to the one through duality by evaluation against test functions. In particular, in their seminal work [SS09, SS13], Schramm and Sheffield introduced the notion of *level lines* and *height gap* of the GFF. The picture is the following. The continuum 2D GFF is spanned by cliffs. On one side of the cliff the GFF has some value, and on the other side a different one. The difference between the two is a universal constant, depending only on the normalization of the GFF. This is the *height gap*, often denoted 2λ. With the normalization in Definition 1.2,

$$2\lambda = \sqrt{\pi/2}. \tag{2.1}$$

The height gap can be interpreted as originating from an entropic repulsion. As for the cliffs themselves, these are the *level lines*, although cliff lines would perhaps have been a better name. A typical level line is an SLE_4-type curve.

Let $D \subsetneq \mathbb{C}$ be an open simply connected domain. Let $x \neq y \in \partial D$ such that x and y divide ∂D into two connected components, $\partial_{\mathrm{L}} D$ and $\partial_{\mathrm{R}} D$ (left and right). Let $a, b \in (-\lambda, \lambda)$, where λ is the half-height gap (2.1). Set

$$\rho_{\mathrm{L}} = -1 + \frac{b}{\lambda}, \qquad \rho_{\mathrm{R}} = -1 + \frac{a}{\lambda}.$$

Let η be the chordal $\mathrm{SLE}_4(\rho_{\mathrm{L}}, \rho_{\mathrm{R}})$ process from x to y with force points in x_- and x_+ (infinitesimally left and right from x). If $D = \mathbb{H}$, $x = 0$ and $y = \infty$, then the driving function of η satisfies

$$d\xi_t = 2dW_t + \frac{\rho_{\mathrm{L}} dt}{\xi_t - g_t(0_-)} - \frac{\rho_{\mathrm{R}} dt}{g_t(0_+) - \xi_t},$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion on $\mathbb{R}$ starting at 0. In particular, if $a = b = \lambda$, then $\rho_{\mathrm{L}} = \rho_{\mathrm{R}} = 0$ and η is just an SLE_4 curve. Let D_{L} be the open subset of D delimited by $\partial_{\mathrm{L}} D$ and η, and similarly D_{R} delimited by $\partial_{\mathrm{R}} D$ and η. See Figure 2.3.

Figure 2.3: On the left illustration of the Schramm-Sheffield coupling. On the right illustration of the Miller-Sheffield coupling.

Theorem 2.1 (Schramm-Sheffield). *Let ϕ_{L} and ϕ_{R} be two independent massless GFFs in D_{L}, resp. D_{R}, also independent from η, with the following boundary conditions. ϕ_{L} has a boundary condition b on $\partial_{\mathrm{L}} D$ and λ on the left side of η. ϕ_{R} has a boundary condition $-a$ on $\partial_{\mathrm{R}} D$ and $-\lambda$ on the right side of η. Let ϕ_D de the following field on D:*

$$\phi_D = \mathbf{1}_{D_{\mathrm{L}}} \phi_{\mathrm{L}} + \mathbf{1}_{D_{\mathrm{R}}} \phi_{\mathrm{R}}.$$

Then ϕ_D is distributed as the massless GFF on D with boundary conditions b on $\partial_{\mathrm{L}} D$ and $-a$ on $\partial_{\mathrm{R}} D$. Moreover, in this coupling of $(\phi_D, \eta, \phi_{\mathrm{L}}, \phi_{\mathrm{R}})$ are measurable w.r.t. ϕ_D. η is in this way a level line of ϕ_D.

Also note that Schramm and Sheffield also proved the convergence towards a level line of a discrete interface in the discrete GFF [SS09]. This approximation from discrete is actually the hard part of their work.

Subsequently, Miller and Sheffield provided a coupling between the GFF and the CLE_4. The loops in a CLE_4 are thus interior level lines (not touching the boundary ∂D) of a GFF. See Figure 2.3. For a proof of the Miller-Sheffield coupling, we refer to [WW17, ASW19].

Theorem 2.2 (Miller-Sheffield). *Let $\mathfrak{C}$ be a CLE_4 loop ensemble in a simply connected domain D. For a loop $\wp \in \mathfrak{C}$, let $\mathrm{Int}(\wp)$ denote the interior enclosed by $\wp$. Let $(\phi_\wp)_{\wp \in \mathfrak{C}}$ be a family of generalized fields, conditionally independent given $\mathfrak{C}$, with the conditional distribution of $\phi_\wp$ being that of a massless free field in $\mathrm{Int}(\wp)$ with 0 boundary conditions on $\wp$. Let $(\sigma_\wp)_{\wp \in \mathfrak{C}}$ be a family of conditionally i.i.d. uniform signs in $\{-1, 1\}$, also conditionally independent from the fields $(\phi_\wp)_{\wp \in \mathfrak{C}}$. Let ϕ_D be the field in D given by*

$$\phi_D = \sum_{\wp \in \mathfrak{C}} \mathbf{1}_{\mathrm{Int}(\wp)} (\phi_\wp + \sigma_\wp 2\lambda).$$

Then ϕ_D is distributed as the massless GFF in D with 0 boundary conditions on ∂D. Moreover, in this coupling of $(\phi_D, \mathfrak{C}, (\phi_\wp)_{\wp \in \mathfrak{C}}, (\sigma_\wp)_{\wp \in \mathfrak{C}})$, the family $(\mathfrak{C}, (\phi_\wp)_{\wp \in \mathfrak{C}}, (\sigma_\wp)_{\wp \in \mathfrak{C}})$ is measurable w.r.t. ϕ_D.

2.3 Clusters in a Brownian loop soup, CLE and SLE

The outer boundary of a 2D Brownian trajectory is an $\mathrm{SLE}_{8/3}$-type curve. This is related to the conformal restriction property of the $\mathrm{SLE}_{8/3}$ [LSW03, Wer08]. Actually, one can obtain SLE_κ-type curves for every $\kappa \in (8/3, 4]$ if one considers not one Brownian trajectory but clusters in a Brownian loop soup.

Let $D \subsetneq \mathbb{C}$ be an open simply connected domain. Fix $\alpha > 0$ and let $\mathcal{L}_D^\alpha$ be the Brownian loop soup in D with intensity parameter α; see Definition 1.16. In dimension 2, the Brownian loop soup $\mathcal{L}_D^\alpha$ is conformally invariant in law up to time reparametrization of the loops; see [Law05, Section 5.7]. This is a consequence of the conformal invariance of the 2D Brownian motion up to time change. In particular, if one considers only the ranges of Brownian loops in $\mathcal{L}_D^\alpha$, one gets a full conformal invariance.

Let us consider the clusters of loops in $\mathcal{L}_D^\alpha$. Two Brownian loops $\wp$ and $\tilde{\wp}$ in $\mathcal{L}_D^\alpha$ belong to the same *cluster* if there is a finite chain $\wp_0, \wp_1, \dots, \wp_n$ of Brownian loops in $\mathcal{L}_D^\alpha$ where $\wp_i$ intersects $\wp_{i-1}$, and $\wp_0 = \wp$ and $\wp_n = \tilde{\wp}$. Sheffield and Werner studied these clusters in [SW12] and proved the following.

Theorem 2.3 (Sheffield-Werner)**.** *If $\alpha > 1/2$, then the Brownian loops in $\mathcal{L}_D^\alpha$ form a.s. one single cluster, everywhere dense in D.*

If $\alpha \in (0, 1/2]$, then a.s. $\mathcal{L}_D^\alpha$ contains infinitely many clusters. The outer boundaries of the outermost clusters (not surrounded by others) are distributed as a conformal loop ensemble CLE_κ with the correspondence

$$2\alpha = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}. \tag{2.2}$$

See Figure 2.4.

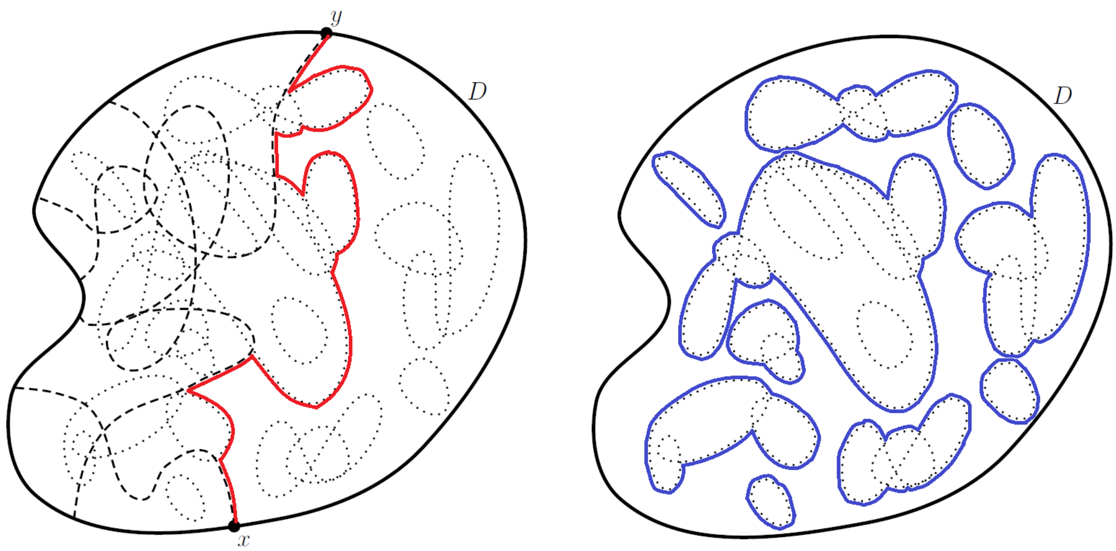

Figure 2.4: Illustration of Theorems 2.4 and 2.3. The Brownian loops are in dotted lines. The boundary excursions are in dashed lines. In red is an $\mathrm{SLE}_\kappa(\rho)$ curve. In blue is the CLE_κ.

In (2.2), the value $\alpha = 1/2$ corresponds to $\kappa = 4$, and the limit $\alpha = 0$ to $\kappa = 8/3$. In particular, one can see that the critical Brownian loop soup $\mathcal{L}_D^{1/2}$ is related to the continuum GFF in two different ways. The first one is through the renormalized Le Jan's isomorphism (Theorem 1.13). It involves the occupation field of $\mathcal{L}_D^{1/2}$. The second relation is via the CLE_4

and the Miller-Sheffield coupling (Theorem 2.2). It involves the interfaces in $\mathcal{L}_D^{1/2}$. Part of my research would be to unify these two descriptions; see Part II.

Further, in [WW13a] Werner and Wu showed that if on top of the Brownian loop soup one takes a Poisson point process (PPP) of boundary-to-boundary Brownian excursions, one can get a chordal $\mathrm{SLE}_\kappa(\rho)$ curve. In the upper half-plane $\mathbb{H}$, a chordal $\mathrm{SLE}_\kappa(\rho)$ curve with one force point at 0_- is given by a driving function satisfying

$$d\xi_t = \sqrt{\kappa} dW_t + \frac{\rho dt}{\xi_t - g_t(0_-)},$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion on $\mathbb{R}$ starting at 0. One has to take $\rho > -2$. In particular, if $\rho = 0$, then this is just an SLE_κ curve. Now if one has a simply connected domain D and $x \neq y \in \partial D$ such that x and y divide ∂D into to connected components, $\partial_{\mathrm{L}} D$ and $\partial_{\mathrm{R}} D$ (left and right), then a chordal $\mathrm{SLE}_\kappa(\rho)$ from x to y with one force point at x_- is obtained from the $\mathrm{SLE}_\kappa(\rho)$ in $\mathbb{H}$ via a conformal mapping $\psi : \mathbb{H} \to D$ with $\psi(0) = x$, $\psi(\infty) = y$ and $\psi(\mathbb{R}_-) = \overline{\partial_L D}$. Given a constant $b > 0$, one can also consider Ξ_D^b a PPP of boundary-to-boundary excursions from $\partial_{\mathrm{L}} D$ to $\partial_{\mathrm{L}} D$ with intensity measure given by

$$\frac{b^2}{2} \iint_{\partial_{\mathrm{L}} D \times \partial_{\mathrm{L}} D} dx\, dy\, \mu_D^{x,y};$$

see Definitions 1.15 and 1.16. One also takes an independent Brownian loop soup $\mathcal{L}_D^\alpha$ and considers the clusters of Brownian paths in $\mathcal{L}_D^\alpha \cup \Xi_D^b$. There are two types of clusters: those that contain both boundary excursions and interior loops, and those that contain only interior loops. Let $\mathcal{A}(\mathcal{L}_D^\alpha, \Xi_D^b)$ be the union clusters that contain both loops and excursions and are thus connected to $\partial_{\mathrm{L}} D$. Let $\partial_{\mathrm{R}} \mathcal{A}(\mathcal{L}_D^\alpha, \Xi_D^b)$ be the rightmost boundary component of $\mathcal{A}(\mathcal{L}_D^\alpha, \Xi_D^b)$ joining x and y. See Figure 2.4. If $\alpha > 1/2$, then by Theorem 2.3, $\partial_{\mathrm{R}} \mathcal{A}(\mathcal{L}_D^\alpha, \Xi_D^b) = \overline{\partial_{\mathrm{R}} D}$. If $\alpha \in (0, 1/2]$, then $\partial_{\mathrm{R}} \mathcal{A}(\mathcal{L}_D^\alpha, \Xi_D^b)$ is a non-trivial curve in $\overline{D}$ from x to y.

Theorem 2.4 (Werner-Wu)**.** *Let $\alpha \in (0, 1/2]$ and $b > 0$. Then $\partial_{\mathrm{R}} \mathcal{A}(\mathcal{L}_D^\alpha, \Xi_D^b)$ is distributed as a chordal $SLE_\kappa(\rho)$ from x to y with one force point at x_-. The κ is given by* (2.2)*. The ρ is given by*

$$b^2 = \frac{\pi}{2} \frac{(\rho + 2)(\rho + 6 - \kappa)}{\kappa}.$$

In the limit case $\alpha = 0$ (no loops, only boundary excursions Ξ_D^b), $\partial_{\mathrm{R}} \mathcal{A}(\Xi_D^b)$ is distributed as an $SLE_{8/3}(\rho)$ curve.

The case $\kappa = 4$, i.e. $\alpha = 1/2$, is again particular. The $SLE_4(\rho)$ is a level line of the continuum GFF for the boundary conditions b on $\partial_{\mathrm{L}} D$ and 0 on $\partial_{\mathrm{R}} D$ (Theorem 2.1). So the gas of Brownian paths $\mathcal{L}_D^{1/2} \cup \Xi_D^b$ is related to the GFF in two different ways. On one hand its (renormalized) occupation field gives the (renormalized) square of the GFF; see Proposition 1.14. On the other hand one of its interfaces, $\partial_{\mathrm{R}} \mathcal{A}(\mathcal{L}_D^{1/2}, \Xi_D^b)$, is a level line of the GFF with the same boundary conditions. Also note that the proofs of both Theorem 2.3 and Theorem 2.4 crucially relied on the simple connectedness of the domain D, while the isomorphism theorems are in a sense blind to the geometry of the domain.

Chapter 3

Clusters of Brownian paths and signs of the fields

In this Chapter are presented the results I obtained during my PhD thesis [13]. I related the signs of the GFF in discrete to clusters of random walk trajectories that appear in the isomorphism theorems (Proposition 1.14). My idea was to replace the discrete edges in an electrical network by continuous lines of length proportional to the resistance, so as to get a so called *metric graph*, and then to interpolate the discrete GFF to this topological object by taking 1D Brownian bridges inside the edge-lines. See Section 3.2. Using this technique I proved certain results, in particular the convergence of random walk clusters to Brownian clusters in a loop soup in dimension 2 (Section 3.3). Both this convergence result and the method of the metric graph GFF will play an important role in my subsequent research; see Parts II and III.

3.1 Continuum 1D setting: clusters of Brownian loops and excursions of the occupation field

In [18] I studied the Brownian loop soups (Definition 1.16) in dimension 1. Take as a domain the half-line $\mathbb{R}_+^* = (0, +\infty)$ and consider the 1D Brownian loop measure $\mu^{\text{loop}}_{\mathbb{R}_+^*}$ (Definition 1.15). The range of a loop $\wp$ will be just a line segment $[\min \wp, \max \wp]$. However a Brownian loop $\wp$ will carry a non-trivial local time process on its range, $(\ell^x(\wp))_{\min \wp \leq x \leq \max \wp}$; see (1.19).

Theorem 3.1 ([18], Section 3.5). *The measure on* $(\min \wp, \max \wp)$ *induced by* $\mu^{\text{loop}}_{\mathbb{R}_+^*}(d\wp)$ *is*

$$\mathbf{1}_{0<a<b} \frac{da\, db}{(b-a)^2}.$$

Conditionally on $(\min \wp, \max \wp) = (a, b)$*, the process* $(2\ell^x(\wp))_{\min \wp \leq x \leq \max \wp}$ *is a square Bessel 4 bridge from* 0 *to* 0 *of length* $b - a$*. In particular, the process* $(\ell^x(\wp))_{\min \wp \leq x \leq \max \wp}$ *is continuous for* $\mu^{\text{loop}}_{\mathbb{R}_+^*}(d\wp)$*-almost every* $\wp$*, is positive on* $(\min \wp, \max \wp)$ *and zero on* $\min \wp$ *and* $\max \wp$*.*

Now consider a 1D Brownian loop soup $\mathcal{L}^{\alpha}_{\mathbb{R}_+^*}$ with intensity measure $\alpha \mu^{\text{loop}}_{\mathbb{R}_+^*}(d\wp)$. Its occupation field

$$\ell^x(\mathcal{L}^{\alpha}_{\mathbb{R}_+^*}) = \sum_{\wp \in \mathcal{L}^{\alpha}_{\mathbb{R}_+^*}} \ell^x(\wp)$$

is a stochastic process in $x \geq 0$, distributed actually as a square Bessel process of dimension 2α [18, Proposition 4.6]. For references on Bessel processes see [RY99, Chapter XI]. The main take away in the study of the 1D Brownian loop soups is that the excursions of $(\ell^x(\mathcal{L}^\alpha_{\mathbb{R}^*_+}))_{x\geq 0}$ above level 0, i.e. the connected components of

$$\{x \geq 0 | \ell^x(\mathcal{L}^\alpha_{\mathbb{R}^*_+}) > 0\},$$

are exactly the ranges of the *clusters of loops* in $\mathcal{L}^\alpha_{\mathbb{R}^*_+}$. Recall that two Brownian loop $\wp$ and $\tilde{\wp}$ in $\mathcal{L}^\alpha_{\mathbb{R}^*_+}$ belong to the same *cluster* if there is a finite chain $\wp_0, \wp_1, \dots, \wp_n$ of Brownian loops in $\mathcal{L}^\alpha_{\mathbb{R}^*_+}$ where $\wp_i$ intersects $\wp_{i-1}$, and $\wp_0 = \wp$ and $\wp_n = \tilde{\wp}$. This is because $\ell^x(\wp)$ is positive on $(\min \wp, \max \wp)$, and so is $\ell^x(\mathcal{L}^\alpha_{\mathbb{R}^*_+})$. Moreover, a.s. for every $\wp \in \mathcal{L}^\alpha_{\mathbb{R}^*_+}$, there is $\tilde{\wp} \in \mathcal{L}^\alpha_{\mathbb{R}^*_+}$ such that $\ell^{\min \wp}(\tilde{\wp}) > 0$. And similarly for $\max \wp$. So $\ell^x(\mathcal{L}^\alpha_{\mathbb{R}^*_+})$ is actually positive on $[\min \wp, \max \wp]$ for every $\wp \in \mathcal{L}^\alpha_{\mathbb{R}^*_+}$. Finally, a.s. for every $x \geq 0$ such that x is not visited by any loop, $\ell^x(\mathcal{L}^\alpha_{\mathbb{R}^*_+}) = 0$. A bit of care is needed in the above argument because there are infinitely countably many loop in a loop soup; see [18, Corollary 5.5].

Theorem 3.2 ([18], Section 4.2)**.** *Fix $\alpha > 0$. The process $(2\ell^x(\mathcal{L}^\alpha_{\mathbb{R}^*_+}))_{x\in 0}$ is a.s. continuous and distributed as a square Bessel process of dimension 2α starting at 0 in $x = 0$. In particular, for $\alpha = 1/2$ it is the square of a Brownian motion. The excursions of $(\ell^x(\mathcal{L}^\alpha_{\mathbb{R}^*_+}))_{x\in 0}$ above level 0 give exactly the clusters of loops in $\mathcal{L}^\alpha_{\mathbb{R}^*_+}$. In particular there is a phase transitions at $\alpha = 1$: for $\alpha \geq 1$ there is a.s. one single cluster.*

3.2 Isomorphisms for the GFF on metric graphs

3.2.1 Brownian paths on metric graphs

In dimension 1 in continuum there is a nice correspondence between clusters of Brownian paths and the connected components of the positive set of their occupation field (Theorem 3.2). One would like to combine this correspondence with a non-trivial geometry for the ranges of Brownian paths, something one does not get on a line. So the idea is to consider *metric graphs*. Consider a discrete electrical network $\mathcal{G} = (V, E)$ with conductances $(C(x, y))_{x\sim y}$, as in Section 1.1.1. We will replace each discrete edge of the graph $\mathcal{G}$ by a continuous line segment, so one can move inside the edge.

Definition 3.3. The *metric graph* $\widetilde{\mathcal{G}}$ associated with the electrical network $\mathcal{G} = (V, E)$ is obtained by replacing each edge $\{x, y\} \in E$ by a continuous line segment $I_{\{x,y\}}$. Topologically, $\widetilde{\mathcal{G}}$ is a one-dimensional simplicial complex, with V being the family of 0-cells and $(I_e)_{e\in E}$ the family of 1-cells. Moreover, $\widetilde{\mathcal{G}}$ is endowed with a metric $d_{\widetilde{\mathcal{G}}}$ by setting the length of each line segment $I_{\{x,y\}}$ to be equal to $C(x, y)^{-1}$. It is also endowed with the corresponding interval-length (Lebesgue) measure $m_{\widetilde{\mathcal{G}}}$.

One can define on the metric graph $\widetilde{\mathcal{G}}$ a natural continuous Markovian diffusion process [BC84]. It is the symmetric Markov process associated with the Dirichlet form

$$\mathcal{E}_{\widetilde{\mathcal{G}}}(f, f) = \int_{\widetilde{\mathcal{G}}} (f'(x))^2 m_{\widetilde{\mathcal{G}}}(dx),$$

where f is a continuous function on $\widetilde{\mathcal{G}}$ which is $\mathcal{C}^1$ inside each edge-line I_e and has bounded derivatives. This is the *metric graph Brownian motion* and will be denoted $(\widetilde{B}_t)_{t\geq 0}$. Inside an edge line I_e, $(\widetilde{B}_t)_{t\geq 0}$ behaves as a one-dimensional Brownian motion. Upon reaching a vertex $x \in V$, the process will perform Brownian excursions in each of the adjacent edges. See [16, Section 2] for details. The process $(\widetilde{B}_t)_{t\geq 0}$ has local times $(\ell^x_t(\widetilde{B}))_{x\in\mathcal{G},t\geq 0}$ continuous in (x,t), characterized by

$$\int_0^t f(\widetilde{B}_s)ds = \int_{\mathcal{G}} f(x)\ell^x_t(\widetilde{B})m_{\widetilde{\mathcal{G}}}(dx), \tag{3.1}$$

for $t\geq 0$ and f a bounded measurable function on $\widetilde{\mathcal{G}}$. It is easy to see that the trace of $(\widetilde{B}_t)_{t\geq 0}$ on the vertices $x\in V$ is the Markovian jump process with jump rates given by the conductances.

Proposition 3.4. *Let $x_0\in V$ and $(\widetilde{B}_t)_{t\geq 0}$ the metric graph Brownian motion staring at x_0. For $u\geq 0$, let be*

$$A_u = \inf\Big\{t>0\Big|\sum_{x\in V}\ell^x_t(\widetilde{B})>u\Big\}. \tag{3.2}$$

Then the process $(\widetilde{B}_{A_u})_{u\geq 0}$ is the Markovian jump process on the discrete electrical network $\mathcal{G}$ with jump rates given by the conductances $C(x,y)$ as in Section 1.2.1.

We will also consider the process $\widetilde{B}_t$ killed upon reaching V_∂. The measures on interior-to-interior excursions (Definition 1.3), interior-to-boundary excursions (Definition 1.5), boundary-to-boundary excursions (Definition 1.7) and on interior loops (Definition 1.9), all have their metric graph equivalents. Moreover, one can recover the discrete-space measures by applying (3.2). In this way, one can consider *metric graph Brownian loop soups* $\mathcal{L}^\alpha_{\widetilde{\mathcal{G}}}$. Note that the loops in $\mathcal{L}^\alpha_{\widetilde{\mathcal{G}}}$ do not hit the boundary V_∂. There are two types of loops in $\mathcal{L}^\alpha_{\widetilde{\mathcal{G}}}$, those that do not visit any vertex in V and stay inside an edge-line I_e, and those that visit vertices in V_{int}. The trace of the latter on the vertices is actually distributed as a continuous-time random walk loop soup $\mathcal{L}^\alpha$ (Definition 1.10). One can also define the occupation field $(\ell^x(\mathcal{L}^\alpha_{\widetilde{\mathcal{G}}}))_{x\in\widetilde{\mathcal{G}}}$ by summing over the metric graph loops the local times (3.1). The analog of Theorem 3.2 holds on metric graphs.

Theorem 3.5 ([16], Section 2)**.** *Fix $\alpha>0$. The field $(\ell^x(\mathcal{L}^\alpha_{\widetilde{\mathcal{G}}}))_{x\in\widetilde{\mathcal{G}}}$ is a.s. continuous in x, and equals* 0 *on V_∂. Moreover, the clusters of loops in $\mathcal{L}^\alpha_{\widetilde{\mathcal{G}}}$ are exactly the connected components of*

$$\{x\in\widetilde{\mathcal{G}}|\ell^x(\mathcal{L}^\alpha_{\widetilde{\mathcal{G}}})>0\}.$$

Now let us see how the clusters of metric graph loops in $\mathcal{L}^\alpha_{\widetilde{\mathcal{G}}}$ and the clusters of discrete-space loops in $\mathcal{L}^\alpha$ are related. First note that for every $x\in V$, $\ell^x(\mathcal{L}^\alpha_{\widetilde{\mathcal{G}}})=\ell^x(\mathcal{L}^\alpha)$. Obviously, each cluster of $\mathcal{L}^\alpha$ is contained in a cluster of $\mathcal{L}^\alpha_{\widetilde{\mathcal{G}}}$. However, a cluster of $\mathcal{L}^\alpha_{\widetilde{\mathcal{G}}}$ can contain strictly more vertices. This is because both the excursions of loops inside the edge-lines I_e and the metric graph loops that stay inside I_e create additional connections that one does not see at the discrete level. For $e\in E$ set

$$\tilde{\omega}_e = \begin{cases} 1 & \text{if } \forall x\in I_e, \ell^x(\mathcal{L}^\alpha_{\widetilde{\mathcal{G}}})>0,\\ 0 & \text{otherwise;}\end{cases} \qquad \omega_e = \begin{cases} 1 & \text{if } e \text{ visited by a loop in } \mathcal{L}^\alpha,\\ 0 & \text{otherwise.}\end{cases} \tag{3.3}$$

An edge $e\in E$ is *open* for $\mathcal{L}^\alpha_{\widetilde{\mathcal{G}}}$, resp. $\mathcal{L}^\alpha$, if $\tilde{\omega}_e=1$, resp. $\omega_e=1$. Clearly, $\tilde{\omega}_e\geq\omega_e$.

Theorem 3.6 ([16], Theorem 1 bis)**.** *Fix $\alpha>0$. If $\alpha\geq 1$, then a.s. $\tilde{\omega}_e=1$ for every edge e not adjacent to a vertex in V_∂, and $\tilde{\omega}_e=0$ for e adjacent to V_∂. If $\alpha\in(0,1)$, then conditionally on $\mathcal{L}^\alpha$, the configuration $(\tilde{\omega}_e)_{e\in E}$ is distributed as follows. For $e=\{x,y\}\in E$, $\tilde{\omega}_e$ is set to* 1 *if $\omega_e=1$. If $\omega_e=0$, then*

$$\mathbb{P}(\tilde{\omega}_e=1|\mathcal{L}^\alpha,\omega_e=0)=\mathbb{P}(\tilde{\omega}_e=1|\ell^x(\mathcal{L}^\alpha),\ell^y(\mathcal{L}^\alpha),\omega_e=0)=p_\alpha(\ell^x(\mathcal{L}^\alpha),\ell^y(\mathcal{L}^\alpha),C(x,y))\in(0,1),$$

where $p_\alpha(\ell^x(\mathcal{L}^\alpha),\ell^y(\mathcal{L}^\alpha),C(x,y))$ can be expressed as the probability that a certain one-dimensional stochastic process does not have zeroes. Moreover, the variables $(\tilde{\omega}_e)_{e\in E,\omega_e=0}$ are conditionally independent given $\mathcal{L}^\alpha$. In the particular case $\alpha=1/2$,

$$p_{1/2}(\ell^x(\mathcal{L}^\alpha),\ell^y(\mathcal{L}^\alpha),C(x,y)) = 1-\exp\Big(-2C(x,y)\sqrt{\ell^x(\mathcal{L}^\alpha)\ell^y(\mathcal{L}^\alpha)}\Big).$$

3.2.2 Metric graph GFF and isomorphisms

The discrete GFF ϕ on $\mathcal{G}$ (Definition 1.1) can be interpolated to a continuous Gaussian field $\tilde{\phi}$ on the metric graph $\widetilde{\mathcal{G}}$ such that ϕ is the restriction of $\tilde{\phi}$ to V, $\phi = \tilde{\phi}_{|V}$.

Definition 3.7. The *massless metric graph GFF* on $\widetilde{\mathcal{G}}$ with boundary condition f on V_∂ is the random Gaussian field $(\tilde{\phi}(x))_{x\in V}$ distributed as follows. The restriction of $\tilde{\phi}$ to vertices, $\tilde{\phi}_{|V}$, is distributed as the massless discrete GFF on $\mathcal{G}$ with boundary condition f on V_∂ (Definition 1.1). Conditionally on $\tilde{\phi}_{|V}$, the fields $(\tilde{\phi}_{|I_e})_{e\in E}$ are independent. For each $\{x,y\} \in E$, $\tilde{\phi}_{|I_{\{x,y\}}}$ is distributed as a 1D Brownian bridge of length $C(x,y)^{-1}$ between the values $\tilde{\phi}(x)$ and $\tilde{\phi}(y)$.

See Figure 3.1, left picture. By construction, $\tilde{\phi}$ is a continuous field on $\widetilde{\mathcal{G}}$, and on V_∂ it equals f, the boundary condition. It also satisfies a strong Markov property, even if one cuts it inside the edge-lines; see [16, Section 3]. Compared to the discrete GFF, it also has the advantage to satisfy the intermediate value property, because of the continuity. Le Jan's isomorphism (Theorem 1.13) extends to the metric graph setting, with the correspondence of Theorem 3.5 on top of that. See Figure 3.1, right picture.

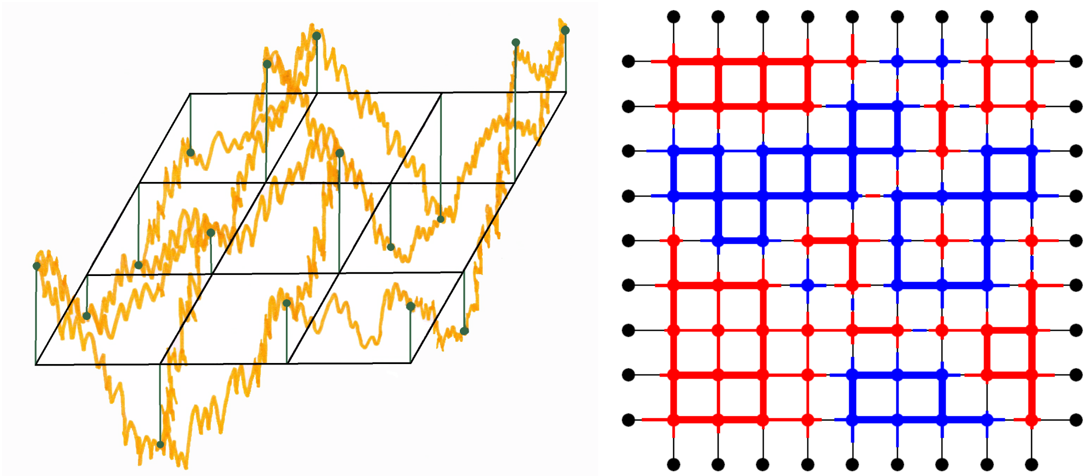

Figure 3.1: On the left, an illustration of the metric graph GFF. In green is the height of the discrete GFF on the vertices. In orange the interpolating Brownian bridges. On the right, the sign clusters of metric graph GFF, positive in red and negative in blue. The bold lines represent the edges visited by the random walk loops. The black dots represent the boundary vertices in V_∂. The boundary condition is 0.

Theorem 3.8 ([16], Theorem 1)**.** *Take $\mathcal{L}^{1/2}_{\widetilde{\mathcal{G}}}$ the metric graph loop soup of intensity parameter $\alpha = 1/2$. Let $(\sigma_{\widetilde{\mathcal{C}}})_{\widetilde{\mathcal{C}} \text{ cluster of } \mathcal{L}^{1/2}_{\widetilde{\mathcal{G}}}}$ be a conditionally i.i.d. family of signs in $\{-1,1\}$ with $\mathbb{P}(\sigma_{\widetilde{\mathcal{C}}} = 1 | \mathcal{L}^{1/2}_{\widetilde{\mathcal{G}}}) = \mathbb{P}(\sigma_{\widetilde{\mathcal{C}}} = -1 | \mathcal{L}^{1/2}_{\widetilde{\mathcal{G}}}) = 1/2$. For $x \in \widetilde{\mathcal{G}}$ such that $\ell^x(\mathcal{L}^{1/2}_{\widetilde{\mathcal{G}}}) > 0$, denote by $\widetilde{\mathcal{C}}(x)$ the cluster covering x. For $x \in \widetilde{\mathcal{G}}$, set*

$$\tilde{\phi}(x) = \sigma_{\widetilde{\mathcal{C}}(x)} \sqrt{2\ell^x(\mathcal{L}^{1/2}_{\widetilde{\mathcal{G}}})}.$$

Then the field $\tilde{\phi}$ is distributed as the metric graph GFF on $\widetilde{\mathcal{G}}$ with 0 boundary condition on V_∂.

Theorem 3.8 also has a version where the metric graph GFF $\tilde{\phi}$ has some non-negative boundary conditions. Then on top of the metric graph Brownian loop soup one has to add an independent PPP of boundary-to-boundary excursions that accounts for the non-zero boundary condition. This will be important in the study of the *first passage sets*; see Chapter 4, in particular Section 4.4.

On the picture on the right on Figure 3.1 there are 3 types of clusters. First there are the clusters of the random walk loops in $\mathcal{L}^{1/2}$ (bold lines). Then there are the clusters of metric graphs loops in $\mathcal{L}^{1/2}_{\tilde{\mathcal{G}}}$, that are in general strictly larger. By Theorem 3.8 the latter also coincide with the sign clusters of the metric graph GFF $\tilde{\phi}$. Finally there are the sign clusters of the discrete GFF ϕ. These can be strictly larger than that of $\tilde{\phi}$. Indeed, given $e \in E$ an edge, the sign at both ends of e may be the same, but the opposite sign may occur on a portion of the corresponding edge-line I_e. Then this change of sign is not seen at the level of the vertices, but seen on the metric graph level. These 3 types of clusters have a natural interpretation in terms of the combinatorics of the Ising model; see Chapter 11.

In dimension 2, it turns out that the clusters of $\mathcal{L}^{1/2}$ and that of $\mathcal{L}^{1/2}_{\tilde{\mathcal{G}}}$ are in the fine mesh limit the same. I proved this in [14]; see Section 3.3. However, the sign clusters of $\tilde{\phi}$ and that of ϕ are predicted to be different in the fine mesh limit. The boundaries of clusters of $\tilde{\phi}$ are in the limit the CLE_4 [14], which is consistent with the Miller-Sheffield coupling (Theorem 2.2). As for the clusters of ϕ, its sign clusters in the limit are predicted to correspond to the *arc loop ensemble* (ALE) [QW18]. The latter, just as the CLE_4, is a collection of SLE_4 type loops, but unlike the CLE_4, the loops in ALE are boundary touching and two different loops may touch each other. So in 2D, the sign clusters of ϕ are macroscopically larger than the sign clusters of $\tilde{\phi}$.

The coupling of Theorem 3.8 provides an upper bound for the probability of two points belonging to the same cluster of $\mathcal{L}^{1/2}$.

Corollary 3.9 ([16], Proposition 5.2)**.** *Let $x, y \in V_{\text{int}}$. Then*

$$\mathbb{P}(x, y \textit{ connected by } \mathcal{L}^{1/2}) \leq \mathbb{P}(x, y \textit{ connected by } \mathcal{L}^{1/2}_{\tilde{\mathcal{G}}}) = \mathbb{E}[\text{sign}(\tilde{\phi}(x))\,\text{sign}(\tilde{\phi}(y))]$$
$$= \mathbb{E}[\text{sign}(\phi(x))\,\text{sign}(\phi(y))] = \frac{2}{\pi}\,\text{Arcsin}\Big(\frac{G(x,y)}{\sqrt{G(x,x)G(y,y)}}\Big). \quad (3.4)$$

3.3 Convergence of clusters of loops in 2D

Consider the discrete upper half-plane $\mathtt{H} = \mathbb{Z} \times \mathbb{N}$ and the random walk loop soups $\mathcal{L}^{\alpha}_{\mathtt{H}}$ on $\mathtt{H}$. A natural question is whether $\mathcal{L}^{\alpha}_{\mathtt{H}}$ contains infinite or only bounded clusters of loops. So one sees $\mathcal{L}^{\alpha}_{\mathtt{H}}$ as a dependent percolation model on the edges of $\mathtt{H}$. Note that it is pointless to consider the random walk loop soup in the whole discrete plane $\mathbb{Z}^2$. Indeed, because of the recurrence of the 2D random walk, $\mathcal{L}^{\alpha}_{\mathbb{Z}^2}$ will a.s. contain one single cluster covering all the vertices, whatever the value of $\alpha > 0$.

I proved that for the clusters of $\mathcal{L}^{\alpha}_{\mathtt{H}}$, the phase transition occurs at the intensity parameter $\alpha = 1/2$, which corresponds to the GFF case (Theorem 1.13) and which is also critical for the 2D Brownian loop soup in continuum (Theorem 2.3). To show that for $\alpha \in (0, 1/2]$ there is no percolation by loops [16] I relied on the upper bound (3.4). To show that for $\alpha > 1/2$ there is indeed percolation [15], I used a comparison between discrete and Brownian loops, and a 1-depend block percolation argument together with the Ligett-Schonmann-Stacey's theorem [LSS97]. One can also consider $\widetilde{\mathtt{H}}$, the metric graph associated with $\mathtt{H}$, and the corresponding metric graph Brownian loop soups $\mathcal{L}^{\alpha}_{\widetilde{\mathtt{H}}}$. For these a similar result holds.

Theorem 3.10 ([16], Theorem 2 and [15], Theorem 1). *For $\alpha \in (0, 1/2]$, both $\mathcal{L}^{\alpha}_{\mathtt{H}}$ and $\mathcal{L}^{\alpha}_{\widetilde{\mathtt{H}}}$ contain only bounded clusters of loops. For $\alpha > 1/2$, both $\mathcal{L}^{\alpha}_{\mathtt{H}}$ and $\mathcal{L}^{\alpha}_{\widetilde{\mathtt{H}}}$ contain one and only one unbounded cluster of loops, the other clusters being bounded.*

Further, for $\alpha \in (0, 1/2]$, one can consider the scaling limits of clusters in $\mathcal{L}^{\alpha}_{\mathtt{H}}$ and $\mathcal{L}^{\alpha}_{\widetilde{\mathtt{H}}}$. It turns out that these are the same. More precisely, in [14] I considered the clusters of loops on the rescaled lattices, i.e. in $\mathcal{L}^{\alpha}_{\frac{1}{N}\mathtt{H}}$ and $\mathcal{L}^{\alpha}_{\frac{1}{N}\widetilde{\mathtt{H}}}$, and proved the following.

Theorem 3.11 ([14], Theorem 1.1). *Let $\alpha \in (0, 1/2]$. As $N \to +\infty$, the outer boundaries of outermost clusters of loops in both $\mathcal{L}^{\alpha}_{\frac{1}{N}\mathtt{H}}$ and $\mathcal{L}^{\alpha}_{\frac{1}{N}\widetilde{\mathtt{H}}}$ converge in law to a CLE$_\kappa$ on the upper half-plane $\mathbb{H}$, with the relation* (2.2) *between α and κ.*

Now let us see the main ideas for the proof of Theorem 3.11. Brug, Camia and Lis proved in [vdBCL16] that if one throws away all the small loops up to some mesoscopic scale (loops with less than N^{θ} jumps for a $\theta \in (16/9, 2)$), then one gets convergence of random walk clusters to Brownian clusters. They relied on a Komlós-Major-Tusnády (KMT) type coupling between random walk an Brownian loops that appeared in [LTF07]. However the KMT coupling fails for small loops, let alone the small loops on the metric graph. So one gets a lower bound, the boundaries in $\mathcal{L}^{\alpha}_{\frac{1}{N}\mathtt{H}}$ and $\mathcal{L}^{\alpha}_{\frac{1}{N}\widetilde{\mathtt{H}}}$ cannot be asymptotically smaller than a CLE$_\kappa$. But one needs an upper bound since small loops (and there are many of them) create additional connections that could *a priori* make the clusters strictly larger even at a macroscopic level. It is actually enough to establish the upper bound for $\alpha = 1/2$ and $\kappa = 4$, as this would imply one for all the smaller values of α. Also, it is enough to consider the metric graph loop soup $\mathcal{L}^{1/2}_{\frac{1}{N}\widetilde{\mathtt{H}}}$ since there the clusters are larger.

To establish the upper bound I considered the following topological events. One fixes four points $x_1 < x_2 < x_3 < x_4 \in \mathbb{R}$, and two values $u, v > 0$. One takes the Brownian loop soup $\mathcal{L}^{1/2}_{\mathbb{H}}$ in $\mathbb{H}$, and on top of it two independent PPPs of boundary-to-boundary Brownian excursions Ξ^{u}_{12} and Ξ^{v}_{34}, with respective intensity measures

$$\frac{u^2}{2} \iint_{(x_1,x_2)^2} \mu^{x,y}_{\mathbb{H}} dx\, dy, \qquad \frac{v^2}{2} \iint_{(x_3,x_4)^2} \mu^{x,y}_{\mathbb{H}} dx\, dy;$$

see (1.18). Let $E(u, v)$ be the event that Ξ^{u}_{12} and Ξ^{v}_{34} are connected, either directly because one excursion from Ξ^{u}_{12} intersects one from Ξ^{v}_{34}, or indirectly trough a chain of loops in $\mathcal{L}^{1/2}_{\mathbb{H}}$; see Figure 3.2. The probability of this event can be computed using the theory of the conformal restriction [LSW03] and is actually

$$\mathbb{P}(E(u, v)) = 1 - \Big(\frac{(x_3 - x_1)(x_4 - x_2)}{(x_4 - x_1)(x_3 - x_2)}\Big)^{-\dfrac{uv}{8\pi}}.$$

A similar event, denoted $E_N(u, v)$, can be defined on the metric graph $\frac{1}{N}\widetilde{\mathtt{H}}$, and by a computation similar in spirit to (3.4) one has

$$\mathbb{P}(E_N(u, v)) = 1 - e^{-C^{\mathrm{eff}}_N uv},$$

where C^{eff}_N is an effective conductance in $\frac{1}{N}\widetilde{\mathtt{H}}$ from (x_1, x_2) to (x_3, x_4). The convergence of C^{eff}_N ensures that

$$\lim_{N\to+\infty} \mathbb{P}(E_N(u, v)) = \mathbb{P}(E(u, v)).$$

This establishes the desired upper bound, as otherwise $\mathbb{P}(E_N(u, v))$ would have been asymptotically larger than $\mathbb{P}(E(u, v))$.

By combining Theorem 3.11 and the coupling of Theorem 3.8, one immediately gets the following.

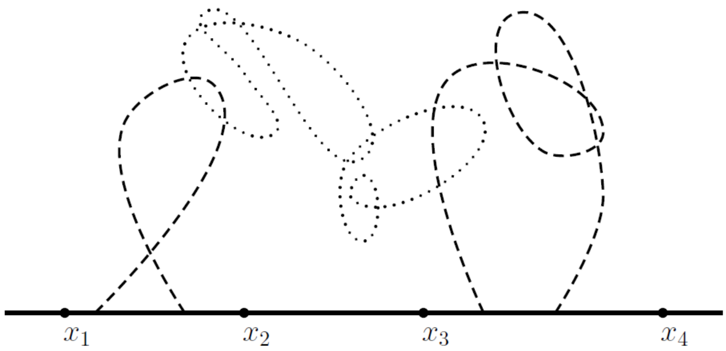

Figure 3.2: Two Brownian boundary excursions (dashed lines) connected by a chain of Brownian loops (dotted lines).

Corollary 3.12 ([14])**.** *Let $\tilde{\phi}_N$ be the massless metric graph GFF on $\frac{1}{N}\widetilde{\mathtt{H}}$ with* 0 *boundary conditions. Then the outer boundaries of outermost sign clusters of $\tilde{\phi}_N$ converge in law to the CLE_4 on the upper half-plane $\mathbb{H}$.*

The above convergence is consistent with the Miller-Sheffield coupling (Theorem 2.2).

Part II

Constructing fields out of Brownian paths in dimension two

Chapter 4

First passage sets of the continuum GFF

In this Chapter will be presented the results obtained in a collaboration of myself with Juhan Aru and Avelio Sepúlveda, that appeared in two companion papers [9, 8]. This is a project that was initiated during my post-doc at ETH Zürich. In [9] we constructed what we called the *first passage sets* (FPS) of the continuum GFF in dimension 2. These are random stopping sets of the continuum GFF. See Section 4.1 for generalities on Markovian decompositions of the latter. An FPS can be informally described as all the points in the domain that can be reached from the boundary by a continuous path along which the GFF is larger than some value. This description does not make immediate sense in continuum, but an FPS can be rigorously constructed through the level lines of the GFF; see Section 4.2. We also prove a characterization of the FPS that is useful to show convergences (Theorem 4.8). An FPS has zero Lebesgue measure, yet the restriction of the GFF to it is non-trivial. It is a positive Radon measure, actually a Minkowski content in the gauge $|\log r|^{1/2}r^2$; see Section 4.3. In [8] we described the FPS as a cluster of Brownian loops and boundary-to-boundary excursions. To get this, one first considers the analogue of the FPS on metric graphs, and uses the isomorphism theorems to show that the FPS is a cluster of loops and excursions on the metric graph; see Section 4.4. Then by convergence from the metric graphs, one establishes the analogous result in continuum; see Section 4.5. Finally, based on this work, Aru, Sepúlveda and myself announced the decomposition of the GFF in the whole domain into excursion sets, by analogy with excursions of the one-dimensional Brownian motion, with the excursion sets being given by the clusters in a Brownian loop soup. This decomposition will be briefly presented in Section 4.6.

4.1 Markov property and local sets of the GFF

In this Section we recall the Markov properties of the continuum GFF and the notion of *local sets*. We will restrict to the 2D setting, however these notions are valid in any dimension.

Let $D \subsetneq \mathbb{C}$ be an open connected subset. For simplicity we assume D bounded. Let ϕ_D be the massless GFF on D with some boundary condition f on ∂D; see Definition 1.2. Let K be a deterministic subset of $\overline{D}$ such that $K \cap D \neq \emptyset$ and $D \setminus K \neq \emptyset$. Let $(\tilde{u}_i)_{i\geq 1}$ be an orthonormal eigenbasis of $-\Delta$ on $D \setminus K$ with 0 boundary conditions. The Dirichlet inner product of ϕ_D against $\tilde{u}_i$ is well defined (see [She07, Section 2.1]):

$$\langle \nabla \phi_D, \nabla \tilde{u}_i \rangle = \int_{D\setminus K} \nabla \phi_D \cdot \nabla \tilde{u}_i.$$

Denote

$$\phi_{D\setminus K} = \sum_{i\geq 1} \langle \nabla\phi_D, \nabla\tilde{u}_i \rangle \tilde{u}_i, \qquad \phi^K = \phi_D - \phi_{D\setminus K}.$$

Next we state the *weak Markov property* of the continuum GFF. We refer to [She07, Section 2.6].

Proposition 4.1 (Weak Markov property). *The random fields $\phi_{D\setminus K}$, and thus ϕ^K, are well defined as generalized functions in $H^{-\varepsilon}(D)$ for $\varepsilon > 0$. The field $\phi_{D\setminus K}$ is Gaussian, distributed as the massless GFF on $D \setminus K$ with 0 boundary conditions. The restriction of the field ϕ^K to $D \setminus K$ is a harmonic function. The fields $\phi_{D\setminus K}$ and ϕ^K are independent.*

The GFF ϕ_D also satisfies a *strong Markov property*. We refer to [SS13, Lemma 3.9].

Proposition 4.2 (Strong Markov property). *Let K be a random compact subset of $\overline{D}$. Assume that K is a stopping set, i.e. for every open deterministic subset $U \subset D$, the event $K \subset U$ is measurable w.r.t. $\mathbf{1}_U\phi_D$. Then ϕ_D admits a decomposition*

$$\phi_D = \phi^K + \phi_{D\setminus K}, \tag{4.1}$$

with the restriction of ϕ^K to $D \setminus K$ being a.s. harmonic, and $\phi_{D\setminus K}$ being distributed, conditionally on (K, ϕ^K), as the massless GFF on $D \setminus K$ with 0 boundary conditions.

Examples of stopping sets include the level lines (Theorem 2.1) and the CLE_4 gasket, i.e. the exterior of the CLE_4 loops (Theorem 2.2). Note however that in these examples the measurability w.r.t. the free field is non-trivial, because the latter is not defined pointwise. Also for that reason one prefers to use a more general notion, that of *local sets*, introduced in [SS13]. One only needs a decomposition (4.1), and does not assume the measurability of K w.r.t. to ϕ_D. In this way, a random compact set independent from ϕ_D will be a local set but not a stopping set. The local sets are also more stable by convergence from discrete.

Definition 4.3. Let K be a random compact subset of $\overline{D}$ coupled to ϕ_D. K is a *local set* of ϕ_D if ϕ_D decomposes

$$\phi_D = \phi^K + \phi_{D\setminus K}, \tag{4.2}$$

where the restriction of ϕ^K to $D \setminus K$ is a.s. harmonic, and conditionally on (K, ϕ^K), $\phi_{D\setminus K}$ is distributed as the massless GFF on $D \setminus K$ with 0 boundary conditions.

There is an important distinction between *thin* and *non-thin* local sets which we present next. We refer to [Sep19].

Definition 4.4. A local set K of ϕ_D is *thin* if in the decomposition (4.2), ϕ^K coincides with its restriction to $D \setminus K$, i.e. it is a random harmonic function on $D \setminus K$. Otherwise, the local set K is said *non-thin.*

The level lines (Theorem 2.1) and the CLE_4 gasket (Theorem 2.2) are thin local sets. In these two examples, ϕ^K is nothing more than the harmonic extension of the boundary values on the SLE_4 type interfaces. If the local set K has non-empty interior with positive probability, then it is obviously non-thin. However, there are more interesting examples of non-thin local sets. The *first passage sets* that will be developed in this Chapter are an example of local sets that a.s. have empty interior and even zero Lebesgue measure, but are still non-thin.

Next we consider continuously growing families of local sets $K(t)$ and how the value of $\phi^{K(t)}(z)$ evolves for a fixed point $z \in D$ away from the $K(t)$. This was first observed for the level lines in [SS13] and then generalized in [MS16, Proposition 6.5]. Also recall that the Green's function on D can be decomposed as

$$G_D(z,w) = \frac{1}{2\pi}\log(1/\|w-z\|) + g_D(z,w),$$

where $g_D(z,w)$ is a continuous function on $D \times D$. If D is simply connected, then

$$g_D(z,z) = \frac{1}{2\pi}\log \mathrm{CR}(z,D),$$

where $\mathrm{CR}(z,D)$ denotes the *conformal radius* of D seen from z (Definition 7.1).

Definition 4.5. Let $(K(t))_{0\leq t\leq T}$ be a non-decreasing random family of compact subsets of $\overline{D}$, coupled to the GFF ϕ_D, and continuous for the Hausdorff metric. Let $(\mathcal{F}_t)_{0\leq t\leq T}$ be the natural filtration induced by $(K(t))_{0\leq t\leq T}$. The process $(K(t))_{0\leq t\leq T}$ is said to be a *local set process* for ϕ_D if for every τ stopping time for $(\mathcal{F}_t)_{0\leq t\leq T}$, $K(\tau)$ is a local set for ϕ_D.

Proposition 4.6. *Let $(K(t))_{0\leq t\leq T}$ be a local set process for the GFF ϕ_D. Fix $z \in D$ and assume that a.s. $z \notin K(T)$. Set $M_t = \phi^{K(t)}(z)$ (recall that $\phi^{K(t)}$ is a.s. harmonic on $D \setminus K(t)$). Then the process $(M_t)_{0\leq t\leq T}$ is a continuous martingale in the natural filtration of $(K(t), \phi^{K(t)})_{0\leq t\leq T}$. Moreover, its quadratic variation is given by*

$$\langle M, M\rangle_t = g_D(z,z) - g_{D\setminus K(t)}(z,z).$$

4.2 First passage sets: construction and characterization

Here we present the notion of *first passage sets* (FPS) of the continuum 2D GFF that was intruduced in [9] by Aru, Sepúlveda and myself. We assume that the domain D is finitely connected, i.e. $\mathbb{C}\setminus D$ has finitely many connected components. Also, each connected component of $\mathbb{C}\setminus D$ is assumed to be non-polar. For simplicity sake, we also take here D to be bounded. The field ϕ_D is a massless GFF on D with a boundary condition f on ∂D which is piecewise constant. Let $-a \in \mathbb{R}$. To avoid technical complications, here we will assume that $-a < \inf_{\partial D} f$, although this restrictive condition in not present in [9]. Informally, the first passage set of ϕ_D of level $-a$ is the subset of all points $z \in \overline{D}$ that can be reached from ∂D by a continuous path along which $\phi_D \geq -a$. Since ϕ_D is not defined pointwise, this does not make immediately sense. However, a similar definition makes perfectly sense in the metric graph setting; see Section 4.4.

Definition 4.7 ([9], Definition 4.1)**.** A local set $\mathbf{A}$ is **a** *first passage set* of ϕ_D of level $-a$ if the following conditions are satisfied.

1. $\phi^{\mathbf{A}} = \mu^{\mathbf{A}} - a\mathbf{1}_D$, where $\mu^{\mathbf{A}}$ is a non-negative Radon measure supported on $\mathbf{A}$.
2. Each connected component of $\mathbf{A}$ a.s. intersects ∂D.

In [9], the first passage sets are constructed by iterating an infinite countable sequence of level lines inside the domain D; see [9, Proposition 4.4]. Each level line is an SLE_4 type curve, as in Theorem 2.1. There is a first generation of level lines that touch ∂D. Then in the smaller domain delimited by the level lines of the first generation, one samples a second generation of level lines. Then one continues with successive generations of level lines sampled inside domains delimited by the previous generations. Finally one takes the topological closure of all the level lines sampled during this process. The values of ϕ_D along these level lines depend on the value $-a$ one aims to. We will denote by $\mathbb{A}_{-a}$ this particular first passage set. Note that by construction it is a stopping set, not just a local set, because the level lines of the GFF are. The crucial result is that $\mathbb{A}_{-a}$ is actually the only first passage set, and thus **the** first passage set of level $-a$.

Theorem 4.8 ([9], Theorem 4.3)**.** *Let* $\mathbf{A}$ *be a local set of the GFF* ϕ_D*. Assume that* $\mathbf{A}$ *is a first passage set of level* $-a$*. Then* $\mathbf{A} = \mathbb{A}_{-a}$ *a.s.*

The proof of this result is done in two steps. The first step consists in showing that $\mathbb{A}_{-a} \subset \mathbf{A}$ a.s. Informally, this is done as follows. Let η be one of the level lines involved in the construction of $\mathbb{A}_{-a}$. The values of the GFF on either side of η are v, resp. $v + 2\lambda$, with $v \geq -a$. If η would have entered a connected O component of $D \setminus \mathbf{A}$, then it would no be able to hit ∂O again because the boundary values of ϕ_D on ∂O are $-a$. Thus, v would not be able to exit O and reach its destination, and would stay trapped in O. This is a contradiction, and thus η does not enter O in the first place. So η has to stay in $\mathbf{A}$.

The second step consists in showing that $\mathbf{A} = \mathbb{A}_{-a}$ a.s. given that $\mathbb{A}_{-a} \subset \mathbf{A}$ a.s. From general considerations on local sets one gets that $\mu^{\mathbf{A}} \geq \mu^{\mathbb{A}_{-a}}$ a.s., and further that

$$\mathbb{E}[\mu^{\mathbf{A}} - \mu^{\mathbb{A}_{-a}} | \mathbb{A}_{-a}] = \mathbb{E}[\phi_{D\setminus\mathbb{A}_{-a}} | \mathbb{A}_{-a}] = 0.$$

Thus $\mu^{\mathbf{A}} = \mu^{\mathbb{A}_{-a}}$ a.s. and $\phi_{D\setminus\mathbb{A}_{-a}} = \phi_{D\setminus\mathbf{A}}$ a.s. Then,

$$\begin{aligned}\iint_{(D\setminus\mathbb{A}_{-a})^2} G_{D\setminus\mathbb{A}_{-a}}(z,w)\,dz\,dw &= \mathbb{E}[\langle \phi_{D\setminus\mathbb{A}_{-a}}, \mathbf{1}_{D\setminus\mathbb{A}_{-a}}\rangle^2 | \mathbb{A}_{-a}] \\ &= \mathbb{E}[\langle \phi_{D\setminus\mathbf{A}}, \mathbf{1}_{D\setminus\mathbb{A}_{-a}}\rangle^2 | \mathbb{A}_{-a}] = \iint_{(D\setminus\mathbf{A})^2} G_{D\setminus\mathbf{A}}(z,w)\,dz\,dw.\end{aligned}$$

This implies that $\mathbb{A}_{-a} = \mathbf{A}$ a.s.

Now consider that the domain D is simply connected, that the boundary condition of ϕ_D is 0 and that $-a = -2\lambda$ (2.1). Then the FPS $\mathbb{A}_{-2\lambda}$ can be constructed out of the nested CLE$_4$; see Figure 4.1. One samples a first generation of the CLE$_4$ inside D. Then inside each CLE$_4$ one samples and independent CLE$_4$ to get the second generation, and than continuous on with the successive generations. For each new CLE$_4$ loop one adds either -2λ or 2λ with equal probability. In this way one gets a branching random walk of step size 2λ, with infinite offspring at each generation. To get the FPS $\mathbb{A}_{-2\lambda}$ one stops along each branch upon hitting the value -2λ. In this way, the connected components of $\partial(D \setminus \mathbb{A}_{-2\lambda})$ are CLE$_4$ loops of different random generations, rather than a fixed one. There are also exceptional branches along which the random walk never hits -2λ. This branches are important too, because they give rise to the measure $\mu^{\mathbb{A}_{-2\lambda}}$.

4.3 The measure on first passage sets

Given an FPS $\mathbb{A}_{-a}$, the GFF ϕ_D decomposes

$$\phi_D = \mu^{\mathbb{A}_{-a}} - a\mathbf{1}_D + \phi_{D\setminus\mathbb{A}_{-a}}, \tag{4.3}$$

where $\mu^{\mathbb{A}_{-a}}$ is a non-negative Radon measure supported on $\mathbb{A}_{-a}$, and conditionally on $\mathbb{A}_{-a}$, $\phi_{D\setminus\mathbb{A}_{-a}}$ is distributed as a massless GFF on $D \setminus \mathbb{A}_{-a}$ with 0 boundary conditions. In this Section we will present the identification of the measure $\mu^{\mathbb{A}_{-a}}$. First note that

$$\mathbb{E}[\mu^{\mathbb{A}_{-a}}] = (h(f) + a)\mathbf{1}_D,$$

where $h(f)$ is the harmonic extension of the boundary condition f. So in particular, one has $\mathbb{P}(\mu^{\mathbb{A}_{-a}} \neq 0) > 0$. Thus, the FPS $\mathbb{A}_{-a}$ is a non-thin local set; recall Definition 4.4. One can actually show that $\mu^{\mathbb{A}_{-a}} \neq 0$ a.s. [9, Proposition 4.6].

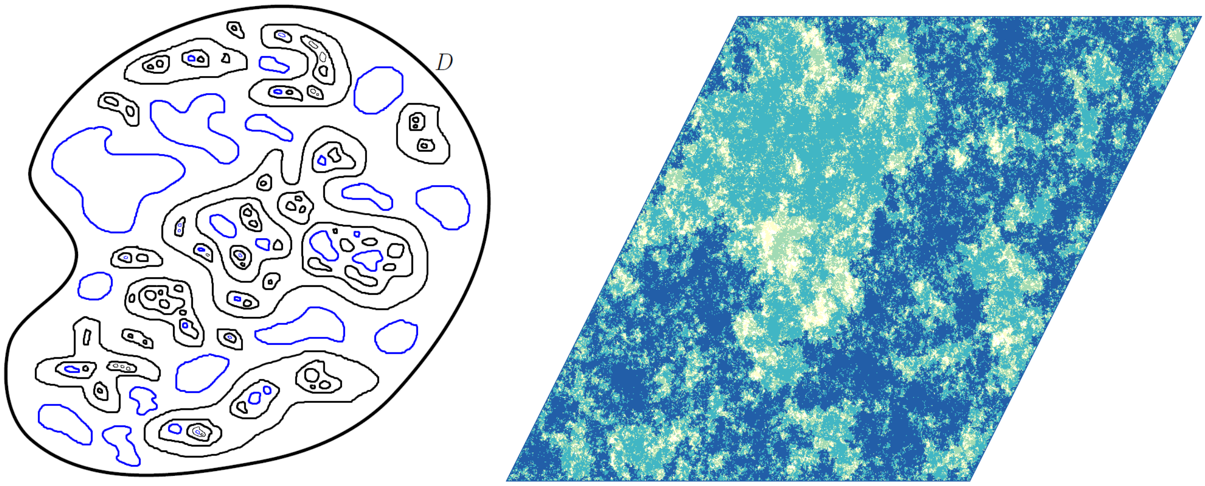

Figure 4.1: On the left: construction of the FPS of level -2λ through the nested CLE_4. The blue loops correspond to the level -2λ and one does not iterate inside. On the right: computer simulations of FPSs of levels $-\lambda$ (dark blue), -2λ, -3λ and -4λ (bright yellow). Done with the help of Brent Werness.

Let $(W_t)_{t\geq 0}$ be a one-dimensional standard Brownian motion starting from 0. For $x < 0$, set

$$\tau_x = \inf\{t \geq 0 | W_t = x\}.$$

Fix $z \in D$. Actually a.s. $z \notin \mathbb{A}_{-a}$. Proposition 4.6 implies that

$$g_D(z, z) - g_{D\setminus\mathbb{A}_{-a}}(z, z) \overset{\text{(law)}}{=} \tau_{-h(f)(z)-a}, \tag{4.4}$$

where $h(f)(z)$ is the harmonic extension of the boundary condition at z. This implies that as $t \to +\infty$,

$$\mathbb{P}(-g_{D\setminus\mathbb{A}_{-a}}(z, z) > t) \asymp t^{-1/2}. \tag{4.5}$$

Further, one can show that there is a constant $C = C(D) > 1$, depending on the domain D, such that a.s. for every $z \in D \setminus \mathbb{A}_{-a}$,

$$d(z, \mathbb{A}_{-a}) \leq e^{2\pi g_{D\setminus\mathbb{A}_{-a}}(z,z)} \leq C(D) d(z, \mathbb{A}_{-a}). \tag{4.6}$$

See [9, Lemma 5.13]. If D is simply connected, one can take $C(D) = 4$, and this follows from Koebe quarter theorem; see [Ahl10, Section 5.1]. For general D, one can apply the Beurling estimate [Law05, Theorem 3.76]. Let $(\mathbb{A}_{-a})_r$ denote the r-neighborhood of $\mathbb{A}_{-a}$. By combining (4.5) and (4.6) one gets that as $r \to 0$,

$$\mathbb{E}[\text{Leb}((\mathbb{A}_{-a})_r)] \asymp |\log r|^{-1/2},$$

where Leb denotes the Lebesgue measure on $\mathbb{C}$. So in particular, $\text{Leb}(\mathbb{A}_{-a}) = 0$ a.s. There is a more precise result.

Theorem 4.9 ([9], Theorem 5.1)**.** *A.s. the measure $\mu^{\mathbb{A}_{-a}}$ is the restriction to D of the weak limit of the measures*

$$\mathbf{1}_{z\in D, d(z,\mathbb{A}_{-a})<r} \frac{1}{2} |\log r|^{1/2} \, dz$$

as $r \to 0$. That is to say, $\mu^{\mathbb{A}_{-a}}$ is a Minkowski content measure in the gauge $|\log r|^{1/2} r^2$.

Next we explain the main ideas of the proof of Theorem 4.9. We go through the Gaussian multiplicative chaos (GMC). The GMC is the renormalized exponential of the GFF. Let $(\phi_D)_\varepsilon(z)$ denote the average value of the GFF ϕ_D on the circle of radius ε with center z. It is defined pointwise and continuous in z. For $\gamma \in (-2,2)$, the GMC measure μ_γ is

$$\mu_\gamma = \lim_{\varepsilon\to 0} \varepsilon^{\gamma^2/2} e^{\sqrt{2\pi}\gamma(\phi_D)_\varepsilon(z)} dz.$$

See Section 6.1 for details. μ_γ is a random Radon measure on D.

Let also denote $\hat{\mu}_\gamma$ the GMC associated with the field $\phi_{D\setminus \mathbb{A}_{-a}}$ (4.3). For $\gamma \in (0,2)$, one can see that

$$\mu_{-\gamma} = e^{\sqrt{2\pi}\gamma a}\hat{\mu}_{-\gamma}. \tag{4.7}$$

See [APS20]. Informally, the idea is that $e^{-\sqrt{2\pi}\gamma\mu^{\mathbb{A}_{-a}}}$ is bounded from above, and converges to 0 once multiplied by the renormalization factor $\varepsilon^{\gamma^2/2}$.

Further, one has

$$\phi_D = -\lim_{\gamma\to 0^+} \frac{1}{\sqrt{2\pi}} \frac{d}{d\gamma}\mu_{-\gamma},.$$

Together with (4.7), one gets

$$\phi_D = -\lim_{\gamma\to 0^+} \frac{1}{\sqrt{2\pi}} \frac{d}{d\gamma}\hat{\mu}_{-\gamma} - a\mathbf{1}_D.$$

By taking the conditional expectation w.r.t. $\mathbb{A}_{-a}$, and after checking that $\mathbb{E}[\cdot|\mathbb{A}_{-a}]$ and $\dfrac{d}{d\gamma}$ commute, one gets that

$$\mathbb{E}[\phi_D|\mathbb{A}_{-a}] = -\lim_{\gamma\to 0^+} \frac{1}{\sqrt{2\pi}} \frac{d}{d\gamma}\mathbb{E}[\hat{\mu}_{-\gamma}|\mathbb{A}_{-a}] - a\mathbf{1}_D.$$

But from the decomposition (4.3) one gets that

$$\mathbb{E}[\phi_D|\mathbb{A}_{-a}] = \mu^{\mathbb{A}_{-a}} - a\mathbf{1}_D.$$

Thus,

$$\mu^{\mathbb{A}_{-a}} = -\lim_{\gamma\to 0^+} \frac{1}{\sqrt{2\pi}} \frac{d}{d\gamma}\mathbb{E}[\hat{\mu}_{-\gamma}|\mathbb{A}_{-a}].$$

Further, the conditional expectation $\mathbb{E}[\hat{\mu}_{-\gamma}|\mathbb{A}_{-a}]$ is an absolutely continuous measure

$$\mathbb{E}[\hat{\mu}_{-\gamma}|\mathbb{A}_{-a}] = \mathbf{1}_{z\in D\setminus\mathbb{A}_{-a}} e^{\gamma^2\pi g_{D\setminus\mathbb{A}_{-a}}(z,z)} dz.$$

Thus,

$$\mu^{\mathbb{A}_{-a}} = -\lim_{\gamma\to 0^+} \sqrt{2\pi}\gamma\mathbf{1}_{z\in D\setminus\mathbb{A}_{-a}} g_{D\setminus\mathbb{A}_{-a}}(z,z) e^{\gamma^2\pi g_{D\setminus\mathbb{A}_{-a}}(z,z)} dz.$$

By applying (4.6), one gets that

$$\mu^{\mathbb{A}_{-a}} = \lim_{\gamma\to 0^+} \sqrt{2\pi}\gamma\mathbf{1}_{z\in D\setminus\mathbb{A}_{-a}, d(z,\mathbb{A}_{-a})<1} |\log d(z,\mathbb{A}_{-a})| d(z,\mathbb{A}_{-a})^{\gamma^2/2}. \tag{4.8}$$

So one gets a first expression for the measure $\mu^{\mathbb{A}_{-a}}$. It is a.s. a weak limit of absolutely continuous measures on $D\setminus\mathbb{A}_{-a}$. In the limit $\gamma\to 0^+$, the right-hand side of (4.8) is actually non-zero, but concentrates on $\mathbb{A}_{-a}$. One can further see that this limit is actually a Minkowski content measure in the gauge $|\log r|^{1/2}r^2$. Indeed, if one studies the function

$$r \mapsto \sqrt{2\pi}\gamma\mathbf{1}_{r\in(0,1)}|\log r| r^{\gamma^2/2},$$

one gets that it attains its maximum in $r_\gamma = e^{-2/\gamma^2}$, and the value of the maximum is

$$\sqrt{2\pi}\gamma \times \frac{2}{\gamma^2}e^{-1} \asymp |\log r_\gamma|^{1/2}.$$

4.4 First passage sets on metric graphs

Consider a discrete electrical network $\mathcal{G} = (V, E)$ with conductances $(C(x, y))_{x\sim y}$, and $\widetilde{\mathcal{G}}$ the associated metric graph; see Section 3.2. Let $\tilde{\phi}$ be a massless metric graph GFF on $\widetilde{\mathcal{G}}$ with a boundary condition f on V_∂; see Definition 3.7. Next we define the first passage sets of $\tilde{\phi}$.

Definition 4.10. Let $-a < \min_{V_\partial} f$. The *first passage set* of $\tilde{\phi}$ of level $-a$ is

$$\widetilde{\mathbb{A}}_{-a} = \{x \in \widetilde{\mathcal{G}} | \exists \wp \text{ continuous path from } V_\partial \text{ to } x \text{ s.t. } \tilde{\phi} \geq -a \text{ on } \wp\}.$$

By construction, $\widetilde{\mathbb{A}}_{-a}$ is a closed subset of $\widetilde{\mathcal{G}}$ containing V_∂, and each connected component of $\widetilde{\mathbb{A}}_{-a}$ is connected to V_∂. For every $x \in \widetilde{\mathbb{A}}_{-a}$, $\tilde{\phi}(x) \geq -a$. It is easy to see that $\widetilde{\mathbb{A}}_{-a}$ is a stopping set for the GFF $\tilde{\phi}$. Moreover, conditionally on $(\widetilde{\mathbb{A}}_{-a}, \tilde{\phi}\mathbf{1}_{\widetilde{\mathbb{A}}_{-a}})$, the field $\tilde{\phi}\mathbf{1}_{\widetilde{\mathcal{G}}\setminus\widetilde{\mathbb{A}}_{-a}}$ is distributed as a massless metric graph GFF on $\widetilde{\mathcal{G}} \setminus \widetilde{\mathbb{A}}_{-a}$ with $-a$ boundary conditions on $\partial(\widetilde{\mathcal{G}} \setminus \widetilde{\mathbb{A}}_{-a})$. The very name *first passage set* has been chosen because of the analogy with the first passage time and first passage bridge for one-dimensional Brownian motion.

The first passage sets on metric graphs were first introduced in a collaboration of Wendelin Werner and myself [11]. See Chapter 8 for details, in particular Section 8.4. There was also observed the metric graph analogue of the identity in law (4.4). Let $G_{\widetilde{\mathcal{G}}}(x, y)$, resp. $G_{\widetilde{\mathcal{G}}\setminus\widetilde{\mathbb{A}}_{-a}}(x, y)$, denote the metric graph Green's function on $\widetilde{\mathcal{G}}$, resp. $\widetilde{\mathcal{G}} \setminus \widetilde{\mathbb{A}}_{-a}$ with 0 boundary conditions. Let $\tilde{h}(f)$ denote the harmonic extension to $\widetilde{\mathcal{G}}$ of the boundary condition f on V_∂. As previously, let $(W_t)_{t\geq 0}$ be a standard one-dimensional Brownian motion with $W_0 = 0$, and $(\tau_x)_{x<0}$ its family of first passage times.

Theorem 4.11 ([11], Corollary 1)**.** *For every* $x \in \widetilde{\mathcal{G}}$,

$$\mathbb{P}(x \in \widetilde{\mathbb{A}}_{-a}) = \mathbb{P}(\tau_{-\tilde{h}(f)(x)-a} > G_{\widetilde{\mathcal{G}}}(x, x)).$$

Moreover,

$$(G_{\widetilde{\mathcal{G}}}(x, x) - G_{\widetilde{\mathcal{G}}\setminus\widetilde{\mathbb{A}}_{-a}}(x, x))\mathbf{1}_{x\notin\widetilde{\mathbb{A}}_{-a}} \overset{(law)}{=} \tau_{-\tilde{h}(f)(x)-a}\mathbf{1}_{-\tau_{\tilde{h}(f)(x)-a}<G_{\widetilde{\mathcal{G}}}(x,x)}.$$

Next we explain how $\widetilde{\mathbb{A}}_{-a}$ can be represented through clusters of metric graph loops and boundary-to-boundary excursions. This is an extension of Theorem 3.8 to the case of non-zero boundary conditions. Let $\mathcal{L}^{1/2}_{\widetilde{\mathcal{G}}}$ be a metric graph Brownian loop soup of intensity parameter $\alpha = 1/2$. Additionally, we consider $\Xi^{f+a}_{\widetilde{\mathcal{G}}}$ an independent Poisson point process of boundary-to-boundary Brownian excursions on $\widetilde{\mathcal{G}}$ with intensity measure

$$\frac{1}{2}\sum_{(x,y)\in V_\partial^2}(f(x) + a)(f(y) + a)\mu^{x,y}_{\widetilde{\mathcal{G}}};$$

see (1.13). We consider the clusters $\widetilde{\mathcal{C}}$ created by $\mathcal{L}^{1/2}_{\widetilde{\mathcal{G}}} \cup \Xi^{f+a}_{\widetilde{\mathcal{G}}}$. We see a cluster $\widetilde{\mathcal{C}}$ as a subset of $\widetilde{\mathcal{G}}$. Let be

$$\mathcal{A}(\mathcal{L}^{1/2}_{\widetilde{\mathcal{G}}}, \Xi^{f+a}_{\widetilde{\mathcal{G}}}) = \bigcup_{\substack{\widetilde{\mathcal{C}} \text{ cluster of } \mathcal{L}^{1/2}_{\widetilde{\mathcal{G}}}\cup\Xi^{f+a}_{\widetilde{\mathcal{G}}} \\ \widetilde{\mathcal{C}}\cap V_\partial\neq\emptyset}} \widetilde{\mathcal{C}}.$$

One takes only the clusters connected to the boundary V_∂, i.e. $\widetilde{\mathcal{C}} \cap V_\partial \neq \emptyset$. Note that these are exactly the clusters that contain at least an excursion from $\Xi^{f+a}_{\widetilde{\mathcal{G}}}$, not just loops in $\mathcal{L}^{1/2}_{\widetilde{\mathcal{G}}}$.

Theorem 4.12 ([8], Proposition 2.5)**.** *The first passage set* $\widetilde{\mathbb{A}}_{-a}$ *has the same distribution as* $\overline{\mathcal{A}(\mathcal{L}^{1/2}_{\widetilde{\mathcal{G}}}, \Xi^{f+a}_{\widetilde{\mathcal{G}}})}$, *where* $\overline{\mathcal{A}(\mathcal{L}^{1/2}_{\widetilde{\mathcal{G}}}, \Xi^{f+a}_{\widetilde{\mathcal{G}}})}$ *is the topological closure of* $\mathcal{A}(\mathcal{L}^{1/2}_{\widetilde{\mathcal{G}}}, \Xi^{f+a}_{\widetilde{\mathcal{G}}})$.

4.5 Convergence from metric graph to continuum and consequences

In [8], Aru, Sepúlveda and myself considered approximations of a continuum domain $D \subset \mathbb{C}$ by metric graphs $\widetilde{D}_N$ obtained from the square lattice with mesh size N^{-1}. On $\widetilde{D}_N$ one has a metric graph GFF $\tilde{\phi}_N$, converging as $N \to +\infty$ to the continuum GFF ϕ_D. Denote by $\widetilde{\mathbb{A}}_{-a,N}$ the first passage sets of $\tilde{\phi}_N$.

Theorem 4.13 ([8], Theorem 4.7)**.** *For every fixed a, one has the joint convergence in law of*

$$(\tilde{\phi}_N, \widetilde{\mathbb{A}}_{-a,N}, (\tilde{\phi}_N + a)\mathbf{1}_{\widetilde{\mathbb{A}}_{-a,N}})$$

towards

$$(\phi_D, \mathbb{A}_{-a}, \mu^{\mathbb{A}_{-a}})$$

as $N \to +\infty$, where the convergence of $\widetilde{\mathbb{A}}_{-a,N}$ is for the Hausdorff metric.

In the above convergence, the tightness of $(\widetilde{\mathbb{A}}_{-a,N})_{N\geq 1}$ is immediate, because of the compactness of the space of compact subsets for the Hausdorff metric. To identify the sub-sequential limits of $(\widetilde{\mathbb{A}}_{-a,N})_{N\geq 1}$ with $\mathbb{A}_{-a}$ we relied on the uniqueness result of Theorem 4.8.

Now, in the continuum domain D consider the 2D Brownian loop soup $\mathcal{L}_D^{1/2}$ (Definition 1.16), of intensity parameter $\alpha = 1/2$. Take also an independent PPP Ξ_D^{f+a} of Brownian excursions in D from ∂D to ∂D (Definition 1.16). Consider the clusters $\mathcal{C}$ of $\mathcal{L}_D^{1/2} \cup \Xi_D^{f+a}$, which we see as subsets of $\overline{D}$.

$$\mathcal{A}(\mathcal{L}_D^{1/2}, \Xi_D^{f+a}) = \bigcup_{\substack{\mathcal{C} \text{ cluster of } \mathcal{L}_D^{1/2} \cup \Xi_D^{f+a} \\ \mathcal{C} \cap \partial D \neq \emptyset}} \mathcal{C}.$$

The clusters $\mathcal{C}$ intersecting ∂D are exactly the clusters containing at least one excursions from Ξ_D^{f+a} and not just loops from $\mathcal{L}_D^{1/2}$. The convergence of clusters from metric graph to 2D continuum also holds. This is a strengthening of the result I proved during my PhD (Theorem 3.11), since one considers the whole cluster and not just the outer boundaries.

Theorem 4.14 ([8], Proposition 4.11)**.** *As $N \to +\infty$, we have the convergence in law of $\overline{\mathcal{A}(\mathcal{L}_{\widetilde{D}_N}^{1/2}, \Xi_{\widetilde{D}_N}^{f+a})}$ towards $\overline{\mathcal{A}(\mathcal{L}_D^{1/2}, \Xi_D^{f+a})}$ for the Hausdorff metric.*

By combining the Theorems 4.12, 4.13 and 4.14 one immediately gets the following representation of an FPS $\mathbb{A}_{-a}$ as a cluster of Brownian loops and excursions connected to the boundary.

Corollary 4.15 ([8], Proposition 5.3)**.** *Let $-a < \inf_{\partial D} f$. Then the FPS $\mathbb{A}_{-a}$ has the same distribution as $\overline{\mathcal{A}(\mathcal{L}_D^{1/2}, \Xi_D^{f+a})}$.*

This also implies that the cluster of Brownian paths $\overline{\mathcal{A}(\mathcal{L}_D^{1/2}, \Xi_D^{f+a})}$ has a non-trivial Minkowski content in the gauge $|\log r|^{1/2} r^2$ (Theorem 4.9). However, each individual Brownian loop or excursion has a 0 Minkowski content in this gauge. For a 2D Brownian path one has to take the Minkowski gauge $|\log r| r^2$. So actually the Minkoswki content on $\overline{\mathcal{A}(\mathcal{L}_D^{1/2}, \Xi_D^{f+a})}$ originates from by the accumulation of small Brownian loops of all possible scales. The large loops and excursions contribute only to the macroscopic shape of $\overline{\mathcal{A}(\mathcal{L}_D^{1/2}, \Xi_D^{f+a})}$, but not to the Minkowski content.

For now we have assumed throughout that $-a < \inf_{\partial D} f$, where f is the boundary condition of the GFF ϕ_D. This was to avoid discussing some technical complications. However, this limitation is not present in [9, 8]. In particularly, Corollary 4.15 holds whenever $-a \leq \inf_{\partial D} f$. By taking f to coincide with $-a$ on some subarcs of ∂D, one gets that some outer boundaries of $\mathcal{A}(\mathcal{L}_D^{1/2}, \Xi_D^{f+a})$ are level lines of the GFF ϕ_D. See [8, Proposition 5.11] and the left picture in Figure 4.2. In a simply connected domain, one can also get in this way the multiple commuting SLE_4 that have been introduced in[Dub07]. The corresponding probabilities for each non-crossing partition were computed in [PW19].

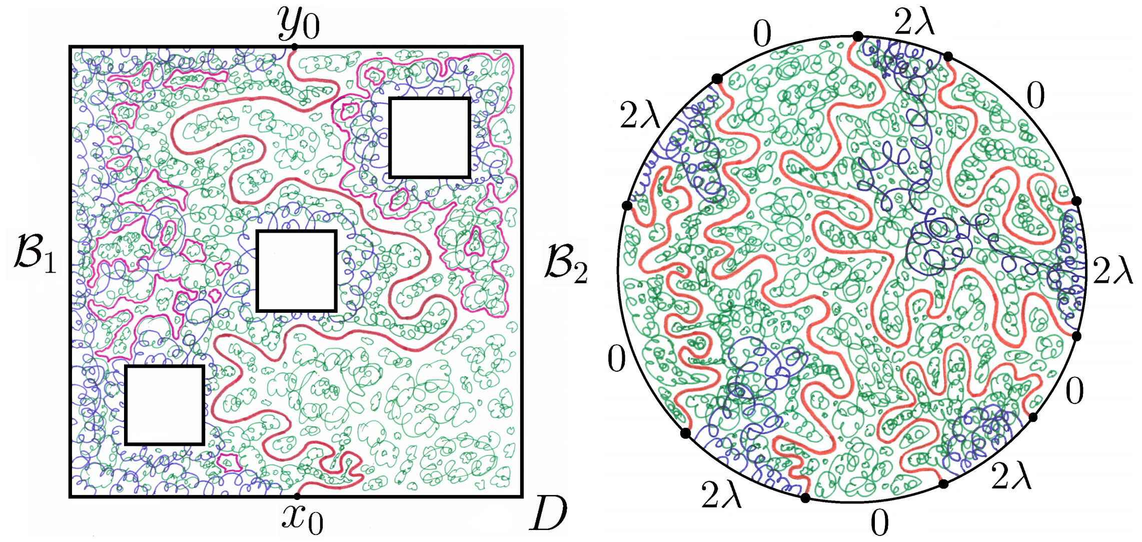

Figure 4.2: Both on left and right, the Brownian loops are represented in green, and the boundary excursions in blue. In red are represented the boundary curves of clusters that join two points on ∂D. On the left is represented a level line of the GFF from x_0 to y_0 in a multiply connected domain. On the right are represented multiple commuting SLE_4 in a disk.

4.6 Decomposition of the continuum GFF into excursion sets

Here we will consider all the clusters in a Brownian loop soup $\mathcal{L}_D^{1/2}$. In [QW19], Qian and Werner proved that in a simply connected domain D, given a cluster $\mathcal{C}$ of $\mathcal{L}_D^{1/2}$, the Brownian loops that touch the outer boundary $\partial_o\mathcal{C}$ of $\mathcal{C}$ can be decomposed into a PPP of Brownian excursions from and to $\partial_o\mathcal{C}$, conditionally on $\partial_o\mathcal{C}$ distributed as $\Xi^{2\lambda}_{\mathrm{Int}(\partial_o\mathcal{C})}$, with $\mathrm{Int}(\partial_o\mathcal{C})$ being the interior enclosed by $\partial_o\mathcal{C}$. The proof of Qian and Werner relies on the Miller-Sheffield coupling (Theorem 2.2), on the isomorphism theorems, in particular the continuum analogue of Theorem 1.8, and on the approximation by metric graphs. In particular they use my convergence result of Theorem 3.11.

So given $\partial_o\mathcal{C}$ the outer boundary of a cluster $\mathcal{C}$ of $\mathcal{L}_D^{1/2}$, there are Brownian loops that touch $\partial_o\mathcal{C}$ and these decompose into a PPP $\Xi^{2\lambda}_{\mathrm{Int}(\partial_o\mathcal{C})}$, and the Brownian loops that are surrounded by $\partial_o\mathcal{C}$ but do not touch it, and these form a conditionally independent Brownian loop soup $\mathcal{L}^{1/2}_{\mathrm{Int}(\partial_o\mathcal{C})}$ in the domain enclosed by $\partial_o\mathcal{C}$. So by further combining the result of Qian and Werner [QW19] with our Corollary 4.15, one can further identify the whole cluster $\mathcal{C}$ with a first passage set inside $\mathrm{Int}(\partial_o\mathcal{C})$. We state this somewhat informally below.

Corollary 4.16 ([8], Corollary 5.4). *Assume that the domain D is simply connected. Conditionally on the outer boundary $\partial_o\mathcal{C}$ of a cluster $\mathcal{C}$ of $\mathcal{L}_D^{1/2}$, the cluster $\mathcal{C}$ is distributed as a first passage set of level* 0 *of a GFF $\phi_{\mathrm{Int}(\partial_o\mathcal{C})}$ inside* $\mathrm{Int}(\partial_o\mathcal{C})$ *with boundary condition* 2λ *on the inner side of $\partial_o\mathcal{C}$. In particular, all the clusters of $\mathcal{L}_D^{1/2}$ can be deterministically obtained out of a labeled nested CLE$_4$ inside D; see Figure 4.1 left.*

Actually, if the domain D is multiply connected, an analogue of Corollary 4.16 still holds. One can use an absolute continuity argument for that. One can consider the domain D^* obtained by filling the inner holes of D. Then the picture on D is absolutely continuous w.r.t that on D^* away from the inner holes of D, both for the GFF and the Brownian loo soup.

What precedes leads to a decomposition of the 2D continuum GFF into excursion sets. It has been announced by Aru, Sepúlveda and myself. At this stage there are some technical details to work out. We present it here nevertheless. Let D be a finitely connected open domain in $\mathbb{C}$ and $\mathcal{L}_D^{1/2}$ a Brownian loop soup in D. For each cluster $\mathcal{C}$ of $\mathcal{L}_D^{1/2}$ sample an independent sign $\sigma_{\mathcal{C}} \in \{-1, 1\}$, with $\mathbb{P}(\sigma_{\mathcal{C}} = -1|\mathcal{L}_D^{1/2}) = \mathbb{P}(\sigma_{\mathcal{C}} = 1|\mathcal{L}_D^{1/2}) = 1/2$. Also, let $\mu^{\mathcal{C}}$ be the measure obtained as the weak limit

$$\mu^{\mathcal{C}} = \lim_{r\to 0} \frac{1}{2}|\log r|^{1/2}\, \mathbf{1}_{d(z,\mathcal{C})<r}\, dz,$$

which a.s. exists as is non-trivial. Set

$$\phi_D = \sum_{\mathcal{C} \text{ cluster of } \mathcal{L}_D^{1/2}} \sigma_{\mathcal{C}}\mu^{\mathcal{C}}. \tag{4.9}$$

Then ϕ_D is distributed as the massless free field on D with 0 boundary conditions. The sum (4.9) is not absolutely convergent. One needs the compensations induced by the signs $\sigma_{\mathcal{C}}$, and in particular the GFF is not a signed measure. The choice of the signs $\sigma_{\mathcal{C}}$ has to be independent of the order of summation. The convergence holds in L^2. Further, the Wick's square of this GFF ϕ_D is given by the renormalized occupation field of $\mathcal{L}_D^{1/2}$ (see Section 1.2.5):

$$\frac{1}{2} : \phi_D^2 :=: \ell(\mathcal{L}_D^{1/2}) : \text{ a.s.}$$

The clusters $\mathcal{C}$ of $\mathcal{L}_D^{1/2}$ form the so-called *excursion sets* of the GFF ϕ_D, and, together with the signs $\sigma_{\mathcal{C}}$, are actually measurable w.r.t. ϕ_D. The decomposition (4.9) is the continuum 2D analogue of the decomposition of Theorem 3.8 on metric graphs.

Now let us present an other heuristic explanation for the Minkowski gauge $|\log r|^{1/2}r^2$. Consider $\widetilde{D}_N$ a metric graph approximation of D. Let C_N be the typical number of vertices in a macroscopic cluster of $\mathcal{L}_{\widetilde{D}_N}^{1/2}$. Let be two points $z, w \in \widetilde{D}_N$, macroscopically far away. Then up to a constant order,

$$\mathbb{P}(x, y \text{ connected by } \mathcal{L}_{\widetilde{D}_N}^{1/2}) \asymp \Big(\frac{C_N}{N^2}\Big)^2.$$

But the identity (3.4) ensures that

$$\mathbb{P}(x, y \text{ connected by } \mathcal{L}_{\widetilde{D}_N}^{1/2}) \asymp \frac{1}{\log N}.$$

Thus,

$$C_N \asymp (\log N)^{-1/2} N^2.$$

So to get convergence, one has to renormalize the counting measure on vertices on a macroscopic cluster of $\mathcal{L}_{\widetilde{D}_N}^{1/2}$ by the factor $(\log N)^{1/2}$.

Chapter 5

Level lines of the continuum GFF with measure-valued boundary conditions

In this Chapter is presented a collaboration with Hao Wu (YMSC, Tsinghua University) [2]. Relying on the results of [8] presented in Section 4.5, we show that the notion of level lines of the GFF can be extended to measure-valued boundary conditions. Previously, more regularity on the boundary conditions was assumed [PW17]. In Section 5.1 we recall the topological background we rely on. In Section 5.2 we introduce chordal SLE_κ-type curves constructed out of CLE_κ and Poisson point processes of boundary-to-boundary excursions, with intensities parametrized by non-negative measures on the boundary. In Section 5.3 we explain how the continuity of in law of these curves with respect to the boundary measures is obtained. This uses the notions of Section 5.1. In Section 5.4 we describe how for the value $\kappa = 4$ this gives the level lines of the GFF with measure-valued boundary conditions.

5.1 Local connection and cut points

In this Section we present the required topological background. For details we refer to [Pom92, Chapter 2]. We start with the notion of *local connection*.

Definition 5.1. Given C a closed non-empty subset of $\mathbb{C}$, and $z, z' \in C$, we say that z and z' are *ε-connected in C* if there is K a compact connected subset of C with $\mathrm{diam}(K) < \varepsilon$ such that $z, z' \in K$.

A closed non-empty subset $C \subset \mathbb{C}$ is *locally connected* if for every $\varepsilon > 0$, there is $\delta > 0$ such that for every $z, z' \in C$ with $|z' - z| < \delta$, the points z and z' are ε-connected in C.

A family of closed non-empty subsets $(C_n)_{n\geq 0}$ of $\mathbb{C}$ is *uniformly locally connected* if for every $\varepsilon > 0$, there is $\delta > 0$ such that for every $n \geq 0$ and every $z, z' \in C_n$ with $|z' - z| < \delta$, the points z and z' are ε-connected in C_n.

Definition 5.2. Given C a closed connected non-empty subset of $\mathbb{C}$, a point $z \in C$ is said to be a *cut point* of C if $C \setminus \{z\}$ is not connected.

Next is the Carathéodory theorem. See [Pom92, Theorem 2.1] and [Pom92, Theorem 2.6].

Theorem 5.3 (Carathéodry)**.** *Let D be an open bounded simply connected domain in $\mathbb{C}$. Let ψ be a conformal map from the unit disk $\mathbb{D}$ to D.*

1. *If $\mathbb{C} \setminus D$ is locally connected, then ψ extends continuously to $\overline{\mathbb{D}}$. In particular ∂D can be parametrized as a continuous closed curve.*

2. *If on top of that, ∂D has no cut points, then ∂D is a Jordan curve, i.e. continuous closed simple curve, and ψ extends to a homeomorphism from $\overline{\mathbb{D}}$ to $\overline{D}$.*

Next we recall the notion of *Carathéodry convergence.*

Definition 5.4. Let D and $(D_n)_{n\geq 0}$ be open non-empty simply connected domains in $\mathbb{C}$, different from $\mathbb{C}$. Let $w \in D$, respectively $w_n \in D_n$. The sequence of marked domains $((D_n, w_n))_{n\geq 0}$ is said to converge to (D, w) in the *Carathéodory sense* if the following holds:

1. $w_n \to w$.

2. For every $z \in D$, there is a neighborhood U of z in D such that

$$U \subset \bigcap_{n\geq m} D_n$$

for m large enough.

3. For every $z \in \partial D$, there exist $z_n \in D_n$ such that $z_n \to z$ as $n \to +\infty$.

Note that the Carathéodory convergence does not imply that D_n converges to D for the Hausdorff distance, even if D is bounded.

For the following theorem we refer to [Pom92, Theorem 1.8], [Pom92, Proposition 2.3] and [Pom92, Corollary 2.4].

Theorem 5.5. *Let D and $(D_n)_{n\geq 0}$ be open simply connected domains in $\mathbb{C}$, different from $\mathbb{C}$ and containing 0. Let ψ, resp. ψ_n, be the conformal map from the unit disk $\mathbb{D}$ to D, resp. D_n, such that $\psi(0) = 0$ and $\psi'(0) > 0$, resp. $\psi_n(0) = 0$ and $\psi'_n(0) > 0$. Assume the following.*

1. *There are $R > r > 0$ such that for every $n \geq 0$, $r\mathbb{D} \subset D_n \subset R\mathbb{D}$.*

2. *The sequence of marked domains $((D_n, 0))_{n\geq 0}$ converges in the Carathéodory sense to $(D, 0)$.*

3. *The family $(\mathbb{C} \setminus D_n)_{n\geq 0}$ is uniformly locally connected.*

Then $\mathbb{C} \setminus D$ is locally connected and ψ_n converges to ψ uniformly on $\overline{\mathbb{D}}$.

5.2 SLE$_\kappa$ type curve as envelop of CLE$_\kappa$ and Brownian excursions

Let $\mathbb{D}$ be the unit disk, $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$. Let A_{L} and A_{R} denote the left and right half-circles on $\partial\mathbb{D}$:

$$A_{\mathrm{L}} = \{z \in \partial\mathbb{D} : \mathrm{Re}(z) < 0\}, \qquad A_{\mathrm{R}} = \{z \in \partial\mathbb{D} : \mathrm{Re}(z) > 0\}.$$

Note that because of the conformal invariance, the particular choice of $\mathbb{D}$ for our simply connected domain, and the particular choice of marked points $-i$ and i do not matter. Let ν be a finite non-negative Radon measure on $\overline{A_{\mathrm{L}}}$. Let $\Xi^{\nu}_{\mathbb{D}}$ be the PPP of Brownian excursion from and to $\overline{A_{\mathrm{L}}}$ of intensity

$$\frac{1}{2} \iint_{\overline{A_{\mathrm{L}}} \times \overline{A_{\mathrm{L}}}} \nu(dx)\nu(dy)\mu^{x,y}_{\mathbb{D}}. \tag{5.1}$$

So compared to (1.18), one replaces a non-negative function f on $\partial\mathbb{D}$ by a measure ν. If ν has no atoms, then (5.1) involves only excursion measures $\mu_{\mathbb{D}}^{x,y}$ for $x \neq y$. If ν has atoms, then there are also measures $\mu_{\mathbb{D}}^{x,x}$ for x an atom. Let $\kappa \in (8/3, 4]$ and let $\mathfrak{C}_\kappa$ be an independent CLE$_\kappa$ in $\mathbb{D}$. Let be $\widehat{\mathfrak{C}}_\kappa$ the subset of $\mathfrak{C}_\kappa$ made of the CLE$_\kappa$ loops that intersect excursions in $\Xi_{\mathbb{D}}^{\nu}$. Denote

$$\mathcal{S}_{\kappa,\nu} = \bigcup_{\wp \in \Xi_{\mathbb{D}}^{\nu}} \mathrm{Range}(\wp) \cup \bigcup_{\hat{\wp} \in \widehat{\mathfrak{C}}_\kappa} \mathrm{Range}(\hat{\wp}).$$

Let $\mathcal{O}_{\kappa,\nu}$ be the connected component of $\mathbb{D} \setminus \mathcal{S}_{\kappa,\nu}$ that is adjacent to A_{R}. By construction, $\mathcal{O}_{\kappa,\nu}$ is an open simply connected subset of $\mathbb{D}$. Set

$$\eta_{\kappa,\nu} = \partial\mathcal{O}_{\kappa,\nu} \setminus A_{\mathrm{R}}.$$

Actually, $\eta_{\kappa,\nu}$ is the rightmost envelop of $\mathcal{S}_{\kappa,\nu}$; see Figure 5.1. Because of the correspondence between Brownian loop soups and the CLE$_\kappa$ (Theorem 2.3), the construction could have been done with Brownian loop soups $\mathcal{L}_{\mathbb{D}}^{\alpha}$ instead of the CLE$_\kappa$. If the measure ν is of form $\nu = \mathbf{1}_{A_{\mathrm{L}}} b$ for a constant $b > 0$, then by Theorem 2.4, $\eta_{\kappa,\nu}$ is distributed as a chordal SLE$_\kappa(\rho)$ curve. In general, let $\psi_{\kappa,\nu}$ denote the conformal mapping from $\mathbb{D}$ to $\mathcal{O}_{\kappa,\nu}$ defined by the following conditions:

$$\psi_{\kappa,\nu}(1) = 1, \psi_{\kappa,\nu}(-i) = -i, \psi_{\kappa,\nu}(i) = i, \psi_{\kappa,\nu}(A_{\mathrm{R}}) = A_{\mathrm{R}}.$$

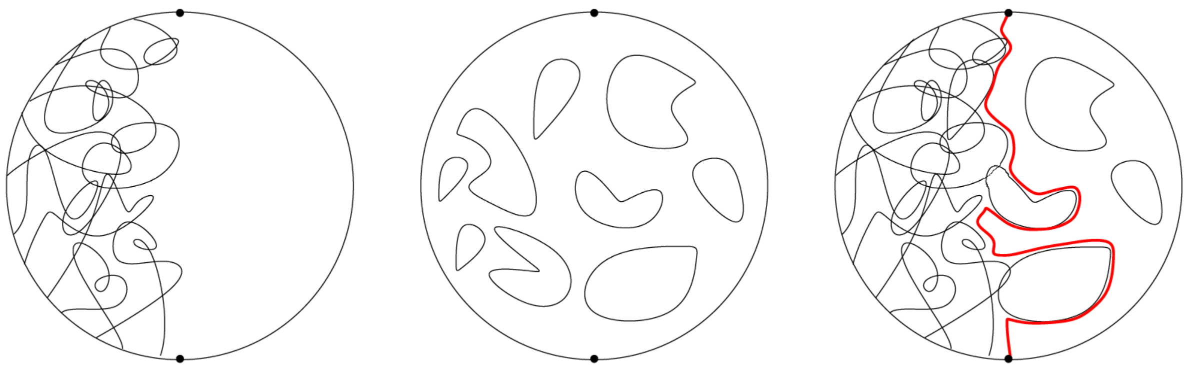

Figure 5.1: On the left, the PPP of boundary excursions $\Xi_{\mathbb{D}}^{\nu}$. In the middle, a CLE$_\kappa$. On the right, the curve $\eta_{\kappa,\nu}$ in red.

Proposition 5.6 ([2], Proposition 1.1)**.** *A.s., the conformal mapping $\psi_{\kappa,\nu}$ extends continuously to $\overline{\mathbb{D}}$. In particular, $\eta_{\kappa,\nu}$ can be parametrized as a continuous curve from $-i$ to i, as image of $\overline{A_{\mathrm{L}}}$ by $\psi_{\kappa,\nu}$.*

We rely on the Carathéodory theorem (Theorem 5.3). One needs to check that $\mathbb{C} \setminus \mathcal{O}_{\kappa,\nu}$ is locally connected. For this it is enough to show that $\partial\mathbb{D} \cup \mathcal{S}_{\kappa,\nu}$ is locally connected. Each excursion in $\Xi_{\mathbb{D}}^{\nu}$ and each CLE$_\kappa$ loop is locally connected, as image of a compact locally connected set (interval $[0, 1]$) by a continuous map; see [New64, Theorem 8.2, Chapter IV] and [Pom92, Section 2.2]. Then one has to deal with the fact that $\partial\mathbb{D} \cup \mathcal{S}_{\kappa,\nu}$ is an infinite countable union of locally connected subsets; see [2, Lemma 2.3] for this.

Further, the curve $\eta_{\kappa,\nu}$ satisfies the following properties.

- For every open subarc $A \subset A_{\mathrm{L}}$, $\mathbb{P}(A \subset \eta_{\kappa,\nu}) > 0$ if and only if $\nu(A) = 0$. Indeed, if $\nu(A) > 0$, then a.s. $\Xi_{\mathbb{D}}^{\nu}$ contains infinitely many excursions with endpoints in A. See Lemma 2.10 and Proposition 3.2 in [2].

- Locally away from the boundary $\overline{A_{\mathrm{L}}}$ the curve $\eta_{\kappa,\nu}$ is absolutely continuous with respect to a chordal SLE$_\kappa$ curve. This follows from Theorem 2.4, by comparing with the case when $\nu = \mathbf{1}_{A_{\mathrm{L}}} b$ for a constant $b > 0$. For the precise formulation and details we refer to [2, Theorem 3.7].

- If the measure ν has no atoms, the the curve $\eta_{\kappa,\nu}$ is a.s. simple, i.e. does not have multiple points [2, Lemma 3.3]. For this we use that for each excursion $\wp \in \Xi^{\nu}_{\mathbb{D}}$ its rightmost boundary is a simple curve without cut points, actually a chordal SLE$_{8/3}(2/3)$ curve joining the two endpoints of $\wp$; see [LSW03, Corollary 8.5]. If ν has atoms, then in some cases an atom of ν can be a double point of $\eta_{\kappa,\nu}$ with positive probability. See [2, Proposition 3.4] for a discussion on this.

- If the compact support of ν is the whole half-circle $\overline{A_{\mathrm{L}}}$, then the curve $\eta_{\kappa,\nu}$ can be parametrized as a Loewner chain with continuous driving function [2, Proposition 1.3].

5.3 Continuity with respect to the boundary conditions

Fix $\kappa \in (8/3, 4]$ and consider a sequence $(\nu_n)_{n\geq 0}$ of finite non-negative Radon measure on $\overline{A_{\mathrm{L}}}$, converging weakly to ν. We will see that the curves η_{κ,ν_n} converge in law to $\eta_{\kappa,\nu}$ in a precise sense.

For $\varepsilon > 0$, denote

$$\mathbb{D}_\varepsilon = \{z \in \mathbb{D} | d(z, A_{\mathrm{L}}) < \varepsilon\}.$$

Given $\wp$ an excursion, recall that $T(\wp)$ denotes its total duration. For an excursion $\wp$ visiting $\mathbb{D} \setminus \mathbb{D}_\varepsilon$, denote

$$T^{\mathrm{f}}_\varepsilon(\wp) = \inf\{t \in [0, T(\wp) | \wp(t) \in \mathbb{D} \setminus \mathbb{D}_\varepsilon\}, \qquad T^{\mathrm{l}}_\varepsilon(\wp) = \sup\{t \in [0, T(\wp) | \wp(t) \in \mathbb{D} \setminus \mathbb{D}_\varepsilon\}.$$

Denote Ξ^{ν}_{ε}:

$$\Xi^{\nu}_{\varepsilon} = \{(\wp(t + T^{\mathrm{f}}_\varepsilon(\wp))_{0\leq t \leq T^{\mathrm{l}}_\varepsilon(\wp) - T^{\mathrm{f}}_\varepsilon(\wp)}) | \wp \in \Xi^{\nu}_{\mathbb{D}}, \wp \text{ visits } \mathbb{D} \setminus \mathbb{D}_\varepsilon\}.$$

So one keeps only the middle part $\wp([T^{\mathrm{f}}_\varepsilon(\wp), T^{\mathrm{l}}_\varepsilon(\wp)])$.

Proposition 5.7 ([2], Proposition 4.1). *One can couple on the same probability space the PPPs $\Xi^{\nu}_{\mathbb{D}}$ and all the $\Xi^{\nu_n}_{\mathbb{D}}$ such that for every $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$, such that for every $n \geq n_\varepsilon$, $\Xi^{\nu_n}_{\varepsilon} = \Xi^{\nu}_{\varepsilon}$.*

The above Proposition uses the fact that for every $\varepsilon > 0$, Ξ^{ν}_{ε} and all the $\Xi^{\nu_n}_{\varepsilon}$ are a.s. finite PPPs, and the intensity measure of $\Xi^{\nu_n}_{\varepsilon}$ converges as $n \to +\infty$ to that of Ξ^{ν}_{ε} in total variation. Using this coupling we deduce the following.

Theorem 5.8 ([2], Theorem 1.2). *Assume the PPPs $\Xi^{\nu}_{\mathbb{D}}$ and all the $\Xi^{\nu_n}_{\mathbb{D}}$ are coupled as in Proposition 5.7. Also take a CLE$_\kappa$ independent from $(\Xi^{\nu}_{\mathbb{D}}, (\Xi^{\nu_n}_{\mathbb{D}})_{n\geq 0})$. Then in this coupling, a.s. the sequence of conformal mappings $(\psi_{\kappa,\nu_n})_{n\geq 0}$ converges to $\psi_{\kappa,\nu}$ uniformly on $\overline{\mathbb{D}}$. In particular, the curves η_{κ,ν_n} a.s. converge uniformly to $\eta_{\kappa,\nu}$ when parametrized by $\overline{A_{\mathrm{L}}}$ via ψ_{κ,ν_n}.*

The proof done through the following steps.

1. First one shows that a.s., for every $w \in \mathcal{O}_{\kappa,\nu}$, the marked domains $(\mathcal{O}_{\kappa,\nu_n}, w)$ converge to $(\mathcal{O}_{\kappa,\nu}, w)$ in the Carathéodory sense (Definition 5.4). See [2, Lemma 4.8].

2. Then one shows that a.s., the family $(\mathbb{C} \setminus \mathcal{O}_{\kappa,\nu_n})_{n\geq 0}$ is uniformly locally connected (Definition 5.1). For this it is enough to show that the family $(\partial\mathbb{D} \cup \mathcal{S}_{\kappa,\nu_n})_{n\geq 0}$ is uniformly locally connected. See [2, Lemma 4.9].

3. Finally, one concludes by using Theorem 5.5.

One can further show that if the measure ν and all the measures ν_n has full compact support on $\overline{A_{\mathrm{L}}}$, then the driving functions of the Loewner chains for η_{κ,ν_n} converge to that of $\eta_{\kappa,\nu}$ [2, Proposition 1.4].

5.4 A level line of the GFF

The level lines of a GFF with piecewise constant boundary conditions are known to be $\mathrm{SLE}_4(\rho)$ curves with multiple force points. In [PW17] Powell and Wu extend the notion of level lines to regulated boundary conditions. The regulated functions are exactly the uniform limits of piecewise constant functions. Additionally, Powell and Wu require a threshold condition that ensures that the level lines do not hit the boundary too often. For instance, their framework does not cover the boundary condition $f(e^{i\theta}) = \mathbf{1}_{\frac{\pi}{2}<\theta<\frac{3\pi}{2}}(\frac{\pi}{2} - |\theta - \pi|)$ for $\theta \in (0, 2\pi]$. This boundary condition is certainly regulated, since it is continuous, but it does not satisfy the threshold condition in [PW17], and the method of [PW17] does not provide a control of the corresponding level line near $-i$ and i.

In my work [2] in collaboration with Wu we show that one can take positive measures, not just regulated functions, as boundary conditions, and we also remove the threshold assumption. Our proof goes through isomorphism theorems. In the construction of the curves $\eta_{\kappa,\nu}$ presented in Section 5.2, one takes $\kappa = 4$. We show the following.

Theorem 5.9 ([2], Theorems 1.7 and 1.8)**.** *Given ν a positive finite Radon measure with full support on $\overline{A_{\mathrm{L}}}$ and without atoms, then one can couple the curve $\eta_{4,\nu}$ and a GFF $\phi_{\mathbb{D}}$ in $\mathbb{D}$ with boundary condition given by ν, such that $\eta_{4,\nu}$ is a level line of $\phi_{\mathbb{D}}$, with value* 0 *to the right of $\eta_{4,\nu}$ and 2λ to the left of $\eta_{4,\nu}$. Moreover, in this coupling, $\eta_{4,\nu}$ is deterministically determined by the field $\phi_{\mathbb{D}}$.*

The result for ν a piecewise constant function has been shown in my article [8] with Aru and Sepúlveda; see Section 4.5. For general ν, we approximate it weakly by piecewise constant functions and use the convergence result of Theorem 5.8. As for ν not having atoms, actually a weaker condition is used in [2], which we do not detail here. We only need that a.s. $\eta_{4,\nu}$ does not hit an atom of ν, which still allows ν to have certain types of atoms.

Importantly, in the proof of Theorem 5.9 we do not rely on the theory of Loewner flows ans SLE processes alone. From abstract considerations, the curve $\eta_{4,\nu}$ has a continuous driving function, and one can even write down an equation for this driving function on the time intervals when $\eta_{4,\nu}$ is away from $\overline{A_{\mathrm{L}}}$ [2, Section 5.3]. However, it is not clear how to describe the evolution of the curve on the time intervals when $\eta_{4,\nu}$ hits $\overline{A_{\mathrm{L}}}$. And actually for some ν-s, $\eta_{4,\nu}$ can hit the boundary $\overline{A_{\mathrm{L}}}$ a lot, which we formalize below.

Theorem 5.10 ([2], Proposition 3.10)**.** *There are continuous non-negative functions f on $\overline{A_{\mathrm{L}}}$ such that the curve $\eta_{4,f}$ (i.e. $\eta_{4,\nu}$ with $\nu = f$) satisfies the following. On one hand, a.s. $\eta_{4,f} \cap \overline{A_{\mathrm{L}}}$ has empty interior, i.e. does not contain open subarcs. On the other hand, with positive probability, $\eta_{4,f} \cap \overline{A_{\mathrm{L}}}$ has a non-zero Lebesgue measure.*

Chapter 6

Brownian loop soups and multiplicative chaoses

In this Chapter are described the results obtained in collaboration with Elie Aïdékon (Sorbonne Université), Nathanaël Berestycki and Antoine Jégo (University of Vienna) [1]. Out of 2D Brownian loop soups we construct random measures that have many similarities with Gaussian multiplicative chaoses and satisfy a conformal covariance property. For the particular intensity parameter $\alpha = 1/2$ of the loop soup we recover the renormalized hyperbolic cosine of the continuum GFF. This comes from the Le Jan's isomorphism (Theorem 1.13). Our construction works however for any intensity parameter $\alpha > 0$, and for $\alpha \notin \frac{1}{2}\mathbb{N}$ it provides new non-Gaussian multiplicative chaoses. In Section 6.1 we recall the notion of the multiplicative chaos of the GFF and its main properties. In Section 6.2 we recall the notion of Brownian multiplicative chaos for finitely many independent 2D Brownian trajectories that was constructed in [BBK94, AHS20, Jé20]. In Section 6.3 we present the multiplicative chaos of the Brownian loop soup constructed in [1]. Compared to the case of finitely many Brownian trajectories it requires additional renormalizations due to the ultraviolet divergence in the loop soup. In Section 6.4 we present the martingale method that was used in the proofs. In Section 6.5 we present the approximations from discrete, and deduce for $\alpha = 1/2$ the identity in law with the renormalized cosh of the GFF.

6.1 Gaussian multiplicative chaos

Here is a quick review on the *Gaussian multiplicative chaos* (GMC). For details, we refer to [RV14, Ber17]. Here by GMC we will mean the random positive measure obtained as the renormlized exponential for the 2D continuum GFF, $e^{\sqrt{2\pi}\gamma\phi_D}$, $\gamma \in (0, 2)$. In the L^2 regime, i.e. $\gamma \in (0, \sqrt{2})$, this is actually a Wick's renormalization. The interacting bosonic field

$$\frac{1}{Z_\gamma} \exp\Big(- \frac{1}{2} \int_D \|\nabla\varphi\|^2 - \int_D e^{\sqrt{2\pi}\gamma\varphi} \Big) \mathcal{D}\varphi$$

has been constructed for $\gamma \in (0, \sqrt{2})$ by Høegh-Krohn in [HK71]; see Section 1.1.4. Independently, Kahane constructed in [Kah85] the GMC measures for the whole range of parameters and for more general Gaussian logarithmically correlated fields.

Let $D \subset \mathbb{C}$ be an open bounded domain. For simplicity, we assume that it is simply connected, however this is unimportant. Let ϕ_D be the continuum massless GFF on D with 0 boundary conditions. Let $(\phi_D)_\varepsilon(z)$ denote the average value of the GFF ϕ_D on the circle of radius ε with center z. It is defined pointwise and continuous in z. Fix $\gamma \in (0, 2)$. The measures

$$\mu_\gamma^\varepsilon = \mathbf{1}_{z\in D}\varepsilon^{\gamma^2/2} e^{\sqrt{2\pi}\gamma(\phi_D)_\varepsilon(z)} dz \tag{6.1}$$

converge weakly in probability as $\varepsilon \to 0$ towards a random Radon measure μ_γ on D that is a.s. finite and positive. The latter is the *Gaussian multiplicative chaos* (GMC). The construction through circle averages has been used by Duplantier and Sheffield in [DS11]. One can extend the definition to $\gamma \in (-2, 0)$, since $-\phi_D$ has the same law as ϕ_D, and to $\gamma = 0$ by setting $\mu_\gamma = \mathbf{1}_{z\in D} dz$. For every u continuous function on $\overline{D}$, the function $\gamma \mapsto \langle \mu_\gamma, u\rangle$ is a.s. analytic on $(-2, 2)$, and can be extended to a random holomorphic function by considering complex γ-s.

The range $\gamma(-\sqrt{2}, \sqrt{2})$ corresponds to the L^2 regime.

Proposition 6.1. *We have that* $\mathbb{E}[\langle \mu_\gamma, 1\rangle^2] < +\infty$ *if and only if* $|\gamma| < \sqrt{2}$.

By applying the Cameron-Martin formula for the GFF, one immediately gets the following description of the μ_γ-typical points.

Proposition 6.2. *Fix* $\gamma \in (-2, 2) \setminus \{0\}$. *Let* F *be a bounded measurable functional on couples point-field. Then*

$$\mathbb{E}\Big[\int_D F(z, \phi_D)\mu_\gamma(dz)\Big] = \int_D \mathrm{CR}(z, D)^{\gamma^2/2}\mathbb{E}[F(z, \phi_D + 2\pi\gamma G_D(z, \cdot))]dz,$$

where $\mathrm{CR}(z, D)$ *denotes the conformal radius (Definition 7.1) and* $G_D(z, \cdot)$ *is the function* $w \mapsto G_D(z, w)$. *In particular, a.s. for* $\mu_\gamma(dz)$ *almost every* $z \in D$,

$$\lim_{\varepsilon\to 0}(\phi_D)_\varepsilon(z)/|\log \varepsilon| = \gamma. \tag{6.2}$$

Note that the behavior (6.2) is very untypical from a GFF standpoint. Typically, for fixed $z \in D$ and $\varepsilon > 0$ small, $|(\phi_D)_\varepsilon(z)|$ is of order $|\log \varepsilon|^{1/2}$. The points $z \in D$ satisfying (6.2) are called, following [HMP10], *thick points* of the GFF. Hu, Miller and Peres also identified in [HMP10] the Hausdorff dimension of the thick points.

Theorem 6.3 (Hu-Miller-Peres)**.** *Fix* $\gamma \in (-2, 2)\setminus\{0\}$. *The Hausdorff dimension of the random subset of* D *defined by the condition* (6.2) *is a.s.* $2 - \gamma^2/2$.

The measures μ_γ satisfy the following conformal covariance property. Let $\widehat{D}$ be an other simply connected domain and $\psi : D \to \widehat{D}$ a conformal mapping. Let $\hat{\mu}_\gamma$ be the GMC measure associated with the GFF $\phi_{\widehat{D}}$ on $\widehat{D}$.

Proposition 6.4. *Fix* $\gamma \in (-2, 2)$. *The following identity in law holds:*

$$(\phi_D \circ \psi^{-1}, \psi_*\mu_\gamma) \overset{(law)}{=} (\phi_{\widehat{D}}, |\psi' \circ \psi^{-1}|^{-(2+\gamma^2/2)}\hat{\mu}_\gamma),$$

where $\psi_*\mu_\gamma$ *is the image measure of* μ_γ *by* ψ.

Biskup and Louidor proved in [BL19] the following approximation of the GMC from discrete. Let D_N be a discrete approximation of the domain D on the lattice $\frac{1}{N}\mathbb{Z}^2$. Let ϕ_N be the discrete massless GFF on D_N with 0 boundary conditions. For $\gamma > 0$, let $\mu_{N,\gamma}$ be the measure

$$\mu_{N,\gamma} = \gamma\frac{(\log N)^{1/2}}{N^{2-\gamma^2/2}}\sum_{z\in D_N}\mathbf{1}_{\phi_N(z)\geq(2\pi)^{-1/2}\gamma\log N}\delta_z,$$

where δ_z is the Dirac measure at z.

Theorem 6.5 (Biskup-Louidor)**.** *There are universal constants* $C, c_0 > 0$, *such that for every* $\gamma \in (0, 2)$ *one has the joint convergence in law of* $(\phi_N, \mu_{N,\gamma})$ *towards* $(\phi_D, Cc_0^{\gamma^2/2}\mu_\gamma)$.

6.2 Brownian multiplicative chaos

Here we present random measures, called *Brownian multiplicative chaoses*, supported on points of infinite multiplicity of a 2D Brownian trajectory, that share many similarities with the Gaussian multiplicative chaos. There are three different constructions for these measures, first by Bass, Burdzy and Khoshnevisan [BBK94], then recently by Aïdékon, Hu and Shi [AHS20], and by Jégo [Jé20]. This is also closely related to the problem of points visited exceptionally often by the 2D random walk, studied by Erdös and Taylor [ET60], and more recently by Dembo, Peres, Rosen and Zeitouni [DPRZ01]. Here we will present the construction due to Jégo [Jé20], as it is the one used in our joint work [1].

Let $D \subset \mathbb{C}$ be an open bounded simply connected domain. Again, the simple connectedness in non-essential, just for simplicity. Fix $z_0 \in D$ and let $(B_t)_{t\geq 0}$ be a Brownian motion in $\mathbb{C}$, with $B_0 = z_0$. Let $T_{\partial D}$ be the first hitting time of ∂D. Let ℓ^B denote the occupation measure of $(B_t)_{0\leq t\leq \partial D}$:

$$\ell^B(A) = \int_0^{T_{\partial D}} \mathbf{1}_{B_t \in A} dt.$$

For $\varepsilon > 0$ and $z \in D$, let $\ell^B_\varepsilon(z)$ denote the circle average of ℓ^B on a circle of radius ε around z. The function $(z, \varepsilon) \mapsto \ell^B_\varepsilon(z)$ is a.s. well defined and Hölder continuous [Jé20, Proposition 1.1]. For $\mathtt{a} > 0$, let $\mathcal{M}^B_{\mathtt{a},\varepsilon}$ denote the following measure:

$$\mathcal{M}^B_{\mathtt{a},\varepsilon} = (4\pi\mathtt{a})^{-1/2}\mathbf{1}_{z\in D}|\log\varepsilon|^{1/2}\varepsilon^{\mathtt{a}}e^{(4\pi\mathtt{a}\ell^B_\varepsilon(z))^{1/2}}dz. \tag{6.3}$$

Jégo proves in [Jé20, Theorem 1.1] the following.

Theorem 6.6 (Jégo)**.** *Fix* $\mathtt{a} \in (0, 2)$*. As* $\varepsilon \to 0$*, the measure* $\mathcal{M}^B_{\mathtt{a},\varepsilon}$ *converges weakly in probability to a random Radon measure* $\mathcal{M}^B_{\mathtt{a}}$ *supported on the range of* $(B_t)_{0\leq t\leq T_{\partial D}}$*. The measure* $\mathcal{M}^B_{\mathtt{a}}$ *is a.s. non-zero and finite.*

The measure $\mathcal{M}^B_{\mathtt{a}}$ above coincides a.s. with the measures constructed differently by Bass-Burdzy-Khoshnevisan [BBK94] and by Aïdékon-Hu-Shi [AHS20]. In [BBK94, AHS20] one uses the crossings by the Brownian motion rather than the occupation measure ℓ^B.

The measure $\mathcal{M}^B_{\mathtt{a}}$ is a Brownian analogue of the Gaussian multiplicative chaos, and therefore it has been called *Brownian multiplicative chaos.* The presence of $\ell^B_\varepsilon(z)^{1/2}$ in (6.3) is natural in view of the isomorphism theorems of Section 1.2, which relate ℓ^B to square-Gaussians. The relation between the parameter γ of the GMC μ_γ and the parameter $\mathtt{a}$ (*thickness* parameter) of $\mathcal{M}^B_{\mathtt{a}}$ is

$$\mathtt{a} = \frac{\gamma^2}{2}. \tag{6.4}$$

There are two major differences between μ_γ and $\mathcal{M}^B_{\mathtt{a}}$. First, the compact support of μ_γ is the whole $\overline{D}$, while that of $\mathcal{M}^B_{\mathtt{a}}$ is just the range of the Brownian motion. Moreover, in (6.3) there is an extra renormalization factor $|\log\varepsilon|^{1/2}$ that does not appear in (6.1).

The analogue of Proposition 6.1 holds for the Brownian multiplicative chaos.

Proposition 6.7. *We have that* $\mathbb{E}[\langle\mathcal{M}^B_{\mathtt{a}}, 1\rangle^2] < +\infty$ *if and only if* $\mathtt{a} < 1$*.*

For $z \in D$, let $\Xi^z_{\mathtt{a}}$ denote the Poisson point process of Brownian excursions from z to z in D with intensity measure $2\pi\mathtt{a}\mu^{z,z}_D$, where $\mu^{z,z}_D$ is given by (1.17) (Definition 1.15). By a slight abuse of notation, $\Xi^z_{\mathtt{a}}$ will also denote the loop obtained by concatenation of all these excursions. For $z \neq z' \in D$, let $\wp^{z,z'}$ denote a Brownian excursion from z to z' in D with distribution $\mu^{z,z'}_D/G_D(z, z')$ (1.17). The analogue of Proposition 6.2 holds for the Brownian multiplicative chaos.

Theorem 6.8 (Bass-Burdzy-Khoshnevisan, Aïdékon-Hu-Shi, Jégo)**.** *Fix* $\mathsf{a} \in (0, 2)$*. Let* F *be a bounded measurable functional on couples point-trajectory. Then*

$$\mathbb{E}\Big[\int_D F(z, (B_t)_{0\leq t\leq T_{\partial D}})\mathcal{M}_{\mathsf{a}}^B(dz)\Big] = \int_D G_D(z_0, z)\,\mathrm{CR}(z, D)^{\mathsf{a}}\mathbb{E}\big[F(z, \wp^{z_0,z}\wedge\Xi_{\mathsf{a}}^z\wedge(B_t^z)_{0\leq t\leq T_{\partial D}})\big]\,dz,$$

where B^z *is a Brownian motion starting at* z*, the symbol* $\wedge$ *denotes the concatenation of paths, and the three paths* $\wp^{z_0,z}$*,* Ξ_{a}^z *and* $(B_t^z)_{0\leq t\leq T_{\partial D}}$ *are independent.*

In particular, a.s. $\mathcal{M}_{\mathsf{a}}^B(dz)$ *almost every* z *is a point of infinite multiplicity for the Brownian path* $(B_t)_{0\leq t\leq T_{\partial D}}$*. Moreover, for* $\mathcal{M}_{\mathsf{a}}^B(dz)$ *almost every* z*,*

$$\lim_{\varepsilon\to 0} \ell_\varepsilon^B(z)/|\log\varepsilon|^2 = \mathsf{a}. \tag{6.5}$$

Let N_ε^z *denote the number of excursions of* $(B_t)_{0\leq t\leq T_{\partial D}}$ *from* z *to the circle of center* z *and radius* ε*. Then a.s. for* $\mathcal{M}_{\mathsf{a}}^B(dz)$ *almost every* z*,*

$$\lim_{\varepsilon\to 0} N_\varepsilon^z/|\log\varepsilon| = \mathsf{a}. \tag{6.6}$$

The points satisfying either (6.5) or (6.6) are called a-*thick points* of the Brownian motion. It is further shown in [AHS20] that the carrying dimension of the measure $\mathcal{M}_{\mathsf{a}}^B$ is $2 - \mathsf{a}$, which is also consistent with Theorem 6.3 and (6.4).

In [Jé19] Jégo proved a convergence towards the Brownian multiplicative chaos from discrete, which is the Brownian analogue of Theorem 6.5. Let D_N be a discrete approximation of the domain D on the lattice $\frac{1}{N}\mathbb{Z}^2$. Let $(X_t^{(N)})_{t\geq 0}$ be a Markov jump process on $\frac{1}{N}\mathbb{Z}^2$ normalized to converge in law towards a Brownian motion on $\mathbb{C}$ as $N \to +\infty$. Let $T_{D_N^{\mathsf{c}}}$ denote its first exit time from D_N. For $z \in D_N$, let $\ell_N(z)$ be the renormalized local time

$$\ell_N(z) = N^2 \int_0^{T_{D_N^{\mathsf{c}}}} \mathbf{1}_{X_t^{(N)}=z}\,dt.$$

Let $\mathcal{M}_{\mathsf{a}}^{(N)}$ be the measure

$$\mathcal{M}_{\mathsf{a}}^{(N)} = \frac{\log N}{N^{2-\mathsf{a}}} \sum_{z\in D_N} \mathbf{1}_{\ell_N(z)\geq(2\pi)^{-1}\mathsf{a}(\log N)^2}\delta_z.$$

Theorem 6.9 (Jégo)**.** *Assume that* $X_0^{(N)}$ *converges as* $N \to +\infty$ *towards* z_0*. Then there is a universal constant* $c_0 > 0$*, such that for every* $\mathsf{a} \in (0, 2)$ *one has the joint convergence in law of* $((X_t^{(N)})_{0\leq t\leq T_{D_N^{\mathsf{c}}}}, \mathcal{M}_{\mathsf{a}}^{(N)})$ *towards* $((B)_{0\leq t\leq T_{\partial D}}, c_0^{\mathsf{a}}\mathcal{M}_{\mathsf{a}}^B)$*.*

The notion of Brownian multiplicative chaos can be extended to a finite number of independent Brownian trajectories, $B^{(1)}, B^{(2)}, \ldots, B^{(n)}$; see [AHS20, Section 7.2] and [Jé19, Section 1.4]. Let be

$$\ell^{B^{(1)},\ldots,B^{(n)}} = \ell^{B^{(1)}} + \ell^{B^{(2)}} + \cdots + \ell^{B^{(n)}}.$$

The measure $\mathcal{M}_{\mathsf{a}}^{B^{(1)},\ldots,B^{(n)}}$ is the weak limit in probability of

$$(4\pi\mathsf{a})^{-1/2}\mathbf{1}_{z\in D}|\log\varepsilon|^{1/2}\varepsilon^{\mathsf{a}}e^{(4\pi\mathsf{a}\ell_\varepsilon^{B^{(1)},\ldots,B^{(n)}}(z))^{1/2}}dz.$$

It can be decomposed according to the number of trajectories that generate a given a-thick point:

$$\mathcal{M}_{\mathsf{a}}^{B^{(1)},\ldots,B^{(n)}} = \sum_{\substack{J\subset\{1,\ldots,n\}\\ J\neq\emptyset}} \int_{S(J,\mathsf{a})} d\mathsf{a}_J \mathcal{M}_{(\mathsf{a}_j)_{j\in J}}^{\bigcap_{j\in J} B^{(j)}}, \tag{6.7}$$

where $S(J, \mathtt{a})$ is the simplex

$$S(J, \mathtt{a}) = \Big\{ (\mathtt{a}_j)_{j\in J} \in [0, \mathtt{a}]^J \Big| \sum_{j\in J} \mathtt{a}_j = \mathtt{a} \Big\},$$

$d\mathtt{a}_J$ is the Lebesgue measure on $S(J, \mathtt{a})$, and $\mathcal{M}^{\bigcap_{j\in J} B^{(j)}}_{(\mathtt{a}_j)_{j\in J}}$ is a measure on $\mathtt{a}$-thick points generated by the intersection of $(B^{(j)})_{j\in J}$, where each $B^{(j)}$ brings a partial thickness $\mathtt{a}_j$.

6.3 Multiplicative chaos of a Brownian loop soup: presentation

In the collaboration [1] by Aïdékon, Berestycki, Jégo and myself we consider the Brownian loop soups $\mathcal{L}^\alpha_D$ (Definition 1.16) rather than a finite family of Brownian trajectories, and construct the corresponding multiplicative chaoses. Because of the ultraviolet divergence in the loop soup (too many small loops), this requires an additional layer of renormalization compared to the construction of Section 6.2. Indeed, one can apply an ultraviolet cutoff to $\mathcal{L}^\alpha_D$ so as to keep finitely many loops, and then take their multiplicative chaos. But as one lowers the cutoff, the corresponding measure diverges a.s. in every open subset of D. So one needs to apply a renormalization factor to tame this divergence.

So let $D \subset \mathbb{C}$ be an open bounded simply connected domain. Let us first present the cutoff that is used in [1]. The most natural cutoff would have been by the diameter or the time duration of a Brownian loop. However, in order to reduce the technical complexity a different one was used. For each loop $\wp \in \mathcal{L}^\alpha_D$ we associate a uniform r.v. $U_\wp$ on $(0, 1)$, with the conditional distribution of $(U_\wp)_{\wp\in\mathcal{L}^\alpha_D}$ being i.i.d. Given a constant $K > 0$, denote

$$\mathcal{L}^{\alpha,K}_D = \{\wp \in \mathcal{L}^\alpha_D | e^{-KT(\wp)} \leq U_\wp\},$$

where $T(\wp)$ is the total time duration of a loop $\wp$. The complementary $\mathcal{L}^\alpha_D \setminus \mathcal{L}^{\alpha,K}_D$ is the so called massive Brownian loop soup, with K being the square mass or equivalently the killing rate; see also Section 1.1.3 and Section 1.2.4. The subset $\mathcal{L}^{\alpha,K}_D$ is non-decreasing in K. Note that $\mathcal{L}^{\alpha,K}_D$ contains a.s. infinitely many loops. However, the density of small loops is rarefied compared to $\mathcal{L}^\alpha_D$. Indeed, as $\varepsilon \to 0$.

$$\mathbb{E}[\#\{\wp \in \mathcal{L}^\alpha_D | \operatorname{diam}(\wp) > \varepsilon\}] \asymp \varepsilon^{-2}, \qquad \mathbb{E}[\#\{\wp \in \mathcal{L}^{\alpha,K}_D | \operatorname{diam}(\wp) > \varepsilon\}] \asymp |\log \varepsilon|.$$

The reason for the choice of the cutoff through the square mass K will be detailed in Section 6.4.

Despite $\mathcal{L}^{\alpha,K}_D$ being a.s. infinite, one can define a Brownian multiplicative chaos of $\mathcal{L}^{\alpha,K}_D$ by using the construction of Section 6.2, without requiring additional renormalization. For $\mathtt{a} \in (0, 2)$, let $\mathcal{M}^{\alpha,K}_\mathtt{a}$ denote the multiplicative chaos measure of thickness $\mathtt{a}$ generated by $\mathcal{L}^{\alpha,K}_D$. A.s., $\mathcal{M}^{\alpha,K}_\mathtt{a}(dz)$ almost every z is intersection of finitely many loops in $\mathcal{M}^{\alpha,K}_\mathtt{a}$. Moreover, a decomposition similar to (6.7) holds:

$$\mathcal{M}^{\alpha,K}_\mathtt{a} = \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{\wp_1,\dots,\wp_n \in \mathcal{L}^{\alpha,K}_D \\ \wp_j\text{-s distinct}}} \int_{S(n,\mathtt{a})} d\mathtt{a}_{\{1,\dots,n\}} \mathcal{M}^{\bigcap_{1\leq j\leq n} \wp_j}_{(\mathtt{a}_j)_{1\leq j\leq n}},$$

where $S(n, \mathtt{a}) = S(\{1, \dots, n\}, \mathtt{a})$ and $\mathcal{M}^{\bigcap_{1\leq j\leq n} \wp_j}_{(\mathtt{a}_j)_{1\leq j\leq n}}$ is the measure on $\mathtt{a}$-thick points generated by the intersection of the n Brownian loops $\wp_1, \dots, \wp_n$, with each $\wp_j$ contributing a thickness $\mathtt{a}_j$. In [1] we prove the following.

Theorem 6.10 ([1], Theorem 1.1). *Fix $\alpha > 0$ and $\mathtt{a} \in (0,2)$. Then as $K \to +\infty$, the measure $(\log K)^{-\alpha}\mathcal{M}_{\mathtt{a}}^{\alpha,K}$ converges weakly in probability to a random Radon measure $\mathcal{M}_{\mathtt{a}}^{\alpha}$ on D. The measure $\mathcal{M}_{\mathtt{a}}^{\alpha}$ satisfies the following properties.*

1. *The measure $\mathcal{M}_{\mathtt{a}}^{\alpha}$ is a.s. finite.*
2. *A.s., for every O open subset of D, $\mathcal{M}_{\mathtt{a}}^{\alpha}(O) > 0$.*
3. *The measure $\mathcal{M}_{\mathtt{a}}^{\alpha}$ is conformally covariant in law. Let $\widehat{D}$ be an other simply connected domain and $\psi : D \to \widehat{D}$ a conformal mapping. Let $\widehat{\mathcal{M}}_{\mathtt{a}}^{\alpha}$ be the measure on $\widehat{D}$. Then $\psi_*\mathcal{M}_{\mathtt{a}}^{\alpha} \overset{(law)}{=} |\psi' \circ \psi^{-1}|^{-(2+\mathtt{a})}\widehat{\mathcal{M}}_{\mathtt{a}}^{\alpha}$.*
4. *One has $\mathbb{E}[\langle \mathcal{M}_{\mathtt{a}}^{\alpha}, 1\rangle^2] < +\infty$ if and only if $\mathtt{a} < 1$.*

The limit measure $\mathcal{M}_{\mathtt{a}}^{\alpha}$ is the multiplicative chaos of the Brownian loop soup $\mathcal{L}_D^{\alpha}$. Its conformal covariance is a consequence of the conformal covariance of the 2D Brownian loop soup; see Section 2.3. The way $\mathcal{M}_{\mathtt{a}}^{\alpha}$ is constructed, it is *a priori* measurable with respect to $\mathcal{L}_D^{\alpha}$ and the values of the uniform r.v.s $(U_{\wp})_{\wp\in\mathcal{L}_D^{\alpha}}$. However, it is easy to check that $\mathcal{M}_{\mathtt{a}}^{\alpha}$ is actually independent from $(U_{\wp})_{\wp\in\mathcal{L}_D^{\alpha}}$ and thus measurable with respect to $\mathcal{L}_D^{\alpha}$; see [1, Section 9.1]. The main ideas in the proof of Theorem 6.10 will be presented in Section 6.4.

Next we describe how the Brownian loop soup looks like viewed from a $\mathcal{M}_{\mathtt{a}}^{\alpha}$-typical point. It is to be compared with Proposition 6.2 and and Theorem 6.8. Recall that $\Xi_{\mathtt{a}}^z$ denotes the loop rooted at z obtained by concatenating the excursions from z to z in a PPP of intensity $2\pi\mathtt{a}\mu_D^{z,z}$.

Theorem 6.11 ([1], Theorem 1.8 and Theorem 1.11). *Fix $\alpha > 0$ and $\mathtt{a} \in (0,2)$. Let F be bounded measurable functional on couples point-collection of loops. Then*

$$\mathbb{E}\Big[\int_D F(z,\mathcal{L}_D^{\alpha})\mathcal{M}_{\mathtt{a}}^{\alpha}(dz)\Big] = \frac{1}{2^{\alpha}\mathtt{a}^{1-\alpha}\Gamma(\alpha)}\int_D \mathrm{CR}(z,D)^{\mathtt{a}}\mathbb{E}\Big[F(z,\mathcal{L}_D^{\alpha}\cup\{\Xi_{\mathtt{a}_j}^z, j\geq 1\})\Big]dz,$$

where on the right-hand side the $\mathtt{a}_j$-s satisfy $\sum_{j\geq 1}\mathtt{a}_j = \mathtt{a}$, and are distributed according to a Poisson-Dirichlet partition $PD(\alpha)$ of $(0,\mathtt{a})$, independent from $\mathcal{L}_D^{\alpha}$; the family of loops $(\Xi_{\mathtt{a}_j}^z)_{j\geq 1}$ is independent from $\mathcal{L}_D^{\alpha}$ and the $\Xi_{\mathtt{a}_j}^z$-s are conditionally independent given $(\mathtt{a}_j)_{j\geq 1}$. In particular, a.s. $\mathcal{M}_{\mathtt{a}}^{\alpha}(dz)$ almost every z is an intersection of infinitely many Brownian loops in $\mathcal{L}_D^{\alpha}$, and is a point of infinite multiplicity for each of these loops.

For $z \in D$, let N_r^z be the number of crossings in $\mathcal{L}_D^{\alpha}$ from the circle of center z and radius r to the circle of center z and radius er. Then a.s. for $\mathcal{M}_{\mathtt{a}}^{\alpha}(dz)$ almost every z,

$$\lim_{n\to+\infty}\frac{1}{n^2}N_{e^{-n}}^z = \mathtt{a}. \tag{6.8}$$

Moreover, the dimension of the random subset of D defined by (6.8) is a.s. $2-\mathtt{a}$.

Note that prior to the construction of the measures $\mathcal{M}_{\mathtt{a}}^{\alpha}$ it was not known that the 2D Brownian loop soup contains points that are intersection of infinitely many loops. The Poisson-Dirichlet partitions appear already in the discrete setting of continuous-time random walk loop soups; see Theorem 1.12. The behavior (6.8) is very different from that for a fixed deterministic $z \in D$. Indeed, for the latter, $N_{e^{-n}}^z$ is of order n and not n^2; see [1, Section 7.2].

6.4 The crucial martingale

The proof of Theorem 6.10 relies on a martingale method. So let us introduce the martingale. Recall that for $K > 0$, $G_{D,K}$ denotes the massive Green's function. Set

$$C_K(z) = 2\pi(G_D - G_{D,K})(z, z),$$

which is well defined and continuous in $z \in D$. Let $\alpha > 0$ and $\mathtt{a} \in (0, 2)$ be fixed. For $K > 0$, let be

$$\widetilde{\mathcal{M}}_{\mathtt{a}}^{\alpha,K} = \mathbf{1}_{z\in D} \frac{1}{2^\alpha \Gamma(\alpha) \mathtt{a}^{1-\alpha}} \operatorname{CR}(z, D)^{\mathtt{a}} e^{-\mathtt{a} C_K(z)} dz \\ + \int_0^{\mathtt{a}} du \frac{1}{2^\alpha \Gamma(\alpha)(\mathtt{a} - u)^{1-\alpha}} \operatorname{CR}(z, D)^{\mathtt{a}-u} e^{-(\mathtt{a}-u)C_K(z)} \mathcal{M}_u^{\alpha,K}.$$

Let $\mathcal{F}_K$ be the σ-algebra of $\mathcal{L}_D^{\alpha,K}$, so that $(\mathcal{F}_K)_{K\geq 0}$ is a filtration.

Theorem 6.12 ([1], Proposition 3.4)**.** *The process $(\widetilde{\mathcal{M}}_{\mathtt{a}}^{\alpha,K})_{K\geq 0}$ is a measure-valued martingale in the filtration $(\mathcal{F}_K)_{K\geq 0}$.*

Let us explain how the martingale $(\widetilde{\mathcal{M}}_{\mathtt{a}}^{\alpha,K})_{K\geq 0}$ has been obtained. One uses a reverse engineering approach. One wants to construct the measure $\mathcal{M}_{\mathtt{a}}^\alpha$ (Theorem 6.10), and for that one starts by constructing its conditional expectation given $\mathcal{F}_K$, which is actually $\widetilde{\mathcal{M}}_{\mathtt{a}}^{\alpha,K}$. The measure $\widetilde{\mathcal{M}}_{\mathtt{a}}^{\alpha,K}$ is a sum of two terms. The first one is deterministic and is the expectation of the multiplicative chaos generated by $\mathcal{L}_D^\alpha \backslash \mathcal{L}_D^{\alpha,K}$. The second one, which is random, is the conditional expectation of the multiplicative chaos generated by the interaction of $\mathcal{L}_D^\alpha \setminus \mathcal{L}_D^{\alpha,K}$ and $\mathcal{L}_D^{\alpha,K}$, with u being the thickness brought by $\mathcal{L}_D^{\alpha,K}$, and $\mathtt{a} - u$ the thickness brought by $\mathcal{L}_D^\alpha \backslash \mathcal{L}_D^{\alpha,K}$. The expectation of $\mathcal{M}_u^{\alpha,K}$ can be expressed through Kummer's confluent hypergeometric functions [1, Proposition 3.1], and the martingale property of $(\widetilde{\mathcal{M}}_{\mathtt{a}}^{\alpha,K})_{K\geq 0}$ follows from a functional equation on these hypergeometric functions [1, Lemma 5.5]. The martingale $(\widetilde{\mathcal{M}}_{\mathtt{a}}^{\alpha,K})_{K\geq 0}$ is the main reason of the choice of the ultraviolet cutoff in $\mathcal{L}_D^\alpha$ though the square-mass K. For a different cutoff, say by diameter or the duration of loops, the conditional expectation of $\mathcal{M}_{\mathtt{a}}^\alpha$ is much less explicit and more difficult to analyze than $\widetilde{\mathcal{M}}_{\mathtt{a}}^{\alpha,K}$.

Since the martingale $(\widetilde{\mathcal{M}}_{\mathtt{a}}^{\alpha,K})_{K\geq 0}$ is non-negative, it has an a.s. limit

$$\mathcal{M}_{\mathtt{a}}^\alpha = \lim_{K\to+\infty} \widetilde{\mathcal{M}}_{\mathtt{a}}^{\alpha,K}. \tag{6.9}$$

Theorem 6.13 ([1], Section 3)**.** *The martingale $(\widetilde{\mathcal{M}}_{\mathtt{a}}^{\alpha,K})_{K\geq 0}$ is uniformly integrable. In particular, the convergence of $\widetilde{\mathcal{M}}_{\mathtt{a}}^{\alpha,K}$ to $\mathcal{M}_{\mathtt{a}}^\alpha$ also holds in L^1.*

Furthermore for all Borel sets $A \subset D$,

$$\lim_{K\to+\infty} \mathbb{E}\big[\big|\widetilde{\mathcal{M}}_{\mathtt{a}}^{\alpha,K}(A) - (\log K)^{-\alpha} \mathcal{M}_{\mathtt{a}}^{\alpha,K}(A)\big|\big] = 0.$$

The convergence in Theorem 6.10 essentially follows from Theorem 6.13 above. To prove the latter, one distinguishes between the L^2 regime ($\mathtt{a} \in (0, 1)$) and the non-L^2 regime ($\mathtt{a} \in [1, 2)$). In the L^2 regime we use the second moments and show that

$$\sup_{K>0} \mathbb{E}[\langle \widetilde{\mathcal{M}}_{\mathtt{a}}^{\alpha,K}, 1\rangle^2] < +\infty, \qquad \lim_{K\to+\infty} \mathbb{E}\big[\big(\widetilde{\mathcal{M}}_{\mathtt{a}}^{\alpha,K}(A) - (\log K)^{-\alpha} \mathcal{M}_{\mathtt{a}}^{\alpha,K}(A)\big)^2\big] = 0.$$

The convergence (6.9) is then also in L^2.

The non-L^2 regime is significantly more technical. We use a method inspired by the approach of Berestycki [Ber17] to the non-L^2 regime of the GMC. For $z \in D$, $r > 0$ and $K > 0$, denote by $N_r^{z,K}$ the number of crossings by $\mathcal{L}_D^{\alpha,K}$ from the circle of center z and radius r to the circle of center z and radius er. For $b \in (\mathtt{a}, 2)$ and $n \geq 0$, denote $A_{b,n}^K$ the random subset of D

$$A_{b,n}^K = \{z \in D | \forall k \geq n, N_{e^{-k}}^{z,K} \leq bk^2\}.$$

We consider the restricted measures $\mathbf{1}_{A_{b,n}^K} \mathcal{M}_a^{\alpha,K}$. Then for all $n \geq 0$ and $b \in (\mathtt{a}, 2)$ with b close enough to $\mathtt{a}$,

$$\sup_{K \geq 2} (\log K)^{-2\alpha} \mathbb{E}\big[\langle \mathbf{1}_{A_{b,n}^K} \mathcal{M}_{\mathtt{a}}^{\alpha,K}, 1 \rangle^2\big] < +\infty.$$

Thus one can use the second moment method for the restricted measures $(\log K)^{-\alpha} \mathbf{1}_{A_{b,n}^K} \mathcal{M}_{\mathtt{a}}^{\alpha,K}$. Moreover, for $b \in (\mathtt{a}, 2)$ with b close enough to $\mathtt{a}$,

$$\lim_{n \to +\infty} \sup_{K \geq 2} (\log K)^{-\alpha} \mathbb{E}\big[\langle (1 - \mathbf{1}_{A_{b,n}^K}) \mathcal{M}_{\mathtt{a}}^{\alpha,K}, 1 \rangle\big] = 0.$$

For details, see [1, Section 7].

6.5 Convergence from discrete and relation to the Gaussian multiplicative chaos

We further prove that the measure $\mathcal{M}_{\mathtt{a}}^{\alpha}$ can be approximated from discrete. Let D_N be a discrete approximation of the domain D on the lattice $\frac{1}{N}\mathbb{Z}^2$. Consider $\mathcal{L}_{D_N}^{\alpha}$ the continuous time random walk loop soup on D_N (Definition 1.10), renormalized to converge in law to the Brownian loop soup $\mathcal{L}_D^{\alpha}$. For $\wp \in \mathcal{L}_{D_N}^{\alpha}$, let $\ell_N^z(\wp)$ be the rescaled occupation time at z:

$$\ell_N^z(\wp) = N^2 \int_0^{T(\wp)} \mathbf{1}_{\wp(t)=z} dt,$$

and let be

$$\ell_N^z(\mathcal{L}_{D_N}^{\alpha}) = \sum_{\wp \in \mathcal{L}_{D_N}^{\alpha}} \ell_N^z(\wp).$$

Let $\mathcal{M}_{\mathtt{a}}^{N,\alpha}$ be the measure

$$\mathcal{M}_{\mathtt{a}}^{N,\alpha} = \frac{(\log N)^{1-\alpha}}{2^\alpha N^{2-\mathtt{a}}} \sum_{z \in D_N} \mathbf{1}_{\ell_N^z(\mathcal{L}_{D_N}^{\alpha}) \geq (2\pi)^{-1} \mathtt{a} (\log N)^2} \delta_z.$$

Theorem 6.14 ([1], Theorem 1.12)**.** *For a universal constant $c_0 > 0$, and for every $\alpha > 0$ and $\mathtt{a} \in (0, 2)$, the couple $(\mathcal{L}_{D_N}^{\alpha}, \mathcal{M}_{\mathtt{a}}^{N,\alpha})$ converges in distribution as $N \to +\infty$ towards $(\mathcal{L}_D^{\alpha}, c_0^{\mathtt{a}} \mathcal{M}_{\mathtt{a}}^{\alpha})$.*

To prove the convergence above, we introduce in discrete the cutoff by the square-mass K (suitably renormalized) as we did in continuum, and consider $\mathcal{L}_{D_N}^{\alpha,K} \subset \mathcal{L}_{D_N}^{\alpha}$, with $\mathcal{L}_{D_N}^{\alpha,K}$ converging to $\mathcal{L}_D^{\alpha,K}$ as $N \to +\infty$. Set

$$\mathcal{M}_{\mathtt{a}}^{N,\alpha,K} = \frac{\log N}{2^\alpha N^{2-\mathtt{a}}} \sum_{z \in D_N} \mathbf{1}_{\ell_N^z(\mathcal{L}_{D_N}^{\alpha,K}) \geq (2\pi)^{-1} \mathtt{a} (\log N)^2} \delta_z.$$

Note that the renormalization factors in $\mathcal{M}_{\mathtt{a}}^{N,\alpha}$ and $\mathcal{M}_{\mathtt{a}}^{N,\alpha,K}$ differ by a $(\log N)^{-\alpha}$.

By relying on Theorem 6.9, we first show that for every $K > 0$, $(\mathcal{L}_{D_N}^{\alpha,K}, \mathcal{M}_{\mathsf{a}}^{N,\alpha,K})$ converges in law towards $(\mathcal{L}_D^{\alpha,K}, c_0^{\mathsf{a}}\mathcal{M}_{\mathsf{a}}^{\alpha,K})$. Then we show that for every Borel subset $A \subset \mathbb{C}$,

$$\lim_{K\to+\infty} \limsup_{N\to+\infty} \mathbb{E}\big[\big|\mathcal{M}_{\mathsf{a}}^{N,\alpha}(A) - (\log K)^{-\alpha}\mathcal{M}_{\mathsf{a}}^{N,\alpha,K}\big|\big] = 0.$$

By combining the two, one gets Theorem 6.14. For $\mathsf{a} \in (0,1)$, the proof of the second step uses the second moment method. For $\mathsf{a} \in [1,2)$ one additionally considers the restrictions of the measures $\mathcal{M}_{\mathsf{a}}^{N,\alpha,K}$ to "good" subsets, similarly to what has been done in continuum (Section 6.4).

For the particular value of the intensity parameter of the loop soup $\alpha = 1/2$, the Le Jan's isomorphism (Theorem 1.13) ensures that the field $(\ell_N^z(\mathcal{L}_{D_N}^{1/2}))_{z\in D_N}$ is distributed as half the square of a discrete GFF on D_N. By combining the convergences of Theorem 6.14 and Theorem 6.5 one gets the following identification in the continuum limit.

Theorem 6.15 ([1], Theorem 1.5)**.** *Let be $\alpha = 1/2$ and $\mathsf{a} \in (0,2)$. The measure $\mathcal{M}_{\mathsf{a}}^{1/2}$ has the same distribution as*

$$\frac{1}{2\gamma}(\mu_\gamma + \mu_{-\gamma}),$$

where $\gamma = \sqrt{2\mathsf{a}}$ and μ_γ and $\mu_{-\gamma}$ are GMC-s associated with the same GFF ϕ_D.

The measure $\mu_\gamma + \mu_{-\gamma}$ can be interpreted as the renormalized cosh (hyperbolic cosine) of the GFF. It is also the renormalized exponential of the absolute value:

$$\mu_\gamma + \mu_{-\gamma} = \lim_{\varepsilon\to 0} \mathbf{1}_{z\in D}\varepsilon^{\gamma^2/2}e^{\sqrt{2\pi}\gamma|(\phi_D)_\varepsilon(z)|}dz.$$

Here is also something that is not proven in [1] but which we plan to do in the future: the measure $\mathcal{M}_{\mathsf{a}}^{1/2}$ is actually the cosh of precisely the GFF given by (4.9). By restricting $\mathcal{M}_{\mathsf{a}}^{1/2}$ to clusters $\mathcal{C}$ such that $\sigma_{\mathcal{C}} = 1$, one gets simply $\frac{1}{2\gamma}\mu_\gamma$.

One can also show that for an intensity parameter of form $\alpha = k/2$ with $k \in \mathbb{N} \setminus \{0\}$, the measure $\mathcal{M}_{\mathsf{a}}^{k/2}$ corresponds to the renormalized exponential of the Euclidean norm of a vector-valued GFF with k i.i.d. components. If α is not half-integer ($\alpha \notin \frac{1}{2}\mathbb{N}$), then the measures $\mathcal{M}_{\mathsf{a}}^{\alpha}$ appear to be new objects. There are many similarities with the multiplicative chaos of the GFF, including the carrying dimensions ($2 - \mathsf{a} = 2 - \gamma^2/2$) and the conformal covariance. So these are non-Gaussian multiplicative chaoses.

Part III

Other works related to the two-dimensional setting

Chapter 7

Extremal distance and conformal radius of a CLE_4 loop

In this Chapter will be presented the results obtained in a collaboration of myself with Juhan Aru and Avelio Sepúlveda, that appeared in the preprint [4]. There we consider the loop in a CLE_4 ensemble surrounding a fixed point in a simply connected domain and determine the joint law of the corresponding conformal radius and extremal distance. The latter two notions are recalled in Section 7.1. In Section 7.2 we state our results. The joint law of conformal radius and extremal distance is expressed in terms of stopping times and last passage times of a one-dimensional Brownian motion. The law of the conformal radius alone was previously known [She09, SSW09], but not that of the extremal distance. Our derivation of this joint law relies on the coupling between the CLE_4 and the GFF (see Theorem 2.2). In Section 7.3 we explain how to derive the law of the extremal distance alone, without the joint law. It relies on the idea that one can discover the same interface of the 2D continuum GFF by Markovian exploration both from outside and from inside. We called this property *reversibility*. An additional argument is required to establish the joint law, and we present it in Section 7.4. We look at how the law of an interface changes when one modifies the boundary conditions for the GFF.

7.1 Conformal radius and extremal distance

Here we recall the notion of *conformal radius*. We have already encountered it in Sections 4.3 and 6.1.

Definition 7.1. Let $D \subset \mathbb{C}$ be an open simply connected domain with $D \neq \mathbb{C}$. Fix $z \in D$. The *conformal radius* of D seen from z, denoted $\mathrm{CR}(z, D)$, is given by

$$\mathrm{CR}(z, D) = |\psi'(0)|,$$

where ψ is any conformal map from the unit disk $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$ to D with $\psi(0) = z$.

If D is a disk with center in z, then $\mathrm{CR}(z, D)$ is just its radius. In general, for $D \subset \mathbb{C}$ an open simply connected domain with $D \neq \mathbb{C}$, the following distortion bounds hold:

$$d(z, D) \leq \mathrm{CR}(z, D) \leq 4d(z, D). \tag{7.1}$$

The lower bound in (7.1) comes from monotonicity, since D contains the disk of center z and radius $d(z, D)$. The upper bound is actually the Koebe quarter theorem; see [Ahl10, Section 5.1].

Next we recall the notions of *extremal distance* between the two boundaries of an annular domain. For details we refer to [Ahl10, Chapter 4]. Let $A \subset \mathbb{C}$ be an open connected domain such that $\mathbb{C} \setminus A$ has two connected components, both not reduced to one point, and of which one is unbounded and the other bounded. We call such a domain *annular domain*. Such a domain is doubly-connected (one hole). Its boundary ∂A has two connected components, the outer boundary $\partial_o A$ and the inner boundary $\partial_i A$.

Definition 7.2. Let u be the harmonic function on A with boundary conditions 0 on $\partial_o A$ and 1 on $\partial_i A$. The *extremal distance* (or *extremal length*) $\mathrm{ED}(\partial_o A, \partial_i A)$ between $\partial_o A$ and $\partial_i A$ is

$$\mathrm{ED}(\partial_o A, \partial_i A) = \Big(\int_A \|\nabla u\|^2\Big)^{-1}.$$

The extremal distance $\mathrm{ED}(\partial_o A, \partial_i A)$ is actually an effective resistance (Ohms). Indeed, one sees the harmonic function u as en electric potential (Volts). Then the quantity $\int_A \|\nabla u\|^2$ is the corresponding electric power (Watts). The resistance is given by the Ohm's law

$$R = \frac{V}{I} = \frac{V^2}{P},$$

where R denotes the resistance, V the voltage, I the electric current and P the electric power.

The extremal distance is a conformal invariant. Two annular domains A and A' are conformally equivalent if and only if

$$\mathrm{ED}(\partial_o A, \partial_i A) = \mathrm{ED}(\partial_o A', \partial_i A').$$

Let $\mathbb{D}$ denote the unit disk, $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$. If A is an actual annulus of form $r_o\mathbb{D} \setminus (r_i\overline{\mathbb{D}})$, then

$$\mathrm{ED}(r_o\partial\mathbb{D}, r_i\partial\mathbb{D}) = \frac{1}{2\pi}\log(r_o/r_i).$$

Thus, every annular domain A is conformally equivalent to an annulus $\mathbb{D} \setminus (r_i\overline{\mathbb{D}})$ with

$$r_i = e^{-2\pi\,\mathrm{ED}(\partial_o A, \partial_i A)}.$$

If $\wp$ is a Jordan curve in $\mathbb{D}$ surrounding 0 and $\mathrm{Int}(\wp)$ is the interior enclosed by $\wp$, then

$$\mathrm{ED}(\partial\mathbb{D}, \wp) \leq -\frac{1}{2\pi}\log \mathrm{CR}(0, \mathrm{Int}(\wp)). \tag{7.2}$$

Denote

$$r_-(\wp) = d(0, \wp), \qquad r_+(\wp) = \max\{|z| | z \in \wp\}. \tag{7.3}$$

One has the following distortion bounds:

$$e^{-2\pi\,\mathrm{ED}(\partial\mathbb{D},\wp)} \leq r_+(\wp) \leq 4e^{-2\pi\,\mathrm{ED}(\partial\mathbb{D},\wp)}. \tag{7.4}$$

The lower bound in (7.4) comes from monotonicity, since the domain delimited by $\partial\mathbb{D}$ and $\wp$ contains an annulus $\mathbb{D} \setminus (r_+(\wp)\overline{\mathbb{D}})$. The upper bound follows from Grötzsch's theorem [Ahl10, Section 4-11]. For details, see [4, Proposition 2.5]. By combining (7.1) and (7.4), we get

$$e^{-2\pi\,\mathrm{ED}(\partial\mathbb{D},\wp)}\,\mathrm{CR}(0, \mathrm{Int}(\wp))^{-1} \leq \frac{r_+(\wp)}{r_-(\wp)} \leq 16e^{-2\pi\,\mathrm{ED}(\partial\mathbb{D},\wp)}\,\mathrm{CR}(0, \mathrm{Int}(\wp))^{-1}. \tag{7.5}$$

The constant 16 above is likely non-optimal.

7.2 Presentation of results

Here we present the results obtained by Aru, Sepúlveda and myself in [4].

Consider the unit disk $\mathbb{D}$ and a CLE_4 conformal loop ensemble inside $\mathbb{D}$. Let $\wp_0$ be the CLE_4 loop surrounding 0, which exists a.s. We determined the joint law of $(\mathrm{ED}(\partial\mathbb{D}, \wp_0), \mathrm{CR}(0, \mathrm{Int}(\wp_0)))$. Because of the conformal invariance in law of the CLE_4, the particular choice of $\mathbb{D}$ and 0 does not matter, and one could have taken any other simply connected domain D and any point $z \in D$. The law of the conformal radius $\mathrm{CR}(0, \mathrm{Int}(\wp_0))$ alone has been determined by Sheffield in [She09]. He also gives the laws of the conformal radii for all the CLE_κ with $\kappa \in (8/3, 4]$. A further description of these laws appeared in [SSW09]. However, the laws for the corresponding extremal distances were unknown. So we answered this question for $\kappa = 4$, by relying on the coupling with the Gaussian free field.

Let $(W_t)_{t\geq 0}$ be a one-dimensional standard Brownian motion starting at 0. For $x > 0$, denote

$$\tau_{\pm x} = \inf\{t \geq 0 | |W_t| = x\}, \qquad \bar{\tau}_{\pm x} = \sup\{t \in [0, \tau_{\pm x}] | W_t = 0\}.$$

Theorem 7.3 ([4], Theorem 1.1)**.** *The following identity in law holds:*

$$\Big(\mathrm{ED}(\partial\mathbb{D}, \wp_0), -\frac{1}{2\pi}\log \mathrm{CR}(0, \mathrm{Int}(\wp_0))\Big) \overset{(law)}{=} (\bar{\tau}_{\pm 2\lambda}, \tau_{\pm 2\lambda}),$$

where $2\lambda = \sqrt{\pi/2}$ *is the height gap of the 2D continuum GFF* (2.1)*.*

So one can read $(\mathrm{ED}(\partial\mathbb{D}, \wp_0), \mathrm{CR}(0, \mathrm{Int}(\wp_0)))$ on a single 1D Brownian trajectory. Note that the fact that $\bar{\tau}_{\pm 2\lambda} < \tau_{\pm 2\lambda}$ a.s. is consistent with (7.2).

Recall the notations $r_-(\wp_0)$ and $r_+(\wp_0)$ (7.3). The tail asymptotic for $r_-(\wp_0)$ was known from the distortion bounds (7.1):

$$\lim_{r\to 0} \frac{\log \mathbb{P}(r_-(\wp_0) \leq r)}{|\log r|} = \lim_{r\to 0} \frac{\log \mathbb{P}(\mathrm{CR}(0, \mathrm{Int}(\wp_0)) \leq r)}{|\log r|} = \lim_{t\to +\infty} \frac{1}{2\pi t} \log \mathbb{P}(\tau_{\pm 2\lambda} \geq t) = -\frac{1}{8}.$$

Similar results hold for $r_+(\wp_0)$, resp. $r_+(\wp_0)/r_-(\wp_0)$, by applying the bounds (7.4), resp. (7.5), in combination with the tail asymptotics for $\bar{\tau}_{\pm 2\lambda}$, resp. $\tau_{\pm 2\lambda} - \bar{\tau}_{\pm 2\lambda}$. We have that

$$\lim_{t\to +\infty} \frac{1}{2\pi t} \log \mathbb{P}(\bar{\tau}_{\pm 2\lambda} \geq t) = \lim_{t\to +\infty} \frac{1}{2\pi t} \log \mathbb{P}(\tau_{\pm 2\lambda} \geq t) = -\frac{1}{8}.$$

As for $\tau_{\pm 2\lambda} - \bar{\tau}_{\pm 2\lambda}$, it is distributed as first hitting time of a Bessel 3 process starting at 0. Using that, one gets

$$\lim_{t\to +\infty} \frac{1}{2\pi t} \log \mathbb{P}(\tau_{\pm 2\lambda} - \bar{\tau}_{\pm 2\lambda} \geq t) = -\frac{1}{2}.$$

Corollary 7.4 ([4], Corollary 1.2)**.** *One has that*

$$\lim_{r\to 0} \frac{\log \mathbb{P}(r_+(\wp_0) \leq r)}{|\log r|} = -\frac{1}{8}, \qquad \lim_{r\to +\infty} \frac{\log \mathbb{P}(r_+(\wp_0)/r_-(\wp_0) \geq r)}{\log r} = -\frac{1}{2}.$$

We would like to mention that the situation for several nested CLE_4-s around 0 is more complicated. Say on has the first CLE_4 loop $\wp_0$ around 0. Then one samples a conditionally independent CLE_4 inside $\mathrm{Int}(\wp_0)$, and takes the loop surrounding 0, denoted $\wp_0^*$. There are then 5 r.v.s: $\mathrm{CR}(0, \mathrm{Int}(\wp_0))$, $\mathrm{CR}(0, \mathrm{Int}(\wp_0^*))$, $\mathrm{ED}(\partial\mathbb{D}, \wp_0)$, $\mathrm{ED}(\partial\mathbb{D}, \wp_0^*)$ and $\mathrm{ED}(\wp_0, \wp_0^*)$. Given the 1D Brownian motion $(W_t)_{t\geq 0}$ as above, consider the following times:

$$\tau = \tau_{\pm 2\lambda}, \bar{\tau} = \bar{\tau}_{\pm 2\lambda}, \qquad \tau^* = \inf\{t \geq \tau | |W_t - W_\tau| = 2\lambda\}, \qquad \bar{\tau}^* = \sup\{t \in [\tau, \tau^*] | W_t = W_\tau\}.$$

By iterating Theorem 7.3, we get the following identity in law:

$$\Big(\mathrm{ED}(\partial\mathbb{D},\wp_0), -\frac{1}{2\pi}\log \mathrm{CR}(0,\mathrm{Int}(\wp_0)), \mathrm{ED}(\wp_0,\wp_0^*), -\frac{1}{2\pi}\log \mathrm{CR}(0,\mathrm{Int}(\wp_0^*))\Big) \overset{(\text{law})}{=} (\bar{\tau},\tau,\bar{\tau}^*-\tau,\tau^*).$$

One can also show an other identity in law, which does not follow directly from Theorem 7.3 and requires additional inputs (see [4, Section 5.4]):

$$\Big(\mathrm{ED}(\partial\mathbb{D},\wp_0), \mathrm{ED}(\wp_0,\wp_0^*), \mathrm{ED}(\partial\mathbb{D},\wp_0^*), -\frac{1}{2\pi}\log \mathrm{CR}(0,\mathrm{Int}(\wp_0^*))\Big) \overset{(\text{law})}{=} (\bar{\tau},\bar{\tau}^*-\tau,\bar{\tau}^*,\tau^*).$$

So one can read two different combinations of 4 out of 5 r.v.s out of a single 1D Brownian trajectory. However, one can not have all the 5. Indeed, the couple

$$\Big(-\frac{1}{2\pi}\log \mathrm{CR}(0,\mathrm{Int}(\wp_0)), \mathrm{ED}(\partial\mathbb{D},\wp_0^*)\Big)$$

does not have the same joint law as $(\tau,\bar{\tau}^*)$, although the one-dimensional marginals match. Indeed, $\tau<\bar{\tau}^*$ a.s., whereas

$$\mathbb{P}\Big(-\frac{1}{2\pi}\log \mathrm{CR}(0,\mathrm{Int}(\wp_0)) > \mathrm{ED}(\partial\mathbb{D},\wp_0^*)\Big) > 0.$$

Indeed, with positive probability, both $\mathrm{CR}(0,\mathrm{Int}(\wp_0))$ and $\mathrm{ED}(\partial\mathbb{D},\wp_0^*)$ are simultaneously small, as on Figure 7.1 on the left.

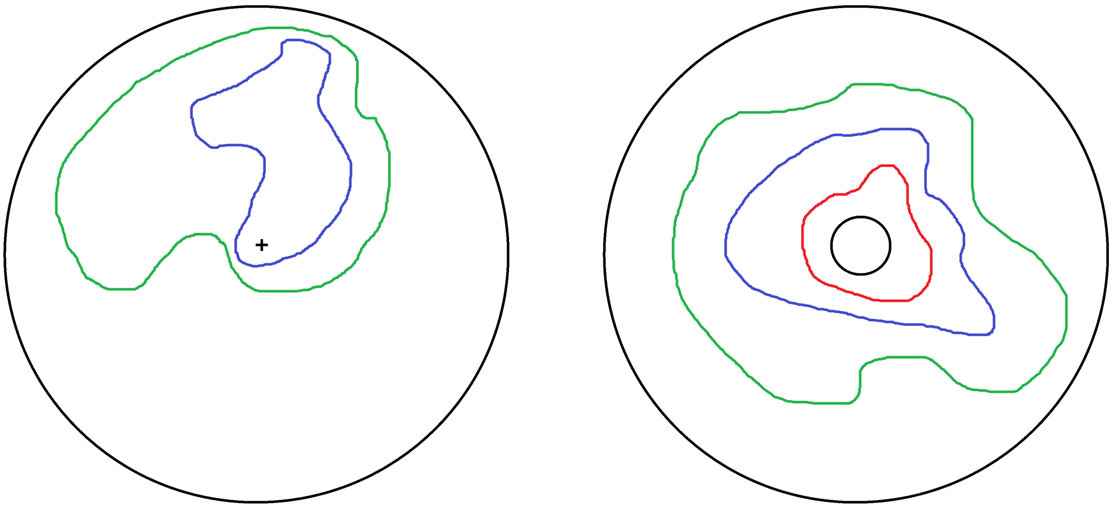

Figure 7.1: On the left, $\wp_0$ in green and $\wp_0^*$ in blue. On the right, some non-contractible interfaces of ϕ_{A_r}.

7.3 The law of the extremal distance alone

Let us first recall how the law of the conformal radius $\mathrm{CR}(0,\mathrm{Int}(\wp_0))$ can be obtained through the coupling of CLE_4 with the GFF (Theorem 2.2). Actually, there is a local set process (Definition 4.5) of the GFF discovering the CLE_4.

Let $\phi_{\mathbb{D}}$ be the massless GFF on $\mathbb{D}$ with 0 boundary conditions. Take an arbitrary $x\in\partial\mathbb{D}$. One takes $(\eta(t))_{t\geq 0}$ a radial $\mathrm{SLE}_4(-2)$ process in $\mathbb{D}$ starting from x an targeted at 0 (see [ASW19, Section 4] and [4, Section 2.6.1]), parametrized by the log conformal radius

$$t = -\frac{1}{2\pi}\log \mathrm{CR}(0,\mathbb{D}\setminus\eta([0,t])). \tag{7.6}$$

The process $(\eta(t))_{t\geq 0}$ can be coupled to $\phi_{\mathbb{D}}$ as a local set process. It draws CLE_4 loops in $\mathbb{D}$, up to a time t_0 when it draws the loop surrounding 0, that is to say $\wp_0$. The values of $\phi_{\mathbb{D}}$ discovered by $(\eta(t))_{t\geq 0}$ up to time t_0 belong to $\{-2\lambda, 0, 2\lambda\}$, and on the inner side of a loop closed by $(\eta(t))_{t\geq 0}$ the value is either -2λ or 2λ. With the notations of Section 4.1, $\phi^{\eta([0,t])}(0)$ denotes the value of the harmonic extension in 0 of the values of $\phi_{\mathbb{D}}$ discovered by $\eta([0,t])$. For $t \leq t_0$, $\phi^{\eta([0,t])}(0)$ is a mixture of -2λ, 0 and 2λ, and thus $\phi^{\eta([0,t])}(0) \in [-2\lambda, 2\lambda]$. For $t = t_0$, $\phi^{\eta([0,t_0])}(0)$ equals either -2λ or 2λ. According to Proposition 4.6, the process $(\phi^{\eta([0,t])}(0))_{t\geq 0}$ is distributed as a standard 1D Brownian motion starting at 0, and so t_0 has the same distribution $\tau_{\pm 2\lambda}$. Because of (7.6), we get the law of

$$\mathrm{CR}(0, \mathbb{D} \setminus \eta([0,t_0])) = \mathrm{CR}(0, \mathrm{Int}(\wp_0)).$$

To get the law of the extremal distance, the idea is to discover the CLE_4 loop $\wp_0$ by a local set process from inside rather than from outside. To make this rigorous, in [4] we cut a small hole around 0 and work on the annulus

$$A_r = \mathbb{D} \setminus (r\overline{\mathbb{D}}). \tag{7.7}$$

Eventually we let r tend to 0. Let

$$T(r) = \mathrm{ED}(\partial_o A_r, \partial_i A_r) = \frac{1}{2\pi}|\log r|.$$

Let ϕ_{A_r} be the massless GFF on A_r with 0 boundary conditions. First one takes a local set process $(\hat{\eta}(t))_{0\leq t\leq T(r)}$ for ϕ_{A_r} starting from $\partial_o A_r$ and ending on $\partial_i A_r$, parametrized by the extremal distance

$$t = \mathrm{ED}(\partial_o A_r \cup \hat{\eta}([0,t]), \partial_i A_r).$$

This local set process is a kind of annulus version of the radial $\mathrm{SLE}_4(-2)$. It discovers some non-contractible interfaces $\hat{\wp}^1, \hat{\wp}^2, \dots, \hat{\wp}^{\widehat{N}}$ ($\widehat{N}$ random) of ϕ_{A_r}. See Figure 7.1 on the right. These interfaces correspond to the jumps of -2λ or 2λ in the values of the GFF ϕ_{A_r}. Each $\hat{\wp}^j$ is a random SLE_4-type loop that separates $\partial_o A_r$ from $\partial_i A_r$, and each $\hat{\wp}^{j+1}$ is surrounded by $\hat{\wp}^j$. Moreover, as $r \to 0$ (i.e. A_r tends to $\mathbb{D}$), the law of the interface $\hat{\wp}^1$ (the first non-contractible interface discovered) converges in total variation to that of the CLE_4 loop $\wp_0$. By using a generalization of Proposition 4.6 (see [4, Proposition 2.18]), one gets the joint law of the extremal distances $(\mathrm{ED}(\hat{\wp}^j, \partial_i A_r))_{1\leq j\leq \widehat{N}}$ in terms of stopping times of a 1D Brownian bridge from 0 to 0 of duration $T(r)$. Let $(\widehat{W}_t)_{0\leq t\leq T(r)}$ be this 1D Brownian bridge associated with the local set process $(\hat{\eta}(t))_{0\leq t\leq T(r)}$ and let

$$\hat{\tau} = \inf\{t \in [0, T(r)] | |\widehat{W}_t| = 2\lambda\}.$$

The event that $(\widehat{W}_t)_{0\leq t\leq T(r)}$ does not exit $(-2\lambda, 2\lambda)$ coincides with the event that $\widehat{N} = 0$, i.e. $(\hat{\eta}(t))_{0\leq t\leq T(r)}$ does not discover any non-contractible interface. On the event that $(\widehat{W}_t)_{0\leq t\leq T(r)}$ exits $(-2\lambda, 2\lambda)$, we have that $T(r) - \hat{\tau} = \mathrm{ED}(\hat{\wp}^1, \partial_i A_r)$. Moreover $T(r) - \mathrm{ED}(\hat{\wp}^1, \partial_i A_r)$ converges in law as $r \to 0$ towards $-\frac{1}{2\pi}\log \mathrm{CR}(0, \mathrm{Int}(\wp_0))$. So one can rederive in this way the law of $\mathrm{CR}(0, \mathrm{Int}(\wp_0))$.

This is not however what we want. So instead we interchange the roles of $\partial_o A_r$ and $\partial_i A_r$ and take a local set process $(\check{\eta}(t))_{0\leq t\leq T(r)}$ from $\partial_i A_r$ to $\partial_o A_r$. It successively discovers non-contractible interfaces for ϕ_{A_r}: $\check{\wp}^1, \check{\wp}^2, \dots, \check{\wp}^{\check{N}}$ ($\check{N}$ random). The local set process $(\check{\eta}(t))_{0\leq t\leq T(r)}$ is again associated with a 1D Brownian bridge $(\check{W}_t)_{0\leq t\leq T(r)}$ from 0 to 0, and one can jointly read the extremal distances $(\mathrm{ED}(\partial_o A_r, \check{\wp}^j))_{1\leq j\leq \check{N}}$ on this bridge. In particular, for the last non-contractible interface discovered, $\check{\wp}^{\check{N}}$, we have that

$$\mathrm{ED}(\partial_o A_r, \check{\wp}^{\check{N}}) = T(r) - \check{\tau},$$

where

$$\check{\tau} = \inf\{t \in [0, T(r)] | \check{W}_t = 0, (\check{W}_s)_{0\leq s\leq t} \text{ exits } (-2\lambda, 2\lambda), (\check{W}_s)_{t\leq s\leq T(r)} \text{ stays in } (-2\lambda, 2\lambda)\}.$$

Now the key point is the following.

Theorem 7.5 ([4], Theorem 4.4)**.** *A.s.* $\check{N} = \widehat{N}$ *and*

$$(\check{\wp}^1, \check{\wp}^2, \dots, \check{\wp}^{\check{N}}) = (\hat{\wp}^{\widehat{N}}, \hat{\wp}^{\widehat{N}-1}, \dots, \hat{\wp}^1).$$

The theorem above says that one can discover the same non-contractible interfaces of the GFF ϕ_{A_r} by local set processes both from the outer boundary $\partial_o A_r$ and from the inner boundary $\partial_i A_r$. We call this *reversibility*. This property is both intuitive and rather tricky to prove rigorously, because the 2D continuum GFF is a generalized function not defined pointwise, so one does not really observe its interfaces. The proof of the reversibility property constitutes actually the main hard point in [4].

Since one knows the law of $\mathrm{ED}(\partial_o A_r, \check{\wp}^{\check{N}})$ and $\check{\wp}^{\check{N}} = \hat{\wp}^1$ a.s., one gets the law of $\mathrm{ED}(\partial_o A_r, \hat{\wp}^1)$. By letting $r \to 0$, one also gets the law of $\mathrm{ED}(\partial_o A_r, \wp_0)$. However, this method does not provide the joint law of $(\mathrm{ED}(\partial_o A_r, \hat{\wp}^1), \mathrm{ED}(\hat{\wp}^1, \partial_i A_r))$, just the one-dimensional marginals. This is because the Brownian bridge $(\check{W}_t)_{0\leq t\leq T(r)}$ associated with $(\check{\eta}(t))_{0\leq t\leq T(r)}$ is **not** a.s. equal to the time-reversal of the bridge $(\widehat{W}_t)_{0\leq t\leq T(r)}$ associated with $(\hat{\eta}(t))_{0\leq t\leq T(r)}$. So an additional input is required to get the joint law in Theorem 7.3.

7.4 The joint law of conformal radius and extremal distance

At this stage we obtained the law of $\mathrm{CR}(0, \mathrm{Int}(\wp_0))$ and that of $\mathrm{ED}(\partial\mathbb{D}, \wp_0)$, but not yet the joint law. We will again work on the annulus A_r (7.7), but we will require an additional degree of freedom. We will consider constants $v \in \mathbb{R}$, and GFFs $\phi_{A_r}^{(v)}$ on A_r with boundary condition 0 on $\partial_o A_r$ and v on $\partial_i A_r$. With some positive probability, $\phi_{A_r}^{(v)}$ has a non-contractible interface $\hat{\wp}^{1,v}$, with a value $a^{(v)} \in \{-2\lambda, 2\lambda\}$ on the inner side of $\hat{\wp}^{1,v}$. If $v = 0$, this interface is just $\hat{\wp}^1$. Let $(\widehat{W}_t^{(v)})_{0\leq t\leq T(r)}$ be a 1D Brownian bridge of duration $T(r)$ from 0 to v, and denote

$$\hat{\tau}^{(v)} = \inf\{t \in [0, T(r)] | |\widehat{W}_t^{(v)}| = 2\lambda\}, \qquad \hat{\hat{\tau}}^{(v)} = \sup\{t \in [0, \hat{\tau}^{(v)}] | \widehat{W}_t^{(v)} = 0\}.$$

Similarly to the case $v = 0$, we have that

$$\mathbb{P}(\text{The interface } \hat{\wp}^{1,v} \text{ exists}) = \mathbb{P}(\widehat{W}^{(v)} \text{ exits } (-2\lambda, 2\lambda)),$$

$$(\mathrm{ED}(\hat{\wp}^{1,v}, \partial_i A_r), a^{(v)}) \overset{(\text{law})}{=} (T(r) - \hat{\tau}^{(v)}, \widehat{W}_{\hat{\tau}^{(v)}}^{(v)}), \qquad (\mathrm{ED}(\partial_o A_r, \hat{\wp}^{1,v}), a^{(v)}) \overset{(\text{law})}{=} (\hat{\hat{\tau}}^{(v)}, \widehat{W}_{\hat{\tau}^{(v)}}^{(v)}),$$

but we do net get directly the joint law of $(\mathrm{ED}(\partial_o A_r, \hat{\wp}^{1,v}), \mathrm{ED}(\hat{\wp}^{1,v}, \partial_i A_r), a^{(v)})$. The idea is to look at how the law of the interface $\hat{\wp}^{1,v}$ and the value $a^{(v)}$ change changes depending on v. Let $\mathbb{Q}^{(v)}$ denote the measure

$$\mathbb{Q}^{(v)}(F) = \mathbb{E}\big[F(\hat{\wp}^{1,v}, a^{(v)})\mathbf{1}_{\hat{\wp}^{1,v} \text{ exists}}\big].$$

The measure $\mathbb{Q}^{(v)}$ is absolutely continuous w.r.t. $\mathbb{Q}^{(0)}$ with the following Radon-Nikodym derivative.

Proposition 7.6 ([4], Proposition 3.3)**.** *Let $v \in \mathbb{R}$. The following absolute continuity holds:*

$$\frac{d\mathbb{Q}^{(v)}}{d\mathbb{Q}^{(0)}} = \exp\Big(-\frac{v^2}{2}(\mathrm{ED}(\hat{\wp}^{1,0}, \partial_i A_r)^{-1} - T(r)^{-1}) + a^{(0)} v\,\mathrm{ED}(\hat{\wp}^{1,0}, \partial_i A_r)^{-1}\Big). \tag{7.8}$$

The important point is that the Radon-Nikodym derivative (7.8) is a function of only $\mathrm{ED}(\hat{\wp}^{1,0}, \partial_i A_r)$ and $a^{(0)}$. This immediately implies the following.

Corollary 7.7. *The conditional law of $\hat{\wp}^{1,v}$ given $(\mathrm{ED}(\hat{\wp}^{1,v}, \partial_i A_r), a^{(v)})$ is the same whatever the value of v. In particular, the conditional law of $\mathrm{ED}(\partial_o A_r, \hat{\wp}^{1,v})$ given $(\mathrm{ED}(\hat{\wp}^{1,v}, \partial_i A_r), a^{(v)})$ is the same whatever the value of v.*

Then we use the following characterization of the joint law, which is a result purely on 1D Brownian bridges.

Proposition 7.8 ([4], Proposition 5.4)**.** *Assume that for every $v \in \mathbb{R}$ there is a triple of random variables $(\rho_o^{(v)}, \rho_i^{(v)}, \theta^{(v)})$ with $\rho_o^{(v)} \in [0, T(r)]$, $\rho_i^{(v)} \in [0, T(r)]$, $\rho_o^{(v)} + \rho_i^{(v)} \leq T(r)$ and $\theta^{(v)} \in \{-2\lambda, 2\lambda\}$. Assume moreover that the following conditions are satisfied:*

1. *For every $v \in \mathbb{R}$, the joint law of $(\rho_i^{(v)}, \theta^{(v)})$ on the event $\rho_i^{(v)} > 0$ is the same as that of $(T(r) - \hat{\tau}^{(v)}, \widehat{W}_{\hat{\tau}^{(v)}})$ on the event $\hat{\tau}^{(v)} < T(r)$.*
2. *For every $v \in \mathbb{R}$, the joint law of $(\rho_o^{(v)}, \theta^{(v)})$ on the event $\rho_i^{(v)} > 0$ is the same as that of $(\hat{\hat{\tau}}^{(v)}, \widehat{W}_{\hat{\tau}^{(v)}})$ on the event $\hat{\tau}^{(v)} < T(r)$.*
3. *For every $\sigma \in \{-1, 1\}$ and Lebesgue almost every $\rho \in (0, T(r))$, the conditional law of $\rho_o^{(v)}$ given that $\rho_i^{(v)} = \rho$ and $\theta^{(v)} = \sigma 2\lambda$ is the same whatever $v \in \mathbb{R}$.*

Then for every $v \in \mathbb{R}$, the joint law of $(\rho_o^{(v)}, \rho_i^{(v)}, \theta^{(v)})$ on the event $\rho_i^{(v)} > 0$ is the same as that of $(\hat{\hat{\tau}}^{(v)}, T(r) - \hat{\tau}^{(v)}, \widehat{W}_{\hat{\tau}^{(v)}})$ on the event $\hat{\tau}^{(v)} < T(r)$.

The above characterization implies that for every $v \in \mathbb{R}$,

$$(\mathrm{ED}(\partial_o A_r, \hat{\wp}^{1,v}), \mathrm{ED}(\hat{\wp}^{1,v}, \partial_i A_r), a^{(v)}) \overset{\text{(law)}}{=} (\hat{\hat{\tau}}^{(v)}, \hat{\tau}^{(v)}, \widehat{W}^{(v)}_{\hat{\tau}^{(v)}}).$$

By letting $r \to 0$ with v fixed, we get Theorem 7.3.

Chapter 8

Lévy-type transformation for the GFF

In this Chapter is presented an article written in collaboration with Wendelin Werner [11]. The motivation for our work is explained in Section 8.1. It has to do with a second coupling between the CLE_4 and the 2D continuum GFF (Theorem 8.1), different but related to the Miller-Sheffield coupling (Theorem 2.2). In [11] we show that the relation between the two couplings has a natural analogue on metric graphs. This is explained in Section 8.2. We show that the Lévy transformation for one-dimensional Brownian motions can be generalized to massless metric graph GFFs, and that the law of the latter is invariant under this transformation (Theorem 8.2). We further conjecture that in the 2D setting, this Lévy transformation has a continuum limit, which actually relates the two couplings between the CLE_4 and the 2D continuum GFF. In Section 8.3 are explained the ideas behind the proof of Theorem 8.2. It mainly relies on stochastic calculus on metric graphs. In Section 8.4 are additionally presented some exact laws related to the Lévy transformation on metric graphs.

8.1 Motivation: a second coupling between the CLE_4 and the GFF

Let us first explain the motivation behind the work in [11]. Let $D \subset \mathbb{C}$ be an open simply connected domain, $D \neq \mathbb{C}$. We have already seen a coupling due to Miller and Sheffield (Theorem 2.2) between the CLE_4 conformal loop ensemble in D and the massless continuum GFF on D with 0 boundary conditions. There is actually an other coupling between these two objects, and we will refer to it as the *second coupling*. It is due to Werner and Wu [WW13b].

There is a conformally invariant way to explore all the CLE_4 loops from the boundary ∂D through a growth dynamic introduced in [WW13b]. In this way each CLE_4 loop $\wp$ comes with a random time $t_\wp > 0$, which is the moment at which the growth dynamic reaches it. The labeled family $(\wp, t_\wp)_{\wp \in CLE_4}$ is conformally invariant in law. The label $t_\wp$ is a sort of distance between $\wp$ and ∂D; not the Euclidean distance of course, but a random one depending on the whole CLE_4 loop family.

Theorem 8.1 (Werner-Wu)**.** *Let $\mathfrak{C}$ be a CLE_4 loop ensemble in a simply connected domain D. Consider the random labels $t_\wp$ for $\wp \in \mathfrak{C}$ obtained through the growth dynamic of [WW13b]. Let $(\phi_\wp)_{\wp \in \mathfrak{C}}$ be a family of generalized fields, conditionally independent given $(\wp, t_\wp)_{\wp \in \mathfrak{C}}$, with the conditional distribution of $\phi_\wp$ being that of a massless free field in* $\mathrm{Int}(\wp)$ *with* 0 *boundary* *conditions on $\wp$. Let ϕ_D be the field in D given by*

$$\phi_D = \sum_{\wp \in \mathfrak{C}} \mathbf{1}_{\mathrm{Int}(\wp)} (\phi_\wp + 2\lambda - t_\wp),$$

where 2λ is the height gap 2.1. Then ϕ_D is distributed as the massless GFF in D with 0 boundary conditions on ∂D.

The coupling above looks similar to the Miller-Sheffield coupling (Theorem 2.2), with the difference that instead of the boundary condition -2λ or 2λ on the inner side of $\wp$, with probability $1/2$ each, one takes for boundary condition $2\lambda - t_\wp$. For details, see also [WW17, Section 3]. We shall see in the next section that this second coupling has a natural interpretation through the metric graphs.

8.2 Lévy transformation on metric graphs

The relation between the Miller-Sheffield coupling (Theorem 2.2) and the second coupling (Theorem 8.1) is analogous the the Lévy transformation for the 1D Brownian motion. Let us explain this analogy.

Let $(W_t)_{t\geq 0}$ be a standard 1D Brownian motion with $W_0 = 0$. Let $\ell_t^x(W)$ denote the local times of $(W_t)_{t\geq 0}$; see [RY99, Chapter VI]. Let $(\bar{W}_t)_{t\geq 0}$ be the following process:

$$\bar{W}_t = |W_t| - \ell_t^0(W). \tag{8.1}$$

The procedure (8.1) is known as the *Lévy transformation*, and the resulting process $(\bar{W}_t)_{t\geq 0}$ is again a Brownian motion; see [RY99, Chapter VI, Theorem 2.3]. For every $t \geq 0$,

$$\inf_{[0,t]} \bar{W} = -\ell_t^0(W). \tag{8.2}$$

If $(W_t)_{t_1 \leq t \leq t_r}$ is an excursion of W away from 0, positive or negative, then $(\bar{W}_t)_{t_1 \leq t \leq t_r}$ is a positive excursion of $\bar{W}$ above the negative level $\inf_{[0,t_1]} \bar{W}$. So in a sense one takes the excursions of W away from 0, reflects the negative ones so as to make them positive, and glues the excursions in the same order but in a different way, above negative levels given by the process (8.2). This is analogous to taking boundary conditions $\pm 2\lambda$ in Theorem 2.2, reflecting -2λ to 2λ, and then subtracting a level $t_\wp$, so as to get the second coupling of Theorem 8.1.

The Lévy transformation (8.1) has a generalization to 1D Brownian bridges. Let $T > 0$ and let $(\widehat{W}_t)_{0\leq t \leq T}$ be a standard Brownian bridge from 0 to 0 of duration T. Let $\ell_t^x(\widehat{W})$ be the local times of $(\widehat{W}_t)_{0\leq t\leq T}$:

$$\ell_t^x(\widehat{W}) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{|\widehat{W}_s - x| < \varepsilon} ds.$$

For $t \in [0, T]$, denote

$$\hat{\delta}_t = \ell_t^0(\widehat{W}) \wedge (\ell_T^0(\widehat{W}) - \ell_t^0(\widehat{W})).$$

Then the process $(|\widehat{W}_t| - \hat{\delta}_t)_{0\leq t\leq T}$ is again distributed as a 1D Brownian bridge from 0 to 0. For a proof, see [BP94, Theorem 4.1].

In the article [11] Werner and myself showed that the Lévy transformation admits a generalization to GFFs on any metric graph. Consider a discrete electrical network $\mathcal{G} = (V, E)$ with conductances $(C(x,y))_{x\sim y}$, as in Section 1.1.1. Recall that V_∂ is a special subset of V considered as the boundary. Let $\widetilde{\mathcal{G}}$ be the metric graph associated with $\mathcal{G}$; see Definition 3.3. Let $f : V_\partial \to \mathbb{R}_+$ be a **non-negative** boundary condition. Let $\tilde{\phi}$ be the massless metric graph

GFF on $\widetilde{\mathcal{G}}$ with boundary condition f; see Definition 3.7. Given $(\wp(t))_{0\leq t\leq T(\wp)}$ a continuous path in $\widetilde{\mathcal{G}}$ such that its derivative $\wp'(t)$ is defined besides at most finitely many times t for which $\wp(t)\in V$, and is bounded, one can define

$$\ell^0(\tilde{\phi},\wp) = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^{T(\wp)} \mathbf{1}_{|\tilde{\phi}(\wp(t))|<\varepsilon}|\wp'(t)|dt.$$

Indeed, as $\wp(t)$ moves inside an edge-line, the field $\tilde{\phi}$ on the edge-line is locally more or less a 1D Brownian motion, and it admits a local time process. So in $\ell^0(\tilde{\phi},\wp)$ one essentially sums local times at level 0 of different 1D Brownian motions. Note that $\ell^0(\tilde{\phi},\wp)$ is invariant under reparametrizations of the path $\wp$. Given $x,y\in\widetilde{\mathcal{G}}$, define

$$\tilde{\delta}(x,y) = \inf_{\substack{\wp \text{ path in } \widetilde{\mathcal{G}} \\ \text{from } x \text{ to } y}} \ell^0(\tilde{\phi},\wp).$$

Then $\tilde{\delta}$ defined in this way is a random pseudo-metric on $\widetilde{\mathcal{G}}$. Given $x,y\in\widetilde{\mathcal{G}}$, one has $\tilde{\delta}(x,y)=0$ if and only if either $x=y$ or there is a continuous path $(\wp(t))_{0\leq t\leq T(\wp)}$ joining x and y such that $\tilde{\phi}(\wp(t))\neq 0$ for every $t\in(0,T(\wp))$. In other words, the pseudo-metric $\tilde{\delta}$ identifies all the points in the topological closure of a sign cluster of $\tilde{\phi}$.

Further, for $x\in\widetilde{\mathcal{G}}$, define

$$\tilde{\delta}(x,V_\partial) = \inf_{y\in V_\partial}\tilde{\delta}(x,y).$$

So $\tilde{\delta}(x,V_\partial)$ is the pseudo-distance from x to the boundary V_∂. Note that the function $x\mapsto\tilde{\delta}(x,V_\partial)$ is continuous on $\widetilde{\mathcal{G}}$ and constant on each sign cluster of $\tilde{\phi}$. Define $\hat{\phi}$ to be the following field on $\widetilde{\mathcal{G}}$:

$$\hat{\phi}(x) = |\tilde{\phi}(x)| - \tilde{\delta}(x,V_\partial). \tag{8.3}$$

We see (8.3) as a Lévy transformation of the metric graph GFF, generalizing the Lévy transformation for a 1D Brownian motion or Brownian bridge. See Figure 8.1. In [11] we prove the following.

Theorem 8.2 ([11], Proposition 1). *Let $\tilde{\phi}$ be a massless metric graph GFF on $\widetilde{\mathcal{G}}$ with a* ***non-negative*** *boundary condition f on V_∂. Then the field $\hat{\phi}$ defined by* (8.3) *has the same distribution as $\tilde{\phi}$.*

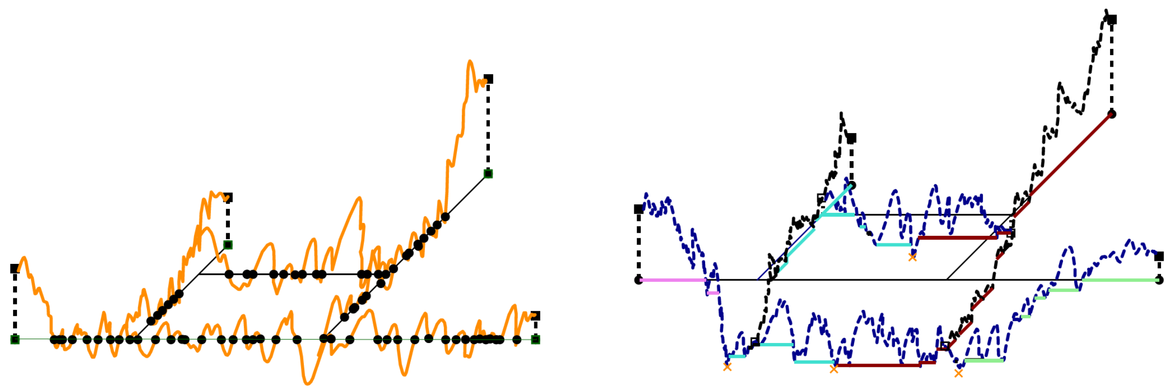

Figure 8.1: On the left in orange, the metric graph GFF $\tilde{\phi}$. The dots represent its zero set. On the right, the field $\hat{\phi}$ obtained through Lévy transformation of $\tilde{\phi}$.

We would like to emphasize that Theorem 8.2 works on any metric graph and there is no planarity involved. The ideas behind the proof of Theorem 8.2 will be presented in the next section. For now, let us discuss how Theorem 8.2 enlightens the coupling of Theorem 8.1. Let $D \subset \mathbb{C}$, $D \neq \mathbb{C}$, be an open simply connected domain. Let $\widetilde{D}_N$ be a metric graph approximation of D obtained from the square lattice wish mesh size N^{-1}. Let $\tilde{\phi}_N$ be the massless metric graph GFF on $\widetilde{D}_N$ with 0 boundary conditions, approximating a continuum GFF on D. Let $\tilde{\delta}_N$ the level 0 local time pseudo-metric on $\widetilde{D}_N$ associated with $\tilde{\phi}_N$ and let $\hat{\phi}_N$ denote the Lévy transformation of $\tilde{\phi}_N$. Given $\widetilde{\mathcal{C}}$ a sign cluster of $\tilde{\phi}_N$, we have that for every $x, y \in \widetilde{\mathcal{C}}$, $\tilde{\delta}_N(x, \partial\widetilde{D}_N) = \tilde{\delta}_N(y, \partial\widetilde{D}_N)$, and thus we can denote by $\tilde{\delta}_N(\widetilde{\mathcal{C}}, \partial\widetilde{D}_N)$ this common value. We know that the outermost boundaries of the outermost sign clusters of $\tilde{\phi}_N$ converge in law to the CLE_4 in D (Theorem 3.11). The quantity $\tilde{\delta}_N(\widetilde{\mathcal{C}}, \partial\widetilde{D}_N)$ is the metric graph analogue of the random labels $t_\wp$ for $\wp$ a loop in the CLE_4. More precisely, we formulate in [11] the following conjecture.

Conjecture 8.3 ([11], Conjecture 15)**.** *Let $\mathsf{C}_{N,o}$ denote the collection of outermost sign clusters of $\tilde{\phi}_N$, i.e. not surrounded by other. For $\widetilde{\mathcal{C}} \in \mathsf{C}_{N,o}$, let $\partial_o\widetilde{\mathcal{C}}$ denote the outer boundary of $\widetilde{\mathcal{C}}$. Then as $N \to +\infty$, the family $(\partial_o\widetilde{\mathcal{C}}, \tilde{\delta}_N(\widetilde{\mathcal{C}}, \partial\widetilde{D}_N))_{\widetilde{\mathcal{C}}\in\mathsf{C}_{N,o}}$ converges in law towards the labeled family $(\wp, t_\wp)_{\wp\in\mathfrak{C}}$ where $\mathfrak{C}$ is a CLE_4 in D. Moreover, the whole pseudo-metric $\tilde{\delta}_N$ converges in law to a random pseudo-metric δ_D on $\overline{D}$, coupled to a continuum GFF ϕ_D, where δ_D identifies the points on a same excursion set of ϕ_D.*

The proof of the conjecture above is somewhere on my to-do list. It tells that the Lévy transformation extends to the continuum setting in dimension 2.

8.3 Method: stochastic calculus on metric graphs

Our proof of Theorem 8.2 relies on stochastic calculus on metric graphs. So let $\widetilde{\mathcal{G}}$ be this metric graph and $\tilde{\phi}$ the GFF on it. We will consider continuous *Markovian explorations* of $\tilde{\phi}$, analogous to local set processes used in continuum (Definition 4.5). Let $(\widetilde{K}(t))_{0\leq t\leq T}$ be a family of continuously growing random compact subsets of $\widetilde{\mathcal{G}}$. We assume moreover the following.

- $\widetilde{K}(0) = V_\partial$.
- $\widetilde{K}(T) = \widetilde{\mathcal{G}}$.
- For every $t \in [0, T]$, each connected component of $\widetilde{K}(t)$ intersects V_∂.
- For every $t \in [0, T]$ and every U deterministic open subset of $\widetilde{\mathcal{G}}$, the event $\widetilde{K}(t) \subset U$ is measurable w.r.t. $\mathbf{1}_U\tilde{\phi}$.

Let $\partial\widetilde{K}(t)$ denote the boundary of $\widetilde{K}(t)$ as a subset of $\widetilde{\mathcal{G}}$. Note that $\partial\widetilde{K}(0) = \widetilde{K}(0) = V_\partial$ and $\partial\widetilde{K}(T) = \partial\widetilde{\mathcal{G}} = \emptyset$, and that the subset $\partial\widetilde{K}(t)$ is always finite. Let $\mathcal{T}$ be the subset of $[0, T]$ made of moments t at which $\widetilde{K}(t)$ finishes exploring one or several edge-lines I_e for $e \in E$, which happens either because a particle in $\widetilde{K}(t)$ finishes crossing I_e and reaches its end-vertex, or because two different particles in $\widetilde{K}(t)$ traveling from opposite sides of I_e meet somewhere inside I_e. Note that by construction, the subset $\mathcal{T}$ is a.s. finite, with $|\mathcal{T}| \leq |E|$. For intervals of time $J \subset [0, T] \setminus \mathcal{T}$, one can enumerate the elements of $\partial\widetilde{K}(t)$ (with $t \in J$): $(X_1(t), X_2(t), \dots, X_{n(t)}(t))$, with each $X_i(t)$ evolving continuously and $n(t)$ constant on J. Each $X_i(t)$ evolves inside an edge-line I_{e_i}, and we will denote by $r_i(t)$ the length of the interval $I_{e_i} \cap \widetilde{K}(t)$ which is non-decreasing in t. Next we write down the stochastic differential equation satisfied by $(\tilde{\phi}(X_1(t)), \tilde{\phi}(X_2(t)), \dots, \tilde{\phi}(X_{n(t)}(t)))$ for $t \in J$.

Theorem 8.4 ([11], Section 2.2). *With the notations above, let $\tilde{\phi}$ be a massless metric graph GFF on $\widetilde{\mathcal{G}}$ and $(\widetilde{K}(t))_{0\le t\le T}$ a Markovian exploration of $\tilde{\phi}$. Let $(\mathcal{F}_t)_{0\le t\le T}$ be the natural filtration of $(\widetilde{K}(t), \mathbf{1}_{\widetilde{K}(t)}\tilde{\phi})_{0\le t\le T}$. Then on the time intervals $J \subset [0,T]\setminus\mathcal{T}$, the following SDE is satisfied:*

$$d\tilde{\phi}(X_i(t)) = dM_i(t) + \sum_{\substack{1\le j\le n(t)\\ j\ne i}} H_{\widetilde{\mathcal{G}}\setminus\widetilde{K}(t)}(X_i(t), X_j(t))(\tilde{\phi}(X_j(t)) - \tilde{\phi}(X_i(t)))dr_i(t), \quad 1 \le i \le n(t), \tag{8.4}$$

where $H_{\widetilde{\mathcal{G}}\setminus\widetilde{K}(t)}$ denotes the boundary Poisson kernel on $\widetilde{\mathcal{G}} \setminus \widetilde{K}(t)$, and each M_i is a continuous martingale in the filtration $(\mathcal{F}_t)_{0\le t\le T}$, with the quadratic covariations given by

$$d\langle M_i, M_j\rangle_t = \mathbf{1}_{i=j} dr_i(t).$$

Conversely, if $\tilde{\phi}$ is a continuous random field on $\widetilde{\mathcal{G}}$ that admits a Markovian exploration $(\widetilde{K}(t))_{0\le t\le T}$ along which the SDE (8.4) is satisfied, then $\tilde{\phi}$ is a massless metric graph GFF on $\widetilde{\mathcal{G}}$.

However, an arbitrary Markovian exploration $(\widetilde{K}(t))_{0\le t\le T}$ of the GFF $\tilde{\phi}$ is not necessarily a Markovian exploration of $\hat{\phi}$. Actually, the variables $\tilde{\delta}(X_i(t), V_\partial)$ are not necessarily $\mathcal{F}_t$-measurable, since there may be a $\tilde{\delta}$-shorter path from $X_i(t)$ to V_∂ than that discovered by $\widetilde{K}(t)$. So the idea is to take a Markovian exploration $(\widetilde{K}(t))_{0\le t\le T}$ of $\tilde{\phi}$ satisfying the following additional property:

$$\forall t \in [0,T], \forall x, y \in \partial\widetilde{K}(t), \tilde{\delta}(x, V_\partial) = \tilde{\delta}(y, V_\partial).$$

The construction of such explorations is explained in [11, Section 2.3]. If moreover the boundary conditions of $\tilde{\phi}$ on V_∂ are non-negative, then it is easy to see that $(\widetilde{K}(t))_{0\le t\le T}$ is a Markovian exploration for the field $\hat{\phi}$ too, and that the natural filtration of $(\widetilde{K}(t), \mathbf{1}_{\widetilde{K}(t)}\tilde{\phi})_{0\le t\le T}$ and that of $(\widetilde{K}(t), \mathbf{1}_{\widetilde{K}(t)}\hat{\phi})_{0\le t\le T}$ coincide. Moreover, for every $t \in [0,T]$ and every $x, y \in \partial\widetilde{K}(t)$, such that $\tilde{\phi}(x) \ne 0$, $\tilde{\phi}(y) \ne 0$, the following holds:

$$\operatorname{sign}(\tilde{\phi}(x)) = \operatorname{sign}(\tilde{\phi}(y)).$$

Thus, by Tanaka's formula [RY99, Chapter VI, Theorem 1.2], on time intervals $J \subset [0,T] \setminus \mathcal{T}$, we have that

$$\begin{aligned} d\hat{\phi}(X_i(t)) &= \operatorname{sign}(\tilde{\phi}(X_i(t)))dM_i(t) \\ &\quad + \operatorname{sign}(\tilde{\phi}(X_i(t))) \sum_{\substack{1\le j\le n(t)\\ j\ne i}} H_{\widetilde{\mathcal{G}}\setminus\widetilde{K}(t)}(X_i(t), X_j(t))(\tilde{\phi}(X_j(t)) - \tilde{\phi}(X_i(t)))dr_i(t) \\ &= \operatorname{sign}(\tilde{\phi}(X_i(t)))dM_i(t) + \sum_{\substack{1\le j\le n(t)\\ j\ne i}} H_{\widetilde{\mathcal{G}}\setminus\widetilde{K}(t)}(X_i(t), X_j(t))(|\tilde{\phi}(X_j(t))| - |\tilde{\phi}(X_i(t))|)dr_i(t) \\ &= d\widehat{M}_i(t) + \sum_{\substack{1\le j\le n(t)\\ j\ne i}} H_{\widetilde{\mathcal{G}}\setminus\widetilde{K}(t)}(X_i(t), X_j(t))(\hat{\phi}(X_j(t)) - \hat{\phi}(X_i(t)))dr_i(t), \quad 1 \le i \le n(t), \end{aligned}$$

where $\widehat{M}_i$ are martingales satisfying

$$d\widehat{M}_i(t) = \operatorname{sign}(\tilde{\phi}(X_i(t)))dM_i(t).$$

By applying Theorem 8.4, we get that the field $\hat{\phi}$ is distributed as a GFF on $\widetilde{\mathcal{G}}$. This proves Theorem 8.2.

8.4 Some exact identities and invariances under rewiring

So one has a metric graph GFF $\tilde{\phi}$ on $\widetilde{\mathcal{G}}$, with non-negative boundary conditions on V_∂, and its Lévy transformation $\hat{\phi}$ (8.3). Moreover $\hat{\phi} \overset{\text{(law)}}{=} \tilde{\phi}$. Let $a > 0$. By construction,

$$\{x \in \widetilde{\mathcal{G}} | \tilde{\delta}(x, V_\partial) \leq a\} = \{x \in \widetilde{\mathcal{G}} | \exists \wp \text{ continuous path from } V_\partial \text{ to } x \text{ s.t. } \hat{\phi} \geq -a \text{ on } \wp\}. \quad (8.5)$$

On the left appears the closed a-neighborhood of V_∂ for the pseudo-metric $\tilde{\delta}$. On the right is nothing else than the first passage set of level $-a$ of the field $\hat{\phi}$; see Definition 4.10. So in Theorem 4.11 we have already given the explicit law of the electric resistance from a fixed point $x \in \widetilde{\mathcal{G}}$ to the subset (8.5), including the value of the probability that x belongs to the subset (8.5). It is remarkable that this law takes as parameter only the quantity $G_{\widetilde{\mathcal{G}}}(x, x)$, where $G_{\widetilde{\mathcal{G}}}$ is the Green's function on the metric graph $\widetilde{\mathcal{G}}$. The quantity $G_{\widetilde{\mathcal{G}}}(x, x)$ can be interpreted in electrical terms as the effective resistance from the point x to V_∂. So it is invariant under potential-preserving rewirings of the electrical network, such as the star-triangle transformation, very classical in electrical engineering; see Figure 8.2. To prove Theorem 4.11, we again use in [11] the stochastic calculus on metric graphs. See [11, Section 3] for details.

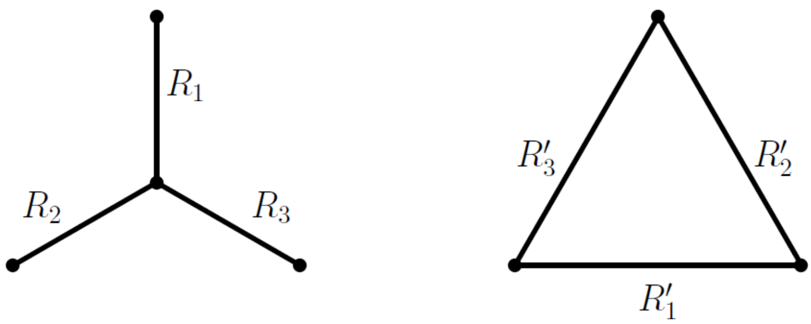

Figure 8.2: Star triangle transformation. R_i-s and R'_i-s denote resistances. The condition to obtain an equivalent electrical circuit is $R_i R'_i = R_i R_j + R_j R_k + R_k R_i$, $1 \leq i \leq 3$.

We also obtain in [11] an other explicit law that is invariant under potential-preserving rewirings of the electrical network, too. Assume that the boundary V_∂ is made of two disjoint parts,

$$V_\partial = V_{\partial,1} \amalg V_{\partial,2},$$

where $V_{\partial,1}$ and $V_{\partial,2}$ are both non-empty. The boundary condition f is assumed to have constant sign (positive or negative) on $V_{\partial,1}$ and constant sign on $V_{\partial,2}$. f is also allowed to have some 0 values on part or whole $V_{\partial,1}$ and on part or whole $V_{\partial,2}$. Denote

$$\tilde{\delta}(V_{\partial,1}, V_{\partial,2}) = \inf_{\substack{x \in V_{\partial,1} \\ y \in V_{\partial,2}}} \tilde{\delta}(x, y).$$

In other words, $\tilde{\delta}(V_{\partial,1}, V_{\partial,2})$ is the distance between $V_{\partial,1}$ and $V_{\partial,2}$ for the pseudo-metric $\tilde{\delta}$. Let $(H(x, y))_{x,y \in V_\partial}$ be the boundary Poisson kernel (1.12) on the discrete graph $\mathcal{G}$. For $x \neq y$, this is also the boundary Poisson kernel on the metric graph $\widetilde{\mathcal{G}}$. It is invariant under potential-preserving rewirings of the electrical network, such as the star-triangle transformation on Figure 8.2. Denote

$$C^f(V_{\partial,1}, V_{\partial,2}) = \sum_{x \in V_{\partial,1}} \sum_{y \in V_{\partial,2}} f(x) H(x, y) f(y).$$

The law of $\tilde{\delta}(V_{\partial,1}, V_{\partial,2})$ is entirely explicit.

Theorem 8.5 ([11],Proposition 10)**.** *Under the assumptions above, for every $a \geq 0$,*

$$\mathbb{P}(\tilde{\delta}(V_{\partial,1}, V_{\partial,2}) \geq a) = \prod_{\substack{x \in V_{\partial,1} \\ y \in V_{\partial,2}}} \exp\Big(-\frac{1}{2} H(x,y)(|f(x)| + |f(y)| + a)^2 + \frac{1}{2} H(x,y)(f(y) - f(x))^2 \Big).$$

In particular, if the sign of f on $V_{\partial,1}$ and $V_{\partial,2}$ is the same, then

$$\mathbb{P}(\tilde{\delta}(V_{\partial,1}, V_{\partial,2}) = 0) = 1 - e^{-2C^f(V_{\partial,1}, V_{\partial,2})}. \tag{8.6}$$

The event $\tilde{\delta}(V_{\partial,1}, V_{\partial,2}) = 0$ happens if and only if the GFF $\tilde{\phi}$ has a crossing from $V_{\partial,1}$ to $V_{\partial,2}$ along which it does not hit 0. The field $\tilde{\phi}$ has to be non-zero also on the endpoints of the crossing. The following corollary is just a restatement of (8.6), but a useful one.

Corollary 8.6. *Assume that the boundary V_∂ is made of three disjoint parts,*

$$V_\partial = \bar{V}_{\partial,0} \amalg \bar{V}_{\partial,1} \amalg \bar{V}_{\partial,2},$$

with $\bar{V}_{\partial,1}$ and $\bar{V}_{\partial,2}$ both non empty and $\bar{V}_{\partial,0}$ being allowed to be empty. Assume that the boundary condition f is positive on both $\bar{V}_{\partial,1}$ and $\bar{V}_{\partial,2}$, and 0 on $\bar{V}_{\partial,0}$. Then

$$\mathbb{P}(\exists \wp \text{ crossing from } \bar{V}_{\partial,1} \text{ to } \bar{V}_{\partial,2} \text{ s.t. } \tilde{\phi} > 0 \text{ on } \wp) = 1 - e^{-2C^f(\bar{V}_{\partial,1}, \bar{V}_{\partial,2})},$$

where

$$C^f(\bar{V}_{\partial,1}, \bar{V}_{\partial,2}) = \sum_{x \in \bar{V}_{\partial,1}} \sum_{y \in \bar{V}_{\partial,2}} f(x) H(x,y) f(y).$$

The above corollary is obtained by applying (8.6) to $V_{\partial,1} = \bar{V}_{\partial,1} \cup \bar{V}_{\partial,0}$ and $V_{\partial,2} = \bar{V}_{\partial,2}$, and noting that a positive crossing joining $V_{\partial,1}$ and $V_{\partial,2}$ cannot end at $\bar{V}_{\partial,0}$, and thus is a positive crossing from $\bar{V}_{\partial,1}$ to $\bar{V}_{\partial,2}$.

Consider a box $\Lambda = (0, L) \times (0, l)$ and $\widetilde{\Lambda}_N$ its metric graph approximations with a square lattice of mesh size N^{-1}. Let $\partial_{\mathrm{l}} \widetilde{\Lambda}_N$, $\partial_{\mathrm{r}} \widetilde{\Lambda}_N$, $\partial_{\mathrm{t}} \widetilde{\Lambda}_N$, resp. $\partial_{\mathrm{b}} \widetilde{\Lambda}_N$ denote the left, right, top, resp. bottom side of the boundary $\partial \widetilde{\Lambda}_N$. Fix $a > 0$ and consider metric graph GFFs $\tilde{\phi}_N$ on $\widetilde{\Lambda}_N$ with boundary conditions a on $\partial_{\mathrm{l}} \widetilde{\Lambda}_N$ and $\partial_{\mathrm{r}} \widetilde{\Lambda}_N$, and 0 on $\partial_{\mathrm{t}} \widetilde{\Lambda}_N$ and $\partial_{\mathrm{b}} \widetilde{\Lambda}_N$. Then Corollary 8.6 implies that as $N \to +\infty$, the probability of a left to right crossing in $\widetilde{\Lambda}_N$ by the positive set of $\tilde{\phi}_N$ is of constant order, bounded away from 0 and 1. Moreover, it has a limit in $(0, 1)$. This limit probability has a natural interpretation in terms of level lines of the continuum GFF on Λ. Curiously enough, if the boundary condition mixes both positive and negative values, say a on $\partial_{\mathrm{l}} \widetilde{\Lambda}_N$ and $\partial_{\mathrm{r}} \widetilde{\Lambda}_N$ and $-a$ on $\partial_{\mathrm{t}} \widetilde{\Lambda}_N$ and $\partial_{\mathrm{b}} \widetilde{\Lambda}_N$, then there is no known expression for the probability of positive crossing on metric graphs. It is also not known whether this probability is invariant under potential-preserving rewirings of the electrical network. However, one can still show that this probability has a limit in $(0, 1)$ as $N \to +\infty$. For this, one can combine Corollary 8.6 with the convergence of first passage sets (Theorem 4.13 and [8, Theorem 4.7]).

Part IV

Isomorphism theorems and topological expansion

Chapter 9

Isomorphisms between random walks and matrix-valued fields

In this Chapter is presented my article [6]. There I considered Gaussian fields of real symmetric, complex Hermitian or quaternionic Hermitian matrices over an electrical network, and described how the isomorphisms between these fields and random walks give rise to topological expansions encoded by ribbon graphs. I further considered matrix-valued Gaussian fields twisted by an orthogonal, unitary or symplectic (quaternionic unitary) connection. In this case the isomorphisms involve traces of holonomies of the connection along random walk loops parametrized by boundary cycles of ribbon graphs. In Section 9.1 I recall how the topological expansion works for a single random matrix. Section 9.2 contains a brief overview of the notion of *connection* on vector bundles on top of a graph, of *holonomy* and of *gauge transformation*. In Section 9.3 are presented some results of Kassel and Lévy [KL21], who considered vector-valued GFFs twisted by a connection and showed how the random walk representations of these fields involve holonomies along random walk paths. In Section 9.4 are presented my results that appeared in [6].

9.1 One-matrix integrals and topological expansion

In this section we will recall how the topological expansion for random matrices works. Here $E_{\beta,n}$ will denote the space of $n \times n$ matrices that are real symmetric for $\beta = 1$, complex Hermitian for $\beta = 2$, and quaternionic Hermitian for $\beta = 4$. Consider the Gaussian distribution on $E_{\beta,n}$ with the following density:

$$\frac{1}{Z_{\beta,n}} e^{-\frac{1}{2}\operatorname{Tr}(M^2)}. \tag{9.1}$$

$\langle \cdot \rangle_{\beta,n}$ will denote the exception w.r.t. (9.1). The distribution of the ordered family of eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of (9.1) is

$$\frac{1}{Z^{\mathrm{ev}}_{\beta,n}} \mathbf{1}_{\lambda_1 > \lambda_2 > \cdots > \lambda_n} \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^{\beta} e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2)}\, d\lambda_1\, d\lambda_2 \dots d\lambda_n.$$

This is the Gaussian Orthogonal Ensemble GOE(n) ($\beta = 1$), the Gaussian Unitary Ensemble GUE(n) ($\beta = 2$), resp. the Gaussian Symplectic Ensemble GSE(n) ($\beta = 4$). We will use the common encompassing notation GβE(n). For more on random matrices, see [Meh04].

Given a family $\nu = (\nu_1, \nu_2, \dots, \nu_m)$ of positive integers, $m(\nu)$ will denote m (the number of integers) and $|\nu|$ will denote

$$|\nu| = \sum_{i=1}^{m(\nu)} \nu_i.$$

Let us recall the combinatorics behind the moments

$$\Big\langle \prod_{k=1}^{m(\nu)} \mathrm{Tr}(M^{\nu_k}) \Big\rangle_{\beta,n}. \tag{9.2}$$

One needs only to consider $|\nu|$ even, since for $|\nu|$ odd (9.2) equals 0. This combinatorial structure is known as the *topological expansion*, because one sums over the maps on 2D surfaces. For background, see [tH74, BIPZ78, BIZ80, IZ80, Zvo97, LZ04, MW03, BP09, Eyn16, EKR18].

So given ν with $|\nu|$ even, one first considers $m(\nu)$ vertices, where each vertex has adjacent *ribbon half-edges*: ν_1 half-edges for the first vertex, ν_2 for the second, etc. A ribbon half-edge is a two-dimensional object and carries an orientation. Also, the ribbon half-edges around each vertex are ordered in a cyclic way. The ribbon half-edges are numbered from 1 to $|\nu|$. See Figure 9.1 for an illustration with $\nu = (4, 3, 1)$.

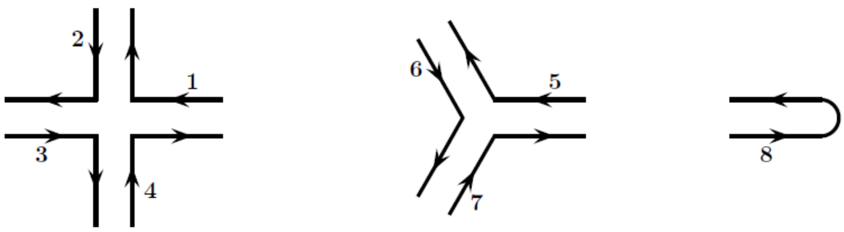

Figure 9.1: Ribbon half-edges in the case of $\nu = (4, 3, 1)$.

Since the total number of half-edges, $|\nu|$, is even, one can pair them to obtain a *ribbon graph* (not necessarily connected), with $m(\nu)$ vertices and $|\nu|/2$ *ribbon edges*. Each time we pair two half-edges, we can glue the corresponding ribbons in two different ways. Either the orientations of the two ribbon half-edges match, or are opposite. In the first case we get a *straight ribbon edge*, in the second a *twisted ribbon edge*. See Figure 9.2. We call such a pairing of ribbon half-edges that keeps straight or twists the ribbons a *ribbon pairing*. Let $\mathcal{R}_\nu$ be the set of all possible ribbon pairings associated to ν. The number of different ribbon pairings is

$$\mathrm{Card}(\mathcal{R}_\nu) = \frac{|\nu|!}{2^{|\nu|/2}(|\nu|/2)!} 2^{|\nu|/2} = \frac{|\nu|!}{(|\nu|/2)!}.$$

Figure 9.3 displays an example of a ribbon pairing with only straight edges, and Figure 9.4 an example with both straight and twisted edges.

Figure 9.2: A straight ribbon edge on the left and a twisted ribbon edge on the right.

A ribbon pairing $\rho \in \mathcal{R}_\nu$ induces a partition in pairs of $\{1, \dots, |\nu|\}$, denoted $\mathbf{p}_\nu(\rho)$. The pairs correspond to the labels of ribbon half-edges associated into an edge. Conversely, given $\mathbf{p}$ a partition in pairs of $\{1, \dots, |\nu|\}$, $\mathcal{R}_{\nu,\mathbf{p}}$ will denote the subset of $\mathcal{R}_\nu$ made of all ribbon pairings ρ such that $\mathbf{p}_\nu(\rho) = \mathbf{p}$ ($\mathrm{Card}(\mathcal{R}_{\nu,\mathbf{p}}) = 2^{|\nu|/2}$).

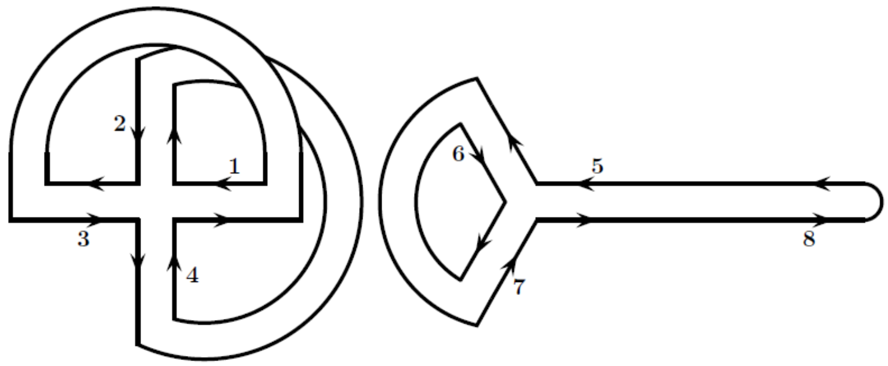

Figure 9.3: A ribbon pairing in the case of $\nu = (4,3,1)$ with only straight edges. The induced partition in pairs is $\mathbf{p}_\nu(\rho) = \{\{1,3\},\{2,4\},\{5,8\},\{6,7\}\}$.

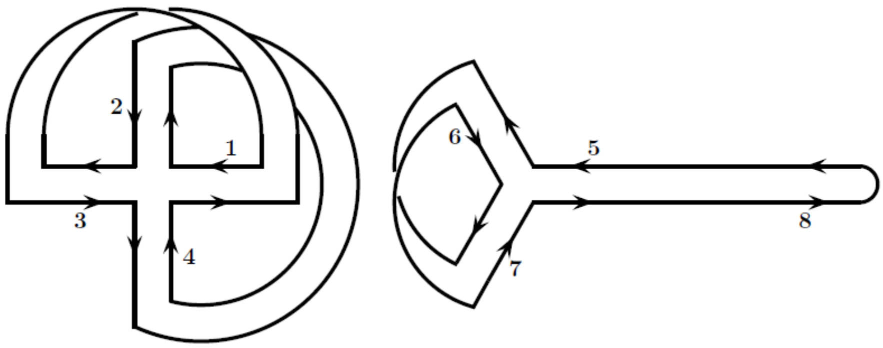

Figure 9.4: A ribbon pairing in the case of $\nu = (4,3,1)$ with straight and twisted edges. The induced partition in pairs $\mathbf{p}_\nu(\rho)$ is the same as on Figure 9.3.

Given a ribbon pairing $\rho \in \mathcal{R}_\nu$, one can see the corresponding ribbon graph as a two-dimensional compact bordered surface (not necessarily connected). Let $f_\nu(\rho)$ denote the number of the connected components of the boundary, that is to say the number of distinct cycles formed by the borders of ribbons. On Figure 9.3, $f_\nu(\rho) = 3$, and on Figure 9.4, $f_\nu(\rho) = 2$. Then, one can glue along each connected component of the boundary a disk ($f_\nu(\rho)$ disks in total), and obtain in this way a two-dimensional compact surface (not necessarily connected) without boundary. We will denote it $\Sigma_\nu(\rho)$, and consider it up to diffeomorphisms. On the example of Figure 9.3, $\Sigma_\nu(\rho)$ has two connected components, a torus on the left and a sphere on the right. On the example of Figure 9.4, $\Sigma_\nu(\rho)$ has again two connected components, a Klein bottle on the left and a projective plane on the right. Observe that if all the edges are straight, the surfaces that appear are orientable. Let $\chi_\nu(\rho)$ denote the Euler's characteristic of $\Sigma_\nu(\rho)$. According to Euler's formula,

$$\chi_\nu(\rho) = m(\nu) - \frac{|\nu|}{2} + f_\nu(\rho).$$

If one contracts the ribbons in a ribbon graph (i.e. reduces the width of the ribbons to 0), then one gets a map drawn on the surface $\Sigma_\nu(\rho)$, that is to say a graph whose faces ($f_\nu(\rho)$ in total) are topological disks.

Given a ribbon pairing $\rho \in \mathcal{R}_\nu$, we associate to it a weight $w_{\nu,\beta}(\rho)$ depending on β:

$$w_{\nu,\beta=1}(\rho) = \frac{1}{2^{|\nu|/2}}, \quad w_{\nu,\beta=2}(\rho) = 1_{\rho \text{ has only straight edges}}, \quad w_{\nu,\beta=4}(\rho) = (-2)^{\chi_\nu(\rho)} 2^{-2m(\nu)+|\nu|/2}.$$

In all three cases $\beta \in \{1,2,4\}$, for every $\mathbf{p}$ partition in pairs of $\{1,\dots,|\nu|\}$,

$$\sum_{\rho\in\mathcal{R}_{\nu,\mathbf{p}}} w_{\nu,\beta}(\rho) = 1.$$

Theorem 9.1 (Brézin-Itzykson-Parisi-Zuber [BIPZ78], Mulase-Waldron [MW03])**.** *For $\beta \in \{1,2,4\}$ and $|\nu|$ even, the value of the matrix integral* (9.2) *is given by*

$$\Big\langle \prod_{k=1}^{m(\nu)} \mathrm{Tr}(M^{\nu_k}) \Big\rangle_{\beta,n} = \sum_{\rho\in\mathcal{R}_\nu} w_{\nu,\beta}(\rho) n^{f_\nu(\rho)}. \tag{9.3}$$

Note that the right-hand side of (9.3) is a polynomial in n, the size of the matrices. Also note that in the quaternionic case the coefficients $w_{\nu,\beta=4}(\rho)$ may take negative values.

9.2 Connections, gauge equivalence and Wilson loops

Let $\mathbb{H}$ denote here the division ring of quaternions. Le be a discrete electrical network $\mathcal{G}=(V,E)$ as in Section 1.1.1. Let $n\in\mathbb{N}$, $n\geq 2$. Let $\mathbb{U}$ be the group of either $n\times n$ orthogonal matrices $O(n)$, or unitary matrices $U(n)$, or quaternionic unitary matrices $U(n,\mathbb{H})$. We consider that each undirected edge in E consists of two directed edges of opposite direction. We consider a family of matrices in $\mathbb{U}$, $(U(x,y))_{\{x,y\}\in E}$, with

$$U(y,x) = U(x,y)^* = U(x,y)^{-1}, \quad \forall \{x,y\}\in E.$$

$(U(x,y))_{\{x,y\}\in E}$ is our *connection* on the vector bundle with base space $\mathcal{G}$ and fiber respectively $\mathbb{R}^n$, $\mathbb{C}^n$ or $\mathbb{H}^n$. In Physics literature, U is called a *gauge field.*

Given a nearest-neighbor oriented discrete path $\wp = (y_1,y_2,\dots,y_j)$, the *holonomy* of U along $\wp$ is the product

$$\mathsf{hol}^U(\wp) = U(y_1,y_2)U(y_2,y_3)\dots U(y_{j-1},y_j).$$

If $\wp$ is a nearest-neighbor path parametrized by continuous time, and does only a finite number of jumps, the holonomy $\mathsf{hol}^U(\wp)$ is defined as the holonomy along the discrete skeleton of $\wp$. We will denote by $\overleftarrow{\wp}$ the time-reversal of a path $\wp$. We have that

$$\mathsf{hol}^U(\overleftarrow{\wp}) = \mathsf{hol}^U(\wp)^* = \mathsf{hol}^U(\wp)^{-1}. \tag{9.4}$$

Given a nearest-neighbor oriented discrete closed path (i.e. a loop) $\wp = (y_1,y_2,\dots,y_j)$, with $y_j = y_1$, we will consider the observable $\mathrm{Tr}(\mathsf{hol}^U(\wp))$ in the orthogonal and unitary case, and $\mathrm{Re}(\mathrm{Tr}(\mathsf{hol}^U(\wp)))$ in the quaternionic unitary case. Such observables are called *Wilson loops* [Wil74]. Note that the Wilson loop observable does not depend on where the loop $\wp$ is rooted. Indeed, if $\tilde{\wp}$ is the loop visiting $(y_i,\dots,y_j,y_1,\dots,y_{i-1},y_i)$ $(i\in\{2,\dots,j\})$, and if $\wp'$ is the path visiting $(y_1,\dots,y_i)$ then

$$\mathsf{hol}^U(\tilde{\wp}) = \mathsf{hol}^U(\wp')^{-1}\mathsf{hol}^U(\wp)\mathsf{hol}^U(\wp').$$

Given another family of matrices in $\mathbb{U}$, $(\mathfrak{U}(x))_{x\in V}$, this time on top of vertices, it induces a *gauge transformation* on the connection U:

$$(U(x,y))_{\{x,y\}\in E} \longmapsto (\mathfrak{U}(x)^{-1}U(x,y)\mathfrak{U}(y))_{\{x,y\}\in E}.$$

Two connections related by a gauge transformation are said to be *gauge equivalent.* A connection is *trivial* if it is gauge equivalent to the identity connection. A criterion for triviality is that along any nearest-neighbor loop $\wp = (y_1,y_2,\dots,y_j)$, with $y_j = y_1$, $\mathsf{hol}^U(\wp) = I_n$. In general, any two gauge equivalent connections have the same Wilson loop observables. The converse is also true (but non-obvious): the collection of all possible Wilson loop observables characterizes a connection up to gauge transformations [Gil81, Sen94, Lé04].

9.3 BFS-Dynkin isomorphism for the Gaussian free field twisted by a connection

In [KL21] Kassel and Lévy introduced the vector-valued GFF twisted by an orthogonal/unitary connection, and generalized the isomorphisms with random walks to this case. They relied on a covariant Feynman-Kac formula ([BFS79] and [KL21, Theorem 3.1]).

Let us consider on top of the electrical network $\mathcal{G} = (V, E)$ an orthogonal connection $(U(x,y))_{\{x,y\}\in E}$, $U(x,y) \in O(n)$. The Green's function G^U associated to the connection U is a function from $V \times V$ to $\mathcal{M}_n(\mathbb{R})$ (i.e. the $n \times n$ matrices with real entries), with the entries given by

$$G^U_{ij}(x,y) = \int_\gamma \mathsf{hol}^U_{ij}(\gamma)\mu^{x,y}(d\wp), \quad x,y \in V,\ i,j \in \{1,\dots,n\},$$

where the measure on paths $\mu^{x,y}(d\wp)$ is given by Definition 1.3. Since the image of $\mu^{x,y}$ by time reversal is $\mu^{y,x}$, and because of (9.4), we have that

$$G^U_{ij}(x,y) = G^U_{ji}(y,x), \qquad G^U_{ij}(x,x) = G^U_{ji}(x,x).$$

i.e. $G^U(x,y)^\mathsf{T} = G^U(y,x)$, (where $^\mathsf{T}$ denotes the transpose) and $G^U(x,x)$ is symmetric. One can see G^U as a symmetric linear operator on $(\mathbb{R}^n)^V$. It is positive definite (see [KL21, Proposition 2.15]). We will denote by $\det G^U$ the determinant of this operator.

The $\mathbb{R}^n$-valued Gaussian free field on $\mathcal{G}$ twisted by the connection U (massless, with 0 boundary condition) is a random Gaussian function $\hat\phi : V \to \mathbb{R}^n$ ($\hat\phi(x) = (\hat\phi_1(x),\dots,\hat\phi_n(x))$) with the distribution given by

$$\frac{1}{Z^U_{\text{GFF}}} \exp\Big(-\frac{1}{2} \sum_{\{x,y\}\in E} C(x,y)\|\hat\varphi(x) - U(x,y)\hat\varphi(y)\|^2\Big) \prod_{x\in V_{\text{int}}} \prod_{i=1}^{n} d\hat\varphi(x)_i,$$

where $\|\cdot\|$ is the usual L^2 norm on $\mathbb{R}^n$ and

$$Z^U_{\text{GFF}} = ((2\pi)^{n|V|}\det G^U)^{\frac{1}{2}}.$$

Note that if $\{x,y\} \in E$,

$$\|\hat\varphi(x) - U(x,y)\hat\varphi(y)\|^2 = \|\hat\varphi(y) - U(y,x)\hat\varphi(x)\|^2.$$

We have that $\mathbb{E}[\hat\phi] \equiv 0$. As for the covariance structure, it is given by

$$\mathbb{E}[\hat\phi_i(x)\hat\phi_j(y)] = G^U_{ij}(x,y);$$

see [KL21, Proposition 4.1]. If $(\mathfrak{U}(x))_{x\in V}$ is a gauge transformation, then $(\mathfrak{U}(x)\hat\phi(x))_{x\in V}$ is the Gaussian free field related to the connection $(\mathfrak{U}(x)^{-1}U(x,y)\mathfrak{U}(y))_{\{x,y\}\in E}$. In particular, if the connection U is trivial, the field $\hat\phi$ can be reduced to n i.i.d. copies of a scalar GFF (massless, with 0 boundary condition).

In [KL21], Theorems 5.1 and 7.3, Kassel and Lévy gave a BFS-Dynkin-type isomorphism for GFFs twisted by connections, thus generalizing Theorem 1.4.

Theorem 9.2 (Kassel-Lévy). *Let $k \in \mathbb{N} \setminus \{0\}$, $x_1, x_2, \dots, x_{2k} \in V_{\mathrm{int}}$, $J(1), J(2), \dots, J(2k) \in \{1, \dots, n\}$ and F a bounded measurable function $\mathbb{R}^V \to \mathbb{R}$. Then*

$$\mathbb{E}\Big[\prod_{i=1}^{2k} \hat{\phi}_{J(i)}(x_i) F(\|\hat{\phi}\|^2/2)\Big]$$
$$= \sum_{\substack{(\{a_j,b_j\})_{1\leq j\leq k} \\ \text{partition in pairs} \\ \text{of } \{1,2,\dots,2k\}}} \int \mathbb{E}\Big[F\big((\|\hat{\phi}(x)\|^2/2 + \sum_{j=1}^{k} \ell^x(\wp_j))_{x\in V}\big)\Big] \prod_{j=1}^{k} \mathsf{hol}^U_{J(a_j)J(b_j)}(\wp_j) \mu^{x_{a_j},x_{b_j}}(d\wp_j),$$

where the sum runs over the $(2k)!/(2^k k!)$ partitions of $\{1, \dots, 2k\}$ in pairs.

Note that $\mathsf{hol}^U(\wp_j)$ is an $n \times n$ matrix in $O(n)$, and $\mathsf{hol}^U_{J(a_j)J(b_j)}(\wp_j)$ means that one takes the entry $J(a_j)J(b_j)$. Also note that this entry may be negative.

9.4 Matrix valued fields, isomorphisms and topological expansion

In my article [6] I observed that the topological expansion (Theorem 9.1) combines well with the BFS-Dynkin isomorphism (1.4) provided one considers matrix-valued fields, that is to say spin systems on a graph where the spins are matrices.

First we will present the case without connection (gauge field). Fix $\beta \in \{1, 2, 4\}$. On top of each vertex $x \in V$, one considers a matrix $M(x) \in E_{\beta,n}$ ($n \times n$ real symmetric, complex Hermitian or quaternionic Hermitian depending on the value of β), with $M(x) = 0$ for $x \in V_\partial$. The distribution of the matrix-valued field is given by

$$\frac{1}{Z^{\mathcal{G}}_{\beta,n}} \exp\Big(-\frac{1}{2}\sum_{\{x,y\}\in E} C(x,y) \operatorname{Tr}\big((M(y) - M(x))^2\big)\Big) \prod_{x\in V_{\mathrm{int}}} dM(x). \tag{9.5}$$

This is a matrix-valued GFF. We will denote by $\langle\cdot\rangle^{\mathcal{G}}_{\beta,n}$ the expectation with respect to (9.5). For every $x \in V_{\mathrm{int}}$, $M(x)/G(x,x)^{\frac{1}{2}}$ under $\langle\cdot\rangle^{\mathcal{G}}_{\beta,n}$ follows the distribution (9.1).

Given $\nu = (\nu_1, \nu_2, \dots, \nu_{m(\nu)})$ a family of positive integers, $\mathbf{k}_\nu$ will denote the map from $\{1, \dots, |\nu|\}$ to $\{1, \dots, m(\nu)\}$ such that $\mathbf{k}_\nu^{-1}(k) = \{\nu_1 + \dots + \nu_{k-1} + 1, \dots, \nu_1 + \dots + \nu_{k-1} + \nu_k\}$.

Theorem 9.3 ([6], Theorem 3.1). *Let $\beta \in \{1, 2, 4\}$. Let ν be a finite family of positive integers with $|\nu|$ even. Let $x_1, x_2, \dots, x_{m(\nu)} \in V_{\mathrm{int}}$. Then for any F bounded measurable function on $\mathbb{R}^V$,*

$$\Big\langle\Big(\prod_{k=1}^{m(\nu)} \operatorname{Tr}(M(x_k)^{\nu_k})\Big) F\Big(\Big(\frac{1}{2}\operatorname{Tr}(M(x)^2)\Big)_{x\in V}\Big)\Big\rangle^{\mathcal{G}}_{\beta,n} =$$
$$\sum_{\substack{p=(\{a_j,b_j\})_{1\leq j\leq |\nu|/2} \\ \text{partition in pairs} \\ \text{of } \{1,2,\dots,|\nu|\}}} \Big(\sum_{\rho\in\mathcal{R}_{\nu,p}} w_{\nu,\beta}(\rho) n^{f_\nu(\rho)}\Big)$$
$$\times \int \Big\langle F\Big(\Big(\frac{1}{2}\operatorname{Tr}(M(x)^2) + \sum_{j=1}^{|\nu|/2} \ell^x(\wp_j)\Big)_{x\in V}\Big)\Big\rangle^{\mathcal{G}}_{\beta,n} \prod_{j=1}^{|\nu|/2} \mu^{x_{\mathbf{k}_\nu(a_j)},x_{\mathbf{k}_\nu(b_j)}}(d\wp_j).$$

In other words, one sums over the ribbon graphs associated to the family of positive integers ν. As in Theorem 9.1, each ribbon graph ρ comes with a weight $w_{\nu,\beta}(\rho)n^{f_\nu(\rho)}$, which may be negative in the case $\beta = 4$. Each vertex of ρ with ν_k ribbon edges corresponds to a point x_k in V_{int}. To each ribbon edge between $x_{\mathbf{k}_\nu(a_j)}$ and $x_{\mathbf{k}_\nu(b_j)}$ is associated a nearest-neighbor path $\wp_j$ on $\mathcal{G}$ between $x_{\mathbf{k}_\nu(a_j)}$ and $x_{\mathbf{k}_\nu(b_j)}$. For instance, the contribution of the ribbon graph appearing on Figure 9.3, resp. Figure 9.4, would be

$$w_{\nu=(4,3,1),\beta}(\rho)n^3\mu^{x_1,x_1}(d\wp_1)\mu^{x_1,x_1}(d\wp_2)\mu^{x_2,x_2}(d\wp_3)\mu^{x_2,x_3}(d\wp_4),$$

$$\text{resp.} \qquad w_{\nu=(4,3,1),\beta}(\rho)n^2\mu^{x_1,x_1}(d\wp_1)\mu^{x_1,x_1}(d\wp_2)\mu^{x_2,x_2}(d\wp_3)\mu^{x_2,x_3}(d\wp_4).$$

Note that for the ribbon graph on Figure 9.3, the coefficients $w_{\nu=(4,3,1),\beta}(\rho)$ are

$$w_{\nu=(4,3,1),\beta=1}(\rho) = \frac{1}{16}, \qquad w_{\nu=(4,3,1),\beta=2}(\rho) = 1, \qquad w_{\nu=(4,3,1),\beta=4}(\rho) = 1;$$

and in the case of Figure 9.4,

$$w_{\nu=(4,3,1),\beta=1}(\rho) = \frac{1}{16}, \qquad w_{\nu=(4,3,1),\beta=2}(\rho) = 0, \qquad w_{\nu=(4,3,1),\beta=4}(\rho) = -\frac{1}{2}.$$

Now let us consider the setting with a connection. Let $(U(x,y))_{\{x,y\}\in E}$ be such a connection, orthogonal for $\beta = 1$, unitary for $\beta = 2$ and quaternionic unitary for $\beta = 4$. Consider the distribution

$$\frac{1}{Z_{\beta,n}^{\mathcal{G},U}} \exp\Big(-\frac{1}{2}\sum_{\{x,y\}\in E} C(x,y)\operatorname{Tr}\big(\big(M(y)-U(y,x)M(x)U(x,y)\big)^2\big)\Big)\prod_{x\in V_{\text{int}}} dM(x). \tag{9.6}$$

and let $\langle\cdot\rangle_{\beta,n}^{\mathcal{G},U}$ denote the expectation with respect to (9.5). Note that if $\{x,y\}\in E$, then

$$\operatorname{Tr}((M(x)-U(x,y)M(y)U(y,x))^2) = \operatorname{Tr}((M(y)-U(y,x)M(x)U(x,y))^2).$$

Also, if the connection U is non-trivial, $M(x)/G(x,x)^{\frac{1}{2}}$ under $\langle\cdot\rangle_{\beta,n}^{\mathcal{G},U}$ does no longer follow in general the distribution (9.1). As for the distribution of $(\frac{1}{\sqrt{n}}\operatorname{Tr}(M(x)))_{x\in V}$, it is the same whatever the connection U, and it is that of a scalar GFF [6, Remark 3.9].

Theorem 9.4 ([6], Theorem 3.4)**.** *Let $\beta\in\{1,2,4\}$. Let ν be a finite family of positive integers with $|\nu|$ even. Let $x_1,x_2,\dots,x_{m(\nu)}\in V_{\text{int}}$. Then for any F bounded measurable function on $\mathbb{R}^V$,*

$$\Big\langle\Big(\prod_{k=1}^{m(\nu)}\operatorname{Tr}(M(x_k)^{\nu_k})\Big)F\Big(\Big(\frac{1}{2}\operatorname{Tr}(M(x)^2)\Big)_{x\in V}\Big)\Big\rangle_{\beta,n}^{\mathcal{G},U} =$$

$$\sum_{\substack{p=(\{a_j,b_j\})_{1\leq j\leq|\nu|/2}\\ \text{partition in pairs}\\ \text{of } \{1,2,\dots,|\nu|\}}}\ \sum_{\rho\in\mathcal{R}_{\nu,p}} w_{\nu,\beta}(\rho)\int\Big\langle F\Big(\Big(\frac{1}{2}\operatorname{Tr}(M(x)^2)+\sum_{j=1}^{|\nu|/2}\ell^x(\wp_j)\Big)_{x\in V}\Big)\Big\rangle_{\beta,n}^{\mathcal{G},U}$$

$$\times\operatorname{Tr}_\beta(\mathsf{hol}_{\nu,\rho}^U(\wp_1,\wp_2,\dots,\wp_{|\nu|/2}))\prod_{j=1}^{|\nu|/2}\mu^{x_{k_\nu(a_j)},x_{k_\nu(b_j)}}(d\wp_j),$$

where $\operatorname{Tr}_\beta(\mathsf{hol}_{\nu,\rho}^U(\wp_1,\wp_2,\dots,\wp_{|\nu|/2}))$ is a holonomy factor explained below.

The factor $\mathrm{Tr}_\beta(\mathsf{hol}^U_{\nu,\rho}(\wp_1, \wp_2, \dots, \wp_{|\nu|/2}))$ is a product of $f_\nu(\rho)$ Wilson loops, i.e. traces of holonomies along closed loops for $\beta = 1$ or $\beta = 2$, real parts of traces for $\beta = 4$. The loops are formed by concatenation of some of the paths among $\wp_1, \wp_2, \dots, \wp_{|\nu|/2}$, and one can read them on the ribbon graph ρ. A ribbon edge is associated to a path $\wp_j$. As one follows one of the two boundaries of the ribbon edge, one moves along $\wp_j$, or in the opposite direction, along $\overleftarrow{\wp}_j$. So if one completes a turn along one of the $f_\nu(\rho)$ boundary cycles of the ribbon graph ρ, then the random walk paths that one has followed form concatenated a loop, and one takes the holonomy of U along this loop. So, in the example of Figure 9.3,

$$\mathrm{Tr}_\beta(\mathsf{hol}^U_{\nu=(4,3,1),\rho}(\wp_1, \wp_2, \wp_3, \wp_4)) =$$
$$\mathrm{Tr}_\beta(\mathsf{hol}^U(\wp_1)\mathsf{hol}^U(\wp_2)\mathsf{hol}^U(\wp_1)^{-1}\mathsf{hol}^U(\wp_2)^{-1})\times\mathrm{Tr}_\beta(\mathsf{hol}^U(\wp_3)\mathsf{hol}^U(\wp_4)\mathsf{hol}^U(\wp_4)^{-1})\times\mathrm{Tr}_\beta(\mathsf{hol}^U(\wp_3)^{-1}),$$

and in the example of Figure 9.4,

$$\mathrm{Tr}_\beta(\mathsf{hol}^U_{\nu=(4,3,1),\rho}(\wp_1, \wp_2, \wp_3, \wp_4)) =$$
$$\mathrm{Tr}_\beta(\mathsf{hol}^U(\wp_1)\mathsf{hol}^U(\wp_2)^{-1}\mathsf{hol}^U(\wp_1)^{-1}\mathsf{hol}^U(\wp_2)^{-1}) \times \mathrm{Tr}_\beta(\mathsf{hol}^U(\wp_3)^2\mathsf{hol}^U(\wp_4)\mathsf{hol}^U(\wp_4)^{-1}).$$

Note that each path $\wp_j$ appears in total twice. This is because a ribbon edge has two borders. Also note that if the connection U is trivial, then $\mathrm{Tr}_\beta(\mathsf{hol}^U_{\nu,\rho}(\wp_1, \wp_2, \dots, \wp_{|\nu|/2}))$ is just $n^{f_\nu(\rho)}$.

Note that the fact that the holonomies appear only inside Wilson loops (traces along closed paths) is not a surprise. Indeed, the fields of eigenvalues of $(M(x))_{x\in V}$ are invariant in law under gauge transformations. Indeed, if the new gauge equivalent connection is $(\mathfrak{U}(x)^{-1}U(x,y)\mathfrak{U}(y))_{\{x,y\}\in E}$, then only needs to apply the conjugation

$$(M(x))_{x\in V} \mapsto (\mathfrak{U}(x)^{-1}M(x)\mathfrak{U}(x))_{x\in V}$$

to get the field for this new connection. So the law of the fields of eigenvalues only depends on the gauge equivalence class of U, which is characterized by the Wilson loop observables [Gil81, Sen94, Lé04].

Next are some comments on the proofs, without going into details. The traces $\mathrm{Tr}(M(x_k)^{\nu_k})$ can be expressed though the coefficients of the matrices $M(x_k)$, which are Gaussian. So Theorem 9.3 is a consequence of Theorem 1.4 and Theorem 9.4 is a consequence of Theorem 9.2. However, as one expands the product of traces and applies the BFS-Dynkin (Theorem 1.4) or Kassel-Lévy isomorphism (Theorem 9.2), one gets plenty of terms, many of which given identical contributions, many give zero contribution, and many give contributions that get canceled out by other terms. So Theorems 9.3 and 9.4 essentially present a way to organize the final result in a combinatorially meaningful way. A brute force approach through direct computation can be carried out in the case $\beta = 1$ and $\beta = 2$, but for $\beta = 4$ it becomes particularly arduous because of the non-commutativity of quaternions, and even more so in the presence of a gauge field. So instead I relied in [6] on an induction on $|\nu|/2$, the number of ribbon edges, inspired by Bryc and Pierce [BP09]. This approach covers all the three cases $\beta \in \{1, 2, 4\}$.

Chapter 10

Isomorphisms between 1D Brownian local time and β-Dyson's Brownian motion

In this Chapter is persented my article [3]. There I show that the β-Dyson's Brownian motions for general values of β satisfy both a Le Jan type (Theorem 1.13) and BFS-Dynkin type (Theorem 1.4) isomorphism with the local times of one-dimensional Brownian trajectories. The Le Jan type isomorphism involves a whole range of intensity parameters α for the 1D Brownian loop soup, not just α half-integer. In Section 10.1 is explained the motivation behind this work. In Section 10.2 is recalled the notion of Gaussian beta ensembles. In Section 10.3 are detailed the isomorphism theorems for the continuum GFF in dimension one. In Section 10.4 is explained the Le Jan type isomorphism for the β-Dyson's Brownian motion. In Section 10.5 is presented the BFS-Dynkin type isomorphism for the β-Dyson's Brownian motion. In Section 10.6 are presented important open questions that motivated this whole work. In essence, the question is whether the Gaussian beta ensembles have natural generalizations to arbitrary electrical networks.

10.1 Motivation

Here is explained the motivation behind my work [3]. It is primarily contained in the open questions presented in Section 10.6.

The relation between the GFF and the loop soup (random walk or Brownian) of intensity parameter $\alpha = 1/2$ (Theorem 1.13 and Theorem 3.8) provides a powerful tool to study both the GFF and the loop soup itself, in particular in dimension 2 (see Part II). Some aspects are easier to see though the GFF, some other through the loop soup, and the isomorphisms between the two enable a transfer of results from one object to the other.

In dimension 2 in continuum, the Brownian loop soups are of major interest for any intensity parameter α, not just $\alpha = 1/2$. This is because of the conformal invariance, the relation to the conformal loop ensembles CLE_κ (Theorem 2.3), and the non-Gaussian multiplicative chaoses constructed out of loop soups (Chapter 6). However, for $\alpha \neq 1/2$, the Brownian loop soups are much less understood and the picture is overall more complicated. So the idea is to obtain for $\alpha \neq 1/2$ some isomorphism Theorems similar to Le Jan's (Theorem 1.13) which would relate the loop soups to some random fields, preferably satisfying some integrability/exact-solvability.

If α is half-integer, $\alpha = d/2$, then there is an isomorphism with a vector-valued GFF with d i.i.d. components. So for general α, what one looks at is a natural notion of non-integer dimension. It turns out that there is a notion of non-integer dimension in the theory of Gaussian beta ensembles GβE(n); see Section 10.2 and (10.2). Further, the β-Dyson's Brownian motion is a one-dimensional field version of the GβE(n), where there is a sample on the GβE(n) on top of each point on the line; see Section 10.4. It turns out that there is indeed a Le Jan type isomorphism between the β-Dyson's Brownian motion and a 1D Brownian loop soup, with an intensity parameter α depending on β and n and that may be non half-integer (Corollary 10.4). On top of that, the β-Dyson's Brownian motion also satisfies a BFS-Dynkin type isomorphism (Theorem 10.9).

Now it would be interesting to have the same thing on any electrical network. These are the open questions presented in Section 10.6. One would like to have a sample of GβE(n) on top of each vertex of a graph, everything being correlated in a non-trivial way, so as to interpolate and extrapolate the fields of eigenvalues in matrix-valued GFFs for $\beta \in \{1, 2, 4\}$. We would also like to emphasize that we are not interested in the large n (number of "eigenvalues") limit that is usually studied random matrix theory. What we are interested in is for fixed values of n, including small ones, 2,3,4,5, etc. Most importantly, the pseudo-dimension $d(\beta, n)$ (10.2) has to be fixed. It is the electrical network that eventually has to become large.

10.2 Gaussian beta ensembles

For references on Gaussian beta ensembles, see [DE02, For15], [EKR18, Section 1.2.2], and [AGZ09, Section 4.5]. Fix $n \geq 2$. A Gaussian beta ensemble GβE(n) follows the distribution on $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$,

$$\frac{1}{Z^{\rm ev}_{\beta,n}} \mathbf{1}_{\lambda_1 > \lambda_2 > \dots > \lambda_n} \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^\beta e^{-\frac{1}{2}(\lambda_1^2 + \lambda_2^2 + \dots + \lambda_n^2)} \, d\lambda_1 \, d\lambda_2 \dots d\lambda_n, \tag{10.1}$$

where the partition function $Z^{\rm ev}_{\beta,n}$ is given by

$$Z^{\rm ev}_{\beta,n} = \frac{(2\pi)^{\frac{n}{2}}}{n!} \prod_{j=1}^{n} \frac{\Gamma\big(1 + j\frac{\beta}{2}\big)}{\Gamma\big(1 + \frac{\beta}{2}\big)}.$$

This distribution is well defined for every $\beta > -2/n$, but in the literature one usually is interested in the large n limit and considers only $\beta > 0$. For $\beta = 1$, 2, resp. 4, one gets the GOE(n), GUE(n), resp. GSE(n); see Section 9.1. For $n = 0$, one gets the reordered family of n i.i.d. $\mathcal{N}(0, 1)$ Gaussian r.v.s. In the limit $\beta \to -2/n$, the GβE(n) converges in law to

$$\Big(\frac{1}{\sqrt{n}}\xi, \frac{1}{\sqrt{n}}\xi, \dots, \frac{1}{\sqrt{n}}\xi\Big),$$

where ξ follows $\mathcal{N}(0, 1)$.

The brackets $\langle \cdot \rangle_{\beta,n}$ will denote the expectation w.r.t. (10.1). For $q \geq 1$, $p_q(\lambda)$ will denote the q-th power sum polynomial

$$p_q(\lambda) = \sum_{j=1}^{n} \lambda_j^q.$$

By convention, $p_0(\lambda) = n$. Let $d(\beta, n)$ denote

$$d(\beta, n) = n + n(n-1)\frac{\beta}{2}. \tag{10.2}$$

One can see $d(\beta, n)$ as a kind of pseudo-dimension. For $\beta \in \{1, 2, 4\}$, $d(\beta, n)$ is the dimension of the corresponding space of matrices.

Next are some elementary properties of GβE.

Proposition 10.1. *The following holds.*

1. *For every $\beta > -2/n$, $\frac{1}{\sqrt{n}}p_1(\lambda)$ under $G\beta E$ has for distribution $\mathcal{N}(0,1)$.*
2. *For every $\beta > -2/n$, $p_2(\lambda)/2$ under $G\beta E$ has for distribution* $\mathrm{Gamma}(d(\beta,n)/2,1)$.
3. *$p_1(\lambda)$ and $\lambda - \frac{1}{n}p_1(\lambda)$ under $G\beta E$ are independent.*
4. *$\frac{1}{2}\big(p_2(\lambda) - \frac{1}{n}p_1(\lambda)^2\big) = \frac{1}{2}p_2\big(\lambda - \frac{1}{n}p_1(\lambda)\big)$ under $G\beta E$ has for distribution* $\mathrm{Gamma}((d(\beta,n)-1)/2,1)$.

Let $\nu = (\nu_1, \nu_2, \dots, \nu_m)$ be a finite family of positive integers. We will denote

$$m(\nu) = m, \qquad |\nu| = \sum_{k=1}^{m(\nu)} \nu_k.$$

Let $p_\nu(\lambda)$ denote

$$p_\nu(\lambda) = \prod_{k=1}^{m(\nu)} p_{\nu_k}(\lambda).$$

By convention, we set $p_\emptyset(\lambda) = 1$ and $|\emptyset| = 0$. Note that $p_\emptyset(\lambda) \neq p_0(\lambda)$. We are interested in the expression of the moments $\langle p_\nu(\lambda)\rangle_{\beta,n}$. These are 0 if $|\nu|$ is odd. For $|\nu|$ even, these moments are given by a recurrence known as *loop equation or Schwinger-Dyson equation* ([LC09, Lemma 4.13], [LC13, slide 3/15] and [EKR18, Section 4.1.1]). This generalizes Theorem 9.1 to $\beta \notin \{1,2,4\}$.

Proposition 10.2 (Schwinger-Dyson equation)**.** *For every $\beta > -2/n$ and every ν as above with $|\nu|$ even,*

$$\begin{aligned}\langle p_\nu(\lambda)\rangle_{\beta,n} \;=\;& \frac{\beta}{2}\sum_{i=1}^{\nu_{m(\nu)}-1} \langle p_{(\nu_r)_{r\neq m(\nu)}}(\lambda) p_{i-1}(\lambda) p_{\nu_{m(\nu)}-1-i}(\lambda)\rangle_{\beta,n} \\ &+\Big(1-\frac{\beta}{2}\Big)(\nu_{m(\nu)}-1)\langle p_{(\nu_r)_{r\neq m(\nu)}}(\lambda) p_{\nu_{m(\nu)}-2}(\lambda)\rangle_{\beta,n} \\ &+\sum_{k=1}^{m(\nu)-1} \nu_k \langle p_{(\nu_r)_{r\neq k,m(\nu)}}(\lambda) p_{\nu_k+\nu_{m(\nu)}-2}(\lambda)\rangle_{\beta,n},\end{aligned} \tag{10.3}$$

where $p_0(\lambda) = n$. The recurrence (10.3) *and the initial condition $p_0(\lambda) = n$ determine all the moments $\langle p_\nu(\lambda)\rangle_{\beta,n}$.*

10.3 Isomorphism theorems in continuum in dimension one

Let $K > 0$. Let $(\phi(x))_{x\in\mathbb{R}}$ be the massive continuum GFF on $\mathbb{R}$ with square-mass K. This is nothing else than a stationary Ornstein-Uhlenbeck process, satisfying the SDE

$$d\phi(x) = \sqrt{2}dW(x) - \sqrt{2K}\phi(x)dx,$$

where $dW(x)$ is a white noise on $\mathbb{R}$. Here we will think of x as a one-dimension space variable rather than a time variable. The covariance function of ϕ is given by the Green's function $G_{\mathbb{R},K}$ of $-\frac{1}{2}\frac{d^2}{dx^2} + K$:

$$G_{\mathbb{R},K}(x,y) = \frac{1}{\sqrt{2K}} e^{-\sqrt{2K}|y-x|}.$$

Let $(B_t)_{t\geq 0}$ denote the standard Brownian motion on $\mathbb{R}$. Let $p_{\mathbb{R}}(t,x,y)$ denote its transition densities (heat kernel) and $\mathbb{P}^{x,y}_{\mathbb{R},t}$ the 1D Brownian bridge measures. The massive measure on 1D Brownian loops is

$$\mu^{\text{loop}}_{\mathbb{R},K} = \int_{\mathbb{R}} dx \int_0^{+\infty} e^{-Kt}\frac{dt}{t} p_{\mathbb{R}}(t,x,x)\mathbb{P}^{x,x}_{\mathbb{R},t}.$$

For $\alpha > 0$, let $\mathcal{L}^{\alpha}_{\mathbb{R},K}$ denote the Poisson point process with intensity measure $\alpha\mu^{\text{loop}}_{\mathbb{R},K}$. This is the massive 1D Brownian loop soup. Given a 1D Brownian path $(\wp(t))_{0\leq t\leq T(\wp)}$, we will denote by $\ell^x(\wp)$ the Brownian local times

$$\ell^x(\wp) = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^{T(\wp)} \mathbf{1}_{|\wp(t)-x|<\varepsilon}dt.$$

Further, $\ell^x(\mathcal{L}^{\alpha}_{\mathbb{R},K})$ will denote

$$\ell^x(\mathcal{L}^{\alpha}_{\mathbb{R},K}) = \sum_{\wp\in\mathcal{L}^{\alpha}_{\mathbb{R},K}} \ell^x(\wp).$$

According to the continuum massive version of the Le Jan's isomorphism (Theorem 1.13), for $\alpha = 1/2$, the field $(\ell^x(\mathcal{L}^{1/2}_{\mathbb{R},K}))_{x\in\mathbb{R}}$ has the same distribution as $(\phi(x)^2/2)_{x\in\mathbb{R}}$. In general, one can see $(\ell^x(\mathcal{L}^{\alpha}_{\mathbb{R},K}))_{x\in\mathbb{R}}$ as a stationary solution to the SDE

$$d\ell^x(\mathcal{L}^{\alpha}_{\mathbb{R},K}) = 2(\ell^x(\mathcal{L}^{\alpha}_{\mathbb{R},K}))^{1/2}dW(x) + 2\alpha dx. \tag{10.4}$$

For $x,y\in\mathbb{R}$, we will denote by $\mu^{x,y}_{\mathbb{R},K}$ the massive measure on Brownian excursions from x to y:

$$\mu^{x,y}_{\mathbb{R},K} = \int_0^{+\infty} dte^{-Kt}p_{\mathbb{R}}(t,x,y)\mathbb{P}^{x,y}_{\mathbb{R},t}.$$

The measures $\mu^{x,y}_{\mathbb{R},K}$ are involved in the BFS-Dynkin isomorphism for the field ϕ (Theorem 1.4). In the 1D setting, the measures $\mu^{x,y}_{\mathbb{R},K}$ can be further decomposed as follows. Let be $x < y \in \mathbb{R}$. Let τ_x be the first hitting time of level x by a Brownian motion $(B_t)_{t\geq 0}$. Let $\check{\mu}^{x,y}_{\mathbb{R},K}$ be the following measure on paths:

$$\int F(\wp)\check{\mu}^{x,y}_{\mathbb{R},K}(d\wp) = \mathbb{E}_{B_0=y}\Big[e^{-K\tau_x}F((B_{\tau_x - t})_{0\leq t\leq\tau_x})\Big].$$

The total mass of $\check{\mu}^{x,y}_{\mathbb{R},K}$ is

$$\mathbb{E}_{B_0=y}\Big[e^{-K\tau_x}\Big] = \frac{G_{\mathbb{R},K}(x,y)}{G_{\mathbb{R},K}(x,x)} = e^{-\sqrt{2K}(y-x)}.$$

Further, the measure $\mu^{x,y}_{\mathbb{R},K}$ is the image of the product measure $\mu^{x,x}_{\mathbb{R},K}\otimes\check{\mu}^{x,y}_{\mathbb{R},K}$ under the concatenation of two paths.

10.4 β-Dyson's Brownian motion and Le Jan type isomorphism

For references on β-Dyson's Brownian motion, see [Dys62, Cha92, RS93, CL97, CL07], [Meh04, Chapter 9] and [AGZ09, Section 4.3]. Let $\beta \geq 0$ and $n \geq 2$. Let $K > 0$. We consider the process $(\lambda(x) = (\lambda_1(x),\dots,\lambda_n(x)))_{x\in\mathbb{R}}$ with $\lambda_1(x)\geq\dots\geq\lambda_n(x)$, satisfying the SDE

$$d\lambda_j(x) = \sqrt{2}dW_j(x) - \sqrt{2K}\,\lambda_j(x) + \beta\sqrt{2K}\sum_{j'\neq j}\frac{dx}{\lambda_j(x)-\lambda_{j'}(x)}, \tag{10.5}$$

the dW_j, $1 \leq j \leq n$, being n i.i.d. white noises on $\mathbb{R}$, and λ being stationary, with $(2K)^{\frac{1}{4}}\lambda(x)$ being distributed according to (10.1). This is the stationary β-Dyson's Brownian motion. For $\beta \in \{1, 2, 4\}$, this is the diffusion of eigenvalues in a stationary matrix-valued Ornstein-Uhlenbeck process. Since we will be interested in the isomorphisms of the β-Dyson's Brownian motion with 1D Brownian local time, we will see x as a one-dimensional space variable rather than a time variable. For $\beta \geq 1$, there is no collision between the $\lambda_j(x)$-s, and for $\beta \in [0, 1)$ two consecutive $\lambda_j(x)$-s can collide, but there is no collision of three or more particles [CL07]. Note that the case $\beta \in (-2/n, 0)$ has not been considered in the literature, although I believe that the β-Dyson's Brownian motion should still exist for this range of β. However, the problem is the extension of the process after a collision of $\lambda_j(x)$-s. If $\beta < 0$, the collision of three or more particles, including all the n together for $\beta < -\frac{2(n-3)}{n(n-1)}$, is no longer excluded.

Next are some elementary properties of the β-Dyson's Brownian motion. See [3, Proposition 4.2].

Proposition 10.3. *The following holds.*

1. *The process* $\left(\frac{1}{\sqrt{n}}p_1(\lambda(x))\right)_{x\in\mathbb{R}}$ *has the same law as* ϕ*; see Section 10.3.*
2. *The process* $(\frac{1}{2}p_2(\lambda(x)))_{x\in\mathbb{R}}$ *is a stationary solution to the SDE*

$$dZ(x) = 2(Z(x))^{1/2}dW(x) + d(\beta, n)dx, \tag{10.6}$$

   where $d(\beta, n)$ *is given by* (10.2)*.*

3. *The processes* $(p_1(\lambda(x)))_{x\in\mathbb{R}}$ *and* $\left(\lambda(x) - \frac{1}{n}p_1(\lambda(x))\right)_{x\in\mathbb{R}}$ *are independent.*
4. *The process* $\left(\frac{1}{2}\left(p_2(\lambda(x)) - \frac{1}{n}p_1(\lambda(x))^2\right)\right)_{x\in\mathbb{R}}$ *is a stationary solution to the SDE*

$$dZ(x) = 2(Z(x))^{1/2}dW(x) + (d(\beta, n) - 1)dx. \tag{10.7}$$

By comparing (10.6) and (10.7) to (10.4), one can observe that the Le Jan's isomorphism (Theorem 1.13) has a generalization to β-Dyson's Brownian motion and it involves a whole range of α-s, not just α half-integers.

Corollary 10.4 ([3], Proposition 4.21)**.** *Take* $\alpha = d(\beta, n)/2$*. The process* $(\frac{1}{2}p_2(\lambda(x)))_{x\in\mathbb{R}}$ *has the same law as the occupation field* $(\ell^x(\mathcal{L}^{\alpha}_{\mathbb{R},K}))_{x\in\mathbb{R}}$ *of a 1D massive Brownian loop soup* $\mathcal{L}^{\alpha}_{\mathbb{R},K}$*. Moreover, the process* $\left(\frac{1}{2}\left(p_2(\lambda(x)) - \frac{1}{n}p_1(\lambda(x))^2\right)\right)_{x\in\mathbb{R}}$ *has the same law as* $(\ell^x(\mathcal{L}^{\alpha-1/2}_{\mathbb{R},K}))_{x\in\mathbb{R}}$*.*

10.5 BFS-Dynkin type isomorphism for β-Dyson's Brownian motion

We observed that the Le Jan's isomorphism has a generalization for the β-Dyson's Brownian motion (Corollary 10.4). Actually, the BFS-Dynkin isomorphism (Theorem 1.4) has such a generalization, too. This is proved in my article [3]. For $\beta \in \{0, 1, 2, 4\}$ this reduces to the Gaussian case; see Theorem 9.3. But for general values of β this requires involved combinatorics, which are not just partitions in pairs as in the Wick's rule.

To begin with, we will present the expression for the symmetric moments of a β-Dyson's Brownian motion. Let the brackets $\langle\cdot\rangle^{\mathbb{R},K}_{\beta,n}$ denote the expectation w.r.t. to a stationary β-Dyson's Brownian motion (10.5). Let $\nu = (\nu_1, \nu_2, \dots, \nu_{m(\nu)})$ be a family of positive integers and consider $m(\nu)$ points $x_1 < x_2 < \cdots < x_{m(\nu)} \in \mathbb{R}$. We are interested in an expression for

$$\Big\langle \prod_{k=1}^{m(\nu)} p_{\nu_k}(\lambda(x_k)) \Big\rangle_{\beta,n}^{\mathbb{R},K}. \tag{10.8}$$

Curiously enough, despite an important literature on the β-Dyson's Brownian motion, an expression for these symmetric moments was nowhere to be found. So I gave one in my paper; see [3, Section 4.2]. It involve a recurrence similar to that of Proposition 10.2. I am not aware whether it has appeared previously elsewhere. The recurrence will be introduced in what follows.

Let $(\mathsf{Y}_{kk})_{k\geq 1}$ denote a family of formal commuting polynomials variables. We will consider finite families of positive integers $\nu = (\nu_1, \nu_2, \ldots, \nu_{m(\nu)})$ with $|\nu|$ even. The order of the ν_k-s will matter. That is to say we distinguish between ν and $(\nu_{\sigma(1)}, \nu_{\sigma(2)}, \ldots, \nu_{\sigma(m(\nu))})$ for σ a permutation of $\{1, \ldots, m(\nu)\}$. We want to construct a family of formal polynomials $Q_{\nu,\beta,n}$ with parameters ν,β and n, where $Q_{\nu,\beta,n}$ has for variables $(\mathsf{Y}_{kk})_{1\leq k\leq m(\nu)}$. To simplify the notations, we will drop the subscripts β, n and just write Q_ν. The polynomials Q_ν will appear in the expression of the symmetric moments (10.8). We will denote by $c(\nu, \beta, n)$ the solutions to the recurrence (10.3), which for $\beta \in (-2/n, +\infty)$ are the moments $\langle p_\nu(\lambda)\rangle_{\beta,n}$ of the GβE(n). By convention, $c((0), \beta, n) = n$ and $c(\emptyset, \beta, n) = 1$. For $k \geq 1$ and Q a polynomial, $Q^{k\leftarrow}$ will denote the polynomial in the variables $(\mathsf{Y}_{k'k'})_{1\leq k'\leq k}$, obtained from Q by replacing each variable $\mathsf{Y}_{k'k'}$ with $k' \geq k+1$ by the variable Y_{kk}. Note that $Q_\nu^{m(\nu)\leftarrow} = Q_\nu$ and that $Q_\nu^{1\leftarrow}$ is an univariate polynomial in Y_{11}. For Y a formal polynomial variable, $\deg_{\mathsf{Y}}$ will denote the partial degree in Y.

See next page.

Definition 10.5 ([3], Definition 4.7)**.** The family of polynomials $(Q_\nu)_{|\nu| \text{ even}}$ is defined by the following.

1. $Q_\nu^{1\leftarrow} = c(\nu,\beta,n)\mathsf{Y}_{11}^{|\nu|/2}$.
2. If $m(\nu) \geq 2$, then for every $k \in \{2,\dots,m(\nu)\}$,

$$
\begin{aligned}
\frac{\partial}{\partial \mathsf{Y}_{kk}} Q_\nu^{k\leftarrow} \quad = \quad & \frac{\beta}{2} \sum_{\substack{k\leq k'\leq m(\nu)\\ \nu_{k'}>2}} \frac{\nu(k')}{2} \sum_{i=2}^{\nu_{k'}-2} Q^{k\leftarrow}_{((\nu_r)_{r\neq k'},i-1,\nu_{k'}-1-i)} \\
&+\frac{\beta}{2} n \sum_{\substack{k\leq k'\leq m(\nu)\\ \nu_{k'}>2}} \nu(k') Q^{k\leftarrow}_{((\nu_r)_{r\neq k'},\nu_{k'}-2)} \\
&+\frac{\beta}{2} n^2 \sum_{\substack{k\leq k'\leq m(\nu)\\ \nu_{k'}=2}} Q^{k\leftarrow}_{(\nu_r)_{r\neq k'}} \\
&+\Big(1-\frac{\beta}{2}\Big) \sum_{\substack{k\leq k'\leq m(\nu)\\ \nu_{k'}>2}} \frac{\nu_{k'}(\nu_{k'}-1)}{2} Q^{k\leftarrow}_{((\nu_r)_{r\neq k'},\nu_{k'}-2)} \\
&+\Big(1-\frac{\beta}{2}\Big) n \sum_{\substack{k\leq k'\leq m(\nu)\\ \nu_{k'}=2}} Q^{k\leftarrow}_{(\nu_r)_{r\neq k'}} \\
&+ \sum_{\substack{k\leq k'<k''\leq m(\nu)\\ \nu_{k'}+\nu_{k''}>2}} \nu_{k'}\nu_{k''} Q^{k\leftarrow}_{((\nu_r)_{r\neq k',k''},\nu_{k'}+\nu_{k''}-2)} \\
&+ n \sum_{\substack{k\leq k'<k''\leq m(\nu)\\ \nu_{k'}=\nu_{k''}=1}} Q^{k\leftarrow}_{(\nu_r)_{r\neq k',k''}}.
\end{aligned}
\tag{10.9}
$$

   If $k = m(\nu)$, then the last two lines of (10.9) vanish.

Proposition 10.6 ([3], Proposition 4.8)**.** *Definition 10.5 uniquely defines a family of polynomials $(Q_\nu)_{|\nu| \text{ even}}$. Moreover, the following properties hold.*

1. *For every A monomial of Q_ν and every $k \in \{2,\dots,m(\nu)\}$,*

$$
2 \sum_{k\leq k'\leq m(\nu)} \deg_{\mathsf{Y}_{k'k'}} A \leq \sum_{k\leq k'\leq m(\nu)} \nu_{k'}, \tag{10.10}
$$

   and

$$
2 \sum_{1\leq k'\leq m(\nu)} \deg_{\mathsf{Y}_{k'k'}} A = |\nu|.
$$

   In particular, Q_ν is a homogeneous polynomial of degree $|\nu|/2$.

2. *For every $k \in \{1,\dots,m(\nu)\}$ and every permutation σ of $\{k,\dots,m(\nu)\}$,*

$$
Q^{k\leftarrow}_{(\nu_r)_{1\leq r\leq k-1},(\nu_{\sigma(r)})_{k\leq r\leq m(\nu)}} = Q^{k\leftarrow}_\nu.
$$

On top of the formal commuting polynomial variables $(\mathsf{Y}_{kk})_{k\geq 1}$ appearing in the polynomials Q_ν, we also consider the family of the formal commuting variables $(\check{\mathsf{Y}}_{k-1\,k})_{k\geq 2}$, also commuting with the first one. A polynomial $P_\nu = P_{\nu,\beta,n}$ will have for variables $(\mathsf{Y}_{kk})_{1\leq k\leq m(\nu)}$ and $(\check{\mathsf{Y}}_{k-1\,k})_{2\leq k\leq m(\nu)}$.

Definition 10.7 ([3], Definition 4.10)**.** Given ν a finite family of positive integers with $|\nu|$ even, let P_ν be the polynomial in the variables $(\mathsf{Y}_{kk})_{1\leq k\leq m(\nu)}, (\check{\mathsf{Y}}_{k-1\,k})_{2\leq k\leq m(\nu)}$ defined by the following.

1. $P_\nu((\mathsf{Y}_{kk})_{1\leq k\leq m(\nu)}, (\check{\mathsf{Y}}_{k-1\,k} = 1)_{2\leq k\leq m(\nu)}) = Q_\nu((\mathsf{Y}_{kk})_{1\leq k\leq m(\nu)})$.
2. For every A monomial of P_ν and every $k \in \{2, \dots, m(\nu)\}$,

$$\deg_{\check{\mathsf{Y}}_{k-1\,k}} A + 2 \sum_{k\leq k'\leq m(\nu)} \deg_{\mathsf{Y}_{k'k'}} A = \sum_{k\leq k'\leq m(\nu)} \nu_{k'}.$$

The property (10.10) ensures that $P_\nu = P_{\nu,\beta,n}$ is well defined. The polynomials P_ν are involved in the expression of the symmetric moments (10.8); see Theorem 10.8 below. As an illustration, we provide below some examples of polynomials P_ν.

$$\begin{aligned} \langle p_2(\lambda)p_1(\lambda)^2\rangle_{\beta,n} &= \frac{\beta}{2}n^3 + \Big(1-\frac{\beta}{2}\Big)n^2 + 2n, \\ P_{(2,1,1)} &= \Big(\frac{\beta}{2}n^3 + \Big(1-\frac{\beta}{2}\Big)n^2\Big)\mathsf{Y}_{11}\mathsf{Y}_{22}\check{\mathsf{Y}}_{23} + 2n\mathsf{Y}_{11}^2\check{\mathsf{Y}}_{12}^2\check{\mathsf{Y}}_{23}, \\ P_{(1,2,1)} &= \Big(\frac{\beta}{2}n^3 + \Big(1-\frac{\beta}{2}\Big)n^2 + 2n\Big)\mathsf{Y}_{11}\check{\mathsf{Y}}_{12}\mathsf{Y}_{22}\check{\mathsf{Y}}_{23}, \\ P_{(1,1,2)} &= \Big(\frac{\beta}{2}n^3 + \Big(1-\frac{\beta}{2}\Big)n^2\Big)\mathsf{Y}_{11}\check{\mathsf{Y}}_{12}\mathsf{Y}_{33} + 2n\mathsf{Y}_{11}\check{\mathsf{Y}}_{12}\mathsf{Y}_{22}\check{\mathsf{Y}}_{23}^2, \end{aligned} \tag{10.11}$$

Theorem 10.8 ([3], Proposition 4.9, Corollary 4.11 and Proposition 4.22)**.** *Let $\nu = (\nu_1, \nu_2, \dots, \nu_{m(\nu)})$ be a family of positive integers, with $|\nu|$ even, and consider $m(\nu)$ points $x_1 < x_2 < \cdots < x_{m(\nu)} \in \mathbb{R}$. Then the moment* (10.8) *is obtained by evaluating the polynomial P_ν as follows:*

$$\Big\langle \prod_{k=1}^{m(\nu)} p_{\nu_k}(\lambda(x_k))\Big\rangle_{\beta,n}^{\mathbb{R},K} = P_\nu((\mathsf{Y}_{kk} = 1/\sqrt{2K})_{1\leq k\leq m(\nu)}, (\check{\mathsf{Y}}_{k-1\,k} = e^{-\sqrt{2K}(x_k-x_{k-1})})_{2\leq k\leq m(\nu)}).$$

The polynomials P_ν are also involved in the BFS-Dynkin type isomorphism for β-Dyson's Brownian motion. Given ν a finite family of positive integers with $|\nu|$ even and $x_1 < x_2 < \cdots < x_{m(\nu)} \in \mathbb{R}$, $\mu_{\mathbb{R},K}^{\nu,x_1,\dots,x_{m(\nu)}}$ (also depending on β and n) will be the measure on finite families of continuous paths obtained by substituting in the polynomial $P_\nu = P_{\nu,\beta,n}$ for each variable Y_{kk} the measure $\mu_{\mathbb{R},K}^{x_k,x_k}$, and for each variable $\check{\mathsf{Y}}_{k-1\,k}$ the measure $\check{\mu}_{\mathbb{R},K}^{x_{k-1},x_k}$; see Section 10.3. Since we will deal with the occupation fields under $\mu_{\mathbb{R},K}^{\nu,x_1,\dots,x_{m(\nu)}}$, the order of the Brownian measures in a product will not matter. For instance, for $\nu = (2,1,1)$, the expression for $P_{(2,1,1)}$ appears in (10.11), and

$$\begin{aligned} \mu_{\mathbb{R},K}^{(2,1,1),x_1,x_2,x_3} &= \Big(\frac{\beta}{2}n^3 + \Big(1-\frac{\beta}{2}\Big)n^2\Big)\mu_{\mathbb{R},K}^{x_1,x_1} \otimes \mu_{\mathbb{R},K}^{x_2,x_2} \otimes \check{\mu}_{\mathbb{R}_+}^{x_2,x_3} \\ &\quad + 2n\mu_{\mathbb{R},K}^{x_1,x_1} \otimes \mu_{\mathbb{R},K}^{x_1,x_1} \otimes \check{\mu}_{\mathbb{R},K}^{x_1,x_2} \otimes \check{\mu}_{\mathbb{R},K}^{x_1,x_2} \otimes \check{\mu}_{\mathbb{R},K}^{x_2,x_3}. \end{aligned}$$

Note that depending on values of n and β, a measure $\mu_{\mathbb{R},K}^{\nu,x_1,\dots,x_{m(\nu)}}$ may be signed.

Theorem 10.9 ([3], Proposition 4.14 and Proposition 4.22)**.** *Let ν be a finite family of positive integers, with $|\nu|$ even and let $x_1 < x_2 < \cdots < x_{m(\nu)} \in \mathbb{R}$. Let F be a bounded measurable functional on $\mathcal{C}(\mathbb{R})$. Then*

$$\Big\langle \prod_{k=1}^{m(\nu)} p_{\nu_k}(\lambda(x_k)) F\Big(\Big(\frac{1}{2}p_2(\lambda(x))\Big)_{x\in\mathbb{R}}\Big)\Big\rangle_{\beta,n}^{\mathbb{R},K} =$$

$$\int \Big\langle F\Big(\Big(\frac{1}{2}p_2(\lambda(x)) + \sum \ell^x(\wp_j)\Big)_{x\in\mathbb{R}}\Big)\Big\rangle_{\beta,n}^{\mathbb{R},K} \mu_{\mathbb{R},K}^{\nu,x_1,\dots,x_{m(\nu)}}((d\wp_j)_j).$$

10.6 Open questions

Here we present the open questions that motivated the paper [3]. The first question is combinatorial. We would like to have the polynomials $P_{\nu,\beta,n}$ given by Definitions 10.5 and 10.7 under a more explicit form. The recurrence on polynomials (10.9) is closely related to the Schwinger-Dyson equation (10.3). Its very form suggests that the polynomials $P_{\nu,\beta,n}$ might be expressible as weighted sums over maps drawn on 2D compact surfaces (not necessarily connected), where the maps associated to ν have $m(\nu)$ vertices with degrees given by $\nu_1, \nu_2, \dots, \nu_{m(\nu)}$, with powers of n corresponding to the number of faces. This is indeed the case for $\beta \in \{1, 2, 4\}$, and this corresponds to the topological expansion of matrix integrals; see Chapter 9.

Question 10.10. *Is there a more explicit expression for the polynomials $P_{\nu,\beta,n}$? Can they be expressed as weighted sums over the maps on 2D surfaces (topological expansion)?*

The second question is whether there is a natural generalization of Gaussian beta ensembles and β-Dyson's Brownian motion to electrical networks.

Question 10.11. *We are in the setting of an electrical network $\mathcal{G} = (V, E)$ as in Section 1.1.1. Given $n \geq 2$ and $\beta > -\frac{2}{n}$, is there a distribution on the fields $(\lambda(x) = (\lambda_1(x), \lambda_2(x), \dots, \lambda_n(x)))_{x\in V}$, with $\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_n(x)$, satisfying the following properties?*

1. *For $\beta \in \{1, 2, 4\}$, λ is distributed as the fields of ordered eigenvalues in a GFF with values into $n \times n$ matrices, real symmetric ($\beta = 1$), complex Hermitian ($\beta = 2$), resp. quaternionic Hermitian ($\beta = 4$) (see Chapter 9).*
2. *For $\beta = 0$, λ is obtained by reordering n i.i.d. scalar GFFs.*
3. *As $\beta \to -\frac{2}{n}$, λ converges in law to*
$$\Big(\frac{1}{\sqrt{n}}\phi, \frac{1}{\sqrt{n}}\phi, \dots, \frac{1}{\sqrt{n}}\phi\Big),$$
where ϕ is a scalar GFF on $\mathcal{G}$.
4. *For every $x \in V_{\text{int}}$, $\lambda(x)/\sqrt{G(x,x)}$ is distributed as the GβE (10.1).*
5. *For every $x, y \in V_{\text{int}}$, the couple $(\lambda(x)/\sqrt{G(x,x)}, \lambda(y)/\sqrt{G(y,y)})$ is distributed as the values of a β-Dyson's Brownian motion (10.5) (with $K = 1/2$) at points 0 and $-\log(G(x,y)/\sqrt{G(x,x)G(y,y)})$.*
6. *The fields $p_1(\lambda)$ and $\lambda - \frac{1}{n}p_1(\lambda)$ are independent.*
7. *The field $\frac{1}{\sqrt{n}}p_1(\lambda)$ is distributed as a scalar GFF.*
8. *The field $\frac{1}{2}\big(p_2(\lambda) - \frac{1}{n}p_1(\lambda)^2\big)$ is distributed as the occupation field of the continuous-time random walk loop soup $\mathcal{L}^{\alpha-\frac{1}{2}}$.*
9. *The symmetric moments*
$$\Big\langle \prod_{k=1}^{m(\nu)} p_{\nu_k}(\lambda(x_k)) \Big\rangle_{\beta,n}^{\mathcal{G}}$$
are linear combination of products
$$\prod_{1\leq k\leq k'\leq m(\nu)} G(x_k, x_{k'})^{a_{kk'}},$$

with $a_{kk'} \in \mathbb{N}$ *and for every* $k \in \{1, \dots, m(\nu)\}$*,*

$$2a_{kk} + \sum_{\substack{1 \leq k' \leq m(\nu) \\ k' \neq k}} a_{kk'} = \nu_k,$$

the coefficients of the linear combination being universal polynomials in β *and* n*, not depending on the electrical network and its parameters; see also Question 10.10.*

10. λ *satisfies a BFS-Dynkin type isomorphism with continuous time random walks.*

In the simplest case $n = 2$, the answer to the question above is yes on any electrical network. I provided a simple construction relying precisely on random walk loop soups in [3, Section 5.2]. For $n \geq 3$ things become complicated. If the graph $\mathcal{G}$ is a tree, the natural generalization λ of the β-Dyson's Brownian motion is straightforward to construct. In absence of cycles, λ satisfies a Markov property, and along each branch of the tree one has the values of a β-Dyson's Brownian motion at different positions. However, if the graph $\mathcal{G}$ contains cycles, constructing λ is not immediate, and I have not encountered such a construction in the literature. One does not expect a Markov property, since already for $\beta \in \{1, 2, 4\}$ one has to take into account the angular part of the matrices.

Part V

Inverting the isomorphism theorems: relation to self-interacting random walks

Chapter 11

GFF and the combinatorics of the Ising model

In this Chapter is explained the relation between between the discrete GFF, the metric graph GFF and the random walk loop soup on one side and the Ising related models, spin Ising, FK-Ising, random currents, on the other side. The latter appear once one conditions by the absolute value of the discrete GFF on vertices. This relation led Wendelin Werner and myself [12] to a probabilistic coupling between the FK-Ising and the random current model; see Proposition 11.4.

11.1 Spin Ising, FK-Ising and random currents

Let $\widehat{\mathcal{G}} = (\widehat{V}, \widehat{E})$ be a finite undirected graph, and for $\{x, y\} \in \widehat{E}$, consider coupling constants $J(x, y) = J(y, x) > 0$.

Definition 11.1. The *spin Ising field* is a random configuration $(\hat{\sigma}(x))_{x\in\widehat{V}} \in \{-1, 1\}^{\widehat{V}}$, such that for every $\sigma \in \{-1, 1\}^{\widehat{V}}$

$$\mathbb{P}(\hat{\sigma} = \sigma) = \frac{1}{Z_{\text{Ising}}} \exp\Big(\sum_{\{x,y\}\in\widehat{E}} J(x, y)\sigma(x)\sigma(y)\Big). \tag{11.1}$$

The *FK-Ising* model is a rendom configuration of edges $(\hat{w}_e)_{e\in\widehat{E}} \in \{0, 1\}^{\widehat{E}}$ (0 for closed and 1 for open) with the following distribution. For every $w \in \{0, 1\}^{\widehat{E}}$,

$$\mathbb{P}(\hat{w} = w) = \frac{1}{Z_{\text{FK}}} 2^{k(w)} \prod_{e\in\widehat{E}} (1 - e^{-2J(e)})^{w_e} (e^{-2J(e)})^{1-w_e},$$

where $k(w)$ is the number of connected components induced by the open edges $\{e \in \widehat{E} | w_e = 1\}$.

Given $S \subset \widehat{V}$ with $|S|$ even, the *random current* with *sources* in S is a random configuration $(\hat{n}_e)_{e\in\widehat{E}} \in \mathbb{N}^{\widehat{E}}$ that is *even* outside S and odd in S in the following sense. For every $x \in \widehat{V} \setminus S$,

$$\sum_{\substack{e\in\widehat{E} \\ e \text{ adjacent to } x}} \hat{n}_e$$

is even, and for every $x \in S$ it is odd. The distribution of $(\hat{n}_e)_{e\in\widehat{E}} \in \mathbb{N}^{\widehat{E}}$ is given by the following. For every $(n_e)_{e\in\widehat{E}} \in \mathbb{N}^{\widehat{E}}$,

$$\mathbb{P}(\hat{n} = n) = \mathbf{1}_{(n_e)_{e\in\widehat{E}} \text{ admissible}} \frac{1}{Z_{\text{RC,S}}} \prod_{e\in\widehat{E}} \frac{J(e)^{n_e}}{n_e!}.$$

If $S = \emptyset$, the random current is said to be *sourceless*.

For more on the FK-Ising model, see [Gri06]. For more on random current model, see [Aiz82, DC16].

Now let us recall how the three models above are related. We start with the Edwards-Sokal coupling between spin Ising and FK-Ising [ES88].

Theorem 11.2 (Edwards-Sokal)**.** *Given an FK-Ising model, sample on each cluster an independent uniformly distributed spin. The spins are then distributed according to the Ising model. Conversely, given a spin configuration $\hat{\sigma}$ following the Ising distribution, consider each edge $\{x,y\} \in \widehat{E}$, such that $\hat{\sigma}(x)\hat{\sigma}(y) < 0$, closed, and each edge $\{x,y\} \in \widehat{E}$, such that $\hat{\sigma}(x)\hat{\sigma}(y) > 0$ open with probability $1 - e^{-2J(x,y)}$. Then this edge configuration is distributed according to the FK-Ising model. The two couplings between FK-Ising and spin Ising are the same.*

Further let us recall the relation between the spin Ising model and the random currents at the level of partition functions. This is elementary and well known, and follows from expanding the exponential in (11.1) into a series and removing the terms that get canceled out.

Proposition 11.3. *The following identity between partition functions holds: $Z_{\text{Ising}} = 2^{|\widehat{V}|} Z_{\text{RC},\emptyset}$. Moreover, given points $x_1, x_2, \dots, x_{2k} \in \widehat{V}$, two by two distinct, set $S = \{x_1, x_2, \dots, x_{2k}\}$. Then*

$$\mathbb{E}[\hat{\sigma}(x_1)\hat{\sigma}(x_2)\dots\hat{\sigma}(x_{2k})] = \frac{Z_{\text{RC,S}}}{Z_{\text{RC},\emptyset}}.$$

Then, in our note [12], Werner and myself observed that there is actually a probabilistic coupling between the random current model and the FK-Ising model. It appears that this has not been known before. This is the Ising version of the idea of opening additional edges that appeared in Theorem 3.6. Sometimes this is referred to as *sprinkling*. See Section 11.2 for more explanations.

Proposition 11.4 ([12])**.** *Let $(\hat{n}_e)_{e\in\widehat{E}} \in \mathbb{N}^{\widehat{E}}$ be a sourceless random current. Let $(\bar{\omega}_e)_{e\in\widehat{E}} \in \{0,1\}^{\widehat{E}}$ be an independent Bernoulli percolation, with*

$$\mathbb{P}(\bar{\omega}_e = 1) = 1 - e^{-J(e)}.$$

Set

$$\hat{\omega}_e = (\hat{n}_e \wedge 1) \vee \bar{\omega}_e,$$

that is to say an edge e is open for the configuration $\hat{\omega}$ is either $\hat{n}_e > 0$, or e is open for $\bar{\omega}$. Then the configuration $\hat{\omega}$ follows an FK-Ising distribution.

The coupling of Proposition 11.4 starts from a random current, and with some additional randomness returns an FK-Ising. It is possible to describe the same coupling by starting from the FK-Ising. This is not in [12], as it has been observed only later. Perhaps it did not appear anywhere. We state this next.

Proposition 11.5. *Let $(\hat{\omega}_e)_{e\in\widehat{E}} \in \mathbb{N}^{\widehat{E}}$ be an FK-Ising. Further, for each $e \in \widehat{E}$, consider a Poisson r.v. $\widehat{N}_e$ with parameter $J(e)$, with different Poisson r.v.s. being independent, and the whole family independent from $\hat{\omega}$. Let $(\hat{n}_e)_{e\in\widehat{E}} \in \mathbb{N}^{\widehat{E}}$ be sampled among all the configurations $(n_e)_{e\in\widehat{E}} \in \mathbb{N}^{\widehat{E}}$ satisfying the following two constraints:*

1. *for every $e \in \widehat{E}$, $n_e \leq \hat{\omega}_e(1+\widehat{N}_e)$;*
2. *for every $x \in \widehat{V}$, $\sum_{\substack{e\in\widehat{E} \\ e \text{ adjacent to } x}} n_e$ is even;*

with weights proportional to

$$\prod_{\substack{e\in\widehat{E} \\ \hat{\omega}_e=1}} \frac{(1+\widehat{N}_e)!}{n_e!(1+\widehat{N}_e-n_e)!}.$$

Then $\hat{n}$ is distributed as a sourceless random current. Moreover, the joint distribution of $(\hat{n},\hat{\omega})$ is the same as in Proposition 11.4.

11.2 Real scalar GFF conditioned on its absolute value and Ising model

Let $\mathcal{G} = (V, E)$ be an electrical network as in Section 1.1.1, with conductances $C(x, y)$. Compared to Section 11.1, let be $\widehat{V} = V_{\text{int}}$ and

$$\widehat{E} = \{\{x,y\} \in E | x, y \in V_{\text{int}}\}.$$

Let $\widetilde{\mathcal{G}}$ be the metric graph associated to $\mathcal{G}$ (Definition 3.3). Let ϕ be a massless discrete GFF on $\mathcal{G}$ with 0 boundary conditions (Definition 1.1) and let be its metric graph extension $\tilde{\phi}$ on $\widetilde{\mathcal{G}}$ (Definition 3.7). Let $\mathcal{L}^{1/2}_{\widetilde{\mathcal{G}}}$ be a metric graph loop soup of intensity parameter $\alpha = 1/2$, coupled to $\tilde{\phi}$ as in Theorem 3.8. Let $\mathcal{L}^{1/2}$ be the random walk loop soup obtained from $\mathcal{L}^{1/2}_{\widetilde{\mathcal{G}}}$ by consider the traces of the latter on the vertices V. Consider the edge configuration $(\tilde{\omega}_e)_{e\in E}$ given by (3.3), with $\alpha = 1/2$:

$$\tilde{\omega}_e = \begin{cases} 1 & \text{if } \forall x \in I_e, \tilde{\phi}(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that for $e \in E \setminus \widehat{E}$, $\tilde{\omega}_e = 0$. For $e \in E$, denote by $N_e(\mathcal{L}^{1/2})$ the total number of crossings of the edge e by the random walk loop soup $\mathcal{L}^{1/2}$, in either of the directions. Note that for $e \in E \setminus \widehat{E}$, $N_e(\mathcal{L}^{1/2}) = 0$. For $\{x, y\} \in \widehat{E}$, denote

$$J_{|\phi|}(x, y) = C(x, y)|\phi(x)\phi(y)|.$$

Next we explain how the isomorphisms for the GFF are related to the combinatorics of the Ising model when one conditions on the absolute value of the GFF on vertices.

Proposition 11.6. *Conditionally on the absolute value on vertices $(|\phi(x)|)_{x\in V}$, the following holds:*

1. *The field $(\text{sign}(\phi(x)))_{x\in\widehat{V}}$ is distributed as a spin Ising field with coupling constants $J_{|\phi|}(x, y)$.*
2. *The configuration of edges $(\tilde{\omega}_e)_{e\in\widehat{E}}$ follows an FK-Ising distribution with coupling constants $J_{|\phi|}(x, y)$.*

3. *The field* $(N_e(\mathcal{L}^{1/2}))_{e\in\widehat{E}}$ *is distributed as a sourceless random current with coupling constants* $J_{|\phi|}(x,y)$.

4. *The fields* $(\text{sign}(\phi(x)))_{x\in\widehat{V}}$ *and* $(\tilde{\omega}_e)_{e\in\widehat{E}}$ *are coupled as in the Edwards-Sokal coupling (Theorem 11.2).*

5. *The fields* $(\tilde{\omega}_e)_{e\in\widehat{E}}$ *and* $(N_e(\mathcal{L}^{1/2}))_{e\in\widehat{E}}$ *are coupled as in Proposition 11.4.*

The point 1. above is obvious from the density of the discrete GFF $(\phi(x))_{x\in V}$. The point 3. has been observed by different people, including Le Jan. The point 2. was first observed by Werner and myself in our note [12]. The point 4. follows immediately from Theorem 3.8. The point 5. follows from Theorem 3.6. Precisely the five observations above led Werner and myself to the coupling of Proposition 11.4 [12].

Chapter 12

Inversion of the isomorphism theorems in discrete: relation to self-repelling jump processes

In this Chapter is presented the conditional law of a random walk loop soup of parameter $\alpha = 1/2$ given a discrete GFF. It appeared in a collaboration of myself with Sabot and Tarrès [10]. This inversion of isomorphism involves self-repelling jump processes, more precisely a variant of the Vertex Diminished Jump Process (VDJP) on FK-Ising type clusters, where the clusters themselves evolve by getting eroded over time. Section 12.1 is a general overview of the Vertex Diminished Jump Process, and its dual the Vertex Reinforced Jump Process. These two processes will also appear in Chapter 13. In Section 12.2 is presented the result of Sabot and Tarrès [ST15a] who gave the conditional law of the loop soup given the square of the GFF (without the signs). This conditional law involves an other variant of VDJP, the magnetized VDJP. In Section 12.3 is presented the result of Sabot, Tarrès end myself, for the signed GFF. It will be also applied in Chapter 13 for the construction of a continuous fine mesh limit of the VDJP in dimension one.

12.1 Vertex Reinforced and Vertex Diminished Jump Processes

In this section we will show a glimpse on two models of self-interacting random walks, the Vertex Reinforced (VRJP) and the Vertex Diminished (VDJP) Jump Processes, since the VDJP will appear in the sequel of this Chapter, and both the VRJP and the VDJP will appear in Chapter 13. This is by no means an exhaustive presentation.

Let $\widehat{\mathcal{G}} = (\widehat{V}, \widehat{E})$ be a finite undirected graph, and for $\{x, y\} \in \widehat{E}$, consider conductances $C(x, y) = C(y, x) > 0$.

Definition 12.1. Let L_0 be a positive function on $\widehat{V}$. Let $x_0 \in \widehat{V}$. The *Vertex Reinforced Jump Process* (VRJP) on $\widehat{\mathcal{G}}$ starting from x_0 with *initial occupation profile* L_0 is the nearest neighbor jump process $(\widehat{X}_s)_{s\geq 0}$, with $\widehat{X}_0 = x_0$, and for x, y two neighbors, the jump rate from x to y at time s being given by $C(x, y)\widehat{L}_s(y)$, where

$$\widehat{L}_s(y) = L_0(y) + \int_0^s \mathbf{1}_{\widehat{X}_{s'}=y} ds'.$$

The *Vertex Diminished Jump Process* (VDJP) on $\widehat{\mathcal{G}}$ starting from x_0 with *initial occupation profile* L_0 is the nearest neighbor jump process $(\check{X}_s)_{s\geq 0}$, with $\check{X}_0 = x_0$, and for x, y two neighbors, the jump rate from x to y at time s being given by $C(x,y)\check{L}_s(y)$, where

$$\check{L}_s(y) = L_0(y) - \int_0^s \mathbf{1}_{\check{X}_{s'}=y} ds'.$$

The VDJP is defined up to the time

$$\check{s}_{\max} = \inf\{s \geq 0 | \exists x \in \widehat{V}, \check{L}_s(x) = 0\},$$

which is finite a.s.

The VRJP, resp. VDJP, is a model of reinforced, resp. self-repelling random walk. The evolution of $(\widehat{X}_s, (\widehat{L}_s(x))_{x\in V})$ and that of $(\check{X}_s, (\check{L}_s(x))_{x\in V})$ is Markovian, but that of $\widehat{X}_s$ or $\check{X}_s$ alone is not.

Note that in the case of the VRJP there is global acceleration of the jump rates over time, and in the case of the VDJP there is a global slowdown of the process. Therefore, sometimes in the literature is used a different time scale which removes the global acceleration, resp. slowdown. For the VRJP, the time change is given by

$$dt = \widehat{L}_s(\widehat{X}_s) ds.$$

We will simply write $\widehat{X}_t$ (instead of $\widehat{X}_s$) for the process in this new time scale, and will further use only this time scale. The jump rate of $\widehat{X}_t$ from x to a neighbor y at time t is given by

$$C(x,y)\sqrt{\frac{L_0(y)^2 + 2\hat{\ell}_t(y)}{L_0(x)^2 + 2\hat{\ell}_t(x)}},$$

where

$$\hat{\ell}_t(z) = \int_0^t \mathbf{1}_{\widehat{X}_{t'}=z} dt'.$$

Similarly, for the VDJP we consider the time change

$$dt = \check{L}_s(\check{X}_s) ds.$$

We will simply write $\check{X}_t$ (instead of $\check{X}_s$) for the process in this new time scale, and will further use only this time scale. The jump rate of $\check{X}_t$ from x to a neighbor y at time t is given by

$$C(x,y)\sqrt{\frac{L_0(y)^2 - 2\check{\ell}_t(y)}{L_0(x)^2 - 2\check{\ell}_t(x)}}, \tag{12.1}$$

where

$$\check{\ell}_t(z) = \int_0^t \mathbf{1}_{\check{X}_{t'}=z} dt'.$$

In this time scale, the VDJP is defined up to the time

$$\check{t}_{\max} = \inf\{t \geq 0 | \exists x \in \widehat{V}, \check{\ell}_t(x) = L_0(x)^2/2\},$$

which is finite a.s.

The VRJP and the VDJP satisfy a remarkable property, the partial exchangeability. Roughly speaking, the infinitesimal weight of the path depends only on total times spent at each vertex and the number of jumps that have occurred along each edge, but not on the order of these jumps. In particular, this partial exchangeability implies, through an extension of de Finetti's theorem [Fre96], that the VRJP has the same distribution as a mixture of Markovian jump processes in a random environment. The explicit law of this random environment was given by Sabot and Tarrès [ST15b]. Further Bauerschmidt, Helmuth and Swan showed in [BHS21] that the VRJP and the VDJP satisfy BFS-Dynkin type isomorphism (Theorem 1.4). Indeed, the VRJP is related through an isomorphism to a random field with values into the hyperbolic space, and the VDJP to a random field with values into the half-spherical space.

12.2 From the squared GFF to the random walk loop soup through the magnetized VDJP

Let $\mathcal{G} = (V, E)$ be an electrical network as in Section 1.1.1, with conductances $C(x, y)$. Let ϕ be a massless discrete GFF on $\mathcal{G}$ with 0 boundary conditions, and $\mathcal{L}^{1/2}$ a random walk loop soup with $\alpha = 1/2$, coupled to ϕ through Le Jan's isomorphism (Theorem 1.13). One can ask the following question. What is the conditional law of $\mathcal{L}^{1/2}$ given $\phi^2/2 = (\ell^x(\mathcal{L}^{1/2}))_{x\in V}$? Let $x_0 \in V_{\text{int}}$. Sabot and Tarrès showed [ST15a] that one can obtain the conditional law of all the loops visiting x_0 through a self-interacting jump process, a magnetized version of the VDJP. We will describe this next.

Fix $x_0 \in V_{\text{int}}$. We will consider a Markovian evolution $(\check{X}_t, (\check{\Phi}_t(x))_{x\in V_{\text{int}}})$ where $\check{X}_t \in V_{\text{int}}$ and $\check{\Phi}_t$ is a positive field on V_{int}. For $\{x, y\} \in E$ with $x, y \in V_{\text{int}}$, denote

$$\check{J}_t(x, y) = C(x, y)\check{\Phi}_t(x)\check{\Phi}_t(y). \tag{12.2}$$

For $x, y \in V_{\text{int}}$, denote

$$\langle\sigma(x)\sigma(y)\rangle_t = \frac{\sum_{\sigma\in\{-1,1\}^{V_{\text{int}}}} \sigma(x)\sigma(y) \sum_{\substack{\{z,w\}\in E\\ z,w\in V_{\text{int}}}} e^{\check{J}_t(z,w)\sigma(z)\sigma(w)}}{\sum_{\sigma\in\{-1,1\}^{V_{\text{int}}}} \sum_{\substack{\{z,w\}\in E\\ z,w\in V_{\text{int}}}} e^{\check{J}_t(z,w)\sigma(z)\sigma(w)}}.$$

In other words, $\langle\sigma(x)\sigma(y)\rangle_t$ is a two-point correlation in a spin Ising field with coupling constants $\check{J}_t$. Note that $\langle\sigma(x)\sigma(y)\rangle_t \geq 0$, as one can see from Proposition 11.3. Further, denote

$$\check{\ell}_t(x) = \int_0^t \mathbf{1}_{\check{X}_{t'}=x} dt'. \tag{12.3}$$

The field $\check{\Phi}_t$ is given by

$$\check{\Phi}_t(x) = \sqrt{\check{\Phi}_0^2 - 2\check{\ell}_t(x)}, \tag{12.4}$$

that is to say

$$\frac{1}{2}\check{\Phi}_t(x)^2 = \frac{1}{2}\check{\Phi}_0(x)^2 - \check{\ell}_t(x).$$

The process $\check{X}_t$ is a nearest neighbor jump process, with jump rates from x to a neighbor y at time t given by

$$C(x, y)\frac{\check{\Phi}_t(y)\langle\sigma(x_0)\sigma(y)\rangle_t}{\check{\Phi}_t(x)\langle\sigma(x_0)\sigma(x)\rangle_t}.$$

By comparing (12.4) to (12.1), we see that the jump rates above are similar to those of a VDJP, but there are additional magnetization factors $\langle\sigma(x_0)\sigma(y)\rangle_t$ and $\langle\sigma(x_0)\sigma(x)\rangle_t$. The process $(\check{X}_t, (\check{\Phi}_t(x))_{x\in V_{\text{int}}})$ is defined up to the time

$$\check{t}_{\max} = \inf\{t \geq 0 | \exists x \in V_{\text{int}}, \check{\ell}_t(x) = \check{\Phi}_0(x)^2/2\}, \tag{12.5}$$

which is finite a.s.

Theorem 12.2 (Sabot-Tarrès). *Fix $x_0 \in V_{\text{int}}$. Let ϕ be a massless discrete GFF on $\mathcal{G}$ with 0 boundary conditions. Let $(\check{X}_t, (\check{\Phi}_t(x))_{x\in V_{\text{int}}})_{0\leq t\leq \check{t}_{\max}}$ be the self-interacting process as above, with initial conditions*

$$\check{X}_0 = x_0, \qquad \forall x \in V_{\text{int}},\ \check{\Phi}_0(x) = |\phi(x)|.$$

Then $\check{X}_{\check{t}_{\max}} = x_0$ a.s. The path $(\check{X}_t)_{0\leq t\leq \check{t}_{\max}}$ and the field $(\check{\Phi}_{\check{t}_{\max}}(x))_{x\in V_{\text{int}}}$ are independent. The path $(\check{X}_t)_{0\leq t\leq \check{t}_{\max}}$ is distributed as the concatenation of all the loops in the random walk loop soup $\mathcal{L}^{1/2}$ that visit x_0 (one roots the loops in x_0). The field $(\check{\Phi}_{\check{t}_{\max}}(x))_{x\in V_{\text{int}}}$ is distributed as the absolute value of a discrete GFF with boundary conditions 0 on $V_\partial \cup \{x_0\}$.

To see why the Ising magnetization factor should appear, check Section 11.2.

12.3 From the GFF with signs to the random walk loop soup through the VDJP on clusters

Consider now that the random walk loop soup $\mathcal{L}^{1/2}$ is coupled to the GFF ϕ through the metric graph extension as in Theorem 3.8. In particular the sign of ϕ is constant of each cluster of $\mathcal{L}^{1/2}$. One can further ask what is the conditional law of $\mathcal{L}^{1/2}$ given ϕ. That is to say one conditions also on the sign of ϕ, not just it absolute value as in Section 12.2. The answer to this question was given in a collaboration of myself with Sabot and Tarrès [10]. The conditional law involves a VDJP on FK-Ising type clusters, where the clusters themselves get eroded over time.

As previously, fix $x_0 \in V_{\text{int}}$. We will consider a Markovian evolution $(\check{X}_t, (\check{\Phi}_t(x))_{x\in V_{\text{int}}}, (\tilde{\omega}_t(e))_{e\in E})$ where $\check{X}_t \in V_{\text{int}}$, $\check{\Phi}_t$ is a positive field on V_{int} and $\tilde{\omega}_t \in \{0,1\}^E$. Specifically, $\check{X}_0 = x_0$ and $\check{X}_t$ will be at all time in the connected component of x_0 induced by the edge configuration $\tilde{\omega}_t$.

We will keep the notations (12.2), (12.3) and (12.5). As previously, we will have $\check{\Phi}_t(x) = \sqrt{\check{\Phi}_0^2 - 2\check{\ell}_t(x)}$, and the whole process will be defined only up to the time $\check{t}_{\max}$ (12.5). Given two neighbors $x, y \in V$, the jump rate of $\check{X}_t$ from x to y at time t is

$$C(x,y)\frac{\check{\Phi}_t(y)}{\check{\Phi}_t(x)}\tilde{\omega}_t(\{x,y\}),$$

that is to say the jump from x to y cannot occur if the edge $\{x,y\}$ is closed for $\tilde{\omega}_t$. Moreover, if $\tilde{\omega}_t(\{x,y\}) = 1$ and $\check{X}_t = x$, then the edge is $\{x,y\}$ is closed (i.e. $\tilde{\omega}_t(\{x,y\})$ set to 0) with rate

$$2C(x,y)\frac{\check{\Phi}_t(y)}{\check{\Phi}_t(x)}(e^{2\check{J}_t(x,y)} - 1)^{-1},$$

and conditionally on the last event,

- if x and y still belong to the same connected component induced by $\tilde{\omega}_t$ after $\{x,y\}$ closed, then $\check{X}_t$ instantaneously jumps to y with probability $1/2$ and stays in x with probability $1/2$;
- otherwise $\check{X}_t$ moves or stays with probability 1 on the unique extremity of $\{x,y\}$ which remains connected to x_0.

Theorem 12.3 ([10], Proposition 3.4). *Fix $x_0 \in V_{\text{int}}$. Let ϕ be a massless discrete GFF on $\mathcal{G}$ with 0 boundary conditions. Let $(\check{X}_t, (\check{\Phi}_t(x))_{x\in V_{\text{int}}}, (\tilde{\omega}_t(e))_{e\in E})_{0\leq t\leq \check{t}_{\max}}$ be the self-interacting process as above, with initial conditions*

$$\check{X}_0 = x_0, \qquad \forall x \in V_{\text{int}},\ \check{\Phi}_0(x) = |\phi(x)|,$$

and conditionally on ϕ, the $(\tilde{\omega}_0(e))_{e\in E}$ being independent, with

$$\mathbb{P}(\tilde{\omega}_0(\{x,y\}) = 1) = \mathbf{1}_{\phi(x)\phi(y)>0}(1 - e^{-2C(x,y)\phi(x)\phi(y)}).$$

Then $\check{X}_{\check{t}_{\max}} = x_0$ a.s. Moreover, the path $(\check{X}_t)_{0\leq t\leq \check{t}_{\max}}$ is independent from $((\check{\Phi}_{\check{t}_{\max}}(x))_{x\in V_{\text{int}}}, (\tilde{\omega}_{\check{t}_{\max}}(e))_{e\in E})$. The path $(\check{X}_t)_{0\leq t\leq \check{t}_{\max}}$ is distributed as the concatenation of all the loops in the random walk loop soup $\mathcal{L}^{1/2}$ that visit x_0 (one roots the loops in x_0). The field $(\check{\Phi}_{\check{t}_{\max}}(x))_{x\in V_{\text{int}}}$ is distributed as the absolute value of a discrete GFF with boundary conditions 0 on $V_\partial \cup \{x_0\}$. On top of that, if one samples i.i.d. uniform signs in $\{-1, 1\}$ for each cluster induced by $\tilde{\omega}_{\check{t}_{\max}}$, one gets a discrete GFF (with its signs) with boundary conditions 0 on $V_\partial \cup \{x_0\}$.

Chapter 13

Continuum limits of the Vertex Reinforced and Vertex Diminished Jump Processes in dimension 1

In this Chapter are presented two articles written in a collaboration of myself with Sabot and Tarrès [7, 5]. There we construct in dimension one the fine mesh limits of the Vertex Reinforced Jump Process (VRJP) [7] and of the Vertex Diminished Jump Process (VDJP) [5]. See Section 12.1 for a presentation of the discrete processes. The limits we constructed can be seen a self-interacting (reinforced and self-repelling) continuous one-dimensional diffusions. Our construction is done through stochastic flows of diffeomorphisms of $\mathbb{R}$ introduced by Bass and Burdzy in [BB99]. Note that Bass and Burdzy introduced these flows for reason completely unrelated to self-interacting processes. In Section 13.1 are presented the heuristics explaining how the Bass-Burdzy flows appear. In Section 13.2 is given the rigorous construction of the self-interacting diffusions out of the Bass-Burdzy flows and the convergence results are stated. In Section 13.3 the reinforced diffusion is presented as a mixture of Langevin motions in random potential, a property that it inherits from the VRJP. In Section 13.4 is presented how the self-repelling diffusion is involved in the inversion of the Ray-Knight identity. This is a one-dimensional continuum version of Theorem 12.3.

13.1 Presentation and heuristic reduction to Bass-Burdzy flows

Let I be an open interval of $\mathbb{R}$, bounded or unbounded. Let L_0 be a continuous positive function on I, satisfying

$$\int_{\inf I} L_0(x)^{-2}dx = +\infty, \qquad \int^{\sup I} L_0(x)^{-2}dx = +\infty. \tag{13.1}$$

The condition above is to avoid an explosion in finite time of the self-interacting processes on I we are going to construct. For $N \geq 1$, denote $I_N = I \cap (\frac{1}{N}\mathbb{Z})$. Consider $\widehat{X}_t^{(N)}$ and $\check{X}_t^{(N)}$ the following self-interacting nearest neighbor jump processes on I_N and denote

$$\hat{\ell}_t^{(N)}(x) = N\int_0^t \mathbf{1}_{\widehat{X}_{t'}^{(N)}=x}dt', \qquad \check{\ell}_t^{(N)}(x) = N\int_0^t \mathbf{1}_{\check{X}_{t'}^{(N)}=x}dt'.$$

The jump rate of $\widehat{X}_t^{(N)}$, resp. $\check{X}_t^{(N)}$, from $x \in I_N$ to $y = x \pm \frac{1}{N}$ is

$$\frac{1}{2}N^2\sqrt{\frac{L_0(y)^2 + 2\hat{\ell}_t^{(N)}(y)}{L_0(x)^2 + 2\hat{\ell}_t^{(N)}(x)}}, \qquad \text{resp.} \qquad \frac{1}{2}N^2\sqrt{\frac{L_0(y)^2 - 2\hat{\ell}_t^{(N)}(y)}{L_0(x)^2 - 2\hat{\ell}_t^{(N)}(x)}}.$$

Thus, $\widehat{X}_t^{(N)}$ is a VRJP on I_N and $\check{X}_t^{(N)}$ is a VDJP on I_N; see Section 12.1. We are interested in the limits in law of $\widehat{X}_t^{(N)}$ and $\check{X}_t^{(N)}$ as $N \to +\infty$.

Let $\widehat{X}_t$ and $\check{X}_t$ denote the limit processes. We see $\widehat{X}_t$ as a reinforced diffusion process, and $\check{X}_t$ as a self-repelling diffusion process. On a purely formal level, without dealing with the convergence or the meaning of the terms involved, one gets the following equations:

$$\begin{aligned} d\widehat{X}_t &= dW_t + \text{"}\frac{1}{2}\partial_x \log(L_0(x)^2 + 2\hat{\ell}_t(x))\Big|_{x=\widehat{X}_t} dt\text{"}, \\ d\check{X}_t &= dW_t + \text{"}\frac{1}{2}\partial_x \log(L_0(x)^2 - 2\check{\ell}_t(x))\Big|_{x=\check{X}_t} dt\text{"}, \end{aligned} \tag{13.2}$$

where dW_t is a white noise, ∂_x denotes the space derivative, and $\hat{\ell}_t(x)$, resp. $\check{\ell}_t(x)$, are the local times of $\widehat{X}_t$, resp. $\check{X}_t$. Note however that the equations (13.2) are not classical SDEs. It is not immediately clear how to make sense of the drift terms $\frac{1}{2}\partial_x \log(L_0(x)^2 + 2\hat{\ell}_t(x))\Big|_{x=\widehat{X}_t} dt$ and $\frac{1}{2}\partial_x \log(L_0(x)^2 - 2\check{\ell}_t(x))\Big|_{x=\check{X}_t} dt$, as $x \mapsto \hat{\ell}_t(x)$ and $x \mapsto \check{\ell}_t(x)$ will not be differentiable for $t > 0$, and moreover there will not be a change of spatial scale under which these will be differentiable for all $t > 0$. So the problem is not only to solve (13.2) by an approximation scheme, the problem is already to give an appropriate meaning to being a solution to (13.2). The equations (13.2) are also somewhat misleading, as it is believed that the solutions are not semi-martingales (see the comment after Proposition 13.2), and in particular the drift terms $\frac{1}{2}\partial_x \log(L_0(x)^2+2\hat{\ell}_t(x))\Big|_{x=\widehat{X}_t} dt$ and $\frac{1}{2}\partial_x \log(L_0(x)^2-2\check{\ell}_t(x))\Big|_{x=\check{X}_t} dt$ are in reality not absolutely continuous w.r.t. dt. However, it turns out that the equations (13.2) are in some sense exactly solvable, and the solutions involve stochastic flows of diffeomorphisms of $\mathbb{R}$ introduced by Bass and Burdzy in [BB99] for unrelated reasons. This is what is described next, in an informal heuristic way to begin with.

We will focus on $\widehat{X}_t$, since the derivations for $\check{X}_t$ are similar. Let $t_0 > 0$ and let $\widehat{X}_t^{(t_0)}$ be the continuous process that coincides with $\widehat{X}_t$ on $[0, t_0]$, and for $t > t_0$ is just a Markovian diffusion with infinitesimal generator

$$\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}\partial_x \log(L_0(x)^2 + 2\hat{\ell}_{t_0}(x))\frac{d}{dx}.$$

Then after time t_0, $\widehat{X}_t^{(t_0)}$ is a scale and time changed Brownian motion. Given $x \mapsto \widehat{S}_{t_0}(x)$ an anti-derivative of $(L_0(x)^2 + 2\hat{\ell}_{t_0}(x))^{-1}$, $\widehat{S}_{t_0}(\widehat{X}_t^{(t_0)})$ is a local martingale for $t \geq t_0$. By further performing the time change

$$du = (L_0(\widehat{X}_t^{(t_0)})^2 + 2\hat{\ell}_{t_0}(\widehat{X}_t^{(t_0)}))^{-2}dt,$$

we get a standard Brownian motion.

Then, it is reasonable to assume that near time t_0, $\widehat{X}_t$ is close to $\widehat{X}_t^{(t_0)}$. The idea is to let the change of the spatial scale depend on time. Assume there is a flow of changes of scales $\widehat{S}_t : I \to \mathbb{R}$, such that $\widehat{S}_t$ is an anti-derivative of $(L_0(x)^2 + 2\hat{\ell}_t(x))^{-1}$, and such that $\widehat{S}_t(\widehat{X}_t)$ is a local martingale. Consider $u(t)$ the change of time given by

$$du = (L_0(\widehat{X}_t)^2 + 2\hat{\ell}_t(\widehat{X}_t))^{-2}dt.$$

and $t(u)$ the inverse time change. Assume that, by analogy with the Markovian case, $(\widehat{S}_{t(u)}(\widehat{X}_{t(u)}))_{u\geq 0}$

is a standard Brownian motion $(B_u)_{u\geq 0}$. Let $x_1 < x_2 \in I$. Then

$$\begin{aligned}
\frac{d}{du}(\widehat{S}_{t(u)}(x_2) - \widehat{S}_{t(u)}(x_1)) &= \frac{dt}{du}\frac{d}{dt}\int_{x_1}^{x_2}(L_0(x)^2 + 2\hat{\ell}_t(x))^{-1}dx \\
&= -(L_0(\widehat{X}_t)^2 + 2\hat{\ell}_t(\widehat{X}_t))^2\frac{d}{dt}\int_0^t \mathbf{1}_{x_1<\widehat{X}_{t'}<x_2}2(L_0(\widehat{X}_{t'})^2 + 2\hat{\ell}_{t'}(\widehat{X}_{t'}))^{-2}dt' \\
&= -2(L_0(\widehat{X}_t)^2 + 2\hat{\ell}_t(\widehat{X}_t))^2(L_0(\widehat{X}_t)^2 + 2\hat{\ell}_t(\widehat{X}_t))^{-2}\mathbf{1}_{x_1<\widehat{X}_t<x_2} \\
&= -2\mathbf{1}_{x_1<\widehat{X}_t<x_2} \\
&= -2\mathbf{1}_{\widehat{S}_{t(u)}(x_1)<B_u<\widehat{S}_{t(u)}(x_2)}.
\end{aligned}$$

This implies that $\frac{d}{du}\widehat{S}_{t(u)}(x)$ is of form

$$\frac{d}{du}\widehat{S}_{t(u)}(x) = -\mathbf{1}_{\widehat{S}_{t(u)}(x)>B_u} + \mathbf{1}_{\widehat{S}_{t(u)}(x)<B_u} + f(u),$$

for some function $f(u)$ not depending on $x \in I$. Further, it is reasonable to assume that the left and the right sides of $\widehat{X}_t$ play symmetric roles, and thus $f(u) \equiv 0$. Then, we get that

$$\forall x \in I,\ \frac{d}{du}\widehat{S}_{t(u)}(x) = -\mathbf{1}_{\widehat{S}_{t(u)}(x)>B_u} + \mathbf{1}_{\widehat{S}_{t(u)}(x)<B_u}. \tag{13.3}$$

This is an equation studied by Bass and Burdzy in [BB99].

In the self-repelling case of $\check{X}_t$ the dynamic change of spatial scale $\check{S}_t$ is given by the anti-derivative of $(L_0(x)^2 - 2\check{\ell}_t(x))^{-1}$, and the change of time by

$$du = (L_0(\check{X}_t)^2 - 2\check{\ell}_t(\check{X}_t))^{-2}dt.$$

The change of spatial scale is governed by the equation

$$\forall x \in I,\ \frac{d}{du}\check{S}_{t(u)}(x) = \mathbf{1}_{\check{S}_{t(u)}(x)>B_u} - \mathbf{1}_{\check{S}_{t(u)}(x)<B_u},$$

that is to say the signs are opposite to those in (13.3).

In the next section the processes $\widehat{X}_t$ and $\check{X}_t$ will be defined rigorously out of the flows of solutions to the Bass-Burdzy equations.

13.2 Construction of a self-interacting diffusions out of Bass-Burdzy flows

Let $(B_u)_{u\geq 0}$ be a standard Brownian motion on $\mathbb{R}$ with $B_0 = 0$. The convergent Bass-Burdzy flow is given by the differential equation

$$\frac{dY_u}{du} = \begin{cases} -1 & \text{if } Y_u > B_u, \\ 1 & \text{if } Y_u < B_u, \end{cases} \tag{13.4}$$

and the divergent Bass-Burdzy flow is given by

$$\frac{dY_u}{du} = \begin{cases} 1 & \text{if } Y_u > B_u, \\ -1 & \text{if } Y_u < B_u, \end{cases} \tag{13.5}$$

As ODEs, (13.4) and (13.5) do not satisfy the usual Cauchy-Lipschitz conditions for the existence and uniqueness of solutions. However, it is shown in [BB99] that given an initial condition, (13.4) and (13.5) each admits a.s. a unique solution defined for all positive times that is Lipschitz continuous. Moreover, these Lipschitz continuous solutions form a flow of increasing $\mathcal{C}^1$ diffeomorphisms of $\mathbb{R}$, both in the case of (13.4) and that of (13.5). We will denote by $(\widehat{\Psi}_u)_{u\geq 0}$ the flow for (13.4) and by $(\check{\Psi}_u)_{u\geq 0}$ the flow for (13.5). For the properties of these flows, we refer to [BB99, HW00, Att10].

Denote

$$\hat{\xi}_u = \widehat{\Psi}_u^{-1}(B_u), \qquad \check{\xi}_u = \check{\Psi}_u^{-1}(B_u).$$

The divergent Bass-Burdzy flow $(\check{\Psi}_u)_{u\geq 0}$ satisfies a bifurcation property: there is a finite random value $y_{\text{bif}} \in \mathbb{R}$, such that for $y > y_{\text{bif}}$, $\check{\Psi}_u(y) > B_u$ for u large enough, and $\lim_{+\infty} \check{\Psi}_u(y) = +\infty$, for $y < y_{\text{bif}}$, $\check{\Psi}_u(y) < B_u$ for u large enough and $\lim_{+\infty} \check{\Psi}_u(y) = -\infty$, and $\{u \geq 0 | \check{\Psi}_u(y_{\text{bif}}) = B_u\}$ is unbounded. Moreover,

$$y_{\text{bif}} = \lim_{u\to+\infty} \check{\xi}_u.$$

The processes $(\hat{\xi}_u)_{u\geq 0}$ and $(\check{\xi}_u)_{u\geq 0}$ are locally $1/2 - \varepsilon$ Hölder continuous for every $\varepsilon > 0$ but are believed not to be semi-martingales [HW00]. See also Proposition 13.2 and the comment that follows it. The process $(\hat{\xi}_u)_{u\geq 0}$, resp. $(\check{\xi}_u)_{u\geq 0}$, admits a family of local times $\hat{\lambda}_u(y)$, resp. $\check{\lambda}_u(y)$, such that for any f bounded Borel-measurable function on $\mathbb{R}$ and $u \geq 0$,

$$\int_0^u f(\hat{\xi}_v)dv = \int_{\mathbb{R}} f(y)\hat{\lambda}_u(y)dy, \qquad \int_0^u f(\check{\xi}_v)dv = \int_{\mathbb{R}} f(y)\check{\lambda}_u(y)dy.$$

Moreover, these local times are related to the spatial derivatives of the flows as follows:

$$\frac{\partial}{\partial y}\widehat{\Psi}_u(y) = 1 - 2\hat{\lambda}_u(y), \qquad \frac{\partial}{\partial y}\check{\Psi}_u(y) = 1 + 2\check{\lambda}_u(y).$$

Note that for all $y \in \mathbb{R}$, $\sup_{u\geq 0} \hat{\lambda}_u(y) \leq 1/2$.

Definition 13.1 ([7], Definition 1.1, and [5], Definition 1.3)**.** Let $x_0 \in I$ and L_0 a positive continuous function on I satisfying (13.1). Let be the change of scale

$$S_0(x) = \int_{x_0}^x L_0(x')^{-2}dx', \tag{13.6}$$

and S_0^{-1} the inverse change of scale.

1. Consider the change of time $t(u)$ from u to t (and $u(t)$ the inverse time change) given by

$$dt = L_0(S_0^{-1}(\hat{\xi}_u))^4(1 - 2\hat{\lambda}_u(\hat{\xi}_u))^{-2}du.$$

Set $\widehat{X}_t = S_0^{-1}(\hat{\xi}_{u(t)})$.

2. Consider the change of time $t(u)$ from u to t (and $u(t)$ the inverse time change) given by

$$dt = L_0(S_0^{-1}(\check{\xi}_u))^4(1 + 2\check{\lambda}_u(\check{\xi}_u))^{-2}du.$$

Let

$$\check{t}_{\max} = \int_0^{+\infty} L_0(S_0^{-1}(\check{\xi}_u))^4(1 + 2\check{\lambda}_u(\check{\xi}_u))^{-2}du,$$

with $\check{t}_{\max} < +\infty$ a.s. Set $\check{X}_t = S_0^{-1}(\check{\xi}_{u(t)})$, for $t \in [0, \check{t}_{\max}]$.

With the definition above, *a posteriori*,

$$\hat{\ell}_t(x) = \frac{1}{2}L_0(x)^2((1-2\check{\lambda}_u(S_0(x)))^{-1}-1), \qquad \check{\ell}_t(x) = \frac{1}{2}L_0(x)^2(1-(1+2\check{\lambda}_u(S_0(x)))^{-1}),$$

$$\check{t}_{\max} = \inf\{t \geq 0 | \exists x \in I, L_0(x)^2 - 2\check{\ell}_t(x) = 0\}, \qquad \check{X}_{\check{t}_{\max}} = S_0^{-1}(y_{\mathrm{bif}}).$$

Next are some regularity properties of the processes $\widehat{X}_t$ and $\check{X}_t$ that appeared in our articles [7, 5].

Proposition 13.2 ([7], Proposition 2.4)**.** *Denote by $(\widehat{\mathcal{F}}_t)_{t\geq 0}$ the natural filtration of $(\widehat{X}_t)_{t\geq 0}$, and by $(\check{\mathcal{F}}_t)_{t\geq 0}$ that of $(\check{X}_{t\wedge\check{t}_{\max}})_{t\geq 0}$. The processes $(\widehat{X}_t)_{t\geq 0}$ and $(\check{X}_{t\wedge\check{t}_{\max}})_{t\geq 0}$ admit adapted decompositions*

$$\widehat{X}_t = \widehat{M}_t + \widehat{R}_t, \qquad \check{X}_{t\wedge\check{t}_{\max}} = \check{M}_t + \check{R}_t.$$

The processes $\widehat{M}_t$, resp. $\check{M}_t$, are martingales w.r.t. $(\widehat{\mathcal{F}}_t)_{t\geq 0}$, resp. $(\check{\mathcal{F}}_t)_{t\geq 0}$, with

$$\widehat{M}_t = \int_0^t (L_0(\widehat{X}_{t'})^2 + 2\hat{\ell}_t(\widehat{X}_{t'}))dB_{u(t')}, \qquad \check{M}_t = \int_0^{t\wedge\check{t}_{\max}} (L_0(\check{X}_{t'})^2 - 2\check{\ell}_t(\check{X}_{t'}))dB_{u(t')},$$

with the quadratic variation

$$\langle \widehat{M}, \widehat{M}\rangle_t = t, \qquad \langle \check{M}, \check{M}\rangle_t = t \wedge \check{t}_{\max},$$

and $\widehat{M}_t$ thus being an $(\widehat{\mathcal{F}}_t)_{t\geq 0}$ Brownian motion. As for the processes $\widehat{R}_t$ and $\check{R}_t$, they are a.s. locally $3/4-\varepsilon$ Hölder continuous, for every $\varepsilon > 0$, and have thus a zero quadratic variation.

The above tells that $(\widehat{X}_t)_{t\geq 0}$ and $(\check{X}_{t\wedge\check{t}_{\max}})_{t\geq 0}$ admit an adapted decomposition into a local martingale and a process with zero quadratic variation, and thus are Dirichlet processes in the sense of Föllmer [Fö81]. It is not shown in our articles [7, 5], but we believe that the exponent $3/4$ is optimal for $\widehat{R}_t$ and $\check{R}_t$, and that the latter do not have a bounded variation and thus, $(\widehat{X}_t)_{t\geq 0}$ and $(\check{X}_{t\wedge\check{t}_{\max}})_{t\geq 0}$ are not semi-martingales.

In [7, 5] we prove that the self-interacting diffusions $\widehat{X}_t$, resp. $\check{X}_t$, are the fine mesh limits of the VRJP $\widehat{X}_t^{(N)}$, resp. of the VDJP $\check{X}_t^{(N)}$.

Theorem 13.3 ([7], Theorem 1.3 and [5], Theorem 1.4)**.** *Assume that*

$$\lim_{N\to+\infty} \widehat{X}_0^{(N)} = \lim_{N\to+\infty} \check{X}_0^{(N)} = x_0 = \widehat{X}_0 = \check{X}_0.$$

Then, as $N \to +\infty$, the process $(\widehat{X}_t^{(N)})_t$ converges in law to the process $(\widehat{X}_t)_t$ and the process $(\check{X}_t^{(N)})_t$ converges in law to the process $(\check{X}_t)_t$.

In both cases, the proof of the convergence in law goes through the construction at the discrete level of something that approximately resembles a Bass-Burdzy flow. We will not detail this here. However, the proof of the convergence also requires to know *a priori* the tightness of the discrete processes $(\widehat{X}_t^{(N)})_t$ and $(\check{X}_t^{(N)})_t$ and of the corresponding local time processes $\hat{\ell}_t^N(x)$ and $\check{\ell}_t^N(x)$. This is achieved differently in the case of the VRJP and in the case of the VDJP. For the VRJP, we use that it is also a mixture of Markov jump processes in random environment; see Section 13.3. For the VDJP we rely on the relation to the Ray-Knight identity; see Section 13.4.

13.3 Reinforced diffusion and random environment

Sabot and Tarrès showed in [ST15b] that the VRJP on any graph corresponds to the annealed description of a Markov jump process in random environment, and gave the distribution of this environment. In the one-dimensional setting this distribution is simpler to describe.

Consider a family of independent r.v.s $(V_N(x, x+1/N))_{x,x+1/N\in I_N}$, with the distribution of $V_N(x,y)$, where $y = x + 1/N$, given by

$$\frac{\sqrt{NL_0(x)L_0(y)}}{2\sqrt{\pi}}\exp\Big(-NL_0(x)L_0(y)\sinh(v/2)^2 + v/2\Big)dv.$$

Define $U_N(x)$ for $x \in I_N$ as follows:

$$U_N(x) = \begin{cases} 0 & \text{if } x = \widehat{X}_0^{(N)}, \\ \sum_{j=1}^{N(x-\widehat{X}_0^{(N)})} V_N(\widehat{X}_0^{(N)} + (j-1)/N, \widehat{X}_0^{(N)} + j/N) & \text{if } x > \widehat{X}_0^{(N)}, \\ \sum_{j=1}^{N(\widehat{X}_0^{(N)}-x)} V_N(\widehat{X}_0^{(N)} - j/N, \widehat{X}_0^{(N)} - (j-1)/N) & \text{if } x < \widehat{X}_0^{(N)}. \end{cases}$$

Note that $U_N(x)$ depends on the initial starting point $\widehat{X}_0^{(N)}$.

Theorem 13.4 (Sabot-Tarrès)**.** *The VRJP $(\widehat{X}_t^{(N)})_t$ has the same distribution as the annealed nearest neighbor Markov jump process on I_N in random environment, with jump rate from x to $y = x \pm 1/N$ given by*

$$\frac{1}{2}N^2\frac{L_0(y)}{L_0(x)}e^{-U_N(y)+U_N(x)}.$$

By passing the random environment $(U_N(x))_{x\in I_N}$ to the limit as $N \to +\infty$, we obtained in [7] a description of the reinforced diffusion $(\widehat{X}_t)_{t\geq 0}$ as a Markovian diffusion in random environment. Let us first describe this environment. Let $(\mathcal{W}(y))_{y\in\mathbb{R}}$ be a bilateral standard Brownian motion, that is to say $(\mathcal{W}(y))_{y\geq 0}$ and $(\mathcal{W}(-y))_{y\geq 0}$ are two independent Brownian motion, with $\mathcal{W}(0) = 0$. Define for $x \in I$

$$U(x) = \sqrt{2}\mathcal{W} \circ S_0(x) + |S_0(x)|,$$

where $S_0(x)$ is given by (13.6).

Theorem 13.5 ([7], Theorem 1.6)**.** *The reinforced diffusion $(\widehat{X}_t)_{t\geq 0}$ given by Definition 13.1 has the same distribution as the annealed Markovian diffusion in random environment on I, with infinitesimal generator*

$$\frac{1}{2}\frac{d^2}{dx^2} + \Big(\frac{d}{dx}(\log(L_0(x)) - U(x))\Big)\frac{d}{dx}. \tag{13.7}$$

A Markovian diffusion with infinitesimal generator (13.7) can be rigorously defined through a change of spatial scale. (13.7) is actually the generator of a Langevin motion in a random potential. Prior to our work [7], no connection between the convergent Bass-Burdzy flow (13.4) and the diffusions in random environment was known. This description through diffusion in random environment implies the following large time asymptotic for $(\widehat{X}_t)_{t\geq 0}$.

Corollary 13.6 ([7], Proposition 4.9)**.** *Take $I = \mathbb{R}$ and $L_0 \equiv 1$. Then*

$$\liminf_{t\to+\infty}\frac{\widehat{X}_t}{\log t} = -\frac{1}{6} \quad a.s., \qquad \limsup_{t\to+\infty}\frac{\widehat{X}_t}{\log t} = \frac{1}{6} \quad a.s.$$

13.4 Self-repelling diffusion and the Ray-Knight identity

Given $a \geq 0$, $(\phi^{(a)}(x))_{x\in\mathbb{R}}$ will denote a massless GFF on $\mathbb{R}$ conditioned to be a at $x = 0$, that is to say $(\phi^{(a)}(x)/\sqrt{2})_{x\geq 0}$ and $(\phi^{(a)}(-x)/\sqrt{2})_{x\geq 0}$ are two independent standard Brownian motions starting from $a/\sqrt{2}$. We recall the Ray-Knight theorem.

Theorem 13.7 (Ray-Knight)**.** *Fix $a > 0$. Let $(\beta_t)_{t\geq 0}$ be a standard Brownian motion starting from 0 and let $\ell_t^\beta(x)$ be its local time process. Let $\tau^\beta_{a^2/2}$ be the stopping time*

$$\tau^\beta_{a^2/2} = \inf\{t \geq 0 | \ell_t^\beta(0) > a^2/2\}.$$

Let $(\phi^{(0)}(x))_{x\in\mathbb{R}}$ be a massless Gaussian free field on $\mathbb{R}$ conditioned to be 0 at $x = 0$, independent from the Brownian motion β. Then the field

$$(\phi^{(0)}(x)^2/2 + \ell^\beta_{\tau^\beta_{a^2/2}}(x))_{x\in\mathbb{R}}$$

has the same law as the field $(\phi^{(a)}(x)^2/2)_{x\in\mathbb{R}}$.

In discrete, the inversion of the Ray-Knight type identities involves a modification of the VDJP where the jump process evolves on clusters that themselves get eroded over time; see Theorem 12.3. It turns out that in dimension one, in a fine mesh limit, this modified VDJP and the ordinary VDJP coincide up to some random macroscopic time. Using this, the following is proved in [5].

Theorem 13.8 ([5], Theorem 1.5)**.** *Let $a > 0$ and $(\phi^{(a)}(x))_{x\in\mathbb{R}}$ be a massless Gaussian free field on $\mathbb{R}$ conditioned to be a at $x = 0$. Let $I(\phi^{(a)})$ be the connected component of 0 in $\{x \in \mathbb{R} | \phi^{(a)}(x) > 0\}$. For $x \in I(\phi^{(a)})$, set $L_0(x) = \phi^{(a)}(x)$. Then a.s. L_0 satisfies the condition (13.1). Let $(\check{X}_t)_{0\leq t\leq \check{t}_{\max}}$ be the process, distributed conditionally on $(\phi^{(a)}(x))_{x\in\mathbb{R}}$, as the self-repelling diffusion on $I(\phi^{(a)})$, starting from $\check{X}_0 = 0$, following Definition 13.1. Let be the triple*

$$(\phi^{(0)}(x)^2, \beta_t, \phi^{(a)}(x)^2)_{x\in\mathbb{R}, 0\leq t\leq \tau^\beta_{a^2/2}},$$

jointly distributed as in the Ray-Knight identity (Theorem 13.7). Let be

$$T^{\beta,a} = \inf\{t \in [0, \tau^\beta_{a^2/2}] | \phi^{(0)}(\beta_t) = 0 \text{ and } \forall t' \in (t, \tau^\beta_{a^2/2}], \beta_{t'} \neq \beta_t\}.$$

Then the couple

$$(\check{X}_t, \phi^{(a)}(x)^2)_{x\in\mathbb{R}, 0\leq t\leq \check{t}_{\max}}$$

has the same distribution as

$$(\beta_t, \phi^{(a)}(x)^2)_{x\in\mathbb{R}, 0\leq t\leq T^{\beta,a}}.$$

In a sense, the self-repelling diffusion $\check{X}_t$ inverts the Ray-Knight coupling in continuum in dimension one. It gives the conditional law of β_t up to time $T^{\beta,a}$ given $\phi^{(a)}$. To get β_t up to time $\tau^\beta_{a^2/2}$ one has to perform a surgery and glue together countably infinitely many processes defined each through a divergent Bass-Burdzy flow (13.5).

Actually, the convergence Theorem 13.3 in the self-repelling case is first proved for L_0 random given by $L_0(x) = \phi^{(a)}(x)$. Then it is extended to the general deterministic L_0 by a change of scale argument.

Part VI

Some further perspectives of research

Chapter 14

Dimension two

14.1 Finalizing the description of the continuum GFF through Brownian loop soup

An important aspect of my research was relating different aspects of the 2D continuum GFF ϕ_D to the Brownian loop soup $\mathcal{L}_D^{1/2}$ and to Brownian trajectories in general; see Parts II and III. There are still things to prove in this direction. These are listed below. To a large extent, these are already works in progress.

- Prove the decomposition of ϕ_D through the clusters of $\mathcal{L}_D^{1/2}$ (4.9) presented in Section 4.6.
- Prove that the multiplicative chaos of the loop soup $\mathcal{L}_D^{1/2}$ constructed in Chapter 6 is exactly the renormalized cosh of the GFF given by (4.9), that is to say that the relation is a.s. and not just in law. Also show that the renormalized exponential is obtained by restricting the renormalized cosh to clusters with positive sign in (4.9).
- Relate the odd Wick powers of ϕ_D to the Brownian loop soup $\mathcal{L}_D^{1/2}$. For the even Wick powers, Le Jan showed [LJ10, LJ11] that these correspond to the renormalized intersection and self-intersection local times of $\mathcal{L}_D^{1/2}$. This however does not work for the odd Wick powers, and the relation should be more in the spirit of (4.9), taking into account the sign of each cluster of $\mathcal{L}_D^{1/2}$.
- Prove Conjecture 8.3 and show that the Lévy transformation (Chapter 8) extends in the fine mesh limit to the 2D continuum GFF.

14.2 The $P(\phi)_2$ fields

Given a bounded domain $D \subset \mathbb{C}$ and P a polynomial of even degree with positive leading term, on can define define a $P(\phi)$ field which is absolutely continuous w.r.t. the GFF ϕ_D on D, with density

$$\frac{1}{Z_{P,D}} e^{-\int_D :P(\phi_D):}, \tag{14.1}$$

where $:P(\phi_D):$ denotes the Wick's renormalization of $P(\phi_D)$. See Section 1.1.4. Note that the $P(\phi)$ fields are no longer conformally invariant in law. The density (14.1) can also be expressed through the Brownian loop soup $\mathcal{L}_D^{1/2}$. Now, one can also take infinite volume limits of a $P(\phi)$ field, i.e. $D \to \mathbb{C}$. See [Sim74, Section VIII.6]. An infinite volume $P(\phi)$ field is not globally absolutely continuous w.r.t. a free field, only locally. Now, what happens to the Brownian loop soup weighted by (14.1) in the infinite volume limit, in particular in the case of *spontaneous symmetry breaking* for $P(\phi)$, or generally when $P(\phi)$ has multiple infinite volume limits? For instance, one can take $P(\phi) = \frac{\lambda}{4!}\phi^4 - \mu^2\phi^2$ (double well potential). Then, depending on the values of λ and μ, there is either only one infinite volume field or several, and in particular two translation invariant ergodic laws related through $\phi \mapsto -\phi$ (symmetry breaking). This is similar to the magnetization in the Ising model. In the symmetry broken phase, does the corresponding Brownian loop soup contain "loops" that are actually excursions from ∞ to ∞, i.e. loops that close only at ∞?

14.3 Fields for $\alpha \in (0, 1/2)$ and relation to SLE processes

As explained in Section 10.1, in dimension 2 the Brownian loop soups $\mathcal{L}_D^\alpha$ are of major interest for every $\alpha \in (0, 1/2]$ because of the conformal invariance in law and the relation to the CLE_κ (Theorem 2.3). So one would like to have conformally invariant fields that play for $\alpha \in (0, 1/2)$ the same role as the GFF for $\alpha = 1/2$, as the GFF was instrumental for many results on $\mathcal{L}_D^{1/2}$. Currently I see two approaches to this problem. The fist one is through the multiplicative chaos measures that have been constructed for every $\alpha > 0$ (Chapter 6). The second approach is more speculative, through the multi-dimensional extensions of the β-Dyson's Brownian motion; see Chapter 10 and Question 10.11. If such hypothetical extensions are possible on a 2D lattice, then one can further consider the fine mesh limits, which have good reasons to be conformally invariant in law. Once such fields constructed, one can consider their level lines or other types of interfaces, and investigate how these are related to different SLE processes.

A particularly intriguing value is $\alpha = 1/4$. For this value, according to Theorem 2.3, the outermost boundaries in a Brownian loop soup are distributed as CLE_3. The CLE_3 also appears in the scaling limit of 2D critical Ising interfaces [CS12]. So the question is whether it is possible to find a deeper connection between the Brownian loop soup with $\alpha = 1/4$ and the 2D critical Ising model. In particular, is there a combinatorial relation between the 2D random walk loop soup in discrete and the planar self-dual Ising model?

Chapter 15

Higher dimensions

15.1 Geometrical description of the continuum GFF

First, let us emphasize once more that the identification of the sign clusters of a metric graph GFF with the clusters of a metric graph loop soup with $\alpha = 1/2$ (Theorem 3.8) is not related to planarity. It holds on any electrical network. So a natural question is what happens in the scaling limit in dimensions $d \geq 3$. In his publication [Wer21], Wendelin Werner presented a series of conjectures on this topic. Let us summarize them.

In dimension $d = 3$ one believes that the picture is pretty similar to that in dimension 2, however more challenging to prove. Loops in a 3D Brownian loop soup can be combined in clusters, since two independent Brownian paths can intersect in dimension 3. All the clusters in the full space $\mathbb{R}^3$ should be bounded. The dimension of each cluster should be 5/2. Let us explain how this 5/2 is obtained. This is similar to the heuristic presented at the end of Section 4.6. On the metric graph of $\mathbb{Z}^3$, the probability that two distant points x and y belong to the same sign cluster of the GFF is, according to Corollary 3.9, of order $\|y - x\|^{-1}$. On the other hand, if δ is the dimension of a cluster in continuum, one expects the above probability to decay as $\|y - x\|^{-2(3-\delta)}$, as one loses $\|y - x\|$ to the codimension $3 - \delta$ at both ends of a cluster. From this one gets $\delta = 5/2$. Further, as for the dimension 2 (Section 4.6 and (4.9)), in dimension 3 each cluster will come with an i.i.d. uniform sign and with a Radon measure supported on the cluster. This Radon measure in likely a Minkowski content measure for the dimension 5/2.

In dimensions 4 and 5, on one hand the Brownian loops cannot intersect, but on the other hand the dimension in the limit of clusters on a metric graph should be $1 + d/2$, that is to say 3 for $d = 4$ and 7/2 for $d = 5$. First of all theses dimensions are strictly larger than 2 (dimension of a Brownian path), and moreover the corresponding codimension is strictly smaller than 2. So the clusters should be able to intersect with positive probability macroscopic Brownian loops. It is conjectured that despite the fact that the Brownian loops do not intersect, there is still a way to regroup them into non-trivial clusters, each cluster being of dimension $1 + d/2$. There is some sort of "glue" that lives outside the Brownian loops and that is inherited from the mesoscopic loops in discrete (which do not correspond to Brownian loops in the scaling limit), and this glue binds different macroscopic loops together. Moreover, Werner conjectures that in dimension $d = 4$ the "glue" is actually measurable w.r.t. the Brownian loop soup, while in dimension $d = 5$ it involves additional randomness.

In dimension $d \geq 7$ the GFF and the loop soup are considered to be independent in the scaling limit. This is because $1 + d/2 < d - 2$, so a typical macroscopic cluster on a metric graph with high probability does not contain macroscopic loops. In fact, on a metric graph, the number of clusters with large loops is of smaller order than the number of large clusters without large loops. The publication [Wer21] contains some proofs in this direction.

The dimension $d = 6$ is not discussed in [Wer21]. It is in some sense critical, because then $1 + d/2 = d - 2$, and there the picture is the hardest even to imagine.

Trying to prove the conjectures above constitutes and exciting and challenging endeavor, which is also of fundamental importance for understanding the intrinsic geometry of the GFF in higher dimensions. Already the dimension $d = 3$ is much harder than $d = 2$, since one does not have the tools related to the planarity. The dimensions 4 and 5 require on top of that the introduction of new mathematical objects, the mysterious "glue".

A related question is whether the Lévy transformation of the GFF (Chapter 8) has a continuum limit in dimensions $d \geq 3$. This is related to the continuum first passage sets. Indeed, on metric graphs the "balls" around the boundary for the pseudo-metric involved in the Lévy transformation are distributed as first passage sets (8.5). Moreover, in the continuum limit these first passage sets should be clusters of Brownian loops and Brownian boundary excursions, which connects to the discussion above.

15.2 Phi 4 field in dimension 3 and the Brownian loop soup

Let D be a bounded domain in $\mathbb{R}^3$. On a purely formal level, the ϕ^4 field on D has a density w.r.t. a GFF ϕ_D on D, which is

$$\frac{1}{Z_D} \exp\Big(- \frac{\lambda}{4!} \int_D \phi_D(x)^4 dx \Big).$$

In the Le Jan's isomorphism (Theorem 1.13), the quantity $\phi_D(x)^4$ corresponds to a self-intersection local time of the Brownian loop soup $\mathcal{L}_D^{1/2}$. This is actually close to the original picture presented by Symanzik [Sym66]. However, the values $\phi_D(x)^4$ do not make sense, and one has to apply a renormalization procedure. In dimension 3 the renormalization procedure is more complicated than the Wick's renormalization in dimension 2, and at the end of the day the renormalized ϕ^4 field in 3D is not absolutely continuous w.r.t. the GFF ϕ_D. See [GJ87, Section 23.1]. A natural question is whether this renormalization procedure has an interpretation in terms of the Brownian loop soup $\mathcal{L}_D^{1/2}$, in particular in terms of the double points of $\mathcal{L}_D^{1/2}$, since heuristically the double points of $\mathcal{L}_D^{1/2}$ carry the self-intersection local times. A double point of $\mathcal{L}_D^{1/2}$ is either a double point for one of the Brownian loops in $\mathcal{L}_D^{1/2}$, or an intersection point of two different loops in $\mathcal{L}_D^{1/2}$. Also note that the 3D Brownian motion, and by extension the 3D Brownian loop soup, does not have points of multiplicity 3 or higher [MP10, Section 9.3].

My publications

[1] Elie Aïdékon, Nathanaël Berestycki, Antoine Jégo, and Titus Lupu. Multiplicative chaos of the Brownian loop soup. arXiv:2107.13340, 103 pages, 2021.

[2] Titus Lupu and Hao Wu. A level line of the Gaussian free field with measure-valued boundary conditions. arXiv:2106.15169, 42 pages, 2021.

[3] Titus Lupu. Isomorphisms of β-Dyson's Brownian motion with Brownian local time. *Electronic Journal of Probability*, 26:1–31, 2021.

[4] Juhan Aru, Titus Lupu, and Avelio Sepúlveda. Extremal distance and conformal radius of a CLE_4 loop. To appear in *The Annals of Probability*, arXiv:2004.13570, 45 pages, 2020.

[5] Titus Lupu, Christophe Sabot, and Pierre Tarrès. Inverting the Ray-Knight identity on the line. *Electronic Journal of Probability*, 26:1–25, 2021.

[6] Titus Lupu. Topological expansion in isomorphism theorems between matrix-valued fields and random walks. To appear in *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, arXiv:1908.06732, 29 pages, 2021.

[7] Titus Lupu, Christophe Sabot, and Pierre Tarrès. Fine mesh limit of the VRJP in dimension one and Bass-Burdzy flow. *Probability Theory and Related Fields*, 177:55–90, 2020.

[8] Juhan Aru, Titus Lupu, and Avelio Sepúlveda. The first passage sets of the 2D Gaussian free field: convergence and isomorphisms. *Communications in Mathematical Physics*, 375:1885–1929, 2020.

[9] Juhan Aru, Titus Lupu, and Avelio Sepúlveda. The first passage sets of the 2D Gaussian free field. *Probability Theory and Related Fields*, 176:1303–1355, 2020.

[10] Titus Lupu, Christophe Sabot, and Pierre Tarrès. Inverting the coupling of the signed Gausssian free field with a loop soup. *Electronic Journal of Probability*, 24:1–28, 2019.

[11] Titus Lupu and Wendelin Werner. The random pseudo-metric on a graph defined via the zero-set of the Gaussian free field on its metric graph. *Probability Theory and Related Fields*, 171:775–818, 2018.

[12] Titus Lupu and Wendelin Werner. A note on Ising random currents, Ising-FK, loop-soups and the Gaussian free field. *Electronic Communications in Probability*, 21:1–7, 2016.

[13] Titus Lupu. *Poissonian ensembles of Markovian loops.* PhD thesis, Université Paris-Sud, 2015.

[14] Titus Lupu. Convergence of the two-dimensional random walk loop-soup clusters to CLE. *Journal of the European Mathematical Society*, 21(4):1201–1227, 2019.

[15] Titus Lupu. Loop percolation on discrete half-plane. *Electronic Communications in Probability*, 21:1–9, 2016.

[16] Titus Lupu. From loop clusters and random interlacements to the free field. *The Annals of Probability*, 44(3):2117–2146, 2016.

[17] Titus Lupu, Jim Pitman, and Wenpin Tang. The Vervaat transform of Brownian bridges and Brownian motion. *Electronic Journal of Probability*, 20:1–31, 2015.

[18] Titus Lupu. *Poisson ensembles of loops of one-dimensional diffusions*, volume 158 of *Mémoires de la SMF*. Société Mathématique de France, 2018.

The publications and preprints listed here can be downloaded

- on HAL `https://hal.archives-ouvertes.fr/search/index/q/*/authIdHal_s/titus-lupu`,
- on arXiv `https://arxiv.org/search/math?query=Lupu%2C+Titus&searchtype=author&abstracts=show&order=-announced_date_first&size=50`.

The papers are listed in the chronological order of their production, from the most recent to the most ancient, and not by date of publication in journals. The articles 14 to 18 were written during my PhD. They are presented in Section 3, with the exception of [17] which is not commented. The reference [13] is the manuscript of my PhD Thesis. The articles and preprints 1 to 12 were written after my PhD. They are all discussed in this memoir.

Bibliography

[ADC21] Michael Aizenman and Hugo Duminil-Copin. Marginal triviality of the scaling limits of critical 4D Ising and ϕ_4^4 models. *Annals of Mathematics*, 194:163–235, 2021.

[AGZ09] Greg W. Anderson, Alice Guionnet, and Ofer Zeitouni. *An introduction to random matrices*, volume 118 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 2009.

[Ahl10] Lars Valerian Ahlfors. *Conformal invariants: topics in geometric function theory*, volume 371. American Mathematical Society, 2010.

[AHS20] Elie Aïdékon, Yueyun Hu, and Zhan Shi. Points of infinite multiplicity of planar Brownian motion: measures and local times. *The Annals of Probability*, 48(4):1785–1825, 2020.

[Aiz82] Michael Aizenman. Geometric analysis of φ^4 fields and Ising model. *Communications in Mathematical Physics*, 86(1):1–48, 1982.

[APS20] Juhan Aru, Ellen Powell, and Avelio Sepúlveda. Liouville measure as a multiplicative cascade via level sets of the Gaussian free field. *Annales de l'Institut Fourier*, 70(1):205–245, 2020.

[ASW19] Juhan Aru, Avelio Sepúlveda, and Wendelin Werner. On bounded-type thin local sets of the two-dimensional Gaussian free field. *Journal of the Institute of Mathematics of Jussieu*, 18(3):591–618, 2019.

[Att10] Stefano Attanasio. Stochastic flow of diffeomorphisms for one-dimensional SDE with discontinuous drift. *Electronic Communications in Probability*, (15):213–226, 2010.

[BB99] Richard F. Bass and Krzysztof Burdzy. Stochastic bifurcation model. *The Annals of Probability*, 27(1):50–108, 1999.

[BBK94] Richard F. Bass, Krzysztof Burdzy, and Davar Khoshnevisan. Intersection local time for points of infinite multiplicity. *The Annals of Probability*, 22(2):566–625, 1994.

[BC84] Jesse Randall Baxter and Rafael van Severen Chacon. The equivalence of diffusions on networks to Brownian motion. In *Conference in Modern Analysis and Probability (New Haven, Conn., 1982)*, volume 26 of *Contempemporary Mathematics*, pages 33–48. American Mathematical Society, 1984.

[BC18] Mikhail Basok and Dmitry Chelkak. Tau-functions à la Dubédat and probabilities of cylindrical events for double-dimers and CLE(4). *arXiv preprint arXiv:1809.00690*, 2018.

[Ber17] Nathanaël Berestycki. An elementary approach to Gaussian multiplicative chaos. *Electronic Communications in Probability*, 22(27):1–12, 2017.

[BFS79] David Brydges, Jürg Fröhlich, and Erhard Seiler. On the construction of quantized gauge fields. I. General results. *Annals of Physics*, 121:227–284, 1979.

[BFS82] David Brydges, Jürg Fröhlich, and Tom Spencer. The random walk representation of classical spin systems and correlation inequalities. *Communications in Mathematical Physics*, 83(1):123–150, 1982.

[BHJ26] Max Born, Werner Heisenberg, and Pascual Jordan. Zur Quantenmechanik II. *Zeitschrift für Physik*, 35(8-9):557–615, 1926.

[BHS21] Roland Bauerschmidt, Tyler Helmuth, and Andrew Swan. The geometry of random walk isomorphism theorems. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 57(1):408–454, 2021.

[BIPZ78] Édouard Brézin, Claude Itzykson, Giorgio Parisi, and Jean-Bernard Zuber. Planar diagrams. *Communications in Mathematical Physics*, 59:35–51, 1978.

[BIZ80] Daniel Bessis, Claude Itzykson, and Jean-Bernard Zuber. Quantum field theory techniques in graphical enumeration. *Advances in Applied Mathematics*, 1(2):109–157, 1980.

[BJ25] Max Born and Pascual Jordan. Zur Quantenmechanik. *Zeitschrift für Physik*, 34(1):858–888, 1925.

[BL19] Marek Biskup and Oren Louidor. On intermediate level sets of two-dimensional discrete Gaussian free field. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 55(4):1948–1987, 2019.

[BP94] Jean Bertoin and Jim Pitman. Path transformations connecting Brownian bridge, excursion and meander. *Bulletin des Sciences Mathématiques*, 118:147–166, 1994.

[BP09] Wlodzimierz Bryc and Virgil Pierce. Duality of real and quaternionic random matrices. *Electronic Journal of Probability*, 14(17):452–476, 2009.

[Cha92] Terence Chan. The Wigner semi-circle law and eigenvalues of matrix-valued diffusions. *Probability Theory and Related Fields*, 93:249–272, 1992.

[CL97] Emmanuel Cépa and Dominique Lépingle. Diffusing particles with electrostatic repulsion. *Probability Theory and Related Fields*, 107:429–449, 1997.

[CL07] Emmanuel Cépa and Dominique Lépingle. No multiple collisions for mutually repelling Brownian particles. In Catherine Donati-Martin, Michel Émery, Alain Rouault, and Christophe Stricker, editors, *Séminaire de Probabilités XL*, volume 1899 of *Lecture Notes in Mathematics*, pages 241–246. Springer, 2007.

[CS12] Dmitry Chelkak and Stanislav Smirnov. Universality in the 2D Ising model and conformal invariance of fermionic observables. *Inventiones Mathematicae*, 189:515–580, 2012.

[DC16] Hugo Duminil-Copin. Random currents expansion of the Ising model. In *7th European Congress of Mathematicians in Berlin*, 2016. arXiv:1607.06933.

[DE02] Ioana Dumitriu and Alan Edelman. Matrix models for beta ensembles. *Journal of Mathematical Physics*, 43(11):5830–5847, 2002.

[DPRZ01] Amir Dembo, Yuval Peres, Jay Rosen, and Ofer Zeitouni. Thick points for planar Brownian motion and the Erdös-Taylor conjecture on random walk. *Acta Mathematica*, 186(2):239–270, 2001.

[DS11] Bertrand Duplantier and Scott Sheffield. Liouville quantum gravity and KPZ. *Inventiones Mathematicae*, 185(2):333–393, 2011.

[Dub07] Julien Dubédat. Commutation relations for Schramm-Loewner evolutions. *Communications on Pure and Applied Mathematics*, 60(12):1792–1847, 2007.

[Dub19] Julien Dubédat. Double dimers, conformal loop ensembles and isomonodromic deformations. *Journal of European Mathematical Society*, 21(1):1–54, 2019.

[Dyn84a] Evgeniy Dynkin. Gaussian and non-Gaussian random fields associated with Markov processes. *Journal of Functional Analysis*, 55:344–376, 1984.

[Dyn84b] Evgeniy Dynkin. Local times and quantum fields. In *Seminar on Stochastic Processes, Gainesville 1983*, volume 7 of *Progress in Probability and Statistics*, pages 69–84. Birkhauser, 1984.

[Dys62] Freeman J. Dyson. A Brownian-motion model for the eigenvalues of a random matrix. *Journal of Mathematical Physics*, 3:1191–1198, 1962.

[Eis95] Nathalie Eisenbaum. Une version sans conditionnement du théorème d'isomorphisme de Dynkin. In *Séminaire de Probabilités XXIX*, volume 1613 of *Lecture Notes in Mathematics*, pages 266–289. Springer-Verlag, 1995.

[EKM$^+$00] Nathalie Eisenbaum, Haya Kaspi, Michael B. Marcus, Jay Rosen, and Zhan Shi. A Ray-Knight theorem for symmetric Markov processes. *The Annals of Probability*, 28(4):1781–1796, 2000.

[EKR18] Bertrand Eynard, Taro Kimura, and Sylvain Ribault. Random matrices. arXiv:1510.04430, 2018.

[ES88] Robert G. Edwards and Alan D. Sokal. Generalization of the Fortuin-Kasteleyn-Swendsen-Wang representation and Monte Carlo algorithm. *Physical Review D*, 38(6):2009–2012, 1988.

[ET60] Pal Erdös and S.J. Taylor. Some problems concerning the structure of random walk paths. *Acta mathematica Academiae Scientiarum Hungaricae*, 11:137–162, 1960.

[Eyn16] Bertrand Eynard. *Counting Surfaces*, volume 70 of *Progress in Mathematical Physics*. Birkhäuser, 2016.

[Fö81] Hans Föllmer. Dirichlet processes. In David Williams, editor, *Stochastic Integrals: Proceedings of the LMS Durham Symposium*, volume 851 of *Lecture Notes in Mathematics*, page 476–478. Springer, 1981.

[Fey50] Richard Feynman. Mathematical formulation of the quantum theory of electromagnetic interaction. *Physical Review*, 80(3):440–457, 1950.

[Fol08] Gerald B. Folland. *Quantum Field Theory. A Tourist Guide for Mathematicians*, volume 149 of *Mathematical Surveys and Monographs.* American Mathematical Society, 2008.

[For15] Peter Forrester. *The Oxford Handbook of Random Matrix Theory*, chapter 20 Beta ensembles, pages 415–432. Oxford Handbooks. Oxford University Press, 1st edition, 2015.

[Fre96] David A. Freedman. De Finetti's theorem in continuous time. *IMS Lecture Notes - Monograph Series*, 30:83–98, 1996.

[Frö82] Jürg Fröhlich. On the triviality of $\lambda\phi_d^4$ theories and the approach to the critical point in $d \geq 4$ dimensions. *Nuclear Physics B*, 200:281–296, 1982.

[Gil81] Roscoe Giles. Reconstruction of gauge potentials from Wilson loops. *Physical Review D*, 24(8):2160–2168, 1981.

[GJ87] James Glimm and Arthur Jaffe. *Quantum Physics. A Functional Integral Point of View.* Springer-Verlag, 2nd edition, 1987.

[Gri06] Geoffrey R. Grimmett. *The random-cluster model*, volume 333 of *Grundlehren der mathematischen Wissenschaften.* Springer, 2006.

[HK71] Raphael Høegh-Krohn. A general class of quantum fields without cut-offs in two space-time dimensions. *Communications in Mathematical Physics*, 21:244–255, 1971.

[HMP10] Xiaoyu Hu, Jason Miller, and Yuval Peres. Thick points of the Gaussian free field. *The Annals of Probability*, 38(2):896–926, 2010.

[HW00] Yueyun Hu and Jon Warren. Ray-Knight theorems related to a stochastic flow. *Stochastic Processes and their Applications*, 86:287–305, 2000.

[IZ80] Claude Itzykson and Jean-Bernard Zuber. The planar approximation. II. *Journal of Mathematical Physics*, 21(3):411–421, 1980.

[Jé19] Antoine Jégo. Characterisation of planar Brownian multiplicative chaos. arXiv:1909.05067, 2019.

[Jé20] Antoine Jégo. Planar Brownian motion and Gaussian multiplicative chaos. *The Annals of Probability*, 48(4):1597–1643, 2020.

[Jan97] Svante Janson. *Gaussian Hilbert Spaces*, volume 129 of *Cambridge Tracts in Mathematics.* Cambridge University Press, 1997.

[Kah85] Jean-Pierre Kahane. Sur le chaos multiplicatif. *Annales des Sciences Mathématiques du Québec*, 9(2):105–150, 1985.

[Ken14] Richard Kenyon. Conformal invariance of loops in the double-dimer model. *Communications in Mathematical Physics*, 326(2):477–497, 2014.

[KL21] Adrien Kassel and Thierry Lévy. Covariant Symanzik identities. *Probability and Mathematical Physics*, 2(3):419–475, 2021.

[Kni63] Frank Knight. Random walks and a sojourn density process of Brownian motion. *Transactions of the American Mathematical Society*, 109(1):56–86, 1963.

[Lé04] Thierry Lévy. Wilson loops in the light of spin networks. *Journal of Geometry and Physics*, 52(4):382–397, 2004.

[Law05] Gregory F. Lawler. *Conformally invariant processes in the plane*, volume 114 of *Mathematical Surveys and Monographs.* American Mathematical Society, 2005.

[LC09] Michael La Croix. *The combinatorics of the Jack parameter and the genus series for topological maps.* PhD thesis, University of Waterloo, 2009.

[LC13] Michael La Croix. β-Gaussian ensembles and the non-orientability of polygonal glueings. Slides available on author's webpage http://www.math.uwaterloo.ca/ malacroi/, 2013.

[LJ10] Yves Le Jan. Markov loops and renormalization. *The Annals of Probability*, 38(3):1280–1319, 2010.

[LJ11] Yves Le Jan. *Markov paths, loops and fields. École d'Été de Probabilités de Saint-Flour XXXVIII – 2008*, volume 2026 of *Lecture Notes in Mathematics.* Springer, 2011.

[LL10] Gregory F. Lawler and Vlada Limic. *Random walk: a modern introduction*, volume 123 of *Cambridge studies in advanced mathematics.* Cambridge University Press, 2010.

[LSS97] Thomas M. Liggett, Roberto H. Schonmann, and Alan M. Stacey. Domination by product measures. *The Annals of Probability*, 25(1):71–95, 1997.

[LSW03] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. Conformal restriction: the chordal case. *Journal of the American Mathematical Society*, 16(4):917–955, 2003.

[LTF07] Gregory F. Lawler and José A. Trujillo-Ferreras. Random walk loop soup. *Transactions of American Mathematical Society*, 359(2):767–787, 2007.

[LW04] Gregory F. Lawler and Wendelin Werner. The Brownian loop soup. *Probability Theory and Related Fields*, 128:565–588, 2004.

[LZ04] Sergei Lando and Alexander Zvonkin. *Graphs on Surfaces and Their Applications*, volume 141 of *Encyclopaedia of Mathematical Sciences*. Springer, 2004.

[Meh04] Madan Lal Mehta. *Random Matrices*, volume 142 of *Pure and Applied Mathematics*. Academic Press, 3rd edition, 2004.

[MP10] Peter Mörters and Yuval Peres. *Brownian Motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2010.

[MS16] Jason Miller and Scott Sheffield. Imaginary geometry I: interacting SLEs. *Probability Theory and Related Fields*, 164(3-4):553–705, 2016.

[MW03] Motohico Mulase and Andrew Waldron. Duality of orthogonal and symplectic matrix integrals and quaternionic Feynman graphs. *Communications in Mathematical Physics*, 240:553–586, 2003.

[New64] Maxwell H.A. Newman. *Elements of the topology of plane sets of points*. Cambridge University Press, 1964.

[Pom92] Christian Pommerenke. *Boundary behaviour of conformal maps*, volume 299 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, 1992.

[PW17] Ellen Powell and Hao Wu. Level lines of the Gaussian free field with general boundary data. *Annales de l'Institut Henri Poincaré Probabilités et Statistiques*, 53(4):2229–2259, 2017.

[PW19] Eveliina Peltola and Hao Wu. Global multiple SLEs for $\kappa \leq 4$ and connection probabilities for level lines of GFF. *Communications in Mathematical Physics*, 366, 2019.

[QW18] Wei Qian and Wendelin Werner. Coupling the Gaussian free fields with free and with zero boundary conditions via common level lines. *Communications in Mathematical Physics*, 361:53–80, 2018.

[QW19] Wei Qian and Wendelin Werner. Decomposition of Brownian loop-soup clusters. *Journal of the European Mathematical Society*, 21(10):3225–3253, 2019.

[Ray63] Daniel Ray. Sojourn times of diffusion processes. *Illinois Journal of mathematics*, 7(4):615–630, 1963.

[RS93] L.C.G. Rogers and Zhan Shi. Interacting Brownian particles and the Wigner law. *Probability Theory and Related Fields*, 95:555–570, 1993.

[RV14] Rémi Rhodes and Vincent Vargas. Gaussian multiplicative chaos and applications: a review. *Probability Surveys*, 11:315–392, 2014.

[RY99] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der mathematischen Wissenschaften*. Springer, 3rd edition, 1999.

[Sch00] Oded Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel Journal of Mathematics*, 118:221–288, 2000.

[Sch07] Oded Schramm. Conformally invariant scaling limits : an overview and a collection of problems. In *Proceedings of the International Congress of Mathematicians (ICM), Madrid, Spain, August 22–30, 2006. Volume I : Plenary lectures and ceremonies*, pages 513–543. European Mathematical Society, 2007.

[Sen94] Ambar Sengupta. Gauge invariant functions of connections. *Proceedings of the American Mathematical Society*, 121(3):897–905, 1994.

[Sep19] Avelio Sepúlveda. On thin local sets of the Gaussian free field. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 55(3):1797–1813, 2019.

[She07] Scott Sheffield. Gaussian free fields for mathematicians. *Probability Theory and Related Fields*, 139(3):521–541, 2007.

[She09] Scott Sheffield. Exploration tree and conformal loop ensembles. *Duke Mathematical Journal*, 147(1):79–129, 2009.

[Sim74] Barry Simon. *The $P(\phi)_2$ Euclidean (quantum) field theory.* Princeton Series in Physics. Princeton University Press, 1974.

[Smi09] Stanislav Smirnov. Critical percolation in the plane. I. Conformal invariance and Cardy's formula. II. Continuum scaling limit. arXiv:0909.4499, 2009.

[SS09] Oded Schramm and Scott Sheffield. Contour lines of the two-dimensional discrete Gaussian free field. *Acta Mathematica*, 202(1):21, 2009.

[SS13] Oded Schramm and Scott Sheffield. A contour line of the continuum Gaussian free field. *Probability Theory and Related Fields*, 157(1-2):47–80, 2013.

[SSW09] Oded Scramm, Scott Sheffield, and David B. Wilson. Conformal radii for conformal loop ensembles. *Communications in Mathematical Physics*, 288(1):43–53, 2009.

[ST15a] Christophe Sabot and Pierre Tarrès. Inverting Ray-Knight identity. *Probability Theory and Related Fields*, 165(3):559–580, 2015.

[ST15b] Christophe Sabot and Pierre Tarrès. Edge-reinforced random walk, vertex-reinforced jump process and the supersymmetric hyperbolic sigma model. *Journal of the European Mathematical Society*, 17(9):2353–2378, 2015.

[SW12] Scott Sheffield and Wendelin Werner. Conformal loop ensembles: the Markovian characterization and the loop-soup construction. *Annals of Mathematics*, 176(3):1827–1917, 2012.

[Sym65] Kurt Symanzik. *Euclidean quantum field theory I: Equations for a scalar model.* New York University, 1965.

[Sym66] Kurt Symanzik. Euclidean quantum field theory I. Equations for a scalar model. *Journal of Mathematical Physics*, 7(3):510–525, 1966.

[Sym69] Kurt Symanzik. Euclidean quantum field theory. In *Scuola intenazionale di Fisica Enrico Fermi. XLV Corso*, pages 152–223. Academic Press, 1969.

[Szn12a] Alain-Sol Sznitman. An isomorphism theorem for random interlacements. *Electronic Commununications in Probability*, 17(9):1–9, 2012.

[Szn12b] Alain-Sol Sznitman. *Topics in occupation times and Gaussian free field.* Zurich lectures in advanced mathematics. European Mathemtical Society, 2012.

[tH74] Gerardus 't Hooft. A planar diagram theory for strong interactions. *Nuclear Physics B*, 72:461–473, 1974.

[vdBCL16] Tim van de Brug, Federico Camia, and Marcin Lis. Random walk loop soups and conformal loop ensembles. *Probability Theory and Related Fields*, 166(1):553–584, 2016.

[Wer08] Wendelin Werner. The conformally invariant measure on self-avoiding loops. *Journal of American Mathematical Society*, 21:137–169, 2008.

[Wer21] Wendelin Werner. On clusters of Brownian loops in d dimensions. In Maria Eulália Vares, Roberto Fernández, Luiz Renato Fontes, and Charles M. Newman, editors, *In and Out of Equilibrium 3: Celebrating Vladas Sidoravicius*, volume 77 of *Progress in Probability*, pages 797–817. Birkhäuser, 2021.

[Wil74] Kenneth G. Wilson. Confinement of quarks. *Physical Review D*, 10(8):2445–2459, 1974.

[WW13a] Wendelin Werner and Hao Wu. From CLE(κ) to SLE(κ, ρ). *Electronic Journal of Probability*, 18:no. 36, 1–20, 2013.

[WW13b] Wendelin Werner and Hao Wu. On conformally invariant CLE explorations. *Communications in Mathematical Physics*, 320:637–661, 2013.

[WW17] Menglu Wang and Hao Wu. Level lines of Gaussian free field I: zero-boundary GFF. *Stochastic Processes and their Applications*, 127(4):1045–1124, 2017.

[Zvo97] Alexander Zvonkin. Matrix integrals and map enumeration: an accessible introduction. *Mathematical and Computer Modelling*, 26(8-10):281–304, 1997.