Moments of a Binned Distribution

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Introduction

The random multipole errors of the SSC dipole magnets are expected to occur with a Gaussian distribution ρ_o. In the case of the sextupole error, for example, the error distribution has a σ_o which is larger than desired, and a correction winding will be added to the dipole to cancel this error. Instead of powering every correction winding separately, the proposal has been made to provide power supplies in evenly spaced increments of correction strength. Then each corrector would be connected to whichever power supply came closest to bringing the residual sextupole error to zero. This method is called binning of the errors, and each correction power supply corresponds to a "bin" to which a given corrector may be assigned. The result is a "binned" distribution ρ_b. The object of this note is to calculate the moments of the binned distribution, where the moments χ_n of any distribution ρ are defined as

$$\chi_n = \int_{-\infty}^{\infty} dx\ x^n\ \rho(x)\ . \tag{1}$$

If the corrector supplies are assigned strengths of 0, $\pm 2\Delta$, $\pm 4\Delta$, ..., then the bins have a half width of Δ. Let x_o be the initial error for some particular magnet. Then the residual error x is given by $x = x_o - 2n\Delta$, where $|x_o - 2n\Delta|$ is the minimum for integers n. If each error is assigned to the correct bin then the residual error will have a maximum magnitude of Δ. This would result in a new error distribution ρ_b which is zero for residual errors x with $|x| > \Delta$. In practice, however, there are not an infinite number of bins available, and also there is an error in the measurement of x_o.

The case with an infinite number of bins lends itself to analytical solution, even when there is a Gaussian distribution of measurement error. However, the real figure of merit of the binning technique is the number of bins necessary to achieve a given result. The effect of a finite number of bins is to produce tails extending from the ideal binned distribution, and their effect can be treated approximately.

Analysis

The Fourier transform of the distribution $\rho(x)$ provides a generating function for the moments χ_n. The Fourier transform of $\rho(x)$ is

$$\hat{\rho}(k) = \int_{-\infty}^{\infty} dx \; e^{-ikx} \; \rho(x) \; , \tag{2}$$

and the transform of x^n is $2\pi i^n \; \delta^{(n)}(k)$. Substituting these into equation (1),

$$\begin{aligned} \chi_n &= (-i)^n \int_{-\infty}^{\infty} dk \; \delta^{(n)}(k) \; \hat{\rho}(k) \\ &= i^n \left[\hat{\rho}^{(n)}(k)\right]_{k=0} \; . \end{aligned} \tag{3}$$

Hence, the problem can be simplified by finding the Fourier transform of the actual distribution and differentiating the necessary number of times.

First consider the case with an infinite number of bins. Assume henceforth that all dimensions are scaled to the σ_o of the original error distribution. The original normalized distribution is

$$\rho_o(x) = (2\pi)^{-\frac{1}{2}} \; e^{-\frac{1}{2}x^2} \; , \tag{4}$$

and the ideal binned distribution may be represented as

$$\rho_b(x) = \begin{cases} (2\pi)^{-\frac{1}{2}} \; \sum\limits_n \; e^{-\frac{1}{2}(x-2n\Delta)^2} & |x| < \Delta \; , \\ 0 & |x| > \Delta \; . \end{cases} \tag{5}$$

The binned distribution ρ_b may be represented conveniently by the Fourier series: For $|x| < \Delta$,

$$\rho_b(x) = \sum_m \rho_{bm}\, e^{i\pi m x/\Delta} \,, \tag{6}$$

$$\rho_{bm} = (2\pi)^{-\frac{1}{2}}\, \frac{1}{2\Delta} \int_{-\Delta}^{\Delta} dx\, e^{-i\pi m x/\Delta} \sum_n e^{-\frac{1}{2}(x-2n\Delta)^2}$$

$$= (2\pi)^{-\frac{1}{2}}\, \frac{1}{2\Delta} \int_{-\infty}^{\infty} dx\, e^{-i\pi m x/\Delta}\, e^{-\frac{1}{2}x^2} \tag{6a}$$

$$= \frac{1}{2\Delta}\, e^{-\frac{1}{2}(\pi m/\Delta)^2} \,.$$

From equation (6), the Fourier transform of ρ_b can be found:

$$\hat{\rho}_b(k) = \sum_m e^{-\frac{1}{2}(\pi m/\Delta)^2}\, \frac{\sin(\Delta k - \pi m)}{\Delta k - \pi m} \,. \tag{7}$$

If the distribution has been smeared by Gaussian measurement error in x_o, then the smeared distribution ρ is given by

$$\hat{\rho}_b(k) = e^{-\frac{1}{2}s^2k^2} \sum_m e^{-\frac{1}{2}(\pi m/\Delta)^2}\, \frac{\sin(\Delta k - \pi m)}{\Delta k - \pi m} \,, \tag{8}$$

where s is the RMS measurement error of x_o, in units of σ_o.

Now consider the case with a finite number of bins N. In general, N is an odd integer, and when $N = 1$ the result is the unmodified Gaussian. The correction to $\rho(x)$ can be represented approximately as

$$\rho_t(x) = \begin{cases} -\frac{1}{2\Delta}\, \mathrm{erfc}\left(\frac{N\Delta}{\sqrt{2}}\right) & |x| < \Delta \,, \\ (2\pi)^{-\frac{1}{2}}\, e^{-\frac{1}{2}(|x|+(N-1)\Delta)^2} & |x| > \Delta \,. \end{cases} \tag{9}$$

The correction is approximated as constant for $|x| < \Delta$ in order to normalize the distribution. This approximation is good when either N or Δ is large. For the trivial case of $N = 1$, the error in calculating σ using equations (7), (10), and (3) is at most 4%, and this also represents the worst case for the approximation. The Fourier transform of (9) is then

$$\hat{\rho}_t(k) = -\operatorname{erfc}\left(\frac{N\Delta}{\sqrt{2}}\right)\frac{\sin(\Delta k)}{\Delta k}$$

$$+\frac{1}{2}e^{-\frac{1}{2}k^2}\left[e^{i(N-1)\Delta k}\operatorname{erfc}\left(\frac{N\Delta + ik}{\sqrt{2}}\right) + e^{-i(N-1)\Delta k}\operatorname{erfc}\left(\frac{N\Delta - ik}{\sqrt{2}}\right)\right]. \tag{10}$$

Results

The moments χ_n can be separated into a value for $N = \infty$ calculated from generating function (8), and a contribution from the tails when $N < \infty$, calculated from generating function (10):

$$\chi_n(N) = \chi_{nb} + \chi_{nt}(N)\ . \tag{11}$$

The moment $\chi_2 \equiv \sigma^2$ is calculated as a function of Δ for fixed N. For a given number of bins N, the result will have a minimum for some bin width Δ. The result for an infinite number of bins is

$$\chi_{2b} = \left(\frac{1}{3} + 4f_2(\Delta)\right)\Delta^2 + s^2\ , \tag{12}$$

where f_2 is defined by

$$f_n(\Delta) = \sum_{m=1}^{\infty}(-1)^m\frac{1}{(\pi m)^n}e^{-\frac{1}{2}(\pi m/\Delta)^2}\ . \tag{13}$$

The correction is

$$\chi_{2t}(N) = \left[\left((N-1)^2 - \frac{1}{3}\right)\Delta^2 + 1\right]\operatorname{erfc}\left(\frac{N\Delta}{\sqrt{2}}\right) + \sqrt{\frac{2}{\pi}}(2-N)\Delta e^{-\frac{1}{2}(N\Delta)^2}\ . \tag{14}$$

Note that the measurement smear s contributes to σ in quadrature, so this can be ignored for the present. $\sigma(N)$ is plotted in Figure 1 versus Δ for $s = 0$ and $N = 3, 5, 7, 9, 11$, and ∞.

The moment χ_4 has also been calculated. For $N = \infty$,

$$\chi_{4b} = \left(\frac{1}{5} + 8f_2(\Delta) - 48f_4(\Delta)\right) \Delta^4 + (24f_2(\Delta) + 2)\,\Delta^2 s^2 + 3s^4 \ , \tag{15}$$

and the tail correction is

$$\begin{aligned}\chi_{4t}(N) = &\left[\left((N-1)^4 - \frac{1}{5}\right)\Delta^4 + 6(N-1)^2\Delta^2 + 3\right] \ \mathrm{erfc}\left(\frac{N\Delta}{\sqrt{2}}\right) \\ &+ \sqrt{\frac{2}{\pi}}\Delta\left[(2-N)(N^2-2N+2)\Delta^2 - 5N + 8\right] e^{-\frac{1}{2}(N\Delta)^2} \ .\end{aligned} \tag{16}$$

The quantity $\chi_4^{\frac{1}{4}}$ is plotted versus Δ in Figure 2, for $s = 0$ and the same values of N as were used above.

The values of Δ which optimize σ for a given number of bins N, and the corresponding values of χ_2 and χ_4 are summarized below in Table 1:

Table 1.

N	Δ	σ	$\chi_4^{\frac{1}{4}}$	χ_2	χ_4
3	0.62	0.44	0.62	0.194	0.1478
5	0.43	0.29	0.44	0.084	0.0375
7	0.34	0.22	0.34	0.048	0.0134
9	0.26	0.18	0.32	0.032	0.0105
11	0.23	0.15	0.27	0.023	0.0053

Note that the minima of χ_2 and χ_4 do not coincide, and so the values of χ_4 given here are not the minima.

A program has also been written by E. Forest [1] to calculate σ for a distribution which is originally cut off at σ_{max} and then is binned with $\Delta = \sigma_{max}/N$. The results for $\sigma_{max} = 2$ exactly coincide with the $N = \infty$ graph of Figure 1, for all values of $N > 1$. The results are also fitted by the curve $\sigma = \Delta/\sqrt{3}$. This is summarized in Table 2:

Table 2.

N	Δ	σ	$\Delta/\sqrt{3}$
3	0.67	0.38	0.39
5	0.40	0.23	0.23
7	0.29	0.16	0.16

Comparing Tables 1 and 2 shows a small contribution of the tail of the distribution beyond $\sigma_{max} = 2$.

References

1. E. Forest, private communication.

Figure 1.

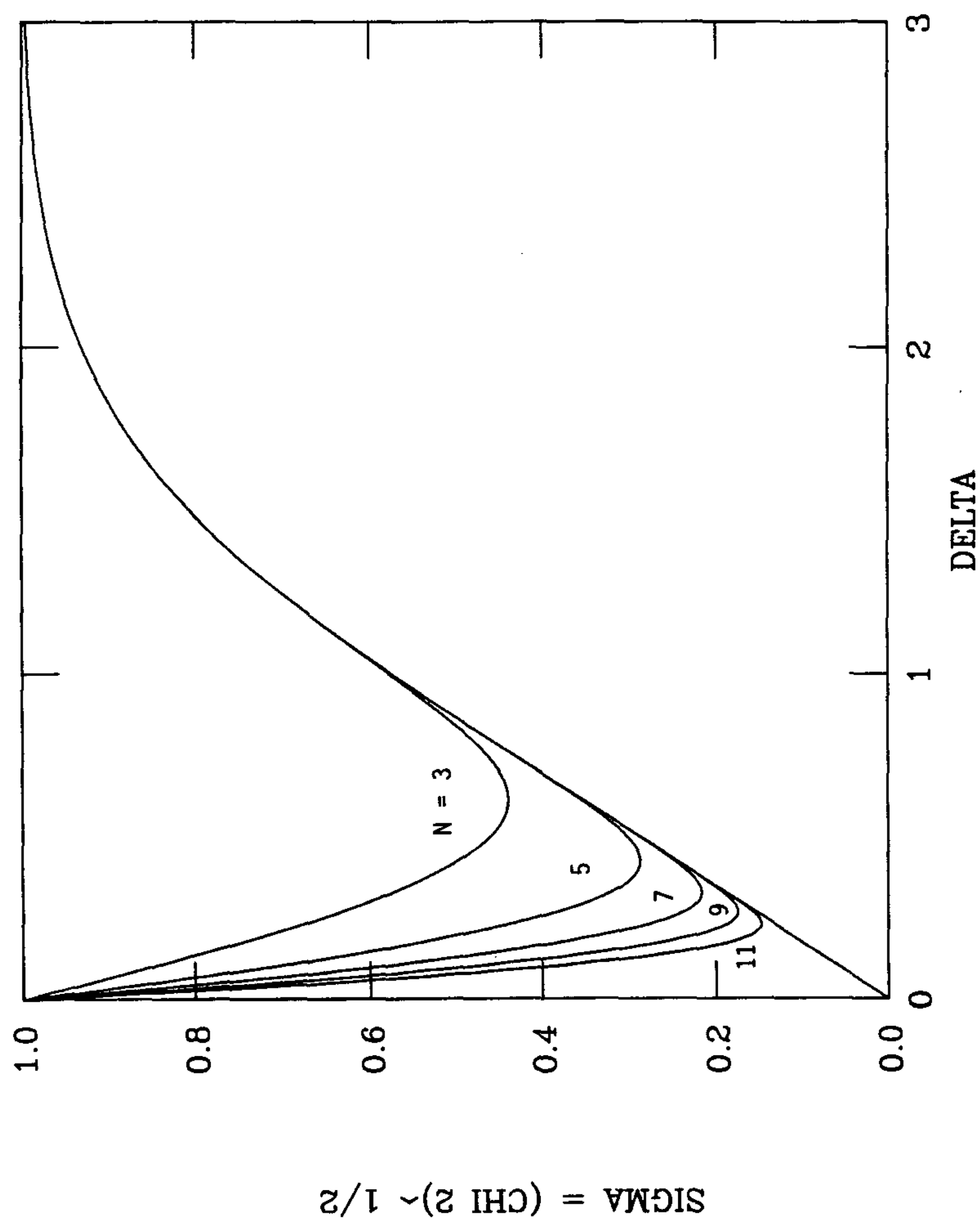

Figure 2.