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Shall I compare thee to a summer's day?
Thou art more lovely and more temperate:
Rough winds do shake the darling buds of May,
And summer's lease hath all too short a date:

Sometime too hot the eye of heaven shines,
And often is his gold complexion dimm'd,
And every fair from fair sometime declines,
By chance or nature's changing course untrimm'd:

But thy eternal summer shall not fade
Nor lose possession of that fair thou ow'st,
Nor shall Death brag thou wander'st in his shade,
When in eternal lines to time thou grow'st:

So long as men can breathe or eyes can see,
So long lives this, and this gives life to thee.

– William Shakespeare, Sonett 18

Für Barbara

Abstract

Starting from a chiral conformal Haag-Kastler net of local observables on two-dimensional Minkowski space-time, we construct associated pointlike localizable charged fields which intertwine between the superselection sectors with finite statistics of the theory.

This amounts to a proof of the spin-statistics theorem, the PCT theorem, the Bisognano-Wichmann identification of modular operators, Haag duality in the vacuum sector, and the existence of operator product expansions.

Our method consists of the explicit use of the representation theory of the universal covering group of $SL(2, \mathbf{R})$. A central role is played by a "conformal cluster theorem" for conformal two-point functions in algebraic quantum field theory.

Generalizing this "conformal cluster theorem" to the n-point functions of Haag-Kastler theories, we can finally construct from a chiral conformal net of algebras a complete set of conformal n-point functions fulfilling the Wightman axioms.

Zusammenfassung

Ausgehend von einem chiralen konformen Haag-Kastler-Netz lokaler Observablen auf einer zweidimensionalen Minkowski-Raumzeit konstruieren wir zugehörige punktartig lokalisierbare geladene Felder, die die unterschiedlichen Superauswahlsektoren mit endlicher Statistik der Theorie miteinander verbinden.

Mit diesem Ergebnis beweisen wir das Spin-Statistik-Theorem, das PCT-Theorem, das Theorem von Bisognano und Wichmann über die Identifikation modularer Operatoren, Haag-Dualität im Vakuumsektor und die Existenz von Operator-Produkt-Entwicklungen.

Für die Beweise benutzen wir explizit die Darstellungstheorie der universellen Überlagerungsgruppe von $SL(2, \mathbf{R})$. Zentrale Bedeutung hat ein "konformes Cluster-Theorem" für konforme Zwei-Punkt-Funktionen in der algebraischen Quantenfeldtheorie.

Mit der Verallgemeinerung dieses "konformen Cluster-Theorems" auf die n-Punkt-Funktionen von Haag-Kastler-Theorien können wir schließlich aus einem chiralen konformen Netz von Algebren einen vollständigen Satz konformer n-Punkt-Funktionen konstruieren, die die Wightman-Axiome erfüllen.

Contents

1 Introduction

Und die Fische, sie verschwinden!
Doch zum Kummer des Gerichts:
Man zitiert am End den Haifisch
Doch der Haifisch weiß von nichts.

Und er kann sich nicht erinnern
Und man kann nicht an ihn ran
Denn ein Haifisch ist kein Haifisch
Wenn man's nicht beweisen kann.

– Bertolt Brecht, Moritat von Mackie Messer

Quantum field theory is the unification of classical physics, quantum physics, and the theory of special relativity. Its most important field of application has been in elementary particle physics. Using refined methods of perturbation theory and computer-aided numerical calculations, several important theoretical predictions have been made in quantum field theory that turned out to be in full agreement with experiments at large particle accelerators. Hence, quantum field theory has proved to be a successful concept in physics, and it is widely expected to continue to produce new results in the future. It has to be stated, though, that quantum field theory has not yet found an accepted mathematical formulation and theoretical framework that combines conceptual consistency with mathematical rigour on the one side and applicability to typical experimental situations on the other side.

In mathematical physics, two main approaches have been developed for a general theory of quantized fields:

Streater and Wightman formalized the long-practiced use of operator-valued distributions

$$f \longmapsto \varphi(f) = \int dx\, f(x)\, \varphi(x)$$

and formulated "Wightman axioms" for pointlike localizable quantized fields ("PCT, Spin & Statistics, and All That", [StW]). This framework is known as Wightman quantum field theory.

The second approach is based on ideas of Haag and Kastler [HaK]. The basic assumption is that all physical information must already be encoded in the structure of the local observables. Haag and Kastler introduced a mathematical structure for the set of observables of a physical system by proposing "Haag-Kastler axioms" for nets of C^*-algebras

$$\mathcal{A} : \mathcal{O} \longmapsto \mathcal{A}(\mathcal{O})$$

on bounded regions of Minkowski space-time. This approach is called algebraic quantum field theory ("Local quantum physics", [Haag]).

The aim of the studies presented in this dissertation is a better understanding of the mutual relation of these two mathematical frameworks for a general theory of quantized fields.

For the investigation of general structures in quantum field theory the formulation in terms of Haag-Kastler nets of local observables has turned out to be well suited:

The selfadjoint elements of the C^*-algebras $\mathcal{A}(\mathcal{O})$ represent the observables that can be measured in the space-time region $\mathcal{O}$. Algebras localized in bounded space-time regions are called local, the norm closure of the union of all local algebras is called the algebra of quasilocal observables.

In a causal relativistic theory, measurements in spacelike separated regions of space-time must not influence each other, i.e. Einstein causality holds. Einstein causality is implemented in algebraic quantum field theory by the postulate that algebras with spacelike separated localization region commute. This postulate is called locality of the net of algebras.

A quantum field theory is supposed to be invariant under an appropriate group of symmetry transformations of space-time. In the algebraic approach, the symmetry group is represented by automorphisms of the algebra of quasilocal observables such that the action of the automorphisms respects the local structure of the net.

Physical states of a system are defined by positive, normed functionals of the algebra of quasilocal observables. By the GNS-construction (cf. [BrR]), we can associate to any state a concrete realization of the (abstract) algebra of quasilocal observables on a Hilbert space and a cyclic vector in this Hilbert space.

If this cyclic vector is invariant under a positive energy representation of the translation group and unique up to a free phase, we call it a vacuum vector and the associated state a vacuum state.

Given a vacuum state on a net of local observables, we have to determine the set of physically realizable states in order to completely characterize the physical system. This is done by appropriate selection criteria (cf. [Bor1, DHR1-4, BuF2]). The structure of the set of physically realizable states then determines the set of charges appearing in a physical system. The properties of the charges in a theory are described by the superselection sectors of a theory, i.e. by the equivalence classes of irreducible representations of the algebra of quasilocal observables associated with physically realizable states.

Doplicher, Haag and Roberts [DHR1-4] have shown that, with an appropriate selection criterion, important experimentally verified phenomena in elementary particle physics like charge addition, anticharge, antiparticles, pair creation, pair annihilation, and quantum statistics of particles can be derived in this mathematical framework. Confer also [Fre1] and [BuF2].

The conceptual strength of algebraic quantum field theory is given by the fact that all constructions and results can directly be reduced to the knowledge of locally observable physical quantities. The weakness of algebraic quantum field theory is the lack of control on pointlike limits and the missing connections to classical field theory in this approach. Therefore, the discussion of concrete models in terms of Haag-Kastler nets and the explicit construction of algebraic quantum field theories apart from free fields meet severe problems and difficulties.

Consequently, the discussion of concrete models is mostly done in terms of pointlike localized fields. In order to be in a precise mathematical framework, these fields might be assumed to obey the Wightman axioms [StW]. Wightman quantum field theory is an axiomatic framework for (charged) pointlike localizable quantum fields. The fields are given as covariantly transforming tempered distributions with values in closable unbounded operators on the physical Hilbert space. Einstein causality is implemented by appropriate commutation relations. The common domain of definition is assumed to be dense in the Hilbert space, invariant with respect to the symmetry group, and stable under the action of field operators.

The Wightmanian notion of quantum fields provides us with a strong intuition for the solution of problems in concrete models because of its analogy to classical field theory. By canonical quantization, the Hamiltonian and Langrangian formalism can formally be transferred from classical physics to quantum field theory. In the theories of electro-weak and strong interaction, experimentally measurable physical quantities can be calculated by perturbation series of Feynman graphs using the concept of pointlike localized fields.

The disadvantages of Wightman quantum field theory compared with the algebraic approach turn out in the analysis of general structures of relativistic quantum physics in a model-independent context:

To begin with, the "primary objects" in the Wightman approach are charge-carrying fields, i.e. non-observable objects. This leads to redundancy in description and might cause inconsistencies, which would be difficult to control.

In addition to that, the quantized fields have, in general, to be defined as unbounded operators on a dense set of vectors in the physical Hilbert space. Hence, "domain problems" cannot be avoided. Even worse, we do not know how to associate these severe and yet unsolved mathematical problems to the corresponding physical problems.

We noticed that the two different approaches to a general theory of quantized fields have complementary advantages and disadvantages. It is this complementarity of Wightman theory and algebraic quantum field theory that makes the investigation of the interrelation of both approaches interesting and important:

Given an equivalence of both frameworks, one could switch between the "algebraic picture" of Haag and Kastler and the "distribution picture" of Streater and Wightman in order to combine the strengths of both approaches and to circumvent the weaknesses.

This motivates the question of whether, how, and under which conditions Wightman fields can be constructed from Haag-Kastler nets and vice versa. The constructed theory should then describe the same physical system with the same physical properties as the original theory.

Heuristically, Wightman fields are constructed out of Haag-Kastler nets by some scaling limit, which, however, is difficult to formulate in an intrinsic way (cf. [Buc2]). In a dilation invariant theory scaling is well-defined, and in the presence of massless particles the construction of a pointlike field was performed in [BuF1].

Generally, the interrelation between both concepts is not yet completely understood. See [BaW, BoY] for the present stage and a survey of the developments in this field of research. A detailed list of literature relevant to this subject can also be found in [Yng] and [DSW].

Several conditions restricting regularity, locality, and type of the pointlike localized operator-valued distributions guarantee that an associated net of von Neumann algebras can be canonically constructed and that the closures of the field operators are affiliated to the associated local von Neumann algebras (cf. [DrF, BoY, Wic, Bucl]). The invariants of this field algebra under the gauge group that is generated by the set of charges in the theory would then be, in accordance with [DHR1-4], the vacuum representation of the Haag-Kastler net of local observables we have been looking for.

Examining the opposite direction, Fredenhagen and Hertel [FrH, Her] and later Rehberg, Wollenberg, Kern and Summer [ReW, Wol1, Ker, Sum] found necessary but not sufficient criteria for the existence of affiliated Wightman fields to a given Haag-Kastler net.

Wollenberg [Wol2, Wol3] could derive a sufficient criterion for the existence of so-called pre-fields. Pre-fields transform covariantly, but generally do not fulfill the Wightman axioms, since invariance and stability of the domain of definition and a product structure are not assumed.

The short summary above outlined some of the problems of the relation of algebraic quantum field theory and Wightman fields. In this thesis, we will study the possibly simplest situation: Haag-Kastler nets in two-dimensional Minkowski space with trivial translations in one light-cone direction ("chirality") and covariant under the real Möbius group ("conformal symmetry group") which acts on the other lightlike direction. Starting from an algebraic formulation of chiral conformal field theory, we will try to find an equivalent formulation of this theory in terms of pointlike localized conformal fields.

Conformal quantum field theory in two dimensions has had its main application in statistical physics. Confer [BPZ] as the fundamental paper in this field and [Gin] as an introduction to that approach. Two-dimensional conformal field theory has strong connections to string theory and to other two-dimensional physical theories (cf., e.g., [LPS]). In this thesis, we will take two-dimensional conformally covariant quantum field theory as a theory of interest in itself with high symmetry on a simple geometry. Structural properties found at first to be inherent to conformal field theories have, however, often shown to be properties of more general classes of field theories. Hence, we hope that the results in this thesis on chiral conformal quantum field theory will help to find similar results on theories with lower symmetries and on more complicated geometries.

Conformal symmetry transformation are, by definition, those transformations that leave the absolute value and the orientation of arbitrary angles in space-time unchanged. Hence, the "classical" conformal symmetry group in two-dimensional Minkowski space-time is given as the product of two groups of orientation-preserving diffeomorphisms on the real numbers (cf., e.g., [Sch]). Since the vacuum has to be invariant under (a positive energy representation of the group of) symmetry transformations, the conformal symmetry group reduces after quantization to

$$G_2 \; = \; SO(2,2)/\mathbf{Z}_2 \, .$$

Introducing light-cone coordinates $x_+ := x + t$ and $x_- := x - t$, one can observe that G_2 factorizes into Möbius groups:

$$G_2 \; = \; SL(2,\mathbf{R})/\mathbf{Z}_2 \; \times \; SL(2,\mathbf{R})/\mathbf{Z}_2 \, .$$

The factors act independently on the single light-cones:

$$x_\pm \; \longmapsto \; \frac{ax_\pm + b}{cx_\pm + d} \, , \quad a,b,c,d \in \mathbf{R} \, , \; ad - bc = 1 \, .$$

Local conformal quantum fields in two dimensions generally do not transform irreducibly under the center of the universal covering group of the conformal symmetry group. They can, however, be decomposed in non-local parts that transform irreducibly under the center. These irreducibly transforming non-local parts separate into light-cone fields. The light-cone fields form an exchange algebra that obeys braid group statistics (cf. [RSc]). Confer as well the solution of the "causality paradox" in [LüM, ScS, SSV].

This line of argument motivates to consider conformal Haag-Kastler nets on double cones in two-dimensional Minkowski space-time that separate into tensor products of light-cone nets:

$$\mathcal{A}(I \times J) \; = \; \mathcal{A}(I) \, \otimes \, \mathcal{A}(J) \, , \quad I, J \subset \mathbf{R} \, .$$

Hereby, $I \times J$ denotes the double cone in Minkowski space-time determined by intervals I and J on the respective light-cones. Locality is implemented in the net of light-cone algebras by the postulate that algebras associated with disjoint intervals commute.

These chiral light-cone nets that transform covariantly under the action of the Möbius group $SL(2,\mathbf{R})$ are the starting point of the discussion in this thesis. In order to provide orientation for the reader of the following chapters, we will now present our procedure and the content of the thesis:

In the vacuum representation pointlike localized fields can be constructed (cf. [FrJ]). Their smeared linear combinations are affiliated to the original net of local observables and generate it. We do not know at the moment whether they satisfy all Wightman axioms, since we have not yet found an invariant and stable domain of definition.

This idea can be generalized to the charged sectors of a theory in the algebraic framework (cf. [Jör3]). We construct pointlike localized fields carrying arbitrary charge with finite statistics and therefore intertwining between the different superselection sectors of the theory. (In conformal field theory, these objects are known as "Vertex Operators".) We obtain the unbounded field operators as limits of elements of the reduced field bundle (cf. [FRS1, FRS2]) associated with the net of local observables of the theory.

Our method consists of an explicit use of the representation theory of the universal covering group of $SL(2, \mathbf{R})$ combined with a conformal cluster theorem in the vacuum sector (cf. [FrJ]) and its generalization to the case of arbitrary charge with finite statistics (cf. [Jör3]).

As a consequence of the existence of charged pointlike localized fields, we can prove the spin-statistics theorem[1], the PCT theorem for the full theory, and the generalization of the Bisognano-Wichmann property (cf. [BiW]) for charged sectors.

The existence of operator product expansions in the Wightman framework has been postulated by Wilson [Wil]. According to Wilson, the product of local fields $\varphi_i(\cdot)$ should admit an asymptotic expansion at short distances x of the form

$$\varphi_i(\frac{x}{2})\,\varphi_j(-\frac{x}{2}) \;=\; \sum_k C_{ijk}(x)\,\psi_k(0)\,.$$

Hereby, $\psi_i(\cdot)$ denote appropriate derivatives of local fields and $C_{ijk}(\cdot)$ are singular functions with values in $\mathbf{C}$. Especially in two-dimensional conformal field theory, this assumption has turned out to be very fruitful. The existence of a convergent expansion of the product of two fields on the vacuum could be derived from conformal covariance:

$$\varphi_i(x)\,\varphi_j(y)\,\Omega \;=\; \sum_k \int dz\; B_{kij}(x,y,z)\,\varphi_k(z)\,\Omega$$

with kernels B_{kij} that can explicitly be calculated. In the proof of this result, however, the existence of the associated local fields has to be postulated [Lüs, Mac1, SSV].

In the Haag-Kastler framework, the existence of an operator product expansion might be formulated as the existence of a sufficient number of

[1] After the completion of [Jör3], we received a paper by Guido and Longo [GLo] that gives an independent proof of the conformal spin-statistics theorem.

Wightman fields such that their linear span applied to the vacuum is dense in the Hilbert space. Actually, we are able to derive a stronger result: We prove an expansion of local observables into field operators with local coefficients and show that this expansion converges $*$-strongly on a dense domain in the vacuum Hilbert space.

Finally, we start again from a chiral conformal Haag-Kastler net of local observables and present a canonical construction of n-point functions that can be shown to fulfill the Wightman axioms. We proceed by generalizing the conformal cluster theorem (cf. [FrJ]) to higher n-point functions and by examining the momentum space limit of the algebraic n-point functions of local observables at $p = 0$.

We are not able to prove that these Wightman fields can be identified with the pointlike localized fields constructed above (cf. [FrJ, Jör3]), nor can we derive (generalized) H-bounds (cf. [FrH, BoY]) for the Wightman fields, which would be sufficient to prove that the closures of the Wightman field operators are affiliated to the net of local observables we have been starting from.

2 First Steps

To follow the technical parts in the following chapters of this thesis, the reader needs detailed information about our assumptions. Here, we present an explicit formulation of the setting from which the calculations and considerations in this thesis start.

In the first section of this chapter, we introduce a formulation of the Haag-Kastler axioms for nets of local algebras that is adapted to the physical situation we consider throughout this thesis: chiral conformal nets on the one-dimensional light-cone. The assumptions in this section on the set of local observables are the basic definition for the whole thesis.

The reader might also find helpful to (re)read, in the second section of this chapter, some (well-known) general remarks on chiral nets which we will repeatedly use in the following chapters of this thesis. We point out which consequences follow directly from the assumptions, we introduce important mathematical structures, and we review results of [FröG] and [BGL] on chiral conformal nets. Finally, the proof of additivity of chiral conformal nets (cf. [FrJ]) is presented.

2.1 Assumptions

Let $\mathcal{A} = (\mathcal{A}(I))_{I \in \mathcal{K}_0}$ be a family of von Neumann algebras on a separable Hilbert space H. $\mathcal{K}_0$ denotes the set of non-empty bounded open intervals on $\mathbf{R}$. $\mathcal{A}$ is assumed to satisfy the following Haag-Kastler conditions.

i) Isotony:

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2) \quad \text{for} \quad I_1 \subset I_2\,, \quad I_1, I_2 \in \mathcal{K}_0\,. \tag{2.1}$$

ii) Locality:

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2)' \quad \text{for} \quad I_1 \cap I_2 = \emptyset\,, \quad I_1, I_2 \in \mathcal{K}_0 \tag{2.2}$$

($\mathcal{A}(I_2)'$ is the commutant of $\mathcal{A}(I_2)$).

iii) Conformal Covariance:

There exists a strongly continuous unitary representation $U(\cdot)$ of the Möbius group $G = SL(2, \mathbf{R})$ in H with $U(-1) = 1$ and

$$U(g)\,\mathcal{A}(I)\,U(g)^{-1} = \mathcal{A}(gI)\,, \qquad I, gI \in \mathcal{K}_0 \tag{2.3}$$

$(SL(2,\mathbf{R}) \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on $\mathbf{R} \cup \{\infty\}$ by the mapping

$$x \mapsto \frac{ax+b}{cx+d} \tag{2.4}$$

with the appropriate interpretation for $x, gx = \infty$).

iv) Stability:

The conformal Hamiltonian $\mathbf{H}$, which generates the restriction of $U(\cdot)$ to $SO(2)$, has a non-negative spectrum.

v) Uniqueness of the Vacuum:

There is a unique (up to a phase) U-invariant unit vector $\Omega \in H$.

vi) Cyclicity of the Vacuum:

H is the smallest closed subspace containing the vacuum Ω which is invariant under $U(g)$, $g \in SL(2,\mathbf{R})$, and $A \in \mathcal{A}(I)$, $I \in \mathcal{K}_0$. [2]

Thereby, we have defined a vacuum representation of a chiral conformal Haag-Kastler net of local observables on $\mathbf{R}$. Following the line of argument in the introduction, this Haag-Kastler net can be interpreted as the algebraic formulation of a quantized theory of conformal light-cone fields in terms of von Neumann algebras of local observables.

2.2 Generalities on Chiral Nets

It is convenient to extend the net to intervals I on the circle $\mathbf{S^1} = \mathbf{R} \cup \{\infty\}$ by setting

$$\mathcal{A}(I) = U(g)\, \mathcal{A}(g^{-1}I)\, U(g)^{-1}, \qquad g^{-1}I \in \mathcal{K}_0,\, g \in SL(2,\mathbf{R}). \tag{2.5}$$

The covariance property guarantees that $\mathcal{A}(I)$ is well-defined for all intervals I of the form $I = gI_0$, $I_0 \in \mathcal{K}_0$, $g \in SL(2,\mathbf{R})$, i.e. for all non-empty non-dense open intervals on $\mathbf{S^1}$ (we denote the set of these intervals by $\mathcal{K}$).

[2] This assumption is seemingly weaker than cyclicity of Ω with respect to the algebra of local observables on $\mathbf{R}$.

We first want to note (see [FrJ]) that Ω is cyclic and separating for all $\mathcal{A}(I)$, $I \in \mathcal{K}$ (Reeh-Schlieder property, cf. [ReS, Bor2]). Namely, let us look at $I = \mathbf{R}_+$ (without restriction of generality). $\mathbf{R}_+$ is mapped into itself by translations

$$x \mapsto x + a\,, \quad a > 0\,, \tag{2.6}$$

and special conformal transformations

$$x \mapsto \frac{x}{-cx+1}\,, \quad c < 0\,. \tag{2.7}$$

Both one-parameter groups have a positive generator under the representation $U(\cdot)$. Hence, by the usual Reeh-Schlieder argument, for both one-parameter groups $(g_t)_{t\in\mathbf{R}}$ the following relations hold:

$$\begin{aligned} H_0 &:= \overline{\mathcal{A}(\mathbf{R}_+)\,\Omega} \\ &= \overline{\bigcup_{t\in\mathbf{R}} U(g_t)\,\mathcal{A}(\mathbf{R}_+)\,\Omega}\,, \end{aligned} \tag{2.8}$$

where the overlined expressions denote the norm closure. So H_0 is invariant under $\mathcal{A}(I)$ for all $I \in \mathcal{K}_0$ and under translations and special conformal transformations, thus under $U(SL(2,\mathbf{R}))$. By assumption vi) we conclude $H_0 = H$, i.e. Ω is cyclic for $\mathcal{A}(\mathbf{R}_+)$, and, by locality, separating for $\mathcal{A}(\mathbf{R}_-)$. (Cf. [FröG] for a similar argument.)

In the following paragraph, we review results of [FröG] and [BGL]. We start with the modular structure. The modular involution J_I and the modular operator Δ_I are obtained by polar decomposition of the closure S_I of the operator defined by the mapping

$$A\,\Omega \mapsto A^*\,\Omega\,, \quad A \in \mathcal{A}(I)\,. \tag{2.9}$$

That means,

$$S_I = J_I\,\Delta_I^{1/2}, \qquad I \in \mathcal{K}\,. \tag{2.10}$$

J_I implements an antiisomorphism between $\mathcal{A}(I)$ and $\mathcal{A}(I)'$ and Δ_I^{it}, $t \in \mathbf{R}$, automorphisms of $\mathcal{A}(I)$ [Tak1]. Borchers has shown [Bor3] that every unitary strongly continuous one-parameter group $(U(t))_{t\in R}$ with a positive generator and Ω as a fixed point which induces endomorphisms of $\mathcal{A}(I)$ for $t > 0$ satisfies the commutation relations

$$\Delta_I^{it}\,U(a)\,\Delta_I^{-it} = U(e^{-2\pi t}a) \tag{2.11}$$

$$J_I\, U(a)\, J_I = U(-a)\,. \tag{2.12}$$

Applying this to $I = \mathbf{R}_+$ and to the one-parameter groups considered above, we find that the operators

$$Z(s) = \Delta^{is}_{\mathbf{R}_+}\, U\begin{pmatrix} e^{\pi s} & 0 \\ 0 & e^{-\pi s} \end{pmatrix}, \qquad s \in \mathbf{R}\,, \tag{2.13}$$

commute with $U(g)$ for all $g \in SL(2,\mathbf{R})$ (in particular, $s \mapsto Z(s)$ is a one-parameter group). Moreover,

$$J_{\mathbf{R}_+}\, U(g)\, J_{\mathbf{R}_+} = U(g_\vartheta)\,, \tag{2.14}$$

where g_ϑ is defined as the matrix $\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$ if g is given as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

With $\vartheta := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2,\mathbf{R})$, we have $\mathbf{R}_- = \vartheta\, \mathbf{R}_+$. Hence,

$$J_{\mathbf{R}_-} = U(\vartheta)\, J_{\mathbf{R}_+}\, U(\vartheta)^{-1}\,. \tag{2.15}$$

So, inserting $g = \vartheta$ in equation (2.14) we obtain that $J_{\mathbf{R}_-}$ coincides with $J_{\mathbf{R}_+}$. Thus, by locality and by the properties of modular involutions, we obtain

$$\begin{aligned} \mathcal{A}(\mathbf{R}_-) &\subset \mathcal{A}(\mathbf{R}_+)' \\ &= J_{\mathbf{R}_+}\, \mathcal{A}(\mathbf{R}_+)\, J_{\mathbf{R}_+} \\ &\subset J_{\mathbf{R}_-}\, \mathcal{A}(\mathbf{R}_-)'\, J_{\mathbf{R}_-} \\ &= \mathcal{A}(\mathbf{R}_-)\,. \end{aligned} \tag{2.16}$$

This shows Haag duality for half lines

$$\mathcal{A}(\mathbf{R}_-) = \mathcal{A}(\mathbf{R}_+)' \tag{2.17}$$

and, by conformal covariance, Haag duality for every $I \in \mathcal{K}$:

$$\mathcal{A}(I')' = \mathcal{A}(I)\,. \tag{2.18}$$

Now we compute

$$
\begin{aligned}
Z(s) &= \Delta^{is}_{\mathbf{R}_+} U \begin{pmatrix} e^{\pi s} & 0 \\ 0 & e^{-\pi s} \end{pmatrix} \\
&= U \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Delta^{is}_{\mathbf{R}_+} U \begin{pmatrix} e^{\pi s} & 0 \\ 0 & e^{-\pi s} \end{pmatrix} U \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \Delta^{is}_{\mathbf{R}_-} U \begin{pmatrix} e^{-\pi s} & 0 \\ 0 & e^{\pi s} \end{pmatrix} \\
&= \Delta^{-is}_{\mathbf{R}_+} U \begin{pmatrix} e^{-\pi s} & 0 \\ 0 & e^{\pi s} \end{pmatrix} \\
&= Z(-s)\,. \qquad (2.19)
\end{aligned}
$$

Here we used essential duality and the fact that the modular operator of $\mathcal{A}(\mathbf{R}_+)'$ is $\Delta^{-1}_{\mathbf{R}_+}$. Since Z is a one-parameter group, it must be trivial. Therefore, we obtain

$$
\Delta^{is}_{\mathbf{R}_+} = U \begin{pmatrix} e^{-\pi s} & 0 \\ 0 & e^{\pi s} \end{pmatrix}. \qquad (2.20)
$$

Moreover, we show that the antiunitary involution $J_{\mathbf{R}_+} =: \Theta$ is a PCT operator which acts on $\mathcal{A}$ by

$$
\Theta\, \mathcal{A}(I)\, \Theta = \mathcal{A}(-I)\,, \quad I \in \mathcal{K}\,, \qquad (2.21)
$$

and on $U(SL(2,\mathbf{R}))$ by

$$
\Theta\, U(g)\, \Theta = U(g_\vartheta)\,. \qquad (2.22)
$$

The commutation relations of $J_{\mathbf{R}_+}$ with $U(SL(2,\mathbf{R}))$ have already been determined; now choose $I \in \mathcal{K}$ and $g \in SL(2,\mathbf{R}))$ such that $gI = \mathbf{R}_+$. Thus,

$$
J_{\mathbf{R}_+} \mathcal{A}(I)\, J_{\mathbf{R}_+} \;=\; J_{\mathbf{R}_+} U(g)\, \mathcal{A}(\mathbf{R}_+)\, U(g)^{-1} J_{\mathbf{R}_+}
$$

$$= U(g_\vartheta)\, J_{\mathbf{R}_+}\, \mathcal{A}(\mathbf{R}_+)\, J_{\mathbf{R}_+}\, U(g_\vartheta)^{-1}$$

$$= U(g_\vartheta)\, \mathcal{A}(\mathbf{R}_-)\, U(g_\vartheta)^{-1}$$

$$= \mathcal{A}(-I)\,. \tag{2.23}$$

Note that $g_\vartheta \mathbf{R}_- = -I$ follows from $gI = \mathbf{R}_+$.

We now use the results of [FröG] and [BGL], presented above, to give the proof that the net is automatically additive (cf. [FrJ]). Consider an open covering of $I \in \mathcal{K}$:

$$I = \bigcup_\alpha I_\alpha\,, \quad I_\alpha \in \mathcal{K}\,. \tag{2.24}$$

Choose $I_0 \in \mathcal{K}$ such that $\overline{I_0} \subset I$, where $\overline{I_0}$ denotes the closure of I_0. Then there is a finite number of intervals $I_{\alpha_1}, \dots, I_{\alpha_n}$ which already cover I_0. According to the Bisognano-Wichmann result above, $\Delta_{I_0}^{it}$ implements the one-parameter subgroup $(g_t)_{t\in\mathbf{R}}$ of $SL(2,\mathbf{R})$ which has the boundary points of I_0 as fixed points. There is a sufficiently small interval $I_1 \in \mathcal{K}$, $I_1 \subset I_0$, such that for all $t \in \mathbf{R}$ the interval $g_t(I_1)$ is contained in one of the intervals I_{α_i}, $i = 1, \dots, n$. The algebra

$$\mathcal{A}_{I_1}(I_0) := \bigvee_{t\in\mathbf{R}} \alpha_{g_t}(\mathcal{A}(I_1)) \subset \mathcal{A}(I_0) \tag{2.25}$$

is invariant under the modular automorphism $\mathrm{Ad}\Delta_{I_0}^{it} = \alpha_{g_t}$ of $\mathcal{A}(I_0)$ and has Ω as a cyclic vector, hence coincides with $\mathcal{A}(I_0)$ (cf. [Tak2]). Thus, $\mathcal{A}(I_0)$ is contained in $\bigvee_\alpha \mathcal{A}(I_\alpha)$ if $\overline{I_0} \subset I$. But a conformally covariant net is continuous from below (cf., e.g., [Jörl]),

$$\mathcal{A}(I) = \bigvee_{\bar{I}_0 \subset I} \mathcal{A}(I_0)\,, \tag{2.26}$$

which implies additivity.

3 From Conformal Nets to Pointlike Neutral Fields

As has been motivated in the introduction, we are interested in the relation and the interplay between the different possibilities to formulate chiral conformal quantum field theory. We start from a chiral conformal Haag-Kastler net on two-dimensional Minkowski space-time and present the construction of a physically and mathematically equivalent theory in terms of pointlike localized fields. The content of this chapter covers the calculations and results in the vacuum sector of the theory. I.e., it deals with local observables and neutral fields without charge.

We proceed as follows: First we present the conformal cluster theorem, a result as interesting in itself as crucial for the rest of this thesis. We then construct pointlike localized neutral fields, show their properties, and derive the consequences of their existence. Finally, we prove an operator product expansion for this field theory.

3.1 Conformal Cluster Theorem

In this section, we derive a bound on conformal two-point functions in algebraic quantum field theory (see [FrJ]). This bound specifies the decrease properties of conformal two-point functions in the algebraic framework to be exactly those known from conformal field theories with pointlike localization.

Conformal Cluster Theorem (see [FrJ]): Let $(\mathcal{A}(I))_{I\in\mathcal{K}_0}$ be a conformally covariant Haag-Kastler net of local observables on $\mathbf{R}$. Let $a, b, c, d \in \mathbf{R}$ and $a < b < c < d$. Let $A \in \mathcal{A}(\,(a,b)\,)$, $B \in \mathcal{A}(\,(c,d)\,)$, $n \in \mathbf{N}$, and

$$P_k\, A\, \Omega = P_k\, A^*\, \Omega = 0\,,\; k < n\,. \tag{3.1}$$

P_k here denotes the projection on the subrepresentation of $U(SL(2,R))$ with conformal dimension k. The conformal dimension is defined as the scaling dimension of the subrepresentation of dilations and uniquely characterizes irreducible representations of the conformal symmetry group with positive energy up to unitary equivalence.

We then have

$$|(\,\Omega, B\,A\,\Omega\,)| \leq \left(\frac{(b-a)\,(d-c)}{(c-a)\,(d-b)}\right)^n \;\|A\|\,\|B\|\,. \tag{3.2}$$

Proof: Choose $R > 0$. We consider the following one-parameter subgroup of $SL(2,\mathbf{R})$:

$$g_t \,:\, x \longmapsto \frac{x\cos\frac{t}{2} + R\sin\frac{t}{2}}{-\frac{x}{R}\sin\frac{t}{2} + \cos\frac{t}{2}}. \tag{3.3}$$

Its generator $\mathbf{H}_R$ is within each subrepresentation of $U(SL(2,R))$ unitarily equivalent to the conformal Hamiltonian $\mathbf{H}$. Therefore, the spectrum of $A\,\Omega$ and $A^*\,\Omega$ with respect to $\mathbf{H}_R$ is bounded from below by n. Let $0 < t_0 < t_1 < 2\pi$ such that

$$g_{t_0}(b) = c \tag{3.4}$$

and

$$g_{t_1}(a) = d. \tag{3.5}$$

We now define

$$F(z) = \begin{cases} (\,\Omega,\, B\,z^{-\mathbf{H}_R}\,A\,\Omega\,), & |z| > 1, \\ (\,\Omega,\, A\,z^{\mathbf{H}_R}\,B\,\Omega\,), & |z| < 1, \\ (\,\Omega,\, A\,\alpha_{g_t}(B)\,\Omega\,), & z = e^{it},\ t \notin [t_0, t_1], \end{cases} \tag{3.6}$$

a function analytic in its domain of definition, and then

$$G(z) = (z - z_0)^n\,(z^{-1} - z_0^{-1})^n\,F(z), \quad z_0 = e^{i(t_0+t_1)/2}. \tag{3.7}$$

(Confer the idea in [Fre2].) At $z = 0$ and $z = \infty$ the function $G(\cdot)$ is bounded because of the bound on the spectrum of $\mathbf{H}_R$ and can therefore be analytically continued. As an analytic function it reaches its maximum at the boundary of its domain of definition, which is the interval $[e^{it_0}, e^{it_1}]$ on the unit circle:

$$\begin{aligned} \sup|G(z)| &\le \|A\|\,\|B\|\,|e^{it_0} - e^{i(t_0+t_1)/2}|^{2n} \\ &= \|A\|\,\|B\|\,|2\sin\frac{t_0 - t_1}{4}|^{2n}. \end{aligned} \tag{3.8}$$

This leads to

$$\begin{aligned} |(\,\Omega,\, B\,A\,\Omega\,)| &= |F(1)| \\ &= |G(1)|\,|1 - e^{i(t_0+t_1)/2}|^{-2n} \\ &= |G(1)|\,|2\sin\frac{t_0+t_1}{4}|^{-2n} \\ &\le \sup|G(\cdot)|\,|2\sin\frac{t_0+t_1}{4}|^{-2n} \\ &\le \|A\|\,\|B\|\,\left|\frac{\sin\frac{t_0-t_1}{4}}{\sin\frac{t_0+t_1}{4}}\right|^{2n}. \end{aligned} \tag{3.9}$$

Determining t_0 and t_1, we obtain

$$\lim_{R\to\infty} R\,t_0 = 2(c-b) \tag{3.10}$$

and

$$\lim_{R\to\infty} R\,t_1 = 2(d-a)\,. \tag{3.11}$$

We now assume $a-b=c-d$ and find

$$\left(\frac{t_0-t_1}{t_0+t_1}\right)^2 = \frac{(a-b)\,(c-d)}{(a-c)\,(b-d)} =: r\,. \tag{3.12}$$

Since the bound on $|(\,\Omega,\,B\,A\,\Omega\,)|$ can only depend on the conformal cross ratio r, we can drop the assumption and the theorem is proven. □

3.2 Construction of Pointlike Localized Fields

The idea for the definition of conformal fields is the following (see [FrJ, Jör3]): Let A be a local observable,

$$A \in \bigcup_{I\in\mathcal{K}_0} \mathcal{A}(I)\,, \tag{3.13}$$

and P_τ the projection onto an irreducible subrepresentation τ of $U(\cdot)$. The vector $P_\tau\,A\,\Omega$ may then be thought of as $\varphi_\tau(h)\,\Omega$ where φ_τ is a conformal field of scaling dimension $n_\tau \in \mathbf{N}$ and h is an appropriate function on $\mathbf{R}$. The relation between A and h, however, is unknown at the moment, up to the known transformation properties under $SL(2,R)$,

$$U(g)\,P_\tau\,A\,\Omega = \varphi_\tau(h_g^{(n_\tau)})\,\Omega \tag{3.14}$$

with

$$h_g^{(n_\tau)}(x) = (cx-a)^{2n_\tau-2}\;h\left(\frac{dx-b}{-cx+a}\right) \tag{3.15}$$

for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,R)\,. \tag{3.16}$$

We may now scale the vector $P_\tau\,A\,\Omega$ by dilations

$$D(\lambda) := U\begin{pmatrix} \lambda^{\frac{1}{2}} & 0 \\ 0 & \lambda^{-\frac{1}{2}} \end{pmatrix} \tag{3.17}$$

and find

$$D(\lambda)\,P_\tau\,A\,\Omega = \lambda^{n_\tau}\,\varphi_\tau(h_\lambda)\,\Omega \tag{3.18}$$

with

$$h_\lambda(x) := \lambda^{-1}\,h\left(\frac{x}{\lambda}\right). \tag{3.19}$$

Hence, we formally obtain for $\lambda \downarrow 0$

$$\lambda^{-n_\tau}\,D(\lambda)\,P_\tau\,A\,\Omega \longrightarrow \int_{\mathbf{R}} dx\, h(x)\,\varphi_\tau(0)\,\Omega\,. \tag{3.20}$$

In order to obtain a Hilbert space vector in the limit, we smear over the group of translations

$$T(b) := U\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \tag{3.21}$$

with some test function f and formally get

$$\lim_{\lambda\downarrow 0} \lambda^{-n_\tau} \int_{\mathbf{R}} db\, f(b)\; T(b)\, D(\lambda)\, P_\tau\, A\,\Omega = \int_{\mathbf{R}} dx\, h(x)\,\varphi_\tau(f)\,\Omega\,. \tag{3.22}$$

We now interpret the left-hand side as a definition of a conformal field φ_τ on the vacuum and try to obtain densely defined operators with the correct localization by defining

$$\varphi_\tau^I(f)\,A'\,\Omega = A'\,\varphi_\tau^I(f)\,\Omega\,,\quad f\in\mathcal{D}(I)\,,\; A'\in\mathcal{A}(I)'\,,\; I\in\mathcal{K}\,. \tag{3.23}$$

In the following, we want to make this formal construction meaningful. There are two problems to overcome.

The first one is that the limit on the left-hand side of equation (3.22) does not exist in general if $A\,\Omega$ is replaced by an arbitrary vector in H. This corresponds to the possibility that the function h on the right-hand side might not be integrable. We will show that, after smearing the operator A with a smooth function on $SL(2,R)$, the limit is well-defined. Such operators, that are smeared-out by a $\mathcal{C}^\infty$-function with compact support in $SL(2,R)$, will be called regularized.

The second problem is to show that the smeared field operators $\varphi_\tau^I(f)$ are closable in spite of the non-local nature of the projections P_τ. This problem can be solved using the fact that the modular operators coincide, as explained above, with conformal transformations [Bor3]. An independent argument without recourse to Borchers' theorem is based on the conformal cluster theorem and will be outlined, together with its consequences, at the end of this section.

Existence of Pointlike Field Vector Limits

In order to investigate the limit in equation (3.22), we use that $P_\tau H$ can be identified with $L^2(\mathbf{R}_+, p^{2n_\tau - 1}\, dp)$, where $G = SL(2,R)$ acts according to

$$\left(U \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Phi \right)(p) \tag{3.24}$$
$$= \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi} \int_{\mathbf{R}} dx \int_{\mathbf{R}_+} dq\; e^{-ip(x+i\varepsilon)+iqg^{-1}(x+i\varepsilon)}\, (a - c(x+i\varepsilon))^{2n_\tau - 2}\, \Phi(q)$$

(cf. [KRY, GGV]).

Now let $\Phi \in P_\tau H$ be smeared-out with a test function on $SL(2,R)$ such that Φ is $\mathcal{C}^\infty$, i.e. $g \mapsto U(g)\,\Phi$ is an infinite number of times differentiable. We will show below that such functions $\Phi(\cdot)$ are continuous and bounded in p. Straightforward calculation then leads to

$$\left(\int_{\mathbf{R}} db\, f(b)\, T(b)\, D(\lambda)\, \lambda^{-n_\tau}\, \Phi \right)(p) = \tilde{f}(p)\, \Phi(\lambda p) \tag{3.25}$$

and

$$\int_{\mathbf{R}_+} dp\, p^{2n_\tau - 1}\, |\tilde{f}(p)|^2\, |\Phi(\lambda p) - \Phi(0)|^2 \longrightarrow 0 \tag{3.26}$$

for $\lambda \downarrow 0$, showing the convergence in equation (3.22).

It remains to be shown that the function $\Phi(\cdot)$ is continuous and bounded. From the above assumption it can be derived that the Hilbert space vector Φ is in the domain of definition of all powers of the conformal Hamiltonian $\mathbf{H}$. Hence, in an expansion of Φ into eigenvectors of $\mathbf{H}$,

$$\Phi = \sum_{k \geq n_\tau} c_k\, \Phi_k\,, \quad \mathbf{H}\,\Phi_k = k\,\Phi_k\,,\ \|\Phi_k\| = 1\,, \tag{3.27}$$

the sequence c_k is strongly decreasing.
Normalized eigenfunctions of $\mathbf{H}$ are of the form

$$\Phi_k^{(n)}(p) = L_{n+k-1}^{2n-1}(2p)\; e^{-p}\,,\ k > n\,, \tag{3.28}$$

with the normalized associated Laguerre polynomials L_{n+k-1}^{2n-1}. In the appendix, we show

$$\sup_p |\Phi_k^{(n)}(p)| \leq C\, k^n + D \tag{3.29}$$

with appropriate constants C and D. This directly implies continuity and boundedness of the function $\Phi(\cdot)$.

We thus obtained for each irreducible subrepresentation τ and each vector $\Phi \in P_\tau H \cap \mathcal{C}^\infty$ with the complex number $\Phi(0) \neq 0$ a multiple of a unitary map

$$V_{\tau,\Phi} \; : \; L^2(\mathbf{R}_+, p^{2n_\tau - 1} dp) \longrightarrow P_\tau H \tag{3.30}$$

which is defined on the dense set $\{\tilde{f}|_{\mathbf{R}_+} \mid f \in \mathcal{D}(\mathbf{R})\}$ by

$$V_{\tau,\Phi} \; : \; \tilde{f}|_{\mathbf{R}_+} \longmapsto \Phi(0) \; |(\tilde{f}|_{\mathbf{R}_+})> \; := \; \lim_{\lambda \downarrow 0} \lambda^{-n_\tau} \int_{\mathbf{R}} db\, f(b)\, T(b)\, D(\lambda)\, \Phi \tag{3.31}$$

and intertwines the irreducible representations of $G = SL(2, R)$. In order to verify that this construction has non-trivial results, we will show later, how a sufficient number of vectors $\Phi \in P_\tau H \cap \mathcal{C}^\infty$ with non-vanishing $\Phi(0) \neq 0$ can be generated.

Definition of Pointlike Localized Field Operators

We now turn to the definition of pointlike localized fields. Choose a regularized local observable $A \in \mathcal{A}(I_0)$, $I_0 \in \mathcal{K}_0$, such that $g \mapsto \alpha_g(A)$ is $\mathcal{C}^\infty$ in the strong operator topology. Let τ be an irreducible subrepresentation of $U(\cdot)$. Then the vector $P_\tau A \Omega$ is $\mathcal{C}^\infty$. Hence, we may define operator-valued distributions $\varphi^I_{\tau,A}$ on $\mathcal{D}(I)$, $I \in \mathcal{K}$, by

$$\varphi^I_{\tau,A}(f) \; B' \, \Omega \; := \; B' \; V_{\tau, P_\tau A\Omega} \; \tilde{f}|_{\mathbf{R}_+} \, , \tag{3.32}$$

for

$$f \in \mathcal{D}(I) \, , \; B' \in \mathcal{A}(I)' \, . \tag{3.33}$$

We point out that these field operators have a dense domain of definition.

Properties of the Pointlike Localized Fields

It is easy to see that the fields transform covariantly,

$$U(g) \; \varphi^I_{\tau,A}(f) \; U(g)^{-1} \; = \; \varphi^{gI}_{\tau,A}(f^{(n_\tau)}_g) \, , \tag{3.34}$$

where $f^{(n_\tau)}_g$ shall be defined in analogy to equation (3.15).

The main problem consists in proving closability of the operators $\varphi^I_{\tau,A}(f)$. This is equivalent to the existence of densely defined adjoint operators. Let us introduce the notation $\bar{\tau}(\cdot) := \Theta \, \tau(\cdot) \, \Theta$. Here and in the following, $\bar{f}$ shall denote the complex conjugate of f. We show that the natural candidates

$\varphi^I_{\bar\tau,A^*}(\bar f)$ are indeed restrictions of the adjoint operators. This amounts to the relation

$$(B'\Omega\,,\,\varphi^I_{\tau,A}(f)\,C'\Omega\,) = (\,\varphi^I_{\bar\tau,A^*}(\bar f)\,B'\Omega\,,\,C'\Omega\,)\,,\quad B',C'\in\mathcal{A}(I)'\,. \tag{3.35}$$

Since we have

$$\int_{\mathbf{R}} db\, f(b)\,T(b)\,D(\lambda)\,A\,D(\lambda)^*\,T(b)^* \in \mathcal{A}(I) \tag{3.36}$$

for local observables $A \in \bigcup_{I\in\mathcal{K}_0}\mathcal{A}(I)$, test functions $f \in \mathcal{D}(I)$, and sufficiently small $\lambda > 0$, it is sufficient to show that

$$(\,B'\Omega\,,\,P_\tau\,A\Omega\,) = (\,P_{\bar\tau}\,A^*\Omega\,,\,B'^*\Omega\,)\,,\quad A\in\mathcal{A}(I)\,,\; B'\in\mathcal{A}(I)'\,. \tag{3.37}$$

But this follows from the established relation between modular operators and conformal transformations,

$$\begin{aligned}(\,P_{\bar\tau}\,A^*\Omega\,,\,B'^*\Omega\,) &= (\,P_{\bar\tau}\,J_I\,\Delta_I^{1/2}\,A\,\Omega\,,\,J_I\,\Delta_I^{-1/2}\,B'\,\Omega\,)\\ &= (\,\Delta_I^{-1/2}\,B'\Omega\,,\,J_I\,P_{\bar\tau}\,J_I\,\Delta_I^{1/2}\,A\,\Omega\,)\\ &= (\,B'\Omega\,,\,P_\tau\,A\,\Omega\,)\,.\end{aligned} \tag{3.38}$$

Moreover, we find

$$\varphi^I_{\tau,A}(f)\;\Omega\in D(\Delta_I^{1/2})\,. \tag{3.39}$$

With Proposition 2.5.9 in [BrR] we can then conclude that the unique closure of $\varphi^I_{\tau,A}(f)$ is affiliated to $\mathcal{A}(I)$:

$$\varphi^I_{\tau,A}(f)^{**}\;\eta\;\mathcal{A}(I)\,. \tag{3.40}$$

That means, the von Neumann algebra generated by polar and spectral decomposition of the closed unbounded operators $\varphi^I_{\tau,A}(f)^{**}$, $f\in\mathcal{D}(I)$, is included in $\mathcal{A}(I)$.

Affiliation of the closed field operators to von Neumann algebras of local observables can also be shown with a more explicit argument (cf. [Jör1]): Using in this proof the notations $\varphi := \varphi^I_{\tau,A}(f)$ and $\bar\varphi$ for the closure of φ, it suffices to prove for all $B'\in\mathcal{A}(I)'$

$$\bar\varphi\,B' \;\supset\; B'\,\bar\varphi \tag{3.41}$$

or, equivalently, for all $B'\in\mathcal{A}(I)'$ and $\Psi\in D(\bar\varphi)$

$$B'\,\Psi\in D(\bar\varphi) \tag{3.42}$$

and

$$\bar{\varphi}\, B'\, \Psi = B'\, \bar{\varphi}\, \Psi \tag{3.43}$$

(cf. [DSW]).
Let now $\Psi \in D(\bar{\varphi})$ and $B' \in \mathcal{A}(I)'$. Let $(\varphi, D(\varphi))$ be the graph of φ, i.e. the set of pairs of image and domain vectors of φ. Since $(\varphi, D(\varphi)) \subset H \oplus H$ is by definition of $\bar{\varphi}$ dense in $(\bar{\varphi}, D(\bar{\varphi}))$, we can choose a sequence of appropriate vectors

$$\Psi_n = B'_n\, \Omega \in D(\varphi) = \mathcal{A}(I)'\, \Omega \tag{3.44}$$

such that

$$\begin{aligned} \Psi &= \lim_{n\to\infty} B'_n\, \Omega \\ &= \lim_{n\to\infty} \Psi_n \end{aligned} \tag{3.45}$$

and

$$\lim_{n\to\infty} \varphi\, \Psi_n = \bar{\varphi}\, \Psi\,. \tag{3.46}$$

With $B'\, \Psi_n \in D(\bar{\varphi})$ we get

$$\lim_{n\to\infty} B'\, \Psi_n = B'\, \Psi \tag{3.47}$$

and

$$\begin{aligned} \lim_{n\to\infty} \bar{\varphi}\, B'\, \Psi_n &= \lim_{n\to\infty} \varphi\, B'\, \Psi_n \\ &= \lim_{n\to\infty} B'\, \varphi\, \Psi_n \\ &= B'\, \bar{\varphi}\, \Psi\,. \end{aligned} \tag{3.48}$$

Since $\bar{\varphi}$ is closed, this implies

$$B'\, \Psi \in D(\bar{\varphi}) \tag{3.49}$$

and

$$\bar{\varphi}\, B'\, \Psi = B'\, \bar{\varphi}\, \Psi\,. \tag{3.50}$$

Hence, affiliation is proven.

It remains to be shown that for each irreducible subrepresentation τ there is a non-zero field $\varphi^I_{\tau,A}$ obtained by this construction. Let $g_y = \begin{pmatrix} y^{-1} & 0 \\ 1 & y \end{pmatrix} \in SL(2,\mathbf{R})$, $y \neq 0$. Using the realization of $P_\tau H$ as $L^2(\mathbf{R}_+, p^{2n-1}dp)$ we find

$$\left(P_\tau\, \alpha_{g_y}(A)\, \Omega\right)(0) = \int_0^\infty dp\, e^{ipy}\, p^{2n-1}\, (P_\tau\, A\, \Omega)\, (p)\,. \tag{3.51}$$

The left-hand side is in y the boundary value of an analytic function in the upper half plane. Therefore, it cannot vanish on an open set if $P_\tau A\Omega \neq 0$. Hence, an accidental vanishing of $P_\tau A\Omega(0)$ can be avoided by a small conformal transformation of A.

We conclude that the spaces $\varphi_\tau^I(f)\Omega$, τ irreducible and $f \in \mathcal{D}(I)$, are dense in H. But the algebra generated by polar and spectral decomposition of all $\varphi_\tau^I(f)^{**}$, τ irreducible and $f \in \mathcal{D}(I)$, is invariant under the modular automorphisms $\mathrm{Ad}\Delta_I^{it}$, hence coincides with $\mathcal{A}(I)$ (cf. [Tak2, Rig]).

This last result implies that we have full equivalence between the formulation of conformal chiral quantum field theory in terms of von Neumann algebras of local observables on the one hand and in terms of unbounded field operators with pointlike localization on the other hand. One can switch between the "algebraic picture", we have been starting from, and the "distribution picture", we have constructed in this section, without loosing information.

An Alternative Approach

Here, we sketch an alternative procedure (see [FrJ]) for the construction of pointlike localized fields. The existence and properties of conformal fields and the results on the structure of conformal Haag-Kastler nets in [FrJ, FröG, BGL] can be derived without making explicit use of Borchers' theorem [Bor3]. The basic input in this approach will be the conformal cluster theorem.

Deviating from the approach in the beginning of this section, we now use P_n instead of P_τ in the definition of the conformal fields. P_τ is the projection on the irreducible subrepresentation τ of $U(SL(2,R))$, P_n denotes the projection on the (reducible) subrepresentation of $U(SL(2,R))$ with conformal dimension n.

In analogy to equation (3.32) and with a regularized local observable $A \in \mathcal{A}(I_0)$, $I_0 \in \mathcal{K}_0$, we obtain well-defined conformal fields

$$\varphi_{n,A}^I(f)\, B'\,\Omega := B'\, V_{n,P_nA\Omega}\, \tilde{f}|_{\mathbf{R}_+} \tag{3.52}$$

for

$$f \in \mathcal{D}(I)\,,\ B' \in \mathcal{A}(I)'\,,\ I \in \mathcal{K}\,. \tag{3.53}$$

The next theorem gives the proof of the closability of the so-defined field operators.

Theorem (see [FrJ]): Let $n \in \mathbf{N}$, $I \in \mathcal{K}$, $f \in \mathcal{D}(I)$, $B', C' \in \mathcal{A}(I)'$, and let $A \in \mathcal{A}(I_0)$, $I_0 \in \mathcal{K}_0$, be a regularized local observable. We then have

$$(B'\Omega, \varphi^I_{n,A}(f)\, C'\Omega) = (\varphi^I_{n,A^*}(\bar{f})\, B'\Omega, C'\Omega). \tag{3.54}$$

Thereby, we get

$$\varphi^I_{n,A}(f)^\dagger := \varphi^I_{n,A}(f)^* \,|_{\mathcal{A}(I)'\Omega} = \varphi^I_{n,A^*}(\bar{f}). \tag{3.55}$$

Hence, $\varphi^I_{n,A}(f)$ is closable because $\varphi^I_{n,A}(f)^*$ has a dense domain.

Proof: The Casimir operator C_G associated with the representation $U(\cdot)$ of the Lie group $G = SL(2,R)$ is known to have the following spectral decomposition [Lang]:

$$C_G = \sum_{i=1}^{\infty} i(i-1)\, P_i. \tag{3.56}$$

Let B_1 and B_2 be regularized observables localized in disjoint intervals. As C_G is a second order differential operator in G, we obtain a commutation relation for B_1 and B_2:

$$(B_1\,\Omega, C_G\, B_2\,\Omega) = (C_G\, B_2^*\,\Omega, B_1^*\,\Omega). \tag{3.57}$$

This argument does not hold for a single projection P_n, since P_n is, in contrast to C_G, not a local operator.

Some algebraic transformations lead to

$$\begin{aligned}
&(B'\Omega, \varphi^I_{n,A}(f)\, C'\Omega) \\
&= \lim_{\lambda\downarrow 0} \int_{\mathbf{R}} dx\, f(x)\, (C'^*B'\Omega, T(x)\, D(\lambda)\, \lambda^{-n} P_n A\Omega) \\
&= \lim_{\lambda\downarrow 0} \int_{\mathbf{R}} dx\, f(x) \\
&\qquad (C'^*B'\Omega, T(x)\, D(\lambda)\, \lambda^{-n} \left(\prod_{i=1}^{n-1} \frac{C_G - i(i-1)}{n(n-1) - i(i-1)} \right) P_n A\Omega) \\
&= \lim_{\lambda\downarrow 0} \int_{\mathbf{R}} dx\, f(x) \\
&\qquad (C'^*B'\Omega, T(x)\, D(\lambda)\, \lambda^{-n} \left(\prod_{i=1}^{n-1} \frac{C_G - i(i-1)}{n(n-1) - i(i-1)} \right) (1 - \sum_{i=1}^{n-1} P_i)\, A\Omega)
\end{aligned}$$

(Because of equation (3.56), the polynomial in C_G has the property to act as the identity operator on P_n and as the zero operator on all $P_i, i < n$. By the conformal cluster theorem the contribution of conformal energies greater than or equal $n+1$ vanishes in the limit $\lambda \to 0$.)

$$\begin{aligned}
&= \lim_{\lambda \downarrow 0} \int_{\mathbf{R}} dx\, f(x)\, (C'^{*}B'\Omega, T(x)\, D(\lambda)\, \lambda^{-n} \left(\prod_{i=1}^{n-1} \frac{C_G - i(i-1)}{n(n-1) - i(i-1)}\right) A\Omega) \\
&= \lim_{\lambda \downarrow 0} \int_{\mathbf{R}} dx\, f(x)\, (T(x)\, D(\lambda)\, \lambda^{-n} \left(\prod_{i=1}^{n-1} \frac{C_G - i(i-1)}{n(n-1) - i(i-1)}\right) A^{*}\Omega, B'^{*}C'\Omega) \\
&= (\varphi^{I}_{n,A^{*}}(\bar{f})\, B'^{*}\Omega, C'\Omega) \qquad (3.58)
\end{aligned}$$

and the theorem is proven. □

Since in this approach arbitrary multiplicities of irreducible representations in $P_n H$ might appear, we have to ensure the existence of a sufficient number of orthogonal fields with conformal dimension $n \in \mathbf{N}$. Because of the cyclicity of the vacuum vector Ω with respect to the set of regularized local observables, appropriate operators A_i, $i \in M \subset \mathbf{N}$, can be found to construct a dense set of vectors

$$\{\varphi^{I}_{n,A_i}(f)\,\Omega \mid I \in \mathcal{K}_0,\ f \in \mathcal{D}(I),\ (A_i)_{i\in M}\} \qquad (3.59)$$

in $P_n H$. Since the conformal two-point function is determined up to a complex constant, we know

$$(\varphi_{n,A_i}(x)\,\Omega,\ \varphi_{n,A_j}(y)) = c_{ij}\,(y - x + i\varepsilon)^{-2n} \qquad (3.60)$$

with suitable $c_{ij} \in \mathbf{C}$. According to Schmidt's orthogonalization procedure, the matrix $(c_{ij})_{i,j}$ can then be transformed into a diagonal matrix $(\tilde{c}_{ij})_{i,j}$. Applying this procedure to the observables A_i, $i \in M$, we obtain new regularized local observables $\tilde{A}_i$, $i \in M$, giving rise to a set of conformal fields $(\varphi^{I}_{n,\tilde{A}_i}(\cdot))_{i\in M}$ that are orthogonal as vacuum field vectors $\varphi^{I}_{n,\tilde{A}_i}(\cdot)\,\Omega$.

The proof of the remaining properties of the field operators can be transferred immediately from the first approach, which uses P_τ in the definition of the fields, to the alternative approach, which uses P_n instead of P_τ. I.e., the field operators $\varphi^{I}_{n,A_i}(f)$ transform covariantly, their closures are affiliated to the local observable algebras and the net of local observables can be fully reconstructed from the pointlike localizable field operators (cf. [FrJ]).

Consequences of the Alternative Approach

Once having shown the existence of conformal fields, one can use it to derive important structures of the original conformal Haag-Kastler net, again and in contrast to [FröG, BGL] without making explicit use of Borchers' theorem. The mere existence of conformal fields will be enough to establish independent proofs of the Bisognano-Wichmann identification of modular structures, Haag duality, PCT covariance, and the possibility to reconstruct the algebras from the fields.

The Bisognano-Wichmann result can be derived using an idea of [BS-M]: Let $S_+ = J_+ \triangle_+^{1/2}$ be the modular operators of Tomita-Takesaki theory assigned to the vacuum vector Ω and the half line algebra $\mathcal{A}_+ := \mathcal{A}(\mathbf{R}_+)$ (cf. [Tak1] and the introduction of the modular structure in chapter 2). Let $V(\cdot)$ be the representation of the dilation group. Let $n \in \mathbf{N}$ and A be an appropriate local observable. Let $\mathbf{R}_+ \supset I \in \mathcal{K}\,,\ f \in \mathcal{D}(I)$. In order to find a candidate for the PCT operator, we define an appropriate antiunitary operator:

$$\Theta\, \varphi_{n,A}(f(\cdot))\, \Omega := (-1)^n\, \varphi_{n,A^*}(\bar{f}(-\cdot))\, \Omega\,. \tag{3.61}$$

This is a definition of an operator which commutes PCT-covariantly with $U(SL(2,R))$. Let $A \in \mathcal{A}_+$ be in the domain of $V(i\pi)$ and $\varphi(\cdot) := \varphi_{n,A}(\cdot)$. Positivity of the energy, locality, analyticity properties and dilation covariance lead to the following relation (cf. [BS-M]):

$$\begin{aligned} (V(i\pi)\, A\, \Omega, \varphi(f)\, \Omega) &= (A\, \Omega, V(i\pi)\, \varphi(f)\, \Omega) \\ &= (-1)^n\, (A\, \Omega, \varphi(f(-\cdot))\, \Omega) \\ &= (\Theta\, A^*\, \Omega, \varphi(f)\, \Omega)\,. \end{aligned} \tag{3.62}$$

This suffices to derive

$$\begin{aligned} \Theta\, V(i\pi) &= \overline{\Theta\, V(i\pi)\, |_{\mathcal{A}_+\Omega}} \\ &= \overline{S_+\, |_{\mathcal{A}_+\Omega}} \\ &= S_+ \\ &= J_+\, \triangle_+^{1/2}\,, \end{aligned} \tag{3.63}$$

where the overlined expressions denote the closures of the respective operators. Hence, we get

$$J_+ = \Theta \tag{3.64}$$

and

$$\triangle_+^{1/2} = V(i\pi)\,. \tag{3.65}$$

The proof for the negative half line needs only trivial modifications. The modular conjugation of the half lines is their reflection; the modular automorphism group is the dilation group.

We can now prove Haag duality without any a priori knowledge of PCT covariance, just exploiting the identity $J_{\mathbf{R}_+} = J_{\mathbf{R}_-}$ (cf. the procedure in [Bor3]):
Using the abbreviations $\mathcal{A}_+, \mathcal{A}_-, J_+, J_-$ as above, we get by straightforward calculations

$$\begin{aligned} \mathcal{A}'_- &= J_-\mathcal{A}_-J_- \\ &\subset J_-\mathcal{A}'_+J_- \\ &= J_-J_+\mathcal{A}_+J_+J_- \\ &= \mathcal{A}_+ \\ &\subset \mathcal{A}'_- , \end{aligned} \tag{3.66}$$

which proves

$$\mathcal{A}_+ = \mathcal{A}'_- \tag{3.67}$$

and

$$A_- = \mathcal{A}'_+ . \tag{3.68}$$

Conformal covariance and some elementary geometry then imply Haag duality.

Having proved Haag duality, we can then verify that Θ acts geometrically on the net of local observables:
Let $I \in \mathcal{K}$, $g \in SL(2,R)$ with $g(\mathbf{R}_+) = I$, and $\tilde{g} \in SL(2,R)$ such that $\tilde{g}(\mathbf{R}_-) = -I$. We already got

$$\Theta\, U(g) = U(\tilde{g})\, \Theta \tag{3.69}$$

and

$$J_\pm = \Theta . \tag{3.70}$$

This proves the theorem

$$\begin{aligned} \Theta\, \mathcal{A}(I)\, \Theta &= U(\tilde{g})\, \Theta\, \mathcal{A}(\mathbf{R}_+)\, \Theta\, U(\tilde{g})^{-1} \\ &= U(\tilde{g})\, \mathcal{A}(\mathbf{R}_-)\, U(\tilde{g})^{-1} \\ &= \mathcal{A}(-I) . \end{aligned} \tag{3.71}$$

We then easily derive the PCT covariance of the fields

$$\Theta\, \varphi_{n,A}(f)\, \Theta = (-1)^n\, \varphi_{n,A^*}(\bar{f}(-\cdot)), \tag{3.72}$$

with $I \in \mathcal{K}$, $f \in D(I)$, $A \in \mathcal{A}(I)$, and $n \in \mathbf{N}$.

Haag duality also establishes the equivalence of the "field picture" and the "algebra picture" in conformal field theory:
Haag duality expresses maximality of the local algebras. The net that can be constructed from the closures of field operators is by affiliation included in the original net of observables. Both nets fulfill Haag duality and must therefore be identical.

Remark on the Construction of Neutral Field Operators: It should be pointed out that in both approaches we have constructed pointlike localized fields with a covariantly transforming domain of definition. We have not found a dense domain of definition that is invariant and stable under the action of the field operators. Hence, we do not know how to define products of the field operators and cannot prove these pointlike localized neutral fields to fulfill the Wightman axioms.

3.3 Operator Product Expansions

The existence of a sufficient number of fields such that their linear span applied to the vacuum is dense in the Hilbert space might be an appropriate formulation of the existence of an operator product expansion in the Haag-Kastler framework. In this section, a stronger result is presented. We derive an expansion (see [FrJ]) with local coefficients, which is covariant with respect to the modular $*$-operation S. Any local observable can be expressed as a converging sum of the pointlike localizable field operators constructed above with explicitly calculable test functions.

Theorem (see [FrJ]): Let $I \in \mathcal{K}$ and $A \in \mathcal{A}(I)$. Let $\varphi_\tau^I(\cdot)$ denote the normalized conformal field associated to the subrepresentation τ of $U(SL(2,R))$. Then for each irreducible τ the two simultaneous conditions

$$P_\tau\, A\, \Omega = \varphi_\tau(f_{\tau,A})\, \Omega \tag{3.73}$$

and

$$P_{\bar{\tau}}\, A^*\, \Omega = \varphi_\tau(f_{\tau,A})^*\, \Omega \tag{3.74}$$

together fully determine the test function $f_{\tau,A}$:

$$
\begin{aligned}
& f_{\tau,A}(x) \\
= \; & \frac{1}{i^{2n_\tau - 1}} \int_{-\infty}^{x} dy_1 \int_{-\infty}^{y_1} dy_2 \cdots \int_{-\infty}^{y_{2n_\tau - 2}} dy_{2n_\tau - 1} \, (\,\Omega,\, [\,\varphi_{\bar\tau}(y_{2n_\tau - 1})^*, A\,]\,\Omega\,) \\
= \; & \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \, e^{ipx} \, \frac{\int_{-\infty}^{\infty} dy \, e^{-ipy} \, (\,\Omega,\, [\,\varphi_{\bar\tau}(y)^*, A\,]\,\Omega\,)}{p^{2n_\tau - 1}} ,
\end{aligned} \tag{3.75}
$$

where n_τ denotes the conformal dimension of τ . We used the abbreviation $\bar\tau(\cdot) := \Theta\,\tau(\cdot)\,\Theta$.
In particular, the support property of the test function is given by

$$
\mathrm{supp} f_{\tau,A} \subset I \, . \tag{3.76}
$$

Therefore, we obtain a local expansion

$$
A \; = \; \sum_\tau \varphi_\tau^I(f_{\tau,A}) \tag{3.77}
$$

which converges $*$-strongly on $\mathcal{A}(I)'\,\Omega$, i.e. with respect to the $*$-strong topology (cf. [BrR]) that is defined by the seminorms

$$
A \mapsto \|A\psi\| \, + \, \|A^*\psi\| \, , \; \psi \in \mathcal{A}(I)'\,\Omega \, . \tag{3.78}
$$

Proof: The two conditions

$$
P_\tau \, A\,\Omega = \varphi_\tau(f_{\tau,A})\,\Omega \tag{3.79}
$$

and

$$
P_{\bar\tau} \, A^*\,\Omega = \varphi_\tau(f_{\tau,A})^*\,\Omega \tag{3.80}
$$

together determine the positive and negative energy content of $f_{\tau,A}$, respectively. We find

$$
\widetilde{f_{\tau,A}}(p) \;=\; \begin{cases} P_\tau \, A\,\Omega\,(p) \, , & p > 0 \, , \\ P_{\bar\tau} \, A^*\,\Omega\,(-p) \, , & p < 0 \, . \end{cases} \tag{3.81}
$$

Let us introduce

$$g_1(x) \ := \ (\,\varphi_\tau(x)\,\Omega,\ A\,\Omega\,) \tag{3.82}$$

and

$$g_2(x) \ := \ (\,A^*\,\Omega,\ \varphi_\tau(x)^*\,\Omega\,)\,. \tag{3.83}$$

One easily finds

$$\tilde{g}_1(p) \ = \ \begin{cases} \widetilde{f_{\tau,A}}(p)\; p^{2n_\tau-1}\,, & p>0\,, \\ 0\,, \ p<0\,. \end{cases} \tag{3.84}$$

Straightforward calculation leads to

$$\begin{aligned} g_2(x) \ &= \ (\,A^*\,\Omega,\ \varphi_\tau(x)^*\,\Omega\,) \\ &= \ (\,\varphi_\tau(f_{\tau,A})^*\,\Omega,\ \varphi_\tau(x)^*\,\Omega\,) \\ &= \ (\,\varphi_{\bar\tau}(\overline{f_{\tau,A}})\,\Omega,\ \varphi_\tau(x)^*\,\Omega\,) \\ &= \ (-1)^{n_\tau}\,(\,\Theta\ \varphi_\tau(f_{\tau,A}(-\cdot))\,\Omega,\ \varphi_\tau(x)^*\,\Omega\,) \\ &= \ (-1)^{n_\tau}\,(\,\Theta\ \varphi_\tau(x)^*\,\Omega,\ \varphi_\tau(f_{\tau,A}(-\cdot))\,\Omega\,) \\ &= \ (\,\varphi_\tau(-x)\,\Omega,\ \varphi_\tau(f_{\tau,A}(-\cdot))\,\Omega\,)\,, \end{aligned} \tag{3.85}$$

where $\overline{f_{\tau,A}}$ denotes the complex conjugate of $f_{\tau,A}$. We now find

$$\tilde{g}_2(p) \ = \ \begin{cases} 0\,, \ p>0\,, \\ -\,\widetilde{f_{\tau,A}}(p)\; p^{2n_\tau-1}\,, & p<0\,. \end{cases} \tag{3.86}$$

Hence, we have proved

$$\widetilde{f_{\tau,A}}(p) \ = \ \frac{\tilde{g}_1(p)-\tilde{g}_2(p)}{p^{2n_\tau-1}}\,, \quad p\in\mathbf{R}\,. \tag{3.87}$$

The two formulas for $f_{\tau,A}$ in equation (3.75) follow directly from this result: The second expression in equation (3.75) is the Fourier transform of the right-hand side of equation (3.87). In order to derive the first expression in equation (3.75) with the multiple integral, we consider the Fourier transform of the product of $\widetilde{f_{\tau,A}}(p)$ and $p^{2n_\tau-1}$. Using equation (3.87) one obtains a differential equation for $f_{\tau,A}(x)$ which can be transformed into the integral equation given in the theorem.
The conformal cluster theorem proves the two-point functions

$$g_1(x) \ = \ (\,\varphi_\tau(x)\,\Omega,\ A\,\Omega\,) \tag{3.88}$$

and

$$g_2(x) \;=\; (\,A^*\,\Omega,\,\varphi_\tau(x)^*\,\Omega\,) \tag{3.89}$$

to decrease in position space as x^{-2n_τ} for $|x| \to \infty$. In momentum space, these distributions are then $2n_\tau - 2$ times continuously differentiable. Hence, the Fourier transform

$$G(p) \;=\; \tilde{g_1}(p) \;-\; \tilde{g_2}(p) \tag{3.90}$$

of the commutator function

$$(\,\Omega,\,[\,\varphi_\tau(x)^*, A\,]\,\Omega\,) \tag{3.91}$$

can be specified to be of the form

$$G(p) \;=\; p^{2n_\tau - 1}\,H(p) \tag{3.92}$$

with an appropriate analytic function $H(p)$.
Therefore, using the Paley-Wiener theorem ([Tre], theorem 29.2), we see that the support of $f_{\tau,A}(x) = \check{H}(x)$ is included in the support of the commutator function. Hence, it is included in I.
The local expansion in equation (3.77) then follows directly from $\mathrm{supp} f_{\tau,A} \subset I$ and the definition of the field operators (cf. (3.32)). □

Remark: Since this local expansion is proven for any local operator A, we see a posteriori that in the construction of pointlike localized fields the regularization of the local operators we have been starting from was not actually necessary. The properties of $\widetilde{f_{\tau,A}}$ suffice for any local operator A without further regularization to control the pointlike limit in the field construction.

4 From Conformal Nets to Pointlike Charged Fields

Up to this point the calculations and results of this thesis were restricted to the vacuum sector of the theory. We have been dealing so far with local observables and neutral fields without charge. We now want to consider charged fields that relate different superselection sectors of the theory. The restriction to the vacuum sector has to be lifted; the reduced field bundle formalism (cf. [FRS1, FRS2]) will show to be the appropriate framework for a theory including sectors with arbitrary charge and finite statistics.

Again starting from a chiral conformal Haag-Kastler net of local observables on two-dimensional Minkowski space-time, we construct in this chapter associated charged pointlike localized fields which intertwine between arbitrary superselection sectors with finite statistics of the theory (see [Jör3]). This amounts to a proof of the spin-statistics theorem, the PCT theorem and a generalized Bisognano-Wichmann property.

Wherever possible, we proceed in analogy to the reasoning in the vacuum case in the last chapter. Sometimes the argumentation used in the vacuum sector can easily be transferred to the charged case, sometimes new arguments have to be found.

4.1 Reduced Field Bundle Formalism

In order to be able to describe charge-carrying objects (i.e. "charged fields") intertwining between the superselection sectors with finite statistics of the theory, we consider the reduced field bundle $\mathcal{F}_{red} = (\mathcal{F}_{red}(I))_{I \in \mathcal{K}_0}$ associated with the net of observables $\mathcal{A} = (\mathcal{A}(I))_{I \in \mathcal{K}_0}$ (cf. [FRS1, FRS2] for an explicit introduction).

The reduced field bundle $\mathcal{F}_{red}$ is an algebra densely spanned by operators $F = F(e, A)$, linear in the local degree of freedom $A \in \mathcal{A}$, and with a multi-index e. This multi-index refers to the charge carried by F as well as to the source sector and the range sector between which F interpolates according to the "fusion rules" of the theory. The elements of the reduced field bundle act on H_{red}, a realization of the physical Hilbert space. H_{red} is the direct sum of copies of the vacuum Hilbert space, one for each superselection sector with finite statistics, i.e. with non-vanishing statistical phase k_ρ (cf. [FRS2]). The direct sum of the representations of the universal covering group of the conformal group on the single superselection sectors will be denoted $U(\widetilde{SL(2,R)})$.

In the following, we give an outline of the definition of the reduced field bundle and present the properties of this construction that will be needed in this thesis:

We choose an representative Doplicher-Haag-Roberts endomorphism ρ_α of $\mathcal{A}$ (cf. [DHR1-4, FRS1, FRS2]) for every superselection sector $[\alpha]$ with finite statistics and define a representation of the observable algebra $\mathcal{A}$ on a copy $H_\alpha := (\alpha, H)$ of the vacuum Hilbert space H by

$$\pi_\alpha(A)\,(\alpha,\Psi) := (\alpha, \pi_0(\rho_\alpha(A))\,\Psi) \tag{4.1}$$

for $\Psi \in H$, $A \in \mathcal{A}$, and with π_0 denoting the vacuum representation. For given representatives ρ, ρ_α and ρ_β we choose a basis $\{T_e\}$ of the space of local intertwiners from ρ_β to $\rho_\alpha\rho$. We then define the elements of the reduced field sector as

$$F(e,A)\,(\alpha,\Psi) := (\beta, \pi_0(T_e^*\rho_\alpha(A))\,\Psi) \tag{4.2}$$

for $\Psi \in H$ and $A \in \mathcal{A}$. Multi-indices e with source sector $[\alpha]$, range sector $[\beta]$, and charge $[\rho]$ will be denoted as field bundle indices of type (α,ρ,β). $[\rho]$ is the superselection sector induced by ρ. Operators $F(e,A)$ will be said to carry charge $[\rho]$. Operators of trivial charge coincide with observables.

It has been shown in [FRS2] that the algebraic relations satisfied by the elements of the reduced field bundle are the bounded operator analogue of the exchange algebra introduced in [RSc] in the context of conformal quantum field theory on the light-cone. We have

$$F(e_2,A_2)\,F(e_1,A_1) = \sum_{f_1\circ f_2} R^{e_2\circ e_1}_{f_1\circ f_2}(+/-)\,F(f_1,A_1)\,F(f_2,A_2) \tag{4.3}$$

with structure constants R, whenever $F(e_1,A_1)$ is localized in the right/left complement of the localization domain of $F(e_2,A_2)$ (cf. [FRS2]).

The superselection sectors have a well-known conjugation structure $\rho \mapsto \bar{\rho}$, which is realized by maps between intertwiner spaces. These maps give rise to numerical matrices ("coupling constants") η, θ, and ζ (cf. [FRS1, FRS2]). Related to this conjugation structure, algebraic structures of the reduced field bundle can be established. Additionally to the (ordinary) operator adjoint we introduce the linear operator reversal $F \mapsto \hat{F}$ and the antilinear charge conjugation operation $F \mapsto \bar{F}$.

The linear operator reversal is defined in [FRS2] by

$$F\widehat{(e,A)} := (d_\beta/d_\alpha)^{1/2} \sum_{\hat{e}} \theta^e_{\hat{e}} F(\hat{e}, A) \tag{4.4}$$

for $A \in \mathcal{A}$ and multi-indices e of arbitrary type (α, ρ, β). Here, the multi-index $\hat{e}$ is of the "reversed" type $(\bar{\beta}, \rho, \bar{\alpha})$ and d_α, d_β denote the respective statistical dimensions of $[\alpha]$ and $[\beta]$ (cf. [FRS2]).

The antilinear charge conjugation operation is given as

$$\bar{F} := (\hat{F})^* = (\widehat{F^*}), \quad F \in \mathcal{F}_{red}. \tag{4.5}$$

In this thesis we will often make use of a specific property of the elements of the reduced field bundle, called "weak locality":
Let F and G be two local elements of the reduced field bundle, F leading from the vacuum sector to a charged sector $[\rho]$, G leading back from $[\rho]$ to the vacuum sector. In [FRS2] it has been proved that

$$GF = (\frac{1}{k_\rho})^{(+/-)1} \hat{F}\hat{G}, \tag{4.6}$$

whenever F is localized in the left/right complement of the localization domain of G. "Weak locality" is the reminiscent of the Haag-Kastler axiom "locality" in the exchange algebra of the reduced field bundle in low-dimensional quantum field theory.

4.2 Conformal Cluster Theorem in Charged Sectors

In this section, we present the generalization of the conformal cluster theorem from the vacuum sector (cf. chapter 3) to the full theory with charged sectors and finite statistics.

Theorem (see [Jör3]): Let $(\mathcal{A}(I))_{I \in \mathcal{K}_0}$ be a conformally covariant Haag-Kastler net of local observables on $\mathbf{R}$. Let $a, b, c, d \in \mathbf{R}$ and $a < b < c < d$. Let $F \in \mathcal{F}_{red}(\,(a,b)\,)$ and $G \in \mathcal{F}_{red}(\,(c,d)\,)$ be elements of the reduced field bundle, and choose an appropriate conformal dimension $m \in \mathbf{R}_+$ such that

$$P_k\, F\, \Omega = P_k\, \bar{F}\, \Omega = 0\,, \quad k < m\,. \tag{4.7}$$

P_k here denotes the projection on the subrepresentation of $U(\widetilde{SL(2,R)})$ with conformal dimension k. $\bar{F}$ is the charge conjugated operator of F introduced above.

We then have

$$|(\Omega, G F \Omega)| \leq \left(\frac{(b-a)(d-c)}{(c-a)(d-b)}\right)^m \|F\| \, \|G\| \,. \tag{4.8}$$

Proof: (Confer the proof of the conformal cluster theorem in the vacuum sector. The argumentation has to be generalized to charged sectors, but it is possible to follow the line of reasoning known from the vacuum case.) Choose $R > 0$. Let us now consider the following one-parameter subgroup of $SL(2, \mathbf{R})$:

$$g_t \,:\, x \longmapsto \frac{x \cos\frac{t}{2} + R \sin\frac{t}{2}}{-\frac{x}{R}\sin\frac{t}{2} + \cos\frac{t}{2}} \,. \tag{4.9}$$

Its generator $\mathbf{H}_R$ is within each subrepresentation of $U(\widetilde{SL(2,R)})$ unitarily equivalent to the conformal Hamiltonian $\mathbf{H}$. Therefore, the spectrum of $F\Omega$ and $\bar{F}\Omega$ with respect to $\mathbf{H}_R$ is bounded from below by m. Let $-\pi < t_0 < t_1 < \pi$ such that $g_{t_0}(b) = c$ and $g_{t_1}(a) = d$. Because of the conformal covariance of the reduced field bundle, the function

$$M(z) := (\,\Omega,\, G\, \alpha_{g_t}(F)\, \Omega\,) \tag{4.10}$$

with

$$z = e^{it},\ -\pi < t < \pi,\ t \notin [t_0, t_1]\,, \tag{4.11}$$

is well-defined in its domain of definition. We now consider the analytical properties of

$$N(z) :=\ (z - z_0)^m \, (z^{-1} - z_0^{-1})^m \, M(z) \tag{4.12}$$

with

$$z_0 := e^{i(t_0+t_1)/2} \,. \tag{4.13}$$

Using the condition of positive energy and weak locality, $N(z)$ can be continued analytically: We find singularities at $z = 0$, $z = \infty$, and on (the copies of) the interval $[e^{it_0}, e^{it_1}]$; possibly branch-cuts (with arbitrary position) have to be introduced which connect the singularities. Hence, we obtain a Riemann surface as a natural domain of definition for the analytical continuation of $N(\cdot)$. In order to apply the maximum principle of complex analysis to a relatively compact subset of this domain of definition, we consider a compactification of the Riemann surface of $N(\cdot)$ at the singularities $z = 0$ and $z = \infty$ (see [For, Str, JSi]). In our case, this can be carried out with the

Alexandroff-compactification of $N(\cdot)$ with respect to the points at $z = 0$ and $z = \infty$. The Alexandroff-compactification [Str] is constructed such that a neighbourhood of the point to be added to the Riemann surface is given by the union of the additional point and the complement of a compact set of the original Riemann surface.

We proceed with the remark that in neighbourhoods of $z = 0$ and $z = \infty$ the function $N(\cdot)$ is bounded because of the bound on the spectrum of $\mathbf{H}_R$. Hence, $N(\cdot)$ can be continued analytically to the compactification (cf. [For]). As an analytic function on the compactified Riemann surface the continuation of $N(\cdot)$ reaches its maximum on the boundary of its domain of definition, i.e. on (the copies of) the "interval" $[e^{it_0}, e^{it_1}]$ (cf. [Str]). Therefore, we obtain the bound:

$$\begin{aligned}\sup|N(\cdot)| &\leq \|F\|\,\|G\|\,|e^{it_0} - e^{i(t_0+t_1)/2}|^{2m} \\ &= \|F\|\,\|G\|\,|(2\sin\frac{t_0 - t_1}{4}|^{2m}\,. \end{aligned} \tag{4.14}$$

This leads, as in the vacuum sector, to

$$\begin{aligned}|(\Omega, GF\Omega)| &= |M(1)| \\ &= |N(1)|\,|1 - e^{i(t_0+t_1)/2}|^{-2m} \\ &= |N(1)|\,|2\sin\frac{t_0 + t_1}{4}|^{-2m} \\ &\leq \sup|N(\cdot)|\,|2\sin\frac{t_0 + t_1}{4}|^{-2m} \\ &\leq \|F\|\,\|G\|\,\left|\frac{\sin\frac{t_0-t_1}{4}}{\sin\frac{t_0+t_1}{4}}\right|^{2m}\,. \end{aligned} \tag{4.15}$$

The completion of the argument is known from the reasoning in the vacuum sector. We determine t_0 and t_1 in the limit $R \to \infty$ in terms of a, b, c, d. One obtains

$$\lim_{R\to\infty} R\,t_0 = 2(c - b) \tag{4.16}$$

and

$$\lim_{R\to\infty} R\,t_1 = 2(d - a)\,. \tag{4.17}$$

Assuming finally $a - b = c - d$ we find

$$\left(\frac{t_0 - t_1}{t_0 + t_1}\right)^2 = \frac{(a-b)\,(c-d)}{(a-c)\,(b-d)} =: r\,. \tag{4.18}$$

We know that the bound on $|(\Omega, G F \Omega)|$ can only depend on the conformal cross ratio r. Hence, we can drop the assumption and the theorem is proven. □

4.3 Construction of Pointlike Localized Charged Fields

The following idea for the definition of pointlike localizable charged conformal fields starting from an algebraic theory of local observables (see [Jör3]) is a direct generalization of the idea for the case of neutral fields in the vacuum sector. We present the construction in detail, although our procedure and the line of argument in the full theory with charged sectors will turn out to be to a large extent parallel to the argumentation used in the chapter above for the case of the vacuum sector:

Let A be a local observable, $A \in \mathcal{A}(I_0)$, $I_0 \in \mathcal{K}_0$. Choose a localized and transportable irreducible endomorphism ρ of $\mathcal{A}$ (cf. [DHR1-4, FRS2]) inducing a charged sector with finite statistics and let e be a field bundle multi-index of type $(0, \rho, \rho)$. Then $F = F(e, A)$ is a local element of the reduced field bundle. Now let $m \in \mathbf{R}_+$ be an appropriate conformal dimension and P_m be the projection on the subspace of conformal dimension m in H_{red}. We can think of $P_m F \Omega$ as a vector of the form $\varphi_m(h)\,\Omega$ where φ_m is a conformal field with charge $[\rho]$ of scaling dimension m and h is an appropriate function on $\mathbf{R}$. As in the case of the vacuum sector, the exact relation between F and h is a priori unknown. All we have are the transformation properties under $\widetilde{SL(2,R)}$:

$$U(\tilde{g})\, P_m\, F\, \Omega = \varphi_m(h_{\tilde{g}}^{(m)})\,\Omega \tag{4.19}$$

with

$$h_{\tilde{g}}^{(m)}(x) = (cx - a)^{2m-2}\, h\left(\frac{dx - b}{-cx + a}\right) \tag{4.20}$$

and the covering projection $\tilde{g} \mapsto g$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, R)$. Scaling the vector $P_m\, F\, \Omega$ by dilations $D(\lambda)$ we find

$$D(\lambda)\, P_m\, F\, \Omega = \lambda^m\, \varphi_m(h_\lambda)\,\Omega \tag{4.21}$$

with $h_\lambda(x) = \lambda^{-1}\, h(\frac{x}{\lambda})$. Hence, we formally obtain for $\lambda \downarrow 0$

$$\lambda^{-m}\, D(\lambda)\, P_m\, F\, \Omega \longrightarrow \int_{\mathbf{R}} dx\, h(x)\, \varphi_m(0)\,\Omega\,. \tag{4.22}$$

We smear over the group of translations $T(b)$ with some test function f and obtain in the formal limit a Hilbert space vector in $H_\rho \subset H_{red}$, i.e. in the copy of the vacuum Hilbert space H in the physical Hilbert space H_{red} associated with the superselection sector $[\rho]$,

$$\lim_{\lambda \downarrow 0} \lambda^{-m} \int_{\mathbf{R}} db\, f(b)\, T(b)\, D(\lambda)\, P_m\, F\, \Omega = \int_{\mathbf{R}} dx\, h(x)\, \varphi_m(f)\, \Omega\,. \tag{4.23}$$

The left-hand side can now be interpreted as a definition of a conformal field φ_m with charge $[\rho]$ and scaling dimension m on the vacuum vector Ω. Writing down

$$\varphi_m^I(f)\, A'\, \Omega = \pi_\rho(A')\, \varphi_m^I(f)\, \Omega \tag{4.24}$$

for

$$f \in \mathcal{D}(I)\,,\ A' \in \mathcal{A}(I')\,,\ I \in \mathcal{K}_0\,, \tag{4.25}$$

we obtain operators with a domain of definition that is dense in the vacuum Hilbert space H and range in H_ρ. In this chapter, I' always denotes the complement of I with respect to $\mathcal{K}_0$.

As in the case of neutral fields in the vacuum sector, we have to solve two main problems in order to make this formal construction meaningful.

The first problem is related to the convergence of equation (4.23). If we replace the vector $F\,\Omega$ on the left-hand side of equation (4.23) by an arbitrary vector in H_{red} , the limit $\lambda \downarrow 0$ does not exist in general. Correspondingly, the formal integral over h on the right-hand side is not well-defined, if the (a priori unknown) function h is not integrable. This convergence problem is solved, in analogy to the vacuum sector, by a "regularization" of the operator F. After smearing the operator F with a smooth function with compact support on $\widetilde{SL(2,R)}$, the existence of the limit can be controlled. Such operators from the reduced field bundle, that have been integrated with $U(\tilde{g})$ and a test function on $\widetilde{SL(2,R)}$, will be called regularized in this thesis.

The second problem is to prove closability of the smeared field operators $\varphi_m^I(f)$, in spite of the non-local nature of the projections P_m appearing in the definition of the field operators. The closability of the charged conformal fields will be solved by an argument based on the properties of the Casimir operator associated with $U(\widetilde{SL(2,R)})$ and on the generalization of the conformal cluster theorem to the charged case.

Existence of Pointlike Field Vector Limits in Charged Sectors

$P_m H_\rho$ can be identified with copies of $L^2(\mathbf{R}_+, p^{2m-1}\, dp)$, where $\widetilde{SL(2,R)}$ acts according to

$$\left(U \left(\widetilde{\begin{matrix} a & b \\ c & d \end{matrix}} \right) \Phi \right) (p) \tag{4.26}$$

$$= \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi} \int_{\mathbf{R}} dx \int_{\mathbf{R}_+} dq \; e^{-ip(x+i\varepsilon)+iqg^{-1}(x+i\varepsilon)} \, (a - c(x+i\varepsilon))^{2m-2} \, \Phi(q)$$

(cf. [KRY, GGV, Lüs]).

This realization can be used to investigate the limit in equation (4.23). We will again proceed in full analogy to the construction of neutral fields in the vacuum sector:

Choose a vector $\Phi \in P_m H_\rho$ that has been smeared-out with a test function on $\widetilde{SL(2,R)}$ such that $\tilde{g} \mapsto U(\tilde{g})\,\Phi$ is $\mathcal{C}^\infty$. In the appendix, we have proved in the vacuum case that smeared-out functions $\Phi(\cdot)$ are continuous and bounded in p. This argument uses an expansion into normalized associated Laguerre polynomials and can be fully transferred to charged sectors. Having proved continuity and boundedness, a straightforward calculation leads to

$$\left(\int_{\mathbf{R}} db\, f(b)\, T(b)\, D(\lambda)\, \lambda^{-m}\, \Phi \right)(p) = \tilde{f}(p)\, \Phi(\lambda p) \tag{4.27}$$

and

$$\int_{\mathbf{R}_+} dp\, p^{2m-1}\, |\tilde{f}(p)|^2\, |\Phi(\lambda p) - \Phi(0)|^2 \longrightarrow 0 \tag{4.28}$$

for $\lambda \downarrow 0$, showing the convergence in equation (4.23). We thus obtained for each m and each $\Phi \in P_m H_\rho \cap \mathcal{C}^\infty$ with (the tensor product of complex numbers) $\Phi(0) \neq 0$ a multiple of a unitary map

$$V_{m,\Phi} \; : \; L^2(\mathbf{R}_+, p^{2m-1} dp) \longrightarrow P_m H_\rho \tag{4.29}$$

which is defined on the dense set $\{\tilde{f}|_{\mathbf{R}_+} \mid f \in \mathcal{D}(\mathbf{R})\}$ by

$$V_{m,\Phi} \; : \; \tilde{f}|_{\mathbf{R}_+} \longmapsto \Phi(0)\; |(\tilde{f}|_{\mathbf{R}_+})\rangle := \lim_{\lambda \downarrow 0} \lambda^{-m} \int_{\mathbf{R}} db\, f(b)\, T(b)\, D(\lambda)\, \Phi \tag{4.30}$$

and intertwines the irreducible representations of $\widetilde{SL(2,R)}$.

Definition of Pointlike Localized Charged Field Operators
We now come to the definition of pointlike localized charged fields. First, we define fields with charge $[\rho]$ mapping the vacuum sector into charged sectors. Then, we generalize this construction to charged fields with arbitrary superselection sectors as source sector.

Take a local observable $A \in \mathcal{A}(I_0)$, $I_0 \in \mathcal{K}_0$. Choose a localized and transportable irreducible endomorphism ρ of $\mathcal{A}$ (cf. [DHR1-4, FRS2]) inducing a charged sector with finite statistics and let e be a field bundle multi-index of type $(0, \rho, \rho)$. Then $F = F(e, A)$ is a local element of the reduced field bundle. We want F to be regularized, i.e. we choose F such that $\tilde{g} \mapsto \alpha_{\tilde{g}}(F)$ is $\mathcal{C}^\infty$ in the strong operator topology. Now let $m \in \mathbf{R}_+$ be an appropriate conformal dimension and P_m be the projection on the subspace of conformal dimension m in H_{red}.

Then the vector $P_m F \Omega$ is $\mathcal{C}^\infty$. Hence, we may define operator-valued distributions $\varphi^I_{m,F}$ on $\mathcal{D}(I)$, $I \in \mathcal{K}_0$, with a domain of definition dense in the vacuum Hilbert space by

$$\varphi^I_{m,F}(f)\; B'\,\Omega \;:=\; \pi_\rho(B')\; V_{m,P_m F\Omega}\; \tilde{f}|_{\mathbf{R}_+}\,, \tag{4.31}$$

with

$$f \in \mathcal{D}(I)\,,\; B' \in \mathcal{A}(I')\,. \tag{4.32}$$

Now, let e be a field bundle multi-index of arbitrary type (α, ρ, β). Then $F = F(e, A)$ is a local element of the reduced field bundle mapping H_α, the copy of H associated with the superselection sector $[\alpha]$, into H_β, the copy of H associated with the superselection sector $[\beta]$. We have seen above that in the reduced field bundle the following exchange algebra relations with structure constants R hold:

$$F(e_2, A_2)\, F(e_1, A_1) = \sum_{f_1 \circ f_2} R^{e_2 \circ e_1}_{f_1 \circ f_2}(+/-)\, F(f_1, A_1)\, F(f_2, A_2)\,, \tag{4.33}$$

whenever $F(e_1, A_1)$ is localized in the right/left complement of the localization domain of $F(e_2, A_2)$.

Hence, we can define operator-valued distributions $\varphi^I_{m,F}$ on $\mathcal{D}(I)$, $I \in \mathcal{K}_0$, with a domain of definition dense in the Hilbert space H_α by

$$\varphi^I_{m,F(e,A)}(f)\; F(e', B')\,\Omega \;:=\; \sum_{g' \circ g} R^{e \circ e'}_{g' \circ g}(+/-)\; F(g', B')\; \varphi^I_{m,F(g,A)}(f)\,\Omega \tag{4.34}$$

for $f \in \mathcal{D}(I)$ and field bundle elements $F(e', B') \in \mathcal{F}_{red}(I')_\pm$, whenever they are localized in the right/left complement of I with respect to $\mathcal{K}_0$. (We introduce the notation $\mathcal{F}_{red}(I')_\pm$ for the span of the elements of $\mathcal{F}_{red}(I')$ that are localized in just one of generally two complements of I with respect to $\mathcal{K}_0$.)

For reasons of simplicity, we have chosen to consider in this chapter the reduced field bundle formalism on the "punctured circle" $\mathbf{R}$ with a fixed point at ∞. As a consequence, the domain of the definition above of unbounded field operators with multi-index of arbitrary type, $\mathcal{F}_{red}(I')_\pm\,\Omega$, is possibly smaller than $\mathcal{F}_{red}(I')\,\Omega$, since $\mathcal{F}_{red}(I')$ with a non-connected localization domain I' typically need not to be spanned by elements that are localized in the right or left complement of I. One could either circumvent this complication by choosing the fixed point of the circle ("∞") at the boundary of the considered interval, i.e. one would use conformal covariance and reduce the problem to situations with $I = \mathbf{R}_+$ and $I' = \mathbf{R}_-$. Or one could consider the "universal" reduced field bundle formalism on $\mathbf{S}^1$ and its more abstract adaption to the universal covering $\mathbf{R}$ of $\mathbf{S}^1$ (cf. [FRS2]), which is essentially the original field bundle introduced by [DHR1-4].

Properties of the Pointlike Localized Charged Field Operators
First, it shall be mentioned that the charged fields transform covariantly:

$$U(\tilde{g})\;\varphi^I_{m,F}(f)\;U(\tilde{g})^{-1} \;=\; \varphi^{gI}_{m,F}(f^{(m)}_{\tilde{g}}), \tag{4.35}$$

with $m \in \mathbf{R}$, F localized in $I_0 \in \mathcal{K}_0$ and regularized, $\tilde{g} \in \tilde{G}$, with the covering projection $\tilde{g} \mapsto g$, $I, gI \in \mathcal{K}_0$, and $f \in D(I)$. $f^{(m)}_{\tilde{g}}$ shall be defined in analogy to equation (4.19).

Next, we prove the closability of the operators $\varphi^I_{m,F}(f)$. We start with the case of local operators F with a field bundle multi-index e of type $(0,\rho,\rho)$.

Theorem (see [Jör3]): Let ρ be a localized and transportable irreducible endomorphism of $\mathcal{A}$ (cf. [DHR1-4, FRS2]) inducing a charged sector with finite statistics and e a field bundle multi-index of type $(0,\rho,\rho)$. Choose $m \in \mathbf{R}_+$, $I \in \mathcal{K}_0$, $f \in \mathcal{D}(I)$, $B' \in \mathcal{A}(I')$, and $G' = F(e, C') \in \mathcal{F}_{red}(I')_\pm$. Let $F = F(e, A)$ be a regularized local element of the reduced field bundle.

With the linear operator reversal $F \mapsto \hat{F}$, the antilinear charge conjugation operation $F \mapsto \bar{F}$, and the statistical phase k_ρ we then have

$$\begin{aligned}(\,G'\,\Omega,\,\varphi^I_{m,F}(f)\,B'\,\Omega\,) \;&=\; (\frac{1}{k_\rho})^{(+/-)1}\,(\,\hat{G}'\,\varphi^I_{m,\bar{F}}(\bar{f})\,\Omega,\,B'\,\Omega\,)\\ &=\; (\,\varphi^I_{m,F^*}(\bar{f})\,G'\,\Omega,\,B'\,\Omega\,),\end{aligned} \tag{4.36}$$

whenever G' is localized in the right/left complement of I. Thereby, we conclude as in the vacuum case

$$\begin{aligned} \varphi^I_{m,F}(f)^\dagger &= \varphi^I_{m,F}(f)^*|_{\mathcal{F}_{red}(I')_\pm\Omega} \\ &= \varphi^I_{m,F^*}(\bar{f}) \,. \end{aligned} \tag{4.37}$$

Hence, $\varphi^I_{m,F}(f)$ is closable because $\varphi^I_{m,F}(f)^*$ has a dense domain of definition.

Proof: (Confer the argumentation in the vacuum sector.) The Casimir operator $C_{\tilde{G}}$ associated with the representation $U(\cdot)$ of the universal covering of the Lie group $SL(2,R)$ has the following spectral decomposition [Lang]:

$$C_{\tilde{G}} = \sum_{0<i}^{i\in\mathbf{Z}+m} i(i-1)\,P_i \,. \tag{4.38}$$

Since a conformal rotation by 2π leaves the observable algebra invariant, the conformal energies in a superselection sector can only differ by integers. The Casimir operator $C_{\tilde{G}}$ acts as a second order differential operator on $\widetilde{SL(2,R)}$. Hence, it is a local operator in contrast to the single projectors $P_i\,, i\in\mathbf{R}_+$. Some algebraic transformations lead, in a similar manner as in the vacuum sector, to

$$\begin{aligned} &(G'\Omega, \varphi^I_{m,F}(f)\,B'\Omega) \\ &= \lim_{\lambda\downarrow 0}\int_{\mathbf{R}} dx\, f(x)\,(\pi_\rho(B')^*G'\Omega, T(x)\,D(\lambda)\,\lambda^{-m}P_mF\Omega) \\ &= \lim_{\lambda\downarrow 0}\int_{\mathbf{R}} dx\, f(x) \\ &\quad (\pi_\rho(B')^*G'\Omega, T(x)\,D(\lambda)\,\lambda^{-m}\left(\prod_{0<i<m}^{i\in\mathbf{Z}+m}\frac{C_{\tilde{G}}-i(i-1)}{m(m-1)-i(i-1)}\right)P_mF\Omega) \\ &= \lim_{\lambda\downarrow 0}\int_{\mathbf{R}} dx\, f(x)\,(\pi_\rho(B')^*G'\Omega, T(x)\,D(\lambda) \\ &\quad \lambda^{-m}\left(\prod_{0<i<m}^{i\in\mathbf{Z}+m}\frac{C_{\tilde{G}}-i(i-1)}{m(m-1)-i(i-1)}\right)(1-\sum_{0<i<m}^{i\in\mathbf{Z}+m}P_i)\,F\Omega) \end{aligned}$$

(Because of equation (4.38), the polynomial in $C_{\tilde{G}}$ has the property to act as the identity operator on P_m and as the zero operator on all $P_i\,, i<m$. As a consequence of the conformal cluster theorem, the contribution of conformal

energies greater than or equal $m+1$ vanishes in the limit $\lambda \to 0$.)

$$
\begin{aligned}
&= \lim_{\lambda \downarrow 0} \int_{\mathbf{R}} dx\, f(x) \\
&\quad (\pi_\rho(B')^* G'\Omega, T(x)\, D(\lambda)\, \lambda^{-m} \left(\prod_{0<i<m}^{i\in \mathbf{Z}+m} \frac{C_{\bar{G}} - i(i-1)}{m(m-1) - i(i-1)} \right) F\Omega) \\
&= \lim_{\lambda \downarrow 0} \int_{\mathbf{R}} dx\, f(x)\, (\frac{1}{k_\rho})^{(+/-)1} \\
&\quad (T(x)\, D(\lambda)\, \lambda^{-m} \left(\prod_{0<i<m}^{i\in \mathbf{Z}+m} \frac{C_{\bar{G}} - i(i-1)}{m(m-1) - i(i-1)} \right) \bar{F}\Omega, \bar{G}' B'\Omega) \\
&= (\frac{1}{k_\rho})^{(+/-)1}\, (\hat{G}'\, \varphi^I_{m,\bar{F}}(\bar{f})\Omega, B'\Omega) \\
&= (\varphi^I_{m,F^*}(\bar{f})\, G'\Omega, B'\Omega) \qquad (4.39)
\end{aligned}
$$

with the definition of φ^I_{m,F^*} and G' localized in the right/left complement of I. □

Next, we argue that the closability theorem can be generalized to fields with a field bundle multi-index e of arbitrary type (α, ρ, β):
The reduced field bundle only considers superselection sectors with finite statistics. The definition of charged fields of arbitrary type contains a finite direct sum of orthogonal closable field bundle operators. Hence, the closability of charged field operators of arbitrary type follows by straightforward calculation from the theorem above. We have, however, not yet been able to give an explicit expression for the adjoint of field operators with field bundle indices of arbitrary type. The necessary calculation turned out to be very complicated, since in the general case of operators with a multi-index of arbitrary type we have got to take into account the full exchange algebra instead of just weak locality in the proof of the theorem above for field operators of type $(0, \rho, \rho)$.

The closures of the charged field operators are affiliated to the local von Neumann algebras of the reduced field bundle. The commutant of the von Neumann algebra of the reduced field bundle localized in $I \in \mathcal{K}_0$ is given by the algebra of observables in I' represented on the Hilbert space of the full theory H_{red}. Therefore, the proof for neutral fields can be transferred to the case of charged fields:
By the proof of proposition 2.5.9 in [BrR], we obtain for closed field operators

φ^I localized in $I \in \mathcal{K}_0$ the relation

$$\varphi^I \mathcal{F}_{red}(I)' \subseteq \mathcal{F}_{red}(I)' \varphi^I . \tag{4.40}$$

(Confer the explicit argument in the case of neutral fields in the vacuum sector.) This is equivalent to the claimed affiliation of the closed field operators to the local von Neumann algebras (cf. [BrR, DSW]):

$$\varphi^I \ \eta \ \mathcal{F}_{red}(I) . \tag{4.41}$$

We have to verify the cyclicity of the vacuum vector with respect to the constructed set of pointlike localized field operators. That means, we must check whether "field vectors" of the form $\varphi^I_{m,F}(f)\,\Omega$ span a dense subset of H_{red}. Using the analyticity argument carried out for neutral fields (cf. equation (3.51)), we see that for each $P_m\, F\,\Omega \neq 0$ a non-zero (charged) field can be constructed. Thereby, the existence of a sufficient number of charged field operators such that their linear span applied to the vacuum vector is dense in the Hilbert space H_{red} is proven. Moreover, elements F_i of the reduced field bundle with a field bundle multi-index e of arbitrary type (α, ρ, β) can be chosen such that the fields φ_{m,F_i} are non-zero and, using Schmidt's orthogonalization procedure, orthogonal as field vectors (cf. the definition in equation (4.34)).

Hence, for every irreducible subrepresentation τ of $U(\widetilde{SL(2,R)})$ a non-vanishing charged field $\varphi^I_{\tau,0}$, $I \in \mathcal{K}_0$, with a dense domain of definition in the vacuum sector can be introduced. Analogously, non-vanishing charged fields $\varphi^I_{\tau,\alpha}$, $I \in \mathcal{K}_0$, defined on an arbitrary source sector $[\alpha]$ with finite statistics can be constructed for every irreducible subrepresentation τ of $U(\widetilde{SL(2,R)})$ using the exchange algebra in the reduced field bundle formalism and definition (4.34).

The algebra generated by polar and spectral decomposition of all operators $\varphi^I_{\tau,\alpha}(f)^{**}$ for irreducible subrepresentations τ of $U(\widetilde{SL(2,R)})$, arbitrary superselection sectors $[\alpha]$ with finite statistics, and $f \in \mathcal{D}(I)$ is invariant under the generalized modular automorphisms $\mathrm{Ad}\Delta^{it}_I$ introduced above and has the vacuum Ω as a cyclic vector. Hence, it coincides with $\mathcal{F}_{red}(I)$. (Confer [Tak2] and the argument in the vacuum sector.)

Thereby, we have proved the equivalence of the formulation in terms of nets of von Neumann algebras of the reduced field bundle and in terms of unbounded charged field operators with pointlike localization. Without any loss of information one can switch between the "algebraic picture" and the "distribution picture".

Remark on the Construction of Charged Field Operators: As in the vacuum sector, we have constructed pointlike localized fields with a covariantly transforming domain of definition starting from a theory of local observables. We have not found a common domain of definition that is dense, invariant, and stable under the action of the field operators. Hence, we again do not know how to construct products of the unbounded field operators and cannot prove the pointlike localized charged fields to fulfill the Wightman axioms.

4.4 PCT, Spin & Statistics, and All That

The co-existence of the formulation of quantum field theory in terms of nets of von Neumann algebras of the reduced field bundle on the one hand and in terms of unbounded charged field operators with pointlike localization one the other hand can be used to derive important structural results of the theory. Here, we derive for all charged sectors with finite statistics of a conformally invariant theory in 1+1 dimensions a generalized Bisognano-Wichmann property, the PCT theorem, the spin-statistics theorem, and additivity of the nets (cf. [Jör3]).

We start with the proof of the spin-statistics theorem for conformally invariant quantum field theory in 1+1 dimensions. We use the line of argument of [FRS2].

Spin-Statistics Theorem (see [Jör3]): Let $[\rho]$ be an arbitrary superselection sector with finite statistics. The statistical phase k_ρ and the spectrum of chiral scaling dimensions m_ρ of conformal fields with charge $[\rho]$ then fulfill the relation

$$e^{2\pi i m_\rho} = e^{2\pi i m_{\bar\rho}} = k_\rho\,. \tag{4.42}$$

Here, $m_{\bar\rho}$ denotes a chiral scaling dimension of a (charge conjugated) field carrying the conjugated charge $[\bar\rho]$.

Remark: After the completion of this proof we received a paper by Guido and Longo [GLo] that gives an independent proof of the conformal spin-statistics theorem.

Proof: (Confer the proof of the spin-statistics theorem in [FRS2]. That proof was "on the premises that on-vacuum pointlike limits yield fields which

generate a dense subspace of H_{red} from the vacuum vector Ω. We have proved that premises in this thesis (cf. [Jör3]) and can therefore follow the line of argument in [FRS2].)
Let F be a regularized element of the reduced field bundle of type $(0,\rho,\rho)$ localized in disjoint intervals and $m \in \mathbf{R}$ such that $P_m F \Omega \neq 0$. With weak locality, we obtain

$$(\varphi_{m,F}(x)\,\Omega,\ \varphi_{m,F}(y)\,\Omega) = k_\rho^{\mathrm{sign}(y-x)}\,(\varphi_{m,\bar F}(y)\,\Omega,\ \varphi_{m,\bar F}(x)\,\Omega)\,. \tag{4.43}$$

Here, we used the antilinear charge conjugation operation $F \mapsto \bar F$, that has been introduced above.

Conformal two-point functions of pointlike localized quantum fields are explicitly known distributions. Considering an appropriate normalization, we find

$$(\varphi_{m,F}(x)\,\Omega,\ \varphi_{m,F}(y)\,\Omega) = e^{-\pi i m}\,(y-x+i\varepsilon)^{-2m} \tag{4.44}$$

and

$$(\varphi_{m,\bar F}(y)\,\Omega,\ \varphi_{m,\bar F}(x)\,\Omega) = e^{-\pi i m}\,(x-y+i\varepsilon)^{-2m}\,. \tag{4.45}$$

Hence,

$$(\varphi_{m,F}(x)\,\Omega,\ \varphi_{m,F}(y)\,\Omega) = e^{2\pi i m\,\mathrm{sign}(y-x)}\,(\varphi_{m,\bar F}(y)\,\Omega,\ \varphi_{m,\bar F}(x)\,\Omega)\,. \tag{4.46}$$

A comparison of the phases in equation (4.43) and equation (4.46) yields $e^{2\pi i m} = k_\rho$. Chiral scaling dimensions m_ρ of arbitrary fields with charge $[\rho]$ or $[\bar\rho]$ differ from m only by integers. Hence, the theorem is proven.

□

Next, we derive a generalization of the Bisognano-Wichmann result to the reduced field bundle formalism. We have seen that the Tomita-Takesaki theory [Tak1, BrR] assigns to every pair of a von Neumann algebra $\mathcal{A}$ and a cyclic and separating vector Ψ, a closable, antilinear operator:

$$S_0 : A\,\Psi \mapsto A^*\,\Psi\,, \qquad A \in \mathcal{A}\,. \tag{4.47}$$

S, the closure of S_0, has a polar decomposition $S = J\Delta^{1/2}$ and its components fulfill the following relations:

$$J = J^*\,, \tag{4.48}$$
$$J^2 = \mathbf{1}\,, \tag{4.49}$$
$$\Delta^{-1/2} = J\,\Delta^{1/2}\,J\,, \tag{4.50}$$
$$J\,\mathcal{A}\,J = \mathcal{A}'\,. \tag{4.51}$$

The set of operators $\triangle^{it}$, $t \in \mathbf{R}$, generate the group of modular automorphisms:

$$\triangle^{it} \mathcal{A} \triangle^{-it} = \mathcal{A}. \tag{4.52}$$

If Ψ is chosen to be the vacuum vector Ω and $\mathcal{A}$ an algebra of local observables in a conformally covariant Haag-Kastler net in 1+1 dimensions, [FröG, BGL] have identified J as the geometric reflection of the localization domain onto its complement on the circle and the modular automorphism group as the subgroup of conformal transformations which leaves the localization domain invariant.

On the Hilbert space H_{red} of all charged sectors with finite statistics, we consider a generalized modular structure based on the charge conjugation operation $F \mapsto \bar{F}$ instead of the operator adjoint $F \mapsto F^*$. In the following,

$$S_I = J_I \, \Delta_I^{1/2} \tag{4.53}$$

is defined as the closure of the operator defined by the mapping

$$F \, \Omega \mapsto \bar{F} \, \Omega \tag{4.54}$$

with $F \in \mathcal{F}_{red}(I)$ for $I \in \mathcal{K}_0$ (cf. [FRS2]).

In order to find a candidate for a PCT operator on H_{red}, we define an appropriate antilinear operator.

Definition:

$$\Theta \, \varphi_{m,F}(x) \, \Omega := (-1)^m \, \varphi_{m,\bar{F}}(-x) \, \Omega \,, \tag{4.55}$$

for m and $x \in \mathbf{R}$, and $F \in \mathcal{F}_{red}$ regularized.

It can easily be seen that Θ can be extended to an antiunitary operator and commutes PCT-covariantly with the representation $U(\widetilde{SL(2,R)})$. In order to prove that Θ acts geometrically on the reduced field bundle and on the charged field operators, we first derive the following generalization of the Bisognano-Wichmann result to charged sectors with finite statistics.

Bisognano-Wichmann Theorem (see [Jör3]): Let $k^{1/2}$ be the operator defined by its eigenvalues $k_\rho^{1/2}$ on the Hilbert spaces H_ρ associated with the superselection sectors $[\rho]$. Let $V(\cdot)$ be the dilation subrepresentation of $U(\cdot)$, and let Θ and S_I be defined as above. We get as a generalization of the result of Bisognano and Wichmann

$$S_{\mathbf{R}_+} = k^{1/2} \, \Theta \, V(i\pi) \tag{4.56}$$

and

$$S_{\mathbf{R}_-} = k^{-1/2}\,\Theta\,V(-i\pi)\,. \tag{4.57}$$

Proof: Our line of argument to prove this Bisognano-Wichmann theorem is based on an idea of [BS-M]. Since we consider theories with arbitrary charge and finite statistics in this thesis, weak locality has to be used instead of locality. We proceed in analogy to [FRS2]:
Let $x > 0$ and $F \in \mathcal{F}_{red}(\mathbf{R}_+)$. Without restriction of generality, we assume F to be regularized such that $V(t)\,F\,\Omega$ can be continued analytically in t. With an appropriate field

$$\varphi(\cdot) := \varphi_{m,F}(\cdot) \tag{4.58}$$

and with the antilinear charge conjugation operation for bounded and unbounded operators we then get the following relations by straightforward calculations:

$$\begin{aligned} (\,V(i\pi)\,F\,\Omega,\,\varphi(x)\,\Omega\,) &= (\,F\,\Omega,\,V(i\pi)\,\varphi(x)\,\Omega\,) \\ &= (-1)^m\,k_\rho^{1/2}\,(\,F\,\Omega,\,\varphi(-x)\,\Omega\,) \\ &= (-1)^m\,k_\rho^{-1/2}\,(\,\bar\varphi(-x)\,\Omega,\,\bar F\,\Omega\,) \\ &= k_\rho^{-1/2}\,(\,\Theta\,\varphi(x)\,\Omega,\,\bar F\,\Omega\,) \\ &= k_\rho^{-1/2}\,(\,\Theta^*\,\bar F\,\Omega,\,\varphi(x)\,\Omega\,)\,. \end{aligned} \tag{4.59}$$

We may conclude

$$V(i\pi)\,F\,\Omega \;=\; k_\rho^{1/2}\,\Theta^*\,\bar F\,\Omega\,, \quad F \in \mathcal{F}_{red}(\mathbf{R}_+)\,. \tag{4.60}$$

This proves the first relation claimed in the theorem, since the domain of definition is a core for the operator $V(i\pi)$. The relation for $\mathbf{R}_-$ can be derived by a trivial modification of the proof above. □

As a consequence of the identification of a generalized modular structure with objects of well-known geometrical meaning in the Bisognano-Wichmann theorem above, we are able to derive PCT covariance in the full theory.

PCT Theorem (see [Jör3]): Θ acts geometrically on the reduced field bundle

$$\Theta\;\mathcal{F}_{red}(\mathbf{R}_+)\;\Theta = \mathcal{F}_{red}(\mathbf{R}_-) \tag{4.61}$$

and on the charged field operators

$$\Theta\;\varphi_{m,F}(x)\;\Theta \;=\; (-1)^m\;\varphi_{m,\bar F}(-x)\,, \tag{4.62}$$

with m and $x \in \mathbf{R}$, and $F \in \mathcal{F}_{red}$ regularized.

Proof: The geometrical action of Θ on the vacuum realization of the net of local observables has already been shown [Bor3]. Confer the independent argument in the alternative approach to the construction of field operators in this thesis, first presented in [FrJ]. We now consider the full reduced field bundle. The generalized modular structure based on the antilinear charge conjugation operation $F \mapsto \bar{F}$ (cf. [FRS2]) has been identified by [Iso] as the "relative modular structure" introduced by Araki (cf. [Ara]). Then, Araki's results on relative modular operators (cf. [Ara]) directly imply the geometrical action of $\Theta = k^{-1/2} J_{\mathbf{R}_+} = k^{1/2} J_{\mathbf{R}_-}$ on the reduced field bundle:

$$\Theta \; \mathcal{F}_{red}(\mathbf{R}_+) \; \Theta = \mathcal{F}_{red}(\mathbf{R}_-) \,. \tag{4.63}$$

The geometrical action of Θ on the charged field operators then follows by straightforward calculation:

$$\Theta \; \varphi_{m,F}(x) \; \Theta \;=\; (-1)^m \; \varphi_{m,\bar{F}}(-x) \,, \tag{4.64}$$

with $m\,, x \in \mathbf{R}$ and $F \in \mathcal{F}_{red}$ regularized. Hence, the PCT theorem is proven for the full theory. □

Remark: The PCT theorem can be obtained more directly. A PCT operator can be constructed on the physical Hilbert space H_{red} in a natural manner:
We already had a PCT operator on the vacuum Hilbert space, and we know how the PCT operation should intertwine between the different copies of the vacuum Hilbert space. This so-defined PCT operator, too, can be shown to act geometrically on the reduced field bundle and on the charged field operators.

Finally, we want to point out that the conformally covariant net of von Neumann algebras $\mathcal{F}_{red}(I)\,, \; I \in \mathcal{K}_0\,,$ in the reduced field bundle formalism can be proved to be additive.

Theorem (see [Jör3]): Consider an open covering of $I \in \mathcal{K}_0$:

$$I = \bigcup_{\alpha} I_\alpha \,, \tag{4.65}$$

with $I_\alpha \in \mathcal{K}_0$. Then the following equation holds:

$$\mathcal{F}_{red}(I) = \bigvee_{\alpha} \mathcal{F}_{red}(I_\alpha) \,, \tag{4.66}$$

where $\bigvee$ denotes the generated von Neumann algebra.

Proof: Additivity of the net of algebras in the reduced field bundle formalism with arbitrary charge and finite statistics can be proved exactly as in the case of observables in the vacuum sector. Using the Bisognano-Wichmann identification of the generalized modular automorphism group with well-known geometrical objects, the argumentation used in section 2.2 for the vacuum sector can directly be transferred to the reduced field bundle net in a theory with arbitrary charge and finite statistics.

5 From Conformal Nets to Wightman Functions

Starting from a chiral conformal Haag-Kastler net of local observables, pointlike localized fields have been constructed in the chapters above. Their smeared linear combinations are affiliated to the original net and generate it. We do not know at the moment whether these fields satisfy all Wightman axioms, since we have not found an invariant domain of definition.

In this chapter, we construct in a canonical manner a complete set of pointlike localized correlation functions from the net of von Neumann algebras of local observables we have been starting from. We proceed by generalizing the conformal cluster theorem to higher n-point functions and by examining the momentum space limit of an appropriate sequence of algebraic n-point functions of local observables at $p = 0$. The so-defined set of pointlike localized canonically constructed correlation functions can be shown to fulfill the conditions for Wightman functions (cf. [StW] and [Jos]). Hence, we can construct, starting from local observables, an associated field theory with pointlike localization fulfilling the Wightman axioms.

We are not able to prove that these Wightman fields can be identified with the pointlike localized fields constructed in the chapters above [FrJ]. Neither do we know how the Haag-Kastler theory we have been starting from can be reconstructed from the Wightman theory. The Wightman fields are canonically constructed from the original Haag-Kastler net, but possibly the field operators cannot be realized in the same Hilbert space as the algebraic theory of local observables.

Such phenomena have been investigated by Borchers and Yngvason (see [BoY]). Starting from a Wightman theory, they could not rule out in general the possibility that the associated local net has to be defined in an enlarged Hilbert space.

5.1 Conformal Two-Point Functions

First, we will determine the general form of (truncated) algebraic two-point functions

$$(\Omega,\, B\,U(x)\,A\,\Omega) \tag{5.1}$$

of local observables A and B in a chiral theory. Throughout this chapter, $U(\cdot)$ denotes the representation of the translation group.

The Fourier transform F of an algebraic two-point function of a chiral local net has support on the positive half line in momentum space. Hence, it is (cf., e.g., [Jörl]) fully determined by the Fourier transform G of the associated commutator function

$$(\Omega,\, [B,\, U(x)\, A\, U(x)^{-1}]\, \Omega)\,. \tag{5.2}$$

Since A and B are local observables, the commutator function has compact support and an analytic Fourier transform $G(p)$. The restriction

$$F(p) \;=\; \Theta(p)\, G(p) \tag{5.3}$$

of this analytic function to the positive half line is then the Fourier transform of $(\Omega,\, B\, U(x)\, A\, \Omega)$.

In the conformally covariant case with

$$P_k\, A\, \Omega = P_k\, A^*\, \Omega = 0\;,\quad k < n\,,\; n \in \mathbf{N}\,, \tag{5.4}$$

the conformal cluster theorem implies that the algebraic two-point function $(\Omega,\, B\, U(x)\, A\, \Omega)$ decreases as x^{-2n} for $|x| \to \infty$:

$$|(\Omega,\, B\, U(x)\, A\, \Omega)| \leq c\, x^{-2n}\,, \tag{5.5}$$

$c > 0$ appropriate (cf. chapter 3 and [FrJ]). Therefore, its Fourier transform $F(p)$ is $2n-2$ times continuously differentiable and can be written as

$$F(p) \;=\; \Theta(p)\, p^{2n-1}\, H(p) \tag{5.6}$$

with an appropriate analytic function $H(p)$.

Using this result, we are able to present a sequence of canonically scaled two-point functions of local observables converging as distributions to the two-point function known from conventional conformal field theory (cf. [ChH, Reh]):

$$\begin{aligned}
&\lim_{\lambda \downarrow 0} \lambda^{-2n}\, (\Omega,\, B\, U(\lambda^{-1}x)\, A\, \Omega) \\
&= \lim_{\lambda \downarrow 0} \lambda^{-2n}\, \mathcal{F}_{p \to x}\, \Theta(\lambda p)\, (\lambda p)^{2n-1}\, H(\lambda p)\, \lambda\, dp \\
&= \lim_{\lambda \downarrow 0} \mathcal{F}_{p \to x}\, \Theta(p)\, p^{2n-1}\, H(\lambda p)\, dp \\
&= H(0)\, (x + i\varepsilon)^{-2n}\,.
\end{aligned} \tag{5.7}$$

The coefficient of the pointlike two-point function can explicitly be determined. Considering local observables A and B with $P_k A\Omega = P_k A^*\Omega = 0$, $k < n$, as above and using straightforward calculations known from the proof of the operator product expansions in chapter 3, we get a couple of relations expressing $H(0)$ as an explicit function of A and B:

$$\begin{aligned}
&H(0) \\
&= \mathcal{F}_{x\to p}\, \frac{1}{i^{2n-1}} \int_{-\infty}^{x} dy_1 \int_{-\infty}^{y_1} dy_2 \cdots \int_{-\infty}^{y_{2n-2}} dy_{2n-1} \\
&\qquad\qquad (\Omega,\, [B,\, U(y_{2n-1})\, A\, U(y_{2n-1})^{-1}]\,\Omega)\Big|_{p=0} \\
&= \frac{\int_{-\infty}^{\infty} dx\, e^{-ipx}\, (\Omega,\, [B,\, U(x)\, A\, U(x)^{-1}]\,\Omega)}{p^{2n-1}}\Bigg|_{p=0} \\
&= \frac{1}{(2n-1)!}\left(\frac{\partial}{\partial p}\right)^{2n-1} \int_{-\infty}^{\infty} dx\, e^{-ipx}\, (\Omega,\, [B,\, U(x)\, A\, U(x)^{-1}]\,\Omega)\Bigg|_{p=0} \\
&= \frac{1}{(2n-1)!} \int_{-\infty}^{\infty} dx\, (-ix)^{2n-1}\, e^{-ipx}\, (\Omega,\, [B, U(x)\, A\, U(x)^{-1}]\,\Omega)\Bigg|_{p=0} \\
&= \frac{1}{(2n-1)!} \int_{-\infty}^{\infty} dx\, (-ix)^{2n-1}\, (\Omega,\, [B,\, U(x)\, A\, U(x)^{-1}]\,\Omega)\,. \qquad (5.8)
\end{aligned}$$

5.2 Conformal Three-Point Functions

We consider the properties of chiral algebraic three-point functions

$$(\Omega,\, A_1\, U(x_1 - x_2)\, A_2\, U(x_2 - x_3)\, A_3\, \Omega) \qquad (5.9)$$

of local observables A_i, $i = 1, 2, 3$. The general form of a (truncated) chiral three-point function of local observables is restricted by locality and by the condition of positive energy. The Fourier transform of an algebraic three-point function can be shown to be the sum of the restrictions of analytic functions to disjoint open wedges in the domain of positive energy:
By straightforward calculations we get

$$\Theta(p)\,\Theta(q-p)\,\mathcal{F}_{\substack{x_2-x_1\to p\\ x_3-x_2\to q}}\,(\,\Omega,\ A_1\,U(x_1-x_2)\,A_2\,U(x_2-x_3)\,A_3\,\Omega\,) \tag{5.10}$$

$$= \ \Theta(p)\,\Theta(q-p)\,\mathcal{F}_{\substack{x_2-x_1\to p\\ x_3-x_2\to q}} \\ (\,\Omega,\ [\,U(x_1)^{-1}\,A_1\,U(x_1),\ [\,U(x_2)^{-1}\,A_2\,U(x_2),\ U(x_3)^{-1}\,A_3\,U(x_3)\,]\,]\,\Omega\,)$$

and

$$\Theta(q)\,\Theta(p-q)\,\mathcal{F}_{\substack{x_2-x_1\to p\\ x_3-x_2\to q}}\,(\,\Omega,\ A_1\,U(x_1-x_2)\,A_2\,U(x_2-x_3)\,A_3\,\Omega\,) \tag{5.11}$$

$$= \ \Theta(q)\,\Theta(p-q)\,\mathcal{F}_{\substack{x_2-x_1\to p\\ x_3-x_2\to q}} \\ (\,\Omega,\ [\,[\,U(x_1)^{-1}\,A_1\,U(x_1),\ U(x_2)^{-1}\,A_2\,U(x_2)\,],\ U(x_3)^{-1}\,A_3\,U(x_3)\,]\,\Omega\,)$$

on the respective wedges

$$W_+ = \{(p,q) \in \mathbf{R}^2 \,|\, p \geq 0,\ q \geq p\} \tag{5.12}$$

and

$$W_- = \{(p,q) \in \mathbf{R}^2 \,|\, q \geq 0,\ p \geq q\}\,. \tag{5.13}$$

Using the Jacoby identity, one can see that the double-commutator functions have compact support in position space. Hence, their Fourier transforms are analytic functions in momentum space. If F now denotes the Fourier transform of $(\,\Omega,\ A_1\,U(\cdot)\,A_2\,U(\cdot)\,A_3\,\Omega\,)$, we have as a first result

$$F(p,q) \ = \ \Theta(p)\,\Theta(q-p)\,G^+(p,q) \ + \ \Theta(q)\,\Theta(p-q)\,G^-(p,q) \tag{5.14}$$

with appropriate analytic functions G^+ and G^-.

In the case of conformal covariance the general form of these algebraic three-point functions is even more restricted by the following generalization of the conformal cluster theorem [FrJ]:

Theorem: Let $(\mathcal{A}(I))_{I\in\mathcal{K}_0}$ be a conformally covariant local net on $\mathbf{R}$. Let $a_i, b_i \in \mathbf{R}$, $i = 1,2,3$, and $a_1 < b_1 < a_2 < b_2 < a_3 < b_3$. Let $A_i \in \mathcal{A}(\,(a_i, b_i)\,)$, $n_i \in \mathbf{N}$, $i = 1,2,3$, and

$$P_k\,A_i\,\Omega = P_k\,A_i^*\,\Omega = 0\,,\ \ k < n_i\,. \tag{5.15}$$

P_k here denotes the projection on the subrepresentation of $U(SL(2,R))$ with conformal dimension k. We then have the following bound:

$$|(\Omega, A_1A_2A_3\Omega)| \leq \left|\frac{(a_1-b_1)+(a_2-b_2)}{(a_2-a_1)+(b_2-b_1)}\right|^{(n_1+n_2-n_3)} \left|\frac{(a_1-b_1)+(a_3-b_3)}{(a_3-a_1)+(b_3-b_1)}\right|^{(n_1+n_3-n_2)} \left|\frac{(a_2-b_2)+(a_3-b_3)}{(a_3-a_2)+(b_3-b_2)}\right|^{(n_2+n_3-n_1)} \|A_1\|\,\|A_2\|\,\|A_3\|\,. \tag{5.16}$$

If we additionally assume

$$a_1-b_1 = a_2-b_2 = a_3-b_3\,, \tag{5.17}$$

we get

$$|(\Omega, A_1A_2A_3\Omega)| \leq r_{12}^{(n_1+n_2-n_3)/2}\, r_{23}^{(n_2+n_3-n_1)/2}\, r_{13}^{(n_1+n_3-n_2)/2}\, \|A_1\|\,\|A_2\|\,\|A_3\|\,, \tag{5.18}$$

with the conformal cross ratios

$$\frac{(a_i-b_i)(a_j-b_j)}{(a_i-a_j)(b_i-b_j)} =: r_{ij}\,,\ i,j=1,2,3\,. \tag{5.19}$$

Proof: This proof follows, wherever possible, the line of argument of the proof in chapter 3 of the conformal cluster theorem for two-point functions (cf. [FrJ]).
Choose $R > 0$. Let us consider the following one-parameter subgroup of $SL(2,\mathbf{R})$:

$$g_t : x \longmapsto \frac{x\cos\frac{t}{2} + R\sin\frac{t}{2}}{-\frac{x}{R}\sin\frac{t}{2} + \cos\frac{t}{2}}\,. \tag{5.20}$$

Its generator $\mathbf{H}_R$ is within each subrepresentation of $U(SL(2,R))$ unitarily equivalent to the conformal Hamiltonian $\mathbf{H}$. Therefore, the spectrum of $A_i\,\Omega$ and $A_i^*\,\Omega$ with respect to $\mathbf{H}_R$ is bounded from below by n_i, $i=1,2,3$. Let $0 < t_{ij}^- < t_{ij}^+ < 2\pi$ such that

$$g_{t_{ij}^-}(b_i) = a_j \tag{5.21}$$

and

$$g_{t_{ij}^+}(a_i) = b_j \tag{5.22}$$

for $i,j = 1,2,3$, $i < j$. We now define

$$F(z_1,z_2,z_3) := (\Omega,\, A_{i_1}\,(\frac{z_{i_1}}{z_{i_2}})^{\mathbf{H}_R} A_{i_2}\,(\frac{z_{i_2}}{z_{i_3}})^{\mathbf{H}_R} A_{i_3}\Omega\,) \tag{5.23}$$

in a domain of definition given by

$$|z_{i_1}| < |z_{i_2}| < |z_{i_3}| \tag{5.24}$$

with permutations (i_1,i_2,i_3) of $(1,2,3)$. This definition can uniquely be extended to certain boundary values with $|z_j| = |z_k|$, $j,k = 1,2,3$, $j \neq k$: F shall be continued to this boundary of its domain of definition if

$$t_{jk} := -i\log\frac{z_j}{z_k} \notin [t_{jk}^-, t_{jk}^+] + 2\pi\mathbf{Z} \tag{5.25}$$

or equivalently if

$$g_{t_k}([a_k,b_k]) \cap g_{t_j}([a_j,b_j]) \neq \emptyset\,, \tag{5.26}$$

using the notation

$$t_i := -i\log z_i\,,\ i = 1,2,3\,. \tag{5.27}$$

Thereby, boundary points with coinciding absolute values are included in the domain of definition. The definition of F is chosen in analogy to the analytic continuation of general Wightman functions (cf., e.g., [StW, Jos]) such that the edge-of-the-wedge theorem for distributions with several variables [StW] proves F to be an analytic function:
Permuting the local observables A_i, $i = 1,2,3$, we have six three-point functions

$$(\Omega,\, A_{i_1}\, U(x_{i_1} - x_{i_2})\, A_{i_2}\, U(x_{i_2} - x_{i_3})\, A_{i_3}\Omega\,)\,. \tag{5.28}$$

These six functions have by locality identical values on a domain

$$E := \{(y_1,y_2) \in \mathbf{R}^2 \,|\, |y_1| > c_1,\ |y_2| > c_2,\ |y_1+y_2| > c_3\} \tag{5.29}$$

with appropriate $c_1,c_2,c_3 \in \mathbf{R}_+$. Each single function can be continued analytically by the condition of positive energy to one of the six disjoint subsets in

$$U := \mathbf{R}^2 + iV := \{(z_1,z_2) \in \mathbf{C}^2 \,|\, \mathrm{Im}z_1 \neq 0 \neq \mathrm{Im}z_2\,,\ \mathrm{Im}z_1 + \mathrm{Im}z_2 \neq 0\}\,. \tag{5.30}$$

In this geometrical situation, the edge-of-the-wedge theorem (cf. [StW], theorem 2.14) proves the assumed analyticity of F.
With the abbreviation

$$z_{ij}^0 := e^{i(t_{ij}^- + t_{ij}^+)/2}\,,\ i,j = 1,2,3\,, \tag{5.31}$$

we then define

$$G(z_1, z_2, z_3) \tag{5.32}$$
$$:= \; F(z_1, z_2, z_3) \prod_{(i,j,k)\in T(1,2,3)} \left(\frac{z_i}{z_j} - z_{ij}^0\right)^{(n_i+n_j-n_k)/2} \left(\frac{z_j}{z_i} - z_{ji}^0\right)^{(n_i+n_j-n_k)/2},$$

where $T(1,2,3)$ denotes the set $\{(1,2,3),\ (1,3,2),\ (2,3,1)\}$. The added polynomial in z_i, $i = 1,2,3$, is constructed such that the degree of the leading terms are restricted by the assumption on the conformal dimensions of the three-point function F. Also, using the binomial formula, it can be controlled by straightforward calculations that no half odd integer exponents appear after multiplication of the product. Hence, at $z_i = 0$ and $z_i = \infty$, $i = 1,2,3$, the function G is bounded because of the bound on the spectrum of $\mathbf{H}_R$ and can therefore be analytically continued. We can find estimates on G by the maximum principle for analytic functions. In order to get the estimate needed in this proof, we do not use the maximum principle for several complex variables [BoM]. Instead, we present an iteration of the maximum principle argument used in the proof of the conformal cluster theorem [FrJ] for the single variables z_i, $i = 1,2,3$, of $G(\cdot,\cdot,\cdot)$ and derive a bound on $G(1,1,1)$:

Applying the line of argument known from the case of the two-point functions now to $G(\cdot,1,1)$, we get the estimate

$$\begin{aligned} |G(1,1,1)| &\leq \sup_{z_1} |G(z_1,1,1)| \\ &= \sup_{z_1 \in B_{\cdot,1,1}} |G(z_1,1,1)|\,. \end{aligned} \tag{5.33}$$

The boundary of the domain of definition of the maximal analytical continuation of $G(\cdot,1,1)$ is here denoted by

$$B_{\cdot,1,1} := \{e^{it}\,|\, t \notin [t_{12}^-, t_{12}^+] \cup [t_{13}^-, t_{13}^+] + 2\pi\mathbf{Z}\}\,. \tag{5.34}$$

Applying this argument to $G(z_1,\cdot,1)$, we analogously get the estimate

$$\begin{aligned} |G(z_1,1,1)| &\leq \sup_{z_2} |G(z_1,z_2,1)| \\ &= \sup_{z_2 \in B_{z_1,\cdot,1}} |G(z_1,z_2,1)| \end{aligned} \tag{5.35}$$

with $B_{z_1,\cdot,1}$ denoting the boundary of the domain of definition of the maximal analytical continuation of $G(z_1,\cdot,1)$. Applying this argument finally to $G(z_1,z_2,\cdot)$, we analogously get the estimate

$$\begin{aligned} |G(z_1,z_2,1)| &\leq \sup_{z_3} |G(z_1,z_2,z_3)| \\ &= \sup_{z_3 \in B_{z_1,z_2,\cdot}} |G(z_1,z_2,z_3)| \end{aligned} \tag{5.36}$$

with $B_{z_1,z_2,\cdot}$ denoting the boundary of the domain of definition of the maximal analytical continuation of $G(z_1,z_2,\cdot)$. Having iterated this maximum principle argument for the single variables z_i, $i=1,2,3$, we can combine the derived estimates and get

$$|G(1,1,1)| \leq \sup_{t_{jk}=-i\log\frac{z_j}{z_k}\notin[t_{jk}^-,t_{jk}^+]+2\pi\mathbf{Z},\ j\neq k} |G(z_1,z_2,z_3)|\,. \tag{5.37}$$

Hence, the boundary values of G have to be evaluated on the domain described by

$$g_{t_k}([a_k,b_k]) \cap g_{t_j}([a_j,b_j]) \neq \emptyset \tag{5.38}$$

with $t_i=-i\log z_i$, $i=1,2,3$. We find the supremum with the same calculation as in the proof of the conformal cluster theorem above (cf. [FrJ]):

$$\begin{aligned} |G(1,1,1)| &\leq \|A_1\|\,\|A_2\|\,\|A_3\| \prod_{(i,j,k)\in T(1,2,3)} |e^{it_{ij}^-} - e^{i(t_{ij}^-+t_{ij}^+)/2}|^{n_i+n_j-n_k} \\ &= \|A_1\|\,\|A_2\|\,\|A_3\| \prod_{(i,j,k)\in T(1,2,3)} |2\sin\frac{t_{ij}^- - t_{ij}^+}{4}|^{n_i+n_j-n_k} \end{aligned} \tag{5.39}$$

This leads to another estimate:

$$\begin{aligned} |(\Omega, A_1A_2A_3\Omega)| &= |F(1,1,1)| \\ &= |G(1,1,1)| \prod_{(i,j,k)\in T(1,2,3)} |1-e^{i(t_{ij}^-+t_{ij}^+)/2}|^{n_i+n_j-n_k} \\ &= |G(1,1,1)| \prod_{(i,j,k)\in T(1,2,3)} |2\sin\frac{t_{ij}^- + t_{ij}^+}{4}|^{n_i+n_j-n_k} \\ &\leq \|A_1\|\,\|A_2\|\,\|A_3\| \prod_{(i,j,k)\in T(1,2,3)} \left|\frac{\sin\frac{t_{ij}^- - t_{ij}^+}{4}}{\sin\frac{t_{ij}^- + t_{ij}^+}{4}}\right|^{n_i+n_j-n_k} \end{aligned} \tag{5.40}$$

Determining t_{ij}^- and t_{ij}^+, we obtain for $i,j=1,2,3$

$$\lim_{R\to\infty} R\,t_{ij}^- = 2(a_j-b_i) \tag{5.41}$$

and

$$\lim_{R\to\infty} R\,t_{ij}^+ = 2(b_j-a_i) \tag{5.42}$$

and the first bound in the theorem is proven. If we now assume

$$a_1-b_1 = a_2-b_2 = a_3-b_3\,, \tag{5.43}$$

we find

$$\left(\frac{t_{ij}^- - t_{ij}^+}{t_{ij}^- + t_{ij}^+}\right)^2 = \frac{(a_i - b_i)(a_j - b_j)}{(a_i - a_j)(b_i - b_j)} = r_{ij}\,, \quad i,j = 1,2,3\,, \tag{5.44}$$

and the theorem is proven. □

This theorem can be used to get deeper insight in the form of the Fourier transforms of algebraic three-point functions. As in the case of the two-point functions, we proceed by transferring the decrease properties of the function in position space into regularity properties of the Fourier transform in momentum space.

In conventional conformal field theory, the three-point function with conformal dimensions n_i, $i = 1,2,3$, is known up to multiplicities as

$$\begin{aligned} f_{n_1 n_2 n_3}(x_1, x_2, x_3) &= (x_1 - x_2 + i\varepsilon)^{-(n_1+n_2-n_3)} \\ &\quad (x_2 - x_3 + i\varepsilon)^{-(n_2+n_3-n_1)} \\ &\quad (x_1 - x_3 + i\varepsilon)^{-(n_1+n_3-n_2)} \end{aligned} \tag{5.45}$$

(cf. [ChH, Reh]).

Its Fourier transform

$$\tilde{f}_{n_1 n_2 n_3}(p,q) =: \Theta(p)\,\Theta(q)\,Q_{n_1 n_2 n_3}(p,q) \tag{5.46}$$

can be calculated to be a sum of the restrictions of homogeneous polynomials $Q^+_{n_1 n_2 n_3}$ and $Q^-_{n_1 n_2 n_3}$ of degree $n_1 + n_2 + n_3 - 2$ to the disjoint open wedges W_+ and W_- in the domain of positive energy (cf. [Reh]). The wedges W_+ and W_- have been introduced above in the present section.

By the bound in the cluster theorem above, we know that a conformally covariant algebraic three-point function $(\Omega,\ A_1\,U(x_1-x_2)\,A_2\,U(x_2-x_3)\,A_3\Omega)$ of local observables A_i with minimal conformal dimensions n_i, $i = 1,2,3$, decreases in position space at least as fast as the associated pointlike three-point function $f_{n_1 n_2 n_3}(x_1, x_2, x_3)$ known from conventional conformal field theory. Hence, the Fourier transform $F_{A_1 A_2 A_3}(p,q)$ of this algebraic three-point function has to be at least as regular in momentum space as the Fourier transform $\tilde{f}_{n_1 n_2 n_3}(p,q)$ of the associated pointlike three-point function known from conventional conformal field theory:
Technically, we use a well-known formula from the theory of Fourier transforms,

$$\mathcal{F}(\mathrm{Pol}(X)S) = \mathrm{Pol}(\frac{\partial}{\partial Y})\mathcal{F}S\,, \tag{5.47}$$

for arbitrary temperate distributions S and polynomials $\mathrm{Pol}(\cdot)$ with a (multidimensional) variable X in position space and an appropriate associated differential operator $\frac{\partial}{\partial Y}$ in momentum space. $\mathcal{F}$ denotes the Fourier transformation from position space to momentum space.

Let now S be the conformally covariant algebraic three-point function of local observables A_i with minimal conformal dimensions n_i, $i = 1,2,3$:

$$S := (\Omega,\, A_1\, U(x_1 - x_2)\, A_2\, U(x_2 - x_3)\, A_3\, \Omega) \tag{5.48}$$

and X be a pair of two difference variables out of $x_i - x_j$, $i,j = 1,2,3$. By the cluster theorem proved above, we can now choose an appropriate homogeneous polynomial $\mathrm{Pol}(X)$ of degree $n_1 + n_2 + n_3 - 4$ such that the product $\mathrm{Pol}(X)\, S$ is still absolutely integrable in position space. Using the formula given above, we see that $\mathrm{Pol}(\frac{\partial}{\partial Y})\mathcal{F}S$ is continuous and bounded in momentum space. Furthermore, we have already derived the form of the Fourier transform F of an arbitrary (truncated) algebraic three-point function in a chiral theory to be

$$F(p,q) = \Theta(p)\,\Theta(q-p)\,G^+(p,q) + \Theta(q)\,\Theta(p-q)\,G^-(p,q) \tag{5.49}$$

with appropriate analytic functions G^+ and G^-. Thereby, we see that in the case of conformal covariance with minimal conformal dimensions n_i, $i = 1,2,3$, the analytic function G^+ (G^-) can be expressed as the product of an appropriate homogeneous polynomial P^+ (P^-) of degree $n_1 + n_2 + n_3 - 2$ restricted to the wedge W_+ (W_-) and an appropriate analytic function H^+ (H^-). Hence, we have proved that the Fourier transform $F_{A_1A_2A_3}$ of the algebraic three-point function $(\Omega,\, A_1\, U(x_1 - x_2)\, A_2\, U(x_2 - x_3)\, A_3\, \Omega)$ can be written as

$$F_{A_1A_2A_3}(p,q) = \Theta(p)\,\Theta(q)\,P_{A_1A_2A_3}(p,q)\,H_{A_1A_2A_3}(p,q) \tag{5.50}$$

with an appropriate homogeneous function $P_{A_1A_2A_3}(p,q)$ of degree $n_1 + n_2 + n_3 - 2$ and an appropriate continuous and bounded function $H_{A_1A_2A_3}(p,q)$.

These results suffice to control the pointlike limit of the considered correlation functions. Scaling an algebraic three-point function in a canonical manner, we construct a sequence of distributions that converges to the three-point function of conventional conformal field theory:

$$\lim_{\lambda\downarrow 0} \lambda^{-(n_1+n_2+n_3)} \, (\Omega,\, A_1\, U(\frac{x_1-x_2}{\lambda})\, A_2\, U(\frac{x_2-x_3}{\lambda})\, A_3\, \Omega\,)$$

$$= \lim_{\lambda\downarrow 0} \lambda^{-(n_1+n_2+n_3)} \, \mathcal{F}_{\substack{p\to x_1-x_2\\ q\to x_2-x_3}} \, F_{A_1A_2A_3}(\lambda p, \lambda q)\, \lambda^2\, dp\, dq$$

$$= \lim_{\lambda\downarrow 0} \lambda^{-(n_1+n_2+n_3)}$$

$$\mathcal{F}_{\substack{p\to x_1-x_2\\ q\to x_2-x_3}} \, \Theta(p)\, \Theta(q)\, \lambda^{n_1+n_2+n_3-2}\, P_{A_1A_2A_3}(p,q)\, H_{A_1A_2A_3}(\lambda p, \lambda q)\, \lambda^2\, dp\, dq$$

$$= (x_1-x_2+i\varepsilon)^{-(n_1+n_2-n_3)} \, (x_2-x_3+i\varepsilon)^{-(n_2+n_3-n_1)} \, (x_1-x_3+i\varepsilon)^{-(n_1+n_3-n_2)}\, H_{A_1A_2A_3}(0,0)\,. \qquad (5.51)$$

The coefficient $H_{A_1A_2A_3}(0,0)$ in the pointlike localized three-point function can explicitly be determined, as well:
We consider local observables A_i with $P_k\, A_i\, \Omega = P_k\, A_i^*\, \Omega = 0\,,\ k < n_i\,,\ i = 1,2,3\,,$ as above. We introduce the abbreviations

$$[A_1,[A_2,A_3]](x_1,x_2,x_3) := (\,\Omega,\, [\,U(x_1)^{-1}\, A_1\, U(x_1),\, [\,U(x_2)^{-1}\, A_2\, U(x_2),\, U(x_3)^{-1}\, A_3\, U(x_3)\,]\,]\,\Omega\,) \qquad (5.52)$$

and

$$[[A_1,A_2],A_3](x_1,x_2,x_3) := (\,\Omega,\, [\,[\,U(x_1)^{-1}\, A_1\, U(x_1),\, U(x_2)^{-1}\, A_2\, U(x_2)\,],\, U(x_3)^{-1}\, A_3\, U(x_3)\,]\,\Omega\,)\,. \qquad (5.53)$$

Straightforward calculations then lead to relations that express the free coefficient $H_{A_1A_2A_3}(0,0)$ as explicit functions of $A_i\,,\ i = 1,2,3$:

$$H_{A_1A_2A_3}(0,0) = \left.\frac{G^+_{A_1A_2A_3}(p,q)}{P^+_{A_1A_2A_3}(p,q)}\right|_{\substack{p=0\\ q=0}} \qquad (5.54)$$

$$= \left.\frac{\mathcal{F}_{\substack{x_2-x_1\to p\\ x_3-x_2\to q}}\, [A_1,[A_2,A_3]](x_1,x_2,x_3)}{Q^+_{n_1n_2n_3}(p,q)}\right|_{\substack{p=0\\ q=0}} \qquad (5.55)$$

or equivalently

$$H_{A_1A_2A_3}(0,0) = \left.\frac{G^-_{A_1A_2A_3}(p,q)}{P^-_{A_1A_2A_3}(p,q)}\right|_{\substack{p=0\\ q=0}} \qquad (5.56)$$

$$= \left. \frac{\mathcal{F}_{\substack{x_2-x_1\to p \\ x_3-x_2\to q}} [[A_1,A_2],A_3](x_1,x_2,x_3)}{Q^-_{n_1n_2n_3}(p,q)} \right|_{\substack{p=0\\q=0}} . \tag{5.57}$$

$Q^+_{n_1n_2n_3}$ and $Q^-_{n_1n_2n_3}$ have been defined above as the two polynomials appearing in the Fourier transform of the three-point function in conventional conformal field theory.

These relations can easily be transformed by standard analysis into similar integral and differential equations. Since this has been presented in detail in the case of two-point functions, we do not repeat it here.

5.3 Conformal n-Point Functions

Since the notational expenditure increases strongly as we come to the construction of higher n-point functions, we concentrate on qualitatively new aspects not occurring in the case of two-point functions and three-point functions. These qualitatively new aspects in the construction of higher n-point functions are related to the fact that in conventional field theory the form of higher n-point functions is not fully determined by conformal covariance. In conventional conformal field theory conformal covariance restricts the form of correlation functions of field operators $\varphi_i(x_i)$, $i = 1, 2, \dots, n$, with conformal dimension n_i in the following manner (cf. [ChH, Reh]):

$$(\Omega, \left(\prod_{1\le i\le n} \varphi_i(x_i) \right) \Omega) = \left(\prod_{1\le i<j\le n} \frac{1}{(x_j - x_i + i\varepsilon)^{c_{ij}}} \right) f(r^{v_1 s_1}_{t_1 u_1}, \dots, r^{v_{n-3} s_{n-3}}_{t_{n-3} u_{n-3}}) . \tag{5.58}$$

Here, $f(\cdot, \dots, \cdot)$ denotes an appropriate function depending on n–3 algebraicly independent conformal cross ratios

$$r^{vs}_{tu} := \frac{(x_v - x_s)\,(x_t - x_u)}{(x_v - x_t)\,(x_s - x_u)} . \tag{5.59}$$

The exponents c_{ij} must fulfill the consistency conditions

$$\sum_{\substack{j=1\\ j\ne i}}^{n} c_{ij} = 2n_i\, ,\ c_{ij} = c_{ji}\, ,\ 1 \le i \le n . \tag{5.60}$$

These conditions do not fully determine the exponents c_{ij} in the case of $n \geq 4$. Hence, in conventional conformal field theory four-point functions and higher n-point functions are not fully determined by conformal covariance.

In the case of conformal two-point functions and conformal three-point functions, our strategy to construct pointlike localized correlation functions was the following: First, we proved that the algebraic correlation functions decrease in position space as fast as the associated correlation functions in conventional field theory, which are uniquely determined by conformal covariance. Then, we transferred this property by Fourier transformation into regularity properties in momentum space. Finally, we were able to prove that the limit $\lambda \downarrow 0$ of canonically scaled algebraic correlation functions converges to (a multiple of) the associated pointlike localized correlation functions in conventional conformal field theory.

In the case of four-point functions and higher n-point functions, the situation has changed and we cannot expect to be able to fully determine the form of the pointlike localized limit in this construction, since for $n > 4$ the correlation functions in conventional conventional field theory are not any longer uniquely determined by conformal covariance.

Beginning with the discussion of the general case with $n \geq 4$, we consider algebraic n-point functions

$$(\Omega, \left(\prod_{1 \leq i \leq n} U(-x_i)\, A_i\, U(x_i)\right) \Omega) \tag{5.61}$$

of local observables A_i with minimal conformal dimensions n_i, $i = 1, 2, \ldots, n$, in a chiral theory with conformal covariance. We want to examine the pointlike limit of canonically scaled correlation functions

$$\lim_{\lambda \downarrow 0} \lambda^{-\left(\sum_{1 \leq i \leq n} n_i\right)} (\Omega, \left(\prod_{1 \leq i \leq n} U(-\frac{x_i}{\lambda})\, A_i\, U(\frac{x_i}{\lambda})\right) \Omega). \tag{5.62}$$

Our procedure in the construction of pointlike localized n-point functions for $n \geq 4$ will be the following: We consider all possibilities to form a set of exponents c_{ij} fulfilling the consistency conditions

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} c_{ij} = 2n_i\,, \; c_{ij} = c_{ji}\,, \; i = 1, 2, 3, \ldots, n\,. \tag{5.63}$$

For each consistent set of exponents a bound on algebraic n-point functions in position space can be proved. Each single bound on algebraic n-point functions in position space can be transferred into a regularity property of algebraic n-point functions in momentum space. We can use the same techniques as in the case of three-point functions. Finally, we will control the canonical scaling limit in (5.62) and construct pointlike localized conformal n-point functions.

We present the following generalization of the conformal cluster theorem proved above (cf. [FrJ]) to algebraic n-point functions of local observables:

Theorem: Let $(\mathcal{A}(I))_{I\in\mathcal{K}_0}$ be a conformally covariant local net on $\mathbf{R}$. Let $a_i, b_i \in \mathbf{R}$, $i = 1, 2, 3, \ldots, n$, and $a_i < b_i < a_{i+1} < b_{i+1}$ for $i = 1, 2, 3, \ldots, n-1$. Let $A_i \in \mathcal{A}((a_i, b_i))$, $n_i \in \mathbf{N}$, and

$$P_k\, A_i\, \Omega = P_k\, A_i^*\, \Omega = 0\,,\ k < n_i\,,\ i = 1, 2, 3, \ldots, n\,. \tag{5.64}$$

P_k here denotes the projection on the subrepresentation of $U(SL(2, R))$ with conformal dimension k. We then have for each set of exponents c_{ij} fulfilling the consistency conditions

$$\sum_{\substack{j=1\\ j\neq i}}^{n} c_{ij} = 2n_i\,,\ c_{ij} = c_{ji}\,,\ i = 1, 2, 3, \ldots, n\,, \tag{5.65}$$

the following bound:

$$\begin{aligned} &|(\,\Omega, \left(\prod_{1\leq i\leq n} A_i\right) \Omega\,)| \\ &\leq \left(\prod_{1\leq i<j\leq n} \left|\frac{(a_i - b_i) + (a_j - b_j)}{(a_j - a_i) + (b_j - b_i)}\right|^{c_{ij}}\right) \prod_{1\leq i\leq n} \|A_i\|\,. \end{aligned} \tag{5.66}$$

If we additionally assume

$$a_1 - b_1 = a_2 - b_2 = \ldots = a_n - b_n\,, \tag{5.67}$$

we can introduce conformal cross ratios and get

$$\begin{aligned} &|(\,\Omega, \left(\prod_{1\leq i\leq n} A_i\right) \Omega\,)| \\ &\leq \left(\prod_{1\leq i<j\leq n} \left(\frac{(a_i - b_i)\,(a_j - b_j)}{(a_i - a_j)\,(b_i - b_j)}\right)^{c_{ij}/2}\right) \prod_{1\leq i\leq n} \|A_i\|\,. \end{aligned} \tag{5.68}$$

Proof: If we pay attention to the obvious modifications needed for the additional variables, we can use in this proof the assumptions, the notation, and the line of argument introduced in the proof of the cluster theorem in the case of three-point functions.
We choose an arbitrary set of exponents c_{ij} fulfilling the consistency conditions

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} c_{ij} = 2n_i \, , \;\; c_{ij} = c_{ji} \, , \;\; i = 1,2,3,\dots,n \, . \tag{5.69}$$

Let $R > 0$. We consider the generator $\mathbf{H}_R$ of the following one-parameter subgroup of $SL(2,\mathbf{R})$:

$$g_t \; : \; x \longmapsto \frac{x\cos\frac{t}{2} + R\sin\frac{t}{2}}{-\frac{x}{R}\sin\frac{t}{2} + \cos\frac{t}{2}} \, . \tag{5.70}$$

We know that $\mathbf{H}_R$ is within each subrepresentation of $U(SL(2,R))$ unitarily equivalent to the conformal Hamiltonian $\mathbf{H}$. Therefore, the spectrum of $A_i\,\Omega$ and $A_i^*\,\Omega$ with respect to $\mathbf{H}_R$ is bounded from below by n_i, $i = 1,2,\dots,n$. Let $0 < t_{ij}^- < t_{ij}^+ < 2\pi$ such that

$$g_{t_{ij}^-}(b_i) = a_j \tag{5.71}$$

and

$$g_{t_{ij}^+}(a_i) = b_j \, , \tag{5.72}$$

for $i,j = 1,2,\dots,n$, $i < j$. We introduce

$$F(z_1,\dots,z_n) \;:=\; (\,\Omega, \left(\prod_{i=1}^{n} z_{p(i)}^{-\mathbf{H}_R} A_{p(i)}\, z_{p(i)}^{\mathbf{H}_R}\right) \Omega\,) \tag{5.73}$$

in a domain of definition given by

$$|z_{p(1)}| < |z_{p(2)}| < \dots < |z_{p(n)}| \tag{5.74}$$

with permutations $(\,p(1),p(2),\dots,p(n)\,)$ of $(\,1,2,\dots,n\,)$. This definition can uniquely be extended in analogy to the case of three-point functions to boundary points with $|z_j| = |z_k|$, $j,k = 1,2,\dots,n$, $j \neq k$, if

$$g_{t_k}([a_k,b_k]) \cap g_{t_j}([a_j,b_j]) \neq \emptyset \, , \tag{5.75}$$

thereby introducing

$$t_i := -i \log z_i \,, \; i = 1, 2, \dots, n \,. \tag{5.76}$$

The line of argument presented above in the case of three-point functions and developed for general Wightman functions in [StW, Jos] proves that this continuation is still an analytic function. We then define

$$G(z_1, \dots, z_n) \; := \; F(z_1, \dots, z_n) \prod_{1 \leq i < j \leq n} \left(\frac{z_i}{z_j} - z^0_{ij}\right)^{c_{ij}/2} \left(\frac{z_j}{z_i} - z^0_{ji}\right)^{c_{ji}/2}, \tag{5.77}$$

using the abbreviation

$$z^0_{ij} := e^{i(t^-_{ij} + t^+_{ij})/2} \,, \; i, j = 1, 2, \dots, n \,. \tag{5.78}$$

This function is constructed such that with the consistency conditions for c_{ij} and with the bound on the spectrum of $\mathbf{H}_R$ we get the following result in analogy to the cluster theorem for three-point functions: At the boundary points $z_i = 0$ and $z_i = \infty$, $i = 1, 2, \dots, n$, the function G is bounded and can therefore be analytically continued. As in the case of three-point functions, we get with the maximum principle for analytic functions further estimates on G: Iterating the well-known maximum principle argument for the single variables, one obtains

$$|G(1, \dots, 1)| \; \leq \; \sup_B |G(z_1, \dots, z_n)| \,, \tag{5.79}$$

where B denotes the set of boundary points

$$B \; := \; \{ \, |z_j| = |z_k| \mid g_{t_k}([a_k, b_k]) \cap g_{t_j}([a_j, b_j]) \neq \emptyset \,, \; j \neq k \} \tag{5.80}$$

with $t_i = -i \log z_i$, $i = 1, 2, \dots, n$. The supremum of the boundary values of G can be calculated in full analogy to the case of the three-point functions and to the proof of the conformal cluster theorem (cf. [FrJ]). We obtain straightforward:

$$|(\Omega, \left(\prod_{1 \leq i \leq n} A_i \right) \Omega)| \; \leq \; \left(\prod_{1 \leq i \leq n} \|A_i\| \right) \prod_{1 \leq i < j \leq n} \left| \frac{\sin \frac{t^-_{ij} - t^+_{ij}}{4}}{\sin \frac{t^-_{ij} + t^+_{ij}}{4}} \right|^{c_{ij}} . \tag{5.81}$$

This estimate converges in the limit $R \downarrow 0$ with

$$\lim_{R \to \infty} R t^-_{ij} = 2(a_j - b_i) \tag{5.82}$$

and

$$\lim_{R\to\infty} R\,t_{ij}^{+} = 2(b_j - a_i) \tag{5.83}$$

for $i, j = 1, 2, \dots, n$ to the first bound asserted in the theorem. If we assume

$$a_1 - b_1 = a_2 - b_2 = \dots = a_n - b_n\,, \tag{5.84}$$

we find

$$\left(\frac{t_{ij}^{-} - t_{ij}^{+}}{t_{ij}^{-} + t_{ij}^{+}}\right)^2 = \frac{(a_i - b_i)\,(a_j - b_j)}{(a_i - a_j)\,(b_i - b_j)} = r_{ij}\,, \quad i, j = 1, 2, \dots, n\,, \tag{5.85}$$

and get the second bound. Hence, the theorem is proven. □

For each consistent set of exponents c_{ij} , $i, j = 1, 2, 3, \dots, n$, we have proved a different bound on conformal four-point functions of chiral local observables. Hence, we know that the algebraic n-point function

$$(\,\Omega, \left(\prod_{1\leq i\leq n} U(-x_i)\,A_i\,U(x_i)\right)\,\Omega\,) \tag{5.86}$$

decreases in position space at least as fast as the set of associated pointlike n-point functions known from conventional conformal field theory. Therefore, the Fourier transform of the algebraic n-point function has to be at least as regular in momentum space as the Fourier transforms of the associated pointlike n-point functions known from conventional conformal field theory.

Technically, we follow the line of argument in the case of three-point functions and use the formula

$$\mathcal{F}(\mathrm{Pol}(X)S) = \mathrm{Pol}\left(\frac{\partial}{\partial Y}\right)\mathcal{F}S \tag{5.87}$$

for arbitrary temperate distributions S and polynomials $\mathrm{Pol}(\cdot)$ with a (multi-dimensional) variable X in position space and an appropriate associated differential operator $\frac{\partial}{\partial Y}$ in momentum space. $\mathcal{F}$ denotes the Fourier transformation from position space to momentum space.

Now, we choose S to be an algebraic n-point function

$$(\,\Omega, \left(\prod_{1\leq i\leq n} U(-x_i)\,A_i\,U(x_i)\right)\,\Omega\,) \tag{5.88}$$

of local observables A_i with minimal conformal dimensions n_i , $i = 1, 2, \dots, n$, and X to be a tuple of $n-1$ algebraicly independent difference variables out of $x_i - x_j$, $i, j = 1, 2, \dots, n$. The estimates in the cluster theorem proved above imply, that appropriate homogeneous polynomials $\mathrm{Pol}(X)$ of degree

$$\deg(\mathrm{Pol}) = \left(\sum_{i=1}^{n} n_i\right) - 2n + 2 \tag{5.89}$$

can be found such that the product $\mathrm{Pol}(X)\, S$ is still absolutely integrable in position space. We then see that $\mathrm{Pol}(\frac{\partial}{\partial Y})\mathcal{F}S$ is continuous and bounded in momentum space. By locality and the condition of positive energy, the Fourier transform F of an arbitrary (truncated) algebraic n-point function is known to be of the form

$$F(p_1, \dots, p_{n-1}) = G(p_1, \dots, p_{n-1}) \prod_{i=1}^{n-1} \Theta(p_i)\,, \tag{5.90}$$

where G denotes a sum of restrictions of appropriate analytic functions to subsets of momentum space (cf. the case of three-point functions in the section above). One can now proceed in analogy to the argumentation in the case of three-point functions: In a situation with conformal covariance and minimal conformal dimensions n_i , $i = 1, 2, \dots, n$, the function G can be expressed as the product of an appropriate homogeneous polynomial P of degree

$$\deg(P) = \left(\sum_{i=1}^{n} n_i\right) - n + 1 \tag{5.91}$$

and an appropriate function H , where H denotes another sum of restrictions of analytic functions to subsets of momentum space. Hence, we have proved that the Fourier transform of the algebraic n-point function

$$(\Omega, \left(\prod_{1 \leq i \leq n} U(-x_i)\, A_i\, U(x_i)\right) \Omega) \tag{5.92}$$

can be written as

$$F(p_1, \dots, p_{n-1}) = P(p_1, \dots, p_{n-1})\; H(p_1, \dots, p_{n-1}) \prod_{i=1}^{n-1} \Theta(p_i) \tag{5.93}$$

with an appropriate homogeneous function P of degree

$$\deg(P) = \left(\sum_{i=1}^{n} n_i\right) - n + 1 \tag{5.94}$$

and an appropriate continuous and bounded function H.

Using this result, we can now show in full analogy to the procedure in the last section that by canonically scaling an algebraic n-point function we construct a sequence of distributions that converges to an appropriate pointlike localized n-point function of conventional conformal field theory:

$$\begin{aligned}
&\lim_{\lambda\downarrow 0}\lambda^{-\left(\sum_{1\le i\le n} n_i\right)}\,(\Omega,\left(\prod_{1\le i\le n} U(-\frac{x_i}{\lambda})\,A_i\,U(\frac{x_i}{\lambda})\right)\Omega)\\
&= \lim_{\lambda\downarrow 0}\lambda^{-\left(\sum_{1\le i\le n} n_i\right)}\,\mathcal{F}_{p_i\to x_i-x_i+1}\,F(\lambda p_1,\dots,\lambda p_{n-1})\,\lambda^{n-1}\prod_{1\le i\le n} dp_i\\
&= \lim_{\lambda\downarrow 0}\,\mathcal{F}_{p_i\to x_i-x_i+1}\,P(p_1,\dots,p_{n-1})\,H(\lambda p_1,\dots,\lambda p_{n-1})\prod_{1\le i\le n}\Theta(p_i)\,dp_i\\
&= \left(\prod_{1\le i<j\le n}\frac{1}{(x_j-x_i+i\varepsilon)^{c_{ij}}}\right) f(r^{v_1 s_1}_{t_1 u_1},\dots,r^{v_{n-3}s_{n-3}}_{t_{n-3}u_{n-3}})\,.
\end{aligned}\tag{5.95}$$

Again, $f(\cdot,\dots,\cdot)$ denotes an appropriate function depending on $n-3$ algebraicly independent conformal cross ratios

$$r^{vs}_{tu} := \frac{(x_v-x_s)\,(x_t-x_u)}{(x_v-x_t)\,(x_s-x_u)}\,.\tag{5.96}$$

The exponents c_{ij} must fulfill the consistency conditions

$$\sum_{\substack{j=1\\ j\neq i}}^{n} c_{ij} = 2n_i\,,\ c_{ij}=c_{ji}\,,\ 1\le i\le n\,,\tag{5.97}$$

which do not fully determine the exponents. Hence, the general form of the pointlike localized conformal correlation functions constructed from algebraic quantum field theory has been determined to be exactly the general form of the n-point functions known from conventional conformal field theory. In both approaches conformal covariance does not fully determine the form of n-point functions for $n>4$.

5.4 Wightman Axioms and Reconstruction Theorem

The most common axiomatic system for pointlike localized quantum fields is the formulation of Wightman axioms given in [StW] and [Jos]. (If braid group statistics has to be considered and the Bose-Fermi alternative does not hold in general, the classical formulation of [StW] and [Jos] has to be modified for the charged case by introducing the axiom of weak locality instead of locality [FRS1, FRS2].)

The construction of pointlike localized correlation functions in the last sections uses sequences of algebraic correlation functions of local observables. The algebraic correlation functions obviously fulfill positive definiteness, conformal covariance, locality, and the spectrum condition. Hence, if the sequences converge, the set of pointlike limits of algebraic correlation functions fulfills the Wightman axioms (see [StW]) by construction. By the reconstruction theorem in [StW] and [Jos], the existence of Wightman fields associated with the Wightman functions is guaranteed and this Wightman field theory is unique up to unitary equivalence.

We do not know at the moment whether the Wightman fields can be identified with the pointlike localized field operators constructed in chapter 3 (cf. [FrJ]) from the Haag-Kastler theory. We do not know either whether the Wightman fields are affiliated to the associated von Neumann algebras of local observables and how the Haag-Kastler net we have been starting from can be reconstructed from the Wightman fields. Possibly, the Wightman fields cannot even be realized in the same Hilbert space as the Haag-Kastler net of local observables.

We do know, however, that the Wightman theory associated with the Haag-Kastler theory is non-trivial: The two-point functions of this Wightman fields are, by construction, identical with the two-point functions of the pointlike localized field operators constructed in chapter 3 (cf. [FrJ]). And we have already proved that those pointlike field vectors can be chosen to be non-vanishing and that the vacuum vector is cyclic for a set of all field operators localized in an arbitrary interval.

It shall be pointed out again that those pointlike fields constructed in the chapters above could not be proved to fulfill the Wightman axioms, since we were not able to find a domain of definition that is stable under the action of the field operators.

To summarize this chapter, we state that starting from a chiral conformal Haag-Kastler theory we have found a canonical construction of non-trivial Wightman fields. The reconstruction of the original net of von Neumann algebras of local observables from the Wightman fields could not explicitly be presented, since we do not know whether the Wightman fields can be realized in the same Hilbert space as the Haag-Kastler net.

Actually, Borchers and Yngvason [BoY] have investigated similar situations and have shown that such problems can occur in quantum field theory. In [BoY] the question is discussed under which conditions a Haag-Kastler net can be associated with a Wightman theory. The condition for the locality of the associated algebra net turned out be a property of the Wightman fields called "central positivity". Central positivity is fulfilled for Haag-Kastler nets and is stable under pointlike limits [BoY]. Hence, the Wightman fields constructed in this thesis fulfill central positivity. The possibility, however, that the local net has to be defined in an enlarged Hilbert space could not be ruled out in general by [BoY].

Furthermore, it has been proved in [BoY] that Wightman fields fulfilling generalized H-bounds (cf. [DSW]) have associated local nets of von Neumann algebras that can be defined in the same Hilbert space. The closures of the Wightman field operators are then affiliated to the associated local algebras. We could not prove generalized H-bounds for the Wightman fields constructed in this thesis. Actually, we suppose that the criterion of generalized H-bounds is too strict for general conformal – and therefore massless – quantum field theories. (Generalized) H-bounds have been proved, however, for massive theories, i.e. for models in quantum field theory with massive particles (cf. also [DrF, FrH, Sum, Bucl]).

6 Results

This chapter summarizes in a compact form and as generally as possible the results derived in this thesis:

In the first section, we review the results of chapter 3 of this thesis. This chapter deals with the vacuum sector of the theory, i.e. with algebras of local observables and with the construction of pointlike localizable neutral fields without charge.

The content of chapter 4 is the generalization of the procedure and the calculations above from the vacuum sector to the full theory with all superselection sectors carrying arbitrary charge and finite statistics. The results of chapter 4 on the construction of pointlike localizable charged fields from the net of algebras in the reduced field bundle formalism, on the spin-statistics theorem, and on the PCT theorem can be found in the second section.

Finally, the results of chapter 5 on the canonical construction of Wightman n-point functions starting from the algebraic framework are reviewed in the third section of this chapter.

6.1 On the Construction of Pointlike Neutral Fields

Let $(\mathcal{A}(I))_{I\in\mathcal{K}_0}$ be a conformally covariant local net on $\mathbf{R}$ fulfilling the Haag-Kastler axioms in the formulation given in the section "Assumptions" at the beginning of this thesis. We consider the extension of this net to the set $\mathcal{K}$ of non-empty non-dense intervals on $\mathbf{S}^1 = \mathbf{R} \cup \{\infty\}$ (cf. equation (2.5)).

In [FröG] and [BGL] it has been shown that for such nets Haag duality and the Bisognano-Wichmann property can be proved. Therefore, we have for every $I \in \mathcal{K}$ the duality relation

$$\mathcal{A}(I')' = \mathcal{A}(I)\,. \tag{6.1}$$

(I' here denotes the complement of I with respect to $\mathcal{K}$) and, furthermore, the identification of the modular operators of the theory as well-known geometrical objects (cf. [BiW]). Hence, there is always an antiunitary involution Θ (the PCT operator) which acts on $\mathcal{A}$ by

$$\Theta\,\mathcal{A}(I)\,\Theta = \mathcal{A}(-I) \tag{6.2}$$

and on $U(SL(2,\mathbf{R}))$ by

$$\Theta\,U(g)\,\Theta = U(g_\vartheta)\,. \tag{6.3}$$

Here g_ϑ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ means $\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$.

As a further consequence of the results of [FröG, BGL], we have shown that the net is automatically additive, i.e. if $I = \bigcup_\alpha I_\alpha$ with $I, I_\alpha \in \mathcal{K}$ then

$$\mathcal{A}(I) = \bigvee_\alpha \mathcal{A}(I_\alpha)\,, \tag{6.4}$$

where $\bigvee$ denotes the generated von Neumann algebra.

We have been able to prove a conformal cluster theorem for chiral conformal Haag-Kastler nets that specifies the decrease properties of conformal two-point functions in the algebraic framework to be exactly those known from conformal field theories with pointlike localization (cf. [FrJ]):
Let $a, b, c, d \in \mathbf{R}$ and $a < b < c < d$. Let $A \in \mathcal{A}((a,b))$, $B \in \mathcal{A}((c,d))$, $n \in \mathbf{N}$, and $P_k A\Omega = P_k A^*\Omega = 0$, $k < n$. P_k here denotes the projection on the subrepresentation of $SL(2,R)$ with conformal dimension k. We then have

$$|(\Omega, BA\Omega)| \leq \left(\frac{(b-a)(d-c)}{(c-a)(d-b)}\right)^n \|A\|\,\|B\|\,. \tag{6.5}$$

We now proceed to the main result in this section, the proof of the existence of pointlike localized conformal fields (cf. [FrJ]):

Due to the condition of positive energy, the representation $U(\cdot)$ is completely reducible into elements of the "discrete series" of $SL(2,R)$ (cf. [Lang]), and the irreducible components τ are (up to unitary equivalence) uniquely characterized by the conformal dimension $n_\tau \in \mathbf{N}_0$ (n_τ is the lower bound of the spectrum of the conformal Hamiltonian $\mathbf{H}$ in the representation τ).

Associated with each irreducible subrepresentation τ of $U(\cdot)$, we find for each $I \in \mathcal{K}$ a densely defined operator-valued distribution φ_τ^I on the space $\mathcal{D}(I)$ of Schwartz functions with support in I such that the following statements hold for all $f \in \mathcal{D}(I)$:

i) The domain of definition of $\varphi_\tau^I(f)$ is given by $\mathcal{A}(I)'\Omega$.

ii)
$$\varphi_\tau^I(f)\,\Omega \in H_\tau\,. \tag{6.6}$$

iii)
$$U(g)\,\varphi_\tau^I(x)\,U(g)^{-1} = (cx+d)^{-2n_\tau}\varphi_\tau^{gI}(gx) \tag{6.7}$$

with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{R})\,,\; I, gI \in \mathcal{K}\,.$

iv)

$$\varphi_\tau^I(f)^* \supset \varphi_{\bar\tau}^I(\bar f) \tag{6.8}$$

with $\bar\tau(\cdot) = \Theta\,\tau(\cdot)\,\Theta$ and $\bar f$ the complex conjugate of f; in particular, $\varphi_\tau^I(f)$ is closable.

v) The closure of $\varphi_\tau^I(f)$ is affiliated to $\mathcal{A}(I)$.

vi) $\mathcal{A}(I)$ is the smallest von Neumann algebra to which all operators $\varphi_\tau^I(f)$ are affiliated.

We have not been able to find a domain of definition for the fields that is conformally invariant and stable under the action of field operators. Therefore, we can neither define products of neutral field operators nor prove that the Wightman axioms are fulfilled by the pointlike localized neutral fields.

Being interested in operator product expansions for conformal field theories, we have been able to derive the following result:
For each interval $I \in \mathcal{K}$ and each observable $A \in \mathcal{A}(I)$ there is a local expansion

$$A \;=\; \sum_\tau \varphi_\tau^I(f_{\tau,A}) \tag{6.9}$$

into a sum over all irreducible modules τ of $U(SL(2,R))$ with explicitly calculable testfunction $f_{\tau,A}$ fulfilling the support property

$$\mathrm{supp} f_{\tau,A} \subset I\,. \tag{6.10}$$

This expansion can be shown to converge $*$-strongly on $\mathcal{A}(I)'\,\Omega$ (cf. the definition in [BrR]).

6.2 On the Construction of Pointlike Charged Fields

Having presented so far the results of the calculations in the vacuum sector of the theory, we now look at what has been proved for the full theory with arbitrary charge and finite statistics (cf. [Jör3]). In order to be able to describe charge-carrying objects (i.e. "charged fields") intertwining between the superselection sectors with finite statistics of the theory, we consider the reduced field bundle $\mathcal{F}_{red} = (\mathcal{F}_{red}(I))_{I \in \mathcal{K}_0}$ (cf. [FRS1, FRS2]) associated to the net of observables $\mathcal{A} = (\mathcal{A}(I))_{I \in \mathcal{K}_0}$.

In chapter 3 of this thesis, we have presented the proof of the conformal cluster theorem in the vacuum sector (cf. [FrJ]). In chapter 4, we have proved the generalization of this cluster theorem to superselection sectors carrying arbitrary charge and finite statistics (cf. [Jör3]). By this theorem, the decrease properties of charged conformal two-point functions in the algebraic framework have been specified to be exactly those known from conventional conformal field theory with pointlike localization:
Let $a, b, c, d \in \mathbf{R}$ and $a < b < c < d$. Let $F \in \mathcal{F}_{red}((a,b))$, $G \in \mathcal{F}_{red}((c,d))$, $m \in \mathbf{R}_+$, and $P_k F \Omega = P_k \bar{F} \Omega = 0$, $k < m$. P_k here denotes the projection on the subrepresentation of $U(\widetilde{SL(2,R)})$ with conformal dimension k. $\bar{F}$ is given by the antilinear charge conjugation operation in the reduced field bundle. We then have

$$|(\Omega, GF\Omega)| \leq \left(\frac{(b-a)(d-c)}{(c-a)(d-b)} \right)^m \|F\| \, \|G\| . \tag{6.11}$$

Generalizing the construction of pointlike localized neutral fields in the vacuum sector, we have been able to prove the existence of pointlike localized charged fields associated with the chiral conformal Haag-Kastler net $\mathcal{A}$ and the reduced field bundle $\mathcal{F}_{red}$. The charged field operators intertwine between the superselection sectors of the theory with finite statistics.

Due to the condition of positive energy, again, the representation of the universal covering of the conformal group $U(\widetilde{SL(2,R)})$ is completely reducible into irreducible subrepresentations. The irreducible components τ are (up to unitary equivalence) uniquely characterized by the conformal dimension $m_\tau \in \mathbf{R}_+$ (m_τ is the lower bound of the spectrum of the conformal Hamiltonian $\mathbf{H}$ in the representation τ).

Associated with each irreducible subrepresentation τ of $U(\widetilde{SL(2,R)})$ and each superselection sector $[\alpha]$ with finite statistics, we find for each $I \in \mathcal{K}_0$ a densely defined operator-valued distribution $\varphi^I_{\tau,\alpha}$ on the space $\mathcal{D}(I)$ of Schwartz functions with support in I such that the following statements hold for all $f \in \mathcal{D}(I)$:

i) The domain of definition of $\varphi^I_{\tau,\alpha}(f)$ is given by $P_\alpha \, \mathcal{F}_{red}(I')_\pm \, \Omega$. P_α here denotes the projector on H_α, i.e. on the copy of the vacuum Hilbert space H in H_{red} associated with the superselection sector $[\alpha]$. $\mathcal{F}_{red}(I')_\pm$

has been defined to be the span of elements of $\mathcal{F}_{red}(I')$ localized in just one of generally two complements of I with respect to $\mathcal{K}_0$.

ii)

$$\varphi^I_{\tau,0}(f)\,\Omega \in P_\tau H_{red} \tag{6.12}$$

with P_τ denoting the projector on the module of τ.

iii)

$$U(\tilde{g})\,\varphi^I_{\tau,\alpha}(x)\,U(\tilde{g})^{-1} = (cx+d)^{-2m_\tau}\varphi^{gI}_{\tau,\alpha}(\tilde{g}x) \tag{6.13}$$

with the covering projection $\tilde{g} \mapsto g$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{R})$ for $I, gI \in \mathcal{K}_0$.

iv) $\varphi^I_{\tau,\alpha}(f)$ is closable.

v) The closure of any $\varphi^I_{\tau,\alpha}(f)$ is affiliated to $\mathcal{F}_{red}(I)$.

vi) $\mathcal{F}_{red}(I)$ is the smallest von Neumann algebra to which all operators $\varphi^I_{\tau,\alpha}(f)$ are affiliated.

As in the vacuum sector, we have not found a domain of definition for the fields that is stable under the action of the field operators. Hence, we cannot define products of charged field operators and do not know how to prove that the pointlike localized charged fields fulfill the Wightman axioms.

With the existence of pointlike localized fields we are able to give a proof of a generalized Bisognano-Wichmann property. This theorem identifies objects in the representation of the conformal group with well-known geometrical meaning as generalized modular operators associated with the charge conjugation operation in the reduced field bundle (cf. [Ara]).

As a consequence, we obtain a PCT operator on H_{red} proving the PCT theorem for the full theory.

Moreover, the existence of pointlike localized fields gives a proof of the hitherto unproven spin-statistics theorem for conformal Haag-Kastler nets of local observables in 1+1 dimensions.

6.3 On the Construction of Wightman Functions

We do not know at the moment whether the pointlike localized neutral and charged fields constructed in chapter 3 and 4 of this thesis satisfy all Wightman axioms, since we have not yet found an invariant domain of definition. We have been able to prove, though, that to each chiral conformal Haag-Kastler net of local observables a Wightman field theory can be canonically associated. We now present the construction of a complete set of pointlike localized correlation functions from the net of algebras of local observables we have been starting with.

Consider algebraic n-point functions

$$(\Omega, \left(\prod_{1\leq i\leq n} A_i \right) \Omega) \tag{6.14}$$

of local observables A_i with minimal conformal dimensions n_i, $i = 1,2,\dots,n$, in a chiral theory with conformal covariance. We have been able to present the following generalization of the conformal cluster theorem (cf. [FrJ]) to algebraic n-point functions:

Let $(\mathcal{A}(I))_{I\in\mathcal{K}_0}$ be a conformally covariant local net on $\mathbf{R}$. Let $a_i, b_i \in \mathbf{R}$, $i = 1,2,3,\dots,n$, and $a_i < b_i < a_{i+1} < b_{i+1}$ for $i = 1,2,3,\dots,n-1$. Let $A_i \in \mathcal{A}((a_i, b_i))$, $n_i \in \mathbf{N}$, and $P_k A_i \Omega = P_k A_i^* \Omega = 0$, $k < n_i$ and $i = 1,2,3,\dots,n$. P_k here denotes the projection on the subrepresentation of $U(G)$ with conformal dimension k. We then have for each set of exponents c_{ij} fulfilling the consistency conditions $\sum_{j=1, j\neq i}^{n} c_{ij} = 2n_i$, $c_{ij} = c_{ji}$, $i = 1,2,3,\dots,n$, the following bound:

$$\begin{aligned} &|(\Omega, \left(\prod_{1\leq i\leq n} A_i \right) \Omega)| \\ \leq\ & \left(\prod_{1\leq i<j\leq n} \left| \frac{(a_i - b_i) + (a_j - b_j)}{(a_j - a_i) + (b_j - b_i)} \right|^{c_{ij}} \right) \prod_{1\leq i\leq n} \|A_i\| . \end{aligned} \tag{6.15}$$

If we additionally assume $a_1 - b_1 = \dots = a_n - b_n$, we can introduce conformal cross ratios and get

$$|(\Omega, \left(\prod_{1\leq i\leq n} A_i \right) \Omega)|$$

$$\leq \left(\prod_{1 \leq i < j \leq n} \left(\frac{(a_i - b_i)\,(a_j - b_j)}{(a_i - a_j)\,(b_i - b_j)} \right)^{c_{ij}/2} \right) \prod_{1 \leq i \leq n} \|A_i\| \,. \tag{6.16}$$

Using this result, we can control the pointlike limit of canonically scaled correlation functions

$$\lim_{\lambda \downarrow 0} \lambda^{-\left(\sum_{1 \leq i \leq n} n_i\right)} \left(\Omega, \left(\prod_{1 \leq i \leq n} U(-\frac{x_i}{\lambda})\, A_i\, U(\frac{x_i}{\lambda}) \right) \Omega \right), \tag{6.17}$$

where $U(\cdot)$ denotes the representation of the translation group. This sequence of algebraic correlation functions converges to a pointlike localized n-point function known from conventional conformal field theory. The limit has been shown to be of the form

$$\left(\prod_{1 \leq i < j \leq n} \frac{1}{(x_j - x_i + i\varepsilon)^{c_{ij}}} \right) f(r^{v_1 s_1}_{t_1 u_1}, \ldots, r^{v_{n-3} s_{n-3}}_{t_{n-3} u_{n-3}}) \,. \tag{6.18}$$

Here, $f(\cdot, \ldots, \cdot)$ denotes an appropriate function depending on n–3 algebraicly independent conformal cross ratios

$$r^{vs}_{tu} := \frac{(x_v - x_s)\,(x_t - x_u)}{(x_v - x_t)\,(x_s - x_u)} \,. \tag{6.19}$$

The exponents c_{ij} must fulfill the consistency conditions

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} c_{ij} = 2n_i \,, \; c_{ij} = c_{ji} \,, \; 1 \leq i \leq n \,. \tag{6.20}$$

These conditions do not fully determine the exponents c_{ij}.

The algebraic correlation functions can easily be shown to fulfill positive definiteness, conformal covariance, locality, and the spectrum condition. Hence, the pointlike limit of such algebraic correlation functions fulfills the Wightman axioms (see [StW]) by construction. By the reconstruction theorem in [StW] and [Jos], the existence of Wightman fields associated with a complete set of Wightman functions is then guaranteed. The realization of this Wightman field theory is unique up to unitary equivalence.

We have not been able to prove that the Wightman fields can be identified with the pointlike localizable field operators constructed in chapter 3 of this thesis from the Haag-Kastler theory. We do not know either whether the

Wightman fields are affiliated to the associated von Neumann algebras and how the Haag-Kastler net of local observables we have been starting from can be reconstructed from the Wightman fields. It has been shown that the Wightman theory canonically associated with the Haag-Kastler theory has the same two-point functions as the pointlike localized fields constructed in chapter 3 and is therefore non-trivial.

7 Conclusion and Outlook

The aim of this thesis has been to investigate the relation between the formulation of quantum field theory in terms of Haag-Kastler nets of local observables on the one hand and the formulation by Wightman in terms of pointlike localized fields on the other hand.

We restricted our investigation to chiral conformally covariant theories. Indeed, we heavily used the strong consequences of the assumption of this high symmetry and simple geometry in our constructions and lines of argument.

Hence, the results of this thesis, presented in detail in the last chapter, only apply to a small class of quantum field theories. This class might be basic and mathematically interesting, it might even be generic for more general classes of quantum field theories – it is, however, still idealized and far from describing complex physical situations.

In order to enlarge the domain of validity of the results proved in this thesis, the following two strategies could be chosen:

Firstly, one might reduce the assumption on the symmetry of the theory. If we consider dilation invariant theories, scaling is well-defined and the definition of pointlike localized fields in this thesis (cf. equation (3.22)) could be expected to hold since the formula does not contain special conformal transformations. However, our control of the existence of the pointlike limit in the physical Hilbert space depends on our explicit knowledge of the representation theory of the conformal group. Even more, the construction of Wightman functions in this thesis relies on conformal covariance as it makes use of the conformal cluster theorem. Nevertheless, we suppose that dilation covariant Haag-Kastler nets can be shown to have associated pointlike localized fields.

Secondly, more complex geometrical situations could be investigated. One might, e.g., consider Minkowski space-time in higher dimensions. The representation theory of the conformal group on Minkowski space-time in higher dimensions is well-known (cf. [Mac2]) and has structural similarities to the representation theory of the Möbius group, that has been used in this thesis. This group-theoretical parallelism between two and more space-time dimensions does not hold, however, in the context of conventional conformal field theory (cf., e.g., [Mac3]). The existence of "conformal families" with "primary fields" transforming covariantly with respect to the Virasoro algebra is, e.g., a particular property of the situation in two-dimensional space-time (cf. [FST]). The infinite-dimensional symmetry group of primary fields is a relict of the infinite-dimensional group of conformal transformations in classical physics on two-dimensional Minkowski space-time and has no obvious correspondence on Minkowski space-time in higher dimensions. Since in this thesis neither the existence of "primary fields" in two-dimensional conformal field theory nor covariance with respect to the infinite-dimensional Virasoro algebra have been used, we expect that our results can be transferred from $1+1$ space-time dimensions to theories on Minkowski space-time in higher dimensions.

We like to conclude by pointing out some open questions concerning the relation between conventional conformal field theory on the one hand and the approaches of Haag-Kastler and Wightman to a general theory of quantized fields on the other hand. It would be important and interesting to formulate the axioms of conventional conformal field theory in terms of more natural "first principles" (cf. [Sch, Fuc, Was]). Actually, we were not very successful in deriving specific features assumed to be true in conventional conformal field theory from the more abstract approaches. The local expansion of observables in terms of pointlike localized fields that has been proved in this thesis is still quite different from the formulation of operator product expansions known from conventional conformal field theory (cf. [FST]). We assume that it would be valuable and fruitful to be able to combine the knowledge and experience from conventional conformal field theory with the structural insight of algebraic quantum field theory.

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APPENDIX

A The Bound for $\sup_p |\Phi_k^{(n)}(p)|$

We present an argument (see [FrJ]) using complex analysis and some algebraic transformations to derive a bound on

$$\sup_p |\Phi_k^{(n)}(p)| \tag{A.1}$$

needed in the construction of pointlike localized fields (cf. equation (3.29):

The eigenfunction of the conformal Hamiltonian for the eigenvalue $k \geq n$ is well-known and can be written in position space as

$$f_k^{(n)}(x) = (1+ix)^{n-k-1}(1-ix)^{n+k-1}\,. \tag{A.2}$$

Its Fourier transform may be computed by the theorem of residues from complex analysis and turns out to be

$$\begin{aligned} \tilde{f}_k^{(n)}(p) &= \int_{\mathbf{R}} dx\, e^{ipx}\, f_k^{(n)}(x) \\ &= N_k^{(n)}\, L_{n+k-1}^{2n-1}(2p)\, e^{-p}\,, \quad p > 0\,. \end{aligned} \tag{A.3}$$

Hereby, we have used the normalized associated Laguerre polynomials L_{n+k-1}^{2n-1} and constants

$$N_k^{(n)} = 2^{n-1}\left(\frac{(k+n-1)!}{(k-n)!}\right)^{1/2}(-1)^{k+n+1}\,. \tag{A.4}$$

For a bound on its modulus we choose as integration path a circle with center iR and radius $R \geq 1$. We obtain

$$\tilde{f}_k^{(n)}(p) = \int_0^{2\pi} d\varphi\; iRe^{i\varphi}\,(1-R+iRe^{i\varphi})^{n-k-1}\,(1+R-iRe^{i\varphi})^{n+k-1}\,e^{-pR+ipRe^{i\varphi}} \tag{A.5}$$

and find

$$\begin{aligned} |\tilde{f}_k^{(n)}| &\leq 2\pi R\, \sup_\varphi \left[\frac{1+2R(R+1)(1+\sin\varphi)}{1+2R(R-1)(1+\sin\varphi)}\right]^{k/2} \\ &\quad \sup_\varphi \left[(1+2R(R-1)\,(1+\sin\varphi))\,(1+2R(R+1)\,(1+\sin\varphi))\right]^{\frac{n-1}{2}} \\ &\leq 2\pi R\,(1+4R^2)^{n-1}\left(\frac{R+1}{R-1}\right)^{k/2}\,. \end{aligned} \tag{A.6}$$

We insert $R = k + 1$, and find

$$\begin{aligned} |\tilde{f}_k^{(n)}(p)| &\leq 2\pi\,(k+1)\,(5+8k+4k^2)^{n-1}\,(1+\frac{2}{k})^{k/2} \\ &\leq \frac{\pi}{2}e\,(5+4k)^{2n-1}\,. \end{aligned} \tag{A.7}$$

As a bound on the normalized eigenfunctions we obtain

$$\begin{aligned} |\Phi_k^{(n)}(p)| &= \frac{|\tilde{f}_k^{(n)}(p)|}{N_k^{(n)}} \\ &\leq \frac{C\,k^{2n}+D}{k^n} \\ &\leq C\,k^n + D \end{aligned} \tag{A.8}$$

with appropriate constants C and D.

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