Beyond δN formalism for a single scalar field

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Abstract

We develop a theory of nonlinear cosmological perturbations on superhorizon scales for a single scalar field with a general kinetic term and a general form of the potential. We employ the ADM formalism and the spatial gradient expansion approach and show the nonlinear theory called the *beyond* δN-formalism as the next-leading order in the expansion to the so-called δN-formalism as the leading order. We obtain the general solution for a full nonlinear version of the curvature perturbation valid up through the second-order in the expansion and find the solution satisfies a nonlinear second-order differential equation as an extension of the equation for the linear curvature perturbation on the comoving hypersurface. The formalism developed in this paper allows us to calculate the superhorizon evolution of a primordial non-Gaussianity beyond δN-formalism.

1 Introduction

The PLANCK satellite launched last year is expected to bring us much finer data and it is hoped that non-Gaussianity may actually be detected. As a consequence, non-Gaussianity from inflation has been a focus of much attention in recent years. To study possible origins of non-Gaussianity, the δN-formalism [2, 4, 5] turned out to be a powerful tool for the estimation of non-Gaussianity. We investigate a possible origin of non-Gaussianity, namely, non-Gaussianity due to a temporary non-slow roll stage on superhorizon scales. In order to investigate such a case, however, the δN-formalism is not sufficient since it is equivalent to the leading order approximation in the spatial gradient expansion. Thus, to evaluate such situation, it is necessary to develop a nonlinear theory of cosmological perturbations valid up through the next-leading order in the gradient expansion.

2 Beyond δN-Formalism

In this section, we will briefly review the nonlinear theory of cosmological perturbations valid up to $O(\epsilon^2)$ in the spatial gradient expansion and follow the previous works [6, 7], where ϵ is the ratio of the Hubble length scale $1/H$ to the characteristic length scale of perturbations L, used as a small expansion parameter, $\epsilon \equiv 1/(HL)$, of the superhorizon scales. First of all, we show the main result in our formula for the nonlinear curvature perturbation, $\mathcal{R}_c^{\rm NL}$,

$$\mathcal{R}_c^{\rm NL\prime\prime} + 2\frac{z'}{z}\mathcal{R}_c^{\rm NL\prime} + \frac{c_s^2}{4}K^{(2)}[\,\mathcal{R}_c^{\rm NL}\,] = O(\epsilon^4)\,, \tag{1}$$

which shows two full-nonlinear effects;

1. Nonlinear variable: $\mathcal{R}_c^{\rm NL}$ including full-nonlinear curvature perturbation, δN
2. Source term: $K^{(2)}[\mathcal{R}_c^{\rm NL}]$ is a nonlinear function of curvature perturbations.

In (1), the prime denotes conformal time derivative and z is a well-known Mukhanov-Sasaki variable. The explicit forms of both the definition of $\mathcal{R}_c^{\rm NL}$ and the source term $K^{(2)}[X]$, that is the Ricci scalar of

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the metric X, will be also seen later, in (5) and in (8), respectively. Of course, in the linear limit, it can be reduced to the well-known equation for the curvature perturbation on comoving hypersurfaces,$\mathcal{R}_c^{\rm Lin}{}'' + 2\frac{z'}{z}\mathcal{R}_c^{\rm Lin}{}' - c_s^2\,\Delta[\mathcal{R}_c^{\rm Lin}] = 0$.

We will briefly summarize our formula and show the above results in the following. Throughout this paper we consider a minimally-coupled single scalar field described by an action of the form $I = \int d^4x\sqrt{-g}P(X,\phi)$, where $X = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. Note that we do not assume the explicit forms of both kinetic term and its potential, that can be given as arbitrary function of $P(X,\phi)$. We adopt the ADM decomposition and employ the gradient expansion. In the ADM decomposition, the metric is expressed as $ds^2 = -\alpha^2dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$, where α is the lapse function, β^i is the shift vector and Latin indices run over $1,2,3$. We introduce the extrinsic curvature K_{ij} defined by $K_{ij} = -\frac{1}{2\alpha}\left(\partial_t\gamma_{ij} - D_i\beta_j - D_j\beta_i\right)$, where D is the covariant derivative compatible with the spatial metric γ_{ij}. As a result, the basic equations are reduced to the first-order equations for the dynamical variables (γ_{ij},K_{ij}), with the two constraint equations (the so-called Hamiltonian and Momentum constraint). We further decompose them as $\gamma_{ij} = a^2e^{2\zeta}\tilde{\gamma}_{ij}$ and $K_{ij} = a^2e^{2\zeta}\left(\frac{1}{3}K\tilde{\gamma}_{ij} + \tilde{A}_{ij}\right)$ where $a(t)$ is the scale factor of the background FRW universe and $\det\tilde{\gamma}_{ij} = 1$. Next, we will employ the gradient expansion. In this approach we introduce a flat FRW universe $(a(t),\ \phi_0(t))$ as a background. As discussed, we consider the perturbations on superhorizon scales, therefore we consider $\epsilon \equiv 1/(HL)$ as a small expansion parameter and systematically expand equations by ϵ. We assume the condition for the gradient expansion; $\partial_t\tilde{\gamma}_{ij} = O(\epsilon^2)$. This corresponds to assuming the absence of any decaying modes at the leading-order in the expansion. This is justified in most of the inflationary models.

When we focus on a contribution arising from the scalar-type perturbations, we may choose the gauge in which $\tilde{\gamma}_{ij}$ approaches the flat metric,

$$\tilde{\gamma}_{ij}\,(t \to \infty) = \delta_{ij}, \tag{2}$$

where in reality the limit $t \to \infty$ may be reasonably interpreted as an epoch close to the end of inflation. We take the *comoving slicing, time-orthogonal* gauge:

$$\delta\phi_c(t,x^i) = \beta_c^i(t,x^i) = O(\epsilon^3), \tag{3}$$

where $\delta\phi \equiv \phi - \phi_0$ denotes a fluctuation of a scalar field. The subscript c denotes this gauge throughout this paper. Now we turn to the problem of properly defining a nonlinear curvature perturbation to $O(\epsilon^2)$ accuracy. Hereafter we will use the expression $\mathcal{R}_c$ on comoving slices to denote it. Let us consider the linear curvature perturbation which is given as $\mathcal{R}^{\rm Lin} = \left(H_L^{\rm Lin} + \frac{H_T^{\rm Lin}}{3}\right)Y$, where, following the notation in [1], the spatial metric in the linear limit is expressed as $\gamma_{ij} = a^2(\delta_{ij} + 2H_L^{\rm Lin}Y\delta_{ij} + 2H_T^{\rm Lin}Y_{ij})$. These expressions in the linear theory correspond to the metric components in our notation as $\zeta = H_L^{\rm Lin}Y$ and $\tilde{\gamma}_{ij} = \delta_{ij} + 2H_T^{\rm Lin}Y_{ij}$. Notice that the variable ζ_c reduces to $\mathcal{R}_c^{\rm Lin}$ at leading-order in the gradient expansion, but not at second-order and it will be also similar to the nonlinear theory. Thus to define a nonlinear generalization of the linear curvature perturbation, we need nonlinear generalizations of H_LY and H_TY. Our nonlinear ζ is an apparent natural generalization of $H_L^{\rm Lin}Y$ as $H_LY = \zeta$. As for H_TY, however, the generalization is non-trivial. It corresponds to the $O(\epsilon^2)$ part of $\tilde{\gamma}_{ij}$ and we have obtained a general solution of the dynamical equation for $\tilde{\gamma}_{ij}$ as a first-order differential equation in [6, 7] and the time-dependent part includes the following solution; $\tilde{\gamma}_{ij}(t) \ni C_{ij}^{(2)}\int\frac{dt'}{a^3(t')}$ with the Momentum constraint: $e^{3\ell^{(0)}}\partial_iC^{(2)} = 6f_{(0)}^{jk}\partial_j\left[e^{3\ell^{(0)}}C_{ki}^{(2)}\right]$. The explicit forms of solutions can be seen in [6]. Here we attach the superscript (m) to a quantity of $O(\epsilon^m)$, and both $\ell^{(0)}$ and $f_{ij}^{(0)}$ will be denoted as the leading-order metric. Our aim is to derive the scalar-type solution $C^{(2)}$ from the tensor $C_{ij}^{(2)}$ by using the constraint eq. As shown in [7], it can be done by introducing the inverse Laplacian operator Δ^{-1} on the flat background and we defined the nonlinear generalization of H_TY as

$$H_TY = E \equiv -\frac{3}{4}\Delta^{-1}\left[\partial^ie^{-3\ell^{(0)}}\partial^je^{3\ell^{(0)}}(\ln\tilde{\gamma})_{ij}\right]. \tag{4}$$

It is easy to see that $E \ni C^{(2)}$ which we expected. At leading-order, the only non-trivial quantities for the spatial metric, ζ and $\tilde{\gamma}_{ij}$, are given by $\zeta = \ell^{(0)}(x^k) + O(\epsilon^2)$ and $\tilde{\gamma}_{ij} = f_{ij}^{(0)}(x^k) + O(\epsilon^2)$, where $\ell^{(0)}(x^k)$ is an arbitrary function of the spatial coordinates $\{x^k\}$ $(k = 1, 2, 3)$ and $f_{ij}^{(0)}(x^k)$ is a (3×3)-matrix function of the spatial coordinates with a unit determinant, respectively. Throughout this paper, we choose $f_{ij}^{(0)} = \delta_{ij}$ consistent with the gauge condition of (2). On the other hand, $\ell^{(0)}$ represents a conserved comoving curvature perturbation, which is denoted by the so-called δN term from some final uniform density (or comoving) hypersurface to the initial flat hypersurface at $t = t_*$, namely, $\ell^{(0)} = \delta N(t_*, x^i)$.

With these definitions of $H_L Y$ and $H_T Y$, we can define the nonlinear curvature perturbation valid up through $O(\epsilon^2)$ as

$$\mathcal{R}_c^{\rm NL} \equiv \zeta_c + \frac{E_c}{3}\,. \tag{5}$$

It is easy to show that this nonlinear quantity can be reduced to $\mathcal{R}_c^{\rm Lin}$ in the linear limit. As clear from (4), finding $H_T Y$ generally requires a spatially non-local operation, however, in the comoving slicing, time-orthogonal gauge with the asymptotic condition on the spatial coordinates (2), we find it is possible to obtain the explicit form of $H_T Y$ without any non-local operation as seen in [7]. Finally, we can derive a nonlinear second-order differential equation that $\mathcal{R}_c^{\rm NL}$ (5) satisfies at $O(\epsilon^2)$ accuracy by introducing the conformal time η, defined by $d\eta = dt/a(t)$ and the Mukhanov-Sasaki variable $z = \frac{a}{H}\sqrt{\frac{\rho+P}{c_s^2}}$. The result can be reduced to a simple equation of the form (1) as a natural extension of the linear version. We also obtain the solution of the nonlinear equation (1) as

$$\mathcal{R}_c^{\rm NL}(\eta) = \ell^{(0)} + \frac{1}{4}\big[F(\eta) - F_*\big]K^{(2)} + \big[D(\eta) - D_*\big]C^{(2)} + O(\epsilon^4), \tag{6}$$

where

$$D(\eta) = 3\mathcal{H}_* \int_\eta^0 \frac{z^2(\eta_*)}{z^2(\eta')}d\eta'\,, \quad F(\eta) = \int_\eta^0 \frac{d\eta'}{z^2(\eta')}\int_{\eta_*}^{\eta'} z^2 c_s^2(\eta'')d\eta''\,. \tag{7}$$

Here $D_* = D(\eta_*)$, $F_* = F(\eta_*)$ and $\mathcal{H}_*$ denotes the conformal Hubble parameter $\mathcal{H} = d\ln a/d\eta$ at $\eta = \eta_*$ which we take the time as some after the horizon crossing. Note that $t \to \infty$ corresponds to $\eta \to 0$ in the conformal time. Thus the functions D and F vanish asymptotically at late times, $D(0) = F(0) = 0$. Deviation of the solution (6) can be easily understood as follows. The second-order differential equation (1) contains two solutions, i.e. decaying mode and growing mode. We can find that the function $D(\eta)$ satisfies $D'' + 2\frac{z'}{z}D' = 0$ in the long-wavelength limit, i.e. no source term in (1). It corresponds to the decaying mode in the linear theory. On the other hand, the function $F(\eta)$ corresponds to the source term in (1), satisfying $F'' + 2\frac{z'}{z}F' + c_s^2 = 0$, as the $O(\epsilon^2)$ correction to a constant mode at the leading-order, i.e. as the growing mode in the linear theory, which is taken the form $1 + F(\eta)K^{(2)} + O(\epsilon^4)$.

Moreover the equation (1) includes two 'constants' of integration, or arbitrary spatial functions, which in general appear as the initial conditions. Let us consider the spatial functions; $\ell^{(0)}, C^{(2)}$ and $K^{(2)}$. Here the last one is related to the Ricci scalar of the 0th-order spatial metric as

$$K^{(2)}[\ell^{(0)}] = R\left[e^{2\ell^{(0)}}\delta_{ij}\right] = -2(2\Delta\ell^{(0)} + \delta^{ij}\partial_i\ell^{(0)}\partial_j\ell^{(0)})e^{-2\ell^{(0)}}. \tag{8}$$

Then we have the two arbitrary spatial functions: $\ell^{(0)}$ and $C^{(2)}$, which are related to the number of physical degrees of freedom for the initial conditions. Therefore they have to be determined by matching a solution of n-th order perturbation solved inside the horizon to this superhorizon solution at $\eta = \eta_*$.

3 Application

In this subsection, we calculate the bispectrum of our nonlinear curvature perturbation by assuming that $\mathcal{R}_{c,\boldsymbol{k}}^{\rm Lin}(\eta_k)$ is a Gaussian random variable with the horizon crossing time; $\eta_k = -\frac{r}{k}$ ($0 < r \ll 1$). We assume the leading order contribution to the bispectrum comes from the terms second order in $\mathcal{R}_{c,\boldsymbol{k}}^{\rm Lin}(\eta_k)$. The final result can be obtained by

$$\zeta_{\boldsymbol{k}} = G(k)\ \mathcal{R}_{c,\boldsymbol{k}}^{\rm Lin}(\eta_k) + H(k)\ \left\{\int \frac{d^3k'd^3k''}{(2\pi)^3}(4k'^2 - \delta_{ij}k'^i k''^j)\mathcal{R}_{c,\boldsymbol{k}'}^{\rm Lin}(\eta_{k'})\mathcal{R}_{c,\boldsymbol{k}''}^{\rm Lin}(\eta_{k''})\delta^3(-\boldsymbol{k}+\boldsymbol{k}'+\boldsymbol{k}'')\right\}, \tag{9}$$

where $G(k) = 1 + \frac{\tilde{D}_k}{3\mathcal{H}(\eta_k)}\frac{\mathcal{R}_c'}{\mathcal{R}_c}\Big|_{\eta=\eta_k} - k^2\tilde{F}_k$, $H(k) = \frac{1}{2}\tilde{F}_k$, $\tilde{D}_k = \tilde{D}(\eta_k)$ and $\tilde{F}_k = \tilde{F}(\eta_k)$. Here we defined the integrals $\tilde{D}$ and $\tilde{F}$ obtained by replacements η_k with η_* in their definitions of (7). In particular, it can

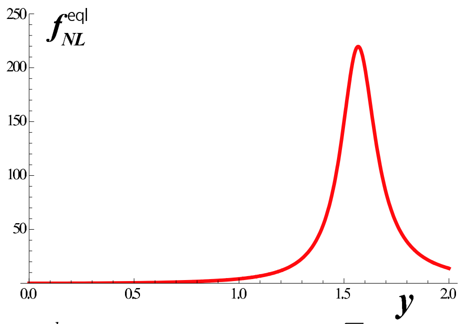

Figure 1: $f_{NL}^{eql}(k)$ as a function of $y = \sqrt{T}k/k_0$ for $T = 10^2$.

deal with the case when there is a temporary violation of slow-roll conditions. As one application of our formalism, we consider Starobinsky's model [3], which is described by the potential having its slope's step as $V(\phi) = V_0 + A_+(\phi - \phi_0)$ for $\phi > \phi_0$ or $V_0 + A_-(\phi - \phi_0)$ for $\phi < \phi_0$, where $A_+ > A_- > 0$ is assumed. The advantage of this model is that it allows analytical treatment of linear perturbations as well as of the background evolution, provided that V_0 dominates in the potential. If $A_+ \gg A_-$, and for ϕ initially large and positive, the slow-roll condition is violated right after ϕ falls below ϕ_0. We find that a large non-Gaussianity can be generated even on superhorizon scales due to this temporary suspension of slow-roll inflation as shown in Fig. 1. We have found that non-Gaussianity can become large if the parameter $T \approx A_+/A_-$, which characterises the ratio of the slope before and after the transition, is large. For $T \gg 1$, we have found that the non-Gaussianity parameter for the bispectrum $f_{NL}(k_1, k_2, k_3)$ is peaked at the wavenumbers forming an equilateral triangle, $k = k_1 = k_2 = k_3$, denoted by $f_{NL}^{eql}(k)$. It is found to be positive and takes the maximum value $f_{NL}^{eql}(k) \simeq 2T$ at $\sqrt{T}k/k_0 \simeq 1.5$ where k_0 is the comoving wavenumber that crosses the horizon at the time when the potential slope changes. This implies that, even for a relatively small T, say for $T = 10$, it is possible to generate a fairly large non-Gaussianity $f_{NL} \sim 20$ at wavenumber $k \simeq 0.5k_0$.

4 Summary

We have developed a theory of nonlinear cosmological perturbations on superhorizon scales for a single scalar field with a general kinetic term and a general form of the potential to the second-order in the spatial gradient expansion. The solution to this order is necessary to evaluate correctly the final amplitude of the curvature perturbation for models of inflation with a temporary violation of the slow-roll condition. We have introduced a reasonable variable that represents the nonlinear curvature perturbation on comoving slices $\mathcal{R}_c^{\rm NL}$, which reduces to the comoving curvature perturbation $\mathcal{R}_c^{\rm Lin}$ in the linear limit. Then we have found that $\mathcal{R}_c^{\rm NL}$ satisfies a nonlinear second-order differential equation, (1), as a natural extension of the linear second-order differential equation. Since the evolution of superhorizon curvature perturbations is genuinely due to the $O(\epsilon^2)$ effect, our formulation can be used to calculate the primordial non-Gaussianity beyond the δN formalism which is equivalent to leading order in the gradient expansion. As one application of our formalism, we have investigated Starobinsky's model [3] in which there is a temporary non-slow-roll stage during inflation due to a sudden change of the potential slope. We have found that non-Gaussianity can become large if the ratio of the slope before and after the transition is large.

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