# On Kerr black hole perfect MHD processes in Doran coordinates

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Doran horizon penetrating coordinates are adopted to study specific perfect MHD processes around a Kerr black hole, focusing in particular on the physical relevance of selected electrodynamical quantities.

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## 1. Introduction

General Relativity is a non linear theory which couples any physical field to gravitation described in terms of the geometry of space-time. This makes the search of exact or approximate analytical solutions for physical relevant problems an extraordinary task. Nowadays, Numerical Relativity has greatly circumvented such a problem by using the huge computational power available[1] although analytical solutions still remain fundamental to have important physical insights. The $60s$ and the $70s$ represented an important moment of rediscovery of Einstein's Theory. At that time, numerical relativity was almost unknown and Mathematical Physics techniques were widely adopted to find exact or approximate solutions. In particular, in the field of Black Holes Physics, the understanding of the electrodynamics and magnetohydrodynamics (MHD) around the recently discovered Kerr rotating black hole solution[2] was central. Due to the aforementioned difficulty associated to the non-linearity of the equations, approximate techniques of perturbative type were applied to this aim as, for instance, in the classical work by Ruffini and Wilson[3] and the one by Damour[4] (RWD) based on the properties of the geodesics in Kerr background studied by Carter.[5] In particular Ruffini and Wilson used a simplified model to describe possible charge separation processes involving the black hole and its magnetosphere. Such a studies have been recently revisited[6] by using coordinates regular on the horizon found decades later by Doran[8] and are here reviewed. Doran's work in particular generalizes classical Painlevé-Gullstrand coordinates for spherical black holes. These coordinates are horizon penetrating and are naturally associated to regular infalling physical observers. They are usually adopted in Numerical Relativity in union with the *excision* technique which allows to extend the computational domain beyond the black hole event horizon, avoiding to impose problematic boundary conditions for the partial differential equations there. Moreover, Painlevé-Gullstrand type coordinates naturally occur in Analogue Gravity in relation with acoustic black holes.[7] Summarizing, what we discuss here is the revisitation of RWD works by using useful coordinates found almost 25 years later. We will show in particular that these coordinates are the most natural ones to describe plasma physics by comoving with the fluid itself.

## 2. The RWD solution

RWD started from the Kerr metric in Boyer-Lindquist coordinates

$$ds^2 = -\left(1-\frac{2Mr}{\Sigma}\right)dt^2 - \frac{4aMr\sin^2\theta}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \left[r^2+a^2+\frac{2Mra^2\sin^2\theta}{\Sigma}\right]\sin^2\theta d\phi^2, \tag{1}$$

with $\Sigma = r^2+a^2\cos^2\theta$, $\Delta = r^2-2Mr+a^2$, $a$ being the specific angular momentum and $M$ the black hole mass, while the outer event horizon is located at $r_+ = M+\sqrt{M^2-a^2}$ and Boyer-Lindquist coordinates are singular there. In the test field approximation (no metric back reaction), neglecting pressure gradients in MHD equations as well as magnetic force terms and imposing a perfect plasma condition $F_{\mu\nu}U^\nu = 0$, the fluid must follow the geodesics on Kerr background studied by Carter:

$$U_{\mu;\nu}U^\nu = 0\,. \tag{2}$$

In particular in their works, RWD consider $U_\phi = 0$ and $U_t = -1$ at infinity and this implies that $U_\theta$ is a constant of motion. The four-velocity geodesic vector is then:

$$\begin{aligned} U^t &= \frac{\Sigma(r^2+a^2)+2Mra^2\sin^2\theta}{\Sigma\Delta}, \quad U^r = -\frac{[-\Delta U_\theta{}^2 + 2Mr(r^2+a^2)]^{\frac{1}{2}}}{\Sigma} \\ U^\theta &= \frac{U_\theta}{\Sigma}, \quad U^\phi = \frac{2MRa}{\Sigma\Delta}\,. \end{aligned} \tag{3}$$

Finally, requiring overall neutral stationary and axisymmetric configuration, the vector potential $A_\mu$ is characterized by the $A_\phi$ component only. RWD solving the simplified MHD equations found in such a a first approximation the analytical solution:

$$A_\phi = A(\theta_\infty) = A_\phi(\theta, r) \tag{4}$$

where

$$\theta_\infty = \theta - U_\theta\,\xi(r) \tag{5}$$

and

$$\xi(r) = \int_r^\infty \frac{dr'}{\sqrt{-(r'^2 - 2Mr' + a^2)U_\theta^2 + 2Mr'(r'^2+a^2)}}\,. \tag{6}$$

From these relations, one case easily reconstruct the entire Maxwell tensor $F_{\mu\nu}$ and obtain the associated four-current $J^\mu$. Great simplification occur by choosing $U_\theta = 0$ because in such a case it results $A_\phi = F(\theta)$ with $F$ a being an arbitrary function. Specifically, in RWD works it has been assumed the simple form $F(\theta) = A_0|\cos\theta|$ where quantity $A_0$ is a constant. Concerning Maxwell invariants, for this solution they have these properties:[6]

$$\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = (\mathbf{B}^2 - \mathbf{E}^2) \geq 0\,, \quad \frac{1}{4}F_{\mu\nu}{}^*F^{\mu\nu} = \mathbf{E}\cdot\mathbf{B} = 0\,, \tag{7}$$

so there must exist a frame in which the observer associated to the geodesics four-velocity measures a magnetic field only, while the electric one vanishes in consequence of the perfect plasma condition. In standard perfect MHD physics, both charge density and electric field disappear in a locally comoving (and corotating) plasma frame. This frame is in general not easy to be found analytically,[9] but for the RWD solution however, by using Doran coordinates, we will successfully obtain it, as now discussed.

## 3. Doran coordinates analysis

The transformation from Boyer-Lindquist (BL) coordinates $(t, r, \theta, \phi)$ to Doran Painlevé-Gullstrand-like (DPG) ones $(T, R, \Theta, \Phi)$ is:

$$T = t - \int^r f(r)dr \ , \qquad R = r \ , \qquad \Theta = \theta \ ,$$
$$\Phi = \phi - \int^r \frac{a}{r^2+a^2} f(r)dr \, , \quad f(r) = -\frac{\sqrt{(2Mr)(r^2+a^2)}}{\Delta} \, , \tag{8}$$

so Kerr solution becomes

$$\begin{aligned} ds^2 =& -\left(1 - \frac{2Mr}{\Sigma}\right) dT^2 + 2\sqrt{\frac{2Mr}{r^2+a^2}} dTdr - \frac{2a(2Mr)}{\Sigma} \sin^2\theta dT d\Phi \\ & + \sin^2\theta \left[r^2 + a^2 + \frac{a^2(2Mr)}{\Sigma}\sin^2\theta\right] d\Phi^2 - 2a\sin^2\theta \sqrt{\frac{2Mr}{r^2+a^2}} dr d\Phi \\ & + \frac{\Sigma}{r^2+a^2} dr^2 + \Sigma d\theta^2 \, . \end{aligned} \tag{9}$$

In the following, due to some of the coordinates' coincidence, we shall denote $R$ with $r$ and $\Theta$ with $\theta$ again. Using the coordinates transformation above, the RWD vector potential becomes

$$A_\mu = \left[0, -a\sqrt{\frac{2Mr}{r^2+a^2}}\frac{F(\theta)}{\Delta}, 0, F(\theta)\right] \, . \tag{10}$$

with $F(\Theta) \equiv F(\theta) = A_0 |\cos\theta|$. Concerning the geodesics, we get

$$U_\mu = \left[-1, -\frac{\sqrt{-\Delta U_\theta^2 + 2Mr(r^2+a^2)}}{\Delta} + \frac{\sqrt{2Mr(r^2+a^2)}}{\Delta}, U_\theta, 0\right] \, , \tag{11}$$

which in the case of interest $U_\theta = 0$ case become

$$U_\mu = [-1, 0, 0, 0] \tag{12}$$

$$U^\mu = \left[1, -\frac{\sqrt{(2Mr)(r^2+a^2)}}{\Sigma}, 0, 0\right] \, . \tag{13}$$

The four-velocity $U^\mu$ above represents the $T = \mathit{const}$ normal geodesic observer. Quantity $T$ is the local proper time of observers in free fall along trajectories characterized by constant $\theta$ and $\Phi$. The DPG electromagnetic tensor of the $U_\theta = 0$ RWD solution field has the only non vanishing component, undefined on the equatorial plane, given by:

$$F_{\Phi\theta} = A_{\Phi,\theta} \equiv \frac{dF(\theta)}{d\theta} = -A_0 \sin\theta \frac{|\cos\theta|}{\cos\theta} \, . \tag{14}$$

Defining the orthonormal locally Lorentzian frame ${e^{(a)}}_\mu$ associated to the DPG normal observer discussed above:

$$
\begin{aligned}
&{e^{(0)}}_\mu = [-1, 0, 0, 0]\ , \quad {e^{(1)}}_\mu = \left[\sqrt{\frac{2Mr}{\Sigma}}, \sqrt{\frac{\Sigma}{r^2+a^2}}, 0, -\sqrt{\frac{2Mr}{\Sigma}} a \sin^2\theta\right] \\
&{e^{(2)}}_\mu = \left[0, 0, \sqrt{\Sigma}, 0\right]\ , \quad {e^{(3)}}_\mu = \left[0, 0, 0, \sqrt{r^2+a^2}\sin\theta\right]\ ,
\end{aligned}
\tag{15}
$$

one can easily show that the Maxwell tensor in the frame $F_{(a)(b)} = {e_{(a)}}^\mu {e_{(b)}}^\mu F_{\mu\nu}$ has the only non vanishing component (giving a magnetic field only):

$$
F_{(3)(4)} = -F_{(4)(3)} \equiv B_{(1)} \equiv B_{\hat{r}} = -\frac{A_0}{\sqrt{(r^2+a^2)\Sigma}} \frac{|\cos\theta|}{\cos\theta}\ . \tag{16}
$$

Moreover, by computing the charge density $J^\mu U_\mu \equiv J_{(0)} = \rho$ measured by the same observer one easily finds that it vanishes everywhere. As anticipated, we have found the natural frame to describe this perfect MHD problem which is the comoving fluid one. Moreover by using the fact that we have horizon penetrating coordinates, we can follow the fields inside the black hole. In particular it is possible to plot the four current lines for this RWD solution given by the numerical integration of the differential equations set $dx^\alpha/d\lambda = J^\alpha$ with $\lambda$ parametrising the curves.[6] To note that while in BL coordinates the current lines whirl infinite times around the event horizon without entering inside, in DPG coordinates (we remind that these are naturally associated to comoving observers) the current lines are not whirled and can be continued regularly in the interior of the black hole. Finally, the use of DPG coordinates allows one to obtain elegant expressions for studying the energetics of the RWD solution. Always in the $U_\theta = 0$ case, one can easily compute an important quasi-local quantity i.e. the electromagnetic energy stored outside the event horizon through a $T = \ const$ cut in space-time as measured by the DPG normal observer. This quantity is given by

$$
E_\sigma(U) = \int_\sigma T^{(\mathrm{em})}_{\mu\nu} U^\mu d\sigma^\nu\ . \tag{17}
$$

Here $\sigma$ is a bounded hypersurface which contains a portion of spacetime while $U^\mu$ represents the normal observer's four-velocity. In the relation above,

$$
\mathcal{E} = T^{(\mathrm{em})}_{\mu\nu} U^\mu U^\nu \equiv \frac{A_0^2}{8\pi(r^2+a^2)\Sigma} \equiv 8\pi\mathcal{F} \geq 0\,, \tag{18}
$$

is the local electromagnetic energy density obtained from the $T^{(\mathrm{em})}_{\mu\nu}$ electromagnetic energy-momentum tensor in DPG coordinates, here proportional to the first Maxwell invariant. Assuming the outer boundary of $\sigma$ being the 2-surface $r = \mathcal{R} = const$, we get the simple relation:

$$
E_{(r_+,\mathcal{R})}(U) = \frac{A_0^2}{2a}\left[\arctan\frac{\mathcal{R}}{a} - \arctan\frac{r_+}{a}\right]\ . \tag{19}
$$

## 4. Conclusions

We have shown that the adoption of recent modern tools is extremely useful for revisiting classical studies of the 60$s$ and 70$s$. We have adopted in particular the most natural object to describe the physics of Ruffini-Wilson and Damour works, represented by Kerr black hole Doran horizon penetrating coordinates with their naturally associated normal observer. We have in particular described plasma physics from the fluid comoving frame and found great simplification for the electrodynamical quantities. The limitation of the analysis here presented however rely on the $U_\theta = 0$ choice which does not allow to obtain possible charge separation processes (charge density is vanishing everywhere in this case in fact), obtained by Ruffini and Wilson by imposing $U_\theta(\theta) = -U_\theta(\pi - \theta) = const$ instead. In order to address such a more complicated problem one should require a further generalization of Doran's work for the $U_\theta \neq 0$ case first. This study is not present in the literature and deserves future studies in order to revisit also the problem of Ruffini and Wilson charge separation.

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