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On the Connection between Nelson's Stochastic Quantum Mechanics and Nottale's Theory of Scale Relativity

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Abstract: In this paper, we review and compare the stochastic quantum mechanics of Nelson and the scale relativity theory of Nottale. We consider both nonrelativistic and relativistic frameworks and include the electromagnetic field. These theories propose a derivation of the Schrödinger and Klein–Gordon equations from microscopic processes. We show their formal equivalence. Specifically, we show that the real and imaginary parts of the complex Lorentz equation in Nottale's theory are equivalent to the Nelson equations, which are themselves equivalent to the Madelung and de Broglie hydrodynamical representations of the Schrödinger and Klein–Gordon equations, respectively. We discuss the different physical interpretations of the Nelson and Nottale theories and stress their strengths and weaknesses. We mention potential applications of these theories to dark matter.

Keywords: stochastic quantum mechanics; theory of scale relativity; Schrödinger equation; Klein–Gordon equation

MSC: 81Pxx; 81P20; 81Qxx; 81Q65; 81Sxx; 81S20; 81Vxx; 81V99; 60Gxx; 60G07

1. Introduction

The Schrödinger [1–4] equation and the Klein–Gordon (KG) [5–7] equation were originally introduced on the basis of very heuristic arguments, either from a dispersive wave equation with a phase velocity determined by the de Broglie relations, from a variational principle (least action principle) deduced from the classical Hamilton–Jacobi equation by writing $S = \hbar \ln \psi$ and forming a quadratic functional to be minimized, or from the correspondence principle (operator prescription) by replacing the energy E and the momentum $\mathbf{p}$ of a classical particle by the operators $i\hbar\partial/\partial t$ and $-i\hbar\nabla$ (see the introduction of [8,9] for a detailed review of the early developments of nonrelativistic and relativistic quantum mechanics). Although very imaginative, these original approaches are essentially ad hoc and cannot be considered as a true derivation of the Schrödinger and KG equations.[1] In addition, the Schrödinger equation has always been regarded as an abstract equation because it describes the evolution of a complex wavefunction $\psi(\mathbf{r}, t)$ whose interpretation is unclear and is still debated. One possibility, initially introduced by Schrödinger [4], is to interpret $\rho_e = e|\psi|^2$ as the electric density.[2] Another possibility, independent of the charge, is the probabilistic interpretation of Born [17–20] who views $\rho = |\psi|^2$ as the probability density of finding the particle in $\mathbf{r}$ at time t. He writes [19]: "A knowledge of ψ enables us to follow the course of a physical process in so far as it is quantum mechanically determinate: not in a causal sense, but in a statistical one". Therefore, quantum mechanics only gives us probabilities. This leads to an abandonment of causality in the classical sense. More precisely [18]: "the motion of the particles follows probabilistic laws, but the probability itself propagates in accord with the law of causality". The probabilistic interpretation of Born was given further support by Heisenberg [21], who discovered the uncertainty principle and argued that indeterminism is necessary. This probabilistic interpretation was also defended by Bohr and is often referred to as the Copenhagen interpretation. However, this probabilistic interpretation did not satisfy many physicists, including Schrödinger,

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Planck, Einstein and de Broglie, who found it hard to abandon causality in the classical sense.[3] These physicists remained skeptical about Born's interpretation as a probability amplitude until the end of their lives. This led to an intense debate between Einstein and Bohr about the foundations of quantum mechanics [23]. Einstein once said: "God does not throw dice". He maintained during all his life that quantum mechanics is incomplete, and believed that the results of quantum calculations should emerge from microscopic phenomena. He argued that quantum constraints should be implemented in a causal theory based on partial differential equations [23]: "I still believe in the possibility of giving a model of reality which shall represent events themselves and not merely the probability of their occurrence". Similarly, he believed that particles would eventually appear as solutions of the equations of a unified theory. According to Einstein, the status of quantum mechanics is like Newton's law before relativity [23]. The Schrödinger and KG equations are probably correct, at least in an approximate manner, but they have not been derived. They may be obtained from a more general approach, like Newton's law is justified as an approximation of general relativity. Therefore, quantum mechanics could be the limiting case of a theory which remains to be discovered.[4]

In order to understand the physical meaning of the Schrödinger equation, it was first important to establish the connection between wave mechanics and classical mechanics. A quasiclassical approximation was developed by Wentzel [24], Brillouin [25] and Kramers [26]. The WKB approximation consists of writing the wave function under the form $\psi = e^{i\mathcal{S}/\hbar}$ and expanding the complex action $\mathcal{S}$ in powers of $\hbar$. At the leading order, it returns the classical Hamilton–Jacobi equation, making a direct connection between quantum and classical mechanics. At the first order, it returns the Bohr–Sommerfeld quantization condition [27,28], making a direct connection between the "old" and the "new" quantum mechanics. Another connection between quantum and classical mechanics was found by Ehrenfest [29], who showed that the equations of motion for the mean values of the physical quantities coincide in form with the classical equations (Ehrenfest's theorem).

There has also been a search for a causal (instead of probabilistic) interpretation of quantum mechanics. In particular, other representations of the Schrödinger equation have been proposed which avoid the wave function. This started with the hydrodynamic approach of Madelung [30], who showed that the Schrödinger equation is formally equivalent to fluid equations for a potential flow formed by a continuity equation and an Euler equation involving a quantum potential proportional to $\hbar^2$. Because of this quantum term, the particle's motion does not follow the laws of classical mechanics. The Madelung transformation suggests the possibility of a semiclassical description of quantum systems through the fluid dynamical viewpoint. At about the same time, de Broglie [31–33] (see also London [34]) independently developed a relativistic hydrodynamical representation of the KG equation. He derived a continuity equation and a relativistic quantum Euler equation that contains a Lorentz invariant quantum potential. The aim of de Broglie was to provide a causal and objective interpretation of wave mechanics, in accordance with the wish expressed many times by Einstein, and in contrast to the purely probabilistic interpretation of quantum mechanics put forward by Born, Bohr, and Heisenberg. This is what he called the "pilot wave theory", because the particle is guided by the wave ψ. This is also called the "double solution theory", because the particles are treated as singular points in a wave process (in this sense, his interpretation fundamentally differs from the extended electron model of Schrödinger).

The results of Madelung and de Broglie were rediscovered by Bohm [35,36] in 1952, who proposed an interpretation of the Schrödinger equation in terms of particle trajectories determined by definite laws analogous to (but not identical to) the classical equations of motion, because of the action of the quantum potential associated with the wave function. This quantum potential, which depends on the density and density gradients of the system, suggests an interpretation of the quantum theory in terms of "hidden" variables, in the same sense that in macroscopic physics the coordinates and momenta of individual atoms are hidden variables which in a large scale system manifest themselves only as statistical averages.

These works were further developed by Takabayasi [37,38], who suggested an alternative formulation of quantum mechanics in terms of a classical picture based on hydrodynamic equations. He emphasized the role of the quantum force, depending on the probability itself, that is responsible for the "blurring" of the classical trajectory. Following Bohm [35,36], he interpreted the diffusion of wave packets, interference effects and tunnel effects in terms of this quantum force. This renewal of interest for a causal interpretation of quantum mechanics stimulated de Broglie to return to the problem again and undertake a fresh examination of his old ideas [39–42].

In a remarkable paper in 1966, Nelson [43] proposed a derivation of the Schrödinger equation from Newtonian mechanics by using a stochastic theory in which a particle of mass m is subject to a classical Brownian motion with diffusion coefficient $\mathcal{D} = \hbar/2m$ and no friction.[5] In ordinary Brownian motion, friction plays an important role. In Nelson's theory, quantum particles experience no friction in order to preserve Galilean invariance. As a result, the quantum Brownian motion is nondissipative and reversible while the usual Brownian motion is dissipative and irreversible. In this stochastic formulation of quantum mechanics, the random motion represents quantum fluctuations due to a sub-quantum medium (a sort of aether). However, Nelson did not specify its physical origin precisely. Since the motion of a Brownian particle is nondifferentiable, Nelson introduced a mean forward velocity $\mathbf{v}_+$ and a mean backward velocity $\mathbf{v}_-$ associated with forward and backward Fokker–Planck equations, defined a mean acceleration $\mathbf{a}$, assumed that Newton's law $\mathbf{F} = m\mathbf{a}$ remains valid in this context, and derived a pair of coupled hydrodynamic equations for the current velocity $\mathbf{v} = (\mathbf{v}_+ + \mathbf{v}_-)/2$ and the osmotic velocity $\mathbf{v}_Q = (\mathbf{v}_+ - \mathbf{v}_-)/2$, the latter equation being equivalent to the continuity equation. Using a form of Madelung transformation (he was apparently not aware of Madelung's paper), he showed that these equations are equivalent to the Schrödinger equation. Nelson's stochastic approach was extended in relativity by several authors [51–54] who derived the KG equation by a similar procedure.

More recently, Nottale [55] developed a theory of scale relativity. He gave up the concept of the differentiability of spacetime and postulated that spacetime is fractal. This is the origin of quantum mechanics in his approach. In a fractal spacetime, the trajectories of the quantum particles are continuous but nowhere differentiable. The velocity is the sum of a differentiable part and a non-differentiable (fractal) part. The new component is an explicit scale-dependent fractal fluctuation. The fractal dimension of the Brownian motion, and more generally of Markov processes which correspond to cases when the various elementary displacements are independent of the previous and next ones, is $D_{\rm F} = 2$ [56,57]. The theory of scale relativity extends Einstein's theory of relativity to scale transformations of resolutions. In this approach, the particles have a stochastic motion that is due to the fractal nature of the spacetime itself. The fractal (non-differentiable) nature of the trajectories leads to introducing twin velocities $\mathbf{v}_+$ and $\mathbf{v}_-$. From these twin velocities, one can form a classical velocity $\mathbf{v} = (\mathbf{v}_+ + \mathbf{v}_-)/2$ and a quantum velocity $\mathbf{v}_Q = (\mathbf{v}_+ - \mathbf{v}_-)/2$ (for a classical— differentiable—motion where $\mathbf{v}_+ = \mathbf{v}_-$, $\mathbf{v}$ coincides with the usual velocity and $\mathbf{v}_Q = \mathbf{0}$). This is similar to Nelson's approach [43] but with a different interpretation. These velocities can be combined together into a complex velocity $\mathbf{V} = \mathbf{v} - i\mathbf{v}_Q$ and this duality (the two-valuedness character of the velocity) is viewed as the fundamental origin of the complex nature of the wave function ψ in quantum mechanics (the wave function is related to the complex velocity by $\mathbf{V} = -i(\hbar/m)\nabla \ln \psi$). By using a principle of scale covariance,[6] Nottale managed to derive the Schrödinger equation from Newton's law. Applying the same procedure in relativity, he derived the KG equation and the Dirac equation.

In addition to atomic and molecular physics, the Schrödinger equation and the KG equation may also have applications in astrophysics and cosmology. They were initially introduced in relation to hypothetical boson stars, then in the context of bosonic dark matter (see the introduction of [8,58–60] for a short account of the history of the subject and an exhaustive list of references). In that case, the Schrödinger equation has to be coupled to the Poisson equation, and the KG equation has to be formulated in a curved spacetime determined by the Einstein equations. Quantum mechanics (Heisenberg's uncertainty principle) may stabilize the system at small scales and prevent gravitational collapse. This may solve notorious difficulties of the cold dark matter (CDM) model, such as the core–cusp problem [61] and the missing satellite problem [62]. For quantum effects to manifest themselves at the galactic scale, dark matter must involve a very light particle or scalar field (axion) with a mass $m \sim 10^{-22}\,\mathrm{eV}/c^2$. This is called fuzzy dark matter (FDM) [63,64] (see [65–72] for some recent reviews on the subject). The scalar field may describe the wavefunction of a Bose-Einstein condensate (BEC) at $T = 0$ where all the bosons are in the same quantum state. Self-interaction between the bosons can be incorporated in a potential, leading to nonlinear Schrödinger equations (e.g., the Gross–Pitaevskii equation) and nonlinear KG equations.[7] The quantum hydrodynamical approaches of Madelung and de Broglie have been applied in this astrophysical context. Actually, the hydrodynamical representation of the Schrödinger and KG equations is more appropriate to describe boson stars and dark matter halos, which can be considered as fluids made of many particles, than to describe a single particle like an electron, as it was originally carried out in quantum mechanics. The literature on FDM is extensive. Usually, the Schrödinger and the KG equations in the FDM model are justified by standard quantum mechanics. However, it is also possible to advocate the alternative theories of Nelson and Nottale. This has been considered recently in [73–76]. The approaches of Nelson and Nottale may provide a new interpretation of quantum mechanics. In Nelson's theory, the origin of quantization is related to the stochastic force acting on the particles, similarly to Brownian motion. The fluctuations leading to quantum mechanics may be due to the stochastic fluctuations of the electromagnetic field (zero-point radiation field), the fluctuations of the gravitational field (gravitational wave background), or the fluctuations of the metric in general relativity (see the conclusion of this paper for more details and references on these topics). In Nottale's theory, the underlying fluctuations may be considered as an intrinsic property of a fractal spacetime. They may also arise from a chaotic dynamics. This may lead to a new expression of the quantum diffusion coefficient in the Schrödinger equation which is not necessarily related to the Planck constant. These ideas may be of potential interest in the context of dark matter and may change the constraints on the mass of the dark matter particle [74].

Therefore, we believe that these rather unconventional interpretations of quantum mechanics are of conceptual interest and potentially fruitful and important. In this paper, we review and compare the stochastic quantum mechanics of Nelson and the scale relativity theory of Nottale. We consider both nonrelativistic and relativistic frameworks and include the electromagnetic field.[8] We show their formal equivalence. We also show their connection to the Madelung and de Broglie hydrodynamical representations of the Schrödinger and KG equations. We discuss the different physical interpretations of the Nelson and Nottale theories and stress their strengths and weaknesses. We mention potential applications of these theories to dark matter.

2. Nonrelativistic Theory

2.1. Basics of Classical Mechanics

In classical mechanics, the energy E and the momentum $\mathbf{p}$ of a nonrelativistic particle of mass m and charge e in the presence of an electromagnetic field can be written as $E = E_{\rm kin} + eU$ and $\mathbf{p} = \mathbf{p}_{\rm kin} + e\mathbf{A} = m\mathbf{v} + e\mathbf{A}$, where $E_{\rm kin}$ and $\mathbf{p}_{\rm kin} = m\mathbf{v}$ are the energy and the momentum of the particle in the absence of magnetic field (free particle). Using the relation $E_{\rm kin} = \mathbf{p}_{\rm kin}^2/2m$ we find that

$$E - eU - m\Phi = \frac{(\mathbf{p} - e\mathbf{A})^2}{2m}. \tag{1}$$

For the sake of generality, we have introduced an external potential Φ. Recalling the relations

$$E = -\frac{\partial S}{\partial t}, \qquad \mathbf{p} = \nabla S, \tag{2}$$

between the energy E, the momentum $\mathbf{p}$, and the action S, we obtain the classical Hamilton–Jacobi equation

$$-\frac{\partial S}{\partial t} - eU - m\Phi = \frac{(\nabla S - e\mathbf{A})^2}{2m}. \tag{3}$$

Taking its gradient and introducing the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, defined by Equation (A65), we obtain the Lorentz equation[9]

$$\frac{d\mathbf{v}}{dt} \equiv \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla\Phi + \frac{e}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{4}$$

It can be written in terms of the momentum $\mathbf{p}$ as

$$\frac{d\mathbf{p}}{dt} \equiv \frac{\partial \mathbf{p}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{p} = -m\nabla\Phi - e\nabla U + e(\mathbf{v} \cdot \nabla)\mathbf{A} + e\mathbf{v} \times (\nabla \times \mathbf{A}). \tag{5}$$

2.2. Schrödinger Equation

The nonrelativistic wave equation of quantum mechanics can be obtained from the classical relation (1) by using the operator prescription[10]

$$E \to i\hbar\frac{\partial}{\partial t}, \qquad \mathbf{p} \to -i\hbar\nabla. \tag{6}$$

This leads to the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\nabla - i\frac{e}{\hbar}\mathbf{A}\right)^2\psi + m\Phi\psi + eU\psi. \tag{7}$$

Expanding the operator in parenthesis, it can be rewritten as

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + \frac{i\hbar e}{m}\mathbf{A} \cdot \nabla\psi + \frac{i\hbar e}{2m}(\nabla \cdot \mathbf{A})\psi + \frac{e^2}{2m}\mathbf{A}^2\psi + m\Phi\psi + eU\psi. \tag{8}$$

We note that the third term on the r.h.s. disappears if we choose the Coulomb gauge from Equation (A77), a choice that we will make in the following.

Multiplying the Schrödinger Equation (8) by ψ^* and subtracting its complex conjugate, we can easily derive the identity

$$\frac{\partial|\psi|^2}{\partial t} + \nabla \cdot \mathbf{J} = 0, \tag{9}$$

where

$$\mathbf{J} = \frac{\hbar}{2im}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{e}{m}|\psi|^2\mathbf{A} \tag{10}$$

is a current. Equation (9) expresses the local conservation of the quantity $|\psi|^2$. The physical interpretation of the wavefunction $\psi(\mathbf{r}, t)$ was a matter of debate in the early days of quantum mechanics (and still is—see the introduction). Schrödinger [4] interpreted the quantity $\rho_e = (e/m)|\psi|^2$ as being the charge density and $\mathbf{J}_e = (e/m)\mathbf{J}$ as being the current of charge, so that Equation (9) expresses the local conservation of the charge. Born [17–20] proposed an interpretation of the wave function independent of the charge. He interpreted $|\psi|^2(\mathbf{r}, t)$ with $\int |\psi|^2 \, d\mathbf{r} = 1$ as giving the probability density of finding the particle in $\mathbf{r}$ at time t. In that case, Equation (9) expresses the local conservation of the normalization condition. The Schrödinger equation has also been introduced in the case of BECs (e.g., in condensed matter physics and in the FDM model) where all the bosons are in the same quantum state described by a single wave function $\psi(\mathbf{r},t)$. In that case, Equation (9) expresses the local conservation of the mass of the system. In order to have the same notations as in our previous works on this subject, we shall normalize the wave function, such that $\rho=|\psi|^2$ represents the mass density of the BEC. Then, $\int|\psi|^2\,d\mathbf{r}=M$ represents the total mass of the BEC.

Remark: The Schrödinger equation with an electromagnetic field can also be obtained from the Schrödinger equation without an electromagnetic field by making the substitutions $\nabla\rightarrow\nabla-i\frac{e}{\hbar}\mathbf{A}$ and $\frac{\partial}{\partial t}\rightarrow\frac{\partial}{\partial t}+i\frac{e}{\hbar}U$ corresponding to the transformations $\mathbf{p}\rightarrow\mathbf{p}-e\mathbf{A}$ and $E\rightarrow E-eU$ in classical mechanics with the operator prescription from Equation (6).

2.3. Madelung Transformation

Making the Madelung [30] transformation, the Schrödinger Equation (7) can be written in the form of hydrodynamic equations. To that purpose, we write the wavefunction as

$$\psi(\mathbf{r},t)=\sqrt{\rho(\mathbf{r},t)}e^{iS(\mathbf{r},t)/\hbar}, \tag{11}$$

where $\rho(\mathbf{r},t)$ is the density and $S(\mathbf{r},t)$ is the action. These quantities can be expressed in terms of ψ as

$$\rho=|\psi|^2,\qquad S=-i\frac{\hbar}{2}\ln\left(\frac{\psi}{\psi^*}\right). \tag{12}$$

Generalizing the Madelung transformation in order to take into account electromagnetic effects, we introduce the velocity field

$$\mathbf{v}=\frac{\nabla S-e\mathbf{A}}{m}. \tag{13}$$

This definition is consistent with the classical expression $\mathbf{p}\equiv m\mathbf{v}+e\mathbf{A}=\nabla S$ of the generalized momentum of a particle in an electromagnetic field (see Section 2.1). Taking the gradient of the action in Equation (12), the velocity can be expressed in terms of ψ as

$$\mathbf{v}=\frac{\hbar}{2im}\frac{\psi^*\nabla\psi-\psi\nabla\psi^*}{|\psi|^2}-\frac{e}{m}\mathbf{A}. \tag{14}$$

We note that

$$\nabla\times\mathbf{v}=-\frac{e}{m}\nabla\times\mathbf{A}=-\frac{e}{m}\mathbf{B}, \tag{15}$$

where we have used Equation (A65) to obtain the second equality. This relation shows that the magnetic field creates a vorticity field. This vorticity is equal to twice the Larmor pulsation $\omega_L=-(e/2m)B$. In the absence of a magnetic field, the velocity field is irrotational ($\nabla\times\mathbf{v}=\mathbf{0}$). Using Equation (15), the well-known identities of fluid mechanics

$$(\mathbf{v}\cdot\nabla)\mathbf{v}=\nabla(\mathbf{v}^2/2)-\mathbf{v}\times(\nabla\times\mathbf{v}),\qquad\Delta\mathbf{v}=\nabla(\nabla\cdot\mathbf{v})-\nabla\times(\nabla\times\mathbf{v}), \tag{16}$$

become

$$(\mathbf{v}\cdot\nabla)\mathbf{v}=\nabla\left(\frac{\mathbf{v}^2}{2}\right)+\frac{e}{m}\mathbf{v}\times\mathbf{B},\qquad\Delta\mathbf{v}=\nabla(\nabla\cdot\mathbf{v})+\frac{e}{m}\nabla\times\mathbf{B}. \tag{17}$$

Substituting Equation (11) into the Schrödinger Equation (8), separating the real and imaginary parts and using Equation (13), we obtain the pair of equations

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0, \tag{18}$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S - e\mathbf{A})^2 = -Q - m\Phi - eU, \tag{19}$$

where

$$Q = -\frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} = -\frac{\hbar^2}{4m}\left[\frac{\Delta\rho}{\rho} - \frac{1}{2}\frac{(\nabla\rho)^2}{\rho^2}\right] = -\frac{\hbar^2}{8m}(\nabla\ln\rho)^2 - \frac{\hbar^2}{4m}\Delta\ln\rho \tag{20}$$

is the quantum potential, taking into account the Heisenberg uncertainty principle. To obtain the last equality we have used the identity

$$\Delta(\ln f) = \frac{\Delta f}{f} - \frac{1}{f^2}(\nabla f)^2. \tag{21}$$

Using Equation (13), we can rewrite Equation (19) as

$$\frac{\partial S}{\partial t} + \frac{1}{2}m\mathbf{v}^2 = -Q - m\Phi - eU. \tag{22}$$

Taking its gradient and using Equations (13) and (A65) we find that

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\frac{\mathbf{v}^2}{2}\right) = -\frac{1}{m}\nabla Q - \nabla\Phi + \frac{e}{m}\mathbf{E}. \tag{23}$$

Using Equation (17), the foregoing equation can be rewritten as

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{m}\nabla Q - \nabla\Phi + \frac{e}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{B}). \tag{24}$$

Introducing the Stokes material derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla, \tag{25}$$

we obtain

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{m}\nabla Q - \nabla\Phi + \frac{e}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{B}). \tag{26}$$

This equation can be written in terms of the momentum $\mathbf{p}$ as

$$\frac{d\mathbf{p}}{dt} = -\nabla Q - m\nabla\Phi - e\nabla U + e(\mathbf{v}\cdot\nabla)\mathbf{A} + e\mathbf{v}\times(\nabla\times\mathbf{A}). \tag{27}$$

The quantum hydrodynamic Equations (18)–(26) have a clear physical interpretation. Equation (18), corresponding to the imaginary part of the Schrödinger Equation (7), is the continuity equation.[11] It accounts for the local conservation of the mass. Equation (19), corresponding to the real part of the Schrödinger Equation (7), is the quantum Hamilton–Jacobi equation or the quantum Bernoulli equation. It involves the quantum potential (20). When $\hbar = 0$ we recover the classical Hamilton–Jacobi equation or the Bernoulli Equation (3). Equation (23), (24) or (26) is the quantum Euler–Lorentz equation. It includes the quantum force $-\nabla Q$, the Lorentz force $e(\mathbf{E} + \mathbf{v}\times\mathbf{B})$, and the external force $-m\nabla\Phi$. In the classical limit $\hbar = 0$ we recover the classical Lorentz Equation (4). The Schrödinger Equation (7) is equivalent[12] to the Madelung Equations (18)–(26) composed of the continuity Equation (18) and the quantum Euler-Lorentz Equation (26).

Remark: We note that the quantum force in Equation (24) can be written as [37]

$$-\frac{1}{m}\nabla Q = -\frac{1}{\rho}\partial_j P_{ij}^Q, \tag{28}$$

where
$$P_{ij}^{Q} = -\frac{\hbar^2}{4m^2}\rho\,\partial_i\partial_j \ln\rho = \frac{\hbar^2}{4m^2}\left(\frac{1}{\rho}\partial_i\rho\partial_j\rho - \partial_i\partial_j\rho\right) \tag{29}$$
is an anisotropic quantum pressure tensor. Therefore, the quantum Euler-Lorentz Equation (24) may be rewritten as
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\partial_j P_{ij}^{Q} - \nabla\Phi + \frac{e}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{B}). \tag{30}$$

Remark: According to Equation (2) we have the very general relation
$$\frac{\partial \mathbf{p}}{\partial t} = -\nabla E. \tag{31}$$
We can check that Equation (27) is equivalent to Equation (31) with the energy
$$E = \frac{1}{2}m\mathbf{v}^2 + Q + m\Phi + eU \tag{32}$$
obtained from Equations (2) and (22).

2.4. Nelson's Stochastic Quantum Mechanics

In his stochastic theory of quantum mechanics, Nelson [43] assumed that a quantum particle, like the electron, in empty space (or in the ether) is subject to Brownian motion without friction. When the trajectory $\mathbf{r}(t)$ of a particle is not differentiable, we need a substitute for the velocity. Following Nelson [43], we define the mean forward and mean backward velocities as follows[13]
$$\mathbf{v}_+ = \frac{d_+\mathbf{r}}{dt} = \lim_{\Delta t\to 0^+}\left\langle \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}\right\rangle, \tag{33}$$
$$\mathbf{v}_- = \frac{d_-\mathbf{r}}{dt} = \lim_{\Delta t\to 0^+}\left\langle \frac{\mathbf{r}(t) - \mathbf{r}(t-\Delta t)}{\Delta t}\right\rangle. \tag{34}$$
If $\mathbf{r}(t)$ is differentiable, then $d_+\mathbf{r}/dt = d_-\mathbf{r}/dt = d\mathbf{r}/dt$, but in general d_- is not the same as d_+. Nelson assumed that the motion of the quantum particle is described by a Markov stochastic process in the coordinate space of the form
$$d\mathbf{r}_\pm = \mathbf{v}_\pm\, dt + d\mathbf{b}_\pm, \tag{35}$$
where $d\mathbf{b}_\pm$ is a Gaussian white noise satisfying
$$\langle d\mathbf{b}_\pm\rangle = \mathbf{0}, \qquad \langle db_{\pm i} db_{\pm j}\rangle = \pm 2\mathcal{D}\delta_{ij}dt, \tag{36}$$
where $\mathcal{D}$ is a sort of "quantum diffusion coefficient" measuring the covariance of the noise.

The probability density $\rho(\mathbf{r}, t)$ of finding the particle in $\mathbf{r}$ at time t satisfies the forward Fokker–Planck equation
$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}_+) = \mathcal{D}\Delta\rho \tag{37}$$
and the backward Fokker–Planck equation
$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}_-) = -\mathcal{D}\Delta\rho. \tag{38}$$

Adding these equations, we obtain the continuity equation
$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0, \tag{39}$$

which involves the "classical" velocity $\mathbf{v}$ defined by

$$\mathbf{v} = \frac{\mathbf{v}_+ + \mathbf{v}_-}{2}. \quad (40)$$

Nelson called it the "current" velocity, since $\rho\mathbf{v}$ is a current of probability.

Subtracting Equations (37) and (38), we obtain the equation

$$\nabla \cdot (\rho \mathbf{v}_Q) = \mathcal{D}\Delta\rho, \quad (41)$$

where we have defined the "quantum" velocity by

$$\mathbf{v}_Q = \frac{\mathbf{v}_+ - \mathbf{v}_-}{2}. \quad (42)$$

Equation (41) can be integrated into

$$\mathbf{v}_Q = \mathcal{D}\nabla \ln \rho. \quad (43)$$

This velocity is similar to the one arising in the Einstein theory of Brownian motion [84], so Nelson called it the "osmotic" velocity. We note that $\mathbf{v}_Q$ defines a potential flow, since it is the gradient of a function. Therefore, this flow is irrotational:

$$\nabla \times \mathbf{v}_Q = \mathbf{0}. \quad (44)$$

Remark: Conversely, from the classical (current) and quantum (osmotic) velocities (40) and (42), we obtain

$$\mathbf{v}_+ = \mathbf{v} + \mathbf{v}_Q, \qquad \mathbf{v}_- = \mathbf{v} - \mathbf{v}_Q. \quad (45)$$

2.4.1. Nelson's First Equation

We can rewrite the continuity Equation (39) as

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla\rho = -\rho\nabla \cdot \mathbf{v} \quad (46)$$

or, equivalently, as

$$\frac{\partial}{\partial t} \ln \rho + \mathbf{v} \cdot \nabla \ln \rho = -\nabla \cdot \mathbf{v}. \quad (47)$$

Multiplying this equation by $\mathcal{D}$ and taking its gradient, we obtain

$$\mathcal{D}\frac{\partial}{\partial t}\nabla \ln \rho + \mathcal{D}\nabla(\mathbf{v} \cdot \nabla \ln \rho) = -\mathcal{D}\nabla(\nabla \cdot \mathbf{v}). \quad (48)$$

Using Equation (43), we find that the equation of continuity can be written as

$$\frac{\partial \mathbf{v}_Q}{\partial t} + \nabla(\mathbf{v} \cdot \mathbf{v}_Q) = -\mathcal{D}\nabla(\nabla \cdot \mathbf{v}). \quad (49)$$

This is Nelson's first equation.

2.4.2. Nelson's Second Equation

Let f be a function of $\mathbf{r}$ and t. To compute df/dt, we expand f in a Taylor series up to terms of order two in $d\mathbf{r}$:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \nabla f \cdot \frac{d\mathbf{r}}{dt} + \frac{1}{2}\sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}\frac{dx_i dx_j}{dt}. \quad (50)$$

Taking the average of this expression and using Equations (35) and (36), we obtain

$$\frac{d_\pm f}{dt} = \frac{\partial f}{\partial t} + \mathbf{v}_\pm \cdot \nabla f \pm \mathcal{D}\Delta f. \tag{51}$$

Nelson defined the mean acceleration by

$$\mathbf{a} = \frac{1}{2}\left(\frac{d_+}{dt}\frac{d_-}{dt} + \frac{d_-}{dt}\frac{d_+}{dt}\right)\mathbf{r}. \tag{52}$$

Using Equations (33) and (34), we have

$$\mathbf{a} = \frac{1}{2}\left(\frac{d_+\mathbf{v}_-}{dt} + \frac{d_-\mathbf{v}_+}{dt}\right). \tag{53}$$

If we apply Equation (51) to $\mathbf{v}_\pm$, we find

$$\frac{d_+\mathbf{v}_-}{dt} = \left(\frac{\partial}{\partial t} + \mathbf{v}_+ \cdot \nabla + \mathcal{D}\Delta\right)\mathbf{v}_- \tag{54}$$

and

$$\frac{d_-\mathbf{v}_+}{dt} = \left(\frac{\partial}{\partial t} + \mathbf{v}_- \cdot \nabla - \mathcal{D}\Delta\right)\mathbf{v}_+. \tag{55}$$

Substituting Equations (54) and (55) into Equation (53), and simplifying the resulting expression with Equations (40) and (42), we find that the mean acceleration can be written as

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} - (\mathbf{v}_Q \cdot \nabla)\mathbf{v}_Q - \mathcal{D}\Delta\mathbf{v}_Q. \tag{56}$$

Nelson assumed that the dynamics of a quantum particle is given by Newton's law

$$\mathbf{F} = m\mathbf{a}, \tag{57}$$

where $\mathbf{a}$ denotes the mean acceleration from Equation (56). This yields

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} - (\mathbf{v}_Q \cdot \nabla)\mathbf{v}_Q - \mathcal{D}\Delta\mathbf{v}_Q = \frac{\mathbf{F}}{m}, \tag{58}$$

which is Nelson's second equation. In the electromagnetic case, the Lorentz force is given by

$$\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{59}$$

Therefore, the second Nelson equation becomes

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} - (\mathbf{v}_Q \cdot \nabla)\mathbf{v}_Q - \mathcal{D}\Delta\mathbf{v}_Q = \frac{e}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \nabla\Phi, \tag{60}$$

where we have introduced an additional potential Φ as before. Since $\mathbf{v}_Q$ is irrotational, we have according to Equation (16):

$$(\mathbf{v}_Q \cdot \nabla)\mathbf{v}_Q = \nabla\left(\frac{\mathbf{v}_Q^2}{2}\right), \qquad \Delta\mathbf{v}_Q = \nabla(\nabla \cdot \mathbf{v}_Q). \tag{61}$$

Therefore, Equation (60) can be rewritten as

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla\left(\frac{\mathbf{v}_Q^2}{2}\right) - \mathcal{D}\nabla(\nabla \cdot \mathbf{v}_Q) = \frac{e}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \nabla\Phi. \tag{62}$$

Defining the quantity

$$Q = -m\frac{\mathbf{v}_Q^2}{2} - \mathcal{D}m\nabla\cdot\mathbf{v}_Q, \tag{63}$$

we obtain

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{m}\nabla Q + \frac{e}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B}) - \nabla\Phi. \tag{64}$$

Using Equation (43), we see that Equation (63) represents the quantum potential from Equation (20) provided that we make the identification

$$\mathcal{D} = \frac{\hbar}{2m}. \tag{65}$$

This is called the Nelson relation. Therefore, Equation (64) is the quantum Euler–Lorentz Equation (26).

Remark: According to Equation (63) we have

$$\nabla Q = -m\nabla\left(\frac{\mathbf{v}_Q^2}{2}\right) - \mathcal{D}m\nabla(\nabla\cdot\mathbf{v}_Q). \tag{66}$$

Using Equation (61), we obtain

$$\nabla Q = -m(\mathbf{v}_Q\cdot\nabla)\mathbf{v}_Q - \mathcal{D}m\Delta\mathbf{v}_Q. \tag{67}$$

We also note that the anisotropic quantum pressure tensor from Equation (29) can be written as

$$P_{ij}^Q = -\frac{\hbar}{4m}\rho\left(\partial_i v_j^Q + \partial_j v_i^Q\right), \tag{68}$$

which is similar to the strain rate tensor in hydrodynamics with the quantum velocity $\mathbf{v}_Q$.

2.4.3. Connection between the Nelson, the Madelung and the Schrödinger Equations

The Nelson Equations (49) and (60) are equivalent to the Madelung Equations (18) and (24), representing the continuity equation and the quantum Euler–Lorentz equation. At that stage we have not assumed that $m\mathbf{v} + e\mathbf{A}$ is a gradient. If we now assume that $\mathbf{v}$ can be written as in Equation (13) and proceed as in Section 2.3 but in the reversed direction, we can show that the Madelung Equations (18) and (24) are equivalent to the Schrödinger equation (7). Therefore, the Nelson Equations (49) and (60), together with the assumption from Equation (13), are equivalent to the Schrödinger equation.

Remark: Starting from the Schrödinger equation (7), the Madelung transformation leads to the quantum Hamilton–Jacobi Equation (19), then to the quantum Euler–Lorentz Equation (24), in addition to the continuity Equation (18). Conversely, the Nelson approach leads to the quantum Euler–Lorentz Equation (24), then, assuming Equation (13), to the quantum Hamilton–Jacobi Equation (19), in addition to the continuity Equation (18).

2.4.4. Other Forms of Nelson Equations Assuming Equation (13)

If we assume Equation (13), we can write the Nelson equations in different forms.

(i) *Nelson's first equation:* Using Equation (17), we can rewrite Equation (49) as

$$\frac{\partial\mathbf{v}_Q}{\partial t} + \nabla(\mathbf{v}\cdot\mathbf{v}_Q) = -\mathcal{D}\Delta\mathbf{v} + \mathcal{D}\frac{e}{m}\nabla\times\mathbf{B}. \tag{69}$$

Recalling the general identity

$$\nabla(\mathbf{v}\cdot\mathbf{v}_Q) = (\mathbf{v}\cdot\nabla)\mathbf{v}_Q + (\mathbf{v}_Q\cdot\nabla)\mathbf{v} + \mathbf{v}\times(\nabla\times\mathbf{v}_Q) + \mathbf{v}_Q\times(\nabla\times\mathbf{v}) \tag{70}$$

and using Equations (15) and (44), we obtain

$$\nabla(\mathbf{v}\cdot\mathbf{v}_Q) = (\mathbf{v}\cdot\nabla)\mathbf{v}_Q + (\mathbf{v}_Q\cdot\nabla)\mathbf{v} - \frac{e}{m}\mathbf{v}_Q\times\mathbf{B}. \tag{71}$$

Therefore, Equations (49) and (69) can be rewritten as

$$\frac{\partial \mathbf{v}_Q}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}_Q + (\mathbf{v}_Q\cdot\nabla)\mathbf{v} - \frac{e}{m}\mathbf{v}_Q\times\mathbf{B} + \mathcal{D}\nabla(\nabla\cdot\mathbf{v}) = \mathbf{0} \tag{72}$$

and

$$\frac{\partial \mathbf{v}_Q}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}_Q + (\mathbf{v}_Q\cdot\nabla)\mathbf{v} - \frac{e}{m}\mathbf{v}_Q\times\mathbf{B} + \mathcal{D}\Delta\mathbf{v} - \mathcal{D}\frac{e}{m}\nabla\times\mathbf{B} = \mathbf{0}. \tag{73}$$

(ii) *Nelson's second equation:* Using Equation (17), we can rewrite Equation (62) as

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla\left(\frac{\mathbf{v}^2}{2}\right) - \nabla\left(\frac{\mathbf{v}_Q^2}{2}\right) - \mathcal{D}\Delta\mathbf{v}_Q = \frac{e}{m}\mathbf{E} - \nabla\Phi. \tag{74}$$

On the other hand, using Equation (15), we can rewrite Equation (60) as

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} - (\mathbf{v}_Q\cdot\nabla)\mathbf{v}_Q - \mathcal{D}\Delta\mathbf{v}_Q = \frac{e}{m}\mathbf{E} - \mathbf{v}\times(\nabla\times\mathbf{v}) - \nabla\Phi. \tag{75}$$

In this manner, the magnetic field $\mathbf{B}$ does not explicitly appear in the equation of motion.

2.5. Nottale's Theory of Scale Relativity

2.5.1. Complex Lorentz Equation

In his theory of scale relativity, Nottale [55] assumed that a quantum particle has a nondifferentiable motion as a result of the fractal structure of spacetime. This motion can be represented by Equations (35) and (36). However, the physical interpretation of these equations is different from the stochastic interpretation given by Nelson [43], in which these equations are considered to be the result of a classical diffusion process by, e.g., a subquantum medium (see Section 7.1). In Nottale's approach, the stochastic (nondifferentiable) part $d\mathbf{b}_\pm$ is a scale-dependent fluctuation which arises from the fractal structure of spacetime. Furthermore, in order to derive the Schrödinger equation, Nottale proceeded differently from Nelson. With the two velocities $\mathbf{v}_+$ and $\mathbf{v}_-$, or equivalently with $\mathbf{v}$ and $\mathbf{v}_Q$, Nottale formed a complex velocity

$$\mathbf{V} = \frac{\mathbf{v}_+ + \mathbf{v}_-}{2} - i\frac{\mathbf{v}_+ - \mathbf{v}_-}{2} = \mathbf{v} - i\mathbf{v}_Q. \tag{76}$$

He also defined a complex derivative operator

$$\frac{D}{Dt} = \frac{d_+ + d_-}{2dt} - i\frac{d_+ - d_-}{2dt} \tag{77}$$

in terms of which [see Equations (33), (34) and (76)]

$$\frac{D\mathbf{r}}{Dt} = \mathbf{V}. \tag{78}$$

Substituting Equation (51) into Equation (77), he obtained the expression of the complex time derivative operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla - i\mathcal{D}\Delta. \tag{79}$$

The complex acceleration is then given by

$$\frac{D\mathbf{V}}{Dt} = \frac{d_+ + d_-}{2dt}\mathbf{V} - i\frac{d_+ - d_-}{2dt}\mathbf{V} \tag{80}$$

or, using Equation (79), by

$$\frac{D\mathbf{V}}{Dt} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} - i\mathcal{D}\Delta\mathbf{V}. \tag{81}$$

Now, the fundamental postulate of Nottale's theory of scale relativity is that the equations of quantum mechanics (non-differentiable trajectories) can be obtained from the equations of classical mechanics (differentiable trajectories) by replacing the standard momentum $\mathbf{p}$ with the complex momentum $\mathbf{P}$ and the standard time derivative d/dt with the complex time derivative D/Dt. In other words, D/Dt plays the role of a "covariant derivative operator", in terms of which the fundamental equations of physics keep the same form in the classical and quantum regimes. This is similar to the principle of covariance in Einstein's theory of relativity, according to which the form of the equations of physics should be conserved under all transformations of the coordinates. We will comment later on the limitations of Nottale's scale covariance principle (see Section 7.2). In particular, there is only a *weak* covariance, in the sense that Nottale's covariance principle does not apply to all the equations of physics, contrary to Einstein's covariance principle in general relativity. We argue that Nottale's covariance principle (in its simplest formulation) only applies to the equations of motion expressed in terms of the momentum $\mathbf{p}$, not in terms of the velocity $\mathbf{v}$. To our knowledge, this important remark has not been made before.

Using the principle of scale relativity covariance for a quantum particle in an electromagnetic field, we obtain the complex Lorentz equation

$$\frac{D\mathbf{P}}{Dt} = -m\nabla\Phi - e\nabla U + e(\mathbf{V} \cdot \nabla)\mathbf{A} + e\mathbf{V} \times (\nabla \times \mathbf{A}), \tag{82}$$

which generalizes the classical Lorentz Equation (5) written in terms of the momentum $\mathbf{p}$. Equations (5) and (82) have the same form under the substitutions $\mathbf{p} \to \mathbf{P}$ and $d/dt \to D/Dt$. We will see that Equation (82) is equivalent to the Schrödinger Equation (7). Using $\mathbf{P} = m\mathbf{V} + e\mathbf{A}$ (see Section 2.5.2) and Equation (A65), and recalling the Nelson relation (65), Equation (82) can be written in terms of the complex velocity $\mathbf{V}$ as

$$m\frac{D\mathbf{V}}{Dt} = -m\nabla\Phi + e(\mathbf{E} + \mathbf{V} \times \mathbf{B}) - \frac{ie\hbar}{2m}\nabla \times \mathbf{B}. \tag{83}$$

We note that Equation (83) does *not* have the same form as Equation (4) under the substitutions $\mathbf{v} \to \mathbf{V}$ and $d/dt \to D/Dt$. Indeed, an extra term proportional to $\hbar$ appears in Equation (83).[14] This is because the left-hand side of Equation (83) involves the derivative of the momentum without electromagnetic field $\mathbf{P}_{\rm kin} = m\mathbf{V}$, not the derivative of the total momentum $\mathbf{P} = m\mathbf{V} + e\mathbf{A}$ as in Equation (82). This shows that the principle of scale covariance applies to the equation of motion written in terms of the momentum $\mathbf{p}$, not to the equation of motion written in terms of the velocity $\mathbf{v}$. Expanding the derivative by using Equation (81), we can rewrite Equation (83) as

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = \frac{i\hbar}{2m}\Delta\mathbf{V} - \nabla\Phi + \frac{e}{m}(\mathbf{E} + \mathbf{V} \times \mathbf{B}) - \frac{ie\hbar}{2m^2}\nabla \times \mathbf{B}. \tag{84}$$

This equation is similar to the viscous Burgers equation of fluid mechanics (with electromagnetic terms included), except that in the present case the velocity field $\mathbf{V}(\mathbf{r}, t)$ is complex and the viscosity $\nu = i\hbar/2m$ is imaginary. Therefore, quantum mechanics may be interpreted as a generalized hydrodynamics involving a complex velocity field and an imaginary viscosity [73].

2.5.2. Derivation of the Schrödinger Equation

Following Nottale, we assume that the relations of classical mechanics relating the energy and the momentum to the action remain valid in the theory of scale relativity for the complex variables. Therefore, taking into account the specificities of the presence of an electromagnetic field, we introduce the complex energy and the complex momentum

$$\mathcal{E} = \mathcal{E}_{\rm kin} + eU, \qquad \mathbf{P} = \mathbf{P}_{\rm kin} + e\mathbf{A} = m\mathbf{V} + e\mathbf{A}, \tag{85}$$

where $\mathcal{E}_{\rm kin}$ and $\mathbf{P}_{\rm kin} = m\mathbf{V}$ are the complex energy and momentum in the absence of electromagnetic field (i.e., for a free particle), and we assume that

$$\mathcal{E} = -\frac{\partial \mathcal{S}}{\partial t}, \qquad \mathbf{P} = \nabla \mathcal{S}, \tag{86}$$

where $\mathcal{S}$ is the complex action. The complex velocity field can be written as

$$\mathbf{V} = \frac{\nabla \mathcal{S} - e\mathbf{A}}{m}. \tag{87}$$

This is the complex generalization of Equation (13). We note that

$$\nabla \times \mathbf{V} = -\frac{e}{m}\nabla \times \mathbf{A} = -\frac{e}{m}\mathbf{B}, \tag{88}$$

where we have used Equation (A65) to obtain the second equality. In the absence of magnetic field, the complex velocity field is irrotational ($\nabla \times \mathbf{V} = \mathbf{0}$). Using Equation (88), the well-known identities of fluid mechanics $(\mathbf{V} \cdot \nabla)\mathbf{V} = \nabla(\mathbf{V}^2/2) - \mathbf{V} \times (\nabla \times \mathbf{V})$ and $\Delta \mathbf{V} = \nabla(\nabla \cdot \mathbf{V}) - \nabla \times (\nabla \times \mathbf{V})$, here written for complex variables, reduce to

$$(\mathbf{V} \cdot \nabla)\mathbf{V} = \nabla\left(\frac{\mathbf{V}^2}{2}\right) + \frac{e}{m}\mathbf{V} \times \mathbf{B}, \qquad \Delta \mathbf{V} = \nabla(\nabla \cdot \mathbf{V}) + \frac{e}{m}\nabla \times \mathbf{B}. \tag{89}$$

The complex Lorentz Equation (84) can then be rewritten as

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} = \frac{i\hbar}{2m}\nabla(\nabla \cdot \mathbf{V}) - \nabla \Phi + \frac{e}{m}(\mathbf{E} + \mathbf{V} \times \mathbf{B}) \tag{90}$$

or, equivalently, as

$$\frac{\partial \mathbf{V}}{\partial t} + \nabla\left(\frac{\mathbf{V}^2}{2}\right) = \frac{i\hbar}{2m}\nabla(\nabla \cdot \mathbf{V}) - \nabla \Phi + \frac{e}{m}\mathbf{E}. \tag{91}$$

We note that the magnetic field $\mathbf{B}$ does not explicitly appear in Equation (91). Using Equations (87) and (A65), and integrating the foregoing equation, we obtain the complex Hamilton–Jacobi equation[15]

$$\frac{\partial \mathcal{S}}{\partial t} + \frac{1}{2m}(\nabla \mathcal{S} - e\mathbf{A})^2 + m\Phi + eU = \frac{i\hbar}{2m}\Delta \mathcal{S}. \tag{92}$$

In the classical limit $\hbar = 0$, we recover Equation (3). We note that the Hamilton–Jacobi equation is not scale-covariant. We cannot simply replace the real quantities in the classical Hamilton–Jacobi Equation (3) with their complex counterpart (see Section 7.2). The complex Hamilton–Jacobi Equation (92) contains an additional term proportional to $\hbar$.

Equation (92) can be viewed as a Hamilton–Jacobi equation for a complex action, or as a Bernoulli equation for a complex potential. Using the analogy with fluid mechanics, it is natural to introduce a function $\psi(\mathbf{r}, t)$ through the complex Cole–Hopf transformation

$$\mathcal{S} = -2im\mathcal{D} \ln \psi. \tag{93}$$

Using the expression of the quantum diffusion coefficient from Equation (65), we obtain

$$\mathcal{S} = -i\hbar \ln \psi. \tag{94}$$

We see that $\psi(\mathbf{r},t)$ represents the wavefunction of quantum mechanics. Indeed, Equation (94) can be rewritten in the standard form

$$\psi = e^{i\mathcal{S}/\hbar} \tag{95}$$

relating the wavefunction to the complex action.[16] Substituting Equation (94) into Equation (92), and using the identity (21), we immediately obtain the Schrödinger Equation (7). Our derivation of the Schrödinger equation (see also [73]) relying on the analogy with fluid mechanics and using the well-known identities of fluid mechanics is substantially simpler than the one given by Nottale [55].[17] We note that the present derivation does not require us to introduce the density ρ, contrary to the approach of Nelson (see Section 7.1). This is similar to the historical approach of Schrödinger [1–4]. He only worked in terms of the wave function ψ. It is only in the second time, in order to give a physical meaning to ψ, that the density ρ was introduced. Schrödinger [4] related $\rho_e = e|\psi|^2$ to the charge density, while Born [17–20] interpreted $\rho = |\psi|^2$ as the probability density of finding the quantum particle in $\mathbf{r}$ at time t.

Remark: Using Equation (94) the complex velocity field $\mathbf{V}$ defined by Equation (87) can be written as

$$\mathbf{V} = -i\frac{\hbar}{m}\nabla \ln \psi - \frac{e}{m}\mathbf{A}. \tag{96}$$

Remark: According to Equation (86), we have the very general relation

$$\frac{\partial \mathbf{P}}{\partial t} = -\nabla \mathcal{E}. \tag{97}$$

We can check that Equation (84) is equivalent to Equation (97) with the complex energy

$$\mathcal{E} = \frac{1}{2}m\mathbf{V}^2 + m\Phi + eU - \frac{i\hbar}{2}\nabla \cdot \mathbf{V} \tag{98}$$

obtained from Equations (86) and (92).

Remark: Using Equation (89), we can rewrite the complex Lorentz Equation (84) as

$$\frac{\tilde{D}\mathbf{V}}{\tilde{D}t} = \frac{e}{m}(\mathbf{E} + \mathbf{V} \times \mathbf{B}) - \nabla\Phi \tag{99}$$

with the new derivative

$$\frac{\tilde{D}\mathbf{V}}{\tilde{D}t} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V} - i\mathcal{D}\nabla(\nabla \cdot \mathbf{V}). \tag{100}$$

In this manner, the extra term in Equation (84) disappears. However, this transformation is a bit artificial, since only the scale covariant derivative from Equation (79) is physically justified.

2.6. Connection between Nottale and Nelson's Theories

2.6.1. Complex Action

We have seen that Nottale's theory leads to the Schrödinger Equation (7). Therefore, if we write the wave function as

$$\psi = \sqrt{\rho}e^{iS/\hbar}, \tag{101}$$

where S is the (real) action and ρ is the density, we obtain the Madelung equations of Section 2.3. Comparing Equation (101) with Equation (95), we see that the complex action is related to the real action and to the density by

$$\mathcal{S} = S - iS_Q \quad \text{with} \quad S_Q = \frac{\hbar}{2}\ln \rho. \tag{102}$$

We note that $\rho = e^{2S_Q/\hbar}$. The complex velocity field $\mathbf{V}$ from Equation (87) can then be written as in Equation (76) with the classical velocity

$$\mathbf{v} = \frac{\nabla S - e\mathbf{A}}{m} \tag{103}$$

and the quantum velocity

$$\mathbf{v}_Q = \frac{\nabla S_Q}{m} = \frac{\hbar}{2m}\nabla \ln \rho. \tag{104}$$

They correspond to the current and osmotic velocities of Nelson (see Section 2.4). We note that Nelson obtained the expression (43) of the osmotic velocity directly from the forward and backward Fokker–Planck Equations (37) and (38) that are not used in Nottale's theory. We also note that the complex velocity from Equations (76) and (96), combining the classical velocity from Equation (103) and the quantum velocity from Equation (104), was first introduced by Madelung in a not well-known paper [85]. It also appeared in Refs. [86,87].

Below, we take the real and imaginary parts of the complex Lorentz Equation (83) and make the connection with the first and second Nelson's Equations (49) and (60).

Remark: According to Equations (86) and (102), we also have

$$\mathbf{P} = \mathbf{p} - i\mathbf{p}_Q, \qquad \mathcal{E} = E - iE_Q, \tag{105}$$

with

$$\mathbf{p} = \nabla S, \qquad \mathbf{p}_Q = \nabla S_Q = \frac{\hbar}{2}\nabla \ln \rho, \tag{106}$$

$$E = -\frac{\partial S}{\partial t}, \qquad E_Q = -\frac{\partial S_Q}{\partial t} = -\frac{\hbar}{2}\frac{\partial \ln \rho}{\partial t}. \tag{107}$$

2.6.2. Real Part of the Complex Lorentz Equation

We can determine the real part of $D\mathbf{V}/Dt$ in two different manners:

(i) Using Equations (45), (76) and (80), we obtain

$$\begin{aligned} \mathrm{Re}\left(\frac{D\mathbf{V}}{Dt}\right) &= \frac{d_+ + d_-}{2dt}\mathbf{v} - \frac{d_+ - d_-}{2dt}\mathbf{v}_Q \\ &= \frac{1}{2}\left[\frac{d_+}{dt}(\mathbf{v}-\mathbf{v}_Q) + \frac{d_-}{dt}(\mathbf{v}+\mathbf{v}_Q)\right] \\ &= \frac{1}{2}\left(\frac{d_+\mathbf{v}_-}{dt} + \frac{d_-\mathbf{v}_+}{dt}\right). \end{aligned} \tag{108}$$

(ii) Using Equations (76) and (81), we find that

$$\mathrm{Re}\left(\frac{D\mathbf{V}}{Dt}\right) = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} - (\mathbf{v}_Q\cdot\nabla)\mathbf{v}_Q - \mathcal{D}\Delta\mathbf{v}_Q. \tag{109}$$

Comparing these results with Equations (53) and (56), we see that the real part of the complex acceleration defined by Nottale coincides with the mean acceleration defined by Nelson:

$$\mathrm{Re}\left(\frac{D\mathbf{V}}{Dt}\right) = \mathbf{a}. \tag{110}$$

On the other hand, the real part of the complex Lorentz Equation (83) reads

$$\mathrm{Re}\left(\frac{D\mathbf{V}}{Dt}\right) = -\nabla\Phi + \frac{e}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{B}). \tag{111}$$

Together with Equation (109), we obtain

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} - (\mathbf{v}_Q\cdot\nabla)\mathbf{v}_Q - \mathcal{D}\Delta\mathbf{v}_Q = -\nabla\Phi + \frac{e}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{B}). \tag{112}$$

Therefore, the real part of the complex Lorentz Equation (83) is equivalent to Nelson's second Equation (60), which is itself equivalent to the quantum Euler–Lorentz equation (24) of Madelung.

Remark: In a sense, the formalism of Nottale [55] justifies the expression of the mean acceleration **a** that Nelson [43] introduced in a rather ad hoc manner.

2.6.3. Imaginary Part of the Complex Lorentz Equation

We can determine the imaginary part of $D\mathbf{V}/Dt$ in two different manners:

(i) Using Equations (45), (76) and (80), we obtain

$$\begin{aligned}\mathrm{Im}\left(\frac{D\mathbf{V}}{Dt}\right) &= -\frac{d_+ + d_-}{2dt}\mathbf{v}_Q - \frac{d_+ - d_-}{2dt}\mathbf{v} \\ &= -\frac{1}{2}\left[\frac{d_+}{dt}(\mathbf{v}_Q + \mathbf{v}) + \frac{d_-}{dt}(\mathbf{v}_Q - \mathbf{v})\right] \\ &= -\frac{1}{2}\left(\frac{d_+\mathbf{v}_+}{dt} - \frac{d_-\mathbf{v}_-}{dt}\right).\end{aligned} \tag{113}$$

(ii) Using Equations (76) and (81), we find that

$$\mathrm{Im}\left(\frac{D\mathbf{V}}{Dt}\right) = -\frac{\partial \mathbf{v}_Q}{\partial t} - (\mathbf{v}\cdot\nabla)\mathbf{v}_Q - (\mathbf{v}_Q\cdot\nabla)\mathbf{v} - \mathcal{D}\Delta\mathbf{v}. \tag{114}$$

The imaginary part of the complex Lorentz Equation (83) reads

$$\mathrm{Im}\left(\frac{D\mathbf{V}}{Dt}\right) = -\frac{e}{m}(\mathbf{v}_Q\times\mathbf{B}) - \frac{e\hbar}{2m^2}\nabla\times\mathbf{B}. \tag{115}$$

Together with Equation (114), we obtain

$$\frac{\partial \mathbf{v}_Q}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}_Q + (\mathbf{v}_Q\cdot\nabla)\mathbf{v} + \mathcal{D}\Delta\mathbf{v} = \frac{e}{m}\mathbf{v}_Q\times\mathbf{B} + \frac{e\hbar}{2m^2}\nabla\times\mathbf{B}. \tag{116}$$

If we assume that **v** is given by Equation (103), then by using the identities (17) and (71), the foregoing equation can be rewritten as

$$\frac{\partial \mathbf{v}_Q}{\partial t} + \nabla(\mathbf{v}\cdot\mathbf{v}_Q) = -\mathcal{D}\nabla(\nabla\cdot\mathbf{v}). \tag{117}$$

Therefore, the imaginary part of the complex Lorentz Equation (83) is equivalent to Nelson's first Equation (49), which is itself equivalent to the equation of continuity (18) of Madelung. It is interesting to note that the continuity equation is contained in the complex Lorentz Equation (83) of Nottale. This assumes, however, that **v** is given by Equation (103). By contrast, in Nelson's approach, the continuity equation is obtained from the start and does not rely on the fact that **v** is given by Equation (103). It results from the forward and backward Fokker–Planck Equations (37) and (38) that are not used in Nottale's theory.

Remark: In the absence of magnetic field, the imaginary part of the complex Lorentz Equation (83) reduces to

$$\mathrm{Im}\left(\frac{D\mathbf{V}}{Dt}\right) = 0. \tag{118}$$

Together with Equation (113), we obtain

$$\frac{d_+\mathbf{v}_+}{dt} = \frac{d_-\mathbf{v}_-}{dt}. \tag{119}$$

This is another formulation of the equation of continuity in the absence of magnetic field (i.e., for a free particle).

2.6.4. Connection between Different Equations

In conclusion, the complex Lorentz Equation (83) of Nottale is equivalent to the Nelson Equations (49) and (60), which are themselves equivalent to the Madelung Equations (18) and (24), representing the continuity equation and the quantum Euler–Lorentz equation. These equations are also equivalent to the Schrödinger Equation (7). We note, however, that the physical interpretation of these equations and the methods of derivation are substantially different (see Section 7.1 for a detailed discussion). For example, the theory of Nelson relies on the forward and backward Fokker–Planck Equations (37) and (38), while these equations are not used in Nottale's theory.[18] On the other hand, it is necessary in some cases to assume that the velocity **v** is given by Equation (103), while this assumption is not necessary in other cases.

Remark: As a corollary of the above results, if we take the real and imaginary parts of the complex Hamilton–Jacobi Equation (92) and use Equation (102), we obtain the quantum Hamilton–Jacobi Equation (19) and the continuity Equation (18).

3. Relativistic Theory

3.1. Basics of Classical Mechanics

In classical mechanics, the quadrimomentum p^μ of a relativistic particle of mass m and charge e in the presence of an electromagnetic field can be written as $p_\mu = p_\mu^{\rm kin} + eA_\mu = mu_\mu + eA_\mu$, where $p_\mu^{\rm kin} = mu_\mu$ is the quadrimomentum of the particle in the absence of magnetic field (free particle).[19] Using the fundamental relation

$$u_\mu u^\mu = c^2, \tag{120}$$

we find that

$$(p_\mu - eA_\mu)(p^\mu - eA^\mu) = m^2c^2. \tag{121}$$

Recalling the relation

$$p_\mu = -\partial_\mu S \tag{122}$$

between the quadrimomentum and the action S, we obtain the classical Hamilton–Jacobi equation

$$(\partial_\mu S + eA_\mu)(\partial^\mu S + eA^\mu) = m^2c^2. \tag{123}$$

Taking the gradient of Equation (120) and introducing the Faraday tensor $F_{\mu\nu}$ defined by Equation (A64), we obtain the Lorentz equation[20]

$$\frac{du_\nu}{d\tau} \equiv u^\mu\partial_\mu u_\nu = -\frac{e}{m}u^\mu F_{\mu\nu}. \tag{124}$$

It can be rewritten as

$$u^\mu\partial_\nu u_\mu = 0 \tag{125}$$

or, equivalently, as

$$u^\mu\left(m\partial_\mu u_\nu + eF_{\mu\nu}\right) = 0. \tag{126}$$

If we write the components of these equations, we can see [9] that Equation (124) corresponds to the Lorentz equation written in terms of the quadrivelocity u^μ, while Equation (126) corresponds to the Lorentz equation written in terms of the quadrimomentum $p^\mu = mu^\mu + eA^\mu$ (the term in parenthesis in Equation (126) can be seen as a form of symmetrized gradient of the quadrimomentum). In this respect, Equations (124) and (126) are the counterparts of Equations (4) and (5) in the nonrelativistic case. This remark will be important in the sequel.

3.2. Klein–Gordon Equation

The wave equation of relativistic quantum mechanics for particles of spin zero (bosons) can be obtained from the classical relation (121) by using the operator prescription

$$p_\mu \to i\hbar\partial_\mu. \tag{127}$$

This leads to the KG equation

$$\left(\partial_\mu + i\frac{e}{\hbar}A_\mu\right)\left(\partial^\mu + i\frac{e}{\hbar}A^\mu\right)\varphi + \frac{m^2c^2}{\hbar^2}\varphi = 0. \tag{128}$$

It can be written as

$$\Box_e\varphi + \frac{m^2c^2}{\hbar^2}\varphi = 0, \tag{129}$$

where $\Box_e = \nabla_\mu\nabla^\mu$ is the d'Alembertian constructed with the gauge covariant derivative $\nabla_\mu = \partial_\mu + i\frac{e}{\hbar}A_\mu$. Expanding the operator in parenthesis, we obtain

$$\partial_\mu\partial^\mu\varphi + i\frac{e}{\hbar}(\partial_\mu A^\mu)\varphi + 2i\frac{e}{\hbar}A_\mu\partial^\mu\varphi - \frac{e^2}{\hbar^2}A_\mu A^\mu\varphi + \frac{m^2c^2}{\hbar^2}\varphi = 0. \tag{130}$$

We note that the second term in the left-hand side disappears if we choose the Lorentz gauge from Equation (A75), a choice that we will make in the following.

Multiplying the KG Equation (128) by φ^* and subtracting its complex conjugate, we can easily derive the identity

$$\partial_\mu J^\mu = 0, \tag{131}$$

where

$$J_\mu = -\frac{m}{2i\hbar}(\varphi^*\partial_\mu\varphi - \varphi\partial_\mu\varphi^*) - \frac{me}{\hbar^2}|\varphi|^2 A_\mu \tag{132}$$

is a current. Since the density J^0/c is not definitely positive, it cannot be interpreted as a probability density like in Born's approach.[21] However, following Schrödinger [4], Gordon [6] and Klein [7] interpreted $\rho_e = eJ^0/mc$ as a charge density associated with a quadricurrent of charge $J_e^\mu = eJ^\mu/m$. Therefore, Equation (131) expresses the local conservation of the charge. In a relativistic BEC, $\rho_m = mn = J^0/c$ can also be interpreted as the rest-mass density of the bosons, provided that anti-bosons are counted negatively [89].

Remark: The KG equation with an electromagnetic field can also be obtained from the KG equation without an electromagnetic field by making the substitution $\partial_\mu \to \partial_\mu + i\frac{e}{\hbar}A_\mu$, corresponding to the transformation $p_\mu \to p_\mu - eA_\mu$ in classical mechanics with the operator prescription from Equation (127).

3.3. De Broglie Transformation

Making the de Broglie [31–33] transformation, the KG Equation (128) can be written in the form of hydrodynamic equations. For that purpose, we write the wavefuction as

$$\varphi(x^\mu) = \frac{\hbar}{m}\sqrt{\rho(x^\mu)}e^{iS(x^\mu)/\hbar}, \tag{133}$$

where $\rho(x^\mu)$ is the pseudo rest-mass density and $S(x^\mu)$ is the action. These quantities can be expressed in terms of $\varphi(x^\mu)$ as

$$\rho = \frac{m^2}{\hbar^2}|\varphi|^2, \qquad S = \frac{\hbar}{2i}\ln\left(\frac{\varphi}{\varphi^*}\right). \tag{134}$$

Following de Broglie [31–33], we introduce the quadrivelocity

$$u_\mu = -\frac{\partial_\mu S + eA_\mu}{m}. \tag{135}$$

This definition is consistent with the classical expression $p_\mu \equiv mu_\mu + eA_\mu = -\partial_\mu S$ of the generalized quadrimomentum of a particle in an electromagnetic field. Taking the gradient of the action in Equation (134), the quadrivelocity can be expressed in terms of φ as

$$u_\mu = -\frac{\hbar}{2im}\frac{\varphi^*\partial_\mu\varphi - \varphi\partial_\mu\varphi^*}{|\varphi|^2} - \frac{e}{m}A_\mu. \tag{136}$$

The electromagnetic field tensor, or Faraday tensor, is defined by Equation (A64). From Equations (135) and (A64), we obtain

$$\partial_\mu u_\nu - \partial_\nu u_\mu = -\frac{e}{m}F_{\mu\nu}. \tag{137}$$

This relation shows that the electromagnetic field creates a vorticity field. In the absence of magnetic field, the velocity field is irrotational ($\partial_\mu u_\nu = \partial_\nu u_\mu$). Using Equation (137), we find that

$$u_\mu\partial_\mu u_\nu = u_\mu\partial_\nu u_\mu - \frac{e}{m}u_\mu F_{\mu\nu}, \qquad \Box u_\nu = \partial_\nu\partial^\mu u_\mu - \frac{e}{m}\partial^\mu F_{\mu\nu}. \tag{138}$$

Substituting Equation (133) into the KG equation (130), separating the real and imaginary parts, and using Equation (135), we obtain the pair of equations

$$\partial_\mu(\rho u^\mu) = 0, \tag{139}$$

$$(\partial_\mu S + eA_\mu)(\partial^\mu S + eA^\mu) - 2mQ - m^2c^2 = 0, \tag{140}$$

where

$$Q = \frac{\hbar^2}{2m}\frac{\Box\sqrt{\rho}}{\sqrt{\rho}} = \frac{\hbar^2}{4m}\left[\frac{\Box\rho}{\rho} - \frac{1}{2\rho^2}\partial_\mu\rho\partial^\mu\rho\right] = \frac{\hbar^2}{4m}\Box\ln\rho + \frac{\hbar^2}{8m}\partial_\mu\ln\rho\partial^\mu\ln\rho \tag{141}$$

is the relativistic covariant quantum potential. To obtain the last equality, we have used the identity

$$\Box(\ln f) = \frac{\Box f}{f} - \frac{1}{f^2}\partial_\mu f\partial^\mu f. \tag{142}$$

Using Equation (135), we can rewrite Equation (140) as

$$u_\mu u^\mu = 2\frac{Q}{m} + c^2. \tag{143}$$

Taking the gradient of Equation (143), we find that

$$u^\mu\partial_\nu u_\mu = \frac{1}{m}\partial_\nu Q. \tag{144}$$

We note that the Faraday tensor does not appear explicitly in this equation. Using the relation from Equation (137), we obtain

$$u^\mu\left(m\partial_\mu u_\nu + eF_{\mu\nu}\right) = \partial_\nu Q. \tag{145}$$

In the classical limit $\hbar \to 0$, we recover Equations (120), (125) and (126). Introducing the material derivative

$$\frac{d}{d\tau} = u^\mu\partial_\mu, \tag{146}$$

we can rewrite Equation (145) as

$$\frac{du_\nu}{d\tau} \equiv u^\mu\partial_\mu u_\nu = \frac{1}{m}\partial_\nu Q - \frac{e}{m}u^\mu F_{\mu\nu}. \tag{147}$$

The quantum hydrodynamic Equations (139)–(147) have a clear physical interpretation. Equation (139), corresponding to the imaginary part of the KG Equation (130), is the continuity equation.[22] It accounts for the local charge (or rest mass) conservation. Equation (140), corresponding to the real part of the KG Equation (130), is the quantum relativistic Hamilton–Jacobi (or Bernoulli) equation with a relativistic covariant quantum potential. When $\hbar = 0$ we recover the classical Hamilton–Jacobi (or Bernoulli) Equation (123). Equation (147) is the quantum relativistic Euler–Lorentz equation. The first term on the right-hand side is the quantum force $\partial_\mu Q$ and the second term is the Lorentz force $F^\mu = eF^{\mu\nu}u_\nu$. In the classical limit $\hbar = 0$, we recover the Lorentz Equation (124). The KG Equation (130) is Equivalent[23] to the de Broglie hydrodynamic Equations (139)–(147) composed of the continuity Equation (139) and the quantum Euler–Lorentz Equation (147).

3.4. Nelson's Stochastic Quantum Mechanics

We now turn to Nelson's stochastic theory of quantum mechanics. Most elements of the nonrelativistic approach remain valid in the relativistic case, with the time t replaced by the proper time τ.[24] When the trajectory $x^\mu(\tau)$ of a particle is not differentiable, we define the mean forward and mean backward quadrivelocities by

$$u^\mu_+ = \frac{d_+x^\mu}{d\tau} = \lim_{\Delta\tau\to 0^+}\left\langle\frac{x^\mu(\tau+\Delta\tau)-x^\mu(\tau)}{\Delta\tau}\right\rangle, \tag{148}$$

$$u^\mu_- = \frac{d_-x^\mu}{d\tau} = \lim_{\Delta\tau\to 0^+}\left\langle\frac{x^\mu(\tau)-x^\mu(\tau-\Delta\tau)}{\Delta\tau}\right\rangle. \tag{149}$$

If $x^\mu(\tau)$ is differentiable, then $d_+x^\mu/d\tau = d_-x^\mu/d\tau$, but in general d_- is not the same as d_+.[25] Following Nelson, we assume that the motion of the quantum particle is described by a Markov stochastic process in coordinate space, defined by

$$dx^\mu_\pm = u^\mu_\pm d\tau + db^\mu_\pm, \tag{150}$$

where $d\mathbf{b}_\pm$ is a Gaussian white noise with

$$\langle db^\mu_\pm\rangle = 0, \qquad \langle db^\mu_\pm db^\nu_\pm\rangle = \mp 2\mathcal{D}\eta^{\mu\nu}d\tau, \tag{151}$$

where $\mathcal{D}$ is the quantum diffusion coefficient measuring the covariance of the noise.

The probability density $\rho(x^\mu)$ satisfies the forward Fokker–Planck equation

$$\partial_\mu(\rho u^\mu_+) = -\mathcal{D}\Box\rho \tag{152}$$

and the backward Fokker–Planck equation

$$\partial_\mu(\rho u^\mu_-) = \mathcal{D}\Box\rho. \tag{153}$$

Adding these equations, we obtain the continuity equation

$$\partial_\mu(\rho u^\mu) = 0, \tag{154}$$

which involves the "classical" quadrivelocity u^μ defined by

$$u^\mu = \frac{u_+^\mu + u_-^\mu}{2}. \tag{155}$$

Following Nelson, we call it the "current" quadrivelocity.

Subtracting Equations (152) and (153), we obtain the equation

$$\partial_\mu(\rho u_Q^\mu) = -\mathcal{D}\Box\rho, \tag{156}$$

where we have defined the "quantum" quadrivelocity by

$$u_Q^\mu = \frac{u_+^\mu - u_-^\mu}{2}. \tag{157}$$

Equation (156) can be integrated into

$$u_\mu^Q = -\mathcal{D}\partial_\mu \ln\rho. \tag{158}$$

Following Nelson, we call it the "osmotic" quadrivelocity. Since it is a gradient, it satisfies the condition

$$\partial_\mu u_\nu^Q = \partial_\nu u_\mu^Q, \tag{159}$$

meaning that the quantum quadrivelocity is irrotational.

Remark: Conversely, from the classical (current) and quantum (osmotic) quadrivelocities (155) and (157), we obtain

$$u_+^\mu = u^\mu + u_Q^\mu, \qquad u_-^\mu = u^\mu - u_Q^\mu. \tag{160}$$

3.4.1. Nelson's First Equation

We can rewrite the continuity equation (154) as

$$-u^\mu\partial_\mu\rho = \rho\partial_\mu u^\mu \tag{161}$$

or, equivalently, as

$$-u^\mu\partial_\mu \ln\rho = \partial_\mu u^\mu. \tag{162}$$

Multiplying this equation by $\mathcal{D}$ and using Equation (158), we obtain

$$u^\mu u_\mu^Q = \mathcal{D}\partial_\mu u^\mu. \tag{163}$$

This is Nelson's first equation. Taking its gradient, we find that

$$\partial_\nu(u^\mu u_\mu^Q) = \mathcal{D}\partial_\nu\partial_\mu u^\mu. \tag{164}$$

3.4.2. Nelson's Second Equation

The equivalent of Equation (51) reads

$$\frac{d_\pm f}{d\tau} = u_\pm^\mu \partial_\mu f \mp \mathcal{D}\Box f, \tag{165}$$

where we have assumed that f does not depend explicitly on the proper time τ. Following Nelson, we define the mean quadriacceleration by

$$a^\mu = \frac{1}{2}\left(\frac{d_+}{d\tau}\frac{d_-}{d\tau} + \frac{d_-}{d\tau}\frac{d_+}{d\tau}\right)x^\mu. \quad (166)$$

Using Equations (148) and (149), we have

$$a^\mu = \frac{1}{2}\left(\frac{d_+ u_-^\mu}{d\tau} + \frac{d_- u_+^\mu}{d\tau}\right). \quad (167)$$

If we apply Equation (165) to $u_\pm^\mu$, we find

$$\frac{d_+ u_-^\mu}{d\tau} = \left(u_+^\lambda \partial_\lambda - \mathcal{D}\Box\right)u_-^\mu \quad (168)$$

and

$$\frac{d_- u_+^\mu}{d\tau} = \left(u_-^\lambda \partial_\lambda + \mathcal{D}\Box\right)u_+^\mu. \quad (169)$$

Substituting Equations (168) and (169) into Equation (167), and simplifying the resulting expression with Equation (160), we find that the mean quadriacceleration can be written as

$$a^\mu = u^\lambda \partial_\lambda u^\mu - u_Q^\lambda \partial_\lambda u_Q^\mu + \mathcal{D}\Box u_Q^\mu. \quad (170)$$

In the electromagnetic case, the classical equation of motion of the particle is given by the Lorentz equation

$$m a^\mu = e F^{\mu\nu} u_\nu. \quad (171)$$

Following Nelson, we assume that this equation remains valid in the quantum case with the mean quadriacceleration from Equation (170). Therefore

$$u^\lambda \partial_\lambda u^\mu - u_Q^\lambda \partial_\lambda u_Q^\mu + \mathcal{D}\Box u_Q^\mu = \frac{e}{m} F^{\mu\nu} u_\nu. \quad (172)$$

This is Nelson's second equation. Using Equation (159), it can be rewritten as

$$u^\mu \partial_\mu u_\nu = u_Q^\mu \partial_\nu u_\mu^Q - \mathcal{D}\partial^\mu \partial_\nu u_\mu^Q - \frac{e}{m} F_{\mu\nu} u^\mu. \quad (173)$$

Defining the quantity

$$Q = \frac{1}{2} m u_\mu^Q u_Q^\mu - m\mathcal{D}\partial_\mu u_Q^\mu, \quad (174)$$

we obtain

$$u^\mu \partial_\mu u_\nu = \frac{1}{m}\partial_\nu Q - \frac{e}{m} F_{\mu\nu} u^\mu. \quad (175)$$

Using Equation (158), we see that Equation (174) represents the quantum potential (141) provided that we make the identification from Equation (65). Therefore, Equation (175) is the quantum Euler–Lorentz Equation (147).

Remark: According to Equation (174) we have

$$\partial_\nu Q = m u_\mu^Q \partial_\nu u_Q^\mu - m\mathcal{D}\partial_\nu \partial_\mu u_Q^\mu. \quad (176)$$

Using Equation (159), we obtain

$$\partial_\nu Q = m u_Q^\mu \partial_\mu u_\nu^Q - m\mathcal{D}\Box u_\nu^Q. \quad (177)$$

3.4.3. Connection between the Nelson, the De Broglie and the Klein–Gordon Equations

The Nelson Equations (164) and (172) are identical to the de Broglie Equations (139) and (147), representing the continuity equation and the quantum Euler–Lorentz equation. At that stage, we had not assumed that $mu^\mu + eA^\mu$ is a gradient. If we now assume that u^μ can be written as in Equation (135) and proceed as in Section 3.3 but in the reversed direction, we can show that the de Broglie Equations (139) and (147) are equivalent to the KG Equation (130). Therefore, the Nelson Equations (164) and (172), together with the assumption from Equation (135) are equivalent to the KG Equation (130).

Remark: Starting from the KG Equation (130), the de Broglie transformation leads to the quantum Hamilton–Jacobi Equation (140), then to the quantum Euler–Lorentz Equation (147), in addition to the continuity Equation (139). Conversely, the Nelson approach leads to the quantum Euler–Lorentz Equation (147), then, assuming Equation (135), to the quantum Hamilton–Jacobi Equation (140), in addition to the continuity Equation (139).

3.4.4. Other Forms of Nelson's Equations Assuming Equation (135)

If we assume Equation (135), we can write the Nelson equations in different forms.

(i) *Nelson's first equation:* Using Equation (137), we can rewrite Equation (164) as

$$\partial_\nu(u^\mu u^Q_\mu) = \mathcal{D}\Box u_\nu + \mathcal{D}\frac{e}{m}\partial^\mu F_{\mu\nu}. \tag{178}$$

Using Equations (137) and (159), we obtain

$$\partial_\nu(u^\mu u^Q_\mu) = u^\mu\partial_\mu u^Q_\nu + u^\mu_Q\partial_\mu u_\nu + \frac{e}{m}u^\mu_Q F_{\mu\nu}. \tag{179}$$

Therefore, Equations (164) and (178) can be rewritten as

$$u^\mu\partial_\mu u^Q_\nu + u^\mu_Q\partial_\mu u_\nu + \frac{e}{m}u^\mu_Q F_{\mu\nu} = \mathcal{D}\partial_\nu\partial_\mu u^\mu \tag{180}$$

and

$$u^\mu\partial_\mu u^Q_\nu + u^\mu_Q\partial_\mu u_\nu + \frac{e}{m}u^\mu_Q F_{\mu\nu} = \mathcal{D}\Box u_\nu + \mathcal{D}\frac{e}{m}\partial^\mu F_{\mu\nu}. \tag{181}$$

(ii) *Nelson's second equation:* Using Equations (137) and (159) we can rewrite Equation (173) as

$$\frac{1}{2}\partial_\nu(u_\mu u^\mu) = u^\lambda_Q\partial_\nu u^Q_\lambda - \mathcal{D}\Box u^Q_\nu. \tag{182}$$

In this manner the Faraday tensor $F_{\mu\nu}$ does not appear explicitly in the equation of motion. Using Equation (137), we can also rewrite Equation (172) as

$$u^\lambda\partial_\lambda u^\mu - u^\lambda_Q\partial_\lambda u^\mu_Q + \mathcal{D}\Box u^\mu_Q = -u_\nu(\partial^\mu u^\nu - \partial^\nu u^\mu). \tag{183}$$

3.5. Nottale's Theory of Scale Relativity

3.5.1. Complex Lorentz Equation

In the theory of scale relativity of Nottale [55], most elements of the nonrelativistic approach remain correct in the relativistic case with the time t replaced by the proper time τ. Not only space, but the full spacetime continuum is considered to be nondifferentiable and, therefore, fractal. The nondifferentiable motion of a particle can be represented by Equations (150) and (151). However, the physical interpretation of these equations is different from the stochastic interpretation given by Nelson (see Section 7.1). Furthermore, in order to derive the KG equation, Nottale proceeded differently from Nelson. With the two quadrivelocities u^μ_+ and u^μ_-, or equivalently with u^μ and u^μ_Q, Nottale formed a complex quadrivelocity

$$U^\mu = \frac{u^\mu_+ + u^\mu_-}{2} - i\frac{u^\mu_+ - u^\mu_-}{2} = u^\mu - iu^\mu_Q. \tag{184}$$

He also defined a complex derivative operator

$$\frac{D}{D\tau} = \frac{d_+ + d_-}{2d\tau} - i\frac{d_+ - d_-}{2d\tau} \tag{185}$$

in terms of which [see Equations (148) and (149)]

$$U^\mu = \frac{Dx^\mu}{D\tau}. \tag{186}$$

Substituting Equation (165) into Equation (185), he obtained the expression for the complex time derivative operator

$$\frac{D}{D\tau} = U^\mu \partial_\mu + i\mathcal{D}\Box. \tag{187}$$

This is the relativistic counterpart of Equation (79).

The complex quadriacceleration is given by

$$\frac{DU^\mu}{D\tau} = \frac{d_+ + d_-}{2dt}U^\mu - i\frac{d_+ - d_-}{2dt}U^\mu \tag{188}$$

or, using Equation (187), by

$$\frac{DU_\nu}{D\tau} = U^\mu \partial_\mu U_\nu + i\mathcal{D}\Box U_\nu. \tag{189}$$

Using the principle of scale relativity covariance for a quantum particle in an electromagnetic field, and using the Nelson relation (65), we obtain the complex Lorentz equation

$$\left(U^\mu + i\frac{\hbar}{2m}\partial^\mu\right)(m\partial_\mu U_\nu + eF_{\mu\nu}) = 0, \tag{190}$$

which generalizes the classical Lorentz Equation (126) written in "momentum form". By writing the components of these equations, we can show that they have the same form under the substitutions $E \to \mathcal{E}$, $\mathbf{p} \to \mathbf{P}$ and $d/d\tau \to D/D\tau$ (see [9] for details).[26] We will see that Equation (190) is equivalent to the KG Equation (128). Equation (190) can be rewritten as

$$\frac{DU_\nu}{D\tau} = -\frac{e}{m}U^\mu F_{\mu\nu} - \frac{ie\hbar}{2m^2}\partial^\mu F_{\mu\nu}. \tag{191}$$

We note that Equation (191) does *not* have the same form as Equation (124) under the substitutions $u^\mu \to U^\mu$ and $d/d\tau \to D/D\tau$. Indeed, an extra term proportional to $\hbar$ appears in Equation (191).[27] This is because the left-hand side of Equation (191) involves the derivative of the quadrimomentum without electromagnetic field $P^\mu_{\rm kin} = mU^\mu$, not the derivative of the total quadrimomentum $P^\mu = mU^\mu + eA^\mu$ as in Equation (190). This shows that the principle of scale covariance applies to the equation of motion written in terms of the quadrimomentum P^μ, not to the equation of motion written in terms of the quadrivelocity U^μ. There is only a weak scale covariance (see Section 7.2). Expanding the derivative in Equation (191) by using Equation (189), we can rewrite it as

$$U^\mu \partial_\mu U_\nu = -i\frac{\hbar}{2m}\Box U_\nu - \frac{e}{m}U^\mu F_{\mu\nu} - \frac{ie\hbar}{2m^2}\partial^\mu F_{\mu\nu}. \tag{192}$$

Remark: In the absence of an electromagnetic field (i.e., for a free particle), Equation (190) can be written as $DU_\nu/D\tau = 0$, which is the (complex) scale covariant version of the equation of classical relativistic mechanics $du_\nu/d\tau = 0$ (geodesic).

3.5.2. Derivation of the Klein–Gordon Equation

Following Nottale, we assume that the relations of classical mechanics relating the quadrimomentum to the action remain valid in the theory of scale relativity for the complex variables. Therefore, taking into account the specificities of the electromagnetic field, we introduce the complex quadrimomentum

$$P^\mu = P^\mu_{\rm kin} + eA^\mu = mU^\mu + eA^\mu, \tag{193}$$

where $P^\mu_{\rm kin} = mU^\mu$ is the quadrimomentum in the absence of electromagnetic field (i.e., for a free particle), and we assume that

$$P_\mu = -\partial_\mu \mathcal{S}, \tag{194}$$

where $\mathcal{S}$ is the complex action. The complex quadrivelocity can be written as

$$U_\mu = -\frac{\partial_\mu \mathcal{S} + eA_\mu}{m}. \tag{195}$$

Using the identity

$$\partial_\mu U_\nu - \partial_\nu U_\mu = -\frac{e}{m} F_{\mu\nu}, \tag{196}$$

obtained from Equations (195) and (A64), we can rewrite the complex Lorentz Equation (192) as

$$\left(U^\mu + i\frac{\hbar}{2m}\partial^\mu\right)\partial_\nu U_\mu = 0, \tag{197}$$

i.e.,

$$U^\mu \partial_\nu U_\mu = -i\frac{\hbar}{2m}\partial^\mu \partial_\nu U_\mu. \tag{198}$$

We note that the Faraday tensor does not explicitly appear in this equation. In the classical limit $\hbar = 0$, we recover Equation (125). Equation (198) can be integrated into

$$U_\mu U^\mu + i\frac{\hbar}{m}\partial_\mu U^\mu = c^2, \tag{199}$$

where the constant of integration has been determined in order to recover the usual relation (120) in the classical limit $\hbar \to 0$.

Using Equation (193), we can rewrite Equation (199) as

$$(P_\mu - eA_\mu)(P^\mu - eA^\mu) - m^2c^2 = -i\hbar\partial_\mu P^\mu. \tag{200}$$

Using Equation (194) we obtain the complex Hamilton–Jacobi equation[28]

$$(\partial_\mu \mathcal{S} + eA_\mu)(\partial^\mu \mathcal{S} + eA^\mu) - m^2c^2 = i\hbar\Box\mathcal{S}. \tag{201}$$

In the classical limit $\hbar = 0$, we recover Equations (121) and (123). We note that the Hamilton–Jacobi equation is not scale-covariant. We cannot simply replace the real quantities in the classical Hamilton–Jacobi Equation (123) with their complex counterparts (see Section 7.2). The complex Hamilton–Jacobi Equation (201) contains an additional term proportional to $\hbar$.

Equation (201) can be viewed as a Hamilton–Jacobi equation for a complex action, or as a Bernoulli equation for a complex potential. Using the analogy with fluid mechanics, it is natural to introduce a function $\varphi(x^\mu)$ through the complex Cole–Hopf transformation

$$\mathcal{S} = -2im\mathcal{D}\ln\varphi. \tag{202}$$

Using the expression of the quantum diffusion coefficient from Equation (65), we obtain

$$\mathcal{S} = -i\hbar \ln \varphi. \tag{203}$$

We see that $\varphi(x^\mu)$ represents the wave function. Indeed, Equation (203) can be rewritten in the standard form

$$\varphi = e^{i\mathcal{S}/\hbar} \tag{204}$$

relating the wavefuction to the complex action. Substituting Equation (203) into Equation (201), and using the identity from Equation (142), we immediately obtain the KG Equation (128). We note that the present derivation does not require us to introduce the density ρ, contrary to the approach of Nelson (see Section 7.1). This is similar to the original derivation of the KG equation (see [9] for a detailed account of the early history of the KG equation).

Remark: Using Equation (203), the complex quadrivelocity U^μ defined by Equation (195) can be written as

$$U_\mu = i\frac{\hbar}{m}\partial_\mu \ln \varphi - \frac{e}{m}A_\mu. \tag{205}$$

Remark: Using Equation (196), we can rewrite the complex Lorentz equation (191) as

$$\frac{\tilde{D}U_\nu}{\tilde{D}\tau} = -\frac{e}{m}U^\mu F_{\mu\nu} \tag{206}$$

with the new derivative

$$\frac{\tilde{D}U_\nu}{\tilde{D}\tau} = U^\mu \partial_\mu U_\nu + i\mathcal{D}\partial_\nu \partial_\mu U^\mu. \tag{207}$$

In this manner, the extra term in Equation (191) disappears. However, this transformation is a bit artificial, since only the scale covariant derivative from Equation (187) is physically justified.

3.6. Connection between Nottale and Nelson's Theories

3.6.1. Complex Action

We have seen that Nottale's theory leads to the KG Equation (128). Therefore, if we write the wave function as

$$\varphi = \frac{\hbar}{m}\sqrt{\rho}e^{iS/\hbar}, \tag{208}$$

where S is the real action and ρ is the pseudo rest-mass density, we obtain the de Broglie equations of Section 3.3. Comparing Equation (208) with Equation (204), we find that the complex action is related to the real action and to the pseudo rest-mass density by

$$\mathcal{S} = S - iS_Q \quad \text{with} \quad S_Q = \hbar \ln\left(\frac{\hbar}{m}\sqrt{\rho}\right). \tag{209}$$

We note that $\rho = (m/\hbar)^2 e^{2S_Q/\hbar}$. The complex quadrivelocity field U^μ from Equation (195) can then be written as in Equation (184) with the classical quadrivelocity

$$u_\mu = -\frac{\partial_\mu S + eA_\mu}{m} \tag{210}$$

and the quantum quadrivelocity

$$u_\mu^Q = -\frac{\partial_\mu S_Q}{m} = -\frac{\hbar}{2m}\partial_\mu \ln \rho. \tag{211}$$

They correspond to the current and osmotic quadrivelocities of Nelson (see Section 3.4). In the Nelson approach, the expression (211) of the osmotic velocity can be directly obtained from the forward and backward Fokker–Planck Equations (152) and (153), which are not used in Nottale's theory.

According to Equations (194) and (209), we also have

$$P_\mu = p_\mu - i p_\mu^Q \tag{212}$$

with

$$p_\mu = -\partial_\mu S, \qquad p_\mu^Q = -\partial_\mu S_Q = -\frac{\hbar}{2}\partial_\mu \ln\rho. \tag{213}$$

3.6.2. Real Part of the Complex Lorentz Equation

We can determine the real part of $DU^\mu/D\tau$ in two different manners:

(i) Using Equations (160), (184) and (188), we obtain

$$\begin{aligned}\mathrm{Re}\left(\frac{DU^\mu}{D\tau}\right) &= \frac{d_+ + d_-}{2d\tau}u^\mu - \frac{d_+ - d_-}{2d\tau}u_Q^\mu \\ &= \frac{1}{2}\left[\frac{d_+}{d\tau}(u^\mu - u_Q^\mu) + \frac{d_-}{d\tau}(u^\mu + u_Q^\mu)\right] \\ &= \frac{1}{2}\left(\frac{d_+ u_-^\mu}{d\tau} + \frac{d_- u_+^\mu}{d\tau}\right).\end{aligned} \tag{214}$$

(ii) Using Equations (184) and (189), we find that

$$\mathrm{Re}\left(\frac{DU_\nu}{D\tau}\right) = u^\mu\partial_\mu u_\nu - u_Q^\mu\partial_\mu u_\nu^Q + \frac{\hbar}{2m}\Box u_\nu^Q. \tag{215}$$

Comparing these results with Equations (167) and (170), we see that the real part of the complex acceleration defined by Nottale coincides with the mean acceleration defined by Nelson:

$$\mathrm{Re}\left(\frac{DU_\nu}{D\tau}\right) = a_\nu. \tag{216}$$

On the other hand, the real part of the complex Lorentz Equation (191) reads

$$\mathrm{Re}\left(\frac{DU_\nu}{D\tau}\right) = -\frac{e}{m}u^\mu F_{\mu\nu}. \tag{217}$$

Together with Equation (215), we obtain

$$u^\mu\partial_\mu u_\nu - u_Q^\mu\partial_\mu u_\nu^Q + \frac{\hbar}{2m}\Box u_\nu^Q = -\frac{e}{m}u^\mu F_{\mu\nu}. \tag{218}$$

Therefore, the real part of the complex Lorentz Equation (191) is equivalent to Nelson's second Equation (172), which is itself equivalent to the quantum Euler–Lorentz Equation (147) of de Broglie.

3.6.3. Imaginary Part of the Complex Lorentz Equation

We can determine the imaginary part of $DU^\mu/D\tau$ in two different manners:

(i) Using Equations (160), (184) and (188), we obtain

$$\begin{aligned}\mathrm{Im}\left(\frac{DU^\mu}{D\tau}\right) &= -\frac{d_+ + d_-}{2d\tau}u_Q^\mu - \frac{d_+ - d_-}{2d\tau}u^\mu \\ &= -\frac{1}{2}\left[\frac{d_+}{d\tau}(u_Q^\mu + u^\mu) + \frac{d_-}{d\tau}(u_Q^\mu - u^\mu)\right] \\ &= -\frac{1}{2}\left(\frac{d_+ u_+^\mu}{d\tau} - \frac{d_- u_-^\mu}{d\tau}\right).\end{aligned} \tag{219}$$

(ii) Using Equations (184) and (189), we find that

$$\mathrm{Im}\left(\frac{DU_\nu}{D\tau}\right) = -u^\mu \partial_\mu u_\nu^Q - u_Q^\mu \partial_\mu u_\nu + \frac{\hbar}{2m}\Box u_\nu. \quad (220)$$

On the other hand, the imaginary part of the complex Lorentz Equation (191) reads

$$\mathrm{Im}\left(\frac{DU_\nu}{D\tau}\right) = \frac{e}{m}u_Q^\mu F_{\mu\nu} - \frac{e\hbar}{2m^2}\partial^\mu F_{\mu\nu}. \quad (221)$$

Together with Equation (220) we obtain

$$u^\mu \partial_\mu u_\nu^Q + u_Q^\mu \partial_\mu u_\nu - \frac{\hbar}{2m}\Box u_\nu = -\frac{e}{m}u_Q^\mu F_{\mu\nu} + \frac{e\hbar}{2m^2}\partial^\mu F_{\mu\nu}. \quad (222)$$

If we assume that u^μ is given by Equation (135), then by using the identities (137) and (159), the foregoing equation can be rewritten as

$$\partial_\nu (u^\mu u_\mu^Q) = \frac{\hbar}{2m}\partial_\nu \partial_\mu u^\mu \quad (223)$$

or, after integration, as

$$u^\mu u_\mu^Q = \frac{\hbar}{2m}\partial_\mu u^\mu. \quad (224)$$

Therefore, the imaginary part of the complex Lorentz Equation (191) is equivalent to Nelson's first Equation (163), which is itself equivalent to the equation of continuity (139) of de Broglie. It is interesting to note that the continuity equation is contained in the complex Lorentz Equation (191) of Nottale. This assumes, however, that u^μ is given by Equation (135). By contrast, in Nelson's approach, the continuity equation is obtained from the start and does not rely on the fact that u^μ is given by Equation (135). It results from the forward and backward Fokker–Planck Equations (152) and (153), which are not used in Nottale's theory.

Remark: In the absence of electromagnetic field, the imaginary part of the complex Lorentz Equation (191) reduces to

$$\mathrm{Im}\left(\frac{DU_\nu}{D\tau}\right) = 0. \quad (225)$$

Together with Equation (219), we obtain

$$\frac{d_+ u_+^\mu}{d\tau} = \frac{d_- u_-^\mu}{d\tau}. \quad (226)$$

This is another formulation of the equation of continuity in the absence of electromagnetic field (i.e., for a free particle).

3.6.4. Connection between Different Equations

In conclusion, the complex Lorentz Equation (191) of Nottale is equivalent to the Nelson Equations (163) and (172), which are themselves equivalent to the de Broglie Equations (139) and (147), representing the continuity equation and the quantum Euler–Lorentz equation. These equations are also equivalent to the KG Equation (128). We note, however, that the physical interpretation of these equations and the methods of derivation are substantially different (see Section 7.1 for a detailed discussion). For example, the theory of Nelson relies on the forward and backward Fokker–Planck Equations (152) and (153), while these equations are not used in Nottale's theory.[29] On the other hand, it is necessary in some cases to assume that the quadrivelocity u^μ is given by Equation (135), while this assumption is not necessary in other cases.

Remark: As a corollary of the above results, if we take the real and imaginary parts of the complex Hamilton–Jacobi Equation (201) and use Equation (209), we obtain the quantum Hamilton–Jacobi Equation (140) and the continuity Equation (139).

4. Photons

There exist remarkable analogies between classical mechanics and geometric optics, which were first noticed by Hamilton (1805–1865) and Jacobi (1804–1851) in the 19th century. These analogies can be best seen through the use of variational principles.[30] In mechanics, the equation of motion of a material particle can be obtained from Hamilton's principle of least action. In the case where the energy is constant, it reduces to the so-called principle of Maupertuis $\delta S = 0$, where

$$S = \int \mathbf{p} \cdot d\mathbf{l} \tag{227}$$

is the action.[31] In geometric optics, the equation of rays is determined by Fermat's principle $\delta\theta = 0$, where

$$\theta = \int \mathbf{k} \cdot d\mathbf{l} \tag{228}$$

is the phase or eikonal (i.e., $\varphi \propto e^{i\theta}$) [77]. In his thesis, de Broglie [93] noted the analogy between these two variational principles and even argued that they are equivalent: "Fermat's principle applied to the phase wave is equivalent to the principle of Maupertuis applied to the particle". This led him to postulate that

$$p^\mu = \hbar k^\mu, \tag{229}$$

where $p^\mu = (E/c, \mathbf{p})$ is the quadrimomentum of the particle and $k^\mu = (\omega/c, \mathbf{k})$ is the quadriwavevector of the associated wave. In components form, this relation can be written as[32]

$$E = \hbar\omega, \qquad \mathbf{p} = \hbar\mathbf{k}. \tag{230}$$

The wave vector $\mathbf{k}$ plays the same role in geometric optics as the momentum $\mathbf{p}$ of the particle in mechanics, while the pulsation ω plays the role of the energy of the particle. As we know, this correspondence was the starting point of wave mechanics, which is similar to wave optics [9]. This led to the Schrödinger [1–4] and KG [5–7] equations.

Later, de Broglie [31–33] introduced a hydrodynamical representation of the KG equation (see Section 3.3). Let us consider what this formalism becomes for particles of mass $m = 0$ (like photons), and by ignoring the electromagnetic terms. As we shall see, there is a deep analogy between relativistic quantum (wave) mechanics and wave optics.[33] First of all, using Equation (229), the relation (122) between the action and the quadrimomentum translates into a relation between the eikonal and the quadriwavenumber:

$$k^\mu = -\partial_\mu \theta, \tag{231}$$

where $\theta = S/\hbar$. In components form, we have:[34]

$$\omega = -\frac{\partial\theta}{\partial t}, \qquad \mathbf{k} = \nabla\theta, \tag{232}$$

which is the counterpart of Equation (2). The equations of rays in optics

$$\dot{\mathbf{k}} = -\frac{\partial\omega}{\partial\mathbf{r}}, \qquad \mathbf{v} = \dot{\mathbf{r}} = \frac{\partial\omega}{\partial\mathbf{k}}, \tag{233}$$

are equivalent to the Hamilton equations in mechanics

$$\dot{\mathbf{p}} = -\frac{\partial E}{\partial\mathbf{r}}, \qquad \mathbf{v} = \dot{\mathbf{r}} = \frac{\partial E}{\partial\mathbf{p}}. \tag{234}$$

These relations are valid for light waves ($m = 0$) and matter waves ($m \neq 0$). As first noted by de Broglie [93], the group velocity of a wave $v_g = \partial\omega/\partial k$ [see Equation (233)] coincides with the ordinary velocity $v = \partial E/\partial p$ [see Equation (234)] of a particle in relativistic mechanics (he presented this result as a theorem). This is one of the reasons that led him to introducing the relations from Equation (230). Now considering $m = 0$, the KG Equation (129) reduces to the ordinary wave equation

$$\Box\varphi = 0. \tag{235}$$

The de Broglie transformation (133) can be written as

$$\varphi = Re^{i\theta}, \tag{236}$$

where $R = (\hbar/m)\sqrt{\rho}$ and $\theta = S/\hbar$. For $m = 0$ the de Broglie Equations (139) and (140) [or Equation (143)] take the form

$$\partial_\mu(R^2 k^\mu) = 0, \tag{237}$$

$$k_\mu k^\mu = \frac{\Box R}{R}, \tag{238}$$

and the equation of motion (144) becomes

$$k^\mu \partial_\mu k_\nu = \frac{1}{2}\partial_\nu \frac{\Box R}{R}. \tag{239}$$

The term $\Box R/R$ is the counterpart of the quantum potential. It is present in wave optics. When the wavelength λ is small or when the phase θ is large, we can neglect the term $\Box R/R$ and we have the geometrical optics, which is analogous to classical mechanics. In that case, Equation (238) reduces to

$$k_\mu k^\mu = 0, \tag{240}$$

which is the eikonal equation. This is the fundamental equation of geometric optics. It can be written as $\partial_\mu\theta\partial^\mu\theta = 0$. This is the counterpart of Equations (120) and (123) with $m = 0$. In components form, it reads $\omega = kc$. This corresponds to the relativistic relation $E^2 = p^2c^2 + m^2c^4$, reducing to $E = pc$ when $m = 0$. Therefore, the magnitude k of the wave vector is related to the frequency by the formula $k = \omega/c$, which is analogous to the relation $p = E/c$ between the momentum and energy of a particle with zero mass and velocity equal to the speed of light. In that case, the phase velocity ($v_\phi = \omega/k$) is equal to the group velocity ($v_g = d\omega/dk$): $v_\phi = v_g = c$. When $\omega = kc$, the Hamilton Equations (233) become $\dot{\mathbf{k}} = \mathbf{0}$ and $\mathbf{v} = c\,\mathbf{n}$, where $\mathbf{n} = \mathbf{k}/k$ is a unit vector along the direction of propagation. In vacuum $\mathbf{k} = (\omega/c)\mathbf{n}$, and, using $d\mathbf{l}\cdot\mathbf{n} = dl$, the Fermat principle (228) becomes $\delta\int dl = 0$, which corresponds to the rectilinear propagation of the rays (geometric optics). In vacuum, the rays are straight lines, along which the light travels with velocity c.

In Nottale's framework, we have the following relations for the complex quadrivectors:

$$P^\mu = \hbar K^\mu \tag{241}$$

and

$$K^\mu = -\partial_\mu\Theta, \tag{242}$$

where $\Theta = \mathcal{S}/\hbar$. When $m = 0$, the complex Hamilton–Jacobi Equation (199) can be written as

$$K_\mu K^\mu + i\partial_\mu K^\mu = 0, \tag{243}$$

and the complex equation of motion (192) becomes

$$K^\mu \partial_\mu K_\nu = -\frac{i}{2}\Box K_\nu. \tag{244}$$

5. General Relativity

The equations derived in the previous sections are valid in special relativity (in the absence of gravitational effects) in a Minkowskian metric. The real equations can be directly extended in general relativity by using Einstein's covariant principle: we just have to replace the derivative ∂_μ by the covariant derivative $D_\mu = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}$ (we recall that $D_\mu A^\alpha = \partial_\mu A^\alpha + \Gamma^\alpha_{\mu\nu}A^\nu$, where $\Gamma^\alpha_{\mu\nu}$ is the Christoffel symbol). For example, the KG Equation (128) becomes

$$\left(D_\mu + i\frac{e}{\hbar}A_\mu\right)\left(D^\mu + i\frac{e}{\hbar}A^\mu\right)\varphi + \frac{m^2c^2}{\hbar^2}\varphi = 0. \tag{245}$$

Similarly, the de Broglie Equations (139) and (143) become

$$D_\mu(\rho u^\mu) = 0, \tag{246}$$

$$u_\mu u^\mu = 2\frac{Q}{m} + c^2, \tag{247}$$

where the quantum potential Q is defined by Equation (141) with the covariant d'Alembertian $\Box = D_\mu D^\mu$. Taking the gradient of Equation (247), or directly applying Einstein's covariant principle to Equation (147), we obtain the quantum Euler–Lorentz equation

$$\frac{Du_\nu}{d\tau} \equiv u^\mu D_\mu u_\nu = \frac{1}{m}\partial_\nu Q - \frac{e}{m}u^\mu F_{\mu\nu}. \tag{248}$$

The covariant advection operator can be written as $Du^\alpha/d\tau = du^\alpha/d\tau + \Gamma^\alpha_{\mu\nu}u^\mu u^\nu$, where the first term is the usual advection operator $d/d\tau = u^\mu\partial_\mu$ and the second term encapsulates the gravitational effects.

Unfortunately, this prescription does not extend to the complex equations. One cannot always obtain the complex equations in general relativity by replacing the derivative ∂_μ by the covariant derivative D_μ. This shows that Nottale's covariance principle is not compatible with Einstein's covariance principle (see Section 7.2). One exception is the complex Hamilton–Jacobi equation in general relativity [8]:

$$U_\mu U^\mu - c^2 = -\frac{i\hbar}{m}D_\mu U^\mu, \tag{249}$$

which has the same form as in special relativity [see Equation (199)] with ∂_μ replaced by D_μ. However, this property is not general. For example, taking the gradient of Equation (249), we obtain the complex Lorentz equation in general relativity [8]:

$$U^\mu D_\mu U_\nu + \frac{i\hbar}{2m}\Box U_\nu - \frac{i\hbar}{2m}R_{\mu\nu}U^\mu = -\frac{e}{m}U^\mu F_{\mu\nu} - \frac{ie\hbar}{2m^2}D^\mu F_{\mu\nu}, \tag{250}$$

where $R_{\mu\nu}$ is the Ricci tensor. This equation is different from the equation that would be obtained from Equation (192) by replacing ∂_μ by D_μ. Indeed, there is an extra term $-(i\hbar/2m)R_{\mu\nu}U^\mu$ that breaks the scale covariance. Therefore, the covariance principle of Nottale is not generally valid.

6. Generalized Schrödinger and KG Equations

It is possible to generalize Nottale's theory of scale relativity in order to take dissipative effects into account. This has been carried out in [73,74] in the nonrelativistic case in the absence of electromagnetic field. The idea is to introduce a friction force $-\mathrm{Re}(\gamma\mathbf{V})$ in the complex equation of dynamics. Interestingly, this yields a generalized Schrödinger equation containing both a friction term *and* a temperature term (with a possibly negative temperature). These terms correspond to the real and imaginary parts of the complex friction coefficient $\gamma = \xi + 2ik_B T/\hbar$. This approach has been generalized in [9] in the presence of an electromagnetic field. It leads to a generalized Schrödinger equation of the form

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\nabla - i\frac{e}{\hbar}\mathbf{A}\right)^2\psi + m\Phi\psi + eU\psi + 2k_B T\ln|\psi|\psi + m\frac{dV}{d|\psi|^2}\psi - i\frac{\hbar}{2}\xi\left[\ln\left(\frac{\psi}{\psi^*}\right) - \left\langle\ln\left(\frac{\psi}{\psi^*}\right)\right\rangle\right]\psi, \quad (251)$$

which involves a logarithmic nonlinearity associated with an isothermal equation of state $P = \rho k_B T/m$ and a logarithmic nonlinearity associated with frictional effects. For the sake of generality, we have added an arbitrary self-interaction potential $V(|\psi|^2)$, i.e., an arbitrary nonlinearity taking into account short-range interactions between the bosons. It is associated with a barotropic equation of state $P(\rho) = \rho V'(\rho) - V(\rho)$. On the other hand, Φ can represent a fixed external potential $\Phi_{\rm ext}(\mathbf{r})$ or a mean field potential $\Phi(\mathbf{r},t) = \int u(|\mathbf{r}-\mathbf{r}'|)|\psi|^2(\mathbf{r}',t)\,d\mathbf{r}'$ produced by the system itself when the interaction between the bosons is long-range. It could be the gravitational potential. In that case, the generalized Schrödinger Equation (251) must be coupled self-consistently to the Poisson equation. If the system is charged, it must also be coupled to the Maxwell equations. We have proposed the generalized Schrödinger Equation (251) as a model of self-gravitating BECs representing dark matter [73,74,96,97]. The temperature and friction terms may be real or effective. In the second case, they could arise as a consequence of a process of violent relaxation and gravitational cooling [98]. When $T = \xi = V = 0$, we recover the FDM model. Interestingly, the dissipative Equation (251) allows us to make a formal connection between the equations of quantum mechanics and the equations of Brownian theory (see Ref. [99] for more details). In certain limits, the self-interaction potential $V(|\psi|^2)$ can be simplified, leading to the Gross–Pitaevskii or Cahn–Hilliard types of equations, as well as the nonlinear Schrödinger equations associated with the Fermi–Dirac or Lynden-Bell statistics [74,98,100]. We have studied these types of equations in [96]. We have considered their hydrodynamical representation, their thermodynamics, their equilibrium states (mass–radius relation, stability, pulsation), their dynamics (by using a Gaussian ansatz, which reduces the problem to the study of a simple mechanical problem for a particle in a one-dimensional potential), their dispersion relation, the Jeans gravitational instability, and the virial theorem.

We have also generalized these results in the relativistic framework [9]. In that case, we obtained a generalized KG equation of the form

$$\Box_e\varphi + \frac{m^2c^2}{\hbar^2}\varphi + \frac{4mk_B T}{\hbar^2}\ln|\varphi|\,\varphi + 2\frac{dV}{d|\varphi|^2}\varphi - i\frac{m}{\hbar}\xi\left[\ln\left(\frac{\varphi}{\varphi^*}\right) - \left\langle\ln\left(\frac{\varphi}{\varphi^*}\right)\right\rangle\right]\varphi = 0. \quad (252)$$

This equation can be applied to dark matter by working in a curved spacetime whose metric is determined by the Einstein equations with the energy–momentum produced by the scalar field φ. If the scalar field is charged, the generalized KG Equation (252) must also be coupled to the Maxwell equations.

7. Discussion

In this section, we discuss the physical interpretations of Nelson and Nottale's theories and point out some difficulties (or limitations) with these theories.

7.1. Fundamental Differences between Nelson and Nottale Theories

The scale relativity theory of Nottale shares some common features with Nelson's stochastic quantum mechanics, but the two theories differ on essential points. Nottale and Nelson both assume that the trajectories of the particles are nondifferentiable. This leads them to introduce two types of velocity $\mathbf{v}_+$ and $\mathbf{v}_-$. However, the origin of the nondifferentiability is different in the two theories.

In Nelson's theory, the nondifferentiability of the trajectories is due to stochastic fluctuations (noise) arising from a sub-quantum medium. The particles are described by a stochastic process of the form of Equations (35) and (36), like in the Brownian theory. They therefore have a classical diffusion motion. Nelson introduces a probability density $\rho(\mathbf{r}, t)$ from the start, which satisfies forward and backward Fokker–Planck equations. He directly obtains the continuity equation involving the current velocity $\mathbf{v}$ from these equations. He also obtains the expression of the osmotic velocity $\mathbf{v}_Q$ in terms of the density. He then postulates a form of mean acceleration (his expression of the acceleration is not justified from first principles) and assumes that the Newtonian equation of motion remains valid with this acceleration. This approach gives him two fundamental equations for $\mathbf{v}$ and $\mathbf{v}_Q$. These equations are equivalent to the Madelung equations, which are themselves equivalent to the Schrödinger equation.

In Nottale's theory, the nondifferentiability of the trajectories is intrinsically due to the fractal structure of spacetime. The equation for the fractal fluctuations is similar to the equation for the fluctuations in a Brownian motion used by Nelson [see Equations (35) and (36)], but this is just an analogy, and the interpretation of this equation is fundamentally different from a classical diffusion.[35] From $\mathbf{v}_+$ and $\mathbf{v}_-$, Nottale introduces a complex velocity $\mathbf{V}$, a complex acceleration $\mathbf{A} = D\mathbf{V}/Dt$, and a principle of scale covariance leading to a complex Newton equation. The complex Newton equation is equivalent to a complex Hamilton–Jacobi equation, which is itself equivalent to the Schrödinger equation and to the Madelung equations.[36] We have also shown that the real and imaginary parts of the complex Newton equation of Nottale are equivalent to the Nelson equations. Therefore, the two theories are formally (but not physically) equivalent.

A crucial difference between the two theories is that Nelson introduces a probability density from the very beginning under a diffusion-like description. This probability density satisfies the forward and backward Fokker–Planck equations that are set as founding equations in stochastic quantum mechanics. It was shown in subsequent works [101,102] that this stochastic approach of quantum mechanics poses fundamental problems. The backward Fokker–Planck equation for $\mathbf{v}_-$ is not well-defined (it does not correspond to any known classical Markovian or non-Markovian process) and Fokker–Planck equations are known to be incompatible with quantum mechanics in general. Moreover, the probability density is used in Nelson's approach to define averages. This leads to contradictions with standard quantum mechanics concerning multitime correlations. The source of the disagreement comes precisely from the Brownian motion (diffusion) interpretation of Nelson's theory, leading to the use of Fokker–Planck equations, and from the wave function reduction.

Contrary to such a diffusion approach, the scale relativity theory of Nottale is not statistical in its essence. The undeterminism of the trajectories is a consequence of the nondifferentiability of spacetime. This is clear from the fact that Nottale does not introduce a probability density in his derivation of the Schrödinger equation. It is only at the end, in order to give a physical interpretation to the wavefuction, that the probability density is introduced. This is similar to the original approach of Schrödinger (see [9] for an historical account). On the other hand, Nottale does not use Fokker–Planck equations, but only the equation of dynamics, properly made scale-covariant. The mean forward and backward Fokker–Planck equations can be derived a posteriori (see the Remark below) from Nottale's formalism, but they have a different interpretation and a different status from that given by Nelson. In the scale relativity approach, $\mathbf{v}_-$ is not defined as a backward (i.e., time-reversal) velocity, but it comes from the breaking of a symmetry, namely, the reflection invariance $dt \to -dt$ of the differential element of time dt. This is one of the most fundamental consequences of the nondifferentiable nature of spacetime. This implies the two-valuedness of the velocity, which can be subsequently shown to be the origin of the complex nature of the wave function in quantum mechanics.[37] Since the Fokker–Planck equations are not used in the derivation of the Schrödinger equation, the problems mentioned above with these equations do not occur in Nottale's theory.

Remark: We have seen that Nottale's theory contains the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{253}$$

and the equation $\mathbf{v}_Q = \mathcal{D}\nabla \ln \rho$, from which we derive

$$\nabla \cdot (\rho \mathbf{v}_Q) = \mathcal{D}\Delta\rho. \tag{254}$$

On the other hand, the forward and backward velocities can be obtained from the relations $\mathbf{v}_+ = \mathbf{v} + \mathbf{v}_Q = \mathbf{v} + \mathcal{D}\nabla \ln \rho$ and $\mathbf{v}_- = \mathbf{v} - \mathbf{v}_Q = \mathbf{v} - \mathcal{D}\nabla \ln \rho$. Adding and subtracting Equations (253) and (254), and using the foregoing relations, we obtain the equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}_+) = \mathcal{D}\Delta\rho, \tag{255}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}_-) = -\mathcal{D}\Delta\rho, \tag{256}$$

which are similar to the forward and backward Fokker–Planck equations of Nelson [see Equations (37) and (38)]. These equations look like classical diffusion equations associated with stochastic processes but, in Nottale's theory, they have a different status. Contrary to Nelson's theory, these equations are not set as founding equations, but they are derived as a very consequence of the nondifferentiable and fractal geometry of spacetime. In particular, the equation for $\mathbf{v}_-$ has no classical equivalent.

7.2. Problems with Nottale's Scale Covariance Principle

The theory of scale relativity of Nottale is based on the concept of a fractal spacetime and the theory of general relativity of Einstein is based on the concept of a curved spacetime. Naturally, it would be desirable to combine these two theories into a single one based on the concept of a fractal curved spacetime. In this manner, all the physics would be contained in the geometrical structure of spacetime. In particular, quantum mechanics and general relativity would emerge from the "texture" of spacetime. A requirement for this unification would be to introduce a notion of scale covariance and show its compatibility with Einstein's covariance principle.

The scale covariance principle of Nottale claims that the equations of quantum mechanics (nondifferentiable trajectories applying to microphysics) can be deduced from the equations of classical mechanics (differential trajectories applying to macrophysics) by replacing the velocity $\mathbf{v}$ by the complex velocity field $\mathbf{V}$ and the standard time derivative d/dt by the scale covariant time derivative D/Dt. This is similar to the covariance principle of Einstein, according to which the equations with gravitation (curved spacetime, Riemannian case) can be deduced from the equations without gravitation (flat spacetime, Minkowskian case) by replacing the partial derivative ∂_μ by the covariant derivative D_μ.

At first sight, the scale covariance principle of Nottale is an attractive idea. Unfortunately, unlike the Einstein covariance principle of general relativity, the scale covariance principle of Nottale is not generally valid, in the sense that it does not systematically apply to *all* the equations of physics. In certain cases, extra terms appear in the equations and break the desired covariance. Therefore, we just have a *weak* covariance principle, valid only for certain equations. This is different from Einstein's covariance principle, which is general.

Originally, the covariance principle of Nottale was applied to the Newton equation of motion for a particle in an external potential. It was shown to lead to the Schrödinger equation. However, if we consider the motion of a particle in an electromagnetic field, we find that the Lorentz equation for the complex velocity $\mathbf{V}$ that leads to the electromagnetic Schrödinger equation does not have the same form as the classical Lorentz equation for the velocity $\mathbf{v}$ [compare Equations (4) and (83)]. In this sense, covariance is broken. However, we have shown in the present paper that covariance is restored if we apply the covariance principle to the equation of motion expressed in terms of the total momentum $\mathbf{p}$ instead of the velocity $\mathbf{v}$ [compare Equations (5) and (82)]. This is an interesting new result of the present paper.[38] This result also extends to the relativistic framework (see [9] for details).

Unfortunately, the covariance principle of Nottale seems to apply only to the equation of motion (82) formulated in terms of the momentum. In other equations, such as the equation of motion (83) expressed in terms of the velocity, the Hamilton–Jacobi Equation (92) and the energy Equation (A47), extra terms appear so that these equations do not have the same form in classical physics (differentiable trajectories, real variables) and in quantum physics (nondifferentiable trajectories, complex variables). In this sense, we only have a *weak* scale covariance principle.

We have also shown that the covariance principle of Nottale is not compatible with the covariance principle of Einstein. Indeed, the complex equation of motion (250) in a curved spacetime (gravity case) does not have the same form as the complex equation of motion (192) in a flat spacetime (in the absence of gravity).

The absence of a general scale covariance is clearly a problem (or a limitation) of Nottale's scale relativity theory.

8. Conclusions

In this paper, we have reviewed and compared the stochastic quantum mechanics of Nelson [43] and the scale relativity theory of Nottale [55]. These theories provide a derivation of the Schrödinger and KG equations from microscopic processes. We have shown the formal equivalence of these two theories, in the sense that they lead to the same equations (the real and imaginary parts of Nottale's complex equation of motion coincide with the Nelson equations). However, the physical interpretation of these theories and the manner for deriving the Schrödinger and KG equations in these two theories are fundamentally different. We have mentioned some problems associated with Nelson's theory–especially his use of Fokker–Planck equations–which are solved (or avoided) by Nottale's approach. However, Nottale's theory also presents difficulties, notably concerning the nonuniversality of his scale covariance principle (contrary to Einstein's covariance principle in general relativity).

The origin of quantization in physics may be fundamentally related to the stochastic interpretations of Nelson [43] and Nottale [55]. However, in the two theories, the source of the stochastic fluctuations leading to quantum mechanics is not clearly specified. In Nelson's theory, the random motion (noise) represents quantum fluctuations due to a sub-quantum medium (a sort of aether), but this idea remains vague. In Nottale's theory, the nondifferentiability of the trajectories at microscale is assumed to be an intrinsic property of the spacetime itself. Therefore, the quantum behavior is understood as a manifestation of the fractal geometry of spacetime. However, the origin of this fractal spacetime is not clearly established. Consequently, in these works, the relation $\mathcal{D} = \hbar/2m$ between the quantum diffusion coefficient and the Planck constant is postulated, not derived, since the nature of the stochastic source remains unidentified. In practice, this relation is imposed in order to recover the usual Schrödinger and KG equations at the end. Nottale's approach is more general than Nelson's approach in the sense that he allows the diffusion coefficient $\mathcal{D}$ to be different from $\hbar/2m$. This may be justified when the quantization is due to a chaotic dynamics, valid beyond the realm of quantum mechanics. This allowed him to treat more general situations, such as the quantized motion of planets around the sun, as well as the morphogenesis of planetary nebulae, and even flowers (!).

Following the paper of Nelson [43], some authors have tried to determine the physical origin of the stochastic fluctuations giving rise to quantum mechanics. As argued by Marshall [103–105] and Boyer [106], and further developed by Puthoff [107] and de la Peña et al. [108], these fluctuations may be due to the electromagnetic zero-point radiation field which creates a fluctuating spacetime. Electromagnetic zero-point radiation could thus be regarded as the cause of quantum motion. This leads to the concept of "stochastic electrodynamics". In this framework, Boyer [106] managed to derive the Planck radiation law for the blackbody radiation spectrum without the formalism of quantum theory. On the other hand, using stochastic electrodynamics, de la Peña and Cetto [109–111] proposed a derivation of the Schrödinger equation different from Nelson, in which the relation $\mathcal{D} = \hbar/2m$ is derived, not postulated. Indeed, the Planck constant $\hbar$ occurs in the noise term from the beginning through the spectral density of the zero-point radiation field. Therefore, two theories have been developed independently with the aim to describe quantum mechanics as a classical stochastic process, namely the stochastic quantum mechanics of Nelson [43] and the stochastic electrodynamics of de la Peña and Cetto [109–111]. Stochastic quantum mechanics (a phenomenological theory) has been conceived as a Brownian-type theory for the particle subject to a white noise from an unidentified source. By contrast, stochastic electrodynamics (a fundamental theory) has been developed as a statistical description for the particle subject to the zero-point radiation field with a colored spectrum. The connection between these two stochastic theories that lead to quantum mechanics is discussed in [112].

A similar idea was developed by Calogero [113], but dealing entirely with gravitational fluctuations instead of electromagnetic ones.[39] He argued that the origin of quantization is due to stochastic gravitational effects or to stochastic fluctuations of the spacetime metric. The possibility to represent the effect of the zero-point field as a fluctuating metric field was already contemplated by Einstein [114] in 1924. On the other hand, Feynman [115] noted that the classical–quantum transition might be a consequence of the existence of fundamental fluctuations of spacetime induced by stochastic gravitational waves present in our galactic environment (the so-called gravitational wave background) [116].

The authors of Refs. [107,108,113] were able to use the idea of a stochastic quantum mechanics to derive the mysterious Eddington relation $\Lambda \sim G^2 m_e^6/\hbar^4$ [117] between the cosmological constant Λ and the mass m_e of the electron (see [118] for a recent account on this topic). Conversely, this relation furnishes a cosmological origin of Planck's constant in the form $\hbar \sim (m_e^3 c^2/\rho R_\Lambda)^{1/2}$ or $\hbar \sim (G m_e^3 R_\Lambda)^{1/2}$, which fixes the scale of the zero-point fluctuations (here $R_\Lambda = 1/\sqrt{\Lambda}$ is the radius of the visible universe and ρ is its density). In this approach, the Planck constant becomes determined by cosmological parameters.

Recently, the alternative approaches of quantum mechanics proposed by Nelson [43] and Nottale [55] have been applied to dark matter by Chavanis [73,74], Cresson et al. [75] and Escobar-Aguilar et al. [76]. These approaches explain the emergence of quantization in terms of (i) the fractal structure of spacetime, (ii) a chaotic dynamics, (iii) stochastic electrodynamics, or (iv) gravitational fluctuations (gravitational wave background). The Schrödinger and KG equations may be the consequence of the fact that the universe evolves in a stochastic background (which may have various origins). This scenario provides an alternative interpretation of the FDM model. This may change the constraints on the mass of the dark matter particle [74]. These ideas certainly deserve further consideration.

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Appendix A. Connection between the Operator Prescription and Nottale's Theory

The operator prescription (or correspondence principle) presented in Sections 2.2 and 3.2 does not have a rigorous foundation, except for yielding the Schrödinger and KG equations in a rather direct manner.[40] In this respect, it is interesting to note that equations similar to Equations (6) and (127) naturally arise in Nottale's theory of scale relativity.

Appendix A.1. Nonrelativistic Theory

According to Equations (86) and (94), the complex momentum and the complex energy can be written in the nonrelativistic case as

$$\mathbf{P} = -i\hbar\nabla \ln \psi, \qquad \mathcal{E} = i\hbar\frac{\partial \ln \psi}{\partial t}. \tag{A1}$$

Therefore, we obtain the relations

$$\mathbf{P}\psi = -i\hbar\nabla\psi, \qquad \mathcal{E}\psi = i\hbar\frac{\partial\psi}{\partial t}, \tag{A2}$$

which are similar to the operator prescription from Equation (6). If we introduce the operators $\hat{\mathbf{p}} = -i\hbar\nabla$ and $\hat{E} = i\hbar\partial/\partial t$, we find that

$$\mathbf{P}\psi = \hat{\mathbf{p}}\psi, \qquad \mathcal{E}\psi = \hat{E}\psi. \tag{A3}$$

However, this connection cannot be considered as a justification of the operator prescription because it is only valid at linear order and ceases to be valid at nonlinear order. Indeed, in terms of the complex quantities, we have

$$\mathbf{P}^2 = -\hbar^2\left(\frac{\nabla\psi}{\psi}\right)^2, \qquad \mathcal{E}^2 = -\hbar^2\left(\frac{1}{\psi}\frac{\partial\psi}{\partial t}\right)^2, \tag{A4}$$

while, in terms of the operators, we have

$$\hat{\mathbf{p}}^2\psi = -\hbar^2\Delta\psi, \qquad \hat{E}^2\psi = -\hbar^2\frac{\partial^2\psi}{\partial t^2}. \tag{A5}$$

Therefore

$$\mathbf{P}^2\psi \neq \hat{\mathbf{p}}^2\psi, \qquad \mathcal{E}^2\psi \neq \hat{E}^2\psi. \tag{A6}$$

Appendix A.2. Relativistic Theory

According to Equations (194) and (203), the complex quadrimomentum can be written as

$$P_\mu = i\hbar\partial_\mu \ln \varphi. \tag{A7}$$

Therefore, we obtain the relation

$$P_\mu\varphi = i\hbar\partial_\mu\varphi, \tag{A8}$$

which is similar to the operator prescription from Equation (127). If we introduce the operator $\hat{p}_\mu = i\hbar\partial_\mu$, we find that

$$P_\mu\varphi = \hat{p}_\mu\varphi. \tag{A9}$$

However, this does not mean that we have justified the operator prescription because this correspondence is not valid at the nonlinear order. Indeed, in terms of the complex quantities, we have

$$P_\mu P^\mu = -\hbar^2\frac{1}{\varphi^2}\partial_\mu\varphi\partial^\mu\varphi, \tag{A10}$$

while, in terms of the operators, we have

$$\hat{p}_\mu\hat{p}^\mu\varphi = -\hbar^2\partial_\mu\partial^\mu\varphi. \tag{A11}$$

Therefore

$$P_\mu P^\mu\varphi \neq \hat{p}_\mu\hat{p}^\mu\varphi. \tag{A12}$$

Appendix B. Energy in the Nonrelativistic Quantum Theory

In this Appendix, we discuss different notions of energy in the nonrelativistic quantum theory. For simplicity, we do not take the electromagnetic field into account.

Appendix B.1. Schrödinger Energy Functional

The energy functional associated with the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + m\Phi\psi \tag{A13}$$

is

$$E_{\rm tot} = \frac{\hbar^2}{2m^2}\int |\nabla\psi|^2\, d\mathbf{r} + \int |\psi|^2\Phi\, d\mathbf{r} = \Theta + W. \tag{A14}$$

It is the sum of the kinetic energy Θ and the potential energy W (here Φ is a fixed external potential). This functional is conserved by the Schrödinger equation.

Remark: This functional (with ψ real) was introduced by Schrödinger [1] at the end of his first communication in a "Zusatz bei der Korrektur" (see also [10,16]).[41] He derived his nonrelativistic wave equation (eigenvalue equation)

$$-\frac{\hbar^2}{2m}\Delta\psi + m\Phi\psi = E\psi \tag{A15}$$

from the minimization of $E_{\rm tot}$ at fixed normalization $\int |\psi|^2\, d\mathbf{r} = M$, where E (eigenenergy) can be interpreted as a Lagrange multiplier (see Appendix F.2 of [8]).

Appendix B.2. Madelung Transformation

Using the Madelung transformation, the kinetic energy can be written as $\Theta = \Theta_c + \Theta_Q$ where

$$\Theta_c = \frac{1}{2}\int \rho\mathbf{v}^2\, d\mathbf{r} \tag{A16}$$

is the classical kinetic energy and

$$\Theta_Q = \frac{\hbar^2}{8m^2}\int \frac{(\nabla\rho)^2}{\rho}\, d\mathbf{r} \tag{A17}$$

is the quantum kinetic energy. The quantum kinetic energy can be alternatively written as (see, e.g., [96])

$$\Theta_Q = \int \rho\frac{Q}{m}\, d\mathbf{r}, \tag{A18}$$

which is similar to a potential energy involving the quantum potential Q. Therefore, we can write the total kinetic energy as

$$\Theta = \frac{1}{2}\int \rho\mathbf{v}^2\, d\mathbf{r} + \int \rho\frac{Q}{m}\, d\mathbf{r}. \tag{A19}$$

Remark: The functional from Equation (A17) was introduced by von Weizsäcker [119] and is related to the Fisher [120] entropy $S_F = (1/m)\int (\nabla\rho)^2/\rho\, d\mathbf{r}$. Actually, the functional (A17) was already introduced by Madelung [30,85] under the equivalent form $\Theta_Q = -(\hbar^2/2m^2)\int \sqrt{\rho}\Delta\sqrt{\rho}\, d\mathbf{r}$.

Appendix B.3. Nelson Theory

From Equation (A17) and Equation (43), we immediately obtain

$$\Theta_Q = \int \rho \frac{\mathbf{v}_Q^2}{2}\, d\mathbf{r}. \tag{A20}$$

This shows that the quantum kinetic energy can be written in the form of a kinetic energy involving the quantum (or osmotic) velocity $\mathbf{v}_Q$. This expression can also be obtained from Equation (A18) as follows. Using Equation (63) the quantum kinetic energy reads

$$\Theta_Q = -\int \rho \frac{\mathbf{v}_Q^2}{2}\, d\mathbf{r} - \mathcal{D}\int \rho \nabla \cdot \mathbf{v}_Q\, d\mathbf{r}. \tag{A21}$$

Integrating by parts we find that

$$\Theta_Q = -\int \rho \frac{\mathbf{v}_Q^2}{2}\, d\mathbf{r} + \mathcal{D}\int \mathbf{v}_Q \cdot \nabla \rho\, d\mathbf{r}. \tag{A22}$$

Using Equation (43) we obtain

$$\Theta_Q = -\int \rho \frac{\mathbf{v}_Q^2}{2}\, d\mathbf{r} + \int \rho \mathbf{v}_Q^2\, d\mathbf{r} = \int \rho \frac{\mathbf{v}_Q^2}{2}\, d\mathbf{r}, \tag{A23}$$

as in Equation (A20). Therefore, we can write the total kinetic energy as

$$\Theta = \frac{1}{2}\int \rho \mathbf{v}^2\, d\mathbf{r} + \int \rho \frac{\mathbf{v}_Q^2}{2}\, d\mathbf{r}. \tag{A24}$$

This functional was first introduced by Nelson [43].

Appendix B.4. Nottale Theory

The total kinetic energy is given by

$$\Theta = \frac{\hbar^2}{2m^2}\int |\nabla \psi|^2\, d\mathbf{r}. \tag{A25}$$

In the absence of electromagnetic field, Equation (96) reduces to

$$\mathbf{V} = -i\frac{\hbar}{m}\nabla \ln \psi. \tag{A26}$$

Therefore, the total kinetic energy can be written as

$$\Theta = \frac{1}{2}\int \rho |\mathbf{V}|^2\, d\mathbf{r}. \tag{A27}$$

It has the form of an ordinary kinetic energy for a complex velocity field. Then, using Equation (76) we find that

$$|\mathbf{V}|^2 = \mathbf{v}^2 + \mathbf{v}_Q^2, \tag{A28}$$

yielding

$$\Theta = \frac{1}{2}\int \rho \mathbf{v}^2\, d\mathbf{r} + \frac{1}{2}\int \rho \mathbf{v}_Q^2\, d\mathbf{r}, \tag{A29}$$

in agreement with the expression (A24) given by Nelson [43].

Appendix B.5. Energy of a Quantum Particle

From Equations (2) and (19) we find that the energy of a quantum particle is

$$E(\mathbf{r},t) = \frac{p^2}{2m} + Q + m\Phi. \tag{A30}$$

In the classical limit $\hbar \to 0$, it reduces to the familiar formula $E = p^2/2m + m\Phi$. We note that [96]

$$E(\mathbf{r},t) = m\frac{\delta E_{\rm tot}}{\delta\rho}. \tag{A31}$$

The average energy $\overline{E} = \frac{1}{M}\int \rho E(\mathbf{r},t)\, d\mathbf{r}$ is

$$N\overline{E} = \Theta + W = \Theta_c + \Theta_Q + W. \tag{A32}$$

This is the sum of the kinetic energy $\Theta = \Theta_c + \Theta_Q$ (including the classical kinetic energy Θ_c and the quantum kinetic energy Θ_Q) and the potential energy W. Thus

$$E_{\rm tot} = N\overline{E}. \tag{A33}$$

At equilibrium, we have $S = -Et$ [73], where E is the eigenenergy, so that $E(\mathbf{r},t) = E = {\rm cst}$ and $\mathbf{v} = \mathbf{0}$. Therefore, Equation (A30) reduces to

$$E = Q + m\Phi. \tag{A34}$$

We also have $\overline{E} = E$ and $\Theta_c = 0$. Therefore, Equation (A32) yields

$$NE = \Theta_Q + W. \tag{A35}$$

Remark: If we consider a static state ($\mathbf{v} = \mathbf{0}$) of the Schrödinger Equation (A13), using Equation (63), we can rewrite Equation (A34) as

$$-\frac{\hbar}{2}\nabla\cdot\mathbf{v}_Q - \frac{1}{2}m\mathbf{v}_Q^2 = E - m\Phi(\mathbf{r}). \tag{A36}$$

This equation can be viewed as a Riccati equation that may be easier to solve than the time-independent Schrödinger Equation (A15) itself.

Appendix C. Energy in the Relativistic Quantum Theory

The quantum Hamilton-Jacobi Equation (140) can be written in components form as

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 - (\nabla S)^2 - 2mQ - m^2c^2 = 0. \tag{A37}$$

From Equations (2) and (A37) we find that the energy $E(\mathbf{r},t)$ of a particle satisfies

$$E^2 = p^2c^2 - 2mc^2Q + m^2c^4. \tag{A38}$$

In the classical limit $\hbar \to 0$, it reduces to the familiar relation $E^2 = p^2c^2 + m^2c^4$.

Appendix D. Complex Energy in Nottale's Nonrelativistic Theory

In this Appendix, we discuss the complex energy in Nottale's nonrelativistic theory. For simplicity, we do not take the electromagnetic field into account.

Appendix D.1. Complex Quantum Potential

The complex Newton equation reads [see Equation (84)]

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = i\frac{\hbar}{2m}\Delta\mathbf{V} - \nabla\Phi. \tag{A39}$$

The term $i(\hbar/2m)\Delta\mathbf{V}$ can be interpreted as a viscous term with an imaginary viscosity $\nu = i\hbar/2m$. Alternatively, using the identity $\Delta\mathbf{V} = \nabla(\nabla\cdot\mathbf{V}) = \nabla(\Delta\mathcal{S}/m)$ valid for a potential flow $\mathbf{V} = \nabla\mathcal{S}/m$, the term $i(\hbar/2m)\Delta\mathbf{V}$ can be interpreted as a complex quantum force by unit of mass

$$\mathcal{F}_Q = -\frac{1}{m}\nabla q \tag{A40}$$

deriving from a complex quantum potential

$$q = -i\frac{\hbar}{2}\nabla\cdot\mathbf{V} = -i\frac{\hbar}{2m}\Delta\mathcal{S} = -\frac{\hbar^2}{2m}\Delta\ln\psi. \tag{A41}$$

In this manner, we can rewrite the complex Newton Equation (A39) as

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = -\frac{1}{m}\nabla q - \nabla\Phi. \tag{A42}$$

Using Equation (76), the complex quantum potential can be written in terms of the classical and quantum velocities as

$$q = -\frac{\hbar}{2}\nabla\cdot\mathbf{v}_Q - i\frac{\hbar}{2}\nabla\cdot\mathbf{v}. \tag{A43}$$

On the other hand, using Equations (13) and (43), and the identity (21), we find that

$$q = -\frac{\hbar^2}{4m}\Delta(\ln\rho) - i\frac{\hbar}{2m}\Delta S = -\frac{\hbar^2}{4m}\left[\frac{\Delta\rho}{\rho} - \frac{(\nabla\rho)^2}{\rho^2}\right] - i\frac{\hbar}{2m}\Delta S. \tag{A44}$$

Comparing Equation (A44) with Equation (20), or Equation (A43) with Equation (63), we note that the real part of the complex quantum potential q is *not* the quantum potential Q. This is because the complex advection operator $(\mathbf{V}\cdot\nabla)\mathbf{V}$ in Equation (A42) involves a term $-(\mathbf{v}_Q\cdot\nabla)\mathbf{v}_Q = -\nabla(\mathbf{v}_Q^2/2)$ proportional to $\hbar^2$ which enters into the quantum potential Q.

Appendix D.2. Complex Energy

The complex Hamilton-Jacobi Equation (92) can be written as

$$\frac{\partial\mathcal{S}}{\partial t} + \frac{(\nabla\mathcal{S})^2}{2m} + m\Phi = \frac{i\hbar}{2m}\Delta\mathcal{S}. \tag{A45}$$

Using Equation (86), we obtain

$$\mathcal{E} = \frac{1}{2}m\mathbf{V}^2 - i\frac{\hbar}{2}\nabla\cdot\mathbf{V} + m\Phi. \tag{A46}$$

Introducing the complex quantum potential from Equation (A41), the complex energy takes the form

$$\mathcal{E} = \frac{1}{2}m\mathbf{V}^2 + q + m\Phi. \tag{A47}$$

It is the sum of the complex kinetic energy, the complex quantum potential and the external potential. As noted by Nottale [55], the theory of scale relativity yields a new contribution $q = -i\frac{\hbar}{2}\nabla\cdot\mathbf{V}$ to the energy that comes from the very geometry of spacetime. Nottale claims that this term is similar to the new contribution mc^2 to the energy in Einstein's theory of relativity. However, this extra term breaks the principle of scale convariance. Indeed, Equation (A47) is not obtained from Equation (1) by simply replacing the velocity by the complex velocity and the energy by the complex energy. Similarly, the complex Hamilton-Jacobi Equation (92) is not obtained from the classical Hamilton-Jacobi Equation (3) by simply replacing the action by the complex action. There is an extra term proportional to $\hbar$ (see Section 7.2).

Using Equation (A47) and the relation $(\mathbf{V}\cdot\nabla)\mathbf{V}=\nabla(\mathbf{V}^2/2)$, we note that the scale covariant equation of dynamics (A39) can be written as

$$m\frac{\partial \mathbf{V}}{\partial t}=-\nabla\mathcal{E}. \tag{A48}$$

This equation also immediately results from Equation (86). Using Equations (76), (A43) and (A47), we find that $\mathcal{E}=E-iE_Q$ with

$$E=\frac{1}{2}m\mathbf{v}^2-\frac{1}{2}m\mathbf{v}_Q^2-\frac{\hbar}{2}\nabla\cdot\mathbf{v}_Q+m\Phi \tag{A49}$$

and

$$E_Q=m\mathbf{v}\cdot\mathbf{v}_Q+\frac{\hbar}{2}\nabla\cdot\mathbf{v}. \tag{A50}$$

Using Equation (63), we can rewrite Equation (A49) as

$$E=\frac{1}{2}m\mathbf{v}^2+Q+m\Phi, \tag{A51}$$

in agreement with Equation (A30). Equations (103), (107) and (A51) return the quantum Hamilton-Jacobi Equation (19). On the other hand, using Equation (104), we can rewrite Equation (A50) as

$$E_Q=\frac{\hbar}{2\rho}\nabla\cdot(\rho\mathbf{v}). \tag{A52}$$

Equations (107) and (A52) return the equation of continuity (18). The average energy $\overline{E}$ is conserved (see Appendix B). This can be interpreted as a consequence of the Ehrenfest theorem [29]. On the other hand, from Equation (A52), we see that $\overline{E}_Q=0$ so it is trivially conserved. As a result, $\overline{\mathcal{E}}=\overline{E}$ is conserved (we note that $\overline{\mathcal{E}}$ is real).

Appendix E. Complex Energy in Nottale's Relativistic Theory

In this Appendix, we discuss the complex energy in Nottale's relativistic theory. For simplicity, we do not take the electromagnetic field into account.

Appendix E.1. Complex Quantum Potential

The complex relativistic equation of motion reads [see Equation (192)]

$$U^\mu\partial_\mu U_\nu=-i\frac{\hbar}{2m}\Box U_\nu. \tag{A53}$$

Using the identity $\Box U_\nu=\partial_\nu(\partial^\mu U_\mu)=-\partial_\nu(\Box\mathcal{S}/m)$ valid when $U_\mu=-\partial_\mu\mathcal{S}/m$, the term $-i\frac{\hbar}{2m}\Box U_\nu$ can be interpreted as a complex quantum force by unit of mass

$$\mathcal{F}_\nu^Q=\frac{1}{m}\partial_\nu q \tag{A54}$$

deriving from a complex quantum potential

$$q=-i\frac{\hbar}{2}\partial_\mu U^\mu=i\frac{\hbar}{2m}\Box\mathcal{S}=\frac{\hbar^2}{2m}\Box\ln\varphi. \tag{A55}$$

In this manner, we can rewrite the complex equation of motion (A53) as

$$U^\mu \partial_\mu U_\nu = \frac{1}{m}\partial_\nu q. \tag{A56}$$

Using Equation (184), the complex quantum potential can be written in terms of the classical and quantum quadrivelocities as

$$q = -\frac{\hbar}{2}\partial_\mu u^\mu_Q - i\frac{\hbar}{2}\partial_\mu u^\mu. \tag{A57}$$

On the other hand, using Equations (195) and (158), and the identity (142), we find that

$$q = \frac{\hbar^2}{4m}\Box(\ln\rho) + i\frac{\hbar}{2m}\Box S = \frac{\hbar^2}{4m}\left[\frac{\Box\rho}{\rho} - \frac{1}{\rho^2}\partial_\mu\rho\partial^\mu\rho\right] + i\frac{\hbar}{2m}\Box S. \tag{A58}$$

Comparing Equation (A58) with Equation (141), or Equation (A57) with Equation (174), we note that the real part of the complex quantum potential q is *not* the quantum potential Q. This is because the complex advection operator $U^\mu\partial_\mu U_\nu$ in Equation (A56) involves a term $-u^\mu_Q\partial_\mu u^Q_\nu = -\partial_\nu(u^\mu_Q u^Q_\mu/2)$ proportional to $\hbar^2$ which enters into the quantum potential Q.

Appendix E.2. Complex Energy

The complex Hamilton-Jacobi Equation (201) can be written in components form as

$$\frac{1}{c^2}\left(\frac{\partial\mathcal{S}}{\partial t}\right)^2 - (\nabla\mathcal{S})^2 - m^2c^2 = i\hbar\left(\frac{1}{c^2}\frac{\partial^2\mathcal{S}}{\partial t^2} - \Delta\mathcal{S}\right). \tag{A59}$$

Using Equation (86), which remains valid in the relativistic framework, we obtain

$$\mathcal{E}^2 - \mathbf{P}^2c^2 - m^2c^4 = -i\hbar\left(\frac{\partial\mathcal{E}}{\partial t} + c^2\nabla\cdot\mathbf{P}\right). \tag{A60}$$

Introducing the complex quantum potential from Equation (A55), the complex energy is determined by the equation

$$\mathcal{E}^2 - \mathbf{P}^2c^2 - m^2c^4 = 2mc^2q. \tag{A61}$$

As in the nonrelativistic case, there is an extra term proportional to $\hbar$ in the complex energy which breaks the principle of scale convariance (see Section 7.2). Indeed, Equation (A61) is not obtained from the classical relation $E^2 = p^2c^2 + m^2c^4$ by simply replacing the momentum by the complex momentum and the energy by the complex energy. Similarly, the complex Hamilton-Jacobi Equation (201) is not obtained from the classical Hamilton-Jacobi Equation (123) by simply replacing the action by the complex action.

Appendix F. Basics of Electrodynamics

In this Appendix we recall basic elements of electrodynamics that are useful for the study developed in the present paper and in our companion paper [9].

Appendix F.1. Electromagnetic Lagrangian

The linear electrodynamics of Maxwell is based on the Lagrangian

$$\mathcal{L} = -\mathcal{F}, \tag{A62}$$

where

$$\mathcal{F} = \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} \tag{A63}$$

is the electromagnetic invariant. The electromagnetic field tensor, or Faraday tensor, can be written as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{A64}$$

where $A^\mu = (U/c, \mathbf{A})$ is the quadripotential. The Faraday tensor is antisymmetric ($F_{\nu\mu} = -F_{\mu\nu}$). From the components of the quadripotential we define the electromagnetic field by

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla U, \qquad \mathbf{B} = \nabla \times \mathbf{A}. \tag{A65}$$

The Faraday tensor (A64) can be expressed in terms of the electromagnetic field as

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}. \tag{A66}$$

Its contravariant components are

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}. \tag{A67}$$

Since $F_{\mu\nu}F^{\mu\nu} = 2B^2 - 2E^2/c^2$, we can rewrite the electromagnetic Lagrangian as

$$\mathcal{L} = \frac{1}{2}\epsilon_0 E^2 - \frac{B^2}{2\mu_0}. \tag{A68}$$

Appendix F.2. Maxwell Equations

The fundamental equations of electromagnetism are the Maxwell equations. In covariant notations, the first pair of Maxwell equations reads

$$\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0, \tag{A69}$$

and the second pair of Maxwell equations reads

$$\partial_\nu F^{\mu\nu} = -\mu_0 J_e^\mu, \tag{A70}$$

where $J_e^\mu = (\rho_e c, \mathbf{J}_e)$ is the quadricurrent of charge. In components form, the first pair of Maxwell equations reads

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \cdot \mathbf{B} = 0, \tag{A71}$$

and the second pair of Maxwell equations reads

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon_0}, \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}_e + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}. \tag{A72}$$

We recall that $\mu_0\epsilon_0 = 1/c^2$.

Appendix F.3. Charge Conservation Equation

The conservation of charge is expressed by the equation of continuity

$$\partial_\mu J_e^\mu = 0. \tag{A73}$$

Therefore, the divergence of the left hand side of Equation (A70) must be zero. This is actually the case because of the antisymmetry of the Faraday tensor. The conservation of charge is therefore included in the Maxwell equations. In components form, we obtain

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \mathbf{J}_e = 0. \tag{A74}$$

Appendix F.4. Lorentz Gauge

The Lorentz gauge corresponds to

$$\partial_\mu A^\mu = 0. \tag{A75}$$

In components form, it can be written as

$$\frac{1}{c^2}\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{A} = 0. \tag{A76}$$

In the nonrelativistic limit, it reduces to the Coulomb gauge

$$\nabla \cdot \mathbf{A} = 0. \tag{A77}$$

Appendix F.5. Wave Equations

Using the Maxwell equations with the Lorentz gauge, we obtain

$$\Box A^\mu = \mu_0 J_e^\mu. \tag{A78}$$

This equation can be written in components form as

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \Delta \mathbf{A} = \mu_0 \mathbf{J}_e, \tag{A79}$$

$$\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} - \Delta U = \frac{\rho_e}{\epsilon_0}. \tag{A80}$$

In the absence of charge and current ($\rho_e = 0$ and $\mathbf{J}_e = \mathbf{0}$), the foregoing equations reduce to the usual wave equation (d'Alembert's equation):

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \Delta \mathbf{A}, \qquad \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = \Delta U. \tag{A81}$$

In electrostatics and magnetostatics, they reduce to

$$\Delta \mathbf{A} = -\mu_0 \mathbf{J}_e, \qquad \Delta U = -\frac{\rho_e}{\epsilon_0}. \tag{A82}$$

Remark: If we introduce a mass term in Equation (A80) we obtain an equation of the form

$$\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} - \Delta U + \frac{\mu^2 c^2}{\hbar^2} U = -g\rho, \tag{A83}$$

which corresponds to a Yukawa type of interaction [121]. This equation can also be interpreted as a KG equation with a source term. When coupled to the Schrödinger equation by using $\rho = |\psi|^2$ we obtain the Schrödinger-Yukawa equations (see, e.g., [122]) which describe a system of conserved scalar nucleons interacting with neutral scalar mesons of mass μ. Here, ψ represents a complex scalar nucleon field and U a real scalar meson field. The constant μ describes the mass of the meson and $g > 0$ is a coupling constant inducing an attractive interaction similar to the gravitational attraction (but with a different amplitude). In the nonrelativistic limit we can replace Equation (A83) by an equation of the form

$$i\hbar\frac{\partial U}{\partial t}=-\frac{\hbar^2}{2\mu}\Delta U+\lambda\rho. \tag{A84}$$

More generally, we could introduce a mass term in Equation (A78) leading to the Proca [123,124] equation for spin 1 bosons

$$\Box A^\mu+\frac{\mu^2c^2}{\hbar^2}A^\mu=-gJ^\mu, \tag{A85}$$

which could be coupled to the KG equation. The solutions of the Schrödinger-Yukawa and KG-Proca equations are spatially localized fields (solitons) which are large in a small spatial region and fall off rapidly as one leaves this region. The Schrödinger-Yukawa equations could provide a model of particles in field theory [125–130] similar to Newtonian boson stars described by the Schrödinger-Poisson equations [131] but with different scales (see the introduction of Ref. [60] for a review on boson stars and on models of extended particles). Interestingly, the Schrödinger-Yukawa equations of particle physics were introduced before the Schrödinger-Poisson equations of boson stars in astrophysics.[42]

Appendix F.6. Electromagnetic Energy

The density of electromagnetic energy is

$$u_e=\frac{1}{2}\epsilon_0\mathbf{E}^2+\frac{\mathbf{B}^2}{2\mu_0}. \tag{A86}$$

Using the identity of vector analysis $\nabla(\mathbf{a}\times\mathbf{b})=\mathbf{b}\cdot(\nabla\times\mathbf{a})-\mathbf{a}\cdot(\nabla\times\mathbf{b})$ and the Maxwell equations, we obtain the equation

$$\frac{\partial u_e}{\partial t}+\nabla\cdot\mathbf{S}_e=-\mathbf{J}_e\cdot\mathbf{E}, \tag{A87}$$

where

$$\mathbf{S}_e=\mathbf{E}\times\frac{\mathbf{B}}{\mu_0} \tag{A88}$$

is the energy current (Poynting vector) [132].

In the electrostatic case, the total electric energy can be written as

$$U_e=\frac{1}{2}\epsilon_0\int\mathbf{E}^2\,d\mathbf{r}=\frac{1}{2}\epsilon_0\int(\nabla U)^2\,d\mathbf{r}=-\frac{1}{2}\epsilon_0\int U\Delta U\,d\mathbf{r}=\frac{1}{2}\int\rho_e U\,d\mathbf{r}, \tag{A89}$$

where we have used the Poisson Equation (A82).

Appendix F.7. Electromagnetic Energy-Momentum Tensor

The energy-momentum tensor is given in terms of the Lagrangian by

$$T_{\mu\nu}=\frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu}} \tag{A90}$$

or, equivalently, by

$$T_{\mu\nu}=2\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}}-g_{\mu\nu}\mathcal{L}. \tag{A91}$$

Under this form, it is manifestly symmetric. For the electromagnetic Lagrangian (A62), we obtain

$$T_{\mu\nu}=\frac{1}{\mu_0}F_{\mu\alpha}F^\alpha{}_\nu-g_{\mu\nu}\mathcal{L} \tag{A92}$$

or, equivalently,

$$T_{\mu\nu} = \frac{1}{\mu_0}\left(F_{\mu\alpha}F^{\alpha}_{\ \nu} + \frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right). \tag{A93}$$

The energy-momentum tensor of the electromagnetic field has the property that $T = T^{\mu}_{\mu} = 0$. Its components are

$$T^{00} = \frac{1}{2}\epsilon_0 E^2 + \frac{B^2}{2\mu_0} = u_e, \tag{A94}$$

$$T^{0i} = T^{i0} = \frac{1}{\mu_0 c}\mathbf{E}\times\mathbf{B} = \frac{\mathbf{S}_e}{c}, \tag{A95}$$

$$T^{ij} = -\epsilon_0 E_i E_j - \frac{1}{\mu_0}B_i B_j + \delta_{ij}\left(\frac{1}{2}\epsilon_0 E^2 + \frac{B^2}{2\mu_0}\right) = -\sigma^{ij}. \tag{A96}$$

We note that T^{00} represents the energy density u_e. On the other hand, $T^{0i}c = \mathbf{S}_e$ represents the energy current (Poynting vector) and $T^{i0}/c = \mathbf{S}_e/c^2$ represents the electromagnetic momentum density. They are equal up to a factor c^2. Finally, T^{ij} is the opposite of the Maxwell stress tensor. It represents the momentum current. The total quadrimomentum is given by $p_e^{\mu} = \frac{1}{c}\int T^{\mu 0} d^3x$. In components form $p_e^{\mu} = (U_e/c, \mathbf{p}_e)$, where $U_e = \int u_e\, d\mathbf{r}$ is the energy and $\mathbf{p}_e = \frac{1}{c}\int T^{i0}\, d\mathbf{r}$ is the momentum. In the absence of charge and current, U_e and $\mathbf{p}_e$ are conserved. The local conservation of momentum and energy reads

$$\partial_{\mu} T^{\mu\nu} = 0. \tag{A97}$$

In components form, we obtain

$$\frac{1}{c}\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{0i}}{\partial x_i} = 0, \tag{A98}$$

$$\frac{1}{c}\frac{\partial T^{i0}}{\partial t} + \frac{\partial T^{ij}}{\partial x_i} = 0. \tag{A99}$$

In the presence of charge and current, one can show that

$$\partial_{\mu} T^{\mu\nu} = -F^{\nu\lambda} J^e_{\lambda}, \tag{A100}$$

where the r.h.s. is minus the Lorentz quadriforce density $f^{\mu} = F^{\mu\lambda}J^e_{\lambda}$. In components form, $f^{\mu} = \left(\frac{1}{c}\mathbf{J}_e\cdot\mathbf{E}, \rho_e\mathbf{E} + \mathbf{J}_e\times\mathbf{B}\right)$. The time component of Equation (A100) reads

$$\frac{\partial u_e}{\partial t} + \nabla\cdot\mathbf{S}_e = -\mathbf{J}_e\cdot\mathbf{E}, \tag{A101}$$

in agreement with Equation (A87), and its space component reads

$$\frac{1}{c^2}\frac{\partial \mathbf{S}_e}{\partial t} - \partial_j\sigma^{ij} = -\rho_e\mathbf{E} - \mathbf{J}_e\times\mathbf{B}. \tag{A102}$$

Appendix F.8. Analogies between the Energy and the Momentum of the Electromagnetic Field and the Energy and the Momentum of a Particle

For a light wave, one can show [77] from the electromagnetic equations that $p_e^{\mu} = Ak^{\mu}$, where the coefficient of proportionality A between the quadrimomentum p_e^{μ} and the quadriwavenumber k^{μ} is a scalar. This shows the analogy between the wave vector of a wave and the momentum of a particle. This result is only based on the Maxwell equations. It could have anticipated the relation (229) found by Planck [94] and Einstein [95] for the radiation and generalized by de Broglie [93] to matter waves (see [9] for an historical account).

For a plane wave, one can show [77] that the Poynting vector (energy current) is given by

$$\mathbf{S}_e = u_e c\, \mathbf{n}, \tag{A103}$$

where u_e is the electromagnetic energy density, in accordance with the fact that the wave propagates with the velocity of light. Recalling that the electromagnetic momentum density is given by $\mathbf{S}_e/c^2$ we find that the momentum of the electromagnetic wave is related to the energy by

$$p_e = \frac{U_e}{c}. \tag{A104}$$

This relation is similar to the relation $E = pc$ characterizing a massless particle moving with the speed of light in the theory of relativity. It reveals the corpuscular nature of light made of photons. This is also a consequence of the dispersion relation $\omega = kc$ combined with Equation (229).

In the model of extended electron developed by Abraham [133] and Lorentz [134], the electron is considered as a spherical charge of radius R (which must be nonzero to avoid infinite energy accumulation) with a charge e uniformly distributed on its surface. It was originally believed that the mass of the electron had a purely electromagnetic nature [135]. The electromagnetic energy of the electron at rest (reducing to its electrostatic energy) is

$$E_0 = \frac{1}{2}\frac{e^2}{R}. \tag{A105}$$

On the other hand, the electromagnetic momentum of the electron (obtained from the Poynting vector) is [136]

$$\mathbf{p} = \frac{2}{3}\frac{e^2}{Rc^2}\frac{\mathbf{v}}{\sqrt{1-v^2/c^2}}. \tag{A106}$$

For slow motion $v \ll c$ the momentum is proportional to the velocity like in the classical mechanical relation $\mathbf{p} = m\mathbf{v}$. This suggests introducing the "electromagnetic mass" of the electron at rest by

$$m_e = \frac{2}{3}\frac{e^2}{Rc^2} = \frac{4}{3}\frac{E_0}{c^2}. \tag{A107}$$

These relations were discovered before Einstein's theory of relativity. Equation (A106) shows that the mass $m = m_e/\sqrt{1-v^2/c^2}$ of the electron depends on the velocity and that the velocity cannot be larger than the speed of light. Actually, Equation (A106) is equivalent to the relation $\mathbf{p} = m_e\mathbf{v}/\sqrt{1-v^2/c^2}$ between the momentum and the velocity of a particle of mass m_e in the theory of relativity. On the other hand, Equation (A107) can be written as $E_0 = \frac{3}{4}m_e c^2$. This relation was first obtained by Wien [137]. It differs from the rest mass energy $E_0 = m_e c^2$ obtained in the theory of relativity by a factor $3/4$ (see Appendix F of [60] for more comments about early models of the extended electron).

Notes

1 Since the wave mechanics of Schrödinger [1–4] is equivalent [10] to the matrix mechanics of Heisenberg–Born–Jordan [11–13], we can consider that the Schrödinger equation stems from matrix mechanics. This is how Dirac [14] introduces the Schrödinger equation in his book, starting from quantum commutation relations. However, matrix mechanics and noncommutative algebra are very formal and rely on some axioms. The Schrödinger equation can also be derived from the path integral approach of Feynman [15], which permits one to correlate quantum mechanics with classical mechanics very graphically. In addition to being pedagogical and intuitive, this approach proposes a very interesting interpretation of wave mechanics and makes a direct connection to classical mechanics by showing that the classical trajectory corresponds to the "most probable" path (the one that minimizes the action) and that all the quantum paths are concentrated around the classical path in the limit $\hbar \to 0$. In this sense, the path integral formalism provides a derivation of the principle of least action (Maupertuis, Hamilton) as the $\hbar \to 0$ limit of quantum mechanics. Furthermore, Feynman [15] derived the time-dependent Schrödinger equation from the path integral theory. However, the Feynman approach also relies on axioms and postulates that are not fully justified.

2 Schrödinger also believed that the electron was "extended" (instead of singular as believed by de Broglie) and that a wave packet represents its actual shape. He wrote [16]: "material points consist of, or are nothing but, wave-systems... the charge of the electron is not concentrated in a point, but is spread out through the whole space, proportional to the quantity $\psi\overline{\psi}$". However, he was bothered by the fact that the wave packet spreads out in time, as if the electron was getting bigger and bigger.

3 Another concern of Einstein regarding quantum mechanics was the problem of nonlocality [22].

4 According to Pais [23], Einstein demanded that the theory be strictly causal, that it shall unify gravitation and electromagnetism, that the particles of physics shall emerge as special solutions of the general field equations, and that the quantum postulates shall be a consequence of the general field equations. He believed that by dealing with microscopic phenomena, the results of quantum calculations would come out by themselves.

5 Prior to his work, several authors proposed an interpretations of the Schrödinger equation in terms of stochastic processes. Schrödinger [44], Fürth [45], Fényes [46], Weizel [47], Bohm and Vigier [48], Kershaw [49] and Comisar [50] tried to describe the motion of quantum particles in terms of a Markov process by analogy with Brownian motion. They pointed out the formal analogy between the Schrödinger equation and the Fokker–Planck equation, and introduced an imaginary diffusion coefficient $D = i\hbar/2m$ (this formula first appeared in the paper of Fürth [45]). In the work of Weizel [47], the random aspects of the motion of a quantum particle are due to the interaction with hypothetical particles that he called zerons. On the other hand, Bohm and Vigier [48] introduced the notion of random fluctuations arising from the interaction with a sub-quantum medium. However, despite some resemblances between quantum mechanical motion and diffusion phenomena as the result of the formal similarity between the Schrödinger equation and the diffusion equation, Takabayasi [37,38] emphasized that the nature of the stochastic process in the two cases is very different. In the quantum theory, the trajectories may have very complicated fluctuations, but these fluctuations are not at random, since, for each individual trajectory, they are completely determined by the density $\rho = |\psi|^2$ appearing in the quantum potential. In contrast, in Brownian theory, the particle experiences a fluctuating force that is uncorrelated at every successive time and independent of the probability distribution ρ.

6 The idea of Nottale's theory of scale relativity is that the trajectories of the particles are intrinsically nondifferentiable. This nondifferentiability manifests itself at small scales (leading to quantum mechanics) but becomes imperceptible at large scales (leading to classical mechanics). This corresponds to the quantum–classical transition. The principle of scale covariance states that the equations of physics properly written should have the same form in the two regimes.

7 Nonlinear Schrödinger equations and nonlinear KG equations, possibly coupled to the Poisson or Einstein equations with a renormalized gravitational constant, have also been used to describe elementary particles (see the introduction of [60] for details and references).

8 Nottale [55] did not consider the electromagnetic case in the nonrelativistic theory. He only introduced the electromagnetic field in the relativistic formalism.

9 The details of the calculations can be found in, e.g., Appendix E of [9] (this amounts to following the steps of Section 2.3 of the present paper with $\hbar = 0$). These equations can also be directly obtained from the least action principle, as explained by Landau and Lifshitz [77].

10 This is sometimes called the correspondence principle.

11 Comparing Equation (14) with the expression of the current in Equation (10), we find that $\mathbf{J} = \rho\mathbf{v}$. As a result, the continuity Equation (18) is equivalent to Equation (9).

12 Actually, the Madelung hydrodynamic equations are not fully equivalent to the Schrödinger equation [78,79]. To achieve perfect equivalence, we must assume that $m\mathbf{v} + e\mathbf{A}$ is equal to a gradient ∇S. Furthermore, we must add by hand a quantization condition $\oint (m\mathbf{v} + e\mathbf{A}) \cdot d\mathbf{l} = nh$, where n is an integer, as in the old Bohr–Sommerfeld quantum theory. This ensures that the wave function is single-valued ($\psi \propto e^{iS/\hbar}$ with $S/\hbar = \theta = \mathrm{mod}[2\pi]$). Using the Stokes theorem, we obtain $\int (\nabla \times \mathbf{v} + \frac{e}{m}\mathbf{B}) \cdot d\mathbf{S} = nh/m$. In the nonelectromagnetic case, the vorticity $\nabla \times \mathbf{v}$ vanishes everywhere except on certain singular points, where it has δ-type singularities. These arguments led Onsager [80] and Feynman [81,82] to conjecture that superfluids can sustain singular point vortices with circulation quantized in units of h/m. These types of arguments (including the electromagnetic field) were also developed by Dirac [83] in his theory of monopoles.

13 More generally, the definition of $d_\pm$ applies to an arbitrary function $f(\mathbf{r}, t)$.

14 This extra term can be expressed in terms of the current of charge $\mathbf{J}_e$ by using the Maxwell Equation (A72).

15 This equation with a complex action $\mathcal{S}$ was obtained by the pioneers of quantum mechanics (see [8,9]), but they did not have the idea to introduce a complex velocity field leading to the complex Lorentz Equation (84).

16 This is essentially how Schrödinger introduced the wave function in his first communication [1] on quantum mechanics, although he thought at that time that the wavefuction was real (he wrote $S = \hbar \ln \psi$ or $\psi = e^{S/\hbar}$). This transformation may have been inspired by the analogy between mechanics and optics developed by Hamilton, Jacobi and de Broglie (see Section 4) that Schrödinger discussed in his second communication [2] (see also [16]). The relation $\psi \propto e^{iS/\hbar}$ between the wavefunction and the action in quantum mechanics is similar to the relation $\varphi \propto e^{i\theta}$ between the wave and the eikonal (phase) in optics with the correspondence $\theta = S/\hbar$ (see Note 34).

17 The usual Cole–Hopf transformation is used in fluid mechanics to transform the viscous Burgers equation (pressureless Navier–Stokes equation) into the diffusion equation. Similarly, the complex Cole–Hopf transformation allows us to transform the complex

viscous Burgers equation (scale covariant equation of dynamics) into the Schrödinger equation, which is similar to a complex diffusion equation (see Appendix A.2 of [73]).

18 This may be an advantage of Nottale's approach, because the validity of these Fokker–Planck equations has been criticized (see Section 7.1).

19 In this paper, we only write the relativistic equations in quadrivectorial form. Their component form is given in our companion paper [9].

20 The details of the calculations can be found in, e.g., Appendix D of [9] (this amounts to following the steps of Section 3.3 of the present paper with $\hbar = 0$). These equations can also be directly obtained from the least action principle, as explained by Landau and Lifshitz [77].

21 This difficulty, together with the fact that the KG equation is a second order differential equation in time, led Dirac [88] to propose another relativistic wave equation (see, e.g., [9] for an historical account).

22 Comparing Equation (136) with the expression of the current in Equation (132), we find that $J^\mu = \rho u^\mu$. As a result, the continuity Equation (139) is equivalent to Equation (131). We can also write the quadricurrent as $J^\mu = \rho_m v^\mu$, where $\rho_m = J^0/c = \rho u^0/c$ is the rest-mass density of the bosons and $v^\mu = (\rho/\rho_m)u^\mu = cu^\mu/u^0$ is the associated quadrivelocity.

23 As in Note 12, in order to achieve perfect equivalence, we must impose the Bohr–Sommerfeld quantization condition $\int (mu_\mu + eA_\mu) d\xi^\mu = nh$ with $n = 0, \pm 1, \pm 2, \ldots$ Using the Stokes theorem, we obtain $\int [m(\partial_\mu u_\nu - \partial_\nu u_\mu) + eF_{\mu\nu}] dS^{\mu\nu} = nh$ [90].

24 We remain here at a basic level. There are indeed mathematical difficulties with the construction of a relativistic stochastic theory, as discussed in [51–54,91].

25 The backward velocity may be connected to antiparticles moving forward in time in Feynman's picture of relativistic quantum mechanics [52,53].

26 Our approach justifies the scale covariant equation of Nottale [55], which was introduced in an ad hoc manner in the electromagnetic case.

27 This extra term can be expressed in terms of the quadricurrent of charge J_e^μ by using the Maxwell Equation (A70).

28 This equation, with a complex action $\mathcal{S}$, was obtained by the pioneers of quantum mechanics (see [8,9]), but they did not have the idea to introduce a complex quadrivelocity field leading to the complex Lorentz Equation (191).

29 See Note 18 above.

30 In the 17th and 18th centuries, several natural philosophers studied the phenomenon of refraction and attempted to derive the Snell–Descartes law from a variational principle [92]. The principle of minimal action was outlined by Fermat (1662), who looked for the "fastest" path, and by Leibniz (1646–1716), who looked for the "easiest" path. It was given a more general form by Maupertuis (1744 and 1746), who elevated the principle of minimal action to the status of a fundamental law of nature. He was convinced that he had discovered a "metaphysical law" established by God: "Nature, in the production of its effects, always acts by the simplest means". He thus argued that his principle of minimal action could cover more general cases than light. The principle introduced by Maupertuis was established in a rigorous way by Euler (1707–1783) and Lagrange (1736–1813). The correct statement of this principle and its precise relationship to geometric optics was established by Hamilton (1805–1865) a century later through the important principle that bears his name.

31 For a free particle, using $\mathbf{p} = m\mathbf{v}$ and $d\mathbf{l} = \mathbf{v}dt$, Maupertuis' action can be rewritten as $S = \int mv^2 dt$, where we recognize the kinetic energy (without the factor 1/2). This is twice the Lagrangian of a free particle ($S = \int L\, dt$ with $L = mv^2/2$).

32 The Planck–Einstein relation is $E = \hbar\omega$ [94,95]. de Broglie [93] noted that energy and momentum are inseparable parts in relativity, so there must be an associated relation $\mathbf{p} = \hbar\mathbf{k}$. This indivision is nicely expressed in quadrivectorial form by Equation (229).

33 We treat the wave function φ here as a complex scalar field. As a result, the particle number is conserved [this is ascertained by the equation of continuity (237)]. In wave optics, the wave function is usually real and the particle (photon) number is not conserved.

34 Fundamentally, the relations $\omega = -\partial\theta/\partial t$ and $\mathbf{k} = \nabla\theta$ can be understood by locally expanding the phase in Taylor series, writing $\theta = \theta_0 + \mathbf{r}\cdot\nabla\theta + t\frac{\partial\theta}{\partial t}$, and comparing this expression with the phase of a plane wave $\theta = \mathbf{k}\cdot\mathbf{r} - \omega t$. Using the de Broglie relations $E = \hbar\omega$ and $\mathbf{p} = \hbar\mathbf{k}$, we obtain the relations $E = -\partial S/\partial t$ and $\mathbf{p} = \nabla S$ of the Hamilton-Jacobi theory with $S = \hbar\theta$. The relation $\varphi \propto e^{i\theta}$ between the wave and the eikonal (phase) in optics then translates into the relation $\psi \propto e^{iS/\hbar}$ between the wavefunction and the action in quantum (wave) mechanics.

35 In his first papers, Nottale considered his fractal spacetime approach as a new formulation of Nelson's stochastic quantum mechanics. However, he then realized that most of the equations and the hypothesis of stochastic mechanics were unnecessary in the scale relativity theory, and that the interpretation, including that of the common fluctuation Equations (35) and (36), was very different (we refer to [55] for a more thorough discussion of these differences).

36 In this paper (see also [9]), following Nottale [55], we have derived the Schrödinger equation from the complex Newton and Hamilton-Jacobi equations. Conversely, we can directly derive the complex Newton and Hamilton-Jacobi equations from the Schrödinger equation. This is the presentation that we have followed in [8]. This reversed presentation is interesting in its own right and shows the fundamental importance of these complex equations.

37 In the scale relativity approach, the elementary description is made in terms of a twin stochastic process, but it is not associated with a classical diffusion interpretation, since it is understood as a direct manifestation of the nondifferentiable geometry of spacetime [55].

38 Nottale [55] proposes another solution, based on the introduction of more general time derivative operators, to cure this problem, but this solution leads to a more complicated formalism that "spoils" the original simplicity of the covariance principle.

39 A problem with the stochastic electrodynamics approach [108] is that it applies only to charged particles. In this sense, the approach of Calogero [113] is more general.

40 As detailed in the introduction, the Schrödinger equation was originally introduced by other arguments (see [9] for a precise historical account).

41 In Ref. [16], Schrödinger mentioned that the integrand of Equation (A14) can be obtained from the ordinary Hamiltonian $H = \mathbf{p}^2/2m + m\Phi$ of a particle by replacing $\mathbf{p}$ by $\hbar\nabla\psi$ and 1 by ψ.

42 We missed this early literature in our review discussion of Ref. [60].

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