# Higher dimensional and supersymmetric extensions of loop quantum gravity

Höherdimensionale und supersymmetrische Erweiterungen der Schleifenquantengravitation

Der Naturwissenschaftlichen Fakultät
der Friedrich-Alexander Universität
Erlangen-Nürnberg

zur Erlangung des Doktorgrades
Dr. rer. nat.

vorgelegt von
Andreas Thurn
aus Kemnath

EMERGING FIELDS INITIATIVE

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
NATURWISSENSCHAFTLICHE FAKULTÄT

ERLANGEN CENTRE FOR ASTROPARTICLE PHYSICS

Als Dissertation genehmigt von der Naturwissenschaftlichen Fakultät der
Friedrich-Alexander Universität Erlangen-Nürnberg

| | |
|---|---|
| Tag der mündlichen Prüfung: | 15. November 2013 |
| Vorsitzender des Promotionsorgans: | Prof. Dr. Johannes Barth |
| Gutachter: | Prof. Dr. Thomas Thiemann |
| | Prof. Dr. Jerzy Lewandowski |

# Abstract

In this work, we extend loop quantum gravity (LQG) both, to higher dimensions and supersymmetry (i.e. supergravity theories), thus overcoming the current limitation to 3+1 dimensions with standard model matter fields. On the one hand, this gives a proof of principle that LQG is in accordance with these two theoretical concepts, and on the other hand hopefully allows contact with superstring/M - theory, which necessarily is supersymmetric and formulated in ten or eleven spacetime dimensions. Symmetry arguments suggest that supergravity theories in the corresponding dimensions constitute the low energy effective field theory limit of superstring/M - theory. This makes a study of the loop quantisation thereof, which we start here, a promising endeavour at the border between the two approaches.

In more detail, our findings are the following: firstly, a new canonical formulation for general relativity in $D+1$ spacetime dimensions ($D \geq 2$) on a Yang Mills theory phase space is presented for the first time, with the core properties that 1. the canonical variables encoding the metric information are a real connection and its real conjugate momentum, in particular satisfying the standard canonical Poisson bracket relations, 2. the gauge group can be chosen to be a compact group (namely $\mathrm{SO}(D+1)$) for both, Lorentzian and Euclidean signature spacetimes, and 3. the system of constraints is first class (in Dirac's terminology). Up to now, such a formulation was only known for $D = 3$ (and $D = 2$), corresponding to Ashtekar Barbero variables, constituting the classical foundation of the loop quantisation programme.

The quantisation procedure itself is formulated almost independently of the number of spacetime dimensions and the choice of compact gauge group, and

therefore the lack of higher dimensional analogues of LQG only was caused by the missing classical canonical formulation satisfying 1. - 3. Thus it is not surprising and we show explicitly that the new formulation we present can be quantised using the methods developed in the loop community straightforwardly to obtain LQG theories in higher dimensions.

The formulation which we present is genuinely new in that it does not reduce to the Ashtekar Barbero formulation for $D = 3$, and furthermore for $D > 2$ comes with an additional set of constraints, the so called simplicity constraints, which pose the only conceptually new challenge when quantising. Interestingly, these constraints are not at all unknown in (quantum) gravity research, and in particular are a standard ingredient in the covariant approach to LQG called spin foam models. The formulation in this sense builds a novel bridge between the covariant and canonical approaches to LQG. The quantum anomalies known for this constraint from spin foams are recovered, which lead to problems when implementing it at the quantum level. We present some new proposals of how to deal with these problems.

In the second part of this work, we give an extension of the above framework to the loop quantisation of a large class of Lorentzian signature supergravities, including in particular the $D + 1 = 4$ $N = 8$, $D + 1 = 11$ $N = 1$ and $D + 1 = 10$ $N = 1$ theories. Concretely, we incorporate standard and also non-standard matter fields, which appear in supergravity theories due to the requirement of supersymmetry, into the afore developed framework of higher dimensional LQG.

Coupling to standard model matter fields has already been achieved in usual LQG and the results obtained there carry over to the case at hand. The only exception is the treatment of Dirac fermions, which needs slight adjustment: coming from an action principle, the Dirac field transforms in the spinor representation of the gauge group $\mathrm{SO}(1, D)$ for the physically relevant Lorentzian theory, but due to the strong similarity of the Lorentzian and the Euclidean Clifford algebras, the gauge group can be exchanged for

SO($D+1$) to fit in with the gravitational degrees of freedom.

Typical non-standard fields appearing in supergravity theories are the spin 3/2 Rarita Schwinger field ("gravitino") on the fermionic side, and (Abelian) higher $p$-form fields as novel bosonic fields (i.e. generalisations of the Maxwell field to higher form degree).

The former usually is a Majorana fermion (i.e. it is its own antiparticle) and therefore belongs to a real representation space of SO($1, D$). In order to formulate supergravities in terms of SO($D+1$) gauge theories, we again have to exchange the gauge group SO($1, D$) with SO($D+1$), but there is no action of SO($D+1$) on these real representation spaces, which hugely complicates the passage when compared to the case of Dirac fermions. We present a solution to this problem and for the first time, to the best of the author's knowledge, provide a background independent Hilbert space representation for the gravitino field.

Concerning novel bosonic fields, we exemplarily treat the three-form field ("three index photon") of $D+1 = 11$ $N = 1$ supergravity. Due to an additional Chern Simons term in the action, this field is not a simple generalisation of the Maxwell field to three-forms, but actually becomes self interacting and the equivalent of the electric field is not gauge invariant. We propose a reduced phase space quantisation with respect to the equivalent of the Gauß constraint, and the background independent representation we use is given by a state of Narnhofer-Thirring type, which already has been used in the loop literature in Thiemann's treatment of the closed bosonic string.

In the third part of this work, as a first application of the new variables, we extend the isolated horizon treatment (a quasi-local notion of black holes) in LQG to higher dimensions. In $D = 3$, the use of Ashtekar Barbero variables induces a Chern Simons theory on the horizon and the quantisation thereof and subsequent state counting led to the derivation of the famous

Bekenstein Hawking entropy formula for black holes from LQG. Here, we study (non-distorted) isolated horizons in $2(n+1)$ dimensional spacetimes and find that using the new variables induces an SO$(2(n+1))$ Chern Simons theory thereon. Since this theory, unlike its $D = 3$ counterpart, has local degrees of freedom, the quantisation and finally rederivation of the entropy formula become significantly more intricate and are left for further research.

We want to stress that several aspects of both, the higher dimensional as well as the supersymmetric extension, definitely deserve further study to actually catch up with the current status of usual canonical LQG. In the non-supersymmetric case, this concerns mainly the implementation of the simplicity constraint and its interplay with the dynamics. In the supersymmetric case, of course the supersymmetry constraint needs intensive study, in particular its role in the quantum super Dirac algebra. We hope that the generalisation of LQG to higher dimensions and supersymmetry achieved in this work will spark further development to clarify the mentioned open problems and finally lead to new interrelations between LQG and superstring/M - theory.

# Zusammenfassung

In dieser Arbeit verallgemeinern wir Schleifenquantengravitation (LQG) sowohl auf höhere Dimensionen als auch auf Supersymmetrie (d.h. Supergravitationstheorien). Damit wird die bestehende Limitation der LQG auf 3+1 Dimensionen und Materiefelder des Standardmodells aufgehoben. Dies beweist einerseits, dass LQG prinzipiell mit diesen beiden theoretischen Konzepten in Einklang gebracht werden kann. Andererseits weckt es Hoffnung, dass neue Anknüpfungspunkte zu Superstring- und M - Theorie ermöglicht werden, da diese Theorien notwendigerweise supersymmetrisch sind und in zehn beziehungsweise elf Raumzeitdimensionen formuliert werden müssen. Symmetrieargumente legen nahe, dass sich diese Theorien im Niederenergielimes effektiv durch Supergravitationstheorien in eben diesen Dimensionen beschreiben lassen. Die Untersuchung der Schleifenquantisierung der entsprechenden Supergravitationstheorien, mit der wir in dieser Arbeit beginnen, stellt daher ein vielversprechendes Unterfangen an der Grenze zwischen den beiden Ansätzen dar.

Präziser formuliert lauten unsere Ergebnisse wie folgt: Wir präsentieren erstmalig eine kanonische Formulierung der allgemeinen Relativitätstheorie in $D+1$ Raumzeitdimensionen ($D \geq 2$) auf einem Yang Mills Phasenraum mit den zentralen Eigenschaften: 1. Die kanonischen Variablen, die die metrische Information tragen, sind ein reeller Zusammenhang und ein dazu konjugierter reeller Impuls, die insbesondere die kanonischen Poissonklammerrelationen erfüllen. 2. Als Eichgruppe kann sowohl für die lorentzsche als auch die euklidische Theorie eine kompakte Gruppe gewält werden (in unserem Fall SO$(D+1)$). 3. Die Zwangsbedingungen sind alle von erster Klasse (in Diracs Terminologie). Eine solche Formulierung war bisher nur in drei und vier Dimensionen bekannt, die Ashtekar Barbero Formulierung,

welche die klassische Basis für LQG darstellt.

Das Programm der Schleifenquantisierung selbst ist fast gänzlich unabhängig von der Anzahl der Raumzeitdimensionen und der Wahl der kompakten Eichgruppe formuliert, weswegen das Fehlen von höherdimensionalen Analoga der LQG alleine dem Nichtvorhandensein der klassischen kanonischen Formulierung zuzuschreiben ist, welche die obigen Anforderungen 1. - 3. erfüllt. Darum ist es nicht verwunderlich, dass die Methoden der Schleifenquantisierung direkt auf die hier präsentierte Formulierung anwendbar sind, um höherdimensionale Schleifenquantengravitationstheorien zu erhalten. Dies arbeiten wir explizit aus.

Die Formulierung, die wir präsentieren, ist insofern wirklich neu, als dass sie sich für die Wahl $D = 3$ nicht auf die bekannte Ashtekar Barbero Formulierung reduziert. Stattdessen finden wir für $D > 2$ eine zusätzliche Zwangsbedingung, die sogenannte „Simplicity“ Zwangsbedingung, die die einzige konzeptionell neue Herausforderung bei der Quantisierung darstellt. Diese Zwangsbedingung ist interessanterweise keineswegs unbekannt in der (Quanten-) Gravitationsforschung und taucht insbesondere generell in Spinschaummodellen auf, die auch kovarianter Ansatz zur LQG genannt werden. In diesem Sinne stellt unsere Formulierung eine neue Verbindung zwischen kovarianter und kanonischer LQG her. Für diese Zwangsbedingung treten Quantenanomalien auf, die schon von den Spinschaummodellen her bekannt sind und die zu Problemen bei der Implementierung der Zwangsbedingung auf Quantenebene führen. Wir stellen einige neue Lösungsansätze hierfür vor.

Den zweiten Teil dieser Arbeit stellt die Erweiterung des obigen Rahmenwerks auf die Schleifenquatisierung einer ganzen Klasse von lorentzschen Supergravitationstheorien dar, die insbesondere die $D + 1 = 4$ $N = 8$, die $D + 1 = 11$ $N = 1$ und die $D + 1 = 10$ $N = 1$ Theorien umfasst. Konkreter untersuchen wir dazu die Kopplung von Standard- und außergewöhnlichen Materiefeldern an die bis dahin untersuchte Vakuumgravitationstheorie, die in Supergravitationstheorien wegen den Anforderungen der Supersymmetrie vorkommen.

Die Kopplung von Standardmaterie wurde für die LQG bereits erforscht und die Ergebnisse aus der vierdimensionalen Theorie sind auch auf die neue Formulierung anwendbar. Die einzige Ausnahme bilden Diracfermionen, bei denen nachgebessert werden muss: Ausgehend von einer Wirkung transformieren sie in der Spinordarstellung der Eichgruppe $SO(1, D)$, aber wegen der starken Ähnlichkeit der lorentzschen und euklidischen Clifford Algebren kann die Eichgruppe gegen $SO(D+1)$ getauscht werden. Das Diracfeld fügt sich so in die Behandlung des gravitativen Anteils der Theorie ein.

Bezüglich der außergewöhnlichen Materiefelder tritt in Supergravitationstheorien im fermionischen Sektor typischerweise das Spin 3/2 Rarita Schwinger Feld („Gravitino") auf und auf bosonischer Seite sind höhere $p$-Form Felder (d.h. Verallgemeinerungen des Maxwellfeldes auf höhere Formgrade) zu finden.

Ersteres ist normalerweise ein Majoranafermion (d.h. es ist sein eigenes Antiteilchen) und gehört damit zu einem reellen Darstellungsraum der $SO(1, D)$. Um nun auch Supergravitationstheorien als $SO(D+1)$ Eichtheorien zu formulieren, muss die Eichgruppe $SO(1, D)$ erneut gegen $SO(D+1)$ getauscht werden. Aber auf den reellen Darstellungsräumen existiert keine Wirkung der Gruppe $SO(D+1)$, was den Eichgruppenwechsel verglichen mit dem Fall des Diracfeldes enorm erschwert. Wir finden eine Lösung für dieses Problem und konstruieren, nach bestem Wissen des Autors erstmalig, eine hintergrundunabhängige Hilbertraumdarstellung für das Gravitino.

Als Beispiel für die neuartigen bosonischen Felder betrachten wir das Dreiformfeld („Dreiindexphoton") der $D+1 = 11$ $N = 1$ Supergravitation. Dieses Feld stellt keine triviale Erweiterung des Maxwellfeldes auf Dreiformen dar, da es wegen eines zusätzlichen Chern Simons Terms in der Wirkung selbstwechselwirkend ist. Das führt unter anderem auch dazu, dass das Äquivalent des elektrischen Feldes nicht eichinvariant ist. Wir führen eine Quantisierung des bezüglich des Pendants der Gauß Zwangsbedingung reduzierten Phasenraumes durch. Eine hintergrundunabhängige Darstellung erhalten wir durch Verwendung eines Zustandes vom Narnhofer-Thirring Typ, wie er in der Literatur zur Schleifenquantisierung bereits von Thiemann in seiner Behandlung des geschlossenen bosonischen Strings benutzt wurde.

Im dritten Teil der Arbeit erweitern wir schließlich als erste Anwendung der neuen Variablen die Behandlung von isolierten Horizonten (einer quasilokalen Beschreibung schwarzer Löcher) in der LQG auf höhere Dimensionen. In vier Raumzeitdimensionen induziert der Gebrauch der Ashtekar Barbero Variablen eine Chern Simons Theorie auf dem Horizont. Eine Quantisierung der entsprechenden Horizontfreiheitsgrade und anschließendes Zählen der Mikrozustände führte zur Herleitung von Bekensteins und Hawkings berühmter Entropieformel für schwarze Löcher innerhalb der LQG. In dieser Arbeit untersuchen wir (nicht-deformierte) isolierte Horizonte in $2(n+1)$-dimensionalen Raumzeiten und finden, dass aus dem Gebrauch der neuen Variablen eine $\mathrm{SO}(D+1)$ Chern Simons Theorie auf dem Horizont resultiert. Diese hat jedoch, im Gegensatz zu ihrem dreidimensionalen Gegenstück, lokale Freiheitsgrade, was die Quantisierung und Herleitung der Entropieformel signifikant erschwert. Beide Punkte müssen in zukünftiger Forschungsarbeit weiter untersucht werden.

Es ist zu betonen, dass einige Aspekte sowohl von der höherdimensionalen als auch von der supersymmetrischen Erweiterung weiterer Forschung bedürfen, um den gleichen Stand wie die aktuelle vierdimensionale LQG zu erreichen. Im nicht supersymmetrischen Fall betrifft dies hauptsächlich die Implementierung der Simplicity-Zwangsbedingung und sein Zusammenspiel mit der Dynamik. Im supersymmetrischen Fall werfen vor allem die Supersymmetrie Zwangsbedingung und insbesondere ihre Rolle in der Quanten-Super-Diracalgebra neue Fragen auf. Wir hoffen, dass die in dieser Arbeit erzielte Verallgemeinerung der LQG auf höhere Dimensionen und Supersymmetrie weitere Forschung zur Klärung dieser Fragen anregt und schließlich zu neuen Anknüpfungspunkten zwischen LQG und Superstring/M - Theorie führt.

To my parents

## Acknowledgements

It is my dear pleasure and duty to dedicate these lines to all those who contributed, directly or indirectly, to this work.

First of all, I would like to thank Prof. Dr. Uli Katz, Prof. Dr. Ulf Peschel, Prof. Dr. Christian Stegmann and Prof. Dr. Werner Wegscheider who taught me experimental physics. Special thanks goes to Prof. Dr. Frieder Lenz, Dr. Olliver Schnetz, Prof. Dr. Klaus Mecke, Prof. Dr. Jaroslav Fabian and Prof. Dr. Michael Thies, who in their excellent theory lectures sparked and nurtured my fascination for theoretical physics.

For enabling my studies in the fast track study programme at the Universities Erlangen-Nuremberg and Regensburg, I owe thanks to the Elite Network of Bavaria and, in particular, to the current programme coordinator Prof. Dr. Klaus Mecke, for his due time spent in shaping and preserving this study course, and for facilitating its various offers.

It is a pleasure to thank Dipl.-Phys. Raoul Gerber, my fellow physics student, long term room mate and friend, for doing such a good job in all these three areas.

Sincere appreciation also goes to my stipend donors, the Max-Weber Programm Bayern, the Leonardo Kolleg of the FAU Erlangen-Nuremberg and "e-fellows.net" during my undergraduate and diploma studies and the German National Merit Foundation and "e-fellows.net" during my PhD phase, for financial as well as ideational support.

I want to thank the AEI Golm for its hospitality during the time of my diploma thesis, as well as Prof. Dr. Alexander Feinstein for accepting my three months visit at the University of Bilbao (UPV/EHU) during the final phase of this work.

I am indebted to Prof. Dr. Karl-Hermann Neeb and Prof. Dr. Toshiyuki Kobayashi for counsel on representation theory, as well as several anonymous referees who gave hints towards further literature and valuable comments, which improved the presentation of our articles and therefore also of this thesis.

Special thanks go to the current and former members and long-term visitors of the Chair for Theoretical Physics III, Prof. Dr. Kristina Giesel, Prof. Dr. Hanno Sahlmann, Prof. Dr. Michael Thies, Prof. Dr. Jurek Lewandowski, Prof. Dr. Jonathan Engle, Prof. Dr. Florian Girelli, Dr. Iñaki Garay, Dr. Enrique F. Borja, Dr. Muxin Han, Dr. Emanuele Alesci, Dr. Francesco Cianfrani, Dr. Derek Wise, Dr. Maïté Dupuis, Dipl.-Phys. Antonia Zipfel, Dipl.-Phys. Suzanne Lanéry, Dipl.-Phys. Christian Fitzner and Eckhard Strobel (M.Sc.) for the pleasant working atmosphere and multiple stimulating discussions, not to forget our secretary Martina Gercke-Viertel for her immediate help in solving all administrative problems. Special mention deserve Dipl.-Phys. Alexander Stottmeister, who always willingly shared his impressive knowledge on mathematics in various discussions and with whom I was happy to collaborate writing two articles, and, of course, Dipl.-Phys. Norbert Bodendorfer. With him, not only did I share all the time of my studies, but also collaborated neatly during the time of my diploma thesis and later in all my research projects. I am sure that, thanks to his constant inspirational input, our joint research output was much higher than double of what I would have been able to do alone.

Above all, I am greatly indebted to my supervisor Prof. Dr. Thomas Thiemann, not only for financial support, but more for his scientific and personal lead, for the opportunity to collaborate with a researcher of that knowledge and reputation, for ingenious solutions when we were stuck, and not least for being a supervisor I was always happy to work with.

Finally, I want to thank my family, my parents Rudolf and Elfrun as well as my brother Christoph and my girlfriend Nicole, for constant support and encouragement.

# Contents

# List of figures

# List of tables

# List of symbols

In the following list, we will give an overview over the symbols appearing repeatedly throughout this thesis. Some "local" notation is missing in this list, but is then defined and explained in the corresponding section.

| **Symbol** | **Meaning** |
|---|---|
| $[\,.\,,\,.\,]$ | commutator |
| $[\,.\,,\,.\,]_+$ | anticommutator |
| $\beta$ | free parameter in the new variables |
| $\gamma$ | (piecewise analytic) graph; Barbero Immirzi parameter |
| $\Lambda$ | cosmological constant |
| $\lceil x \rceil$ | next integer greater than $x$ |
| $_{(\,.\,,\,.)}$ | symmetrisation of indices with total weight one |
| $_{[\,.\,,\,...\,,\,.]}$ | antisymmetrisation of indices; always with total weight one |
| $\mathscr{A}$ | space of smooth connections |
| $\mathscr{P}$ | path ordering symbol; set of piecewise analytic paths with compact support |
| $\mu_H$ | Haar measure |
| $\overline{\mathscr{A}}$ | space of distributional connections |
| $\mathrm{Cyl}(\overline{\mathscr{A}})$ | algebra of cylindrical functions on $\overline{\mathscr{A}}$ |
| $\zeta$ | internal signature |
| $\{\,.\,,\,.\,\}$ | Poisson brackets |
| $\{\,.\,,\,.\,\}_{\mathrm{DB}}$ | Dirac brackets (also $\{\,.\,,\,.\,\}^*$) |
| $c$ | speed of light |
| $D$ | spatial dimension |
| $d = D + 1$ | spacetime dimension |
| $E(\gamma)$ | edge set of a graph $\gamma$ |
| ${f_{\alpha\beta}}^{\gamma}$ | structure constants of a Lie algebra |
| $G$ | Newton's constant |
| $s$ | spacetime signature |
| $V(\gamma)$ | vertex set of a graph $\gamma$ |
| $\alpha$, $\beta$, ... | Lie algebra indices; in part V: $\in \{1, ..., D-1\}$, tensorial indices in $(D-1)$-dimensional subspaces |
| $\underleftarrow{\mu}$, $\underleftarrow{\nu}$, ... | tensorial indices on $\Delta$ |
| $\mu$, $\nu$, ... | $\in \{0, ..., D\}$, spacetime indices |
| $\overline{M}$, $\overline{N}$, ... | $D-3$ multiindices |
| $A$, $B$, ... | Lie algebra indices (only in part V) |
| $a$, $b$, ... | $\in \{1, ..., D\}$, spatial indices |
| $I$, $J$, ... | $\in \{0, ..., D\}$, internal $\mathrm{SO}(D+1)$ or $\mathrm{SO}(1, D)$ indices |
| $i$, $j$, ... | $\in \{1, ..., D\}$, internal $\mathrm{SO}(D)$ indices |
| $[m]$ | gauge orbit of a point $m \in M$ |
| $\alpha_\beta(f)$ | gauge flow with parameter $\beta$ applied to the phase space function $f$ |
| $\Delta$ | $D$-dimensional null surface within $\mathscr{M}$ |
| $\mathscr{M}$ | $(D+1)$-dimensional spacetime manifold |
| $\overline{M}$ | constraint surface in phase space |
| $\partial\mathscr{X}$ | boundary of $\mathscr{X}$ |
| $\sigma$ | $D$-dimensional spatial manifold |
| $\Sigma_t$ | leaves of the foliation of $\mathscr{M}$ |
| $\sigma_{\mathscr{D}}(\overline{M})$ | section in $\overline{M}$ defined by gauge fixing $\mathscr{D} = 0$ |
| $\widehat{M}$ | reduced phase space |
| $M$ | phase space |
| $S^D$ | $D$-sphere |
| $\bar{\eta}_{IJ}$ | transversal projector onto the subspace orthogonal to $n^I$ |
| $\delta_{ij}$ | Kronecker delta |
| $\eta_{IJ}$ | depending on the signature, Minkowski metric or Kronecker delta |
| $B$ | BF theory $B$-field |
| $B^{aj}$ | CDJ magnetic field |
| ${e^\mu}_I$ | spacetime vielbein |
| $E^{bj}$ $(E^{bJ})$ | (hybrid) vielbein of density weight +1 |
| $e^{bj}$ $(e^{bJ})$ | (hybrid) vielbein |
| ${e_a}^i$ $({e_a}^I)$ | spatial (hybrid) co - vielbein |
| ${e_\mu}^I$ | spacetime co - vielbein |
| $E_{bj}$ $(E_{bJ})$ | (hybrid) co - vielbein of density weight -1 |
| $g_{\mu\nu}$ $(g^{\mu\nu})$ | (inverse) metric tensor on the spacetime manifold |
| $h$ | determinant of $h_{\alpha\beta}$ |

| | |
|---|---|
| $h_{\alpha\beta}$ | Riemannian metric on $(D-1)$-dimensional subspaces |
| $h_{\overleftarrow{\mu\nu}}$ | degenerate metric on $\Delta$ |
| $k^b{}_J$ | canonically conjugate variable to $e_a{}^I$ |
| $K_{\mu\nu}$, $K_{ab}$ | extrinsic curvature |
| $K_{aI}$, $P^{(E)}_{aI}$ | canonically conjugate variable to $E^{bJ}$ |
| $N$ ($\underset{\sim}{N}$) | lapse function (of density weight -1) |
| $n^I$, $N^I$ | internal unit normal to the hybrid vielbein |
| $n^\mu$ | future pointing unit normal to the spatial slices |
| $N^\mu$, $N^a$ | shift vector |
| $P^{(N)}(P^{(\underset{\sim}{N})})$ | momentum conjugate to the lapse function (of density weight -1) |
| $P^{(n)}_J$, $P_J$ | canonically conjugate variable to $n^I$, $N^I$, respectively |
| $P^{ab}$ | momentum conjugate to the spatial metric |
| $P^{(\vec{N})}_a$ | momentum conjugate to the shift vector |
| $q$ | determinant of the spatial metric |
| $q_{\mu\nu}$, $q_{ab}$ | spatial metric |
| $T^\mu$ | deformation vector field |
| $\Gamma^{\mathrm{H}}_{aIJ}$ | hybrid vielbein compatible spin connection; superscript sometimes omitted |
| $\Gamma^\rho_{\mu\nu}$ | Christoffel symbols |
| $\Gamma_{\mu IJ}$ | spacetime vielbein compatible connection |
| $\Gamma_{aij}$ | spatial vielbein compatible connection |
| $\Gamma_{aIJ}[\pi]$ | extension of the hybrid spin connection off the simplicity constraint surface |
| ${}^{(d)}e$ | determinant of the co - vielbein; superscript omitted if unambiguous |
| $\mathscr{L}$ | Lie derivative |
| $\nabla^\Gamma_\mu$ | spacetime vielbein compatible covariant derivative |
| $\nabla^A_\mu$ | covariant derivative corresponding to $A_{\mu IJ}$ |
| $\nabla_\mu$ | $g$-compatible torsion free covariant derivative |
| $\overset{(\beta)}{\pi}{}^{bKL}$ | momenta conjugate to $A_{aIJ}$ (and $K_{aIJ}$); often only $\pi^{bKL}$ |
| $A_{\mu\nu\rho}$ ($A_{abc}$) | (spatial components of the) three-form gauge field of $d=11$ supergravity |
| $A_{\mu IJ}$ | so$(D+1)$ or so$(1,D)$ connection |
| $A_{aIJ}$ | spatial components of $A_{\mu IJ}$ |
| $D^A_a$ | covariant derivative corresponding to $A_{aIJ}$ |
| $D^\Gamma_a$ | spatial vielbein compatible covariant derivative |
| $D^{\mathrm{H}}_a$ | hybrid vielbein compatible covariant derivative |
| $D_a$, $D_\mu$ | $q$-compatible torsion free covariant derivative |
| $g$ | determinant of the spacetime metric |
| $h_c(A)$ | holonomy of $A$ along a curve $c$ |
| $K_{aIJ}$ | contortion of $A_{aIJ}$ |
| ${}^{(d)}R$ | Ricci scalar in $d$ dimensions; superscript omitted if unambiguous |
| ${}^{(d)}R_{\mu\nu}$ | Ricci tensor in $d$ dimensions; superscript omitted if unambiguous |
| ${}^{(d)}R_{\mu\nu\rho}{}^\sigma$ | Riemann curvature tensor in $d$ dimensions; superscript omitted if unambiguous |
| $C_{\mu\nu\rho}{}^\sigma$ | Weyl tensor |
| $F_{\mu\nu}{}^{IJ}$ | curvature of $A_{\mu IJ}$ |
| $F_{ab}{}^{IJ}$ | curvature of $A_{aIJ}$ |
| $F_{ab}{}^{ij}$ | curvature of $A_{aij}$ |
| $G_{\mu\nu}$ | Einstein tensor |
| $R^{\mathrm{H}}_{ab}{}^{IJ}$ | curvature of the hybrid spin connection; superscript sometimes omitted |
| $R_{\mu\nu}{}^{IJ}$ | curvature of $\Gamma_{\mu IJ}$ |
| $R_{ab}{}^{ij}$ | curvature of $\Gamma_{aij}$ |
| $T_{\mu\nu}$ | energy momentum tensor |
| $\gamma^I$, $\gamma^i$ ($\gamma^\mu$) | (curved space) gamma matrices |
| $\Psi$ | Dirac field |
| $\psi_a$, $\phi_i$, $\rho_i$, $\sigma$ | variants and components of the Rarita Schwinger field appearing in the Hamiltonian picture |
| $\psi_\mu$ | Rarita Schwinger field |
| $\Sigma^{IJ}$ | representation of the Lie algebra so$(D+1)$ or so$(1,D)$ on spinor space |
| $C$ | charge conjugation matrix |

| | |
|---|---|
| $\chi$ | Euler characteristic |
| $\delta^{(d)}(x-y)$ | $d$-dimensional Dirac delta distribution |
| $\mathbb{P}$, $\mathbb{Q}$ | various projectors |
| $\mathscr{C}$ | diverse constraints |
| $\mathscr{D}^{ab}_{\overline{M}}$ | second class partner to the simplicity constraint |
| $\mathscr{G}_{IJ}$, $\mathscr{G}_{ij}$, $\mathscr{G}^i$ | Gauß constraint |
| $\mathscr{H}$ | Hamiltonian constraint |
| $\mathscr{H}_a$ | spatial diffeomorphism constraint |
| $\mathscr{N}$ | normalisation constraint |
| $\mathscr{S}^{ab}_{\overline{M}}$ | quadratic simplicity constraint |
| $\mathscr{S}^a_{I\overline{M}}$ | linear simplicity constraint |
| $\mathfrak{S}$ | supersymmetry constraint |
| $E^{(d)}$ | Euler topological density in $d$ dimensions |
| $G^{-1}_{ab\ cd}$ | DeWitt supermetric |
| $G^{ab\ cd}$ | inverse DeWitt supermetric |
| $H$ | Hamiltonian |
| $L$ | Lagrangian |
| $l$, $k$, $\{m_I\}$ | generalised null frame |
| $S$ | action |
| $\hat{C}^{\overline{M}}$ | quadratic simplicity constraint operator |
| $\hat{G}$ | Gauß constraint operator |
| $\hat{M}$, $\hat{\boldsymbol{M}}$ | various Master constraint operators |
| $\hat{S}$ | linear simplicity constraint operator |
| $\hat{V}$ | volume operator |
| $\hat{Y}_n(S)$ | flux operator corresponding to a surface $S$ |
| $\widehat{\mathrm{Ar}}$ | area operator |
| $R_{IJ}$ ($L_{IJ}$) | right (left) invariant vector fields on $\mathrm{SO}(D+1)$ or $\mathrm{SO}(1,D)$ |

# Conventions

**General:**

Einstein's summation convention will be used if not specified differently. Dealing with Lorentz metrics, we will use mostly plus signature convention $(-,+,...,+)$. We will mostly use units such that $\kappa := \frac{c^4}{8s\pi G} = 1$.

**Indices:**

Spacetime indices will be denoted by lower Greek letters from the middles of the alphabet, $\mu, \nu, \rho, ... \in \{0, ..., D\}$, whereas indices on the spatial manifold are denoted by lower Latin letters from the beginning of the alphabet $a, b, c, ... \in \{1, ..., D\}$. $D$ denotes the number of spatial dimensions. Internal $\mathrm{SO}(D+1)$ or $\mathrm{SO}(1, D)$ indices will be denoted by capital Latin letters $I, J, K, ... \in \{0, ..., D\}$ and internal $\mathrm{SO}(D)$ indices by lower Latin letters $i, j, k, ... \in \{1, ..., D\}$. If there is the chance that an ambiguity arises, we will put a superscript on tensors to make clear if they refer to a spatial slice or the whole space time manifold. Objects with density weight $+1$ will partly carry a "$\tilde{}$", and with density weight $-1$ a "$\underset{\sim}{}$".

**Covariant derivative, curvature tensor:**

We define $\nabla$ by $\nabla_\mu \phi = \partial_\mu \phi$ for scalar fields $\phi$ and $\nabla_\mu u_\nu := \partial_\mu u_\nu - A^\rho_{\mu\nu}\, u_\rho$ for one forms $u_\mu$. Demanding that $\nabla$ be torsion free and metric compatible, we find that $A = \Gamma$, where $\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)$ are the Christoffel symbols. The Riemann curvature tensor $R_{\mu\nu\rho}{}^\sigma$ is defined by $[\nabla_\mu, \nabla_\nu] u_\rho = R_{\mu\nu\rho}{}^\sigma u_\sigma$ and one finds $R_{\mu\nu\rho}{}^\sigma = -2\partial_{[\mu} \Gamma^\sigma_{\nu]\rho} + 2\Gamma^\lambda_{[\mu|\rho} \Gamma^\sigma_{\nu]\lambda}$. For the Ricci tensor, we use the convention $R_{\mu\nu} = R_{\mu\rho\nu}{}^\rho$.

### Hamiltonian framework

For a given phase space $M$ coordinatised by $\{q^i, p_j\}$ and two phase space functions $f, g$, we define the Poisson bracket of the two functions to be

$$\{f, g\} := \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i},$$

and the obvious generalisation thereof in case of field theories. With this convention, time evolution is obtained by the right action of the Hamiltonian $H = (p_i\dot{q}^i - L)|_{\dot{q}=\dot{q}(q,p)}$,

$$\dot{f} = \{f, H\}.$$

# About this thesis

This work has been supervised by Prof. Dr. Thomas Thiemann and carried out at the Institute for Quantum Gravity, Chair for Theoretical Physics III of the Friedrich Alexander University Erlangen-Nuremberg from December 2009 until January 2013. It is based on a series of papers [1–10] which were written in the course of this PhD work. All these results have been obtained in collaboration with my colleague PhD student Dipl.-Phys. Norbert Bodendorfer and our supervisor Prof. Thiemann.

In this series of papers, for the first time an extension of the loop quantum gravity programme to higher dimensions and supergravity theories was obtained. The PhD thesis of Dipl.-Phys. Bodendorfer is centered around the supersymmetric extension [5–7], while the focus of this thesis lies on the extension to higher dimensions [1–4], and the supersymmetric extension is summarised rather briefly. Parts of this thesis are taken from these papers (with modifications to streamline notation and presentation), which will be again indicated in the main text.

Of these ten articles, nine are published ([1–7] in *Classical and Quantum Gravity*, [8] in *Physics Letters B*, [9] in *Proceedings of Science* (QGQGS 2011)), and the latest [10] is about to be finished.

In the course of this PhD work, two further articles ([11], published in *Classical and Quantum Gravity*, and [12], accepted for publication in *Classical and Quantum Gravity*) were written in collaboration with Dipl.-Phys. Norbert Bodendorfer and Dipl.-Phys. Alexander Stottmeister. Since they are, however, only loosely related to the rest of this work, the corresponding results are not included in this thesis.

# 1

# Introduction

## 1.1 A very brief history of (canonical) loop quantum gravity

More than 25 years have passed since Ashtekar introduced his nowadays famous variables [13, 14] for classical and quantum gravity in 1986. In these papers, extending seminal work by Sen [15], not only did Ashtekar succeed in giving a canonical description of four dimensional general relativity on a Yang Mills phase space, at the same time he found the formulation which up to now features the most simple version of one of the Achilles' heels of quantum gravity research, the so-called Hamiltonian constraint. This usually very complicated initial value constraint of general relativity turns out to be a polynomial function in terms of Ashtekar's variables. However, for the physically relevant Lorentzian signature space times, Ashtekar's original Yang Mills connection takes values in non compact $\mathrm{Sl}(2, \mathbb{C})$ rather than in a compact gauge group and is furthermore subject to complicated reality conditions, and up to now no one succeeded in giving a Hilbert space representation due to these two problems.

Nine years afterwards, Barbero and later Immirzi [16–19] gave a slight generalisation of Ashtekar's original proposal by introducing a free parameter $\gamma$, the so called Barbero Immirzi parameter (or often just Immirzi parameter), into the theory. Upon choosing $\gamma = i$, Ashtekar's original proposal is recovered, while for $\gamma \in \mathbb{R}/\{0\}$, the resulting Yang Mills connection turns out to be real and valued in the compact $\mathrm{SU}(2)$. However, this comes at the cost of a more complicated, non-polynomial Hamiltonian constraint.

In the mean while, enormous progress had been made in giving a mathematically rigorous kinematical framework for background independent quantisation of gauge theories of compact structure groups [20–26], which now could be applied to the SU(2) Ashtekar Barbero formulation to obtain what nowadays is called canonical loop quantum gravity (LQG). In particular, the general quantum solution to two of the three (families of) initial value constraints, the Gauß and spatial diffeomorphism constraints, could be obtained [27], and later on, it was proven that the chosen representation actually is unique (under mild assumptions) [28, 29]. Thiemann finally made groundbreaking progress in giving a mathematically well defined Hamiltonian constraint operator despite the non-polynomiality of the classical constraint [30, 31].

Approximately in the same period of time, the probably most popular results were derived: Riemannian geometric operators like area and volume in LQG were found to have discrete spectrum [32–36], that is, a result of LQG is that spacetime at a fundamental level is discrete; in applications of the LQG framework to cosmological models, so called loop quantum cosmology (LQC), a natural resolution of the big bang singularity was found [37]; and the famous Bekenstein Hawking formula for the black hole entropy was derived from first principles in LQG [38–40]. The latter two, LQC and black holes in LQG, continue to be subject to intense study also nowadays.

The mentioned results have been considerably strengthened since then and new ones were obtained. In LQC, a resolution of various singularities of the classical theory was found in a variety of models and there are results that the LQC effective dynamics favours inflation (cf. [41, 42] and references therein). A mathematically rigorous framework for the derivation of the black hole entropy and a sophisticated counting method were introduced, and also logarithmic corrections to the black hole entropy could be recovered (cf. [43, 44] and references therein). Coupling to standard model matter fields was achieved in [45, 46] and recently the framework was extended to metric theories beyond general relativity [47].

Major open problems are for example the quantum dynamics and recovering of semi-classical physics from the theory as well as a lack of proof that the constraint algebra is faithfully represented at the quantum level. To attack the first problem, a completely new research branch, the so-called spin foam models, was launched, which tries to give loop quantum dynamics using a path integral approach (see e.g. [48, 49] for recent reviews and [50] for a basic introduction). We want to stress the importance of this covariant research branch of LQG, which is at least as active as the canonical line of research we outlined so far, but due to the canonical focus of this thesis will not play the role it deserves.

On the canonical side, recently the focus shifted from Dirac to reduced phase space quantisation and several deparametrised models were introduced [51–54], matter coupled models which, instead of a Hamiltonian constraint, feature a true Hamiltonian and give direct access to the physical Hilbert space. Powerful semiclassical techniques have been introduced ([55] and references therein) in order to extract perturbative quantum field theory on Minkowski (or curved) space time physics from the background independent and non-perturbative quantum theory, but still this task is far from completed.

Finally, regarding the last of the above mentioned major open problems, the quantum constraint algebra has been proven to be non anomalous by Thiemann [56] in the sense that the commutator of two Hamiltonian constraints vanishes on diffeomorphism invariant states, but if the hypersurface deformation algebra actually can be faithfully represented remains to be unclear, see [57–59] for recent literature towards an improvement on this issue. However, in principle this problem can be avoided using the Master constraint method developed in [60].

Of course, this short exposition of LQG cannot be comprehensive and many important aspects of the theory and its development were left unmentioned. For more details, we refer the interested reader to the textbooks [61–63] and references therein or [64] for an introductory textbook suitable for undergraduate students, or to the reviews [65–68].

## 1.2 Motivation

The quantisation procedure developed in the LQG literature is of very general nature, and does neither depend on the spacetime dimension under consideration nor on the compact gauge group chosen, and moreover has been extended to coupling of all matter fields of the standard model[45, 46]. It thus might surprise the unfamiliar reader that LQG was restricted to four spacetime dimensions up to now. However, the problem prohibiting generalisations of LQG to higher dimensions lies not in the quantum theory, but instead appears already at the classical level: the classical canonical formulation of general relativity in Yang Mills type variables suitable for loop quantisation was known for four spacetime dimensions, namely Ashtekar's variables, but not in higher dimensions[1].

Of course, up to now there is no experimental evidence neither for higher dimensions nor for supersymmetry. This justifies the question: should we not be content with the restrictions the theory seems to impose, that up to now extensions to higher dimensions and inclusion of supersymmetry were not possible (see section 1.3 for attempts in both directions)? Should we regard these facts more as a feature than a flaw, perhaps even that "good old" four spacetime dimensions are a prediction of LQG?

We will argue that this is not the case, and that the endeavour of searching for both, the higher dimensional as well as the supersymmetric generalisation, is worth being pursued, out of the following two reasons: Firstly, they constitute a step towards convergence of different approaches in the multi-branched field of quantum gravity research. Secondly, of course the ambition of theoretical physics is not only to give theoretical explanation of experimental data, but also to deduce experimentally falsifiable predictions from a so far untested set of theoretical ideas to gear the development of future experiments. Higher dimensions and supersymmetry are an arguably interesting set of such theoretical ideas and therefore worth studying in their own right. These two relevant reasons will be laid out in more detail in the following.

Today's quantum gravity research is split into several branches, many of which are seemingly unrelated both at a conceptual and technical level. This is generally considered as a problem hindering progress of the field in total, which is underlined by the call

[1] A Yang Mills formulation for general relativity and loop quantisation thereof exists also in three spacetime dimensions, which can be used as a testbed for the four dimensional theory (cf. e.g. [69]).

for convergence appearing regularly in the scope of present-day conferences on (quantum) gravitation (e.g. *Spanish Relativity Meeting: ERE2010*, Sept. 2010, Granda, Spain [70] and *Quantum Theory and Gravitation*, June 2011, Zurich, Switzerland [71]).

Actually, this fact is no novelty at all and seems to be almost as old as quantum gravity research. Already in the early sixties, Peter Bergmann stated regarding the research at his time [72]: "*In view of the great difficulties of this program, I consider it a very positive thing that so many different approaches are being brought to bear on the problem. To be sure, the approaches, we hope, will converge to one goal.*"

Since then, instead of converging, the field seems to have drifted even further apart. At that time, the followed lines of research were mainly the perturbative approach, trying to quantise the metric fluctuations over Minkowski (or another background) spacetime, and the canonical approach which aimed at unravelling and quantising the Hamiltonian theory underlying general relativity. These early approaches known in the sixties were all shortly afterwards shown to fail due to non-renormalisability [73, 74] of general relativity and ill-definedness of the Wheeler-DeWitt equation among other problems (cf. e.g. [75] for an account on the historical development of the field of quantum gravity). To cope with these, very different proposals were made where to modify the existing routes towards quantum gravity, which further fanned out the research branches. Supergravities in diverse dimensions were introduced in the hope of mending the problem of perturbative non-renormalisability, but failed to do so [76–78] (with the possible exception of $d = 4$, $N = 8$ supergravity [79, 80]). Superstring theories [81, 82] and M - theory [83, 84] were introduced, mainly perturbatively defined approaches to quantum gravity which aim at giving a unified description of all forces. Not only do they require spacetime to be ten (superstring theory) or eleven (M - theory) dimensional, they furthermore are necessarily supersymmetric and predict infinitely many new fields.

Almost simultaneously, LQG was developed, following a different philosophy from the outset. It treats the quantisation of the self interacting theory of gravity non perturbatively, and takes the lessons from general relativity seriously, namely that spacetime itself becomes dynamical. This leads to the belief that, at the quantum level, a fundamental theory cannot be a fixed background spacetime and fields on it, but the spacetime itself must be quantised. Spacetime as we perceive it should be an emerging concept, probably only recovered semi-classically. To implement this concept of background independence, new, background independent quantisation techniques were developed. While the quantisation procedure itself is changed in LQG, as we already heard, the approach is rather conservative regarding number of space time dimensions, supersymmetry is no necessary ingredient and therefore was mostly not considered (see, however, section 1.3 for an account on prior work on supersymmetry in LQG), and it also aims not at giving a unified description of the forces.

Of course, there are many more approaches, like non-commutative geometry, regularisation group techniques, twistor theory, causal dynamical triangulations and discrete approaches to quantum gravity, but we will restrict our discussion to the two main lines of research, strings and loops. As we have seen, their respective conceptual basis is fundamentally different. The lack of convergence between these routes to quantum gravity and, in a sense, these "different languages" one speaks make communication of researchers from different branches complicated, and hinders fruitful cross fertilisations. Just to name an example, one of the most astonishing conjectures in string theory in the past years, the AdS/CFT correspondence [85–87] (see also [88] for a basic introduction and [89, 90] for a more recent review on AdS/CFT and its integrability structure), has lead to new developments in other areas like solid state physics and QCD (cf. e.g. [91, 92]), but had, to the best of the author's knowledge, no influence at all on LQG. On the other hand, findings like Thiemann's quantisation of the string [93] with LQG methods, which indicates that the critical dimensions (10, 11 or 26) possibly can be avoided, is hardly acknowledged in the string community (see, e.g. [94] and section 1.3).

In order to stimulate scientific exchange, it would be desirable to conduct research on the boarder between superstring/M - theory and LQG, but the literature on this topic is rather sparse (cf. again section 1.3). To make contact, three different routes suggest themselves.

One possibility is to dimensionally reduce superstring/M - theory down to four dimensions and to break supersymmetry, and to compare the resulting effective theory with an appropriately chosen sector of LQG. However, in order to arrive from a ten at a four dimensional model from string theory, many choices have to be made and associated is a “landscape of string vacua”, and it seems that at least currently there is no preferred model we could compare to (see [95] for a recent review of string theory phenomenology).

The second route then of course is to compare the two approaches directly in ten or eleven dimensions, the natural ones for superstring/M - theory, and to this end, our generalisations of LQG to higher dimensions and supersymmetry are necessary. More concretely, what we propose is to study the loop quantisation of supergravity theories in ten and eleven dimensions, which by symmetry arguments are expected to be the low energy effective field theory limits of superstring/M - theory.

Thirdly, symmetry reduced sectors like cosmology and black holes suggest themselves for the comparison, since cosmological models are well studied [41, 42, 96, 97] and a microscopic derivation of the Bekenstein Hawking entropy formula for black holes is available [43, 44, 98, 99] in both fields. Also to this end, our extensions are interesting, since e.g. in string theory, typically also black holes in higher dimensions are studied, which so far was not possible within LQG.

But irrespective of a possible connection to superstring/M - theory, we believe that higher dimensional and supersymmetric extensions of LQG are of interest because it is thinkable that higher dimensionality and supersymmetry describe real properties of nature. Indications thereof might be about to be found in current experiments like the LHC, and then call for a theoretical explanation. Even if not, they should have their imprint at least in quantum gravity effects and if there is the chance to predict measurable indications of the presence of supersymmetry or higher dimensions in future experiments, this definitely is a subject worth studying in its own right.

Of course, this is a long term project: First, one has to deduce how to break supersymmetry or compactify the excess dimensions at the quantum level, and one might encounter the same problems found in string theory when doing so. Even if this is achieved, obtaining falsifiable predictions in quantum gravity research doubtlessly is an arduous task. Because it was unthinkable for decades to obtain experimental information testing the Planck scale where quantum gravity effects originate from, the only lead for quantum gravity research was conceptual appeal, mathematical rigor, and sticking to well-established theoretical frameworks.

Today, this situation has changed significantly. Thanks to enormous advances in experimental particle and astroparticle physics as well as astronomy, we are about to learn from the Planck scale. Moreover, during the last decade, especially in quantum gravity phenomenological models, experimentally testable quantum gravity effects were thought up (see e.g. [100] for a recent review of LQG phenomenology), although the rigorous derivation of these observable phenomena from a full-fledged proposal for quantum gravity are still mostly missing.

We want to point out one tentative application we have in mind for the LQG generalisations we provide: in a LQG extension of the inflationary scenario all the way down to the Planck scale, it has recently been shown that for generic initial conditions at the big bounce, which is predicted by LQC instead of a big bang, the theory predicts a cosmic microwave background compatible with the seven year WMAP data, with a small window for a quantum gravity imprint on the data which will be measurable by future satellites (see [101] and literature therein). It would be particularly interesting to reexamining this derivation in a higher dimensional and/or supersymmetric model and to see if these predictions deviate. Incidentally, there also have been found first indications that LQC, up to now being a loop quantisation of cosmological models, actually can be obtained from full LQG [102]. A rigorous proof thereof would lift these microwave background imprints to testable predictions from the full theory.

After having motivated the task at hand, we will continue with an outline of the existing literature on higher dimensional Ashtekar variables and supersymmetric extensions.

## 1.3 Position in the existing literature

Despite these promises, the existing literature on higher dimensional extensions of Ashtekar's variables is rather scarce. Peldán [103] sketched a general programme for the extension of Ashtekar's formulation to higher dimensions and for the study of unified theories of general relativity and Yang Mills theory, but only provided further results on the latter [104–106]. Still, we want to remark that the article [103] was rather influential for the work presented here. Later, there was work by Nieto towards an extension of the original complex Ashtekar variables to 7+1 dimensions using octonions [107], and further to dimensions 10+2, 2+2 and 8+0 [108, 109], which however do not (at least straightforwardly) allow for application of the loop quantisation programme. With the extension to higher dimensions which we will present in this work, we therefore enter uncharted territory.

The literature on loop quantisation of supergravity theories is considerably richer. Jacobson was the first to extend the original complex Ashtekar variables to $d = 4$ $N = 1$ supergravity [110]. A similar formulation for $d = 4$ $N = 2$ supergravity was given in [111]. Shortly afterwards, Fülöp realised that in Jacobson's formulation of the $d = 4$ $N = 1$ theory, when splitting the supersymmetry generators into its chiral parts, one chirality can be absorbed into the Gauß constraint to recast the theory in terms of Ashtekar variables for the gauge group Osp(1|2) [112]. Doing this, both, bosonic and fermionic degrees of freedom are combined into a single connection, a feature which is very appealing for a supersymmetric theory in the author's opinion. Based on this work, Osp(1|2) Wilson loops were introduced and a representation through loop variables was discussed [113, 114]. Ling and Smolin introduced spin networks for the groups Osp(1|2$n$) and calculated the spectrum of the area operator for the four dimensional theory [115, 116]. However, these results are only formal since no inner product and therefore no Hilbert space was defined, although some of the necessary mathematical structures for the group Osp(1|2) have been probed [117, 118].

The first canonical formulations of the $d = 4$ $N = 1$ theory in terms of real Ashtekar Barbero variables were given in [119, 120]. At the Lagrangian level, supergravity versions of the Holst action for $d = 4$ $N = 1$ were introduced in [119, 121]. Curiously, while the studies of the loop quantisation of supergravity theories in complex Ashtekar variables we mentioned before were lacking mathematical rigor, the real formulations, which immediately allow for loop quantisation at least in the bosonic sector, were not further studied to the best of the author's knowledge. This is probably related with the reality condition the Rarita Schwinger field ("gravitino") is subject to, being a Majorana fermion. As we will see, they are the core problem when quantising this field, and in particular, prevent it from being treated like the Dirac fermion. This work will succeed in solving the mentioned problems.

Finally, since one of the long term goals is to make contact to string theory, we want to outline the research efforts undertaken in both, the loop and the string community to this end. Firstly, basically all the loop supergravity research we already mentioned contributes to these efforts, but also the work of Nieto on higher dimensions is clearly influenced by superstring/M - theory. Thiemann gave a loop quantisation of the closed bosonic string in [93], which points towards a possible avoidance of a critical dimension and supersymmetry in string theories when using loop methods. This work was criticised in [94] and also in the outside view on LQG [122]. Thiemann answered to both of these papers in [123]. For a recent view on string theory from within the LQG community, see [124]. Similar results regarding the avoidance of a fixed ambient space's dimension were also obtained using different quantisation schemes in [125–127]. Continuing Thiemann's line of research on the loop string, in [128, 129] the canonical analysis of the algebraic string was performed and its relation to the Nambu Goto string was studied. The algebraic formulation of the bosonic string has a lot of similarities with the Ashtekar Barbero formulation, leading to a first class Hamiltonian formulation and allowing for a Barbero Immirzi like parameter. Further studies of this theory at the quantum level were announced in the latter, but did not appear until now. Melosch and Nicolai proposed Ashtekar variables for $d = 11$ $N = 1$ supergravity [130], the conjectured low energy effective field theory of M - theory. Further connections of LQG to M - theory were drawn in [131, 132].

Reference to further literature, of course, will be provided in the main text.

## 1.4 Results in a nutshell

In this thesis, we present a Hamiltonian formulation for Lorentzian and Euclidean general relativity in any spacetime dimension $D + 1$ ($D \geq 2$) which has the following core properties:

1. one of the canonical variables is a connection, in particular Poisson self commuting; this field and its canonically conjugate are real,
2. the gauge group is compact,
3. the theory is free of second class constraints.

It therefore can be loop quantised. Furthermore, various standard and non-standard matter fields can be coupled, which ultimately allows for a loop quantisation of a wide class of supergravity theories in various dimensions. As a first application, combining the new variables with the framework of higher dimensional isolated horizons, we take first steps towards a deduction of Bekenstein's and Hawking's famous entropy formula for black holes also in higher dimensional LQG.

In more detail, we will show that a canonical analysis of the Palatini action in any dimension leads naturally to a formulation which satisfies property 1., but has second class constraints and, in the Lorentzian case, the non-compact gauge group $\mathrm{SO}(1, D)$. We can take care of 3. by means of gauge unfixing, a procedure to turn a second class constraint system into a physically equivalent first class system which we will review in detail. But still we cannot circumvent the non-compactness problem. However, when starting directly from the ADM phase space and enlarging it to the new formulation obtained after gauge unfixing, we will find that actually, the extension is independent of the internal signature, i.e. we can work with the compact gauge group $\mathrm{SO}(D+1)$ irrespective of the spacetime signature. In summary, general relativity for both, Lorentzian and Euclidean spacetimes, in any dimension $D+1$ can be formulated as $\mathrm{SO}(D+1)$ gauge theory with properties 1. - 3. It is, like Ashtekar's theory, subject to Gauß, spatial diffeomorphism and Hamiltonian constraint. Additionally, for $D \geq 3$[1] a new first class constraint, the simplicity constraint already familiar from Plebański theory and covariant LQG in $D = 3$, arises in the canonical theory.

The properties 1. - 3. are all crucial for the applicability of the loop quantisation procedure: second class constraints (3.) should not be quantised and have to be dealt

[1] The case $D = 2$ plays a special role, since the simplicity constraints do not exists in that dimensions and our formulation, unlike in the case $D = 3$, actually coincides with the $D = 2$ Ashtekar formulation (for corresponding choice of internal signature).

with classically, e.g. using the Dirac bracket, which usually spoils property 1., the self-commutativity of the connection. Since the connection acts by multiplication in the representation chosen for loop quantisation, this self-commutativity is crucial. Actually, 1. lies at the heart of the construction of the holonomy flux algebra, on which the quantisation is based. The compactness of the gauge group 2. implies that the holonomies of the connection are valued in a compact set and the probability measure thereon is central when constructing the Ashtekar Lewandowski measure on the space of generalised connections.

Having all these properties fulfilled, the application of the loop quantisation programme, being formulated independent of the number of spacetime dimensions or the compact gauge group, is straightforward, as we will show explicitly, and the rigorous mathematical basis for quantisation carries over to the $\mathrm{SO}(D+1)$ theory. The Hamiltonian constraint is, when compared with the constraint in the Ashtekar Barbero theory, more complicated, since it obtains an extra contribution when applying gauge unfixing, and even more additional terms when choosing the internal and external signature to differ, but these terms can be dealt with at the quantum level. As a new ingredient, we also have to treat the simplicity constraint in the quantum theory. Both, the implementation of the linear and the quadratic version of this constraint will be discussed. We did not succeed in finding a completely satisfactory prescription for the implementation, but give several new ideas how the problem can be attacked and we discuss several ideas of how one could proceed with further research.

Coupling matter to the $\mathrm{SO}(D+1)$ theory is, like in the $\mathrm{SU}(2)$ case, possible for various matter fields. Inclusion of Yang Mills fields and scalar fields works in completely analogy to the four dimensional case. Dealing with Dirac fields is slightly more involved. They transform in the spinor representation of the gauge group $\mathrm{SO}(1, D)$, which seems at first sight in conflict with the gauge group $\mathrm{SO}(D+1)$ we have to use for the gravitational degrees of freedom. Therefore, like in $D = 3$, in the Hamiltonian theory we first break the gauge group down from $\mathrm{SO}(1, D)$ to $\mathrm{SO}(D)$ by choosing time gauge. But then, unlike the $D = 3$ case, a second step is necessary, namely, we have to enlarge the gauge group again to SO(D+1). The Dirac matrices for the $\mathrm{SO}(1, D)$ and $\mathrm{SO}(D + 1)$ Clifford algebras differ, for our sign conventions, by a factor of $\pm i$ in the matrix $\gamma^0$.

That they can be exchanged ultimately is tied to the fact that $\mathrm{SO}(D+1)$ and $\mathrm{SO}(1, D)$ act on the same complex representation spaces. The Hilbert space representation for the fermions then can again be directly taken over from the treatment in $D = 3$. The enlargement to $\mathrm{SO}(D + 1)$ furthermore leads to extra terms in the Hamiltonian constraint, which however are unproblematic both in the classical and in the quantum theory.

While standard matter therefore can be included just like in the case of usual Ashtekar Barbero variables, in order to treat supergravity theories, we need to consider also non-standard matter fields, most prominently, the Rarita-Schwinger field ("gravitino"). This field differs from the Dirac field not only in that it has spin 3/2, but moreover, usually it is a Majorana fermion and therefore its own antiparticle. In particular, for $D + 1 = 4, 10, 11$, when choosing a real representations of the Lorentzian Clifford algebra, it is a real field[1]. Since the Lorentzian Dirac matrices are real in this representation, the real vector space of the Majorana fermions is preserved under the action of $\mathrm{SO}(1, D)$, but when switching the internal signature and using the internal gauge group $\mathrm{SO}(D + 1)$ instead (with necessarily complex Dirac matrices), this no longer is the case. However, it is possible to introduce an additional internal unit vector field $N^I$ to keep track of the complex components the fermionic fields obtain under internal rotations. One can use this field to construct a combined object of both, the fermionic and unit vector field, such that there is an action of $\mathrm{SO}(D+1)$ which respects the reality conditions of the Majorana fermions. Interestingly, the additional, unphysical degrees of freedom introduced by the field $N^I$ and its conjugate momentum can be removed by using the linear simplicity constraint, which interweaves them with the gravitational degrees of freedom. When trying to quantise, one immediately finds that the Hilbert space representation known for Dirac fermions from $D = 3$ cannot be applied in this case. The reason is that the Majorana reality condition the fermions are subject to gives rise to second class constraints which lead to a non-trivial Dirac antibracket. A corresponding Hilbert space representation will be given.

[1] This does not hold in dimensions where there are no real (or imaginary) representations of the Clifford algebra; then, the role of the gravitino is played by anti- or symplectic Majorana fermions, which satisfy a more complicated reality condition. Sometimes, the role of the gravitino is also played by Weyl or Majorana Weyl spinors (cf. [133, 134]).

Having included the gravitino and standard matter already opens the road to loop quantise the easiest supergravity theories, like $d = 4$ $N = 1$ supergravity. However, in many supergravity theories also new bosonic matter fields appear, e.g. Abelian higher $p$-form fields, like the Kalb Ramond two-form field of $d = 10$ supergravities and the three-form field (three index photon) of $d = 11$ $N = 1$ supergravity. As an arguably interesting example, we will treat the latter. Due to an additional Chern Simons term in the corresponding supergravity action [135], this field is not simply a three-form equivalent of the Maxwell field in higher dimensions, but becomes self-interacting and, in particular, the equivalent of the electric field transforms non-trivial under the action of the (equivalent of the) Gauß constraint. We propose a reduced phase space quantisation based on the Weyl algebra generated by the exponentials of certain Dirac observables with respect to the Gauß constraint. Due to the non-standard action of the Gauß constraint, the observable corresponding to the electric field gets an additional contribution proportional to the level of the Chern-Simons theory, which then also shows up in the Weyl relations. These twisted Weyl relations can be computed in closed form and a Hilbert space representation can by given by using a state of Narnhofer-Thirring type [136], which in the LQG literature already appeared in Thiemann's treatment of the closed bosonic string [93].

Finally, as a first application of the variables for higher dimensional LQG, we will take first steps towards the derivation of the famous Bekenstein Hawking formula for the black hole entropy also in higher dimensions. The reproduction of this formula is considered as one of the "benchmarks" of any quantum theory of gravity and already has been met by $D = 3$ LQG (see [43, 44] and references therein). We will work out in detail the boundary symplectic structure arising on $(2n + 1)$-dimensional undistorted non-rotating isolated horizons when using the new variables, which turns out to be the symplectic structure of a higher dimensional $\mathrm{SO}(2(n + 1))$ Chern Simons theory, and provide an appropriate boundary condition connecting bulk and horizon degrees of freedom. However, since Chern Simons theory in higher dimensions, unlike in the three dimensional case, has local degrees of freedom in general [137, 138], its quantisation and the counting of horizon degrees of freedoms done to determine the black hole entropy need further intensive studies.

## 1.5 Outline of the thesis

This thesis is organised as follows: it consists of five parts. Part I is introductory in nature, reviews several Lagrangian and Hamiltonian formulations of general relativity and the relations between them, with an emphasis on those formulations which are relevant for the construction of the new variables. We made an effort to streamline the analysis such that one is lead step by step to the later introduction of the new variables, on the one hand to facilitate access to this work for readers less familiar with canonical formulations of general relativity, and on the other hand to make apparent the various interrelations between them and the new variables and to point out that the new formulation follows rather naturally from existing canonical formulations. The review material is well-known and the familiar reader might directly jump to part II, where we finally will introduce the new variables both from a Hamiltonian (chapter 7) and Lagrangian (chapter 8) point of view. Some extra material on possible extensions of the formulation in terms of the new variables are collected in chapter 9. In part III, we will turn to the quantisation of this theory using LQG methods. A central object of this study is the simplicity constraint. In part IV, we will extend the up to now considered vacuum theory to incorporate (super)matter fields both, at the classical and quantum level. Due to certain properties of the new variables, mainly the inclusion of fermionic fields needs to be revisited thoroughly. We will discuss Dirac fermions in detail, and comment more briefly on Majorana fermions and the three index photon of eleven dimensional supergravity as an example for a higher $p$-form field, the latter two paving the way to treat various supergravity theories. Part V will be dedicated to a first application of the developed framework of higher dimensional LQG, namely black holes in higher dimensions. Concretely, we will derive the isolated horizon boundary degrees of freedom when using the new variables, which constitutes the first step in checking if the famous black hole entropy formula can be derived from LQG also in higher dimensions. Each one of the parts mentioned so far again comes with an introduction and outline of its own, which is the reason why we keep this outline brief. Finally, we will conclude and give an outlook on further research in chapter 18. In the appendices, we provide variational formulae which will be helpful in various calculation throughout this thesis (appendix A), we give details on spatial - temporal decomposition of various tensors used when going from Lagrangian to the

Hamiltonian formulations (appendix B), introduce in detail the vielbein compatible spin connection and generalisations thereof (appendix C), give some details on the Lie algebras so$(D+1)$ and so$(1,D)$ relevant for this work (appendix D), summarise relations satisfied by the Gamma matrices (appendix E), shortly introduce the higher dimensional Newman Penrose formalism (appendix F), and finally in appendix G give calculational details for several lengthy derivations from part V.

# Part I

# Preliminaries: Actions for gravity and corresponding Hamiltonian formulations

We will start with the basics of Lagrangian and Hamiltonian formulations of general relativity. Even if the whole dynamical content of general relativity is encoded in Einstein's famous field equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{1.1}$$

formulating the theory in Lagrangian or Hamiltonian terms gives deeper insight into the theory and, as the reader will notice, increases its aesthetic appeal. Moreover, the passage from a classical to a quantum theory is usually based on Lagrangian or Hamiltonian formulations. Path integral quantisation usually starts from a classical action principle, and canonical quantisation has a classical Hamiltonian formulation as its foundation. Thus, for the aim of quantising gravity, the study of these formulations may be crucial. In (1.1), $G_{\mu\nu}$ and $g_{\mu\nu}$ denote the Einstein and metric tensor, respectively, $\Lambda$ the cosmological constant, $G$ is Newton's constant, $c$ the speed of light and $T_{\mu\nu}$ the energy momentum tensor of the matter fields under consideration. A precise definition of $G_{\mu\nu}$ will be given shortly.

If not made explicit otherwise, we will leave the spacetime dimensions under consideration unspecified except for the requirement $D + 1 > 2$, where $D$ denotes the spatial dimension. Since the focus of the first part of this work lies on a reformulation of the gravitational sector, we will here and in the following only deal with the matter free case, i.e. we choose vanishing energy momentum tensor $T_{\mu\nu}$. Matter fields will be introduced later on in part IV. We will also stick to the case of vanishing cosmological constant $\Lambda = 0$. Moreover, we will neglect boundary terms for the time being, but want to stress that a careful treatment thereof is needed and refer the unfamiliar reader to the standard literature [139–141]. We will treat simultaneously Lorentzian and Euclidean gravity, denoting the spacetime signature by $s = \pm 1$. We will furthermore denote the signature of the internal space with $\zeta = \pm 1$. Starting from an action principle, internal and spacetime signature coincide, $\zeta = s$. However, in the Hamiltonian picture, we have the freedom to choose $s \neq \zeta$. Finally, we want to recommend also the excellent overview over actions for gravity with particular emphasis on 3 and 4 dimensions given in [103].

There are several actions known for general relativity which differ in form and/or in (kinematical) field content, but basically by definition all have to share the same physical degrees of freedom satisfying the same equations of motion, namely Einstein's vacuum field equations (1.1). We will make a distinction between metric and vielbein formulations. A formulation in terms of a vielbein becomes mandatory as soon as one is dealing with fermionic matter, but as we will see, it might also be convenient when working without fermions. A vielbein was, to the best of the author's knowledge, first considered in [142], precisely when coupling spinor fields to general relativity, although the field there is not called vielbein and denoted by $\sqrt{g}^{\nu}{}_{\nu'}$, primed indices being internal ones.

We furthermore differentiate between *first* and *second* order actions of gravity depending on whether the actions are of first or second order in the derivatives of the fields. Typically, in first order formulations, the connection and the metric (or vielbein) are independent kinematical fields, but varying the action with respect to the connection, one obtains equations of motion which require the connection to be metric compatible. Palatini is usually credited for first observing this, and therefore, first order actions are often named after him. Thus, Wald [143] names the first order action depending on a metric and an affine connection "Palatini action", while the first order action depending on a vielbein and a spin connection is called either called "Palatini action" in Ashtekar's book [144] or "Hilbert-Palatini action" in [103, 145, 146]. However, in his 1919 paper [147] (English translation [148]), Palatini is actually not varying with respect to the connection independently. Aiming at unifying general relativity and electromagnetic phenomena, it is Einstein who did this in his 1925 paper [149], and according to [150], where this issue is discussed, no one did before. Despite this comment, we will stick to the nomenclature used by Ashtekar and call the action depending on a vielbein and an independent spin connection "Palatini action" in the following. This also is consistent with [2], one of the articles this thesis is based on.

The first action ever written down for general relativity, however, is a second order one, the Einstein Hilbert action [151]. Surprisingly, this arguably simplest non-trivial, generally covariant action one could possibly write down, the integral over spacetime of the densitised scalar curvature of the Levi-Civita connection, yields Einstein's field equations as Lagrangian equations of motion. We will briefly review this action in section 2.1. The very basic formulas in this and the following sections are supposed to acquaint the reader with the notation and conventions used in this thesis and to lead even the non-experts step by step from the Einstein Hilbert action to the Hamiltonian formulation in terms of the new variables which lies at the heart of this thesis. The experienced reader might want to jump directly to part II and, if needed, consult the list of tables and conventions section at the beginning of this thesis.

After deriving Einstein's field equations by varying the Einstein Hilbert action, we will turn to the corresponding Hamiltonian formulation. Actually, the first Hamiltonian descriptions of general relativity date back to the 1950's [152–155], based on the pioneering methods developed by Dirac [156, 157] and Bergmann and collaborators [158–160] for Hamiltonian formulations of gauge theories like general relativity. We strongly encourage the unfamiliar reader to consult the excellent exposures [157] and [62, section 24], or [161] for a very detailed account on this so-called "Dirac algorithm", which nowadays has become a standard technique in the relativist's toolbox.

In 1960, Arnowitt, Deser and Misner introduced a certain decomposition of the spacetime metric [162, 163], which leads to a very convenient and nowadays frequently used Hamiltonian formulation of general relativity. It is named after its inventors: ADM formulation. We will give the derivation of the ADM formulation starting from the Einstein Hilbert action and discuss it in some detail, since the methods will be needed in any Hamiltonian formulation of general relativity.

After this, we will turn to a neatly related action, by introducing the vielbein and using it instead of the spacetime metric as fundamental degree of freedom in the Einstein Hilbert action. Consequently, when performing the canonical analysis, also in the corresponding Hamiltonian formulation the fundamental role now is played by the spatial co vielbein instead of the spatial metric. We will furthermore present the canonical transformation from the co vielbein to a densitised vielbein, which is related to both, the Ashtekar Barbero formulation and the new formulation of [1, 2]. In this work, we will refer to all of these formulation as extended ADM (eADM) formulation[1].

[1] Note that in some references, these formulations are also just called ADM formulation (cf. e.g. [103]).

Thereafter, we will turn to the already mentioned Palatini action. While it is straightforward to see that the variation leads to Einstein's vacuum field equations, the singular Legendre transformation to the corresponding Hamiltonian formulation is much more intricate and was first derived in full generality for arbitrary dimensions $d > 2$ in [2, 164, 165] (results for spacetime dimensions 3 and 4 are given in [103], and for arbitrary dimensions $d > 2$ in so called time gauge in [166]. Choosing this gauge fixing simplifies the analysis considerably.). We will find that, after solving all second class constraints, we are lead back to the eADM formulation.

In continuation, we will turn to the (real version of the) Plebański formulation [167] of general relativity. Its importance for the LQG community is based on the fact that this action is the starting point for the spin foam models [49, 168]. The Plebański formulation exists for all dimensions $D > 2$[1]. We refrain from displaying the full, rather lengthy canonical analysis of the (real) Plebański action, and refer the interested reader to [170]. Instead, we give the canonical analysis of a certain hybrid version of Plebański and Palatini gravity, which will be the basis for our later Lagrangian access to the new variables in chapter 8.

Finally, we will study some actions peculiar to $D = 3$ which lead to Ashtekar or Ashtekar Barbero variables when passing to the canonical theory. Firstly, in section 6.1, we modify the Palatini action by adding the so-called Holst term [145], which only exists in four spacetime dimensions. We will perform a canonical analysis of this action in section 6.2. This does not directly lead to Ashtekar Barbero variables, but the canonical formulation obtained in this way will be helpful in section 9.3 when reintroducing the Barbero Immirzi parameter $\gamma$ into the framework of the new variables for $D = 3$. Only after solving all second class constraints and choosing time gauge in section 6.2.3, we obtain the famous Ashtekar Barbero formulation [13, 14], which the loop quantisation approach is based on. In section 6.3, we will also describe the canonical transformation which relates this formulation to the ADM formulation, since this route will be mimicked when obtaining the new variables following the Hamiltonian route.

[1]While Plebański worked in $D = 3$, a more general version for $D > 2$ was given in [169].

Secondly, in section 6.4 we will shortly comment on the CDJ [171] action. Like the Ashtekar Barbero formulation on the Hamiltonian side, the CDJ Lagrangian only exists in $D = 2, 3$ dimensions[1], but we will restrict to the $D = 3$ case here. It is an almost pure Lagrangian connection formulation (the metric can be eliminated up to the denisitised lapse function) of general relativity, which for nonzero cosmological constant or certain matter coupling can even be turned into a pure connection formulation [173]. Its Hamiltonian formulation coincides with Ashtekar's original complex (in the Lorentzian case) formulation, i.e. the choice $\gamma^2 = s$.

There are several other actions for gravity, partly particular for four dimensions. Just to name a few, there is a formulation in terms of an affine connection solely, the Lagrangian being the square root of the determinant of the Ricci curvature, which is due to Schrödinger [174]; in another action for vacuum gravity, the fundamental geometrical object is neither the metric nor the vielbein, but the curved space gamma matrices [175]; a formulation by 't Hooft [176] with internal SL(3) or SU(3) symmetry, respectively, for Euclidean or Lorentzian signature, where, instead of a vielbein, a "cube root" of the metric constitutes the metric degrees of freedom; a formulation by Faddeev where Einstein's equations are derived employing the embedding of the spacetime into 10-dimensional linear space [177]; and for some of the Lagrangians we will discuss, there are related complex or self-dual versions, or modifications by topological terms. The selection of actions and Hamiltonian formulations which are going to be presented should by no means indicate that the others do not deserve intense study. Classically, different actions or Hamiltonian formulations are equivalent, but from the quantum gravity perspective, the right classical starting point might be crucial for the endeavour of quantising general relativity. In fact, the successes when quantising general relativity using Ashtekar's variables when compared to the older Wheeler - DeWitt approach [178–180] are an indication in this direction. The fact that some of the formulations mentioned above are rather recent, as is the Hamiltonian connection formulation central to this thesis, shows that this field of research is an active one and it might well be that the right classical starting point still has to be found.

A last comment concerning the expression *corresponding* Hamiltonian formulation in

[1] The extension to $D = 2$ dimensions was achieved in [172].

the title of this part of the thesis: since all actions as well as Hamiltonian formulations of general relativity are physically equivalent, any Hamiltonian formulation in a sense corresponds to any Lagrangian formulation. *Corresponding* in the title therefore only refers to the fact that the Hamiltonian formulations are obtained from the actions under consideration by applying Dirac's standard method in a *straightforward way*, which will be made more explicit in the following sections.

# 2

# Einstein Hilbert action and ADM formulation

## 2.1 Einstein Hilbert action

The Einstein Hilbert action in $D+1$ dimensions is given by

$$S_{EH}[g] := \frac{s}{2\kappa} \int_{\mathscr{M}} d^{D+1}X \ \sqrt{|\det g(X)|} \ ^{(D+1)}R(X). \tag{2.1}$$

Here, $\mathscr{M}$ denotes a $(D+1)$ - dimensional spacetime manifold, $g_{\mu\nu}$ the metric tensor, $\mu, \nu, ... \in \{0, ..., D\}$, $^{(D+1)}R$ the Ricci scalar and $\kappa := \frac{c^4}{8s\pi G}$. In the Lorentzian case, we will restrict to globally hyperbolic spacetimes. The restriction to globally hyperbolic spacetimes is demanded by causality and assures by a theorem due to Geroch [181] that the manifold $(\mathscr{M}, g)$ is topologically isomorphic to $\mathbb{R} \times \sigma$ for some spatial manifold $\sigma$. Quite recently, these results where strengthened by showing that globally hyperbolic manifolds are actually isometric to $(\mathbb{R} \times \sigma, -\beta dt^2 + g^t)$ for a smooth family $(\sigma, g^t)$ of Riemannian manifolds and smooth function $\beta$ on $\mathscr{M}$ [182]. In the Euclidean case, we will as well restrict to manifolds of topology $\mathbb{R} \times \sigma$.

We will choose units such that $\kappa = 1$. The remaining factor of $\frac{s}{2}$ also appeared in front of the Palatini action in [2], and in order to simplify comparison, we will introduce it for most action we will consider. In the following, we will use the short hand notation $g := \det g$ and drop the superscript indicating the dimension from curvature tensors for the time being. Our conventions for the curvature tensors are as follows[1]:

[1] Cf. also the conventions.

for one-forms $u_\rho$, we define the Riemann tensor $R_{\mu\nu\rho}{}^{\sigma}$ to be

$$[\nabla_\mu, \nabla_\nu]\, u_\rho = R_{\mu\nu\rho}{}^{\sigma} u_\sigma, \tag{2.2}$$

where $[A, B] := AB - BA$ denotes the commutator and $\nabla$ the unique torsion free[1], metric compatible connection, $\nabla_\mu u_\nu := \partial_\mu u_\nu - \Gamma^\rho_{\mu\nu} u_\rho$. $\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\nu g_{\mu\sigma} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu}\right)$ denote the Christoffel symbols. The Ricci tensor is defined by $R_{\mu\nu} := R_{\mu\rho\nu}{}^{\rho}$ and the Ricci scalar by $R := R_{\mu\nu} g^{\mu\nu}$.

It is straightforward to see that the variation of this action yields

$$\delta S_{EH} = \frac{s}{2} \int_{\mathscr{M}} d^{D+1}X \ \sqrt{|g|} \ \left[ \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \delta g^{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu} \right], \tag{2.3}$$

and using the formulas given in appendix A for the variation of the Riemann tensor, the contribution from the last summand

$$g^{\mu\nu} \delta R_{\mu\nu} = \nabla^\mu \left( \nabla^\nu \delta g_{\mu\nu} - g^{\nu\rho} \nabla_\mu \delta g_{\nu\rho} \right), \tag{2.4}$$

gives only a boundary term by Stoke's theorem and thus will be neglected[2]. The remaining integrand yields Einstein's famous (vacuum) field equations

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0, \tag{2.5}$$

and any other action we will write down in the following has to reproduce them.

## 2.2 Canonical analysis: ADM formulation

### 2.2.1 $D + 1$ split

While general relativity inherently is a theory of spacetime, in order to obtain a Hamiltonian formulation, one has to make a split in "space" and "time", the so-called $D + 1$ split[3]. To this end, we first introduce a foliation of $\mathscr{M} \,\tilde{=}\, \mathbb{R} \times \sigma$ by a family $\Sigma_t = X_t(\sigma)$ of spacelike hypersurfaces labelled by $t =$ const., where $X_t : \sigma \to \mathscr{M}$

[1] In the following, we assume all affine connections to be torsion free and refer the interested reader to the review [183] for a discussion of the inclusion of torsion.

[2] Note, however, that for this term to vanish, it does not suffice to demand $\delta g|_{\partial\mathscr{M}} = 0$, one furthermore has to demand that the derivates of $\delta g$ vanish at the boundary, or, alternatively, modify the action by a boundary term, which is discussed e.g. in [143, appendix E].

[3] Our exposition will follow [62].

denotes a one-parameter family of embeddings defined by $X_t(x) := X(t,x)$. Here, $X^\mu$ and $x^a$ $(a, b, ... \in \{1, ..., D\})$ denote local coordinates on $\mathscr{M}$ and $\sigma$, respectively. The freedom of choice of foliation can be parametrised by the deformation vector field $T^\mu := \frac{\partial X^\mu(t,x)}{\partial t}|_{X=X(t,x)}$. This vector field describes the flow of "time", and we will interpret Lie derivatives along this vector field as time derivatives. Introducing the future pointing unit normal field $n^\mu$ to the spatial slices $\Sigma_t$, $g_{\mu\nu}n^\mu n^\nu = s$, we can decompose the deformation vector field according to

$$T^\mu = Nn^\mu + N^\mu, \tag{2.6}$$

where $N$ is called lapse function, and $N^\mu$ denotes the shift vector field, which is tangential to the spatial slices, $g_{\mu\nu}n^\mu N^\nu = 0$. We will call tensor fields with the property that their contraction with $n$ vanishes "spatial". For a visualisation, see fig. 2.1. The

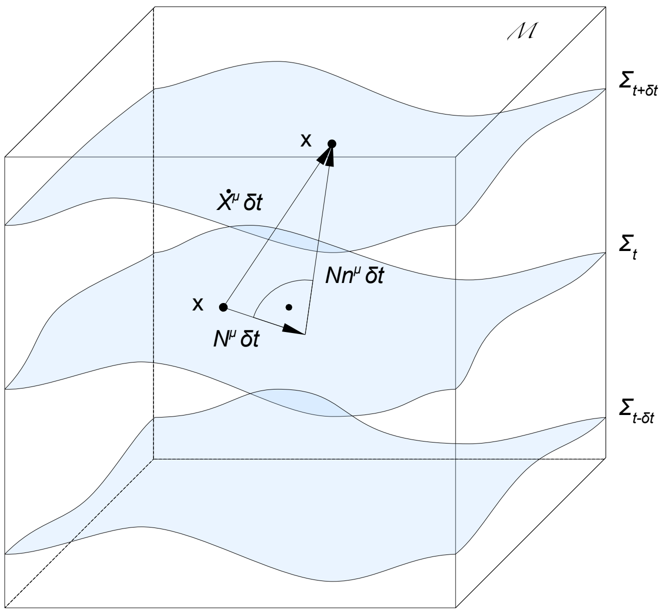


**Figure 2.1: Visualisation of the $D + 1$ split** - Foliation of the spacetime manifold $\mathscr{M}$ into (spatial) leaves $\Sigma_t$ labelled by the value of the time function $t$ on these leaves. The figure illustrates furthermore the meaning of the lapse function $N$ and the shift vector $N^\mu$.

induced spatial metric on $\Sigma_t$ is given by

$$q_{\mu\nu} = g_{\mu\nu} - sn_\mu n_\nu, \tag{2.7}$$

also called first fundamental form. For a Hamiltonian formulation of general relativity, it is convenient to use $q_{\mu\nu}$, $N$, and $N^\mu$ as fundamental fields, which in turn encode the whole information of $g^{\mu\nu}$. To obtain the ADM action, we have to express the Einstein Hilbert action in terms of these fields and their derivatives. We first introduce the torsion free covariant spatial derivative compatible with $q_{\mu\nu}$. For scalar fields $\phi$ and spatial one forms $u_\mu$ on $\Sigma_t$, it is defined by

$$\begin{aligned} D_\mu \phi &= q^\nu_\mu \nabla_\nu \tilde{\phi}, \\ D_\mu u_\nu &= q^\rho_\mu q^\sigma_\nu \nabla_\rho \tilde{u}_\sigma, \end{aligned} \tag{2.8}$$

and extended to general tensor fields by linearity and Leibnitz' rule[1].

It by definition preserves spatial tensors and it is easy to check that $D_\mu q_{\nu\rho} = 0$. The Ricci scalar in the Einstein Hilbert action can be reexpressed using the famous Gauß-Codacci equation (see appendix B for a derivation)

$$^{(D+1)}R = {}^{(D)}R - s\left[K_{\mu\nu}K^{\mu\nu} - K^2\right] + 2s\nabla_\mu\left(n^\nu\nabla_\nu n^\mu - n^\mu\nabla_\nu n^\nu\right), \tag{2.9}$$

where we introduced the extrinsic curvature or second fundamental form

$$K_{\mu\nu} := q^\rho_\mu q^\sigma_\nu \nabla_\rho n_\sigma, \tag{2.10}$$

and denoted its trace by $K := K_{\mu\nu}q^{\mu\nu}$. The final split form of the action is obtained by pulling back (spatial) tensor fields to $\sigma$ using the $D$ spatial vector fields $X^\mu_a(X) := X^\mu_{,a}(x,t)|_{X(x,t)=X}$ (e.g. $q_{ab}(x,t) := (X^\mu_a X^\nu_b q_{\mu\nu})(X(x,t))$ etc., cf. [62, section 1.1] for more details). Neglecting the surface term due to the last term in (2.9), we obtain

$$S = \frac{s}{2}\int dt \int_\sigma d^D x N\sqrt{q}\left({}^{(D)}R - s\left[K_{ab}K^{ab} - K^2\right]\right). \tag{2.11}$$

Here, we again denote the determinant of the spatial metric by $q := \det q$ and furthermore used that $g = sN^2 q$ and therefore $\sqrt{|g|} = |N|\sqrt{q}$. However, since we chose $T^\mu$ future pointing timelike, $N > 0$ classically and we have therefore dropped the absolute value sign in (2.11).

[1] $\tilde{\phi}$ and $\tilde{u}_\mu$ here denote smooth extensions to a neighbourhood of $\Sigma_t$ in $\mathscr{M}$. Note that $D_\mu$ is insensitive to the chosen extension. We will drop the tilde in what follows.

### 2.2.2 Legendre transformation

Following Dirac [157], we start the canonical analysis by calculating the momenta conjugate to the spatial metric, which we will denote by $P^{ab}$. By Frobenius' theorem, $K_{\mu\nu}$ is a symmetric (0,2)-tensor. Using this, one can reexpress $K_{\mu\nu}$ as follows

$$K_{\mu\nu} = \frac{1}{2}(\mathscr{L}_n q)_{\mu\nu} = \frac{1}{2N}(\mathscr{L}_{T-N} q)_{\mu\nu}, \tag{2.12}$$

from which we see that it is related to the time derivative of the spatial metric. In the last step above, we used that $n^\mu q_{\mu\nu} = 0$ and therefore, terms $\propto \partial N$ vanish. Pulling back to $\sigma$, we obtain

$$K_{ab} = \frac{1}{2N}\left(\dot{q}_{ab} - (\mathscr{L}_{N} q)_{ab}\right). \tag{2.13}$$

Using this, a short calculation shows that

$$\begin{aligned} P^{ab}(t,x) &:= \frac{\delta S}{\delta \dot{q}_{ab}(t,x)} \\ &= \frac{s}{4N(t,x)}\frac{\delta S}{\delta K_{ab}(t,x)} \\ &= -\frac{1}{2}\sqrt{q}(t,x)\left[K^{ab}(t,x) - K(t,x)q^{ab}(t,x)\right] \\ &= -\frac{1}{2}\sqrt{q}(t,x)G^{ab\,cd}(t,x)K_{cd}(t,x), \end{aligned} \tag{2.14}$$

where

$$G^{ab\,cd} := q^{a(c|}q^{b|d)} - q^{ab}q^{cd}. \tag{2.15}$$

Since the inverse of $G^{ab\,cd}$ is easily found to be[1]

$$G^{-1}_{ab\,cd} := q_{a(c|}q_{b|d)} - \frac{1}{D-1}q_{ab}q_{cd}, \tag{2.16}$$

we find solving (2.14) for $\dot{q}_{ab}$

$$K_{ab}(q,P) = -\frac{2}{\sqrt{q}}G^{-1}_{ab\,cd}P^{cd} = -\frac{2}{\sqrt{q}}\left(P_{ab} - \frac{1}{D-1}q_{ab}P\right), \tag{2.17}$$

$$\dot{q}_{ab}(q,P,N,\vec{N}) = 2NK_{ab}(q,P) + (\mathscr{L}_{N} q)_{ab}, \tag{2.18}$$

[1] $G^{-1}_{ab\,cd}$ is sometimes referred to as the DeWitt "supermetric", introduced in [178] as a metric in the "superspace" of spatial Riemannian metrics.

where $P := P^{ab}q_{ab}$ denotes the trace of $P^{ab}$. For the momenta conjugate to lapse and shift, we immediately find

$$P^{(N)}(t,x) := \frac{\delta S}{\delta \dot{N}(t,x)} = 0, \tag{2.19}$$

$$P_a^{(\vec{N})}(t,x) := \frac{\delta S}{\delta \dot{N}^a(t,x)} = 0. \tag{2.20}$$

It is obvious that we cannot solve for the velocities $\dot{N}$, $\dot{N}^a$ in terms of $q_{ab}$, $P^{ab}$, $N$, $P^{(N)}$, $N^a$, $P_a^{(\vec{N})}$, and therefore, we see that the Lagrangian we are dealing with is singular. The equations (2.19, 2.20) constitute primary constraints according to Bergmann's terminology,

$$\mathscr{C}(t,x) := P^{(N)}(t,x) = 0, \quad \mathscr{C}_a(t,x) := P_a^{(\vec{N})}(t,x) = 0. \tag{2.21}$$

Now it is straightforward, using (2.13, 2.14, 2.17, 2.18), to obtain the action in canonical form[1]

$$\begin{aligned} S = &\int dt \int_\sigma d^Dx \Big[P^{(N)}\dot{N} + P_a^{(\vec{N})}\dot{N}^a + P^{ab}\dot{q}_{ab} - \lambda\mathscr{C} - \lambda^a\mathscr{C}_a \\ &\quad - \left(P^{ab}\dot{q}_{ab} - \frac{s}{2}N\sqrt{q}\left({}^{(D)}R - sG^{ab\,cd}K_{ab}K_{cd}\right)\right)(P,q,N,\vec{N})\Big] \\ = &\int dt \int_\sigma d^Dx \Big[P^{(N)}\dot{N} + P_a^{(\vec{N})}\dot{N}^a + P^{ab}\dot{q}_{ab} - \lambda\mathscr{C} - \lambda^a\mathscr{C}_a \\ &\quad - \left(P^{ab}(2NK_{ab} + (\mathscr{L}_{\vec{N}}q)_{ab}) - \frac{s}{2}N\sqrt{q}{}^{(D)}R - NP^{ab}K_{ab}\right)(P,q,N,\vec{N})\Big] \\ = &\int dt \int_\sigma d^Dx \Big[P^{(N)}\dot{N} + P_a^{(\vec{N})}\dot{N}^a + P^{ab}\dot{q}_{ab} - \lambda\mathscr{C} - \lambda^a\mathscr{C}_a \\ &\quad -N\left(-\frac{2}{\sqrt{q}}\left(P^{ab}P_{ab} - \frac{1}{D-1}P^2\right) - \frac{s}{2}\sqrt{q}{}^{(D)}R\right) - N^a\left(-2q_{ac}D_bP^{bc}\right) \\ &\quad -2\partial_b\left(N^aq_{ac}P^{bc}\right)\Big]. \end{aligned} \tag{2.22}$$

Dropping the surface term in the last line, we can easily read off the non-vanishing Poisson brackets

$$\{q_{ab}(x), P^{cd}(y)\} = \delta^c_{(a}\delta^d_{b)}\ \delta^{(D)}(x-y), \tag{2.23}$$

$$\{N(x), P^{(N)}(y)\} = \delta^{(D)}(x-y), \tag{2.24}$$

$$\{N^a(x), P_b^{(\vec{N})}(y)\} = \delta^a_b\ \delta^{(D)}(x-y), \tag{2.25}$$

[1]Note that the undetermined velocities $\dot{N}$, $\dot{N}^a$ appear in the Hamiltonian as Lagrange multipliers $\lambda$, $\lambda^a$ according to Dirac's procedure.

and the Hamiltonian

$$\begin{aligned} H &:= \int_\sigma d^D x\,[\lambda\mathscr{C} + \lambda^a \mathscr{C}_a + N\mathscr{H} + N^a\mathscr{H}_a] \\ &=: \mathscr{C}[\lambda] + \mathscr{C}_a[\lambda^a] + \mathscr{H}[N] + \mathscr{H}_a[N^a]. \end{aligned} \tag{2.26}$$

Note the notation $\mathscr{C}[c]$ we introduced for smeared versions of constraints $\mathscr{C}$ with a Lagrange multiplier field $c$. This notation will be used extensively throughout this thesis. We furthermore defined

$$\mathscr{H} := -\frac{2}{\sqrt{q}}\left(P^{ab}P_{ab} - \frac{1}{D-1}P^2\right) - \frac{s}{2}\sqrt{q}\,^{(D)}R, \tag{2.27}$$

$$\mathscr{H}_a := -2q_{ac}D_b P^{bc}, \tag{2.28}$$

which are called Hamiltonian constraint and spatial diffeomorphism constraint, respectively.

### 2.2.3 Constraint analysis

More precisely, (2.27, 2.28) are secondary constraints which arise when we demand that the primary constraints $\mathscr{C}$, $\mathscr{C}_a$ be preserved by the time evolution generated by $H$,

$$0 \overset{!}{=} \dot{\mathscr{C}} = \{\mathscr{C}, H\} = \{P^{(N)}, H\} = -\mathscr{H}\ , \tag{2.29}$$

$$0 \overset{!}{=} \dot{\mathscr{C}}_a = \{\mathscr{C}_a, H\} = \{P_a^{(\vec{N})}, H\} = -\mathscr{H}_a. \tag{2.30}$$

These secondary constraints satisfy the Dirac or hypersurface deformation algebra

$$\begin{aligned} \{\mathscr{H}_a[N^a], \mathscr{H}_b[M^b]\} &= -\mathscr{H}_c[(\mathscr{L}_N M)^c], \\ \{\mathscr{H}_a[N^a], \mathscr{H}[M]\} &= -\mathscr{H}[(\mathscr{L}_N M)], \\ \{\mathscr{H}[N], \mathscr{H}[M]\} &= -s\mathscr{H}_a[q^{ab}(N\partial_b M - M\partial_b N)], \end{aligned} \tag{2.31}$$

and in particular trivially Poisson commute with the primary constraints. Thus, they are preserved by the time evolution and the stability analysis ends here. To verify (2.31), note that $\mathscr{H}_a$ generates spatial diffeomorphisms on all phase space variables,

$$\{q_{ab}[f^{ab}], \mathscr{H}_c[N^c]\} = \left(\mathscr{L}_{\vec{N}} q\right)_{ab}[f^{ab}], \tag{2.32}$$

$$\{P^{ab}[F_{ab}], \mathscr{H}_c[N^c]\} = \left(\mathscr{L}_{\vec{N}} P\right)^{ab}[F_{ab}], \tag{2.33}$$

which readily explains the first two lines in (2.31). Here, the notation indicates the smearing of $q_{ab}$, $P^{ab}$ with smearing field $f^{ab}$, $F_{ab}$ that we already introduced for constraints earlier.

Calculating the Poisson bracket between two Hamilton constraints above is more complicated, but the calculation simplifies if we apply the formula for $\delta^{(D)}R$ given in appendix A and furthermore use that, since the expression is antisymmetric in $M, N$, all terms without derivatives acting on the multipliers vanish. The Hamiltonian constraint can be shown to generate diffeomorphisms in time on shell, i.e. if the equations of motion hold. The derivation thereof is cumbersome and we refer the reader to e.g. [62, section 1.3].

| Variable | Dof | $1^{\text{st}}$ cl. constraints | Dof (count twice) |
|---|---|---|---|
| $q_{ab}$ | $\frac{D(D+1)}{2}$ | $\mathscr{H}$ | 1 |
| $P^{ab}$ | $\frac{D(D+1)}{2}$ | $\mathscr{H}_a$ | $D$ |
| Sum: | $D^2+D$ | Sum: | $2D+2$ |

**Table 2.1:** ADM phase space: counting of degrees of freedom

As one expects for a generally covariant theory like general relativity, we see that the Hamiltonian is constrained to vanish. All constraints are first class in Dirac's terminology, as (2.31) shows (the remaining Poisson brackets are trivial). Again following Dirac's programme, we introduce the extended Hamiltonian, which amounts to adding all secondary first class constraints (in this case, $\mathscr{H}$ and $\mathscr{H}_a$) to the Hamiltonian with arbitrary Lagrange multipliers, say $N'$ and $N^{a\prime}$. But $\mathscr{H}$ and $\mathscr{H}_a$ are already present in the Hamiltonian, and thus get multiplied by $N'' := N + N'$ and $N^{a\prime\prime} := N^a + N^{a\prime}$ in $H$. But $N''$ and $N^{a\prime\prime}$ still are completely arbitrary multipliers. Now we can trivially solve the constraints $\mathscr{C}$ and $\mathscr{C}_a$ by demanding these equations strongly and gauge fixing $N$ and $N^a$. One obtains the canonical ADM action and the ADM Hamiltonian [162, 163]

$$S = \int dt \int_\sigma d^D x \left[ P^{ab}\dot{q}_{ab} - N\mathscr{H} - N^a\mathscr{H}_a \right], \tag{2.34}$$

$$H = \int_\sigma d^D x \left[ N\mathscr{H} + N^a\mathscr{H}_a \right]. \tag{2.35}$$

Finally, counting degrees of freedom (cf. table 2.1), we find $(D+1)(D-2)$ phase space degrees of freedom, which have to be reproduced by any of the following formulations.

# 3

# Einstein Hilbert action with vielbein and eADM formulation

## 3.1 Einstein Hilbert vielbein formulation

In this section, we will consider an action closely related to the Einstein Hilbert action. To this end, we introduce the "square root" of the metric, the co - vielbein $e_\mu{}^I$ satisfying

$$g_{\mu\nu} = e_\mu{}^I e_\nu{}^J \eta_{IJ}, \tag{3.1}$$

and the action we want to consider is given by

$$\begin{aligned} S'_{\text{EH}} &:= \frac{s}{2} \int_{\mathscr{M}} d^{D+1}X \ \det e \ e^\mu{}_I e^\nu_J R_{\mu\nu}{}^{IJ} = \\ &= s \int_{\mathscr{M}} d^{D+1}X \ * (e \wedge e)_{IJ} \wedge R^{IJ}. \end{aligned} \tag{3.2}$$

Here, $e^\mu{}_I$ denotes the vielbein, $e_{\mu I} e^\mu{}_J = \eta_{IJ}$, $\eta := \text{diag}(\zeta, +1, ..., +1)$, and $\det e$ the determinant of the co - vielbein. Of course the internal signature here is dictated by the space time signature, $\zeta = s$, i.e. the gauge group is $\text{SO}(D+1)$ for Euclidean and $\text{SO}(1, D)$ for Lorentzian general relativity. In the following, we will use ${}^{(D+1)}e := \det e$. We furthermore introduced the vielbein compatible spin connection: The equation

$$0 = \nabla^\Gamma_\mu e^\nu{}_I := \partial_\mu e^\nu{}_I + \Gamma^\nu_{\mu\rho} e^\rho{}_I + \Gamma_{\mu I}{}^J e^\nu{}_J, \tag{3.3}$$

can be solved uniquely for

$$\Gamma_{\mu IJ} := e^\nu{}_{[I|} \nabla_\mu e_{\nu|J]}, \tag{3.4}$$

and $R_{\mu\nu}{}^{IJ} := 2\partial_{[\mu}\Gamma_{\nu]}{}^{IJ} + 2\Gamma_{[\mu|}{}^{IK}\Gamma_{|\nu]K}{}^{J}$ above denotes its curvature. Using the formulas in appendix A, we easily find for the variation of this action

$$\delta S'_{\text{EH}} = s \int_{\mathscr{M}} d^{D+1}X \ {}^{(D+1)}e \ \left[\left(e^{\nu}{}_{J}R_{\rho\nu}{}^{KJ} - \frac{1}{2}e_{\rho}{}^{K}e^{\mu}{}_{I}e^{\nu}{}_{J}R_{\mu\nu}{}^{IJ}\right)\delta e^{\rho}{}_{K} \right.$$
$$\left. + e^{\mu}{}_{I}e^{\nu}{}_{J}\nabla^{\Gamma}_{[\mu}\delta\Gamma_{\nu]}{}^{IJ}\right]. \qquad (3.5)$$

Since $\nabla^{\Gamma}_{\mu}$ annihilates $e^{\nu}_{J}$, the term in the last line obviously only contributes a boundary term which we drop, and the field equations read

$$\begin{aligned} 0 &= e^{\nu}{}_{J}R_{\rho\nu}{}^{KJ} - \frac{1}{2}e_{\rho}{}^{K}e^{\mu}{}_{I}e^{\nu}{}_{J}R_{\mu\nu}{}^{IJ} \\ &= e^{\sigma K}\left(R_{\rho\sigma} - \frac{1}{2}g_{\rho\sigma}R\right) \\ &= e^{\sigma K}G_{\rho\sigma}, \end{aligned} \qquad (3.6)$$

where we used $R_{\mu\nu}{}^{IJ} = R_{\mu\nu\rho\sigma}e^{\rho I}e^{\sigma J}$ (cf. appendix C). Since we only consider invertible $e^{\sigma K}$ (otherwise, the metric would be degenerate), $G_{\mu\nu} = 0$ is a necessary consequence, and sufficient to solve the field equations.

This is not too surprising, since the action $S'_{EH}$ coincides with the Einstein Hilbert action considered as a function of the vielbein up to sgn $({}^{(D+1)}e)$. Using the results in appendix C, we have

$$\begin{aligned} S_{EH}[e] &= \frac{s}{2}\int_{\mathscr{M}} d^{D+1}X \ \left[\sqrt{|g|}{}^{(D+1)}R\right](e) \\ &= \frac{s}{2}\int_{\mathscr{M}} d^{D+1}X \ |\det e| \ e^{\mu}{}_{I}e^{\nu}_{J}R_{\mu\nu}{}^{IJ}, \end{aligned} \qquad (3.7)$$

since $\sqrt{|g|} = |\det e|$ and $R = R_{\mu\nu}{}^{IJ}e^{\mu}{}_{I}e^{\nu}{}_{J}$. Variation of (3.7) yields trivially (only using $\delta g_{\mu\nu} = 2e_{(\mu|I}\delta e_{\nu)}{}^{I}$ in (2.3)) the field equations (3.6).

## 3.2 Canonical Analysis: eADM formulation

### 3.2.1 $D+1$ split and Legendre transformation

The Hamiltonian formulation corresponding to this action will be called extended ADM formulation (eADM). To obtain it, we perform the $D+1$ split analogously to the treatment in section 2.2.1. Using the (future pointing unit) normal to the spatial slices

$n^\mu$, we construct $\delta^\mu{}_\nu = (g^\mu{}_\nu - sn^\mu n_\nu) + sn^\mu n_\nu =: q^\mu{}_\nu + sn^\mu n_\nu$ and decompose the vielbein according to $e^\mu_I = q^\mu{}_\nu\, e^\nu_I + s e^\nu_I n_\nu n^\mu =: {}^{\|}e^\mu_I + s n_I n^\mu$. By construction, ${}^{\|}e^\mu_I$ is the spatial part of the vielbein, ${}^{\|}e^\mu_I n_\mu = 0$. From appendix B, we have

$$e^{\mu I} e^{\nu J} R_{\mu\nu IJ} = {}^{\|}e^{\mu I}\,{}^{\|}e^{\nu J} R^{\mathrm{H}}{}_{\mu\nu IJ} - s\left(K_{\mu\nu}K^{\mu\nu} - K^2\right), \tag{3.8}$$

where $R^{\mathrm{H}}{}_{\mu\nu IJ}$ is the curvature of $\Gamma^{\mathrm{H}}{}_\mu{}^{IJ} := {}^{\|}e^{\nu[I} D_\mu {}^{\|}e_\nu{}^{J]} + sn^{[I} D_\mu n^{J]}$ and $K_{\mu\nu} = \frac{1}{2}(\mathscr{L}_n q)_{\mu\nu} = {}^{\|}e_{(\mu|}{}^I (\mathscr{L}_n {}^{\|}e_{\nu)})_I$. Pulling back to the spatial manifold $\sigma$ like in section 2.2, we obtain

$$S = \frac{s}{2}\int dt \int_\sigma d^D x\; N\, {}^{(D)}e \left[ e^{aI} e^{bJ} R^{\mathrm{H}}{}_{abIJ} - s\Big(K_{ab}K^{ab} - K^2\Big)\right], \tag{3.9}$$

where $e^{aI}$ denotes the hybrid vielbein and $R^{\mathrm{H}}{}_{abIJ}$ is the curvature of the hybrid spin connection annihilating the hybrid vielbein introduced by Peldán [103],

$$\Gamma^{\mathrm{H}}{}_a{}^{IJ} := e^{b[I} D_a e_b{}^{J]} + sn^{[I} D_a n^{J]}, \tag{3.10}$$
$$D^{\mathrm{H}}_a e_b{}^I := D_a e_b{}^I + \Gamma^{\mathrm{H}}{}_a{}^I{}_J e_b{}^J = 0. \tag{3.11}$$

For more details, see appendix C. Actually, we could have used a shortcut to arrive here: Starting with the action (3.7), we already know its split form using that this action is equal to $S_{EH}[e]$. The result is given by (2.11), where ${}^{(D)}R$ and $\sqrt{q}$ are now considered as a function of $e_a{}^I$ via

$$q_{ab} = e_a{}^I e_b{}^J \eta_{IJ}, \tag{3.12}$$

and $K_{ab}$ as a function of $e_a{}^I$, $N$ and $N^a$.

Since (3.7) only differs from the action we used above by a factor of $\mathrm{sgn}({}^{(D+1)}e)$, the split form we arrived at has the same property. In what follows, we will neglect this subtle difference, i.e. effectively continue with the split form of the action (3.7), in order to facilitate comparison with the ADM case.

Instead of $P^{ab}$, we now calculate the momenta $k^a{}_I$ conjugate to $e_a{}^I$,

$$\begin{aligned}
k^a{}_I(t,x) &:= \frac{\delta S}{\delta \dot{e}_a{}^I(t,x)} \\
&= \int_\sigma d^D y \frac{\delta \dot{q}_{bc}(t,y)}{\delta \dot{e}_a{}^I(t,x)} \frac{\delta S}{\delta \dot{q}_{bc}(t,y)} \\
&= 2\delta^a_{(c} e_{b)I}(t,x) P^{bc}(t,x) \\
&= -\sqrt{q}(t,x) e_{bI}(t,x) G^{ab\,cd}(t,x) K_{cd}(t,x) \\
&= G^a{}_I{}^b{}_J \, (\dot{e}_b{}^J - (\mathscr{L}_N e)_b{}^J), \qquad (3.13)
\end{aligned}$$

where

$$\begin{aligned}
G^a{}_I{}^b{}_J &:= -\frac{\sqrt{q}}{N} e_{cI} G^{ac\,bd} e_{dJ} \\
&= -\frac{\sqrt{q}}{2N} q^{c[a} e^{d]}{}_I e_{(c}{}^J \delta^b_{d)}, \qquad (3.14)
\end{aligned}$$

and $G^{ab\,cd}$ is defined in (2.15). Solvability of (3.13) for $\dot{e}_a{}^I$ depends on the invertibility of this matrix, and therefore, the primary constraints can be obtained from it. That $G^a{}_I{}^b{}_J$ is not invertible is already clear from the fact that $K_{cd}$ in the second to last line of (3.13) only depends on

$$\dot{q}_{ab} = 2e_{(aI} \dot{e}_{b)}{}^I, \qquad (3.15)$$

and thus we cannot hope for solving for the other components of $\dot{e}_a{}^I$. Contracting the third to last line with $e_{aJ}$ and symmetrising / antisymmetrising in the index pair $I$, $J$, we obtain

$$k^a_{(I} e_{a|J)} = 2e_{b(I} e_{a|J)} P^{ba}, \qquad (3.16)$$

$$\mathscr{G}_{IJ} := 2k^a_{[I} e_{a|J]} = 4e_{b[I} e_{a|J]} P^{ba} = 0, \qquad (3.17)$$

where the factor of 2 in the definition of $\mathscr{G}_{IJ}$ is inserted in order to fit with later results. Contracting the first line with $e^{(c|I} e^{|d)J}$, we obtain

$$k^{(c|I} e^{|d)}{}_I = 2P^{cd}, \qquad (3.18)$$

which can be solved for $\dot{q}_{ab} = 2e_{(aI} \dot{e}_{b)}{}^I$ in analogy to (2.17, 2.18). Note that in this section, $P^{ab}$ of course is given in terms of $e, \dot{e}$ via (2.13, 2.14) and (3.12, 3.15). The analysis of $N$, $N^a$ and their momenta is completely the same as before. We can decompose

${k^a}_I {\dot{e}_a}^I$ according to

$$\begin{aligned} {k^a}_I {\dot{e}_a}^I &= {k^b}_I e_{bJ} e^{aJ} {\dot{e}_a}^I \\ &= {k^b}_I e_{bJ} e^{a(J|} {\dot{e}_a}^{I)} + {k^b}_I e_{bJ} e^{a[J|} {\dot{e}_a}^{I]} \\ &= \frac{1}{2} {k^b}_I e^{aI} \dot{q}_{ab} + \frac{1}{2} e^{a[J|} \left( {\dot{e}_a}^{I]} + s n^{I]} n^K \dot{e}_{aK} \right) \mathscr{G}_{IJ} \\ &= P^{ab} \dot{q}_{ab} + \frac{1}{2} e^{a[J|} \left( {\dot{e}_a}^{I]} + s n^{I]} n^K \dot{e}_{aK} \right) \mathscr{G}_{IJ}. \end{aligned} \tag{3.19}$$

Adding and subtracting ${k^a}_I {\dot{e}_a}^I$ in (2.11) and correspondingly for $N$, $N^a$, and introducing the Lagrange multiplier $\lambda^{IJ} = e^{a[J|} \left( {\dot{e}_a}^{I]} + s n^{I]} n^K \dot{e}_{aK} \right)$ for the velocities we cannot solve for, we obtain

$$\begin{aligned} S = \int dt \int_\sigma d^D x \, \Big[ & P^{(N)} \dot{N} + P^{(\vec{N})}_a \dot{N}^a + {k^a}_I {\dot{e}_a}^I - \lambda \mathscr{C} - \lambda^a \mathscr{C}_a \\ & - \left( P^{ab} \dot{q}_{ab} + \frac{1}{2} \lambda^{IJ} \mathscr{G}_{IJ} - \frac{s}{2} N \sqrt{q} \left( {}^{(D)}R - s G^{ab\,cd} K_{ab} K_{cd} \right) \right) (k, e, N, \vec{N}) \Big] . \end{aligned} \tag{3.20}$$

Clearly, after performing the same manipulations as in the metric case and eliminating $N$, $N^a$ and their momenta, the final form of the action will be

$$S = \int dt \int_\sigma d^D x \left[ {k^a}_I {\dot{e}_a}^I - \frac{1}{2} \lambda^{IJ} \mathscr{G}_{IJ} - N \mathscr{H} - N^a \mathscr{H}_a \right], \tag{3.21}$$

with the non-vanishing Poisson brackets

$$\{ {e_a}^I(x), {k^b}_J(y) \} = \delta^b_a \delta^{(D)}(x-y). \tag{3.22}$$

In (3.21), $\mathscr{G}_{IJ}$ is given by (3.17) and $\mathscr{H}$, $\mathscr{H}_a$ as in (2.27, 2.28) with $q_{ab}$, $P^{cd}$ replaced by ${e_a}^I$, ${k^a}_I$ using (3.12) and (3.18),

$$\begin{aligned} \mathscr{H}_a &= -D_b (k^{bJ} e_{aJ}) + \frac{1}{2} D_b (\mathscr{G}^{IJ} {e^b}_J e_{aI}) \\ &\approx -D_b (k^{bJ} e_{aJ}), \end{aligned} \tag{3.23}$$

$$\begin{aligned} \mathscr{H} &= -\frac{1}{2\sqrt{q}} \left( k^{aI} k^{bJ} - \frac{1}{D-1} k^{aJ} k^{bI} + \frac{1}{2} \mathscr{G}^{KJ} {e^b}_K k^{aI} \right) e_{aJ} e_{bI} - \frac{s}{2} \sqrt{q} {}^{(D)}R \\ &\approx -\frac{1}{2\sqrt{q}} \left( k^{aI} k^{bJ} - \frac{1}{D-1} k^{aJ} k^{bI} \right) e_{aJ} e_{bI} - \frac{s}{2} \sqrt{q} {}^{(D)}R, \end{aligned} \tag{3.24}$$

where it is understood that $q$ and ${}^{(D)}R$ are expressed using $e_{aI}$. $\mathscr{G}_{IJ}$ is called the Gauß constraint.

### 3.2.2 Constraint analysis

It is easy to convince oneself (and it is made explicit below in section 3.2.3) that the Gauß constraint (3.17) generates internal SO$(D+1)$ or SO$(1,D)$ transformations. The commutation relations of the corresponding Lie algebra

$$\{\frac{1}{2}\mathscr{G}_{IJ}[\lambda^{IJ}], \frac{1}{2}\mathscr{G}_{KL}[\omega^{KL}]\} = \frac{1}{2}\mathscr{G}_{MN}[[\lambda,\omega]^{MN}], \tag{3.25}$$

follow. Since $\mathscr{G}^{IJ}$ also strongly Poisson commutes with $\mathscr{H}$, $\mathscr{H}_a$, which transform as scalars under internal rotations, it is a first class constraint. Moreover, with the replacements (3.12, 3.18), it is easy to show that the ADM Poisson brackets (2.23) and therefore also the hypersurface deformation algebra (2.31) are reproduced weakly, i.e. up to terms $\propto \mathscr{G}_{IJ}$ (cf. e.g. [62, section 4.2.1] for calculational details), i.e. we again obtain a first class system.

Interestingly, while we will see shortly that the Gauß constraint arises in the first order formulation as a secondary constraint, here it is a primary constraint. Counting of degrees of freedom results in $(D+1)(D-2)$ coinciding with the ADM counting in the previous section.

| Variable | Dof | $1^{\text{st}}$ cl. constraints | Dof (count twice!) |
|---|---|---|---|
| ${e_a}^I$ | $D(D+1)$ | $\mathscr{H}$ | 1 |
| ${k^b}_J$ | $D(D+1)$ | $\mathscr{H}_a$ | $D$ |
| | | $\mathscr{G}^{IJ}$ | $\frac{D(D+1)}{2}$ |
| Sum: | $2D^2+2D$ | Sum: | $D^2+3D+2$ |

**Table 3.1:** eADM phase space: counting of degrees of freedom

### 3.2.3 From the co vielbein to the densitised vielbein

In later chapters, we will exclusively use the canonical variables $K_{aI}, E^{aI}$. These are related to the ones used here by $\{{e_a}^I, {k^b}_J\} \to$

$$\left\{K_{aI} := \frac{1}{\sqrt{q}}\left(e_{bI}e_{aJ} - \frac{1}{D-1}e_{aI}e_{bJ} - sq_{ab}n_I n_J\right)k^{bJ}, E^{aI} := \sqrt{q}q^{ab}{e_b}^I\right\}, \tag{3.26}$$

which can be easily shown to be a canonical transformation (The above matrix in the definition of $K_{aI}$ is the non-singular part of $-\frac{\delta e_{bJ}(x)}{\delta E^{aI}(y)}$, and the minus sign stems from the fact that we switched momenta and configuration variables). Solving for $k^{bJ} = \sqrt{q}(2e^{a[J}e^{bI]} - sq^{ab}n^I n^J)K_{aI}$, we can rewrite the set of constraints in terms of these variables,

$$\mathscr{G}^{IJ} = 2E^{a[I}K_a{}^{|J]}, \tag{3.27}$$

$$\mathscr{H}_a \approx -D_b\left(K_{aI}E^{bI} - \delta^b_a K_{cI}E^{cI}\right), \tag{3.28}$$

$$\mathscr{H} \approx -\frac{1}{2\sqrt{q}}\left(K_a{}^I K_b{}^J - K_a{}^J K_b{}^I\right)E^{aJ}E^{bI} - \frac{s}{2}\sqrt{q}{}^{(D)}R, \tag{3.29}$$

where in the last two expressions we dropped terms proportional to $2K^{[c}{}_I E^{d]I} = \frac{1}{q}E^{cI}E^{dJ}\mathscr{G}_{IJ}$ and, of course, $\sqrt{q}$ and ${}^{(D)}R$ are now understood as functions of $E^{aI}$. Since this formulation will be more relevant for the rest of this work than the one we had before the canonical transformation, we will at this point shortly discuss the action of these constraints on the phase space variables. The Gauß constraint of course generates internal SO$(D+1)$ or SO$(1,D)$ transformations,

$$\{K_{aI}(x), \frac{1}{2}\mathscr{G}^{KL}[\lambda_{KL}]\} = \lambda_I{}^J K_{aJ}, \tag{3.30}$$

$$\{E^{aI}(x), \frac{1}{2}\mathscr{G}^{KL}[\lambda_{KL}]\} = \lambda^I{}_J E^{aJ}. \tag{3.31}$$

$\mathscr{H}_a$ in this form not only generates spatial diffeomorphisms, but also internal rotations. We have

$$\begin{aligned}\mathscr{H}_a[N^a] &= \int_\sigma d^Dx\ N^a\Big[-\partial_b(E^{bI}K_{aI}) + \Gamma^c_{ba}E^{bI}K_{cI} + (D^{\mathrm{H}}{}_a E^{bI})K_{bI} \\ &\qquad\qquad + E^{bI}\left(\partial_a K_{bI} - \Gamma^c_{ab}K_{cI} + \Gamma^{\mathrm{H}}{}_{aIJ}K_b{}^J\right)\Big] \\ &= \int_\sigma d^Dx\ N^a\left[-\partial_b(E^{bI}K_{aI}) + E^{bI}\partial_a K_{bI} + \frac{1}{2}\Gamma^{\mathrm{H}}{}_{aIJ}\mathscr{G}^{IJ}\right] \\ &= \int_\sigma d^Dx\ \left[E^{bI}(\mathscr{L}_{\vec{N}}K)_{bI} + \frac{1}{2}N^a\Gamma^{\mathrm{H}}{}_{aIJ}\mathscr{G}^{IJ}\right],\end{aligned} \tag{3.32}$$

where in the first step, we only wrote out all terms appearing in $\mathscr{H}_a$, and in the second noted that the terms containing Christoffel symbols cancel due to their symmetry and the hybrid spin connection annihilates $E^{aI}$. In the last step, we dropped a surface term. Therefore, introducing $\tilde{\mathscr{H}_a} := \mathscr{H}_a - \frac{1}{2}\Gamma^{\mathrm{H}}_{aIJ}\mathscr{G}^{IJ}$, we find that this linear combination generates spatial diffeomorphisms

$$\{K_{aI}[f^{aI}], \tilde{\mathscr{H}}_c[N^c]\} = \left(\mathscr{L}_{\vec{N}} K\right)_{aI}[f^{aI}], \tag{3.33}$$

$$\{E^{aI}[F_{aI}], \tilde{\mathscr{H}}_c[N^c]\} = \left(\mathscr{L}_{\vec{N}} E\right)^{aI}[F_{aI}], \tag{3.34}$$

from which we can easily deduce the action of $\mathscr{H}_a$. This implies that, on rotationally invariant observables (i.e. observables w.r.t. $\mathscr{G}$), the action of $\mathscr{H}_a$ still is that of spatial diffeomorphisms. Of course, it can also be worked out that the action of $\mathscr{H}$ on rotationally invariant observables is the same as in the ADM case. The discussion is more intricate and we leave it to the interested reader to work out the details.

### 3.2.4 Time gauge

One often encounters a formulation in the literature similar to the one we considered in section 3.2.3, which, however, only has a SO($D$) gauge symmetry. In $D = 3$, when going from the ADM formulation to the Ashtekar Barbero variables, this formulation usually arises at an intermediate step, as we will see in section 6.3. It can be obtained from the SO($D+1$) or SO($1, D$) formulation we considered in the first remark by choosing time gauge (the time gauge is a canonical gauge, see, for instance, [161]), i.e. setting $n^I = \delta_0^I$ or equivalently $E^{a0} = 0$. Furthermore, this requirement is second class with the boost part of the Gauß constraint, which we therefore also have to solve if we do not want to retain second class constraints. After a straightforward symplectic reduction, we obtain a phase space coordinatised by the canonical pair $\{K_{bj}, E^{ai}\}$, $i, j, ... \in \{1, ..., D\}$, subject to the constraints

$$\mathscr{G}^{ij} = 2E^{a[i} K_a{}^{|j]}, \tag{3.35}$$

$$\mathscr{H}_a \approx -D_b \left(K_{ai} E^{bi} - \delta_a^b K_{ci} E^{ci}\right), \tag{3.36}$$

$$\mathscr{H} \approx -\frac{1}{2\sqrt{q}} \left(K_a{}^i K_b{}^j - K_a{}^j K_b{}^i\right) E^{aj} E^{bi} - \frac{s}{2}\sqrt{q}^{(D)}R. \tag{3.37}$$

## 3.3 eADM via extension of the ADM phase space

Of course, as the name already suggests, the same Hamiltonian formulation can also be obtained via an extension of the ADM phase space. We will be rather brief here and refer the interested reader to [62, section 4.2.1] for a detailed treatment.

We introduce a larger phase space coordinatised by a densitised (hybrid) vielbein $E^{aI} = \sqrt{q} e^{aI}$ and its conjugate $K_{aI}$, and postulate the non-vanishing Poisson brackets $\{K_{aI}(x), E^{bJ}(y)\} = \delta_a^b \delta_I^J \delta^{(D)}(x-y)$. Note that, since we are just adding internal degrees of freedom, we can choose to work with a $(D+1)$-dimensional internal space with signature $\zeta = \pm 1$ irrespective of the space time signature $s$. Moreover, we have another possibility, namely to work with a $D$-dimensional internal space, i.e. a genuine $D$-bein of which the spatial metric is constructed. To treat all three cases at the same time, it will be convenient to use $\zeta = 0$ in the third case and, only in this section, use indices $I, J, ...$ also for $D$-dimensional internal space. We define the map from the extended to the ADM phase space by:

$$qq^{ab}[E] := E^{aI} E^b{}_I, \quad K_{ab}[E,K] := -\frac{1}{\sqrt{q}} E^{cI} q_{c(a} K_{b)I}, \quad P^{ab}[E,K] = -\frac{1}{2}\sqrt{q} G^{ab\,cd} K_{cd}. \tag{3.38}$$

Now we impose constraints demanding that those parts of $K_{aI}$ which do not contribute to $K_{ab}$ vanish. If these constraints define a coisotropic constraint surface (i.e., first class constraints), we expect that modding out by the corresponding gauge orbits will account for the additionally introduced degrees of freedom in the vielbein. The parts which do not contribute are $\mathscr{G}_{ab} := E^{cI} q_{c[a} K_{b]I}$ and $\zeta n^J K_{bJ}$. The requirement that both of them vanish can be conveniently combined into the constraint $\mathscr{G}^{IJ} = 2E^{a[I} K_a{}^{|J]}$, that we already encountered in the last sections. It generates $\mathrm{SO}(D+1)$, $\mathrm{SO}(1,D)$ or $\mathrm{SO}(D)$ transformations and therefore, as we expected, removes the "rotational" degrees of freedom which the vielbein has in addition to the information on the spatial metric. Obviously, the ADM variables as given in (3.38) are Dirac observables with respect to this constraint. If the ADM Poisson brackets are reproduced on the extended phase space, we know that we have a first class algebra of constraints (using the map (3.38) to rewrite the ADM constraints in terms of the coordinates on the extended phase space) and that symplectic reduction with respect to $\mathscr{G}^{IJ}$ leads back to the ADM phase space. Therefore, what is left to check is

$$\{q_{ab}[E](x), P^{cd}[E,K](y)\} = \delta^c_{(a}\delta^d_{b)}\delta^{(D)}(x-y), \tag{3.39}$$

which follows in one line using (A.21) and

$$\delta P^{cd}[E,K] = \frac{q}{2} G^{cd\,ef} E_{eI} \delta K_f{}^I + \left[ \frac{1}{2} G^{cd\,e}{}_f \delta^I_J + \frac{1}{q} G^{-1\,e(d|}{}_{gf} E^{c)}{}_J E^{gI} - \frac{1}{q} G^{-1\,cd}{}_{hf} E^h{}_I E^{eJ} \right] K_e{}^J \delta E^f{}_I. \tag{3.40}$$

To derive (3.40), the formulas in appendix A might be helpful.

Similarly, we find

$$\begin{aligned}
\{P^{ab}[E,K](x), P^{cd}[E,K](y)\} &= \int_\sigma d^D z \, \left[ \frac{\delta P^{ab}[E,K](x)}{\delta K_f{}^I(z)} \frac{\delta P^{cd}[E,K](y)}{\delta E^f{}_I(z)} - ab \leftrightarrow cd \right] \\
&= \left[ \frac{q}{4} G^{ab\,xf} G^{cd\,e}{}_f E_{xI} K_e{}^I \delta^{(D)}(x-y) + \frac{1}{2} G^{ab\,gf} G^{-1\,e(d|}{}_{gf} E^{c)}{}_J K_e{}^J \delta^{(D)}(x-y) \right. \\
&\quad \left. - \frac{1}{2} G^{ab\,hf} G^{-1\,cd}{}_{hf} E^{eJ} K_e{}^J \delta^{(D)}(x-y) \right] - ab \leftrightarrow cd \\
&= \frac{1}{2} G^{ab\,xf} G^{cd\,e}{}_f \mathscr{G}_{xe} + \frac{1}{8} \left( q^{(b|d} \mathscr{G}^{c|a)} + q^{(b|c} \mathscr{G}^{d|a)} \right) \delta^{(D)}(x-y) \approx 0.
\end{aligned} \tag{3.41}$$

From the first to the second line, we only inserted (3.40) to obtain three terms. Due to the antisymmetry in $ab$, $cd$, the first term obviously is antisymmetric in $x, e$ and therefore proportional to $\mathscr{G}_{xe}$, which gives the first term in the fourth line. In the second summand, we only have to contract the matrices, which is trivial since they are inverse to each other, and then rearrange the terms exploiting again the antisymmetry in $ab$, $cd$ to obtain the remaining terms in the fourth line. The third summand of the second step obviously is symmetric in the exchange of $ab$, $cd$ and therefore drops out. As we have seen before, since $\mathscr{G}_{ab} = -\frac{q}{2} E_b{}^I E_c{}^J \mathscr{G}_{IJ}$ is proportional to (a part of) the Gauß constraint, it vanishes weakly, and therefore the ADM Poisson bracket is weakly fulfilled. Note that the remaining Poisson bracket between two metrics is trivially fulfilled. This ends the proof that the symplectic reduction of the eADM constrained Hamiltonian system with respect to the constraint $\mathscr{G}^{IJ}$ gives back the ADM phase space. The explicit form of $\mathscr{H}_a$, $\mathscr{H}$ when expressed in terms of $\{K_{aI}, E^{bJ}\}$ of course is the same as in (3.28, 3.29).

# 4

# Palatini action and corresponding $2^{\text{nd}}$ class constraint system

## 4.1 Palatini action

The action we want to study in this section is a first order action in $D+1$ dimensions, which we will call "Palatini action" as we commented on in the introduction to this chapter. It is given by

$$S_{\mathrm{P}}[e,A] = \frac{s}{2}\int_{\mathscr{M}} d^{D+1}X \; e e^{\mu I} e^{\nu J} F_{\mu\nu IJ}. \tag{4.1}$$

The notation is as before, $e^{\mu I}$ denotes the vielbein and $e$ the determinant of the co-vielbein, and $F_{\mu\nu IJ} := 2\partial_{[\mu}A_{\nu]IJ} + 2A_{[\mu|IK}A_{|\nu]}{}^{K}{}_{J}$ is the field strength of the $\mathrm{SO}(1,D)$ or $\mathrm{SO}(D+1)$ connection $A_{\mu IJ}$ for Lorentzian and Euclidean spacetimes respectively. Using $\delta F_{\mu\nu}{}^{IJ} = 2\nabla^{A}{}_{[\mu}\delta A_{\nu]}{}^{IJ} = 2\partial_{[\mu}\delta A_{\nu]}{}^{IJ} + 4A_{[\mu|}{}^{[I|K}\delta A_{|\nu]K}{}^{|J]}$ (cf. appendix A), we obtain, up to a boundary term, the field equations

$$\nabla^{A}_{\mu}(e e^{[\mu|I}e^{|\nu]J}) = 0, \tag{4.2}$$

$$e^{\nu J}F_{\mu\nu IJ} - \frac{1}{2}e^{\rho I}e^{\sigma J}F_{\rho\sigma IJ}e_{\mu I} = 0. \tag{4.3}$$

The first of these can be easily shown to be the torsion freeness condition $\nabla^{A}{}_{[\mu}e_{\nu]}{}^{I} = 0$, which is solved by $A_{\mu IJ} = \Gamma_{\mu IJ}$, and inserting this into the second field equation and using the relations is appendix C, Einstein's field equations are reproduced.

In the following, we will obtain two different Hamiltonian formulations from this action, one strictly following Dirac's procedure (here), and one where we introduce the so-called quadratic simplicity constraints (section 5.2).

## 4.2 Canonical analysis

### 4.2.1 $D+1$ split and Legendre transformation

The canonical analysis presented here was firstly given in [2, 164, 165]. In our presentation, we will follow [2] and several parts were taken from there with only minor modifications.

The $D+1$ split is performed analogous to section 3.2.1 using

$$e^{\mu I} e^{\nu J} F_{\mu\nu IJ} = {}^{\|}e^{\mu I \|} e^{\nu J} F_{\mu\nu IJ} + 2 s n^I n^{\mu \|} e^{\nu J} F_{\mu\nu IJ}, \tag{4.4}$$

and rewriting the terms according to

$$\begin{aligned} N^{(D)} e^{\|} e^{\mu I} n^J n^\nu F_{\mu\nu IJ} &= -\frac{1}{2} \pi'^{\mu IJ} (T^\nu - N^\nu) F_{\mu\nu IJ} \\ &= \frac{1}{2} \pi'^{\mu IJ} \mathscr{L}_T A_{\mu IJ} + \frac{1}{2} (T^\nu A_{\nu IJ}) \mathscr{G}'^{IJ} - N^\mu \mathscr{H}'_\mu, \end{aligned} \tag{4.5}$$

$$\frac{1}{2} N^{(D)} e^{\|} e^{\mu I \|} e^{\nu J} F_{\mu\nu IJ} = -s \underset{\sim}{N} \mathscr{H}', \tag{4.6}$$

where we introduced

$$\pi'^{\mu IJ} := 2 n^{[I} E^{\mu|J]} := 2^{(D)} e n^{[I \|} e^{\mu|J]}, \tag{4.7}$$

$$\underset{\sim}{N} := N / {}^{(D)}e, \tag{4.8}$$

$$\mathscr{G}'^{IJ} := D^A{}_\mu \pi'^{\mu IJ} := \partial_\mu \pi'^{\mu IJ} + [A_\mu, \pi'^\mu]^{IJ}, \tag{4.9}$$

$$\mathscr{H}' := -\frac{1}{2} \pi'^{\mu IK} \pi'^{\nu J}{}_K F_{\mu\nu IJ}, \tag{4.10}$$

$$\mathscr{H}'_\mu := \frac{1}{2} \pi'^{\nu IJ} F_{\mu\nu IJ}. \tag{4.11}$$

The split is completed by pulling back all spatially projected quantities to the spatial manifold $\sigma$. Note that we changed the lapse function according to (4.8), but we will still refer to $\underset{\sim}{N}$ simply as lapse function (of density weight -1). Introducing the notation $\lambda_{IJ} = -(T \cdot A)_{IJ}$ and adding and subtracting all momenta multiplied with the corresponding velocities, the action reads

$$\begin{aligned} S = & \int dt \int_\sigma d^D x \left( \frac{1}{2} \pi'^{aIJ} \dot{A}_{aIJ} - \underset{\sim}{N} \mathscr{H}' - N^a \mathscr{H}'_a - \frac{1}{2} \lambda_{IJ} \mathscr{G}'^{IJ} \right) \\ = & \int dt \int_\sigma d^D x \left( \frac{1}{2} \pi^{aIJ} \dot{A}_{aIJ} + P^{(E)I}_a \dot{E}^a{}_I + P^{(\underset{\sim}{N})} \dot{\underset{\sim}{N}} + P^{(\vec{N})}_a \dot{N}^a + \frac{1}{2} P^{(\lambda)}_{IJ} \dot{\lambda}^{IJ} \right. \\ & \qquad - \left[ \frac{1}{2} c_{aIJ} \mathscr{S}^{aIJ} + \gamma^a_I P^{(E)I}_a + \alpha P^{(\underset{\sim}{N})} + \alpha^a P^{(\vec{N})}_a + \frac{1}{2} \alpha^{IJ} P^{(\lambda)}_{IJ} \right. \\ & \qquad \left. \left. + \underset{\sim}{N} \mathscr{H}' + N^a \mathscr{H}'_a + \frac{1}{2} \lambda_{IJ} \mathscr{G}'^{IJ} \right] \right) , \end{aligned} \tag{4.12}$$

from which we read off the total Hamiltonian

$$\begin{aligned} H_{\mathrm{T}} = \int_\sigma d^D x \Big( & \underset{\sim}{N} \mathscr{H}' + N^a \mathscr{H}'_a + \frac{1}{2} \lambda_{IJ} \mathscr{G}'^{IJ} + \alpha P^{(\underset{\sim}{N})} + \alpha^a P^{(\vec{N})}_a + \frac{1}{2} \alpha^{IJ} P^{(\lambda)}_{IJ} \\ & + \frac{1}{2} c_{aIJ} \mathscr{S}^{aIJ} + \gamma^a_I P^{(E)I}_a \Big) . \end{aligned} \tag{4.13}$$

Here, the $P$s denote the canonically conjugate momenta to the variables indicated in brackets and $\pi^{aIJ}$ is the momentum to $A_{aIJ}$[1]. We furthermore replaced all the velocities which could not be solved for in the Hamiltonian by Lagrange multipliers, $\alpha = \dot{\underset{\sim}{N}}$, $\alpha^a = \dot{N}^a$, $\alpha_{IJ} = \dot{\lambda}_{IJ}$, $c_{aIJ} = \dot{A}_{aIJ}$ and $\gamma^a_I = \dot{E}^a{}_I$. Furthermore, we introduced a factor of 1/2 in front of term with a trace over the Lie algebra indices. Note that, of course, $P^{(E)I}_a$, $P^{(\underset{\sim}{N})}$, $P^{(\vec{N})}_a$ and $P^{(\lambda)}_{IJ}$ constitute primary constraints, but unlike the case of other constraints, we refrain from introducing calligraphic letters $\mathscr{C} = P \approx 0$ for them, in order not to keep notation simpler. There is one more primary constraint,

$$\mathscr{S}^{aIJ} = \pi^{aIJ} - \pi'^{aIJ} \tag{4.14}$$

called the simplicity constraint. Using it, we can replace all $\pi'^{aIJ}$s by $\pi^{aIJ}$s in the primed constraint $\mathscr{H}'$, $\mathscr{H}'_a$ and $\mathscr{G}'^{IJ}$ (this being equivalent to a redefinition of the Lagrange multiplier of the simplicity constraint) and will call the resulting expressions $\mathscr{H}$, $\mathscr{H}_a$ and $\mathscr{G}^{IJ}$ in the following. The requirement of conservation under the Hamiltonian time evolution of the constraints $P^{(\underset{\sim}{N})}$, $P^{(\vec{N})}_a$ and $P^{(\lambda)}_{IJ}$ shows that $\mathscr{H}$, $\mathscr{H}_a$ and $\mathscr{G}^{IJ}$ are constraints and that the total Hamiltonian is a linear combination of constraints

[1]To avoid confusion, we want to remark that we here break with our previous convention, calling the conjugate variable to $E^{aI}$ $P^{(E)I}_a$ and not $K_{aI}$ like before. This is because later, after solving second class constraints, we will find that we are lead back to the eADM phase space, but $P^{(E)I}_a$ will not exactly coincide with the eADM conjugate momentum $K_{aI}$.

as expected. We take a shortcut at this point and solve $P^{(\underset{\sim}{N})}$, $P_a^{(\vec{N})}$ and $P_{IJ}^{(\lambda)}$ strongly at this point, and will treat lapse, shift and $\lambda_{IJ}$ in the following as Lagrange multipliers.

Note that the (timelike in the Lorentzian case) unit normal $n^I$ appearing in (4.7) is determined (up to sign) by the requirements $E^{aI}n_I = 0$ and $n_I n^I = s$. However, we will see in the following that the constraint analysis is simplified if we introduce $n^I$ together with its conjugate momenta $P_I^{(n)}$ as additional phase space degree of freedom, and add the constraints $E^{aI}n_I \approx 0$ and $n^I n_I - s \approx 0$ as well as the requirement $P_I^{(n)} = 0$ to the total Hamiltonian with Lagrange multipliers $\rho$ and $\rho_a$ and $\gamma^I$, respectively.

The final ingredient we want to introduce before starting with the constraint analysis of this theory are certain projectors. We define the projection transversal to $n^I$ by $\bar{\eta}^I{}_J := \eta^I{}_J - sn^I n_J$ (in the Lorentzian theory, one could speak of rotational components as opposed to boost components along $n^I$, which we, in slight abuse of terminology, will also use in the Euclidean case). Using it, we can decompose Lie algebra valued tensors $X_{IJ}$ according to

$$X_{IJ} = 2n_{[I}\bar{X}_{|J]} + \bar{X}_{IJ}, \tag{4.15}$$

where we defined $\bar{X}_{IJ} := \bar{\eta}^K{}_I \bar{\eta}^L{}_J X_{KL}$ and $\bar{X}_I := -sX_{IJ}n^J$. Applying this decomposition to the Lagrange multiplier of the simplicity constraint, we can split it into a boost and a non-boost part,

$$\frac{1}{2}c_{aIJ}\mathscr{S}^{aIJ} = 0 \quad \Leftrightarrow \quad \begin{aligned} &\frac{1}{2}\bar{c}_{aIJ}\bar{\mathscr{S}}^{aIJ} := \frac{1}{2}\bar{c}_{aIJ}\bar{\pi}^{aIJ} = 0, \\ &s\bar{c}_{aI}\bar{\mathscr{S}}^{aI} := -\bar{c}_{aI}\left(\pi^{aIJ}n_J + s\, E^{aI}\right) = 0. \end{aligned} \tag{4.16}$$

This set of constraints is of course equivalent to the full simplicity constraint: solving $\bar{\mathscr{S}}^{aIJ} = 0$, we have $\bar{\pi}^{aIJ} = 0$ and hence $\pi^{aIJ} = 2n^{[I}B^{a|J]}$ for some $B^{aJ}$. $\bar{\mathscr{S}}^{aI}$ then demands that $B^{aJ} = E^{aJ}$.

We will decompose the rotational components of the simplicity constraint even further into trace and trace free parts with respect to $E^{aI}$. To this end, we define the inverse of $E^{aI}$, $E_{aI} := \frac{1}{q}q_{ab}E^b{}_I$, satisfying $E_{aI}E^{bI} = \delta_a^b$ and $E_{aI}E^a{}_J = \bar{\eta}_{IJ}$ and introduce the decomposition of tensors of the index structure $\bar{X}_{aIJ}$ as

$$\bar{X}_{aIJ} = \bar{X}^{\text{tf}}_{aIJ} + \frac{2}{D-1}E_{a[I}\bar{X}^{\text{tr}}_{J]}, \tag{4.17}$$

where $\bar{X}^{\text{tr}}_J := E^{aI}\bar{X}_{aIJ}$. The superscripts "tr" and "tf" here of course stand for "trace" and "trace free", and indeed one easily verifies that

$$\bar{X}^{\text{tf}}_{aIJ} = \mathbb{P}^{\text{tf}}{}_{aIJ}{}^{bKL}\bar{X}_{bKL} := (\delta^b_a \bar{\eta}^K_{[I}\bar{\eta}^L_{J]} - \frac{2}{D-1}E_{a[I}\bar{\eta}^L_{J]}E^{bK})\bar{X}_{bKL} \tag{4.18}$$

is trace free with respect to $E^{aI}$. $\mathbb{P}^{\text{tf}}$ here denotes the projector on the tracefree part. Applying this decomposition to $\bar{\mathscr{S}}^{aIJ}$, we can as well project the corresponding Lagrange multipliers accordingly.

After these considerations, the total Hamiltonian reads

$$\begin{aligned} H_{\text{T}} = \int_\sigma d^Dx \Big( &\underset{\sim}{N}\mathscr{H} + N^a\mathscr{H}_a + \frac{1}{2}\lambda_{IJ}\mathscr{G}^{IJ} + \frac{1}{2}\bar{c}^{\text{tf}}_{aIJ}\bar{\mathscr{S}}^{aIJ}_{\text{tf}} + \frac{1}{D-1}\bar{c}^{\text{tr}}_J\bar{\mathscr{S}}^J_{\text{tr}} \\ &+ s\bar{c}_{aI}\mathscr{S}^{aI} + \gamma^a_I P^{(E)I}_a + \gamma^I P^{(n)}_I + \rho(n^I n_I - s) + \rho_a(E^{aI}n_I)\Big). \end{aligned} \tag{4.19}$$

The non-vanishing Poisson-brackets can be read off from (4.12),

$$\begin{aligned} \left\{A_{aIJ}(x), \pi^{bKL}(y)\right\} &= 2\delta^b_a \eta^{[K}_I \eta^{L]}_J \delta^{(D)}(x-y), \\ \left\{E^{aI}(x), P^{(E)}_{bJ}(y)\right\} &= \delta^a_b \eta^I_J \delta^{(D)}(x-y), \\ \left\{n^I(x), P^{(n)}_J(y)\right\} &= \eta^I_J \delta^{(D)}(x-y). \end{aligned} \tag{4.20}$$

### 4.2.2 Constraint analysis

In order to perform the constraint analysis of the Hamiltonian $H_{\text{T}}$ given in (4.19), we introduce smeared constraints $\mathscr{C}[f] := \int_\sigma d^Dx\ f\cdot\mathscr{C}$, where the smearing function mirrors the index structure of the constraint $\mathscr{C}$ under consideration, and, following Dirac, check if they are either (at least weakly) preserved by the time evolution generated by $H_{\text{T}}$, or fix some Lagrange multipliers of the set $\{\underset{\sim}{N}, N^a, \lambda_{IJ}, \bar{c}^{\text{tf}}_{aIJ}, \bar{c}_I, \bar{c}_{aI}, \gamma^a_I, \gamma_I, \rho, \rho_a\}$, or lead to new, secondary constraints. We already obtained the secondary constraints $\mathscr{H}$, $\mathscr{H}_a$ and $\mathscr{G}_{IJ}$ in the last section from evolving $P^{(\underset{\sim}{N})}$, $P^{(\vec{N})}_a$ and $P^{(\lambda)}_{IJ}$. In the following, we will investigate the evolution of the remaining primary constraints.

For the constraint demanding that $n^I$ be a (timelike) unit vector, we find

$$0 \overset{!}{\approx} \int_\sigma d^Dx\, f(x)\left\{n^I(x)n_I(x) - s, H_T\right\} = 2\int_\sigma d^Dx\, f\, n^I\gamma_I, \tag{4.21}$$

and therefore, $\gamma^I = \bar{\gamma}^I$. Since $\gamma^I$ multiplies the momenta conjugate to $n^I$, the Hamiltonian flow is such that it does not change the length of $n^I$, exactly as expected. Stability of $\bar{\mathscr{S}}^{aI}$ requires

$$\begin{aligned} 0 \stackrel{!}{\approx} \{\bar{\mathscr{S}}^{aI}[\bar{f}_{aI}], H_{\mathrm{T}}\} &= \int_\sigma d^Dx\, \bar{f}_{aI}(x) \left\{-s\pi^{aIJ}(x)n_J(x) - E^{aI}(x), H_T\right\} \\ &= \int_\sigma d^Dx\, \bar{f}_{aI} \left(-s\{\pi^{aIJ}, H_{\mathrm{T}}\}n_J - s\pi^{aIJ}\gamma_J - \bar{\gamma}^{aI}\right). \end{aligned} \tag{4.22}$$

Notice that in the last term, only the rotational parts of $\gamma^{aI}$ survive since $\bar{f}_{aI}$ is projected accordingly. We will not detail the computation of the left-over Poisson bracket between $\pi^{aIJ}$ and the total Hamiltonian, which is however straightforward. One finds that $\bar{\mathscr{S}}^{aI}$ is stable under the time evolution if we choose

$$\bar{\gamma}^{aI} = \bar{\gamma}_0^{aI} := \bar{\lambda}^{IJ}E^a{}_J - 2(D_bN^{[a})E^{b]I} + N^b\bar{\eta}^I_J D^A{}_bE^{aJ} - \underset{\sim}{N}\left(qq^{ab}\bar{\eta}^I_J - E^{aI}E^b{}_J\right)D^A{}_bn^J. \tag{4.23}$$

Similarly,

$$\begin{aligned} 0 &\stackrel{!}{\approx} \int_\sigma d^Dx\, f_a(x) \left\{E^a{}_I(x)n^I(x), H_T\right\} \\ &= \int_\sigma d^Dx\, f_a \left(\gamma^a{}_In^I + \gamma^IE^a{}_I\right) \end{aligned} \tag{4.24}$$

can be solved by choosing

$$\gamma^a{}_In^I = \bar{\gamma}_0^a := -\gamma^IE^a{}_I. \tag{4.25}$$

For the constraint demanding the vanishing of the momenta conjugate to $E^a{}_I$, we have

$$0 \stackrel{!}{\approx} \int_\sigma d^Dxf^{aI}(x)\left\{P^{(E)}_{aI}(x), H_T\right\} = \int_\sigma d^Dxf^{aI}(s\bar{c}_{aI} - \rho_an_I). \tag{4.26}$$

Decomposing $f^{aI}$ into boost and rotational components, we find that both, $\bar{c}_{aI} = 0 = \rho_a$. With the same method, we find $\rho = 0 = \bar{c}^{\mathrm{tr}}_I$, since

$$0 \stackrel{!}{\approx} \int_\sigma d^Dx\, f^I(x)\left\{P^{(n)}_I(x), H_T\right\} \approx \int_\sigma d^Dx\, f^I\left(-2n_I\rho - \bar{c}^{\mathrm{tr}}_I\right). \tag{4.27}$$

Finally, we come to the trace and trace free parts of the simplicity constraint. It is helpful to notice that we actually never have to calculate the derivatives of the projections we introduced when calculating Poisson brackets, since e.g.

$$\{\bar{\mathscr{S}}^{aIJ}_{\mathrm{tf}}, \,.\,\} = \mathbb{P}_{\mathrm{tf}}^{aIJ}{}_{bKL}\{\mathscr{S}^{bKL}, \,.\,\} + \mathscr{S}^{bKL}\{\mathbb{P}_{\mathrm{tf}}^{aIJ}{}_{bKL}, \,.\,\} \approx \mathbb{P}_{\mathrm{tf}}^{aIJ}{}_{bKL}\{\mathscr{S}^{bKL}, \,.\,\}. \tag{4.28}$$

Therefore, it is sufficient to calculate the time evolution of $\bar{\mathscr{S}}^{aIJ}$ and decompose its multiplier after calculating the Poisson bracket,

$$
\begin{aligned}
0 &\approx \{\frac{1}{2}\bar{\mathscr{S}}^{aIJ}[\bar{f}_{aIJ}], H_{\mathrm{T}}\} = \int_\sigma d^D x\, \frac{1}{2}\bar{f}_{aIJ}(x)\, \{\bar{\pi}^{aIJ}(x), H_T\} \\
&\approx \int_\sigma d^D x\, \bar{f}_{aIJ}\Big(-\underset{\sim}{N} D^A{}_b \left(\pi^{b[I|K}\pi^{a|J]K}\right) \\
&\qquad\qquad + E^{a[J}\left(-\bar{\gamma}^{I]} - n_K\lambda^{K|I]} - s(D_b\underset{\sim}{N})E^{b|I]} + N^b\, D^A{}_b n^{I]}\right)\Big) \\
&\approx \int_\sigma d^D x\, \Big(-s\underset{\sim}{N}\bar{f}_{aIJ}E^{b[I}D^A{}_b E^{a|J]} \\
&\qquad\qquad + \bar{f}_{aIJ}E^{a[J}\left(-\bar{\gamma}^{I]} - n_K\lambda^{K|I]} - s(D_b\underset{\sim}{N})E^{b|I]} + N^b\, D^A{}_b n^{I]}\right)\Big) \\
&\approx \int_\sigma d^D x\, \Big(-s\underset{\sim}{N}\bar{f}^{\mathrm{tf}}_{aIJ}E^{b[I}D^A{}_b E^{a|J]} - \bar{f}^{\mathrm{tr}}_I\Big[-\bar{\gamma}^I - n_K\lambda^{KI} - s(D_b\underset{\sim}{N})E^{bI} \\
&\qquad\qquad + N^b\, D^A{}_b n^I + \frac{s\underset{\sim}{N}}{D-1}(\delta^b_c\eta^I_J - E^{bI}E_{cJ})D^A{}_b E^{cJ}\Big]\Big),
\end{aligned}
\tag{4.29}
$$

where we have used the simplicity constraint several times and in the last step separated trace and trace free components of the multiplier $\bar{f}_{aIJ}$. The corresponding terms have to vanish separately. For the trace components, we simply fix

$$
\bar{\gamma}^I = \bar{\gamma}^I_0 := -n_K\lambda^{KI} - s(D_b\underset{\sim}{N})E^{bI} + N^b\, D^A{}_b n^I + \frac{s\underset{\sim}{N}}{D-1}(\delta^b_c\eta^I_J - E^{bI}E_{cJ})D^A{}_b E^{cJ}.
\tag{4.30}
$$

The trace free part cannot be dealt with by fixing multipliers, since we know that the only possible choice, $\underset{\sim}{N} = 0$, is physically not viable, corresponding to a degenerate spacetime metric. Therefore, we have to introduce an additional constraint,

$$
\bar{\mathscr{D}}^{aIJ}_{\mathrm{tf}} := -2s\mathbb{P}^{aIJ}_{\mathrm{tf}}{}_{bKL}E^{c[K}\, D^A{}_c E^{bL]}.
\tag{4.31}
$$

Note that this transversal and trace free projection of the term in (4.29) is sufficient, since the smearing field is projected accordingly.

This ends the stability analysis of the primary constraints. The total Hamiltonian is reduced to

$$
H_{\mathrm{T}} = \int_\sigma d^D x\Big(\underset{\sim}{N}\mathscr{H} + N^a\mathscr{H}_a + \frac{1}{2}\lambda_{IJ}\mathscr{G}^{IJ} + \frac{1}{2}\bar{c}^{\mathrm{tf}}_{aIJ}\bar{\mathscr{S}}^{aIJ}_{\mathrm{tf}} + \gamma^{aI}_0 P^{(E)}_{aI} + \bar{\gamma}^I_0 P^{(n)}_I\Big),
\tag{4.32}
$$

where $\gamma_0^{aI}$ and $\bar{\gamma}_0^I$ are fixed functions of phase space variables and the remaining free Lagrange multipliers as given in (4.23, 4.25, 4.30). The secondary constraints are $\mathscr{H}$, $\mathscr{H}_a$, $\mathscr{G}^{IJ}$ and $\bar{\mathscr{D}}_{\text{tf}}^{aIJ}$, and the next step consists of analyzing their stability under the Hamiltonian time evolution. As we will see, $\mathscr{S}_{\text{tf}}^{aIJ}$ and $\bar{\mathscr{D}}_{\text{tf}}^{aIJ}$ form a second class pair and the remaining secondary constraints are (or better, can be made) first class and correspond to the first class constraints we already encountered in section 3.2, the Hamiltonian, spatial diffeomorphism and Gauß constraint.

Starting with $\mathscr{G}^{IJ}$, it is easy to verify that its action on the phase space variables $A_{aIJ}, \pi^{bKL}$ is given by so$(1,D)$ or so$(D+1)$ transformations,

$$\left\{A_{aIJ}, \frac{1}{2}\mathscr{G}^{KL}[f_{KL}]\right\} = -D^A{}_a f_{IJ}, \tag{4.33}$$

$$\left\{\pi^{aIJ}, \frac{1}{2}\mathscr{G}^{KL}[f_{KL}]\right\} = [f, \pi^a]^{IJ}. \tag{4.34}$$

It satisfies the commutation relations of the so$(1,D)$ or so$(D+1)$ Lie algebra and Poisson commutes with $\mathscr{H}$ and $\mathscr{H}_a$ since they have no free internal indices,

$$\left\{\frac{1}{2}\mathscr{G}^{IJ}[f_{IJ}], \frac{1}{2}\mathscr{G}^{KL}[\gamma_{KL}]\right\} = \frac{1}{2}\mathscr{G}^{IJ}\left[[f,\gamma]_{IJ}\right], \tag{4.35}$$

$$\left\{\frac{1}{2}\mathscr{G}^{IJ}[f_{IJ}], \mathscr{H}_a[N^a]\right\} = 0, \tag{4.36}$$

$$\left\{\frac{1}{2}\mathscr{G}^{IJ}[f_{IJ}], \mathscr{H}[\underset{\sim}{N}]\right\} = 0. \tag{4.37}$$

It trivially Poisson commutes with all constraints which do neither depend on $A_{aIJ}$ nor $\pi^{bKL}$, but not with the simplicity constraints. However, since both, $P_{aI}^{(E)}$ and $P_I^{(n)}$ are constraints, we can introduce a linear combination of constraints which we will call the "improved" Gauß constraint

$$\hat{\mathscr{G}}_{IJ} := D^A{}_a\pi^a{}_{IJ} + 2P^{(E)}_{a[I}E^a{}_{J]} + 2P^{(n)}{}_{[I}n_{J]}, \tag{4.38}$$

which now generates SO$(1,D)$ or SO$(D+1)$ transformations on all phase space variables, and therefore weakly Poisson commutes with all constraints, in particular is stable under the Hamiltonian time evolution. Since the constraints we added were already stable, we know that also the original constraint $\mathscr{G}^{IJ}$ is, which can also be verified by direct calculation. Since the diffeomorphism constraint $\mathscr{H}_a$ as we defined it here generates spatial diffeomorphisms mixed with internal $\mathrm{SO}(1,D)$ or $\mathrm{SO}(D+1)$ transformations on $A_{aIJ}$ and $\pi^{bKL}$, it is convenient to introduce

$$\tilde{\mathscr{H}}_a := \mathscr{H}_a - \frac{1}{2}A_{aIJ}\mathscr{G}^{IJ} = \frac{1}{2}\pi^{bIJ}\partial_a A_{bIJ} - \frac{1}{2}\partial_b\left(\pi^{bIJ}A_{aIJ}\right). \tag{4.39}$$

$\tilde{\mathscr{H}}_a$ now acts on $A_{aIJ}$ and $\pi^{bKL}$ by generating spatial diffeomorphisms solely,

$$\left\{A_{aIJ}, \tilde{\mathscr{H}}_b[f^b]\right\} = f^b\partial_b A_{aIJ} + (\partial_a f^b)A_{bIJ} = \mathscr{L}_f A_{aIJ}, \tag{4.40}$$

$$\left\{\pi^{aIJ}, \tilde{\mathscr{H}}_b[f^b]\right\} = f^b\partial_b\pi^{aIJ} - (\partial_b f^a)\pi^{bIJ} + (\partial_b f^b)\pi^{aIJ} = \mathscr{L}_f\pi^{aIJ}. \tag{4.41}$$

From this and (4.39), we can deduce

$$\left\{\mathscr{H}_a[f^a], \mathscr{H}_b[N^b]\right\} = \mathscr{H}_a[(\mathscr{L}_f N)^a] - \frac{1}{2}\mathscr{G}^{IJ}[f^a N^b F_{abIJ}], \tag{4.42}$$

$$\left\{\mathscr{H}_a[f^a], \mathscr{H}[\underset{\sim}{N}]\right\} = \mathscr{H}[\mathscr{L}_f\underset{\sim}{N}] + \mathscr{G}^{IJ}[\underset{\sim}{N} f^a\pi^b{}_I{}^K F_{abJK}], \tag{4.43}$$

where $(\mathscr{L}_f N)^a = f^b\partial_b N^a - N^b\partial_b f^a$ and $(\mathscr{L}_f\underset{\sim}{N}) = f^b\partial_b\underset{\sim}{N} - \underset{\sim}{N}\partial_b f^b$ (note that $\underset{\sim}{N}$ is a scalar density of weight $-1$). Like in the case of $\mathscr{G}^{IJ}$, $\tilde{\mathscr{H}}_a$ trivially Poisson commutes with all other constraints except the simplicity constraints, which we can rectify by introducing the "improved" spatial diffeomorphism generator

$$\hat{\mathscr{H}}_a := \frac{1}{2}\pi^{bIJ}\partial_a A_{bIJ} - \frac{1}{2}\partial_b\left(\pi^{bIJ}A_{aIJ}\right) - E^{bI}\partial_a P^{(E)}_{bI} + \partial_b\left(P^{(E)}_{aI}E^{bI}\right) + P^{(n)}_I\partial_a n^I, \tag{4.44}$$

[1]which is, upon smearing the constraint and partially integrating, again a linear combination of constraints, and now generates spatial diffeomorphisms on all constraints. Like $\hat{\mathscr{G}}^{IJ}$, it is therefore first class and in particular stable, and implies stability of $\mathscr{H}_a$. The Hamiltonian constraint at this point already commutes with most of the constraints present in $H_{\mathrm{T}}$, and what is left to study is its Poisson bracket with itself and the transversal trace part of the simplicity constraints. For the former, we have

$$\begin{aligned}\left\{\mathscr{H}[\underset{\sim}{M}], \mathscr{H}[\underset{\sim}{N}]\right\} &= -\frac{1}{2}\mathscr{H}_a\left[(\underset{\sim}{M}\partial_b\underset{\sim}{N} - \underset{\sim}{N}\partial_b\underset{\sim}{M})\pi^{aIJ}\pi^b_{IJ}\right] \\ &\quad + \int_\sigma d^D x\,\frac{3}{2}(\underset{\sim}{M}\partial_a\underset{\sim}{N} - \underset{\sim}{N}\partial_a\underset{\sim}{M})\pi^a_{[IJ}\pi^b_{KL]}\pi^{cIJ}F_{cb}{}^{KL} \\ &\approx -s\mathscr{H}_a\left[(\underset{\sim}{M}\partial_b\underset{\sim}{N} - \underset{\sim}{N}\partial_b\underset{\sim}{M})qq^{ab}\right],\end{aligned} \tag{4.45}$$

[1]The sign difference in the terms containing the hybrid vielbein when compared to (3.32) of course are due to the fact that here, although $E^{aI}$ is a density of weight one, we still treat $P^{(E)}_{aI}$ as momenta, whereas $K_{aI}$ in (3.32) constituted the configuration variable. The same difference in sign of course appeared already in $\hat{\mathscr{G}}^{IJ}$ when compared to (3.27).

which is reminiscent of (2.31), except that $\underset{\sim}{M}$, $\underset{\sim}{N}$ here are densities of weight -1. Again, it is helpful for this calculation that, due to the antisymmetry in $\underset{\sim}{M}$, $\underset{\sim}{N}$, only terms $\propto \underset{\sim}{M}\partial_b\underset{\sim}{N} - \underset{\sim}{N}\partial_b\underset{\sim}{M}$ are non-vanishing, and furthermore, a relation satisfied by the contraction of so$(1,D)$ or so$(D+1)$ structure constants was used, cf. appendix D. The term in the second line vanishes when using the simplicity constraints.

For the Poisson bracket with the total Hamiltonian, we now find in analogy to (4.29)

$$\{\mathscr{H}[\underset{\sim}{f}], H_{\mathrm{T}}\} \approx \{\mathscr{H}[\underset{\sim}{f}], \frac{1}{2}\bar{\mathscr{S}}^{aIJ}_{\mathrm{tf}}[\bar{c}^{\mathrm{ft}}_{aIJ}]\} \approx -\frac{1}{2}\bar{\mathscr{D}}^{aIJ}_{\mathrm{tf}}[\underset{\sim}{f}\bar{c}^{\mathrm{tf}}_{aIJ}] \approx 0. \tag{4.46}$$

The Hamiltonian constraint therefore is also preserved.

Finally, the last secondary constraint to investigate is $\bar{\mathscr{D}}^{aIJ}_{\mathrm{tf}}$. We find

$$\left\{\frac{1}{2}\bar{\mathscr{S}}^{aIJ}_{\mathrm{tf}}[\bar{f}^{\mathrm{tf}}_{aIJ}], \frac{1}{2}\bar{\mathscr{D}}^{bKL}_{\mathrm{tf}}[\bar{f}'^{\mathrm{tf}}_{bKL}]\right\} = \int d^Dx\, \frac{1}{2}\bar{f}^{\mathrm{tf}}_{aIJ}F^{aIJ,bKL}\frac{1}{2}\bar{f}'^{\mathrm{tf}}_{bKL} \tag{4.47}$$

with

$$F^{aIJ,bKL} = 4sE^{a[K}\bar{\eta}^{L][I}E^{b|J]}. \tag{4.48}$$

Any contraction of this matrix with $n^I$ vanishes, it is symmetric in the exchange of the first set of indices with the second set, and although it is not trace free, it preserves the property of trace freeness, in the sense that $\mathbb{P}^{aIJ}_{\mathrm{tf}}{}_{bKL}F^{bKL,cMN}E_{cM} = 0$. Most importantly, it is invertible, and its inverse is given by

$$\left(F^{-1}\right)_{aIJ,bKL} = -sE_{aA}E_{bB}\left(\bar{\eta}^{AB}\bar{\eta}_{K[I}\bar{\eta}_{J]L} - 2\bar{\eta}^{B}_{[I}\bar{\eta}_{J][K}\bar{\eta}^{A}_{L]}\right), \tag{4.49}$$

$$\frac{1}{2}F^{aIJ,bKL}\left(F^{-1}\right)_{bKL,cMN} = 2\delta^a_b\bar{\eta}^{[I}_{M}\bar{\eta}^{J]}_{N}, \tag{4.50}$$

which shows that $\bar{\mathscr{S}}^{aIJ}_{\mathrm{tf}}$ and $\bar{\mathscr{D}}^{bKL}_{\mathrm{tf}}$ are a second class pair. Therefore, to stabilize $\bar{\mathscr{D}}^{bKL}_{\mathrm{tf}}$ we can fix the Lagrange multiplier $\bar{c}^{\mathrm{tf}}_{aIJ}$, because

$$\frac{1}{2}\{\bar{\mathscr{D}}^{bKL}_{\mathrm{tf}}[\bar{f}^{\mathrm{tf}}_{bKL}], H_{\mathrm{T}}\} = \int_\sigma d^Dx\, \frac{1}{2}\bar{f}^{\mathrm{tf}}_{aIJ}\left(-\frac{1}{2}F^{aIJ,bKL}\bar{c}^{\mathrm{tf}}_{bKL} + \bar{\Sigma}^{aIJ}_{\mathrm{tf}}\right) \approx 0, \tag{4.51}$$

where we denoted all contributions which do not stem from the $\bar{\mathscr{S}}^{aIJ}_{\mathrm{tf}}[\bar{c}^{\mathrm{tf}}_{aIJ}]$ term in $H_{\mathrm{T}}$ with $\bar{\Sigma}^{aIJ}_{\mathrm{tf}}$ after partially integrating terms where derivatives acted on $\bar{f}^{\mathrm{tf}}_{aIJ}$. Of course, trace or boost components of the resulting terms contributing to $\bar{\Sigma}^{aIJ}_{\text{tf}}$ vanish because of the projection of the multiplier. Choosing

$$\bar{c}^{\text{tf}}_{aIJ} = \bar{c}^{\text{tf}\,0}_{aIJ} := \left(F^{-1}\right)_{aIJ,bKL} \bar{\Sigma}^{bKL}_{\text{tf}}, \tag{4.52}$$

the constraint $\bar{\mathscr{D}}^{bKL}_{\text{tf}}$ is stabilized. Finally, the total Hamiltonian reads

$$\begin{aligned} H_{\text{T}} &= \int_\sigma d^D x \left( \underset{\sim}{N} \mathscr{H} + N^a \mathscr{H}_a + \frac{1}{2}\lambda_{IJ}\mathscr{G}^{IJ} + \frac{1}{2}\bar{c}^{\text{tf}\,0}_{aIJ}\bar{\mathscr{S}}^{aIJ}_{\text{tf}} + \gamma^{aI}_0 P^{(E)}_{aI} + \bar{\gamma}^I_0 P^{(n)}_I \right) \\ &= \int_\sigma d^D x \left( \underset{\sim}{N} \hat{\mathscr{H}} + N^a \hat{\mathscr{H}}_a + \frac{1}{2}(\lambda_{IJ} + N^a A_{aIJ})\hat{\mathscr{G}}^{IJ} \right). \end{aligned} \tag{4.53}$$

In the last step, we noted that the only free Lagrange multipliers left are $\underset{\sim}{N}$, $N^a$ and $\lambda^{IJ}$, i.e. upon inserting (4.23, 4.25, 4.30, 4.52), $H_{\text{T}}$ must be of the displayed form. Moreover, by general arguments, at the end of the stability analysis the total Hamiltonian is a linear combination of first class constraints. Explicit calculation shows that, indeed, the "improved" generators of spatial diffeomorphisms $\hat{\mathscr{H}}_a$ and $\text{SO}(1, D)$ or $\text{SO}(D+1)$ transformations $\hat{\mathscr{G}}^{IJ}$ given in (4.44, 4.38) appear, as well as a first class Hamiltonian constraint $\hat{\mathscr{H}}$, whose form is rather complicated and we refrain from displaying it explicitly. At this point, we could already by counting degrees of freedom deduce that all other constraints have to be second class. We will work out the second class pairs in the following explicitly and postpone counting of degrees of freedom to the end of the next section.

### 4.2.3 Second class pairs and degrees of freedom

We already found the first class constraints $\hat{\mathscr{H}}$, $\hat{\mathscr{H}}_a$, $\hat{\mathscr{G}}^{IJ}$ at the end of the last section, and in the following will decompose the remaining constraints into second class pairs, i.e. block-diagonalise the Dirac matrix. We expect that $\bar{\mathscr{S}}^{aIJ}_{\text{tf}}$ and $\bar{\mathscr{D}}^{bKL}_{\text{tf}}$ are a second class pair since the corresponding part of the Dirac matrix (4.48) is invertible. Indeed, since all constraints except $\bar{\mathscr{D}}^{bKL}_{\text{tf}}$ Poisson commute with $\bar{\mathscr{S}}^{aIJ}_{\text{tf}}$, we can substitute all remaining constraints by

$$\mathscr{C} \to \mathscr{C} - \int \bar{\mathscr{S}}^{aIJ}_{\text{tf}} \left( \left\{ \bar{\mathscr{D}}_{\text{tf}}, \bar{\mathscr{S}}_{\text{tf}} \right\}^{-1} \right)_{aIJ,bKL} \left\{ \bar{\mathscr{D}}^{bKL}_{\text{tf}}, \mathscr{C} \right\}, \tag{4.54}$$

which then Poisson commute also with $\bar{\mathscr{D}}^{bKL}_{\text{tf}}$. This notation is symbolic (notice that the pointwise Poisson brackets are distributional): the matrix $\{\bar{\mathscr{D}}_{\text{tf}}, \bar{\mathscr{S}}_{\text{tf}}\}$ is ultralocal and what is meant is its non distributional factor.

A further set of pairs is given by

$$\begin{aligned}\left\{P^{(E)}_{aI}[f^{aI}], -\bar{\mathscr{S}}^{aI}[\bar{g}_{aI}] + sE^{aI}n_I[\bar{g}_a]\right\} &= \left\{P^{(E)}_{aI}[f^{aI}], \left(-\bar{\mathscr{S}}^{aI} + sE^{aJ}n_J n^I\right)[g_{aI}]\right\} \\ &=: \left\{P^{(E)}_{aI}[f^{aI}], E'^{aI}[g_{aI}]\right\} \\ &= \int_\sigma d^Dx\, f^{aI}\left[-\delta^b_a \eta^J_I\right] g_{bJ}. \end{aligned} \tag{4.55}$$

Another set of second class pairs is obtained realising that

$$P'^{(n)}_I[\gamma^I] := P^{(n)}_I[\gamma^I] + sP^{(E)}_{aJ}[\gamma_I \pi^{aIJ}] \tag{4.56}$$

Poisson commutes with all the above constraints, and its second class partner is given by

$$\begin{aligned}&\left\{P'^{(n)}_I[f^I], \frac{s}{2}(n^J n_J - s)[\bar{g}] + \frac{1}{D-1}\bar{\mathscr{S}}^J_{\text{tr}}[\bar{g}_J]\right\} = \\ =&\left\{P'^{(n)}_I[f^I], \left(\frac{s}{2}(n^K n_K - s)n^J + \frac{1}{D-1}\bar{\mathscr{S}}^J_{\text{tr}}\right)[g_J]\right\} \\ =:&\left\{P'^{(n)}_I[f^I], n'^J[g_J]\right\} \approx \int_\sigma d^Dx\, f_I\left[-\eta^I_J\right] g^J. \end{aligned} \tag{4.57}$$

For the last two sets, the Dirac matrix (indicated by square brackets) is trivially invertible, and constraints from different sets Poisson commute with each other. Therefore, the determinant of the whole Dirac matrix, being block-diagonal, is given by the product of the three subdeterminants corresponding to the three sets of second class pairs, and since all of them are non-zero, the whole Dirac matrix is invertible. The structure of the Dirac matrix is summarized in table 4.1 and the counting of the degrees of freedom is given in table 4.2. As expected for general relativity, we find $(D-2)(D+1)$ phase space degrees of freedom.

### 4.2.4 Solution of the second class constraints: eADM formulation

The solution of the second class constraints is done analogously to the treatment by Peldán [103]. To solve the second class constraints, we use the ansatz

$$A_{aIJ} = \Gamma_{aIJ} + K_{aIJ}, \tag{4.58}$$

| | ${P'}^{(n)}_I$ | ${n'}^I$ | $P^{(E)}_{aI}$ | ${E'}^{aI}$ | $\bar{\mathscr{S}}^{aIJ}_{\text{tf}}$ | $\bar{\mathscr{D}}^{aIJ}_{\text{tf}}$ |
|---|---|---|---|---|---|---|
| ${P'}^{(n)}_K$ | 0 | $\eta^I_K$ | 0 | 0 | 0 | 0 |
| ${n'}^K$ | $-\eta^K_I$ | 0 | 0 | 0 | 0 | 0 |
| $P^{(E)}_{bK}$ | 0 | 0 | 0 | $\delta^a_b\eta^I_K$ | 0 | 0 |
| ${E'}^{bK}$ | 0 | 0 | $-\delta^b_a\eta^K_I$ | 0 | 0 | 0 |
| $\bar{\mathscr{S}}^{bKL}_{\text{tf}}$ | 0 | 0 | 0 | 0 | 0 | $F^{aIJ,bKL}$ |
| $\bar{\mathscr{D}}^{bKL}_{\text{tf}}$ | 0 | 0 | 0 | 0 | $-F^{bKL,aIJ}$ | 0 |

**Table 4.1:** Palatini theory: structure of the Dirac matrix.

| Variable | Dof | Constraint | Number |
|---|---|---|---|
| $A_{aIJ}$ | $\frac{D^2(D+1)}{2}$ | First class | (count twice!) |
| $\pi^{aIJ}$ | $\frac{D^2(D+1)}{2}$ | $\hat{\mathscr{H}}$ | 1 |
| $n^I$ | $D+1$ | $\hat{\mathscr{H}}_a$ | $D$ |
| $E^{aI}$ | $D(D+1)$ | $\hat{\mathscr{G}}^{IJ}$ | $\frac{D(D+1)}{2}$ |
| $P^{(n)}_I$ | $D+1$ | Second class | |
| $P^{(E)}_{aI}$ | $D(D+1)$ | $\mathscr{S}^{aIJ}$ | $\frac{D^2(D+1)}{2}$ |
| | | $\bar{\mathscr{D}}^{aIJ}_{\text{tf}}$ | $\frac{D^2(D-1)}{2} - D$ |
| | | $E^{aI}n_I$ | $D$ |
| | | $n^I n_I - s$ | 1 |
| | | $P^{(E)}_{aI}$ | $D(D+1)$ |
| | | $P^{(n)}_I$ | $D+1$ |
| Sum: | $D^3+3D^2+4D+2$ | Sum: | $D^3+2D^2+5D+4$ |

**Table 4.2:** Palatini theory: counting of degrees of freedom

where $\Gamma_{aIJ}$ denotes the hybrid spin connection annihilating $E^{aI}$ (cf. appendix C). Further, we decompose $K_{aIJ}$ into $\bar{K}_{aIJ}$ and $2n_{[I}\bar{K}_{a|J]}$ and solve the simplicity constraints strongly, $\pi^{aIJ} = 2n^{[I}E^{aJ]}$. Solving the constraints demanding that $n^I$ is orthogonal to $E^{aI}$ and has unit length leads to

$$n^I = \frac{\epsilon^{IJ_1 \ldots J_D} E^{a_1}_{J_1} \ldots E^{a_1}_{J_D} \epsilon_{a_1 \ldots a_D}}{D! \sqrt{\det E^{aI} E^b_I}}, \tag{4.59}$$

and in the following, it will be understood that $n_I = n_I(E^{aJ})$, and thus that the constraints $n^I n_I - s \approx 0$, $E^{aI} n_I \approx 0$, and $P_{n_I} \approx 0$ are solved strongly. The boost (longitudinal) part of the Gauß constraint becomes using the above ansatz for $A_{aIJ}$

$$n_{[I}\lambda_{J]} D^A{}_a \pi^{aIJ} = s\lambda_I \bar{K}_a{}^{KI} E^a_K = s\lambda_I \bar{K}^I_{\text{tr}}. \tag{4.60}$$

Again using this ansatz, we find for $\bar{\mathscr{D}}^{aIJ}_{\text{tf}}$

$$\begin{aligned} \frac{1}{2} \bar{f}^{\text{tf}}_{aIJ} \bar{\mathscr{D}}^{aIJ}_{\text{tf}} &= -s \bar{f}^{\text{tf}}_{aIJ} E^{b[I|} D^A{}_b E^{a|J]} \\ &= -\frac{1}{2} \bar{f}^{\text{tf}}_{aIJ} F^{aIJ,bKL} \frac{1}{2} \bar{K}^{\text{tf}}_{bKL}. \end{aligned} \tag{4.61}$$

We see that $\bar{\mathscr{D}}^{aIJ}_{\text{tf}}$ demands the vanishing of the transversal trace part of $K_{aIJ}$, and together with the boost part of the Gauß constraint, we see that $K_{aIJ} \approx 2n_{[I}K_{a|J]}$. The latter, however, we will not solve strongly.

Since we solved second class constraints, we have to perform a symplectic reduction and determine the new symplectic structure, which is analogous to the symplectic reduction of the 3+1 Palatini action in [103]. In addition to the above considerations, we set $P^{(E)}_{aI} = 0$. The symplectic potential now reads

$$\begin{aligned} \frac{1}{2} \pi^{aIJ} \dot{A}_{aIJ} &= n^{[I} E^{a|J]} \left( \dot{\Gamma}_{aIJ} + \dot{K}_{aIJ} \right) \\ &= n^I \left( (D^\Gamma{}_a E^a_I)^\cdot - D^\Gamma{}_a \dot{E}^a_I + E^{aJ} \dot{K}_{aIJ} \right) \\ &= -\, \partial_a (n^I \dot{E}^a_I) + n^I E^{aJ} \dot{K}_{aIJ} \\ &= -\, \dot{E}^{aJ} n^I K_{aIJ} - \dot{n}^I E^{aJ} K_{aIJ} \\ &= \dot{E}^{aJ} (-n_J E^I_a \bar{K}^{\text{tr}}_I - s\bar{K}_{aJ}) \\ &=: E^{aJ} \dot{K}_{aJ}, \end{aligned} \tag{4.62}$$

where we have dropped total time derivatives and divergences, and in the second before the last step we used that $\dot{n}^I$ is transversal, i.e. $\dot{n}^I = \bar{\eta}^{IJ}\dot{n}_J = E_a{}^I E^{aJ}\dot{n}_J = -E_a{}^I n_J \dot{E}^{aJ}$[1]. Notice also that we keep the trace part of $\bar{K}_{aIJ}$ since we do not solve first class constraints at this point.

In the last step, we have to express the remaining constraints $\mathscr{H}$, $\mathscr{H}_a$, and $\mathscr{G}^{IJ}$ in terms of the new canonical variables. Note that we do not need to consider their hatted versions, since the difference is a linear combination of second class constraints which are now all solved strongly. The calculation yields

$$\mathscr{G}^{IJ} = 2E^a_{[I}K_{a|J]}, \tag{4.63}$$

$$\mathscr{H}_a \approx -2D_{[a}E^{bJ}K_{b]J}, \tag{4.64}$$

$$\mathscr{H} \approx -\frac{s}{2}E^{aI}E^{bJ}R^{\mathrm{H}}{}_{abIJ} + E^{a[I}E^{b|J]}K_{aI}K_{bJ}, \tag{4.65}$$

which coincides with the constraints (3.27, 3.28, 3.29) up to the subtlety that $\mathscr{H}$ now has density weight two. We have neglected terms proportional to the Gauß constraint in the expressions for $\mathscr{H}_a$ and $\mathscr{H}$. $R^{\mathrm{H}}{}_{abIJ}$ denotes the field strength of the hybrid spin connection. Thus, we arrive at the extended ADM formulation considered before.

Finally, we want to remark that one does not necessarily need to solve all second class constraints: one could also try to only solve one or two of the three sets in table 4.1. The possibility to only solve $P'^{(n)}_I$ and $n^J$ indicates that one can also perform the analysis without introducing $n^I$ as an independent field in the beginning, but it probably becomes more complicated. Furthermore, expressing $E^{aI}$ and $n^I$ appearing in $\bar{\mathscr{S}}^{bKL}_{\mathrm{tf}}$ and $\bar{\mathscr{D}}^{bKL}_{\mathrm{tf}}$ by $\pi^{aIJ}$ and then solving all but this set of second class pairs leads to the formulation we will encounter in the next section.

[1] We also used $D^\Gamma{}_a n^I = 0$ which follows from $E^a_I n^I = n^I n_I - s = D^\Gamma{}_a E^b_I = 0$: we have for the longitudinal part $n_I D^\Gamma{}_a n^I = D^\Gamma{}_a(n_I n^I/2) = 0$ and for the transversal part $E^b_I D^\Gamma{}_a n^I = D^\Gamma{}_a(E^b_I n^I) = 0$.

# 5

# Plebański and related formulations

## 5.1 BF theory and Plebański action

Before coming to the Plebański formulation, we will shortly introduce BF theory. BF theory is a topological field theory in any dimensions, and its name stems from the form of its action, being

$$S_{\mathrm{BF}}[A, B] = \int_{\mathscr{M}} \mathrm{Tr}(B \wedge F), \tag{5.1}$$

where $F$ is the $\mathrm{so}(D+1)$ or $\mathrm{so}(1, D)$ valued curvature two-form of a connection $A$ in the Euclidean and Lorentzian case respectively, and $B$ is a $\mathrm{so}(D+1)$ or $\mathrm{so}(1, D)$ valued $(D-1)$-form field. The trace is taken in the Lie algebra. The field equations are easily found to be

$$F = 0, \tag{5.2}$$

$$d_A B = 0, \tag{5.3}$$

from which immediately follows that there are no local degrees of freedom. Actually, for $D = 2$, general relativity in first order form coincides with $D = 2$ BF theory. The fact that general relativity is topological in $D = 2$ allows the use of a variety of techniques from TQFT and ultimately is the basis for the successes made with its quantisation [184].

In $D = 3$, a basic object in BF theory is thus a two-form field $B_{\mu\nu}$. Plebański [167] was the first to consider not the metric or vielbein as a fundamental object in $D = 3$ general relativity, but instead (self-dual) two-forms, and wrote down conditions needed in order that the vielbein can be recovered from these two-forms. These conditions are now often called "simplicity constraints". He also wrote down an action principle using these two-forms, and remarkably, it is neatly related with BF theory. In essence, gravity in $D = 3$ is "almost" the topological BF theory, more precisely, it can be formulated as BF theory subject to these additional simplicity constraints. This deformation of BF theory is also the classical starting point of spin foam models [185–191], the path integral or "covariant" approach to LQG. The formulation we will present here actually is not due to Plebański, but a generalisation of his ansatz. In slight abuse of terminology, we still named this section "Plebański action" since the idea originates from his work. The two-forms are not assumed to be self-dual, the formulation exists in any $D > 2$, and was introduced by Freidel, Krasnov and Puzio in [169].

The action we want to consider is, in $D + 1$ dimensions, given by [169]

$$S_{\text{FKP}}[A, B, \Phi] := \int_{\mathscr{M}} \left[ \text{Tr}(B \wedge F) + \text{Tr}(B \wedge \Phi(B)) \right], \tag{5.4}$$

where $\Phi$ is a certain Lagrange multiplier filed which can be contracted with the $B$ field in a certain way to yield a $(D-1)$-form denoted by $\Phi(B)$. It is constructed such that the variation of $S_{\text{FKP}}$ with respect to the $\Phi$ field results in the field equations

$$\epsilon^{IJKL\overline{M}} B^{\mu\nu}_{IJ} B^{\rho\sigma}_{KL} = \epsilon^{\mu\nu\rho\sigma\overline{\lambda}} c^{\overline{M}}_{\overline{\lambda}} \tag{5.5}$$

for some coefficients $c^{\overline{M}}_{\overline{\lambda}}$. The indices which are overlined denote totally antisymmetric $(D-3)$ multiindices and $B^{\mu\nu IJ} = \frac{1}{2!(D-1)!} \epsilon^{\mu\nu\rho_1 \ldots \rho_{D-1}} B^{IJ}_{\rho_1 \ldots \rho_{D-1}}$ is the rank two antisymmetric contravariant tensor density (often called "bivector" in the literature) dual to $B_{\mu\nu}$. The equations (5.5) define the simplicity constraints in Lagrangian form. The name simplicity constraints stems from the fact that in the corresponding literature, a bivector which is the exterior product of two vectors is called "simple". This is exactly what the constraint ensures. Namely, one of the central results of [169] is *Theorem 1*, stating that (5.5) demands that $B$ comes from a co-vielbein $e$,

$$B = \pm * (e \wedge e), \tag{5.6}$$

plus an additional degenerate sector of solutions in $D > 3$. Substituting back into the action, we obtain (up to sign) the first order vielbein action of general relativity we considered in chapter 4. Note that the sign ambiguity in (5.6) is of relevance, it leads to opposite sign of the cosmological constant in the corresponding solution sectors. For $D = 3$, two additional, topological solution sectors are present,

$$B = \pm e \wedge e. \tag{5.7}$$

Since, as we already stated, spin foam models are based on the $D = 3$ deformed BF theory, even modern models like the EPRL model [186, 189] suffer from these unwanted (or even unphysical) sectors in that that their semiclassical limit is not given by (the discrete equivalent of) $e^{iS_{EH}}$ solely (see, however, [192] and references therein for a up to date discussion and possible resolution of this problem using projectors).

As we already stated, we refrain from displaying the rather lengthy canonical analysis of the Plebański action and refer the interested reader to [170]. However, we will perform the analysis of a related formulation, which one could call "Palatini formulation with BF type simplicity constraints". The BF simplicity constraints (or more precisely, a slightly simpler version thereof) will play a central role, and are, in fact, at the heart of the new variables we will introduce.

## 5.2 Palatini formulation with BF type simplicity constraints

We will start with the split form of the Palatini action before performing the Legendre transform (cf. (4.12)),

$$S = \int dt \int_\sigma d^D x \left( \frac{1}{2} \pi'^{aIJ} \dot{A}_{aIJ} - \underset{\sim}{N} \mathscr{H}' - N^a \mathscr{H}'_a - \frac{1}{2} \lambda_{IJ} \mathscr{G}'^{IJ} \right), \tag{5.8}$$

where the notation is the same as in section 4.2. We already know that $\underset{\sim}{N}$, $N^a$ and $\lambda_{IJ}$ will become Lagrange multipliers and we can treat them accordingly already at this point. We have to introduce conjugate momenta for $A_{aIJ}$ and $E^{aI}$, $\pi^{aIJ}$ and $P^{(E)}_{aI}$, together with the constraints enforcing that $P^{(E)}_{aI} = 0$ and the simplicity constraints demanding $\pi^{aIJ} = 2n^{[I} E^{a|J]}$. We can free all other constraints of the dependence of $E^{aI}$ as we did in section 4.2 by using the simplicity constraint. Note that if we wrote a simplicity constraint solely in terms of $\pi^{aIJ}$, the action would not depend on

$E^{aI}$ anymore and we could trivially solve the constraint $P^{(E)}_{aI}$ by simply dropping $E^{aI}$ and $P^{(E)}_{aI}$ completely. But this is exactly what the simplicity constraints achieve in BF theory, being formulated a priori without vielbeins. In our Hamiltonian setting, actually a subset of the BF simplicity constraints suffices, since the variables $\pi^a{}_{IJ}$ correspond only to the spatial-temporal components $B^{at}_{IJ}$ of the $B$ field.

### 5.2.1 BF type simplicity constraints

Following this line of thought, the action we want to consider is given by

$$S = \int_{\mathbb{R}} dt \int_\sigma d^D x \left( \frac{1}{2}\pi^{aIJ}\dot{A}_{aIJ} - \underset{\sim}{N}\mathscr{H} - N^a\mathscr{H}_a - \frac{1}{2}\lambda_{IJ}\mathscr{G}^{IJ} - c^{\overline{M}}_{ab}\mathscr{S}^{ab}_{\overline{M}} \right), \tag{5.9}$$

where $c^{\overline{M}}_{ab}$ is a Lagrange multiplier field, symmetric in the index pair $a$, $b$, which enforces the BF-type simplicity constraints

$$\mathscr{S}^{ab}_{\overline{M}} := \frac{1}{2}({}_{\overline{M}} * \pi^a)_{IJ}\pi^{bIJ} := \frac{1}{4}\epsilon_{IJKL\overline{M}}\pi^{aIJ}\pi^{bKL}. \tag{5.10}$$

The other constraints are the same as in section 4.2,

$$\mathscr{G}^{IJ} := D^A{}_a\pi^{aIJ} := \partial_a\pi^{aIJ} + [A_a, \pi^a]^{IJ}, \tag{5.11}$$

$$\mathscr{H} := -\frac{1}{2}\pi^{aIK}\pi^{bJ}{}_K F_{abIJ}, \tag{5.12}$$

$$\mathscr{H}_a := \frac{1}{2}\pi^{bIJ}F_{abIJ}. \tag{5.13}$$

The action is motivated by [169], but as we already noted, we can also arrive at it by taking the action from the previous chapter, dropping the variables $E^{aI}$, $n_I$, and all constraints containing them, and introducing the BF-type simplicity constraint. The theorem which relates $\pi^{aIJ}$ solving $\mathscr{S}^{ab}_{\overline{M}} = 0$ with the vielbein is a special case of *Theorem 1* from [169] for the full BF simplicity constraint,

**Theorem 1.**
*In dimension $D > 3$ a generic field $\pi^{aIJ}$ satisfies the constraints*

$$\mathscr{S}^{ab}_{\overline{M}} = 0 \tag{5.14}$$

*if and only if it comes from a frame field. In other words, a non-degenerate $\pi^{aIJ}$ satisfies the constraints (5.14) if and only if there exist $e^a_I$ such that*

$$\pi^{aIJ} = \pm 2\sqrt{q}n^{[I}e^{a|J]}, \tag{5.15}$$

*where $q$ is the determinant of the inverse matrix of $q^{ab} = e^{aI} e^b{}_I$.*

Like for the full BF simplicity constraint, the theorem also holds for $D = 3$ with the additional appearance of a topological sector which we will neglect in the following. We will provide a short sketch of the proof of this theorem. This proof as well is a special case of the proof given in [169] and we refer the interested reader to the original literature for more details.

The constraints are divided into the categories

$$\begin{aligned} &\text{simplicity:} && \pi^a{}_{[IJ}\pi^a{}_{KL]} = 0 \quad \text{(no summation)}, \\ &\text{intersection:} && \pi^a{}_{[IJ}\pi^b{}_{KL]} = 0 \quad \text{for } a,\, b \text{ distinct.} \end{aligned}$$

We find it convenient for the following considerations to look at $\pi^a_{IJ}$ as a two form $\pi^a_{IJ} dx^I \wedge dx^J$. It can be shown that for a two-form $B_{IJ}$

$$B_{[IJ} B_{KL]} = 0 \;\Leftrightarrow\; B_{IJ} = u_{[I} v_{J]}, \tag{5.16}$$

which corresponds to the "simplicity" part of the simplicity constraints. Therefore, all $\pi^a_{IJ}$ factor into $u^a_{[I} v^a_{J]}$ (no summation). To complete the proof, we have to relate the different $u^a_I$ to each other. For this purpose, it is proved in [169] that for two two-forms $B_{IJ}$ and $B'_{IJ}$,

$$B_{[IJ} B'_{KL]} = 0 \;\Leftrightarrow\; B_{IJ} = u_{[I} v_{J]} \text{ and } B'_{IJ} = u_{[I} w_{J]}, \tag{5.17}$$

meaning that the two two-forms share a common factor which is unique up to scaling. This relation is ensured by the intersection constraint. Combining these two arguments, we realise that $\pi^a_{IJ}$ factors into one-forms with a common factor. Introducing the correct density weight and a suitable normalisation, we obtain

$$\pi^a_{IJ} = \pm 2\sqrt{q}\, n_{[I} e^a_{J]}. \tag{5.18}$$

The sign can be absorbed into $n^I$ for $D + 1$ even, the otherwise appearing signs can be absorbed into the Lagrange multipliers in the Hamiltonian. We remark that in the general case discussed in [169], additional normalisation constraints are necessary and the proof becomes considerably longer.

The property of $n^I$ being time-like in the Lorentzian case will be enforced by another constraint. Namely, since we want the metric to be positive definite, we impose the constraint

$$s\pi^{aIJ}\pi^b_{IJ} \approx 2qq^{ab} > 0, \tag{5.19}$$

where the greater sign means positive definiteness of matrices. In the Lorentzian case, the relation is only satisfied if $n^I$ is time-like, because $E^{aI}E^b_I$ would be indefinite otherwise. This non-holonomic constraint of course does not reduce the degrees of freedom.

### 5.2.2 Constraint analysis

From the above action (5.9) we "read off" the non-vanishing Poisson brackets as $\{A_{aIJ}, \pi^{bKL}\} = 2\delta^b_a\delta^K_{[I}\delta^L_{J]}$. Most of the canonical analysis is the same as in the previous chapter and we will only describe the differences. The Poisson bracket

$$\begin{aligned}\left\{\mathscr{H}[\underset{\sim}{M}], \mathscr{H}[\underset{\sim}{N}]\right\} = &-\mathscr{H}_a\left[(\underset{\sim}{M}\partial_b \underset{\sim}{N} - \underset{\sim}{N}\partial_b \underset{\sim}{M})\frac{1}{2}\pi^{aIJ}\pi^b_{IJ}\right] \\ &+ \mathscr{S}^{ab}_{\overline{M}}\left[(\underset{\sim}{M}\partial_a \underset{\sim}{N} - \underset{\sim}{N}\partial_a \underset{\sim}{M})\frac{s}{2(D-3)!}(^{\overline{M}} * \pi^c)_{IJ}F^{IJ}_{cb}\right]\end{aligned} \tag{5.20}$$

of two Hamiltonian constraints reproduces exactly the BF-simplicity constraint and shows that the theory would be inconsistent without this constraint. The BF-simplicity constraint is stable under spatial diffeomorphisms and internal rotations as reflected by the Poisson brackets

$$\left\{\tilde{\mathscr{H}}_a[N^a], \mathscr{S}^{ab}_{\overline{M}}[c^{\overline{M}}_{ab}]\right\} = -\mathscr{S}^{ab}_{\overline{M}}\left[(\mathscr{L}_N c)^{\overline{M}}_{ab}\right] \tag{5.21}$$

and

$$\left\{\frac{1}{2}\mathscr{G}^{IJ}[\lambda_{IJ}], \mathscr{S}^{ab}_{\overline{M}}[c^{\overline{M}}_{ab}]\right\} = \mathscr{S}^{ab}_{\overline{M}}\left[\sum_{i=1}^{D-3}\lambda^{M_i}{}_{M_i'}c_{ab}^{M_1\ldots M_{i-1}M_i'M_{i+1}\ldots M_{D-3}}\right], \tag{5.22}$$

and trivially commutes with itself. As in the previous chapter, the Poisson bracket with the Hamiltonian constraint

$$\left\{\mathscr{S}^{ab}_{\overline{M}}[c^{\overline{M}}_{ab}], \mathscr{H}[\underset{\sim}{N}]\right\} = \mathscr{D}^{ab}_{\overline{M}}\left[\underset{\sim}{N}c^{\overline{M}}_{ab}\right] + \mathscr{S}^{ab}_{\overline{M}}[\ldots] \tag{5.23}$$

imposes a new constraint

$$\mathscr{D}^{ab}_{\overline{M}} = 2(^{\overline{M}} * \pi^c)_{IJ}\pi^{(a|IK}D^A{}_c\pi^{b)J}{}_K. \tag{5.24}$$

To show its stability, we have to show that the Poisson bracket of this new constraint with the BF-simplicity is invertible. Irrespective of this, we emphasise that $\mathscr{D}^{ab}_{\overline{M}}$ is stable under internal rotations, reflected by

$$\left\{\frac{1}{2}\mathscr{G}^{IJ}[\lambda_{IJ}], \mathscr{D}^{ab}_{\overline{M}}[d^{\overline{M}}_{ab}]\right\} = \mathscr{D}^{ab}_{\overline{M}}\left[\sum_{i=1}^{D-3} \lambda^{M_i}{}_{M_i'} d_{ab}^{M_1\ldots M_{i-1}M_i'M_{i+1}\ldots M_{D-3}}\right]. \tag{5.25}$$

Concerning the diffeomorphism constraint, it is easy to see that we can extend the covariant derivative in $\mathscr{D}^{ab}_{\overline{M}}$ to act on spatial indices via the Christoffel symbols. Namely, adding the corresponding terms to the constraint, we see that, due to the symmetry of the Christoffel symbols in their lower indices, the added terms are proportional to simplicity constraints. $\mathscr{D}^{ab}_{\overline{M}}$ therefore transforms like a scalar density of weight $+3$ under spatial diffeomorphisms and the Poisson bracket with the diffeomorphism constraint has to be proportional to the $\mathscr{D}^{ab}_{\overline{M}}$. Another easy way to do this calculation is to use the Jacobi identity after expressing $\mathscr{D}^{ab}_{\overline{M}}$ as a Poisson bracket. We do not know of any nice way to express the Poisson bracket of $\mathscr{D}^{ab}_{\overline{M}}$ with the Hamiltonian constraint and will leave the discussion of this bracket open, as its value is not important in the following.

A counting of the degrees of freedom which are reduced by the BF-simplicity constraint (i.e. $\pi^{aIJ} \to E^{aI}$), yields $D^2(D-1)/2 - D$ which is for $D > 3$ less than the number of components of the BF-simplicity $\frac{1}{2}D(D+1)\binom{D+1}{4}$. The BF-simplicity constraints are therefore not independent and the matrix formed by calculating the Poisson bracket with $\mathscr{D}^{ab}_{\overline{M}}$ cannot be invertible. The solution to this problem is to find an independent set of BF-simplicity and $\mathscr{D}$ constraints which still enforce the same constraint surface. The constraints of section 4.2 do have this property and lead us to the following ansatz: We choose some internal time-like vector $n^I$ with $n^I n_I = s$ which may vary as a function of the spatial coordinates and decompose $\pi^{aIJ}$ as

$$\pi^{aIJ} = \bar{\pi}^{aIJ} + 2n^{[I}E^{a|J]} \tag{5.26}$$

as in the previous chapter. We also define $E_{aI}$ by $E_{aI}E^{bI} = \delta^b_a$ and $n^I E_{aI} = 0$. Together with the normalisation condition $n^I n_I = s$ this means that $n^I = n^I[E]$ can be considered as a function of $E^a_I$ only and thus does not count as independent degree of freedom. The BF-simplicity constraints plus the non-holonomic constraint are equivalent with $\bar{\pi}^{aIJ} = 0$ and $n_I$ being time-like in the Lorentzian case. However,

$\bar{\pi}^{aIJ}$ has $D^2(D-1)/2$ degrees of freedom and $E^a_I$ has $D(D+1)$ which together yields $D^2(D+1)/2+D$ degrees of freedom while $\pi^{aIJ}$ has only $D^2(D+1)/2$ degrees of freedom. It follows that $\bar{\pi}^{aIJ}$ and $E^a_I$ cannot be considered as independent degrees of freedom, there must be $D$ additional relations among them. Indeed, in section 7.1.2 we will argue[1] that it is always possible to arrange that $\bar{\pi}^{aIJ} = \bar{\pi}^{aIJ}_{\text{tf}}$ is automatically trace free with respect to $E_{aI}$. These would be the missing $D$ relations and now the BF-Simplicity constraints are equivalent with the $D^2(D-1)/2-D$ constraints $\bar{\pi}^{aIJ}_{\text{tf}} = 0$ which in number match with the constraints $\bar{K}^{\text{tf}}_{aIJ} = 0$ to which the constraints $\mathscr{D}^{ab}_{\overline{M}} = 0$ reduce as we will now show below. At the moment we have no proof of this for $D \geq 3$ thus we will make the assumption that $\pi^{aIJ}$ can always be decomposed in this way. In other words, we will only allow $\pi^{aIJ}$ of the following form: There is a tensor $E^{aI}$ with $qq^{ab} = \eta_{IJ}E^{aI}E^{bJ}$ positive definite. Let $n^I[E]$ be the unique normal vector satisfying $E^{aI}n_I = 0$, $n^I n_I = s$. Take any tensor $t^{aIJ}$ and construct from it $\bar{t}^{aIJ}_{\text{tf}}[t,E]$ using $E, n[E]$. Then $\pi^{aIJ} := \bar{t}^{aIJ}_{\text{tf}} + 2n^{[I}E^{aJ]}$ and automatically $E^{aI} = -s\pi^{aIJ}n_J$. For $\pi^{aIJ}$ constructed in this way, we derived a fixed point equation in [2] which has obviously non trivial solutions and the question is whether such $\pi^{aIJ}$ are generic.

Concerning the $\mathscr{D}^{\overline{M}}_{ab}$ constraint, we make the same ansatz as in the previous chapter and set

$$A_{aIJ} = \Gamma_{aIJ} + \bar{K}_{aIJ} + 2n_{[I}\bar{K}_{a|J]}. \tag{5.27}$$

A short calculation yields

$$\begin{aligned}\bar{f}_{(a|IJ}\pi_{|b)KL}\epsilon^{IJKL\overline{M}}\mathscr{D}^{ab}_{\overline{M}} &= \bar{f}_{(a|IJ}\pi_{|b)KL}\epsilon^{IJKL\overline{M}}\epsilon_{ABCD\overline{M}}\pi^{cAB}\pi^{(a|C}{}_E D^A{}_c\pi^{b)DE} \\ &\approx -(D-3)!(D-1)\bar{K}_{aIJ}F^{aIJ,bKL}\bar{f}_{bKL},\end{aligned} \tag{5.28}$$

where $F^{aIJ,bKL}$ denotes the same matrix as in (4.48). As before, we defined $\pi_{bKL} := q^{-1}q_{ab}\pi^a{}_{KL}$, where $q^{-1}q_{ab}$ is the inverse matrix of $\frac{s}{2}\pi^{aIJ}\pi^b{}_{IJ}$, such that $\pi^{aIJ}\pi_{bIJ} = 2s\delta^a_b$. We notice that $\bar{f}_{aIJ}$ can be chosen traceless with respect to $E^{aI}$, since any trace part would drop out in the combination $\bar{f}_{(a|IJ}\pi_{b)KL}\epsilon^{IJKL\overline{M}}$ modulo the BF-Simplicity constraint. The subset of $\mathscr{D}$ constraints parametrised by $\bar{f}_{aIJ} = \bar{f}^{\text{tf}}_{aIJ}$ as above thus sets

[1] This is not trivial: For $D \geq 3$ one cannot use closed formulas for a proof. It is apparently necessary to make use of fixed point theorems.

the trace free part of $\bar{K}_{aIJ}$ to zero. When inserting the solution of the BF-simplicity constraint into the full $\mathscr{D}^{ab}_{\overline{M}}$ constraint, we get

$$\mathscr{D}^{ab}_{\overline{M}} \approx 2s\epsilon_{ABC}{}^{D}{}_{\overline{M}} n^A E^{cB} E^{(a|C} E^{b)E} \bar{K}_{cED} \tag{5.29}$$

and immediately verify that the solution $\bar{K}^{\mathrm{tf}}_{aIJ} = 0$ solves all the $\mathscr{D}$ constraints because the trace part of $\bar{K}_{cED}$ drops out in the above combination.

From these considerations, we realise that it is legitimate to use the Lagrange multipliers displayed in (5.28) and therefore only a subset of the $\mathscr{D}$ constraints. It follows that we only have to check the stability of this subset of constraints. To form the Dirac matrix, we choose similarly a subset of BF-simplicity constraints equivalent to $\bar{\pi}^{aIJ}_{\mathrm{tf}} = 0$ and calculate

$$\int d^Dx \int d^Dy\, [\bar{f}^{\mathrm{tf}}_{(a|IJ}\pi_{b)KL}\epsilon^{IJKL\overline{M}}](x) \left\{ \mathscr{S}^{ab}_{\overline{M}}(x), \mathscr{D}^{cd}_{\overline{N}}(y) \right\} [\bar{g}^{\mathrm{tf}}_{(c|MN}\pi_{d)OP}\epsilon^{MNOP\overline{N}}](y)$$
$$\approx 4(D-1)^2((D-3)!)^2 \int d^Dx\, \bar{f}^{\mathrm{tf}}_{aIJ} F^{aIJ,bKL} \bar{g}^{\mathrm{tf}}_{bKL}. \tag{5.30}$$

We can therefore adjust the multiplier of the BF-simplicity such that the independent subset of $\mathscr{D}$ constraints is stable under time evolution and finish the canonical analysis. Since the Dirac matrix is invertible, the chosen subset of BF-simplicity constraints has to be independent. The number of BF-simplicities in this subset is equivalent to the number of degrees of freedom in a transverse trace free matrix, i.e. $D^2(D-1)/2 - D = D(D+1)(D-2)/2$ and matches the degrees of freedom which are to be taken out of the system by the full BF-simplicity constraints and all BF-simplicity constraints can thus be derived by taking the linear span of this subset.

The solution of the constraints proceeds analogously to the previous chapter, the only difference being that we do not need to solve the momenta associated with $E^{aI}$ and $n^I$. The two formulations presented are therefore equivalent.

### 5.2.3 Degrees of freedom

As in the previous chapter, we check the degrees of freedom of the Hamiltonian system derived using the BF-simplicity constraint. For $\mathscr{H}$ to become a first class constraint, we construct the linear combination (using the same abuse of notation as before)

$$\tilde{\mathscr{H}} := \mathscr{H} - \int \bar{\mathscr{S}}_{\text{tf}}^{aIJ} \left( \left\{ \bar{\mathscr{D}}_{\text{tf}}, \bar{\mathscr{S}}_{\text{tf}} \right\}^{-1} \right)_{aIJ,bKL} \left\{ \bar{\mathscr{D}}_{\text{tf}}^{bKL}, \mathscr{H} \right\}. \tag{5.31}$$

Since the Dirac matrix between the independent BF-simplicity and $\mathscr{D}^{ab}_{\overline{M}}$ constraints is invertible, they are of the second class. The rest of the constraints is of the first class. The difference between the degrees of freedom and the weighted sum of the constraints

| Variable | DoF | Constraint | DoF |
|---|---|---|---|
| $A_{aIJ}$ | $\frac{D^2(D+1)}{2}$ | First class | (count twice!) |
| $\pi^{aIJ}$ | $\frac{D^2(D+1)}{2}$ | $\tilde{\mathscr{H}}$ | 1 |
| | | $\mathscr{H}_a$ | $D$ |
| | | $\mathscr{G}^{IJ}$ | $\frac{D(D+1)}{2}$ |
| | | Second class | |
| | | $\mathscr{S}^{ab}_{\overline{M}}$ | $\frac{D^2(D-1)}{2} - D$ |
| | | $\mathscr{D}^{ab}_{\overline{M}}$ | $\frac{D^2(D-1)}{2} - D$ |
| Sum: | $D^3 + D^2$ | Sum: | $D^3 + D + 2$ |

**Table 5.1:** Palatini theory with BF simplicity constraints: counting of degrees of freedom.

is again $(D+1)(D-2)$ and matches those of general relativity. Solution of the second class constraints is in analogy to the treatment in section 4.2.4 and leads to the extended ADM phase space. Note that, instead of solving the second class constraints, one could as well work with the Dirac bracket, but then the connection $A_{aIJ}$ is not Poisson self-commuting and the loop quantisation programme at least not directly applicable. For research in this direction, which however usually considers $D = 3$ and the Holst modification of the action we will introduce in section 6.1, cf. [193–196].

# 6

# $D = 3$: Holst and CDJ action, Ashtekar (Barbero) formulation

## 6.1 Holst action

Holst [145] was the first to write down an action for Ashtekar Barbero variables, and he also gave a canonical analysis using time gauge. Like the Ashtekar Barbero variables, this action only exists in $D = 3$. Barros e Sá [146] reconsidered the analysis without choosing any gauge fixing. We will follow his work and, only after having solved the second class constraints, choose time gauge to obtain the Ashtekar Barbero formulation.

Holst's action is given by

$$S_{\text{Holst}} = \frac{s}{2} \int_{\mathscr{M}} d^4X \; e e^{\mu}{}_{I} e^{\nu}{}_{J} \overset{(\gamma)}{F}{}_{\mu\nu}{}^{IJ}, \tag{6.1}$$

where the notation is the same as in the previous chapters. We additionally introduced the notation $\overset{(\gamma)}{X}{}^{IJ} := \overset{(\gamma)}{\mathscr{M}}{}^{IJ}{}_{KL} X^{KL}$, where the matrix $\mathscr{M}$ is given by

$$\overset{(\gamma)}{\mathscr{M}}{}^{IJ}{}_{KL} := \eta^{[I}_{K} \eta^{J]}_{L} + \frac{1}{2\gamma} \epsilon^{IJ}{}_{KL}, \tag{6.2}$$

and $\gamma$ denotes the Barbero Immirzi parameter [16–19]. This action coincides with the Palatini action we studied in chapter 4, except for this additional matrix, which amounts to an additional term $\propto \frac{1}{\gamma}$. While the Palatini action, as we have seen, in any dimensions yields general relativity, the additional, so called Holst term only exists for

$D = 3$. The matrix $\overset{(\gamma)}{\mathscr{M}}{}^{IJ}{}_{KL}$ is invertible if $\gamma^2 \neq \zeta$, and its inverse is then given by

$$(\overset{(\gamma)}{\mathscr{M}}{}^{-1})^{IJ}{}_{KL} := \frac{\gamma^2}{\gamma^2 - \zeta}\left(\eta^{[I}_K \eta^{J]}_L - \frac{1}{2\gamma}\epsilon^{IJ}{}_{KL}\right). \tag{6.3}$$

In the following, we will restrict to the invertible case for convenience, but want to remark that the cases we excluded, $\gamma^2 = \zeta$, of course can also be dealt with and, in fact, correspond to Ashtekar's original variables. The corresponding action, which then only depends on the self-dual part of the connection, was written down by Jacobson and Smolin [197] and, in fact, was the first action known to yield (complex) Ashtekar variables when passing to the Hamiltonian picture. Furthermore, the internal signature $\zeta$ of course here coincides with the space time signature $s = \zeta$, but for later convenience, we use to define $\mathscr{M}^{-1}$ as above.

To see that this is a valid action for gravity, it is instructive to vary the action with respect to the so(4) or so(1,3) connection $A_\mu{}^{IJ}$. The calculation is analogous to section 4.1, using $\delta F_{\mu\nu}{}^{IJ} = 2\nabla^A{}_{[\mu}\delta A_{\nu]}{}^{IJ}$ we easily obtain

$$\begin{aligned}\delta S_{\text{Holst}} &= \frac{s}{2}\int_{\mathscr{M}} d^4X\ ee^\mu{}_I e^\nu{}_J\left(\delta F_{\mu\nu}{}^{IJ} + \frac{1}{2\gamma}\epsilon^{IJ}{}_{KL}\delta F_{\mu\nu}{}^{KL}\right) \\ &= -s\int_{\mathscr{M}} d^4X\ \left(\nabla^A{}_\mu ee^\mu{}_I e^\nu{}_J\right)\delta \overset{(\gamma)}{A}{}_\nu{}^{IJ}.\end{aligned} \tag{6.4}$$

Here and in following calculations, it is useful to note that

$$\operatorname{Tr}(\overset{(\gamma)}{X}Y) = \operatorname{Tr}(X\overset{(\gamma)}{Y}), \tag{6.5}$$

$$\operatorname{Tr}(\overset{(\gamma)}{X}YZ) = \operatorname{Tr}(X\overset{(\gamma)}{Y}Z) = \operatorname{Tr}(XY\overset{(\gamma)}{Z}), \tag{6.6}$$

$$[\overset{(\gamma)}{X}, Y]^{IJ} = [X, \overset{(\gamma)}{Y}]^{IJ} = (\overset{(\gamma)}{[X, Y]})^{IJ}. \tag{6.7}$$

Because of the invertibility of $\overset{(\gamma)}{\mathscr{M}}{}^{IJ}{}_{KL}$ for $\gamma^2 \neq s$, one immediately finds the field equation $\nabla^A{}_\mu\left(ee^\mu{}_{[I}e^\nu{}_{|J]}\right) = 0$. The solution is given by $A_\mu{}^{IJ} = \Gamma_\mu{}^{IJ}$, as we already have seen in the Palatini theory. Reinserting into the action, we find that the term proportional to $\frac{1}{\gamma}$ vanishes due to the first Bianchi identity,

$$\epsilon^{IJKL}ee^\mu{}_I e^\nu{}_J R_{\mu\nu KL} = \epsilon^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = 0, \tag{6.8}$$

and the action thus reduces to the second order vielbein formulation we considered in section 3.1.

## 6.2 Canonical analysis: Ashtekar Barbero formulation

### 6.2.1 3+1 split and Legendre transformation

The $3+1$ split and canonical analysis of the Holst action is performed analogous to section 5.2. We find

$$\begin{aligned} S_{\text{Holst}} &= \frac{s}{2}\int_{\mathscr{M}} d^4X\ Ne\left({}^{||}e^{\mu}{}_I{}^{||}e^{\nu}{}_J + 2sn^{\mu}n_I{}^{||}e^{\nu}{}_J\right)\overset{(\gamma)}{F}{}_{\mu\nu}{}^{IJ} \\ &= \frac{s}{2}\int_{\mathscr{M}} d^4X\ Ne\left[{}^{||}e^{\mu}{}_I{}^{||}e^{\nu}{}_J\overset{(\gamma)}{F}{}_{\mu\nu}{}^{IJ} + \frac{2s}{N}(\overset{(\gamma)}{n_{[I}{}^{||}e^{\nu}{}_{J]}})\left(\mathscr{L}_T A_{\nu}{}^{IJ} - D^A{}_{\nu}A_T{}^{IJ} - N^{\mu}F_{\mu\nu}{}^{IJ}\right)\right] \\ &= \int dt\int_{\sigma} d^3x\ \left[\frac{s}{2}Nee^a{}_Ie^b{}_J\overset{(\gamma)}{F}{}_{ab}{}^{IJ} + (\overset{(\gamma)}{en_{[I}e^b{}_{J]}})\left(\dot{A}_b{}^{IJ} - D^A{}_bA_T{}^{IJ} - N^aF_{ab}{}^{IJ}\right)\right] \\ &= \int dt\int_{\sigma} d^3x\ \left[\frac{1}{2}\underset{\sim}{N}\pi^a{}_{IK}\pi^b{}_J{}^K\overset{(\gamma)}{F}{}_{ab}{}^{IJ} + \frac{1}{2}\overset{(\gamma)}{\pi}{}^b{}_{IJ}\left(\dot{A}_b{}^{IJ} - D^A{}_bA_T{}^{IJ} - N^aF_{ab}{}^{IJ}\right)\right], \end{aligned} \tag{6.9}$$

where $\pi^{aIJ} := 2en^{[I}e^{a|J]}$. Holst proceeds by choosing time gauge, but we refrain from doing so. We have three possibilities of how to proceed without choosing any gauge fixing: a) following Dirac, introduce momenta conjugate to $e, A$ and treat all of them as independent phase space degrees of freedom like in section 4.2, b) partially integrate the kinetic term $\pi\dot{A} \to \dot{\pi}A = \dot{e}K'$ and to treat only $e, K'$ as phase space coordinates, c) drop $e$ and instead, introduce the BF type quadratic simplicity constraint of section 5.2 and work on a phase space coordinatised by $A$ and its momenta $\pi$. The easiest is b) and leads to Ashtekar Barbero variables after choosing time gauge. c) reproduces b) after solving the simplicity constraint and its arising second class partner. a) leads to a much larger phase space, which also can be reduced to b). We will follow route c), and can read off from (6.9) the non-vanishing Poisson brackets

$$\{A_{aIJ}(x), \overset{(\gamma)}{\pi}{}^{bKL}(y)\} = 2\eta^K_{[I}\eta^L_{J]}\delta^b_a\delta^{(3)}(x-y) \tag{6.10}$$

and the Hamiltonian

$$H = \int_{\sigma} d^3x\ \underset{\sim}{N}\mathscr{H} + \frac{1}{2}\lambda^{IJ}\mathscr{G}_{IJ} + N^a\mathscr{H}_a + \underset{\sim}{c}_{ab}\mathscr{S}^{ab}, \tag{6.11}$$

where

$$\mathscr{H} = -\frac{1}{2}\pi^a{}_{IK}\pi^b{}_J{}^K \overset{(\gamma)}{F}{}_{ab}{}^{IJ}, \tag{6.12}$$

$$\mathscr{H}_a = \frac{1}{2}\overset{(\gamma)}{\pi}{}^b{}_{IJ}F_{ab}{}^{IJ}, \tag{6.13}$$

$$\mathscr{G}_{IJ} = D^A{}_b \overset{(\gamma)}{\pi}{}^b{}_{IJ}, \tag{6.14}$$

$$\mathscr{S}^{ab} = \frac{1}{2}(*\pi^a)_{IJ}\pi^{bIJ} = \frac{1}{4}\epsilon_{IJKL}\pi^{aIJ}\pi^{bKL}. \tag{6.15}$$

### 6.2.2 Constraint analysis

The constraint algebra is easily obtained using the details on the derivation of the constraint algebra in section 5.2.2. The calculations are equal up to the appearance of the matrices $\overset{(\gamma)}{\mathscr{M}}$, $\underset{(\gamma)}{\mathscr{M}}^{-1}$, which, however, can be easily included using (6.5, 6.6). $\mathscr{G}_{IJ}$ and $\mathscr{H}_a$ again generate internal SO(4) or SO(1, 3) transformations and spatial diffeomorphisms (mixed with internal transformations) respectively, and $\mathscr{S}^{ab}$ trivially Poisson commutes with itself, so we will only display the remaining Poisson brackets. We find

$$\begin{aligned}\left\{\mathscr{H}[\underset{\sim}{M}], \mathscr{H}[\underset{\sim}{N}]\right\} &= -\mathscr{H}_a\left[(\underset{\sim}{M}\partial_b \underset{\sim}{N} - \underset{\sim}{N}\partial_b \underset{\sim}{M})\frac{1}{2}\pi^{aIJ}\pi^b_{IJ}\right] \\ &\quad + s\mathscr{S}^{ab}\left[(\underset{\sim}{M}\partial_a \underset{\sim}{N} - \underset{\sim}{N}\partial_a \underset{\sim}{M})\frac{1}{2}(*\pi^c)_{IJ}\overset{(\gamma)}{F}{}_{cb}{}^{IJ}\right], \end{aligned}\tag{6.16}$$

$$\{\mathscr{S}^{ab}[\underset{\sim}{c}_{ab}], \mathscr{H}[\underset{\sim}{N}]\} = \mathscr{D}^{ab}[\underset{\sim}{N}\underset{\sim}{c}_{ab}], \tag{6.17}$$

where

$$\mathscr{D}^{ab} = 2(*\pi^c)_{IJ}\pi^{(a|I}{}_K D^A{}_c \pi^{b)J}{}_K \tag{6.18}$$

exactly coincides with (5.24) for $D = 3$. Like before, they form a second class pair,

$$\begin{aligned}\{\mathscr{S}^{ab}[c_{ab}], \mathscr{D}^{cd}[d_{cd}]\} = &\int_\sigma d^3x\ c_{ab}F^{ab\,cd}d_{cd} \\ &+ \frac{4\gamma^2}{\gamma^2 - s}\mathscr{S}^{ab}\Big[\mathscr{S}^{cd}(c_{ab}d_{cd} - c_{a(c}d_{d)b}) \\ &\qquad - \frac{1}{\gamma}qq^{cd}(c_{ab}d_{cd} - c_{a(c}d_{d)b} - c_{b(c}d_{d)a} + c_{cd}d_{ab})\Big], \end{aligned}\tag{6.19}$$

where we defined $F^{ab\,cd} := -\frac{4s\gamma^2}{\gamma^2-s}q^2G^{ab\,cd}$, and from (2.16) we know that this matrix is invertible. Like in section 5.2.2, we can therefore stabilise $\mathscr{D}^{ab}$ by fixing the Lagrange multiplier $\underset{\sim}{c}_{ab}$ in the action and end the stability analysis.

| Variable | DoF | Constraint | DoF |
|---|---|---|---|
| $A_{aIJ}$ | 18 | First class | (count twice!) |
| $\overset{(\gamma)}{\pi}{}^{aIJ}$ | 18 | $\tilde{\mathscr{H}}$ | 1 |
| | | $\mathscr{H}_a$ | 3 |
| | | $\mathscr{G}^{IJ}$ | 6 |
| | | Second class | |
| | | $\mathscr{S}^{ab}$ | 6 |
| | | $\mathscr{D}^{ab}$ | 6 |
| Sum: | 36 | Sum: | 32 |

**Table 6.1:** Holst with BF simplicity constraints: counting of degrees of freedom.

### 6.2.3 Solution of the second class constraints, time gauge

To solve the second class constraints, we make the ansatz $A_{aIJ} = \Gamma_{aIJ} + \underset{(\gamma)}{K}_{aIJ}$. Inserting this into the $\mathscr{D}$ constraint and using the solution of the simplicity constraint $\pi^{aIJ} = 2n^{[I}E^{a|J]}$, we find that again the transversal trace free part of $K_{aIJ}$ is fixed, and this time does not have to vanish but is given by

$$\bar{K}^{\mathrm{tf}}_{aIJ} = \frac{1}{\gamma}\mathbb{P}^{\mathrm{tf}}_{aIJ}{}^{bKL}\epsilon_{KLMN}n^M\bar{K}_b{}^N \approx \frac{1}{\gamma}\epsilon_{IJMN}n^M\bar{K}_a{}^N. \tag{6.20}$$

Note that the last expression is weakly trace free since its trace part is proportional to the rotational components of the Gauß constraint expressed in the reduced variables, which we will give below. For the symplectic reduction and the constraints expressed in the reduced variables, we will only give the results. More details on the calculations can be found in section 9.3, where very similar considerations can be found. We find

$$\frac{1}{2}\overset{(\gamma)}{\pi}{}^{aIJ}\dot{A}_{aIJ} = -E^{\dot{a}J}\left(s\bar{K}_{aJ} + \frac{1}{2\gamma}\epsilon_{IJ}{}^{KL}n^I\Gamma_{aKL} \right.$$
$$\left. -n_J E^I_a\left(-\bar{K}^{\mathrm{tr}}_I + \frac{1}{2\gamma}\epsilon_{IK}{}^{LM}E^{bK}\Gamma_{bLM}\right)\right)$$
$$=: E^{aJ}\dot{A}_{aJ}, \tag{6.21}$$

$$\mathscr{G}^{IJ} = 2\left(E^{a[I}A^{J]}_a + \frac{1}{2\gamma}\epsilon^{IJKL}\partial_a\left(n_K E^a_L\right)\right), \tag{6.22}$$

$$\mathscr{H}_a \approx E^{bI}\partial_a A_{bI} - \partial_b\left(E^{bI}A_{aI}\right). \tag{6.23}$$

$$\mathscr{H} \approx E^{[a|I}E^{b]J}\left(A_{aI} - \frac{1}{2\gamma}\epsilon_{MI}{}^{KL}n^M\Gamma_{aKL}\right)\left(A_{bJ} - \frac{1}{2\gamma}\epsilon_{NJ}{}^{AB}n^N\Gamma_{bAB}\right)$$
$$- \frac{s}{2}E^{aI}E^{bJ}R_{abIJ}, \tag{6.24}$$

where in the expression for $\mathscr{H}_a$, we dropped the term $+\frac{1}{2}A_{aIJ}\mathscr{G}^{IJ}$. In $\mathscr{H}$, we used (6.20) and the fact that $\bar{K}^I_{\mathrm{tr}}$, being proportional to the the boost part of the Gauß constraint, weakly vanishes, and therefore $K_{aIJ} \approx 2\overset{(\gamma)}{\mathscr{M}}{}_{IJ}{}^{MN}n_M\bar{K}_{aN}$. Note that $A_{aI}$ cannot transform as a connection under $\mathscr{G}^{IJ}$ at this point. Only after introducing time gauge $n^I = \delta^I_0 \Leftrightarrow E^{a0} = 0$ and solving its second class partner, the boost part of the Gauß constraint $G^{0i} = -E^{ai}A^0_a$, we finally arrive at the Ashtekar Barbero formulation

$$E^{aI}\dot{A}_{aI} \to E'^{ai}\dot{A}'_{ai}, \tag{6.25}$$

$$G^{IJ} \to \frac{1}{2}\epsilon^{ikj}G_{ij} = \partial_a E'^{ak} + \epsilon^{kij}A'_{ai}E'^a_j, \tag{6.26}$$

$$\mathscr{H}_a \to \mathscr{H}_a = E'^{bi}\partial_a A'_{bi} - \partial_b\left(E'^{bi}A'_{ai}\right), \tag{6.27}$$

$$\mathscr{H} \to \mathscr{H} = E^{[a|i}E^{b]j}\left(A_{ai} - \frac{s}{2\gamma}\epsilon_i{}^{kl}\Gamma_{akl}\right)\left(A_{bj} - \frac{s}{2\gamma}\epsilon_j{}^{mn}\Gamma_{bmn}\right) - \frac{s}{2}E^{ai}E^{bj}R_{abij}$$
$$\approx \frac{1}{2}\epsilon_{ijk}F'_{ab}{}^kE'^{ai}E'^{bj} - \frac{1}{2}\left(1 - s\gamma^2\right)\epsilon_{ijk}R_{ab}{}^kE'^{ai}E'^{bj}, \tag{6.28}$$

where terms proportional to the Gauß constraint have been dropped in the expression for the Hamilton constraint. Here, we introduced the primed variables $A'_{ai} := -s\gamma A_{ai}$ and $E'^{bj} := -\frac{s}{\gamma}E^{bj}$ and used the notation $F'_{abij} = \epsilon_{ikj}F'_{ab}{}^k$.

Let us very briefly point out the special features of this formulation:
1.) For the choice $\gamma^2 = s$, the (density weight 2) Hamiltonian constraint (6.28) takes a very simple form. However, for the physically relevant Lorentzian spacetime signature,

the corresponding connection variable is necessarily complex.
2.) For $\gamma \in \mathbb{R}/\{0\}$, the connection and its conjugate momentum are real. They satisfy standard Poisson bracket relations and the gauge group is compact. These properties lie at the heart of the loop quantisation programme, as we will see later. The cost is a more complicated Hamiltonian constraint, which however can be dealt with [30].

| Variable | Dof | $1^{\text{st}}$ cl. constraints | Dof (count twice!) |
|---|---|---|---|
| ${E^a}_i$ | 9 | $\mathscr{H}$ | 1 |
| ${A_b}^j$ | 9 | $\mathscr{H}_a$ | 3 |
| | | $\mathscr{G}^{ij}$ | 3 |
| Sum: | 18 | Sum: | 14 |

**Table 6.2:** Ashtekar Barbero formulation: counting of degrees of freedom

## 6.3 From ADM to Ashtekar Barbero variables: Integrability of the spin connection

Of course, this formulation also can be obtained starting from the ADM phase space, which we want to comment on briefly. Following [62] where this issue is nicely discussed, the passage can be nicely separated in three steps:

1. $\{q_{ab}, P^{cd}; \mathscr{H}, \mathscr{H}_a\} \rightarrow \{K_{ai}, E^{bj}; \mathscr{H}, \mathscr{H}_a, \mathscr{G}^{ij}\}$: Extend the ADM constrained Hamiltonian system by introducing a densitised vielbein, i.e. an SO(3) gauge symmetry.

2. $\{K_{ai}, E^{bj}\} \rightarrow \{{}_{(\gamma)}K_{ai}, {}^{(\gamma)}E^{bj}\}$: Perfom a constant Weyl rescaling with the Barbero Immirzi parameter $\gamma$, ${}^{(\gamma)}E^{bj} = \gamma E^{bj}$ and ${}_{(\gamma)}K_{ai} = \frac{1}{\gamma}K_{ai}$. This transformation is, of course, canonical.

3. $\{{}_{(\gamma)}K_{ai}, {}^{(\gamma)}E^{bj}\} \rightarrow \{A_{ai}, {}^{(\gamma)}E^{bj}\}$: Perform a canonical transformation to so(3) connection variables, where

$$A_{ai} := \Gamma_{ai} + {}_{(\gamma)}K_{ai} = \frac{1}{2}\epsilon_{jik}{\Gamma_a}^{jk} + {}_{(\gamma)}K_{ai}. \tag{6.29}$$

Note that the this last step is clearly singling out $D = 3$: To define $A_{ai}$, we take a linear combination of the variable conjugate to the vielbein with the spin connection $\Gamma_{aij}$.

Now, ${}_{(\gamma)}K_{ai}$, like the vielbein, transforms in the defining representation of so($D$) while $\Gamma_{aij}$ transforms in the adjoint representation. Only for $D = 3$, the defining and the adjoint representation of so($D$) are equivalent, which enables us to define $A_{ai}$ like above.

Step 1. has already been discussed in section 3.3, and the second is trivial. Therefore, what is left to check is if the transformation in step 3. really is canonical. Since the spin connection $\Gamma_{aij}$ (cf. appendix C) is a function of $E^{ai}$ and its derivatives, what is non trivial is the Poisson self commutativity of the newly introduced so(3) connection $A_{ai}$. As we will see later when quantising, this self commutativity is central to the LQG programme, since $A_{ai}$ (or, more precisely, the holonomies) will be represented as multiplication operators, which would be inconsistent otherwise. We have

$$\begin{aligned}\{A_{ai}(x), A_{bj}(y)\} &= \int_\sigma d^3z \left[\frac{\delta A_{ai}(x)}{\delta_{(\gamma)}K_{ck}(z)}\frac{\delta A_{bj}(y)}{\delta^{(\gamma)}E^{ck}(z)} - \frac{\delta A_{ai}(x)}{\delta^{(\gamma)}E^{ck}(z)}\frac{\delta A_{bj}(y)}{\delta_{(\gamma)}K_{ck}(z)}\right] \\ &= \frac{1}{\gamma}\left[\frac{\delta\Gamma_{bj}(y)}{\delta E^{ai}(x)} - \frac{\delta\Gamma_{ai}(x)}{\delta E^{bj}(y)}\right], \end{aligned} \tag{6.30}$$

which is the integrability condition for $\Gamma_{ai}$. In the following, we will prove that $F := \int_\sigma d^3x \Gamma_{ai}E^{ai}$ indeed is a generating functional for $\Gamma_{ai}$, $\delta F = \int_\sigma d^3x \Gamma_{ai}\delta E^{ai}$, and therefore (6.30) vanishes. To this end, consider

$$\begin{aligned}\int_\sigma d^3x\ E^{ai}\delta\Gamma_{ai} &= \frac{1}{2}\int_\sigma d^3x\ \sqrt{q}\epsilon_{jik}e^{ai}\delta\left(e^{bj}D_a e_b{}^k\right) \\ &= \frac{1}{2}\int_\sigma d^3x\ \sqrt{q}\epsilon_{jik}\left(e^{ai}e^{bj}\delta D_a e_b{}^k + e^{ai}(\delta e^{bj})D_a e_b{}^k\right) \\ &= \frac{1}{2}\int_\sigma d^3x\ \sqrt{q}\epsilon_{jik}\left(e^{ai}e^{bj}D_a\delta e_b{}^k + e^{ai}(\delta e_{cl})e^{cj}e^{bk}D_a e_b{}^l\right) \\ &= \frac{1}{2}\int_\sigma d^3x\ \frac{\sqrt{q}}{e}\epsilon^{abc}\left(e_{bk}D_a\delta e_c{}^k + (\delta e_{ck})D_a e_b{}^k\right) \\ &= \frac{1}{2}\int_\sigma d^3x\ \epsilon^{abc}\partial_a\left(\operatorname{sgn} e\ e_{bk}\delta e_c{}^k\right), \end{aligned} \tag{6.31}$$

where in the first step, we just used the definition of $\Gamma_{ai}$ and $E^{ai}$, and in the second step wrote out the two terms stemming from the variation. Note that in the first summand, due to the antisymmetry in $a, b$, the covariant derivative $D_a$ commutes with $\delta$, which explains the first summand in the third line. In the second summand, we used (A.10) to rewrite $\delta e^{bj}$ and $(D_a e_b{}^k)e^{bl} = -(D_a e_b{}^l)e^{bk}$ due to $D_a\delta^{kl} = 0$ and metric compatibility. Finally, we used $\epsilon_{jik}e^{cj}e^{ai}e^{bk} = \frac{1}{e}\epsilon^{cab}$ and that $\operatorname{sgn} e$ classically is a constant. Neglecting the appearing boundary term, we see that the the transformation in step 3 indeed is canonical. Note that if the boundary of $\sigma$ is non-empty, we have to improve the generating functional $F$, cf. e.g. [62, section 4.2] for details. Finally, let us have a look at the constraints (3.35, 3.36, 3.37) when expressed in terms of the new variables. For the Gauß constraint, we have

$$\begin{aligned}\frac{1}{2}\epsilon_{ikj}\mathscr{G}^{ij} &= \epsilon_{ikj}{}^{(\gamma)}E^{ai}{}_{(\gamma)}K_a{}^j \\ &= \epsilon_{ikj}{}^{(\gamma)}E^{ai}{}_{(\gamma)}K_a{}^j + \left(D_a{}^{(\gamma)}E^a{}_k + \epsilon_{kji}\Gamma_a{}^{j(\gamma)}E^{ai}\right) \\ &= D_a{}^{(\gamma)}E^a{}_k + \epsilon_{kji}A_a{}^{j(\gamma)}E^{ai} \\ &=: D^A{}_a{}^{(\gamma)}E^a{}_k =: \mathscr{G}_k, \end{aligned} \tag{6.32}$$

where we added $0 = D^\Gamma{}_a E^a{}_k$ in the second step. $A_{ai}$ now transforms as a connection,

$$\{A_{ai}, \mathscr{G}^k[\lambda_k]\} = -D^A{}_a\lambda_i, \tag{6.33}$$

$$\{{}^{(\gamma)}E^a{}_i, \mathscr{G}^k[\lambda_k]\} = \epsilon_{kij}\lambda^{k(\gamma)}E^{aj}. \tag{6.34}$$

Concerning the spatial diffeomorphism constraint, we already know that it can be rewritten as $\mathscr{H}_a \approx {}^{(\gamma)}E^{bi}\partial_{a(\gamma)}K_{bi} - \partial_b({}_{(\gamma)}K_{ai}{}^{(\gamma)}E^{bi})$ up to terms proportional to the Gauß constraint from (3.32). Replacing ${}_{(\gamma)}K_{ai}$ by $A_{ai}$, we find

$$\begin{aligned}\mathscr{H}_a &\approx {}^{(\gamma)}E^{bi}\partial_a(A-\Gamma)_{bi} - \partial_b((A-\Gamma)_{ai}{}^{(\gamma)}E^{bi}) \\ &= {}^{(\gamma)}E^{bi}\partial_a A_{bi} - \partial_b(A_{ai}{}^{(\gamma)}E^{bi}) - 2{}^{(\gamma)}E^{bi}\partial_{[a}\Gamma_{b]i} + \Gamma_{ai}\partial_b{}^{(\gamma)}E^{bi} \\ &= {}^{(\gamma)}E^{bi}\partial_a A_{bi} - \partial_b(A_{ai}{}^{(\gamma)}E^{bi}) - \frac{1}{2}{}^{(\gamma)}E^b{}_i\epsilon^{jik}R_{ab}{}^{jk} \\ &= {}^{(\gamma)}E^{bi}\partial_a A_{bi} - \partial_b(A_{ai}{}^{(\gamma)}E^{bi}) =: \mathscr{H}'_a, \end{aligned} \tag{6.35}$$

where in the last line, we used that $e^b{}_i\epsilon^{jik}R_{ab}{}^{jk} = 0$ due to the first Bianchi identity. It is easy to see that it generates spatial diffeomorphisms solely. Another form of displaying this constraint often encountered in the literature is

$$\mathscr{H}'_a = {}^{(\gamma)}E^{bi}F_{abi} - A_{ai}\mathscr{G}^i \approx {}^{(\gamma)}E^{bi}F_{abi}. \tag{6.36}$$

Finally, using

$$\begin{aligned}\frac{1}{2}{}^{(\gamma)}E^{ai(\gamma)}E^{bj}F_{abij} &= \frac{1}{2}{}^{(\gamma)}E^{ai(\gamma)}E^{bj}\left(R_{abij} + 2\epsilon_{ikj}D^\Gamma{}_{[a(\gamma)}K_{b]}{}^k + 2_{(\gamma)}K_{[a|j(\gamma)}K_{|b]i}\right) \\ &= \frac{1}{2}{}^{(\gamma)}E^{ai(\gamma)}E^{bj}\left(R_{abij} + 2_{(\gamma)}K_{[a|j(\gamma)}K_{|b]i}\right) + D^\Gamma{}_a({}^{(\gamma)}E^{ai}\mathscr{G}_i), \end{aligned} \tag{6.37}$$

where $F_{abij} = \epsilon_{ikj}(2\partial_{[a}A_{b]}{}^k + \epsilon^{klm}A_{al}A_{bm})$ denotes the curvature tensor of $A_{ai}$, we find for the Hamiltonian constraint

$$\mathscr{H} \approx \frac{1}{2\sqrt{q}}\epsilon_{ijk}{}^{(\gamma)}E^{ai(\gamma)}E^{bj}F_{ab}{}^k - \frac{1}{2\sqrt{q}}\epsilon_{ijk}{}^{(\gamma)}E^{ai(\gamma)}E^{bj}\left(1 - \frac{s}{\gamma^2}\right)R_{ab}{}^k. \tag{6.38}$$

Of course, the constraints (6.32, 6.35, 6.38) coincide with (6.26, 6.27, 6.28) up to the different density weight of $\mathscr{H}$ and inversion of the Barbero Immirzi parameter, $\gamma \leftrightarrow \frac{1}{\gamma}$.

## 6.4 CDJ action and the original Ashtekar variables

The successes with Ashtekar's Yang-Mills type variables on the Hamiltonian side nourished interest in formulations of general relativity in terms of a connection, and culminated on the Lagrangian side in the CDJ formulation [171], a formulation restricted to $D = 2, 3$, but almost purely in terms of a connection. In $D = 3$, it is given by

$$S_{CDJ}[A, \underset{\sim}{\eta}] = \frac{1}{8}\int_{\mathscr{M}} d^4X\ \underset{\sim}{\eta} G^{-1}_{ij\,kl}\tilde{\Omega}^{ij}\tilde{\Omega}^{kl}, \tag{6.39}$$

where $\tilde{\Omega}^{ij} := \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}{}^iF_{\rho\sigma}{}^j$, $F_{\mu\nu}{}^i$ is the curvature tensor of the so$(3,\mathbb{C})$ connection $A_\mu{}^i$, $\underset{\sim}{\eta}$ is a scalar density[1] of weight $-1$, and $G^{-1}_{ij\,kl} = \delta_{i(k}\delta_{l)j} - \frac{1}{2}\delta_{ij}\delta_{kl}$ coincidently has the same form as (2.16) for $D = 3$ and therefore, the form of its inverse is given in (2.15). Like the self-dual Palatini action, this action leads directly to a Hamiltonian formulation in terms of complex Ashtekar variables and is therefore an action of general relativity. Treating $x^0$ as time coordinate, we find using $\tilde{\Omega}^{ij} = 4(\dot{A}_a{}^{(i} - D^A{}_aA_0{}^{(i)})B^{a|j)}$, $B^{aj} = \epsilon^{abc}F_{bc}{}^j$,

$$S_{CDJ} = \int_{\mathscr{M}} d^4X\ \frac{1}{2}\underset{\sim}{\eta} G^{ab}_{ij}(\dot{A}_a{}^i - D^A{}_aA_0^i)(\dot{A}_b{}^j - D^A{}_bA_0^j), \tag{6.40}$$

where $G^{ab}_{ij} := 4G^{-1}_{ik\,jl}B^{ak}B^{bl}$. The canonical momenta are given by

$$p_{\underset{\sim}{\eta}} = 0, \tag{6.41}$$

$$\pi_i = 0, \tag{6.42}$$

$$\pi^a{}_i = \underset{\sim}{\eta} G^{ab}_{ij}(\dot{A}_b{}^j - D^A{}_bA_0^j). \tag{6.43}$$

[1] In presence of a cosmological constant, this degree of freedom can be eliminated [173]) and one is left with a formulation solely in terms of a connection.

and, restricting to non-degenerate $B$-fields and denoting $\dot{A}^{ij} := \dot{A}^i_a B^{aj}$, the last equation can be solved for the 6 components corresponding to the symmetric part

$$\dot{A}^{(ij)} = \frac{1}{4\underset{\sim}{\eta}} G^{ij\,kl} B^{-1}_{ak} \pi^a_l + B^{a(i} D^A{}_a A^{j)}_0, \tag{6.44}$$

whereas $\epsilon_{abc} B^{ci}$ constitute the three zero Eigenvectors of the matrix $G^{ab}_{ij}$. In total, we thus have 7 primary constraints

$$\mathscr{H}_a := \frac{1}{2} \epsilon_{abc} \pi^b_i B^{ci} = 0, \tag{6.45}$$

$$\mathscr{C} := \underset{\sim}{p_\eta} = 0, \tag{6.46}$$

$$\mathscr{C}_i := \pi_i = 0. \tag{6.47}$$

We will denote the corresponding velocities which cannot be eliminated in the Hamiltonian and have to be treated as Lagrange multipliers with $\lambda^{ij} := \dot{A}^{[ij]}$, $\lambda := \underset{\sim}{\dot{\eta}}$, $\lambda^i := \dot{A}^i_0$. The action in Hamiltonian form is given by

$$S_{CDJ} = \int dt \int_\sigma d^3x \; \Bigg[ \underset{\sim}{p_\eta} \underset{\sim}{\dot{\eta}} + \pi^i \dot{A}_{0i} + \pi^{ai} \dot{A}_{ai} - \left( \frac{1}{2} \pi^a_i \dot{A}^i_a + \frac{1}{2} \pi^a_i D^A_a A^i_0 + \lambda^i \mathscr{C}_i + \lambda \mathscr{C} \right)_{\dot{A} = \dot{A}(\pi, A)} \Bigg]. \tag{6.48}$$

For the first term in the second line, we find

$$\begin{aligned}
&\pi^a_i \dot{A}^i_a(\pi, A) \\
&= \pi^a_i B^{-1}_{aj} B^{bj} \dot{A}^i_b(\pi, A) \\
&= \pi^a_i B^{-1}_{aj} \left( \dot{A}^{(ij)}(\pi, A) + \dot{A}^{[ij]} \right) \\
&= \pi^a_i B^{-1}_{aj} \left( \frac{1}{4\underset{\sim}{\eta}} G^{ij\,kl} B^{-1}_{bk} \pi^b_l + B^{b(i} D^A{}_b A^{j)}_0 + \lambda^{ij} \right) \\
&= \pi^a_i B^{-1}_{aj} \left( \frac{1}{4\underset{\sim}{\eta}} \left( B^{-1\,(i}_b \pi^{b|j)} - \delta^{ij} B^{-1}_{bk} \pi^{bk} \right) + B^{bj} D^A{}_b A^i_0 - B^{b[j} D^A{}_b A^{i]}_0 + \lambda^{ij} \right) \\
&= \pi^a_i B^{-1}_{aj} \left( \frac{1}{4\underset{\sim}{\eta}} \left( B^{-1\,[j}_b \pi^{b|i]} - 2\delta^{i[j} B^{-1}_{bk} \pi^{b|k]} \right) + B^{bj} D^A{}_b A^i_0 - B^{b[j} D^A{}_b A^{i]}_0 + \lambda^{ij} \right) \\
&= \pi^a_{[i} B^{-1}_{a|j]} \left( \frac{1}{4\underset{\sim}{\eta}} B^{-1\,[j}_b \pi^{b|i]} - B^{b[j} D^A{}_b A^{i]}_0 + \lambda^{ij} \right) + \pi^a_i D^A{}_a A^i_0 + \frac{1}{2\underset{\sim}{\eta}} \pi^{ai} \pi^{bj} B^{-1}_{a[j} B^{-1}_{b|i]}
\end{aligned}$$

$$= 2N^a \mathscr{H}_a + \pi^a_i D^A{}_a A^i_0 - \frac{1}{4\underset{\sim}{\eta} \det B} \pi^{ai} \pi^{bj} \epsilon_{ijk} \epsilon_{abc} B^{ck}. \tag{6.49}$$

In the third line, we used (6.44) and afterwards only reorganised terms until, in the last step, we used $\left(\frac{1}{4\underset{\sim}{\eta}} B^{-1}_b{}^{[j} \pi^{b|i]} - B^{b[j} D^A{}_b A^{i]}_0 + \lambda^{ij}\right) \pi^a_{[i} B^{-1}_{a|j]} = 2N^a \mathscr{H}_a$. Here, we introduced three new Lagrange multiplier fields $N^a$ equivalent to $\lambda^{ij}$ (note that both have three independent components), explicitly given by $\frac{1}{4\underset{\sim}{\eta}} B^{-1}_b{}^{[j} \pi^{b|i]} - B^{b[j} D^A{}_b A^{i]}_0 + \lambda^{ij} =: -\epsilon^{ijk} B^{-1}_{ak} N^a$, and furthermore made use of $B^{-1}_{ck} = \frac{1}{2! \det B} \epsilon_{abc} \epsilon_{ijk} B^{ai} B^{bj}$. Thus, the Hamiltonian is given by

$$H := \int_\sigma d^3x \left[ -\frac{1}{8\underset{\sim}{\eta} \det B} \pi^{ai} \pi^{bj} \epsilon_{ijk} \epsilon_{abc} B^{ck} + \pi^a_i D^A_a A^i_0 + N^a \mathscr{H}_a + \lambda^i \mathscr{C}_i + \lambda \mathscr{C} \right]. \tag{6.50}$$

The stability analysis for $\mathscr{C}$, $\mathscr{C}_i$ immediately yields the secondary constraints

$$\mathscr{G}_i := D^A_a \pi^a_i = 0, \tag{6.51}$$

$$\mathscr{H} := \frac{1}{4} \pi^{ai} \pi^{bj} \epsilon_{ijk} \epsilon_{abc} B^{ck} = 0. \tag{6.52}$$

Analogous to the ADM case, we can solve $\mathscr{C}$, $\mathscr{C}_i$ and treat $A^i_0$, $\underset{\sim}{\eta}$ as Lagrange multipliers. Using $\underset{\sim}{N} := -\frac{1}{2\underset{\sim}{\eta} \det B}$, $\lambda^i := -A^i_0$ and dropping a boundary term, the final form of the Hamiltonian is

$$H = \int_\sigma d^D x \left[ \underset{\sim}{N} \mathscr{H} + \lambda^i \mathscr{G}_i + N^a \mathscr{H}_a \right], \tag{6.53}$$

and the constraints (6.51, 6.45, 6.52) exactly coincide with those of Ashtekar's original (complex in the Lorentzian case) formulation, i.e. with (6.32, 6.36, 6.38) for the choice $\gamma^2 = s$ (up to the density weight of $\mathscr{H}$).

Several actions we considered here as well as corresponding Hamiltonian formulations with their interrelations and their connection to the new variables are given systematically in figure 6.1, with which we will end the first part of this thesis[1].

[1]To simplify the diagram, always just one direction of the relation of the formulations is indicated. E.g., gauge unfixing can be reversed by gauge fixing, and similarly can all other arrows be reversed.

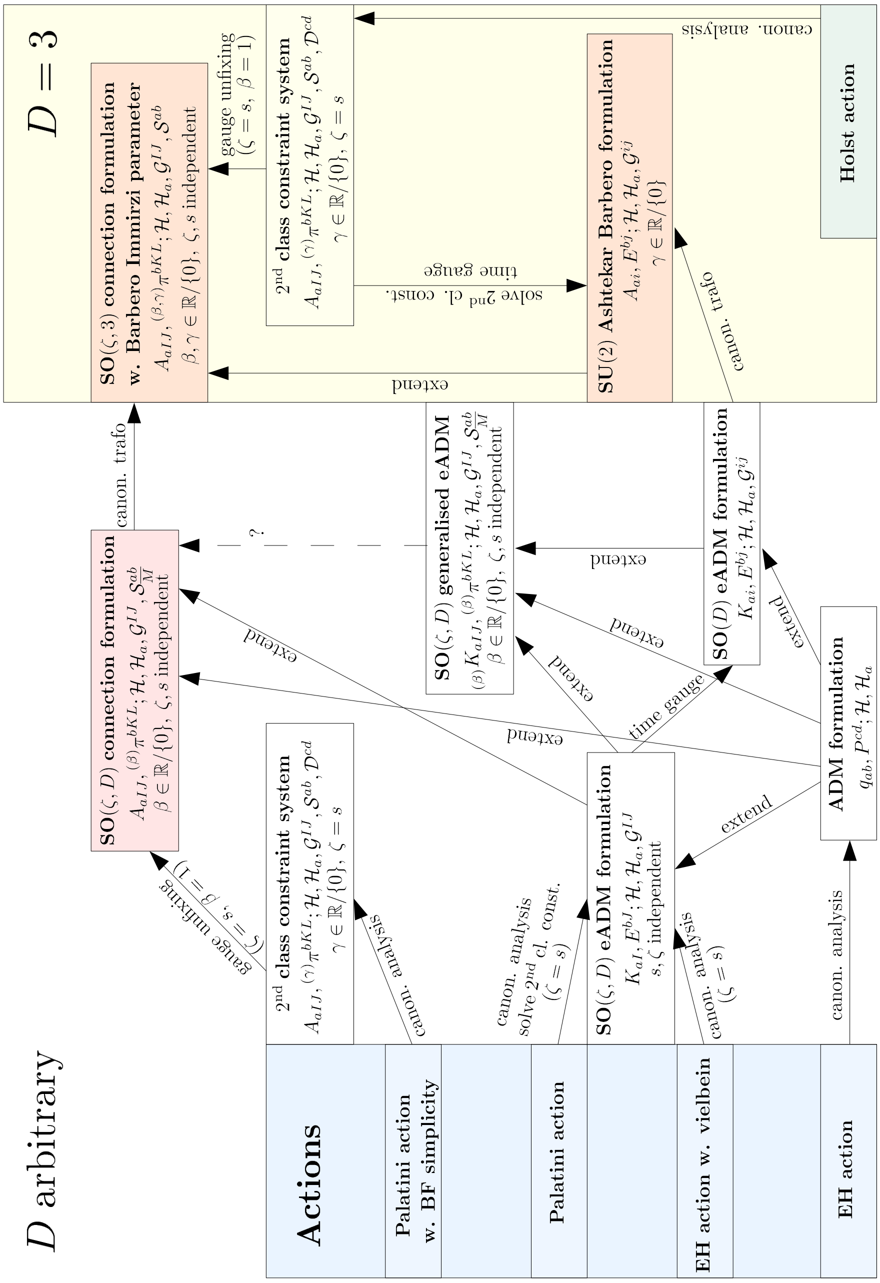


**Figure 6.1: Overview over Hamiltonian formulations** - In this diagram, several actions and Hamiltonian formulations which we studied and relations between them are displayed. In particular, the connection to the new variables are shown. The corresponding canonical variables and constraints are given. First class connection formulations are displayed in a red box, while actions are in a blue background. Formulations restricted to $D = 3$ are coloured in yellow. $\mathrm{SO}(\zeta, D)$ here means $\mathrm{SO}(D+1)$ for $\zeta = 1$ and $\mathrm{SO}(1, D)$ for $\zeta = -1$.

# Part II

# Extension to higher dimensions: The new variables

The following content is taken from [1, 2] with only slight modifications. In chapter 9, some new parts were added.

Let us shortly summarise what we learned so far: As we heard in the introduction 1, the programme of loop quantisation (see e.g. [62] and references therein) requires the gravity theory to be formulated in terms of a gauge theory. The reason for that is that only for theories based on connections and conjugate momenta background independent Hilbert space representations have been found so far, which also support the constraints of the theory as densely defined and closable operators. Of course, a connection formulation is also forced on us if we want to treat fermionic matter as well. A connection formulation for gravity in $D+1>4$ that can be satisfactorily quantised, even in the vacuum case, has not been given so far. For the case $D+1=4$, it was only in 1986 that Ashtekar discovered his new variables for general relativity [13, 14]. The most important property of these variables is that the connection $A$ used has a canonically conjugate momentum $E$ such that $(A, E)$ have standard canonical brackets, in particular the connection Poisson commutes with itself (cf. section 6.4). This is not trivial. Indeed, the naive connection that one would expect from the first order Palatini formulation does not have this crucial property, because the canonical formulation of Palatini gravity suffers from second class constraints as we have seen in section 5.2 and the Palatini connection then has non trivial corresponding Dirac brackets [193].

This prohibited so far to find Hilbert space representations, in particular those of LQG type in which the connection is represented as a multiplication operator, for the Palatini connection (see, however, [194, 198]). The Ashtekar connection does not suffer from this problem because it is the self-dual part of the Palatini connection (or spin connection in the absence of torsion terms). Unfortunately, for the only physically interesting case of Lorentzian signature this Ashtekar connection takes values in the non compact $\mathrm{Sl}(2,\mathbb{C})$ rather than a compact group and again it is very difficult to find Hilbert space representations of gauge theories with non compact structure groups.

As observed by Barbero [16, 17], a possible strategy to deal with this non compactness problem is to use the time gauge and to gauge fix the boost part of $\mathrm{SO}(1,3)$. The resulting connection, which can be seen as the self dual part of the spin connection for

Euclidean signature, is then an SU(2) connection. The price to pay is that the Hamiltonian constraint for Lorentzian signature in terms of these variables is more complicated than in terms of the complex valued ones (cf. section 6.2). However, this does not pose any problems in its quantisation [30]. Using these variables (which also allow a one parameter freedom related to the Barbero Immirzi parameter [16–19]) a rigorous quantisation of general relativity with a unique Hilbert space representation could be derived [20, 21, 28, 29].

A different way to arrive at the same formulation is to start from the geometrodynamics phase space coordinatised by the ADM variables (three metric and extrinsic curvature) and to expand it by introducing (densitised) triads $E$ and conjugate momenta $K$ (basically the extrinsic curvature contracted with the triad, cf. section 3.3). The connection is then the triad spin connection $\Gamma$ plus this conjugate momentum, that is, $A = \Gamma + \gamma K$ where $\gamma$ is the real valued Barbero Immirzi parameter (cf. section 6.3). The first miracle that happens in 3 spatial dimensions is that this is at all possible: While $K$ transforms in the defining representation of SO(3), $\Gamma$ transforms in the adjoint representation of SO(3). But for the case of SO(3), these are isomorphic and enable to define the object $A$. The second miracle that happens in 3 spatial dimensions is that this connection is Poisson self commuting which is entirely non trivial. Notice that in three spatial dimensions, the expansion of the phase space alters the number of configuration degrees of freedom from six per spatial point (described by the three metric tensor) to nine (described by the co-triad). To get back to the original ADM phase space, one therefore has to add three constraints and these turn out to comprise precisely an SU(2) Gauß constraints just as in Yang Mills theory.

It is clear that this strategy can work only in $D = 3$ spatial dimensions: A metric in $D$ spatial dimensions has $D(D+1)/2$ configuration degrees of freedom per spatial point while a $D$-bein has $D^2$. We therefore need $D^2 - D(D+1)/2 = D(D-1)/2$ constraints which is precisely the dimensionality of SO($D$). However, an SO($D$) connection has $D^2(D-1)/2$ degrees of freedom. Requiring that connection and triad have equal amount of degrees of freedom leads to the unique solution $D = 3$. Thus in higher dimensions we need a generalisation of the procedure that works in $D = 3$. Attempts to construct a higher dimensional connection formulation have been undertaken, but few results are available (cf. section 1.3).

In this part, we will derive a connection formulation for higher dimensional general relativity. In the first chapter 7, this will be achieved by using a different extension of the ADM phase space than the one employed in [13, 14]. This new extension of the ADM phase space does not require the time gauge and generalises to any dimension $D > 1$. The result is a Yang Mills theory phase space subject to Gauß, spatial diffeomorphism and Hamiltonian constraint as well as one additional constraint, the simplicity constraint which we already encountered before. The structure group can be chosen to be $\mathrm{SO}(1, D)$ or $\mathrm{SO}(D + 1)$ and the latter choice is preferred for purposes of quantisation. Furthermore, like in the case of Ashtekar Barbero variables, there is a one parameter freedom in choosing the variables. However, in $D = 3$, the new variables and the Ashtekar Barbero variables differ and we will show that the new parameter does not play the role of the Barbero Immirzi parameter.

In chapter 8, we will present how this theory was derived for the first time, which was not by an extension of the ADM phase space but rather by applying the machinery of gauge unfixing [199–202] to the second class constraint system we encountered in section 5.2 when studying the Palatini formulation. Following this line, we can map the second class system to an equivalent first class system which turns out to be identical to the one we obtained following the Hamiltonian route in chapter 7. However, this action based approach has limitations compared to the Hamiltonian approach: There is no Barbero Immirzi like freedom and the structure group is tied to the space time signature, i.e. $\mathrm{SO}(1, D)$ for the physically relevant Lorentzian signature, which makes the approach less favourable with an eye towards quantisation.

Finally, in chapter 9, we will present several possible extensions of the framework we outlined so far: We will show in section 9.1 that the quadratic version of the simplicity constraints can be replaced by the linear version known from spin foam models, which will turn out to be important for supergravity theories later in part IV, in section 9.2 we point out that the theory can be extended to the gauge groups $\mathrm{SO}(p, D + q)$ ($p, q \geq 0$, $p + q \neq 0$) which might be interesting for unified models, in section 9.3 reintroduce the Barbero Immirzi parameter in $D = 3$ to obtain a two parameter family of theories, in section 9.4 study gauge fixing conditions for the simplicity constraints and finally in section 9.5 introduce a first class Hamiltonian formulation with arbitrary internal space and comment on the possibility of turning it into a connection formulation.

# 7

# The new variables - Hamiltonian picture

This chapter is based in part on Peldán's seminal work [103] on the possibility of using higher dimensional gauge groups for gravity as well as on his concept of a hybrid spin connection which naturally appears in the connection formulation of $2+1$ gravity [203]. The idea how to construct a connection formulation also in higher dimensions is the following.

If one starts from the Palatini formulation in $D+1$ spacetime dimensions, then the natural gauge group to consider is $\mathrm{SO}(1,D)$ or $\mathrm{SO}(D+1)$ respectively for Lorentzian or Euclidean gravity respectively. Both groups have dimension $D(D+1)/2$. This motivates to look for a connection formulation of the Hamiltonian framework with a connection $A_{aIJ},\ a=1,..,D;\ I,J=0,..,D$. Such a connection has $D^2(D+1)/2$ degrees of freedom. The corresponding Gauß constraint removes $D(D+1)/2$ degrees of freedom, leaving us with $(D-1)D(D+1)/2$ degrees of freedom. However, a metric in $D$ spatial dimensions has only $D(D+1)/2$ degrees of freedom, which means that we need $D^2(D-1)/2-D$ additional constraints which together with the ADM constraints and the Gauß constraint form a first class system. To discover this constraint, we need an object that transforms in the defining representation of the gauge group. It is given by the spatial (co) vielbein $e_a^I$, $q_{ab}=\eta_{IJ}e_a^I e_b^J$ where $\eta$ has Lorentzian or Euclidean signature respectively. Since the $D$ internal vectors $e_a^I$ are linearly independent, we can complete them to a uniquely defined $(D+1)$-bein by the unit vector $e_0^I=n^I$ where

$\eta_{IJ} e^I_a n^J = 0$. Now the momentum $\pi^{aIJ}$ conjugate to $A_{aIJ}$ is supposed to be entirely determined by $e^I_a$, that is, $\pi^{aIJ} \propto \sqrt{\det(q)} q^{ab} n^{[I} e^{J]}_b$. In other words, $\pi$ is "simple" as in chapters 4 and 5, and we call these constraints therefore simplicity constraints. Since $e^I_a$ has $D(D+1)$ degrees of freedom while $\pi^{aIJ}$ has $D^2(D+1)/2$, these present precisely the required $D^2(D-1)/2 - D$ constraints. Furthermore, from $e^I_a$ one can construct the hybrid spin connection $\Gamma_{aIJ}$ which annihilates $e^I_a$ and the idea, as for Ashtekar's variables, is that $A - \Gamma$ is related to the extrinsic curvature. In order to show that the symplectic reduction of this extension of the ADM phase is given by the ADM phase space, similar to what happens in case of Ashtekar's variables, we need that $\Gamma$ is integrable at least modulo the simplicity constraints which we show to be the case.

It should be stressed that even in $D+1=4$ this extension of the ADM phase space is different from the one employed in LQG: In LQG the Ashtekar-Barbero connection is given by $A^{\text{LQG}}_{ajk} - \Gamma_{ajk} \propto \epsilon_{jkl} K^l_a$, $i,j,k = 1,...,D$, while in our case in the time gauge $n^I = \delta^I_0$ we have $A^{\text{NEW}}_{ajk} - \Gamma_{ajk}$ is pure gauge. Here $\Gamma_{ajk}$ is the spin connection of the corresponding triad. Thus, in the new formulation the information about the extrinsic curvature sits in the $A_{a0j}$ component which is absent in the LQG formulation. We also emphasise that it is possible to have gauge group SO$(D+1)$ even for the Lorentzian ADM phase space. While a Lagrangian formulation is only available when spacetime and internal signature match (cf. 8 or [2]), this opens the possibility to quantise gravity in $D+1$ spacetime dimensions using LQG methods albeit with structure group SO$(D+1)$ and additional (simplicity) constraints.

The chapter is is organised as follows: in section 7.1, we will define the required kinematical structure of a $(D+1)$-dimensional connection formulation of general relativity. We will study in detail the properties of the simplicity constraint and the hybrid spin connection.

In section 7.2, we will postulate an extension of the ADM phase space in terms of a connection and its conjugate momentum subject to the corresponding Gauß constraint and the simplicity constraint discussed before. We will then prove that the symplectic reduction of this extension with respect to both constraints recovers the ADM phase space. There is a one parameter freedom in this extension, similar to but different from the Barbero Immirzi parameter of standard LQG [18].

In section 7.3, we express the spatial diffeomorphism constraint and the Hamiltonian constraint in terms of the new variables and prove that the full set of four types of constraints, namely Gauß, simplicity, spatial diffeomorphism and Hamiltonian constraints, is of first class. This can be done for either choice of $\mathrm{SO}(1, D)$ or $\mathrm{SO}(D+1)$ independently of the spacetime signature. Similar to the situation with standard LQG, the Hamiltonian simplifies when spacetime signature and internal signature match and if one chooses unit Barbero Immirzi like parameter. There is an additional correction term present which accounts for the removal of the pure gauge degrees of freedom affected by the gauge transformations generated by the simplicity constraint.

## 7.1 Kinematical structure of $(D+1)$-dimensional canonical gravity

This section is subdivided into three parts. In the first part we show that simple dimensional counting and natural considerations lead to a unique candidate connection formulation that works in any spacetime dimension $D+1$ and has underlying structure group $\mathrm{SO}(D+1)$ or $\mathrm{SO}(1, D)$ respectively. We also identify the simplicity constraints additional to the Gauß constraint that such a formulation requires and show that, while there is no $D$-bein and no spin connection in such a formulation, there is a generalised $D$-bein and a hybrid connection. The latter is required in order to express the ADM variables in terms of the connection and its conjugate momentum. In the second part we formulate an equivalent expression for the simplicity constraint already known from section 5.2 and discuss its properties and some subtleties. Since we do not assume that the reader necessarily went through part I on preliminaries, we will be rather explicit here. Finally, in the third part we prove a key property of the hybrid connection, namely its integrability modulo simplicity constraints. This will be key to proving in the next section that the symplectic reduction of the extended phase space by Gauß and simplicity constraints recovers the ADM phase space.

### 7.1.1 Preliminaries

Similar to the formulation of standard LQG in $D+1=4$ dimensions, we would like to arrive at a connection formulation of the ADM constrained system which then can be quantised using standard LQG techniques. This requires the corresponding structure group to be compact.

To obtain such a formulation, following Peldán [103], the idea is to extend the ADM phase space by additional degrees of freedom and then to impose additional first class constraints in such a way that the symplectic reduction of the extended system with respect to these constraints coincides with the original ADM phase space. In practical terms, this means that one considers a connection $A_a^\alpha$, i.e. a Lie algebra valued one form with a Lie algebra of dimension $N$ and a conjugate momentum $\pi^a_\alpha$ which is a Lie algebra valued vector density. Here $\alpha, \beta, .. = 1, .., N$. Such a Yang-Mills phase space is subject to a Gauß constraint

$$\mathscr{G}_\alpha = {D^A}_a \pi^a_\alpha = \partial_a \pi^a_\alpha + f_{\alpha\beta}{}^{\gamma}\, A_a^\beta\, \pi^a_\gamma, \tag{7.1}$$

where $f_{\alpha\beta}{}^{\gamma}$ denote the structure constants of the corresponding gauge group. The requirement is then that there is a reduction $(A, \pi) \mapsto q_{ab} := q_{ab}[A, \pi],\ P^{ab} := P^{ab}[A, \pi]$ such that the Poisson brackets of the ADM phase space are reproduced modulo the Gauß constraint and possible additional first class constraints that maybe necessary in order that the correct dimensionality of the reduced phase space is achieved.

The question is of course which group should be chosen depending on $D$ and how to express $q_{ab}, P^{ab}$ in terms of $A_a^\alpha,\ \pi^a_\alpha$. Furthermore, one may ask whether the Gauß constraint is sufficient in order to reduce to the correct number of degrees of freedom or whether there should be additional constraints. Consider first the case that the Gauß constraint is sufficient. Then the extended phase space has $DN$ configuration degrees of freedom of which the Gauß constraint removes $N$. This has to agree with the dimension of the ADM configuration degrees of freedom which in $D$ spatial dimensions is $D(D+1)/2$. It follows $N(D-1) = D(D+1)/2$. Next we need to relate $(A_a^\alpha, \pi^a_\alpha)$ to $(q_{ab}, P^{ab})$. There may be many possibilities for doing so but here we will follow a strategy that is similar to the strategy of standard LQG. We consider some representation $\rho$ of the corresponding Lie group $G$ of dimension $M \geq D$ and introduce generalised $D$-beins $e^I_a$, $I, J, K, ... = 1, .., M$ taking values in this representation with $q_{ab} = e^I_a \eta_{IJ} e^J_b$. The requirement $M \geq D$ is needed in order that $q_{ab}$ can be chosen to be non degenerate and we furthermore require that it is positive definite. Here $\eta$ is a $G$-invariant tensor, i.e. $\rho(g)^I_K \eta_{IJ} \rho(g)^J_L = \eta_{KL}$. The existence of such a tensor already severely restricts the possible choices of $G$ and typically $G$ is simply defined in this way whence $\rho$ will typically be the defining representation of $G$. We extend the covariant derivative $D_a$ to $\rho$ valued objects by asking that $D^\Gamma{}_a$ annihilates the co-$D$-bein

$$D^\Gamma_a e^I_b = \partial_a e^I_b - \Gamma^c_{ab} e^I_c + \Gamma^\alpha_a \, [X^\rho_\alpha]^I{}_J e^J_b = 0, \tag{7.2}$$

with the Levi-Civita connection $\Gamma^c_{ab}$. This equation defines the hybrid (or generalised) spin connection $\Gamma^\alpha_a$ (cf. appendix C). Here the $X^\rho_\alpha$ denote the generators of the Lie algebra of $G$ in the representation $\rho$.

The idea is now that $\tilde{K}_a{}^b := -\frac{1}{2}[A^\alpha_a - \Gamma^\alpha_a]\pi^b_\alpha$ is the expression for the ADM extrinsic curvature $\sqrt{\det(q)} K_a{}^b$, $\quad P_a{}^b = -\frac{1}{2}\sqrt{\det(q)}[K_a{}^b - \delta^b_a K_c{}^c]$, in terms of the new variables. However, there are several caveats. First of all, it is not clear that (7.2) has a non-trivial solution: These are $D^2 M$ equations for $DN$ coefficients $\Gamma^\alpha_a$ and thus the system (7.2) could be overdetermined. Secondly, even if a solution exists, $\Gamma^\alpha_a$ will be a function of $e^I_a$ while we need to express it in terms of the momentum $\pi^a_\alpha$ conjugate to $A^\alpha_a$. If there is no other constraint than the Gauß constraint, then $\pi^a_\alpha$ itself must be already determined in terms of $e^I_a$ which implies that $M = N$: The representation $\rho$ has the same dimension as the adjoint representation of the Lie group. If one scans the classical Lie groups, then the only case where the defining representation and the adjoint representation have the same dimension (and are in fact isomorphic) is SO(3) or SO(1, 2) respectively, whence $N = 3$. In this case, the equation $N(D-1) = D(D+1)/2$ has the solutions $D = 2$ and $D = 3$ which can be shown to be the only solutions to this equation on the positive integers.

In order to go beyond $D = 3$, we therefore need more constraints. We consider now the case of the choice $G = \mathrm{SO}(M+1)$ or $G = \mathrm{SO}(1, M)$ which is motivated by the fact that these Lie groups underly the Palatini formulation of general relativity in $M + 1$ spacetime dimensions. Following Peldán's programme, other choices may be leading, conceivably, to canonical formulations of GUT theories (cf. section 9.2). For this choice, we obtain $N = M(M+1)/2$ and thus (7.2) presents $D^2(M+1)$ equations for $DM(M+1)/2$ coefficients. Explicitly,

$$\partial_a e_b^I - \Gamma^c_{ab} e_c^I + \Gamma_a^{IJ}\, e_{bJ} = 0, \tag{7.3}$$

where all internal indices are moved with $\eta$. Since $\Gamma_{a(IJ)} = 0$, we obtain the consistency condition

$$e_{(cI} \partial_a e^I_{b)} - \Gamma_{(c|a|b)} = 0, \tag{7.4}$$

where $q_{ab} = e_a^I e_{bI}$ was used. It is not difficult to see that (7.4) is in fact identically satisfied. Therefore the $D^2(M+1)$ equations (7.3) are not all independent, there are $D^2(D+1)/2$ identities (7.4) among them, reducing the number of independent equations to $D^2[M+1-\frac{1}{2}(D+1)]$ for $DM(M+1)/2$ coefficients $\Gamma_{aIJ}$. Equating the number of independent equations to the number of equations yields a quadratic equation for $M$ with the two possible roots $M = D$ and $M = D-1$. In the second case $e_a^I$ is an ordinary $D$-bein and $\Gamma_{aIJ}$ its ordinary spin connection. In the former case we obtain the hybrid spin connection mentioned before.

Let us discuss the cases SO($D$) and SO($D+1$) separately (the discussion is analogous for SO($1, D-1$) and SO($1, D$) except that SO($1, D-1$) does not allow for a positive definite $D$ metric and therefore must be excluded anyway). In the case of SO($D$) we have $D^2(D-1)/2$ configuration degrees of freedom and $D(D-1)/2$ Gauß constraints. In order to match the number of ADM degrees of freedom, we therefore need $S = D^2(D-1)/2 - D(D-1)/2 - D(D+1)/2 = D^2(D-3)/2$ additional constraints. These must be imposed on the momentum $\pi^{aIJ}$ conjugate to $A_{aIJ}$ and require that $\pi^{aIJ}$ is already determined by $e_a^I$. Now $e_a^I$ has $D^2$ degrees of freedom while $\pi^{aIJ}$ has $D^2(D-1)/2$ so that exactly $S$ degrees of freedom are superfluous. However, there is no way to to build an object $\pi^{aIJ}$ with $\pi^{a(IJ)} = 0$ from $e_a^I$: In order to match the density weight we can consider $E^{aI} = \sqrt{\det(q)} q^{ab} e_b^I$, but we cannot algebraically build another object $v^I$ from $e_a^I$ without tensor index in order to define $\pi^{aIJ} = v^{[I} E^{a|J]}$.

The only solution is that there are no superfluous degrees of freedom, which leads back to $D = 3$. Now consider SO($D+1$). In this case we have $D^2(D+1)/2$

configuration degrees of freedom and $D(D+1)/2$ Gauß constraints requiring $S = D^2(D+1)/2 - D(D+1)/2 - D(D+1)/2 = D^2(D-1)/2 - D$ additional constraints. The number of superfluous degrees of freedom in $\pi^{aIJ}$ as compared to $e_a^I$ is now also precisely $S = D^2(D+1)/2 - D(D+1)$. In contrast to the previous case, however, now it is possible to construct an object without tensor indices: If we assume that the $D$ internal vectors $e_a^I,\ a = 1, .., D$ are linearly independent then we construct the common normal

$$n_I := \frac{1}{D!} \frac{1}{\sqrt{\det(q)}} \epsilon^{a_1..a_D} \epsilon_{IJ_1..J_D} e_{a_1}^{J_1} \,..\, e_{a_D}^{J_D}, \tag{7.5}$$

which satisfies $e_a^I n_I = 0,\ n_I n^I = \zeta$ where $\zeta = 1$ for SO$(D+1)$ and $\zeta = -1$ for SO$(1, D)$. Notice that $n_I$ is uniquely (up to sign) determined by $e_a^I$. We may now require that

$$\pi^{aIJ} = 2\sqrt{\det(q)} q^{ab} n^{[I} e_b^{J]} =: 2n^{[I} E^{a|J]}. \tag{7.6}$$

These are the searched for constraints on $\pi^{aIJ}$ and constitutes our candidate connection formulation for general relativity in arbitrary spacetime dimensions $D + 1 \geq 3$. Since they require $\pi$ to come from a generalised $D$-bein, we call them *simplicity constraints*. These are indeed exactly the constraints we found in the Palatini theory in chapter 4. Notice that $D^2(D-1)/2 - D = 0$ for $D = 2$. Indeed, $2+1$ gravity is naturally defined as an SO$(1, 2)$ or SO$(3)$ gauge theory.

### 7.1.2 Properties of the simplicity constraints

The form of the constraint (7.6) is not yet satisfactory because the constraint should be formulated purely in terms of $\pi^{aIJ}$. The same requirement applies to the hybrid connection to which we will turn in the next subsection. Of course, the simplicity constraint which we construct here will coincide with the one from section 5.2, but we will give a slightly different view here.

Given $\pi^{aIJ}$ and any unit vector $n_I$ we may define $E^{aI}[\pi, n] := -\zeta\pi^{aIJ} n_J$. This object then automatically satisfies $E^{aI} n_I = 0$. Furthermore we may define the transversal projector

$$\bar{\eta}_J^I[n] := \delta_J^I - \zeta n^I n_J \;\;\Rightarrow\;\; \bar{\eta}_J^I\, n^J = 0 \tag{7.7}$$

and define

$$\bar{\pi}^{aIJ} := \bar{\eta}^I_K[n]\;\bar{\eta}^J_L[n]\;\pi^{aKL}. \tag{7.8}$$

As before, all tensors with purely transversal components will carry an overbar. We obtain the decomposition

$$\pi^{aIJ} = \bar{\pi}^{aIJ} + 2n^{[I}E^{a|J]}. \tag{7.9}$$

It appears that the simplicity constraint now is equivalent to $\bar{\pi}^{aIJ} = 0$. However, there are two subtleties: First, at this point $n^I$ is an extra structure next to $\pi^{aIJ}$ which is required to define (7.8). Therefore the decomposition (7.9) is not intrinsic and $n^I$ appears as an extra degree of freedom. It is therefore necessary to give an intrinsic definition of $n^I$. Next, suppose that we have achieved to do so, then $\bar{\pi}^{aIJ}$ constitute $D^2(D-1)/2$ degrees of freedom rather than the required $D^2(D-1)/2 - D$ while due to $E^a_I n^I = 0$ the $E^a_I$ constitute only $D^2$ degrees of freedom rather than $D(D+1)$.

To remove these subtleties, it is cleaner to adopt the following point of view: we consider $D+1$ vector densities $E^a_I$ to begin with such that the corresponding $D(D+1)$-matrix has maximal rank. From these we can construct the densitised inverse metric

$$qq^{ab}[E] := E^a_I E^b_J \eta^{IJ}, \tag{7.10}$$

which we require to have Euclidean signature as well as their common normal

$$n_I[E] := \frac{1}{D!\,\sqrt{\det(q[E])}^{D-1}}\;\epsilon_{a_1..a_D}\;\epsilon_{IJ_1..J_D}E^{a_1J_1}..E^{a_DJ_D}, \tag{7.11}$$

which is now considered as a function of $E$. Notice that $n_I n^I = \zeta$. Therefore, also $\bar{\eta}^I_J = \bar{\eta}^I_J[E]$ is a function of $E$. We can again apply the decomposition (7.9) and now have cleanly deposited the searched for degrees of freedom into $E^a_I$. However, while $n^I$ is now intrinsically defined via $E^a_I$, the constraints $\bar{\pi}^{aIJ} = 0$ are still $D$ to many. We should remove $D$ additional degrees of freedom from $\bar{\pi}^{aIJ}$. To do so we impose a tracefree condition. Consider the object

$$E^I_a := \frac{1}{q}q_{ab}[E]E^{bI}. \tag{7.12}$$

It follows easily from the definitions that

$$E_a^I E_I^b = \delta_a^b, \quad E_a^I E_J^a = \bar{\eta}_J^I. \tag{7.13}$$

Consider the tracefree, transverse projector

$$\mathbb{P}_{\text{tf}}^{aIJ}{}_{bKL}[E] := \delta_b^a \bar{\eta}_{[K}^I \bar{\eta}_{L]}^J - \frac{2}{D-1} E^{a[I} \, E_{b[K} \, \bar{\eta}_{L]}^{J]}. \tag{7.14}$$

Then for any tensor $\pi^{aIJ}$ we have with $\bar{\pi}_{\text{tf}}^{aIJ} = \mathbb{P}_{\text{tf}}^{aIJ}{}_{bKL}\pi^{bKL}$ that

$$\bar{\pi}^J := E_{aI}\bar{\pi}_{\text{tf}}^{aIJ} = 0 \tag{7.15}$$

and $\bar{\pi}_{\text{tf}}^{aIJ} n_I = 0$. Notice that $\bar{\pi}_{\text{tf}}^{aIJ}$ has only $D^2(D-1)/2 - D$ degrees of freedom independent of $E_I^a$.

We therefore consider in what follows tensors $\pi^{aIJ}$ of the following form

$$\pi^{aIJ}[E, \bar{S}_{\text{tf}}] := \bar{S}_{\text{tf}}^{aIJ} + 2n^{[I}[E] \, E^{a|J]}, \tag{7.16}$$

where $\bar{S}_{\text{tf}}$ and $E$ are considered as independent parameters for $\pi$. Notice that $\bar{S}_{\text{tf}}$ can be constructed as $\mathbb{P} \cdot S$ from an arbitrary tensor $S^{aIJ}$. Such tensors can be intrinsically described as follows: given $\pi$, there exists a normal $n_I[\pi]$ such that the following holds: Define $E_I^a[\pi, n] = -\zeta \pi^{aIJ} n_J$ and $\bar{\pi}^{aIJ}[\pi, n]$ as above. Then automatically

$$\bar{\pi}^J[\pi, n] := \bar{\pi}^{aIJ}[\pi, n] Q_{ab}[\pi, n] E_I^b[\pi, n] = 0. \tag{7.17}$$

This is a set of $D$ independent (since automatically $\bar{\pi}^I n_I = 0$ no matter what $n^I$ is), non-linear equations for the $D$ independent (due to the normalisation $n_I n^I = \zeta$) components of $n^I$. In the original work [2], we studied this non trivial system of equations further and showed that it can possibly be solved by fixed point methods. At present we do not know whether at least tensors $\pi^{aIJ}$ subject to the condition that $\zeta \pi^{aIJ} \pi^b_{IJ}/2$ is positive definite always allow for such a solution $n^I$, however, we know that the number of possible solutions is always finite because we can transform (7.17) into a system of polynomial equations. In what follows, we will assume that the solution $n^I[\pi]$ is in fact unique by suitably restricting the set of allowed tensors $\pi^{aIJ}$. This could imply that the set of such tensors no longer has the structure of a vector space which however does not pose any problems for what follows.

On the other hand, we can prove the following for general $\pi^{aIJ}$:

**Theorem 2.**
*Let $D \geq 3$ and*[1]

$$\mathscr{S}^{ab}_{\overline{M}} := \frac{1}{4}\epsilon_{IJKL\overline{M}}\pi^{aIJ}\pi^{bKL}, \tag{7.18}$$

*where $\overline{M}$ is any totally skew $(D-3)$-tuple of indices in $\{0, 1, .., D\}$. Then*

$$\mathscr{S}^{ab}_{\overline{M}} = 0 \ \forall \ \overline{M}, \ a, \ b \quad \Leftrightarrow \quad \mathbb{P}_{tf}^{aIJ}{}_{bKL}[\pi, n]\, \pi^{bKL} = 0 \tag{7.19}$$

*for any unit vector $n$ where $\mathbb{P}_{tf}^{aIJ}{}_{bKL}[\pi, n] := [\mathbb{P}_{tf}^{aIJ}{}_{bKL}[E]]_{E=E[\pi,n]}$ and $E^{aI}[\pi, n] = -\zeta\pi^{aIJ}n_J$ and where $\mathbb{P}[E]$ is defined in (7.14). Here we assume that $qq^{ab}[\pi, n] := \pi^{aIK}\pi^{bJL}\eta_{IJ}n_K n_L$ is non degenerate for any (timelike for $\zeta = -1$) vector $n_I$.*

This result implies that although $\mathscr{S}^{ab}_{\overline{M}}$ are $D(D+1)/2$ $\binom{D+1}{4}$ equations which exceeds $D^2(D-1)/2 - D$ for $D > 3$ only $D^2(D-1)/2$ of them are independent. The constraint $\mathscr{S}^{ab}_{\overline{M}} = 0$ does not fix $n^I$ and makes no statement about the trace part $\bar{\pi}^J[\pi, n] = \bar{\pi}^{aIJ}[\pi, n]E_{aI}[\pi, n]$. Given that the theorem holds for any $n$ it is natural to fix $n$ such that the trace part vanishes simultaneously as otherwise we would have only that $\bar{\pi}^{aIJ} = 2E^{a[I}\bar{\pi}^{J]}/(D-1)$ and not $\bar{\pi}^{aIJ} = 0$ or $\pi^{aIJ} = 2n^{[I}E^{a|J]}$ on the constraint surface of the simplicity constraint.

*Proof.*
Obviously

$$\mathscr{S}^{ab}_{\overline{M}} = 0 \quad \Leftrightarrow \quad \epsilon^{IJKL\overline{M}}\mathscr{S}^{ab}_{\overline{M}} = \frac{\zeta}{4}\, 4!\, (D-3)!\, \pi^{a[IJ}\, \pi^{bKL]} = 0. \tag{7.20}$$

Given $\pi$, consider any unit vector $n$ and decompose as in (7.9)

$$\pi^{aIJ} = \bar{\pi}^{aIJ}[\pi, n] + 2n^{[I}E^{a|J]}[\pi, n]. \tag{7.21}$$

Inserting into (7.21), we obtain

$$\pi^{a[IJ}\pi^{bKL]} = \bar{\pi}^{a[IJ}\bar{\pi}^{bKL]} + 4n^{[I}E^{(a|J}\bar{\pi}^{b)KL]} = 0. \tag{7.22}$$

Contracting with $n_I$ yields

$$E^{(a[J}\bar{\pi}^{b)KL]} = 0. \tag{7.23}$$

Contracting further with $E_{aJ}$ yields

$$(D-1)\, [\bar{\pi}^{bKL} - \frac{2}{D-1}E^{b[K}\bar{\pi}^{aJ|L]}E_{aJ}] = (D-1)\, \mathbb{P}_{\rm tf}^{bKL}{}_{aIJ}[\pi, n]\pi^{aIJ} = 0. \tag{7.24}$$

We conclude $\pi^{aIJ} = 2v^{[I}E^{a|J]}$, $v^I = (n^I - \frac{1}{D-1}\bar{\pi}^{bJI}E_{bJ})$ and inserting back into (7.20) we see that it is identically satisfied. □

[1] For $D = 2$ no simplicity constraints are needed since $D^2(D-1)/2 - D = 0$.

The theorem therefore says that on the constraint surface $\pi^{aIJ} = 2v^{[I}E^{a|J]}$ for some vector $v$ which is not necessarily normalised and not necessarily normal to $E^{aI}$ but such that $E^{aI}, v^I$ constitute $D+1$ linearly independent internal vectors. We can however draw, for $\zeta = -1$, some additional conclusion from the requirement that $qq^{ab} = \pi^{aIJ}\pi^b_{IJ}/(2\zeta)$ should have Euclidean signature. First of all, $v^I$ cannot be null since otherwise $qq^{ab} \propto (E^a_I v^I)(E^b_J v^J)$ would be degenerate. If $v^I$ would be spacelike then consider $\tilde{E}^a_I = E^a_I - E^a_J v^J v_I/(v^K v_K)$. It follows $\pi^{aIJ} = 2v^{[I}\tilde{E}^{a|J]}$ and $qq^{ab} \propto \tilde{E}^{aI}\tilde{E}^b_I$. Since $v^I, \tilde{E}^a_I$ constitutes a $(D+1)$-bein and $v^I$ is spacelike while $\eta$ is Lorentzian, also $qq^{ab}$ would need to be Lorentzian. Hence $v^I$ must in fact be timelike for $\zeta = -1$.

We may therefore absorb for either signature the normalisation of $v$ into $E^a_I$ and define $n_I := v_I/\sqrt{\zeta v_J v^J}$ as well as $\tilde{E}^{aI} = \sqrt{\zeta v_K v^K} E^{aJ}\bar{\eta}^I_J$. Then $2v^{[I}E^{a|J]} = 2n^{[I}\tilde{E}^{a|J]}$ with $\tilde{E}^{aI} n_I = 0$, $n^I n_I = \zeta$. Therefore, the constraint surface defined via (7.18) is the same as the one given by $\bar{\pi}^{aIJ}_{\text{tf}}$ above, where we assumed that $\pi$ is of the form (7.16) and constitutes the unique decomposition of $\pi^{aIJ}$ with no trace part. In what follows, we will use the simplicity constraint in the form (7.18). However, it will be convenient to have the presentation (7.16) at one's disposal when we work off the constraint surface.

Notice that the proof given above also in the case $D = 3$ does not allow for a "topological sector" $\pi^{aIJ} = \epsilon^{IJKL} n^K E^{aL}$ or "degenerate sector" due to the non degeneracy assumption. This assumption is dropped in the alternative proof in [2] which is based on [169] which is why the topological sector does appear there.

### 7.1.3 Integrability of the hybrid connection modulo simplicity constraint

The hybrid connection is defined via (7.3) on the constraint surface $\mathscr{S}^{ab}_M = 0$. We want to define an extension off the constraint surface such that the resulting expression is integrable, i.e. is the functional derivative $\Gamma_{aIJ} = \delta F/\delta \pi^{aIJ}$ of a generating functional $F = F[\pi]$. To that end, we need the explicit expression of $\Gamma_{aIJ}$ in terms of $e^I_a$, which is given in appendix C. Here, we will provide a detailed derivation thereof.

To begin with, we notice that $D^{\text{H}}_a n^I = 0$. To see this we consider its $D+1$ independent components $n_I D^{\text{H}}_a n^I = \frac{1}{2} D^{\text{H}}_a (n^I n_I) = 0$ and $e^I_b D^{\text{H}}_a n^I = -n^I D^{\text{H}}_a e^I_b = 0$. We decompose

$$\Gamma_{aIJ} = \bar{\Gamma}_{aIJ} + 2n_{[I}\bar{\Gamma}_{a|J]}, \ \bar{\Gamma}_{aI} = -\zeta \Gamma_{aIJ} n^J \tag{7.25}$$

and further

$$\bar{\Gamma}_{aIJ} = \bar{\Gamma}_{abc} e^b_I e^c_J, \ \ \bar{\Gamma}_{aI} = \bar{\Gamma}_{ab} e^b_I, \tag{7.26}$$

where, as before, $e^b_I = q^{ab} e_{bI}, \ q^{ac} q_{cb} = \delta^a_b, \ \ q_{ab} = e^I_a e_{bI}$. We find

$$\bar{\Gamma}_{ab} = -\zeta n_I \partial_a e^I_b, \ \ \bar{\Gamma}_{abc} = \Gamma_{bac} - e_{bI} \partial_a e^I_c, \tag{7.27}$$

where $\Gamma_{bac} = q_{bd} \Gamma^d_{ac}$ is the Levi-Civita connection. Combining these formulae, we obtain

$$\begin{aligned} \Gamma_{aIJ}[E] &= -[\eta_{K[I} + \zeta n_K \ n_{[I]} e^b_{J]} \partial_a e^K_b + \Gamma^b_{ac} e_{b[I} e^c_{J]} \\ &= \zeta n_{[I} \partial_a n_{J]} + e_{b[I} \partial_a e^b_{J]} + \Gamma^b_{ac} e_{b[I} e^c_{J]}, \end{aligned} \tag{7.28}$$

where we used here and will also use frequently later $n_K \partial_a E^{bK} = -E^{bK} \partial_a n_K$, $n^K \partial_a n_K = 0$ and $n_{[I} \bar{\eta}^K_{J]} = n_{[I} \eta^K_{J]}$.

To write $\Gamma_{aIJ}$ in terms of $\pi^{aIJ}$, we notice the following weak identities modulo the simplicity constraint, that is $\pi^{aIJ} \approx 2n^{[I} E^{a|J]}$,

$$\begin{aligned} \pi^{aIJ} \pi^b_{IJ} &\approx 4n^{[I} E^{a|J]} \ n_{[I} E^b_{J]} = 2\zeta E^{aI} E^b_I = 2\zeta q q^{ab}, \\ \frac{1}{q} q_{ab} \pi^{aKI} \pi^b_{KJ} &\approx [n^K E^{aI} - n^I E^{aK}] \ [n_K E_{aJ} - n_J E_{aK}] \\ &= D n^I n_J + \zeta \bar{\eta}^I_J = (D-1) n^I n_J + \zeta \eta^I_J, \\ E^{a[I} n^{J]} &= -\zeta \pi^{a[I|L} \ n^{J]} n_L, \\ \frac{1}{q} q_{bd} \pi^{dK} \ {}_{[I} \ \pi^c_{K|J]} &\approx [n^K E_{b[I} - E^K_b n_{[I]} \ [E^c_{J]} n_K - n_{J]} E^c_K] = \zeta \ E_{b[I} E^c_{J]} = \zeta \ e_{b[I} e^c_{J]}, \\ \frac{1}{q} q_{bc} \pi^{bK} \ {}_{[I} \partial_a \pi^c_{K|J]} &\approx [n^K E_{c[I} - E^K_c n_{[I]} \ \partial_a \ [E^c_{J]} n_K - n_{J]} E^c_K] \\ &= -n^K E_{c[I} \ [n_{J]} (\partial_a E^c_K) - (\partial_a E^c_{J]}) n_K] \\ &\quad + E^K_c n_{[I} \ [(\partial_a n_{J]}) E^c_K - E^c_{J]} (\partial_a n_K)] \\ &= (D-1) n_{[I} (\partial_a n_{J]}) + E_{c[I} n_{J]} E^c_K (\partial_a n^K) + \zeta E_{c[I} (\partial_a E^c_{J]}) \\ &= (D-2) n_{[I} (\partial_a n_{J]}) + \zeta E_{c[I} (\partial_a E^c_{J]}) \\ &= (D-2) n_{[I} (\partial_a n_{J]}) + \zeta e_{c[I} (\partial_a e^c_{J]}), \\ \bar{\eta}^K_I \bar{\eta}^L_J \frac{1}{q} q_{bd} \pi^{dM} \ {}_{[K} \ \partial_a \pi^c_{|M|L]} &\approx \zeta \bar{\eta}^K_{[I} \bar{\eta}^L_{J]} e_{b[K} \partial_a e^b_{L]} \\ &= \zeta e_{b[I} \partial_a e^b_{J]} - n_{[I} \partial_a n_{J]}. \end{aligned} \tag{7.29}$$

Consider the quantities

$$T_{aIJ} := \pi_{bK[I}\partial_a\pi^{bK}{}_{J]}, \;\; T^c_{bIJ} := \pi_{bK[I}\pi^{cK}{}_{J]}, \tag{7.30}$$

where $\pi_{aIJ} = \frac{1}{q}q_{ab}\pi^b_{IJ}$. Then

$$(D-1)n_{[I}\partial_a n_{J]} = T_{aIJ} - \bar{T}_{aIJ}, \;\; (D-1)\zeta e_{b[I}\partial_a e^b_{J]} = T_{aIJ} + (D-2)\bar{T}_{aIJ}. \tag{7.31}$$

Inserting (7.30) and (7.31) into (7.28) then leads to the explicit expression

$$\Gamma_{aIJ}[\pi] = \frac{2\zeta}{D-1}T_{aIJ} + \frac{\zeta(D-3)}{D-1}\bar{T}_{aIJ} + \zeta\Gamma^b_{ac}T^c_{bIJ}. \tag{7.32}$$

Expressing $\Gamma^b_{ac}$ in terms of $qq^{ab} = \pi^{aIJ}\pi^b_{IJ}/(2\zeta)$, this determines $\Gamma_{aIJ}$ completely in terms of $\pi^{aIJ}$ if we simply replace the $\approx$ signs in (7.29) by $=$ signs and take the left hand sides as definitions for the right hand sides.

It transpires that $\Gamma_{aIJ}$ is a rational, homogeneous function of degree zero of $\pi$ and its first derivatives which vanishes at $\pi = 0$. Therefore, if $\Gamma_{aIJ}[\pi]$ has a generating functional, then it is given by[1]

$$F'[\pi] = \int d^Dx\, \pi^{aIJ}\,\Gamma_{aIJ}[\pi]. \tag{7.33}$$

Variation of $F'$ with respect to $\pi^{aIJ}$ yields

$$\begin{aligned}
\delta F' &= \int d^Dx\left(\delta\pi^{aIJ}\,\Gamma_{aIJ}[\pi] + \pi^{aIJ}\delta\Gamma_{aIJ}[\pi]\right) \\
&= \int d^Dx\left(\delta\pi^{aIJ}\,\Gamma_{aIJ}[\pi] + \pi^{aIJ}[\delta\Gamma_{aIJ}[E] + \delta\mathscr{S}'_{aIJ}]\right) \\
&= \delta[\int d^Dx\pi^{aIJ}\mathscr{S}'_{aIJ}] + \int d^Dx\left(\delta\pi^{aIJ}\,\Gamma_{aIJ}[\pi] + 2n^{[I}E^{a|J]}\,\delta\Gamma_{aIJ}[E]\right) \\
&\quad + \int d^Dx\,\left(\mathscr{S}^{aIJ}\,\delta\Gamma_{aIJ}[E] - \delta\pi^{aIJ}\,\mathscr{S}'_{aIJ}\right),
\end{aligned} \tag{7.34}$$

where $\mathscr{S}^{aIJ} := \pi^{aIJ} - 2n^{[I}E^{a|J]}$ and $\mathscr{S}'_{aIJ} := \Gamma_{aIJ}[\pi] - \Gamma_{aIJ}[E]$ both vanish on the constraint surface of the simplicity constraint. We see that $F'$ itself cannot be a generating functional but rather

$$F = F' - \int d^Dx\,\pi^{aIJ}\mathscr{S}'_{aIJ}, \tag{7.35}$$

[1] If a one form $\Gamma_M$ is exact, i.e. has potential $U$ with $\Gamma_M = U_{,M}$ then $U(\pi) - U(\pi_0) = \int_{\gamma_{\pi_0,\pi}} \Gamma$ for any path $\gamma_{\pi_0,\pi}$ between $\pi_0$ and $\pi$. If $\Gamma$ is defined at $\pi_0 = 0$ to vanish then choosing the straight path $t \mapsto t\pi$ yields $U(\pi) = \text{const.} + \int_0^1\, dt\pi^M\Gamma_M(t\pi)$.

i.e. $F'$ has to be corrected by a term that vanishes on the constraint surface of the simplicity constraint, however, its variation does not necessarily vanish on that constraint surface. It follows that $\delta F/\delta\pi^{aIJ} = \Gamma_{aIJ} + \tilde{\mathscr{S}}_{aIJ}$ for some $\tilde{\mathscr{S}}_{aIJ}$ which vanishes on the constraint surface of the simplicity constraint provided that

$$\int d^Dx\, n^{[I}E^{a|J]}\delta\Gamma_{aIJ}[E] = \int d^Dx\sqrt{\det(q)}n^{[I}e^{a|J]}\delta\Gamma_{aIJ}[E] = 0. \tag{7.36}$$

This is the key identity that one has to prove. It is the counterpart to the key identity that is responsible for the fact that the Ashtekar connection is Poisson commuting in $D+1 = 4$. The reason for the correction $F' \to F$ is that $\Gamma_{aIJ}[\pi]$ is not strictly integrable but only modulo terms that vanish on the constraint surface of the simplicity constraint.

We proceed with the proof of (7.36). It is easiest to use (7.25) – (7.27). We have, using $n_K\delta n^K = 0$, $n_K\delta e^K_b = -e^K_b\delta n_K$ and that $\bar{\Gamma}_{a(bc)} = 0$,

$$\begin{aligned} n^{[I}e^{a|J]}\delta(2n_{[I}\bar{\Gamma}_{a|J]}) &= 2n^{[I}e^{a|J]}[n_I(\delta\bar{\Gamma}_{aJ}) + \bar{\Gamma}_{aJ}\delta n_I)] \\ &= \zeta e^{aI}(\delta\bar{\Gamma}_{aI}) = -e^{aI}\delta(n_J(\partial_a e^J_b)e^b_I) \\ &= e^{aI}\delta(e^J_b(\partial_a n_J)e^b_I) = e^{aI}\delta(\bar{\eta}^J_I\partial_a n_J) \\ &= e^{aI}D_a(\delta n_I), \end{aligned} \tag{7.37}$$

$$\begin{aligned} n^{[I}e^{a|J]}\delta\bar{\Gamma}_{aIJ} &= n^Ie^{aJ}\delta(\bar{\Gamma}_{abc}e^b_Ie^c_J) \\ &= n^Ie^{aJ}\bar{\Gamma}_{abc}e^c_J(\delta e^b_I) = q^{ac}\bar{\Gamma}_{acb}e^b_I(\delta n^I) \\ &= -[e^a_J(\partial_a e^J_b) - \Gamma^a_{ab}]e^b_I(\delta n^I) \\ &= -[e^a_J[\partial_a(e^b_Ie^J_b) - e^J_b(\partial_a e^b_I) - \Gamma^a_{ab}e^b_I]\,(\delta n^I) \\ &= -[e^a_J\partial_a(\bar{\eta}^J_I) - (D_a e^a_I)](\delta n^I) = [D_a e^a_I]\,[\delta n^I], \end{aligned} \tag{7.38}$$

where $D_a$ is the torsion free covariant differential annihilating $q_{ab}$ as before (it acts only on tensor indices, not on internal ones). We conclude

$$\int d^Dx\; n^{[I}E^{a|J]}\;\delta\Gamma_{aIJ}[E] = \int d^Dx\;\sqrt{\det(q)}D_a[e^a_I\delta n^I] = \int d^Dx\partial_a(E^a_I\delta n^I) = 0 \tag{7.39}$$

for suitable boundary conditions on $E^a_I$ and its variations[1].

[1] For instance one could impose that $n_I$ deviates from a constant by a function of rapid decrease at spatial infinity. Note that the final expression in (7.39) will not vanish but play a central role in part V.

We therefore have established:

**Theorem 3.**
*There exists a functional $F[\pi]$ such that for $\delta n^I$ vanishing sufficiently fast at spatial infinity, we have*

$$\delta F[\pi]/\delta\pi^{aIJ}(x) = \Gamma_{aIJ}[\pi; x) + \mathscr{S}_{aIJ}[\pi; x), \tag{7.40}$$

*where $\mathscr{S}_{aIJ}$ vanishes on the constraint surface of the simplicity constraint, depending at most on its first partial derivatives and $\Gamma_{aIJ}[\pi]$ is the hybrid connection (7.32).*

## 7.2 New variables and equivalence with ADM formulation

We want to construct a $G = \mathrm{SO}(D+1)$ or $G = \mathrm{SO}(1, D)$ canonical gauge theory over $\sigma$ with connection $A_{aIJ}$ and conjugate momentum $\pi^{aIJ}$. In analogy to the treatment in 6.3, we will present the passage in three steps:

1. $\{q_{ab}, P^{cd}; \mathscr{H}, \mathscr{H}_a\} \to \{K_{aIJ}, \pi^{bKL}; \mathscr{H}, \mathscr{H}_a, \mathscr{G}^{IJ}, \mathscr{S}^{ab}_{M}\}$: Extend the ADM phase space to be coordinatized by a denistized "vielbein" $\pi^{aIJ}$ and conjugate variable $K_{aIJ}$ transforming in the *adjoint* of $G = \mathrm{SO}(D+1)$ or $G = \mathrm{SO}(1, D)$, subject to additional Gauß and simplicity constraints.

2. $\{K_{aIJ}, \pi^{bKL}\} \to \{\underset{(\beta)}{K}_{aIJ}, \overset{(\beta)}{\pi}{}^{bKL}\}$: Perform a constant Weyl rescaling with a free parameter $\beta \in \mathbb{R}/\{0\}$.

3. $\{q_{ab}, P^{cd}; \mathscr{H}, \mathscr{H}_a\} \to \{A_{aIJ}, \overset{(\beta)}{\pi}{}^{bKL}; \mathscr{H}, \mathscr{H}_a, \mathscr{G}^{IJ}, \mathscr{S}^{ab}_{M}\}$: Note that, up to now, each step was in close analogy to the ones of section 6.3. Thus, in the third step, we would like to perform a canonical transformation $\{\underset{(\beta)}{K}_{aIJ}, \overset{(\beta)}{\pi}{}^{bKL}\} \to \{A_{aIJ}, \overset{(\beta)}{\pi}{}^{bKL}\}$ to connection variables $A_{aIJ} = \Gamma_{aIJ}[\pi] + \underset{(\beta)}{K}_{aIJ}$, where $\Gamma_{aIJ}[\pi]$ is the hybrid connection (7.32) constructed from $\pi$. However, as we will see, due problems arising since $\Gamma_{aIJ}[\pi]$ is integrable only up to the simplicity constraint, it is hard to prove that this transformation is canonical. Instead, we will give a proof that the symplectic reduction of this Yang Mills phase space with respect to Gauß and simplicity constraint again leads back to the ADM phase space, and therefore the Yang Mills formulation is valid.

**Step 1:** We introduce the variables $K_{aIJ}$, $\pi^{bKL}$ subject to the canonical brackets

$$\begin{aligned} \{K_{aIJ}(x), \pi^{bKL}(y)\} &= 2\delta^b_a \delta^K_{[I} \delta^L_{J]} \delta^{(D)}(x-y), \\ \{K_{aIJ}(x), K_{bKL}(y)\} &= \{\pi^{aIJ}(x), \pi^{bKL}(y)\} = 0, \end{aligned} \tag{7.41}$$

as well as to the Gauß constraint

$$\mathscr{G}^{IJ} := [K_a, \pi^a]^{IJ} = 2K_a^{[I}{}_K \, \pi^{a|K|J]} \tag{7.42}$$

and the simplicity constraint

$$\mathscr{S}^{ab}_{\overline{M}} = \frac{1}{4}\epsilon_{IJKL\overline{M}} \pi^{aIJ} \pi^{bKL}. \tag{7.43}$$

Internal indices as before are moved by the internal metric $\eta$ which is just the Euclidean metric for SO$(D+1)$ ($\zeta = 1$) and the Minkowski metric for SO$(1, D)$ ($\zeta = -1$). We have for $g \in$ SO$(1, D)$ or SO$(D+1)$ that $g^{IJ} g^{KL} \eta_{KL} = \eta^{IJ}$, $\det((g^{IJ})) = 1$. We define a map from this extended phase space with coordinates $(K_{aIJ}, \pi^{aIJ})$ to the coordinates $(q_{ab}, P^{ab})$ of the ADM phase space by the following formulas

$$\det(q) q^{ab} := \frac{1}{2\zeta} \pi^{aIJ} \, \pi^b{}_{IJ}, \tag{7.44}$$

$$\begin{aligned} P^{ab} &:= \frac{1}{4}\left(q^{a[c} K_{cIJ} \pi^{b]IJ} + q^{b[c} K_{cIJ} \pi^{a]IJ}\right) \\ &= \frac{1}{4} G^{ab\,c}{}_d K_{cIJ} \pi^{dIJ}, \end{aligned} \tag{7.45}$$

which should be compared with (3.38). The central result of this section is:

**Theorem 4.**
*i. Gauß and simplicity constraints obey a first class constraint algebra.*
*ii. The symplectic reduction of the extended phase space defined above with respect to Gauß and simplicity constraints coincides with the ADM phase space. More in detail, the functions $q_{ab}[\pi]$, $P^{ab}[K, \pi]$ defined in (7.44, 7.45) are (weak) Dirac observables with respect to Gauß and simplicity constraints and (weakly) obey the standard Poisson brackets*

$$\{q_{ab}(x), P^{cd}(y)\} = \delta^c_{(a}\delta^d_{b)} \, \delta^{(D)}(x-y), \quad \{q_{ab}(x), q_{cd}(y)\} = \{P^{ab}(x), P^{cd}(y)\} = 0$$

*on the constraint surface defined by simplicity and Gauß constraints.*

*Proof.*
i.
Since $\mathscr{S}^{ab}_{\overline{M}}$ only depends on $\pi^{aIJ}$, it Poisson commutes with itself. The Gauß constraint of course generates $G$ gauge transformations under which $\pi$ transforms as a section in an associated vector bundle under the adjoint representation of $G$ and $K$ accordingly. The Poisson algebra of the smeared Gauß constraints is therefore (anti-)isomorphic with the Lie algebra of $G$

$$\{\frac{1}{2}\mathscr{G}^{IJ}[f_{IJ}], \frac{1}{2}\mathscr{G}^{KL}[f'_{KL}]\} = -\frac{1}{2}\mathscr{G}^{IJ}[[f, f']_{IJ}]. \tag{7.46}$$

Under finite Gauß transformations we have

$$\pi^{aIJ} \mapsto [g\pi^a g^{-1}]^{IJ}. \tag{7.47}$$

Since $G = \mathrm{SO}(1, D)$ or $\mathrm{SO}(D+1)$ is unimodular, we obtain

$$\mathscr{S}^{ab}_{\overline{M}} \mapsto \zeta\, g_{\overline{M}}{}^{\overline{N}} \mathscr{S}^{ab}_{\overline{N}}, \qquad g_{\overline{M}}{}^{\overline{N}} = \prod_{i=1}^{D-3} g_{M_i}{}^{N_i}. \tag{7.48}$$

It follows the first class structure $\{\mathscr{G}, \mathscr{G}\} \propto \mathscr{G},\ \ \{\mathscr{G}, \mathscr{S}\} \propto \mathscr{S},\ \ \{\mathscr{S}, \mathscr{S}\} = 0$. This is, of course, what we expected from section 5.2.
ii.
Since both $\pi^{aIJ},\ K_{aIJ}$ transform in the adjoint representation of $G$ it is clear that $Q^{ab} \propto \mathrm{Tr}(\pi^a\pi^b),\ \ K_a^b \propto \mathrm{Tr}(K_a\pi^b)$ are in fact Gauß invariant, possibly modulo the simplicity constraint, and thus are $q_{ab},\ P^{ab}$. Since $\mathscr{S}^{ab}_{\overline{M}}$ and $q_{ab}$ are both constructed from $\pi^{aIJ}$ alone it is clear that they strictly Poisson commute. As for $P^{ab}$ we notice that it is a linear combination of the objects

$$K_a{}^b := -\frac{1}{2}\, K_{aIJ}\pi^{bIJ}, \tag{7.49}$$

with coefficients that depend only on $q_{ab}$. While the notation already suggests that $K_a{}^b$ is related with the extrinsic curvature, note that as it is defined here, $K_a{}^b$ has density weight one. It is therefore sufficient to show that $\{K_a{}^b, \mathscr{S}^{cd}_{\overline{M}}\} \approx 0$. We compute with the smeared simplicity constraint

$$\begin{aligned}\{K_a{}^b(x), \mathscr{S}^{cd}_{\overline{M}}[f^{\overline{M}}_{cd}]\} &= -\frac{1}{8}\int d^Dy\ f^{\overline{M}}_{cd}(y)\ \pi^{bIJ}(x)\ \epsilon_{ABCD\overline{M}}\ \{K_{aIJ}(x), \pi^{cAB}(y)\pi^{dCD}(y)\} \\ &= -2\ f^{\overline{M}}_{cd}(x)\delta_a^{(c}\mathscr{S}^{d)b}_{\overline{M}}(x).\end{aligned} \tag{7.50}$$

It follows that $P^{ab}$ Poisson commutes with the simplicity constraint on its constraint surface.

It remains to verify the ADM Poisson brackets. Since $q_{ab}(x)$ depends only on $\pi^{aIJ}(x)$ we have trivially $\{q_{ab}(x), q_{cd}(y)\} = 0$. Next, we invoke (A.24),

$$\delta q_{ab} = -\frac{\zeta}{q} G^{-1}_{ab\,cd} \pi^{cIJ} \delta\pi^d{}_{IJ}, \tag{7.51}$$

and since $G^{ab\,cd}$ appears in (7.45), it follows in one line that

$$\begin{aligned}\{q_{ab}(x), P^{cd}(y)\} &= -\frac{\zeta}{q(x)} G^{-1}_{ab\,ef}(x)\pi^{eIJ}(x)\{\pi^f{}_{IJ}(x), K_{gKL(y)}\}\frac{1}{4}G^{cd\,g}{}_h \pi^{hKL} \\ &= \delta^{(D)}(x-y)\delta^c_{(a}\delta^d_{b)}.\end{aligned} \tag{7.52}$$

The last bracket is the most complicated. Again using (A.24), we obtain

$$\begin{aligned}\delta P^{ab} &= \frac{1}{4} G^{ab\,c}{}_d \pi^{dIJ} \delta K_{cIJ} \\ &\quad + \frac{1}{4}\left[G^{ab\,c}{}_d \eta^K_{[I}\eta^L_{J]} + \frac{\zeta}{q}\pi^{(b|KL}G^{a)c}{}_{ed}\pi^e{}_{IJ} - \frac{\zeta}{q}\pi^{cKL}G^{ab}{}_{ed}\pi^e{}_{IJ}\right] K_{cKL}\delta\pi^{dIJ},\end{aligned} \tag{7.53}$$

and using this, analogous to the calculation in section 3.3, we obtain

$$\begin{aligned}\{P^{ab}[K,\pi](x), P^{cd}[K,\pi](y)\} &= \int_\sigma d^D z \left[2\frac{\delta P^{ab}[K,\pi](x)}{\delta K_{fIJ}(z)}\frac{\delta P^{cd}[K,\pi](y)}{\delta \pi^{fIJ}(z)} - ab \leftrightarrow cd\right] \\ &= \left[\frac{q}{8}G^{ab\,xf}G^{cd\,e}{}_f \pi_{xIJ}K_e{}^{IJ}\delta^{(D)}(x-y) + \frac{\zeta}{8}G^{ab\,gf}G^{-1\,e(d|}{}_{gf}\pi^{c)}{}_{IJ}K_e{}^{IJ}\delta^{(D)}(x-y)\right. \\ &\quad \left. - \frac{\zeta}{8}G^{ab\,hf}G^{-1\,cd}{}_{hf}\pi^{eIJ}K_e{}^{IJ}\delta^{(D)}(x-y)\right] - ab \leftrightarrow cd \\ &= \frac{1}{4}G^{ab\,xf}G^{cd\,e}{}_f \mathscr{G}_{xe} + \frac{\zeta}{32}\left(q^{(b|d}\mathscr{G}^{c|a)} + q^{(b|c}\mathscr{G}^{d|a)}\right)\delta^{(D)}(x-y) \approx 0.\end{aligned} \tag{7.54}$$

In the first step, we used (7.53). Due to the antisymmetry in $ab$, $cd$, the first term is antisymmetric in $x, e$ and therefore proportional to $\mathscr{G}_{xe} := \pi^{aIJ}q_{a[x}K_{e]IJ}$. In the second summand of the second line, contracting the matrices and rearranging the terms leads to the remaining terms in the fourth line. The terms in the third line are symmetric in the exchange of $ab$, $cd$ and therefore drop out. We claim that $\mathscr{G}_{ab}$ is constrained to vanish by the Gauß constraint. With the convention $\bar{K}_{aI} := -\zeta K_{aIJ}n^J$ we obtain for the Gauß constraint dropping terms $\propto \mathscr{S}$

$$\begin{aligned}\mathscr{G}_{IJ} &= 2K_{aL[I}\pi^a{}_{J]}{}^L \approx 2K_{aL[I}(n_{J]}E^{aL} - E^a_{J]}n^L) \\ &= -2\zeta\bar{K}_{a[I}E^a_{J]} + 2\bar{K}^{\text{tr}}_{[I}n_{J]} =: \bar{\mathscr{G}}_{IJ} + 2n_{[I}\bar{\mathscr{G}}_{J]},\end{aligned} \tag{7.55}$$

where $\bar{K}^{\mathrm{tr}}{}_I = E^{aL}\bar{K}_{aLI}$ is the trace part of $\bar{K}_{aIJ} = \bar{\eta}^K_{[I}\bar{\eta}^L_{J]}K_{aKL}$, $\bar{\eta}_{IJ} = \eta_{IJ} - \zeta n_I n_J$. It follows that $\bar{K}^{\mathrm{tr}}{}_I = 0$ and $\bar{K}_{a[I}E^a_{J]} = 0$ on the Gauß constraint surface. Now

$$\mathscr{G}^{ab}E_{aI}E_{bJ} \approx 2\zeta E^{[a|K}q^{b]c}\bar{K}_{cK}E_{aI}E_{bJ} = \frac{2\zeta}{q}\bar{K}_{a[I}E^a{}_{J]}. \tag{7.56}$$

Therefore, $\mathscr{G}^{[ab]} = [\mathscr{G}^{[cd]}E_{cI}E_{cJ}]E^{aI}E^{bJ}$ vanishes on the Gauß constraint surface and proves (7.54). □

**Step 2:** Of course, the transformation $\{K_{aIJ}, \pi^{bKL}\} \to \{\underset{(\beta)}{K}_{aIJ} := \beta K_{aIJ}, \overset{(\beta)}{\pi}{}^{bKL} := \frac{1}{\beta}\pi^{bKL}\}$ for $\beta \in \mathbb{R}/\{0\}$ is canonical. We restrict to real $\beta$ in order to retain a real phase space, otherwise we reproduce the problems of the original, complex Ashtekar variables with implementing the reality conditions. Like the usual spin connection, $\Gamma_{aIJ}[\pi]$ is unchanged by constant rescalings, $\Gamma_{aIJ}[\pi] = \Gamma_{aIJ}[\overset{(\beta)}{\pi}]$. The parameter $\beta$ is similar to, but structurally different from the Immirzi parameter in $D = 3$, as we will see.

**Step 3:** Finally, one would like to perform a canonical transformation to connection variables $\{\underset{(\beta)}{K}_{aIJ}, \overset{(\beta)}{\pi}{}^{bKL}\} \to \{A_{aIJ} := \Gamma_{aIJ}[\pi] + \underset{(\beta)}{K}_{aIJ}, \overset{(\beta)}{\pi}{}^{bKL}\}$. The only non-trivial Poisson bracket is the one between two connections $A_{aIJ}$. We have

$$\begin{aligned}\{A_{aIJ}(x), A_{bKL}(y)\} &= \int_\sigma d^Dz \left[2\frac{\delta A_{aIJ}(x)}{\delta \underset{(\beta)}{K}_{cMN}(z)}\frac{\delta A_{bKL}(y)}{\delta \overset{(\beta)}{\pi}{}^{cMN}(z)} - aIJx \leftrightarrow bKLy\right] \\ &= 2\beta\left[\frac{\delta\Gamma_{bKL}[\pi, y]}{\delta\pi^{aIJ}(x)} - \frac{\delta\Gamma_{aIJ}[\pi, x]}{\delta\pi^{bKL}(y)}\right] = 2\beta\left[\frac{\delta\mathscr{S}_{bKL}[\pi, y]}{\delta\pi^{aIJ}(x)} - \frac{\delta\mathscr{S}_{aIJ}[\pi, x]}{\delta\pi^{bKL}(y)}\right],\end{aligned} \tag{7.57}$$

where in the last step, we invoked the key result of the previous section, $\Gamma_{aIJ} = \delta F/\delta\pi^{aIJ} + \mathscr{S}_{aIJ}$ and exploited the commutativity of partial functional derivatives. However, while $\mathscr{S}^{aIJ}$ vanishes on the simplicity constraint surface, its functional derivatives do not vanish necessarily. One would have to study $\mathscr{S}^{aIJ}$ more carefully in order to decide if the transformation is canonical or not[1]. Here, we will proceed differently. We will still perform the transformation $\{\underset{(\beta)}{K}_{aIJ}, \overset{(\beta)}{\pi}{}^{bKL}\} \to \{A_{aIJ} :=$

[1] Using the explicit expression (9.34) for $\Gamma$ in $D = 3$, a direct, rather lengthy computation shows that $\{A_{aIJ}[\alpha^{aIJ}], A_{bKL}[\beta^{bKL}]\} \approx 0$ only if both multiplier fields are chosen such that their transversal tracefree parts vanish and therefore, this particular transformation is not canonical. However, other explicit expressions for $\Gamma$ may differ from the chosen one by $\mathscr{S}$, $\partial\mathscr{S}$, which does not allow for general conclusions.

$\Gamma_{aIJ}[\pi] + \underset{(\beta)}{K}_{aIJ}, \overset{(\beta)}{\pi}{}^{bKL}\}$ and postulate the (non-vanishing) Poisson brackets

$$\{A_{aIJ}(x), \overset{(\beta)}{\pi}{}^{bKL}(y)\} = 2\delta_a^b \delta_{[I}^K \delta_{J]}^L \delta^{(D)}(x-y), \tag{7.58}$$

but then, rather than showing that it is canonical, prove that, like in step 1, the symplectic reduction of the obtained Yang Mills phase space with respect to the simplicity constraint leads to the ADM phase space.

To this end, let us rewrite the Gauß constraint in terms $A_{aIJ}$ and $\overset{(\beta)}{\pi}{}^{bKL}$,

$$\begin{aligned}
\mathscr{G}^{IJ} &= 2\underset{(\beta)}{K}{}_a^{[I}{}_K \overset{(\beta)}{\pi}{}^{a|K|J]} \\
&\approx \partial_a \overset{(\beta)}{\pi}{}^{aIJ} + 2\Gamma_a^{[I}{}_K \overset{(\beta)}{\pi}{}^{a|K|J]} + 2\underset{(\beta)}{K}{}_a^{[I}{}_K \overset{(\beta)}{\pi}{}^{a|K|J]} \\
&= D^A{}_a \overset{(\beta)}{\pi}{}^{aIJ}.
\end{aligned} \tag{7.59}$$

Note that the terms we added vanish on the simplicity constraint surface, since $\Gamma_{aIJ}[\pi]$ weakly annihilates $\pi$. The covariant differential $D^A{}_a$ of $A$ acts only on internal indices. This does not affect the tensorial character of (7.59) because $\pi^{aIJ}$ is a Lie algebra valued vector density of weight one and (7.59) is its covariant divergence which is independent of the Levi-Civita connection. Under this constraint, $A$ transforms as a connection. The map from this Yang-Mills theory phase space to the coordinates $(q_{ab}, P^{ab})$ of the ADM phase space is given by

$$\det(q) q^{ab} := \frac{\beta^2}{2\zeta} \overset{(\beta)}{\pi}{}^{aIJ} \overset{(\beta)}{\pi}{}^{b}{}_{IJ}, \tag{7.60}$$

$$P^{ab} := \frac{1}{4} G^{ab\,c}{}_d [A_{cIJ} - \Gamma_{cIJ}[\pi]] \overset{(\beta)}{\pi}{}^{dIJ}, \tag{7.61}$$

which of course directly follows from (7.44, 7.45).

Now we want to prove

**Theorem 5.**
*i. Gauß and simplicity constraints obey a first class constraint algebra.*
*ii. The symplectic reduction of the Yang-Mills phase space defined above with respect to Gauß and simplicity constraints coincides with the ADM phase space. More in detail, the functions $q_{ab}[\pi]$, $P^{ab}[A,\pi]$ defined in (7.60, 7.61) are (weak) Dirac observables*

*with respect to Gauß and simplicity constraints and (weakly) obey the standard Poisson brackets*

$$\{q_{ab}(x), P^{cd}(y)\} = \delta^c_{(a}\delta^d_{b)}\,\delta^{(D)}(x-y),\;\; \{q_{ab}(x), q_{cd}(y)\} = \{P^{ab}(x), P^{cd}(y)\} = 0$$

*on the constraint surface defined by simplicity and Gauß constraints.*

This theorem is of course the direct analogon of theorem 4:

*Proof.*
i.
analogous to the case before.
ii.
The hybrid spin connection $\Gamma_{aIJ}[E]$ is a $G$ connection by construction. Its extension $\Gamma_{aIJ}[\pi]$ off the simplicity constraint surface therefore transforms as a $G$ connection modulo the simplicity constraint. Since both $\pi^{aIJ},\;\; K_{aIJ} := \frac{1}{\beta}(A_{aIJ} - \Gamma_{aIJ})$ transform in the adjoint representation of $G$ it is clear that $\mathrm{Tr}(\pi^a\pi^b),\;\; \mathrm{Tr}(K_a\pi^b)$ are in fact Gauß invariant, possibly modulo the simplicity constraint, and thus are $q_{ab},\;\; P^{ab}$. That $q_{ab},\;\; P^{ab}$ are also simplicity invariant (modulo the simplicity constraint) follows from the same calculation as before.

Concerning the ADM Poisson brackets, the only calculation that changes is the Poisson bracket between two ADM momenta. Dropping all terms $\propto \mathscr{G}^{ab}$, which already appeared in the previous case and also here vanish weakly, we obtain the additional terms

$$\begin{aligned}&\{P^{ab}(x), P^{cd}(y)\} \approx \\ &\approx -\frac{1}{16} G^{ab\,e}{}_f(x) \overset{(\beta)}{\pi}{}^{fIJ}(x)\; G^{cd\,g}{}_h(y) \overset{(\beta)}{\pi}{}^{hKL}(y)\; [\{A_{eIJ}(x), \Gamma_{gKL}(y)\} - \{A_{gKL}(y), \Gamma_{eIJ}(x)\}].\end{aligned} \tag{7.62}$$

We now again invoke the key result of the previous section and write $\Gamma_{aIJ} = \delta F/\delta\pi^{aIJ} + \mathscr{S}_{aIJ}$ where $\mathscr{S}_{aIJ}$ vanishes on the constraint surface of the simplicity constraint and depends at most on its first partial derivatives. It is therefore given by an expression of the form

$$\mathscr{S}_{gKL} = \lambda^{\overline{M}}_{gKLmn}\mathscr{S}^{mn}_{\overline{M}} + \mu^{\overline{M}p}_{gKLmn}\partial_p\mathscr{S}^{mn}_{\overline{M}} \tag{7.63}$$

for certain coefficients $\lambda, \mu$. First of all, we notice that due to the commutativity of partial functional derivatives

$$\{A_{eIJ}(x), \delta F/\delta\pi^{gKL}(y)\} - \{A_{gKL}(y), \delta F/\delta\pi^{eIJ}(x)\} = 0. \tag{7.64}$$

Next, due to the derivatives involved, the Poisson bracket is not ultralocal, however, what we intend to prove is that $\{P^{ab}[f_{ab}], P^{cd}[f'_{cd}]\} \approx 0$ with the smeared functions $P^{ab}[f_{ab}] = \int d^Dx f_{ab} P^{ab}$. Let $M^e_f = \frac{1}{4} f_{ab} G^{abe}_f$, $M'^g_h = \frac{1}{4} G^{cdg}_h f'_{cd}$, then the contribution from $\mathscr{S}_{gKL}$ in the first term of (7.62) becomes after smearing

$$\begin{aligned}
&\approx \int d^Dx \int d^Dy \; M^e_f \overset{(\beta)}{\pi}{}^{fIJ}(x) \left( M'^g_h \overset{(\beta)}{\pi}{}^{hKL} \lambda^{\overline{M}}_{gKLmn} - [M'^g_h \overset{(\beta)}{\pi}{}^{hKL} \mu^{\overline{M}p}_{gKLmn}]_{,p} \right)(y) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \{A_{eIJ}(x), \mathscr{S}^{mn}_{\overline{M}}(y)\} \\
&= 4 \int d^Dx \; M^e_f \left( M'^g_h \overset{(\beta)}{\pi}{}^{hKL} \lambda^{\overline{M}}_{gKLmn} - [M'^g_h \overset{(\beta)}{\pi}{}^{hKL} \mu^{\overline{M}p}_{gKLmn}]_{,p} \right) \delta^{(m}_e \mathscr{S}^{n)f}_{\overline{M}}(x) \\
&\approx 0.
\end{aligned} \tag{7.65}$$

The calculation for the second term is similar. In conclusion, $\{P^{ab}[f_{ab}], P^{cd}[f'_{cd}]\} \approx 0$ vanishes on the joint constraint surface of the Gauß and the simplicity constraint. □

## 7.3 ADM Constraints in terms of the new variables

It remains to express the ADM constraints in terms of the new variables. Of course we could just substitute for the expressions (7.60, 7.61), however, this is not the most convenient form for the ADM constraints because they involve the hybrid connection which is a complicated expression in terms of $\pi$. We will therefore adopt the strategy familiar from $D+1=4$ and invoke the curvature $F$ of $A$. In the end, we will arrive at expressions $\mathscr{H}_a, \mathscr{H}$ for spatial diffeomorphism and Hamiltonian constraint which differ from their counterparts $\mathscr{H}'_a, \mathscr{H}'$, obtained by naive substitution of $q_{ab}, P^{ab}$ by (7.60, 7.61) in (2.27), (2.28), by terms proportional to Gauß and simplicity constraints. This guarantees that the algebra of Gauß, simplicity, spatial diffeomorphism an Hamiltonian constraints is of first class.

To see this, let us write $\mathscr{H}_a = \mathscr{H}'_a + \mathscr{Z}_a$, $\mathscr{H} = \mathscr{H}' + \mathscr{Z}$ where $\mathscr{Z}_a, \mathscr{Z}$ vanish on the constraint surface of the simplicity and Gauß constraint. We have seen already that $\{\mathscr{S}, \mathscr{S}\} = 0, \{\mathscr{G}, \mathscr{S}\} \propto \mathscr{S}$, $\{\mathscr{G}, \mathscr{G}\} \propto \mathscr{G}$. We also have shown that (7.60, 7.61) are weak Dirac observables with respect to $\mathscr{S}$ and invariant under $\mathscr{G}$. Since $\mathscr{H}'_a, \mathscr{H}'$ are defined in terms of (7.60, 7.61) it follows that $\{\mathscr{S}, \mathscr{H}'_a\} \propto \mathscr{S}, \{\mathscr{S}, \mathscr{H}'\} \propto \mathscr{S}$. Altogether therefore $\{\mathscr{S}, \mathscr{H}_a\}$, $\{\mathscr{S}, \mathscr{H}\}$, $\{\mathscr{G}, \mathscr{H}_a\}$, $\{\mathscr{G}, \mathscr{H}\} \propto \mathscr{S}, \mathscr{G}$ thus $\mathscr{S}, \mathscr{G}$ form an ideal. Next we have $\{\mathscr{H}'_a, \mathscr{H}'_b\} \propto \mathscr{H}'_c, \mathscr{S}, \mathscr{G}$, $\{\mathscr{H}'_a, \mathscr{H}'\} \propto \mathscr{H}', \mathscr{S}, \mathscr{G}$, $\{\mathscr{H}', \mathscr{H}'\} \propto \mathscr{H}'_a, \mathscr{S}, \mathscr{G}$ because the algebra of the variables (7.60, 7.61) is the same as that of the ADM variables modulo $\mathscr{S}, \mathscr{G}$ terms and therefore the algebra of the ADM constraints is reproduced modulo $\mathscr{S}, \mathscr{G}$ terms. Together with what was already said, this implies that $\mathscr{H}_a, \mathscr{H}$ reproduce the ADM algebra of constraints modulo $\mathscr{S}, \mathscr{G}$ terms.

In the following, we will repeatedly use the formulas (C.12, C.13) relating the hybrid and the Riemann curvature. We obtain modulo $\mathscr{S}$ for the Ricci scalar

$$R_{abIJ}\pi^{aIK}\pi^{b}{}_{K}{}^{J} \approx R_{abIJ}[n^I E^{aK} - n^K E^{aI}][n_K E^{bJ} - n^J E^b_K] = -\zeta \det(q) R. \quad (7.66)$$

Next, using (C.12)

$$R_{abIJ}\pi^{bIJ} \approx 2R_{abIJ}n^I E^{bJ} = 2q^{bc}\sqrt{\det(q)}R_{abIJ}n^I e^J_c = -2q^{bc}\sqrt{\det(q)}R_{abc}{}^{d}e_{dI}n^I = 0, \quad (7.67)$$

which is the analog of the algebraic Bianchi identity.

We now expand the curvature

$$F_{abIJ} := 2\partial_{[a}A_{b]IJ} + A_{aIK}\,A_b{}^K{}_J - A_{aJK}\,A_b{}^K{}_I \quad (7.68)$$

of $A = \Gamma + \beta K$ in terms of $\Gamma, K$ and obtain

$$F_{abIJ} = R_{abIJ} + 2\beta D^{\mathrm{H}}{}_{[a}K_{b]IJ} + 2\beta^2 K_{[aIK}\,K_{b]}{}^K{}_J. \quad (7.69)$$

Contracting (7.69) with $\pi^{bIJ}$ we find using (7.67)

$$F_{abIJ}\pi^{bIJ} \approx 2\beta(D_{[a}K_{b]IJ})\pi^{bIJ} - \beta^2\mathrm{Tr}([K_a, K_b]\pi^b). \quad (7.70)$$

The second term is proportional to the Gauß constraint because $\mathrm{Tr}([K_a, K_b]\pi^b) = \mathrm{Tr}(K_a[K_b, \pi^b])$ and remembering (7.59). In the first term we notice that $D^\Gamma{}_a\pi^{bIJ} \approx 0$ so that

$$F_{abIJ}\pi^{bIJ} \approx -4\beta D_{[a}K^b_{b]} = 2\beta D_b[K_a{}^b - \delta^b_a K_c{}^c] = -4\beta D_b P_a{}^b = 2\beta\mathscr{H}_a \quad (7.71)$$

is proportional to the spatial diffeomorphism constraint modulo $\mathscr{S}, \mathscr{G}$. Note that $K^{ab} := -\frac{1}{2}K_{cIJ}q^{ca}\pi^{bIJ}$ is symmetric in $a, b$ modulo the Gauß constraint.

Next, using (7.66)

$$F_{abIJ}\pi^{aIK}\pi^{b}{}_{K}{}^{J} \approx -\zeta\det(q)R - 2\beta D_a\mathrm{Tr}(K_b[\pi^a, \pi^b]) - \beta^2\mathrm{Tr}([K_a, K_b]\pi^a\pi^b). \quad (7.72)$$

The second term is again proportional to the Gauß constraint, since $\mathrm{Tr}(K_b[\pi^a,\pi^b]) = -\mathrm{Tr}(\pi^a[K_b,\pi^b])$. So far all the steps were similar to the $3+1$ situation. The difference comes in when looking at the third term in (7.72)

$$\begin{aligned}-\mathrm{Tr}([K_a,K_b]\pi^a\pi^b) &\approx [K_{aIK}K_b{}^K{}_J - K_{bIK}K_a{}^K{}_J][n^IE^{aL}-n^LE^{aI}][n^LE^{bJ}-n^JE^b_L] \\ &= -\zeta[-(K_{aIK}E^{aI})(K_{bJ}{}^KE^{bJ}) + (K_{bIK}E^{aI})(K_{aJ}{}^KE^{bJ})]. \quad (7.73)\end{aligned}$$

By the Gauß constraint (7.55), we have $\bar{K}^{\mathrm{tr}}{}_I \approx 0$ and therefore $K_{aJI}E^{aJ} = -n_I\bar{K}_{aJ}E^{aJ} = \zeta n_I K^a_a$. Thus the first term in (7.73) is given by $[K^a_a]^2$. However, the second term cannot be written in terms of $K^b_a$. To explore the structure of the disturbing term we notice that from $\bar{K}^{\mathrm{tr}}{}_I = 0$ we have the decomposition

$$K_{aIJ} = \bar{K}^{\mathrm{tf}}{}_{aIJ} + 2n_{[I}\bar{K}_{a|J]}, \quad \bar{K}_{aI} = -\zeta K_{aIJ}n^J. \quad (7.74)$$

Hence

$$\begin{aligned}-\zeta(K_{bIK}E^{aI})(K_{aJ}{}^KE^{bJ}) &= -\zeta(\bar{K}^{\mathrm{tf}}{}_{bIK}E^{aI} - \bar{K}_{bI}E^{aI}n_K)(\bar{K}^{\mathrm{tf}}{}_{aJ}{}^KE^{bJ} - \bar{K}_{aJ}E^{bJ}n^K) \\ &= -\zeta(\bar{K}^{\mathrm{tf}}{}_{bIK}E^{aI})(\bar{K}^{\mathrm{tf}}{}_{aJ}{}^KE^{bJ}) - K^b_aK^a_b, \quad (7.75)\end{aligned}$$

where $\bar{K}_{aI}E^{bI} = -\zeta K_{aIJ}E^{bI}n^J \approx K_{aIJ}\pi^{bIJ}/(2\zeta)$ was used. Altogether,

$$-\mathrm{Tr}([K_a,K_b]\pi^a\pi^b) = -[K^b_aK^a_b - (K^c_c)^2] - \zeta(\bar{K}^{\mathrm{tf}}{}_{bIK}E^{aI})(\bar{K}^{\mathrm{tf}}{}_{aJ}{}^KE^{bJ}). \quad (7.76)$$

The first term in (7.76) has the structure that appears in the Hamiltonian constraint and can be written in terms of $P^{ab}, q_{ab}$, however, the second term does not appear in the Hamiltonian constraint and must be removed. Also notice that the Ricci term in (7.72) has sign $-\zeta$ while the first term has negative sign. If we are interested in Lorentzian gravity then the relative sign between these two terms should be negative which is not the case for the choice of a compact gauge group $\zeta = 1$. Therefore the expression (7.72) fails to yield the Hamiltonian constraint for several reasons.

To assemble the Hamiltonian constraint without making use of $\Gamma$, the idea is to consider covariant derivatives which give access to $A$. Using suitable algebraic combinations then yields the desired expressions. To that end, let again $D^A{}_a$ be the covariant differential of $A$ acting only on internal indices and let ${D'^A}_a$ be its extension by the Levi-Civita connection. Consider

$$D_b{}^a := \pi^{aK}{}_J\,({D^A}_b\pi^{cJL})\,\pi_{cKL} = \pi^{aK}{}_J\,({D'^A}_b\pi^{cJL})\,\pi_{cKL} - 2\pi^{aK}{}_J\,\pi_{cKL}\,\Gamma^{[c}_{bd}\pi^{d]JL}. \tag{7.77}$$

The second term equals modulo $\mathscr{S}$

$$-2[n^K E^a_J - n_J E^{aK}][n_K E_{cL} - n_L E_{cK}]\Gamma^{[c}_{bd}\pi^{d]JL} = -2\zeta\; E^a_J\; E_{cL}\Gamma^{[c}_{bd}\pi^{d]JL} \approx 0 \tag{7.78}$$

and thus vanishes modulo $\mathscr{S}$. Writing ${D'^A}_a = [{D'^A}_a - {D^{\rm H}}_a] + {D^{\rm H}}_a$ and noticing that $D^{\rm H}_a\pi^{cJL} \approx 0$, we obtain

$$\begin{aligned}\pi^{aK}{}_J\,({D^A}_b\pi^{cJL})\,\pi_{cKL} &\approx \zeta\beta E^a_J E_{cL}[K_b{}^J{}_M\pi^{cML} + K_b{}^L{}_M\,\pi^{cJM}] \\ &\approx \zeta\beta E^a_J E_{cL}[K_b{}^J{}_M E^{cL} n^M - K_b{}^L{}_M E^{cJ} n^M] \\ &= -\beta(D-1)E^{aJ}\,K_{bJ} = \zeta\beta(D-1)K_b{}^a. \end{aligned} \tag{7.79}$$

It follows that

$$\frac{1}{(D-1)^2}[D_b{}^a\,D_a{}^b - (D_c{}^c)^2] \approx \beta^2[K_b{}^a\,K_a{}^b - (K^c_c)^2] \tag{7.80}$$

and thus linear combinations of (7.72) and (7.80) can be used in order to produce the correct factor in front of the term quadratic in the extrinsic curvature.

In analogy to (7.77), consider

$$D^{aIJ} := \pi^{b[I}{}_K {D^A}_b \pi^{a|K|J]} = \pi^{b[I}{}_K {D'^A}_b \pi^{a|K|J]} - 2\pi^{b[I}{}_K \Gamma^{[a}_{bc}\pi^{c]|K|J]}. \tag{7.81}$$

The second term equals modulo $\mathscr{S}$

$$2\zeta E^{b[I}\Gamma^{[a}_{bc}E^{c]J]} = (-\zeta\Gamma^c_{bc}E^{b[I})E^{aJ]} \tag{7.82}$$

and thus is pure trace. Since we intend to cancel ${\bar{K}^{\rm tf}}_{aIJ}$ we therefore consider instead of (7.81) its transverse tracefree projection

$$\bar{D}_{\rm tf}{}^{aIJ} := [\mathbb{P}_{\rm tf}\cdot D]^{aIJ}, \tag{7.83}$$

under which (7.82) drops out. The projector $\mathbb{P}_{\rm tf}$ given in (7.14) can be expressed purely in terms of $\pi^{aIJ}$ using (7.29) and

$$E^{a[I}\bar{\eta}^{J]}_{[K}E_{bL]} \approx -\zeta\left(\pi^{aM[I}\bar{\eta}^{J]}{}_{[K}\pi_{bL]M} + \delta^a_b n^{[I}\eta^{J]}{}_{[K}n_{L]}\right). \tag{7.84}$$

We continue using again $D^{\mathrm{H}}{}_a\pi \approx 0$

$$\begin{aligned}\bar{D}_{\mathrm{tf}}{}^{aIJ} &\approx \beta\mathbb{P}_{\mathrm{tf}}\left(\pi^{b[I|K|}\left[K_{bKL}\pi^{a|L|J]} + K_b{}^{J]}{}_L\pi^a{}_K{}^L\right]\right)\\ &\approx -\beta\zeta\,\mathbb{P}_{\mathrm{tf}}\left(E^{b[I}K_b{}^{J]}{}_L E^{aL}\right) = -\beta\zeta\, E^{b[I}\bar{K}^{J]L}_{b\mathrm{tf}}E^a_L.\end{aligned} \tag{7.85}$$

Notice that the last line is indeed tracefree and transverse. We write (7.85) as

$$\bar{D}_{\mathrm{tf}}{}^{aIJ} = -\frac{\beta\zeta s}{4}F^{aIJ,bKL}\,\bar{K}^{\mathrm{tf}}{}_{bKL}, \quad F^{aIJ,bKL} = -4sE^{b[I}\bar{\eta}^{J][L}E^{aK]}. \tag{7.86}$$

The tensor $F^{aIJ,bKL}$ can be seen as bilinear form on transverse tensors of type $\bar{K}_{aIJ}$ and actually coincides with the tensor given in (4.48). Its inverse $(F^{-1})_{aIJ,bKL}$ has already been given in (4.49), $[F\cdot F^{-1}]^{aIJ}_{bKL} = 4\delta^a_b\bar{\eta}^I_{[K}\bar{\eta}^J_{L]}$. Using (7.29) and

$$E_{aI}E_{bJ} \approx \zeta[\pi_{aIM}\pi_{bJ}{}^M - \zeta\frac{1}{q}q_{ab}n_In_J], \tag{7.87}$$

$F^{-1}$ can be completely expressed in terms of $\pi^{aIJ}$. The quadratic combination of $\bar{K}^{\mathrm{tf}}$ to be removed from (7.76) can now be compactly written as

$$\begin{aligned}&E^{bI}\bar{K}^{\mathrm{tf}}{}_{bJM}E^{aJ}\bar{K}^{\mathrm{tf}}{}_{aI}{}^M = E^{b[I}\bar{\eta}^{N][M}E^{aJ]}\bar{K}^{\mathrm{tf}}{}_{bJM}\bar{K}^{\mathrm{tf}}{}_{aIN}\\ &= -\frac{s}{4}F^{aIN,bJM}\bar{K}^{\mathrm{tf}}{}_{aIN}\bar{K}^{\mathrm{tf}}{}_{bJM} = -\frac{s}{\beta^2}(F^{-1})_{aIJ,bKL}\bar{D}_{\mathrm{tf}}{}^{aIJ}\bar{D}_{\mathrm{tf}}{}^{bKL}.\end{aligned} \tag{7.88}$$

| Variable | Dof | $1^{\mathrm{st}}$ cl. constraints | Dof (count twice!) |
|---|---|---|---|
| $A_a{}^{IJ}$ | $\frac{D^2(D+1)}{2}$ | $\mathscr{H}$ | $1$ |
| $\overset{(\beta)}{\pi}{}^b{}_{KL}$ | $\frac{D^2(D+1)}{2}$ | $\mathscr{H}_a$ | $D$ |
| | | $\mathscr{G}^{IJ}$ | $\frac{D(D+1)}{2}$ |
| | | $\mathscr{S}^{ab}_{\overline{M}}$ | $\frac{D^2(D-1)}{2} - D$ |
| Sum: | $D^3 + D^2$ | Sum: | $D^3 + D + 2$ |

**Table 7.1:** The new variables: counting of degrees of freedom

We now have all the pieces we need. The appropriate Hamiltonian constraint for spacetime signature $s$ is displayed in (2.27). We find

$$\begin{aligned}\sqrt{q}\mathscr{H} =& \frac{\zeta s}{2}\Bigg(F_{abIJ}\pi^{aIK}\pi^b{}_K{}^J - s\zeta\bar{D}_{\mathrm{tf}}{}^{aIJ}\,(F^{-1})_{aIJ,bKL}\,\bar{D}_{\mathrm{tf}}{}^{bKL}\\ &+ \frac{1}{(D-1)^2}[D_b{}^a D_a{}^b - (D_c{}^c)^2]\Bigg) - \frac{1}{2\beta^2(D-1)^2}[D_b{}^a D_a{}^b - (D_c{}^c)^2].\end{aligned} \tag{7.89}$$

This expression simplifies for $s = \zeta$ and $\beta = 1$ in which case the terms quadratic in $D_b{}^a$ precisely cancel. This is again similar to the situation in $3+1$ dimensions. This special case can also be obtained more directly starting from the Palatini formulation as we will see in the next chapter 8. Counting of the degrees of freedom is shown in table 7.1 and of course is in agreement with general relativity.

# 8

# The new variables - Lagrangian picture

In the last chapter 7, we developed a higher dimensional connection formulation for general relativity with only first class constraints and Poisson commuting connections in any spacetime dimension $D+1 \geq 3$ by a judicious extension of the $D+1$ ADM phase space supplemented by first class Gauß and simplicity constraints. This approach has the advantage that it is rather simple, allows for $\mathrm{SO}(1,D)$ or $\mathrm{SO}(D+1)$ as structure group irrespective of the spacetime signature and that in addition it admits a free parameter that is, as we have seen, similar to but yet rather different from the Barbero Immirzi parameter in $3+1$ dimensions. However, one may ask whether this connection formulation can be obtained from an action principle, just as the LQG connection formulation can be obtained from the Holst action [145]. Here, we answer this question in the affirmative.

The appropriate action to choose will be simply the $D+1$ Palatini action with BF type simplicity constraints we already studied in section 5.2. However, following this route will not allow for the Immirzi like freedom and the structure group will be tied to the spacetime signature: it is necessarily $\mathrm{SO}(1,D)$ for Lorentzian spacetime signature and $\mathrm{SO}(D+1)$ for Euclidean spacetime signature. This makes this approach less favourable for quantisation of the Lorentzian theory which requires a compact structure group. Yet the efforts of this chapter are not in vain as our results confirm the achievements of the previous chapter via an alternative route. Maybe the most astonishing outcome is that we obtain a pure first class theory while it is well known that the Palatini formulation is plagued by second class constraints, as we have seen in 5.2. The resolution of the apparent contradiction is that we have to apply an additional step in order to arrive at the first class formulation which goes by the name *gauge unfixing*.

In more detail, we do the following: as we have seen in section 5.2, when starting from the Palatini formulation with BF type simplicity constraints of, say, Lorentzian general relativity in $D+1$ spacetime dimensions with structure group $\mathrm{SO}(1,D)$ and following Dirac's canonical analysis, we are naturally lead to an $\mathrm{SO}(1,D)$ connection $A$ and a $\mathrm{so}(1,D)$ valued vector density $\pi$ which is canonically conjugate to the connection. However, in addition to the $\mathrm{SO}(1,D)$ Gauß constraint, the $D$-dimensional spatial diffeomorphism constraint and the Hamiltonian constraint, we had to introduce an additional primary constraint $\mathscr{S}$ which requires the momentum $\pi$ to derive from (the pull back to the leaves of the foliation of) a co-$(D+1)$-bein, called (BF type) simplicity constraint (precisely because it is the temporal spatial part of the simplicity constraint of a higher dimensional Plebański formulation, cf. section 5.1 and [169]). The stability of the constraint $\mathscr{S}$ with respect to the canonical Hamiltonian enforces a secondary constraint $\mathscr{D}$ and $(\mathscr{S}, \mathscr{D})$ form a second class pair. The situation is of course completely the same as in $D = 3$ dimensions. In $D = 3$ dimensions one can now either consider this $\mathrm{SO}(1,3)$ connection formulation and try to quantise the corresponding Dirac bracket [194] with non Dirac bracket commuting connections or one imposes the time gauge and reduces the (Holst modified) theory to a Dirac bracket commuting SU(2) (or SO(3)) connection formulation. In higher dimensions also both possibilities exist, except that imposing the time gauge does not lead to a $\mathrm{SO}(D)$ connection formulation but rather the extended ADM formulation of section 3.2.4, as has already been shown in [166]. Thus the second strategy does not lead to the desired connection formulation with compact $\mathrm{SO}(D)$ precisely due to the dimensional mismatch between $D$ and $D(D-1)/2$. Thus, in order to have a connection formulation only the first possibility remains but then the complication with the Dirac bracket arises. It is at this point where gauge unfixing comes into play: by a systematic, allowed modification of the Hamiltonian constraint which does not alter its first class character, the simplicity constraints $\mathscr{S}$ become Poisson commuting with all but the $\mathscr{D}$ constraints. Remarkably, this modification of the Hamiltonian constraint, which makes it simplicity invariant, involves a correction term which is precisely the one found in the previous chapter which makes sure that the Hamiltonian constraint derived from the Palatini Lagrangian coincides with the ADM Hamiltonian constraint when Gauß and simplicity constraints are satisfied. One can now consider the $\mathscr{D}$ constraints as gauge fixing conditions for the $\mathscr{S}$ constraints and impose only the first class constraints. This way one can map the second class constraint system to an equivalent first class constraint system and replace the complicated Dirac bracket by the simple ordinary Poisson bracket with Poisson commuting connections. In the end, this formulation is identical to the one of the previous chapter for matching spacetime and internal signature as well as unit Barbero Immirzi like parameter.

The chapter is is organised as follows: the canonical analysis of the higher dimensional Palatini theory was already considered in sections 4.2 and 5.2 and the corresponding Hamiltonian formulations have been found to be plagued by second class constraints. Therefore, here we will start by reviewing the procedure of gauge unfixing in section 8.1 and then apply it to the outcome of the canonical analysis of section 5.2. The result is an $\mathrm{SO}(1, D)$ or $\mathrm{SO}(D+1)$ connection formulation for Lorentzian or Euclidean general relativity respectively with first class constraints only and a connection variable which is Poisson self-commuting, the price to pay is one extra term in the Hamiltonian constraint.

## 8.1 Review of gauge unfixing

The name "gauge unfixing" suggests that this is a procedure in some sense inverse to "gauge fixing". To see to what extent this is indeed the case it is useful to recall some facts about gauge fixing first. After that we focus on the gauge unfixing case. This review section can be skipped by readers familiar with gauge (un)fixing although we add a few extra twists to it. We have combined material from several sources: to the best of our knowledge, the pioneering paper on gauge unfixing of second class theories is [199] and the general theory was developed in [201, 202]. Parts of this theory were independently rediscovered from the point of view of a first class theory in [161, 204], see also [205–207].

### 8.1.1 Gauge fixing

Recall that gauge fixing of a *first class system* with first class constraints $\mathscr{S}_I$ (where $I$ takes values in some index set) on a phase space $M$ consists in imposing an equal number of gauge fixing conditions $\mathscr{D}_I$ such that the matrix $F$ with entries $F_{IJ} := \{\mathscr{S}_I, \mathscr{D}_J\}$ is regular. The gauge fixing conditions, modulo the problem of Gribov copies, select a unique point on each gauge orbit of the $\mathscr{S}_I$. Here the gauge orbit of a point $m \in M$ is the set[1]

$$[m] := \{\alpha_\beta(m),\ \beta^I \in \mathbb{R}\}, \quad \alpha_\beta(f) := \exp(\beta^I \{\mathscr{S}_I, .\}) \cdot f, \tag{8.1}$$

where $\alpha_\beta(f)$ is the gauge flow with parameter $\beta$ applied to the (smooth) function $f$ on phase space. To qualify as an admissible gauge fixing condition, at least on the constraint surface

$$\overline{M} := \{m \in M;\ \mathscr{S}_I(m) = 0\ \forall\ I\}, \tag{8.2}$$

it must be possible to reach the selected section

$$\sigma_{\mathscr{D}}(\overline{M}) := \{m \in \overline{M};\ \mathscr{D}_I(m) = 0\ \forall\ I\} \tag{8.3}$$

from any other section of $\overline{M}$.

At least locally, the constraint surface acquires the structure of a fibre bundle where the fibres are given by the gauge orbits (considered as subsets of $\overline{M}$) and the base space is the set of equivalence classes

$$\widehat{M} := \{[m];\ \ m \in \overline{M}\} \tag{8.4}$$

called the reduced phase space. Under the above conditions there is a bijection between $\sigma_{\mathscr{D}}(\overline{M})$ and $\widehat{M}$: given $m \in \sigma_{\mathscr{D}}(\overline{M})$ one obtains $[m] \in \widehat{M}$ via (8.1) and given $[m]$ (considered as a subset of $\overline{M}$) one computes the unique point $m' \in [m]$ such that $\mathscr{D}_I(m') = 0$ for all $I$, that is $m' = [m] \cap \sigma_{\mathscr{D}}(\overline{M})$. However, while the construction of $\widehat{M}$ is *canonical*, i.e. does not use any structure other than $\mathscr{S}_I$ which canonically follow from the Dirac

[1] In case that the first class constraints close with non trivial structure functions only, it maybe necessary to apply several of the Poisson automorphisms $\alpha_\beta$ with different $\beta$ because the $\alpha_\beta$ do not form a group under concatenation in this case.

algorithm applied to the singular Lagrangian in question, the cross section $\sigma_{\mathscr{D}}(\overline{M})$ uses the additional input of $\mathscr{D}$ which, except for the regularity condition on $F$, is rather arbitrary.

The observables of the first class system are the gauge invariant functions evaluated on the constraint surface. These therefore only depend on the equivalence classes $[m]$. It appears that the construction of such gauge invariant functions is generically impossible for sufficiently complicated constraints $\mathscr{S}_I$. This turns out to be correct if one is interested in these observables as functions on $\overline{M}$. However, given a set of gauge fixing conditions $\mathscr{D}_I$, not only can one write an explicit formula for these observables but one can also compute their Poisson algebra. This also then displays the relation between the spaces $\sigma_{\mathscr{D}}(\overline{M})$ and $\overline{M}$ in explicit form. Given a function $f$ on $M$ one can define a weak Dirac observable by the formula

$$O^{(\mathscr{D})}(f) := [\alpha_\beta(f)]_{\alpha_\beta(\mathscr{D})=0}, \tag{8.5}$$

where the superscript $(\mathscr{D})$ is to make it explicit that this formula is not canonical but depends on the chosen gauge fixing. This formula has to be understood in the following way: first one computes the gauge flow of $f$ at $m \in \overline{M}$ with real valued (phase space independent) constants $\beta^I$, that is

$$\alpha_\beta(f) := f + \sum_{n=1}^{\infty} \frac{1}{n!}\, \beta^{I_1}..\beta^{I_n}\, \{\mathscr{S}_{I_1}, \{.., \{\mathscr{S}_{I_n}, f\}..\}\} \tag{8.6}$$

and then one solves the condition $\alpha_\beta(\mathscr{D}_I) = 0$ for all $I$ for $\beta^I = \gamma^I(m)$ and inserts the corresponding phase space dependent function into (8.6). The value $\gamma(m)$ is thus the parameter needed in order to map $m$ to that point on its orbit $[m]$ at which the $\mathscr{D}_I$ vanish. It is not difficult to check that indeed $\{\mathscr{S}_I, O_f\} \approx 0$, and that $O^{(\mathscr{D})}$ preserves the pointwise addition and multiplication of functions

$$O^{(\mathscr{D})}(f+g) = O^{(\mathscr{D})}(f) + O^{(\mathscr{D})}(g), \quad O^{(\mathscr{D})}(fg) = O^{(\mathscr{D})}(f)\, O^{(\mathscr{D})}(g). \tag{8.7}$$

Moreover, the following remarkable formula holds[1]

$$\{O^{(\mathscr{D})}(f), O^{(\mathscr{D})}(g)\} \approx \{O^{(\mathscr{D})}(f), O^{(\mathscr{D})}(g)\}^*_{\mathscr{S},\mathscr{D}} \approx O^{(\mathscr{D})}(\{f,g\}^*_{\mathscr{S},\mathscr{D}}), \tag{8.8}$$

where $\{.,.\}^*_{\mathscr{S},\mathscr{D}}$ is the Dirac bracket of the *second class system* of constraints $\mathscr{S}_I, \mathscr{D}_I$. Since a sufficient number of the $O^{(\mathscr{D})}(f)$ serves as coordinates of $\widehat{M}$ we see that the Poisson bracket on the reduced phase space $\widehat{M}$ is given by the Dirac bracket and $O^{(\mathscr{D})}$ is a Dirac bracket homomorphism from the algebra of smooth functions on $M$ to the one on $\widehat{M}$.

It should be stressed, however, that this algebra of observables is not canonical, it depends on the choice of admissible gauge fixing $\mathscr{D}$ which is an extra input necessary for their very construction. Nevertheless, once we have made such a choice, we see that a first class system $\mathscr{S}$ together with a gauge fixing condition $\mathscr{D}$ is completely equivalent to the second class system $\mathscr{S}, \mathscr{D}$. Namely, for a second class system the reduced phase space consists simply in the constraint manifold

$$\overline{\overline{M}} := \{m \in M;\;\; \mathscr{S}_I(m) = \mathscr{D}_I(m) = 0 \;\forall\; I\} \equiv \sigma_{\mathscr{D}}(\overline{M}), \tag{8.9}$$

which precisely coincides with the gauge section (8.3), and the symplectic structure on $\overline{\overline{M}}$ is given by the Dirac bracket

$$\{f,g\}^*_{\mathscr{S},\mathscr{D}} = \{f,g\} - \{f,\mathscr{C}_\alpha\}\;[F^{-1}]^{\alpha\beta}\;\{\mathscr{C}_\beta, g\}, \tag{8.10}$$

where $\{\mathscr{C}_\alpha\} = \{\mathscr{S}_I, \mathscr{D}_I\}$ and $F_{\alpha\beta} = \{\mathscr{C}_\alpha, \mathscr{C}_\beta\}$ is non degenerate by construction. When restricting $O^{(\mathscr{D})}$ to $\overline{\overline{M}}$ which is in bijection with $\widehat{M}$ as we have seen, it becomes a Dirac bracket isomorphism.

### 8.1.2 Gauge unfixing

We now consider a second class system with constraints $\mathscr{S}_I, \mathscr{D}_I$ with the special structure that $\mathscr{S}_I$ is a first class subalgebra of constraints, that is

$$\{\mathscr{S}_I, \mathscr{S}_J\} = f_{IJ}{}^K\;\mathscr{S}_K \tag{8.11}$$

[1]The first identity holds because the $\mathscr{S}$ constraints form a subalgebra. The Dirac matrix $F_{\alpha\beta} = \{\mathscr{C}_\alpha, \mathscr{C}_\beta\}$, $\{\mathscr{C}_\alpha\} = \{\mathscr{S}_I, \mathscr{D}_I\}$ on the constraint surface therefore has the symbolic structure $F = \begin{pmatrix} 0 & B \\ -B & A \end{pmatrix}$ and its inverse is given by $F^{-1} = \begin{pmatrix} B^{-1}AB^{-1} & -B^{-1} \\ B^{-1} & 0 \end{pmatrix}$ so that the Dirac bracket $\{f,g\}^*$ contains no terms $\propto \{f,\mathscr{D}\}\{g,\mathscr{D}\}$.

for certain structure functions $f_{IJ}{}^{K}$ and $F_{IJ} := \{\mathscr{S}_I, \mathscr{D}_J\}$ is supposed to be non degenerate on the constraint surface

$$\overline{\overline{M}} := \{m \in M;\ \ \mathscr{S}_I(m) = \mathscr{D}_I(m) = 0\ \forall\ I\}, \tag{8.12}$$

which is equipped with the Dirac bracket (8.10). In [202] we find conditions under which linear combinations of a given set of second class constraints can be subdivided into sets $\mathscr{S}_I$ and $\mathscr{D}_I$ subject to (8.11). Here we will simply assume that this has been achieved.

We have seen in the previous section that a first class system $\mathscr{S}_I$ together with additional gauge fixing conditions $\mathscr{D}_I$ is equivalent with the second class system $\mathscr{S}_I, \mathscr{D}_I$. The idea of gauge unfixing is now simply to interpret the given second class system of constraints $\mathscr{S}_I, \mathscr{D}_I$ as just a first class system $\mathscr{S}_I$ to which the particular gauge fixing conditions $\mathscr{D}_I$ have been added.

This point of view has the following advantage towards quantisation: for a first class system of constraints we have two possible quantisation strategies, namely A. Operator Constraint Quantisation and B. Reduced Phase Space Quantisation. The advantage of option A. is that one can work with the simple Poisson bracket algebra on the kinematical phase space $M$ for which Hilbert space representations are typically easy to find and the task is to find those which support the $\mathscr{S}_I$ as densely defined, closable and non anomalous operators. The disadvantage is that one has to equip the joint kernel of the constraints with a new (physical) inner product which carries a representation of the observables of the theory and while there are general tools available such as group averaging, it is generically not possible to determine the physical Hilbert space in closed form. The disadvantage of option B. is that the Dirac bracket algebra on the reduced phase space is typically so complicated that no Hilbert space representations can be found. On the other hand, if one manages to do so, then one has direct access to the physical Hilbert space and the algebra of observables. Now in case that option B. is inhibited due to the complexity of the Dirac bracket algebra which is typically the case for second class systems, option A. appears to be the only possible approach to quantisation. As we will see, one can do even better than that, but let us assume for the moment that we take a second class system $\mathscr{S}_I, \mathscr{D}_I$ with complicated Dirac bracket algebra and therefore drop $\mathscr{D}_I$ and just perform an operator constraint quantisation of $\mathscr{S}_I$.

At first sight, this strategy seems to be false for at least two reasons:

1. From the point of view of the first class system, the gauge fixing conditions $\mathscr{D}_I$ are just one of an infinite number of possible choices, the first class system does not know about the $\mathscr{D}_I$ and therefore one can drop the $\mathscr{D}_I$. However, we are not given a first class system, we are given a second class system and from the point of view of the second class system, the $\mathscr{D}_I$ are *canonical*, they follow canonically from Dirac's stabiliser algorithm applied to the given singular Lagrangian. It seems therefore to be wrong to forget about the special role of the $\mathscr{D}_I$ within the first class system as we would drop information that is forced on us by Dirac's algorithm. However, imposing the $\mathscr{D}_I$ as operators as well in the quantum theory is not possible, that is, the joint kernel of the $\mathscr{D}_I, \mathscr{S}_I$ is just the zero vector.

2. The canonical Hamiltonian $H$ of the second class system as derived via Dirac's stabiliser algorithm is typically not gauge invariant with respect to the $\mathscr{S}_I$ which would not be the case for a true first class system with just the constraints $\mathscr{S}_I$. In fact, in many applications the second class structure $\mathscr{S}_I, \mathscr{D}_I$ arises from primary constraints $\mathscr{S}_I$ and a canonical Hamiltonian of the form

$$H' = H_0 + \lambda^I \, \mathscr{S}_I, \tag{8.13}$$

with nontrivial $H_0$ independent of the $\mathscr{S}_I$ (that is $[H_0]_{\mathscr{S}=0} \neq 0$) and the $\mathscr{D}_I$ arise as secondary constraints from the stability requirement

$$0 \stackrel{!}{=} \{H', \mathscr{S}_I\} \; \approx \; \{H_0, \mathscr{S}_I\} =: \mathscr{D}_I, \tag{8.14}$$

where $\{\mathscr{S}_I, \mathscr{S}_J\} \propto \mathscr{S}_K \approx 0$ was used. The stability of the $\mathscr{D}_I$ fixes the Lagrange multipliers $\lambda^I$

$$0 \stackrel{!}{=} \{H', \mathscr{D}_I\} = \{H_0, \mathscr{S}_I\} + \lambda^J \{\mathscr{S}_J, \mathscr{D}_I\} \;\; \Rightarrow \;\; \lambda^I = -[F^{-1}]^{JI} \, \{H_0, \mathscr{S}_J\} =: \lambda_0^I, \tag{8.15}$$

so that the stabilised, first class Hamiltonian (it weakly commutes with all the constraints $\mathscr{S}_I, \mathscr{D}_I$) reads

$$H = H_0 + \lambda_0^I \, \mathscr{S}_I. \tag{8.16}$$

It is not gauge invariant with respect to just the constraints $\mathscr{S}_I$ since $\{H, \mathscr{S}_I\} \approx \mathscr{D}_I$ so that the constraints $\mathscr{D}_I$ appear again as a consistency condition.

We now explain how both obstacles can be overcome. We deal first with the second issue: we simply make the canonical Hamiltonian $H$ gauge invariant with respect to the $\mathscr{S}_I$ by using the map $O^{(\mathscr{D})}$ displayed in (8.5) that is, we replace $H$ by

$$\tilde{H} := O^{(\mathscr{D})}(H). \tag{8.17}$$

To see that this is an allowed Hamiltonian within the second class system we need to compute $\tilde{H}$ in some detail. As one can show [204, 207] one has explicitly

$$O^{(\mathscr{D})}(H) = H + \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{k=1}^{n} [-\mathscr{D}_{I_k}] \{\mathscr{S}'^{I_1}, ..\{\mathscr{S}'^{I_n}, H\}..\}, \tag{8.18}$$

where $\mathscr{S}'^I = [F^{-1}]^{IJ}\mathscr{S}_J$ so that $\{\mathscr{S}'^I, \mathscr{D}_J\} = \delta^I_J$ modulo $\mathscr{S}$. We have

$$\begin{aligned} \tilde{H} - H &= -\mathscr{D}_I\{\mathscr{S}'^I, H\} + \mathscr{O}(\mathscr{D}^2) = -\mathscr{D}_I \left([F^{-1}]^{IJ}\{\mathscr{S}_J, H\} + \{F^{IJ}, H\}\mathscr{S}_J\right) + \mathscr{O}(\mathscr{D}^2) \\ &= -\mathscr{D}_I \left([F^{-1}]^{IJ}[\mathscr{D}_J + N^K_J \mathscr{S}_K] + \{F^{IJ}, H\}\mathscr{S}_J\right) + \mathscr{O}(\mathscr{D}^2) = \mathscr{O}(\mathscr{D}^2, \mathscr{D}\mathscr{S}) \end{aligned} \tag{8.19}$$

for some $N^K_J$. Therefore $\tilde{H}$ and $H$ differ by terms at least quadratic in the constraints and thus do not spoil the first class structure of $H$. Therefore $\tilde{H}$ is an admissible Hamiltonian for the second class system which is simultaneously weakly invariant with respect to the $\mathscr{S}_I$. This is also the reason why one did not choose $\tilde{H}' = O^{(\mathscr{G})}(H)$ for some gauge fixing conditions $\mathscr{G}_I \neq \mathscr{D}_I$ because by a similar calculation as in (8.19) one would now compute $\tilde{H}' - H = O(\mathscr{D}\mathscr{G}, \mathscr{S}\mathscr{G}, \mathscr{G}^2)$ but $\mathscr{G}_I$ is no constraint and thus $\tilde{H}'$ is not an admissible Hamiltonian for the second class system. Notice also that $H$ and $\tilde{H}$ generate the same equations of motion on $\overline{\overline{M}}$.

We now come to the second issue. The question is: how can it be that the first class constrained Hamiltonian system $(\tilde{H}, \mathscr{S}_I)$ be equivalent to the second class system $(\tilde{H}, \mathscr{S}_I, \mathscr{D}_I)$? The first class system does not know about the $\mathscr{D}_I$. It is true that if we choose the special gauge fixing conditions $\mathscr{G}_I := \mathscr{D}_I = 0$ for the first class system, then the reduced phase spaces of the two systems are indeed isomorphic as we have shown above. However, the choice of $\mathscr{G}_I$ is arbitrary from the point of view of the first class system as long as the matrix with entries $\{\mathscr{S}_I, \mathscr{G}_J\}$ is non degenerate and therefore it appears that one has to still somehow feed the additional information about the special role of the gauge fixing condition $\mathscr{G}_I = \mathscr{D}_I$ into the first class system. However, this is not the case: the point is simply that an arbitrary gauge condition $\mathscr{G}_I = 0$ is related by a gauge transformation generated by the $\mathscr{S}_I$ to the gauge condition $\mathscr{D}_I = 0$. Therefore the observables of the form $O^{(\mathscr{G})}(f)$ are in fact linear combinations, with phase space independent coefficients, of the observables of the form $O^{(\mathscr{D})}(f)$. This follows simply from the fact that for gauge invariant functions $F$ (with respect to the $\mathscr{S}_I$) we have $F \approx O^{(\mathscr{D})}(F)$. Applied to $F = O^{(\mathscr{G})}(f)$ it follows

$$O^{(\mathscr{G})}(f) \approx O^{(\mathscr{D})}\left(O^{(\mathscr{G})}(f)\right). \tag{8.20}$$

Hence any observable of the form $O^{(\mathscr{G})}(f)$ can be written as $O^{(\mathscr{D})}(f')$ for some other function $f' = O^{(\mathscr{G})}(f)$. Since the roles of $\mathscr{G}_I, \mathscr{D}_I$ can be interchanged we see that the range of the maps $O^{(\mathscr{D})}, O^{(\mathscr{G})}$ is the same. Since the algebra of the $O^{(\mathscr{G})}(f)$ and of the $O^{(\mathscr{D})}(f)$ can be computed using the original Poisson bracket on the unreduced phase space we see that the algebra of the $O^{(\mathscr{D})}(f)$ and $O^{(\mathscr{G})}(f)$ are isomorphic, i.e. it does not matter whether we display one and the same observable $F$ in the form $F = O^{(\mathscr{D})}(f)$ or in the form $F = O^{(\mathscr{G})}(f')$.

What is different are of course the maps $O^{(\mathscr{D})}$, $O^{(\mathscr{G})}$ which provide different gauge invariant extensions of a given function $f$. Only the map $O^{(\mathscr{D})}$ yields an isomorphism with the Dirac bracket algebra of the second class system. However, this does not mean that one cannot use $O^{(\mathscr{G})}$ to construct gauge invariant observables. It just means that the identification between the Dirac bracket algebra of functions on $\overline{\overline{M}}$ with the Poisson bracket algebra on $\widehat{M}$ is rather complicated to write down because the correct gauge invariant function is $O^{(\mathscr{D})}(f) \approx O^{(\mathscr{G})}(O^{(\mathscr{D})}(f))$ and not just $O^{(\mathscr{G})}(f)$.

Remarks:

1.

This last observation now is also the key to a reduced phase space quantisation approach to second class systems $(H, \mathscr{S}_I, \mathscr{D}_I)$: after having replaced it by an equivalent first class system $(\tilde{H}, \mathscr{S}_I)$ one can now make use of the local Abelianisation theorem (see e.g. [51] and references therein) and replace the constraints $\mathscr{S}_I$ by an equivalent, strongly Abelian set $\mathscr{S}'_I = \pi_I + h_I(\phi^I; q^a, p_a)$ at least locally in phase space where the system of first class constraints $\mathscr{S}_I$ has been solved for some of the momenta $\pi_I$ in terms of its conjugate variables $\phi^I$ and the remaining canonical pairs $(q^a, p_a)$. Using the natural gauge fixing condition $\mathscr{G}_I = \phi_I$ the algebra of the $Q^a := O^{(\mathscr{G})}(q^a),\ P_a := O^{(\mathscr{G})}(p_a)$ coincides with the algebra of the $q^a, p_a$ because the corresponding Dirac bracket does not affect the subalgebra of functions of $q^a, p_a$. Since the algebra of the $Q^a, P_a$ is simple it can be quantised. This is surprising because we could have chosen to solve the constraints $\mathscr{S}_I = \mathscr{D}_I = 0$ for $\mathscr{S}'_I = \pi_I - \Pi_I(q^a, p_a),\ \mathscr{D}'_I = \phi^I - \Phi^I(q^a, p_a)$ from the outset so that the reduced phase space is parametrised by the $q^a, p_a$ but the corresponding Dirac bracket $\{p_a, q^b\}^* \neq \delta_a^b$ is not simple. The reason is of course that the functions $Q^a, P_a$ are genuinely different from $q^a, p_a$, in fact they are nontrivial functions of $\phi^I, q^a, p_a$ built in such a way that they have a simple Dirac bracket with respect to $\mathscr{S}, \mathscr{D}$. Moreover $\{Q^a, P_b\} = \{Q^a, P_b\}^*_{\mathscr{S},\mathscr{D}}$ due to gauge invariance. This holds for any two pairs of gauge invariant functions, in particular for the Hamiltonian $\tilde{H}$.

2.

For generally covariant systems $H_0$ is not a true Hamiltonian but rather a linear combination of different constraints $H_0 = \mu^A \mathscr{C}'_A$, typically a closed subalgebra of the form $\{\mathscr{C}'_A, \mathscr{C}'_B\} = f_{AB}{}^C\ \mathscr{C}'_C$ such that $\{\mathscr{C}'_A, \mathscr{S}_I\} = f_{AI}{}^J \mathscr{S}_J$ for $A \neq 0$ and $\{\mathscr{C}'_0, \mathscr{S}_I\} = \mathscr{D}_I$ thus $\{H_0, \mathscr{S}_I\} \approx \mu^0 \mathscr{D}_I$. In our applications it will turn out that $\{\mathscr{C}'_A, \mathscr{D}_I\} = \tilde{f}_{AI}{}^K \mathscr{D}_K,\ A \neq 0$ and $\{\mathscr{C}'_0, \mathscr{D}_I\}$ is not weakly zero. The Dirac stabiliser algorithm then replaces $\mathscr{C}'_0$ by $\mathscr{C}_0 = \mathscr{C}'_0 - F^{JI}\{\mathscr{C}'_0, \mathscr{D}_J\} S_I$ so that $\{\mathscr{C}_0, \mathscr{D}_I\} = 0$ while $\mathscr{C}'_A = \mathscr{C}_A$ for $A \neq 0$ and $H'$ is replaced by $H = \mu^A \mathscr{C}_A$. The $\mathscr{C}_A$ now close among themselves modulo $\mathscr{S}_I$. Application of $O^{(\mathscr{D})}$ replaces $H$ by $\tilde{H} = \mu^A \tilde{\mathscr{C}}_A,\ \tilde{\mathscr{C}}_A = O^{(\mathscr{D})}(\mathscr{C}_A)$. Now modulo $\mathscr{S}_I$ constraints

$$\{\tilde{\mathscr{C}}_A, \tilde{\mathscr{C}}_B\} \approx O^{(\mathscr{D})}(\{\mathscr{C}_A, \mathscr{C}_B\}^*_{\mathscr{S},\mathscr{D}}) \tag{8.21}$$

and

$$\begin{aligned}\{\mathscr{C}_A, \mathscr{C}_B\}^*_{\mathscr{S},\mathscr{D}} &\propto \{\mathscr{C}_A, \mathscr{C}_B\},\ \{\mathscr{C}_A, \mathscr{S}_I\}\{\mathscr{C}_B, \mathscr{S}_J\},\ \{\mathscr{C}_A, \mathscr{S}_I\}\{\mathscr{C}_B, \mathscr{D}_J\},\ \{\mathscr{C}_A, \mathscr{D}_I\}\{\mathscr{C}_B, \mathscr{D}_J\},\\ &\propto \mathscr{C}_A, \mathscr{S}_I, \mathscr{D}_I. \end{aligned} \tag{8.22}$$

Since $O^{(\mathscr{D})}(\mathscr{D}_I) \approx 0$ it follows that the $\tilde{\mathscr{C}}_A$ and the $\mathscr{S}_I$ form a first class algebra.

3.

Whether gauge unfixing is feasible depends largely on the question whether the series that determines $\tilde{H}$ terminates. Fortunately, in our application this will be the case.

4.
The formula $O^{(\mathscr{G})}(f) \approx O^{(\mathscr{D})}(O^{(\mathscr{G})}(f))$ does not display the fact that $\mathscr{G}$ can be reached from $\mathscr{D}$ via a gauge transformation. However, using the fact that $O^{(\mathscr{D})}(\mathscr{D}_I) \approx 0$ and that $O^{(\mathscr{G})}(f)$ is a power series in $\mathscr{G}$ we also have

$$O^{(\mathscr{G})}(f) \approx O^{(\mathscr{D})}\left(\left[\exp(\beta_I\{(\tilde{F}^{-1})^{IJ}\mathscr{S}_J,.\})\cdot f]_{\beta=-(\mathscr{G}-\mathscr{D})}\right]\right), \tag{8.23}$$

with $\tilde{F}_{IJ} = \{\mathscr{S},\mathscr{G}_J\}$. Notice that the argument of $O^{(\mathscr{D})}$ on the right hand side of (8.23) is not gauge invariant and that it is the gauge transform of $f$ with respect to the weakly Abelian constraints $\tilde{\mathscr{S}}_I = [\tilde{F}^{-1}]^{IJ}\mathscr{S}_J$ from the gauge $\mathscr{G}=0$ to the gauge $\mathscr{D}=0$ as desired.
5.
An important final comment concerns the dynamics of the theory (we consider for simplicity only one pair of second class constraints but the same discussion applies, with more notational load, to the general case): suppose first that $H' = H_0 + \lambda^I\mathscr{S}_I$, that is $H_0$ is not constrained to vanish. From the point of view of the second class system the Hamiltonian that drives the dynamics of the system is $H$ or equivalently $\tilde{H}$ via the Dirac bracket evaluated on the constraint surface of the second class system $\overline{\overline{M}}$, that is

$$\dot{f}_{|\mathscr{S}=\mathscr{D}=0} = [\{\tilde{H},f\}^*_{\mathscr{S},\mathscr{D}}]_{\mathscr{S}=\mathscr{D}=0} = \{H_{|\mathscr{S}=\mathscr{D}=0}, f_{|\mathscr{S}=\mathscr{D}=0}\}^*_{\mathscr{S},\mathscr{D}}. \tag{8.24}$$

On the other hand, from the point of view of the first class system, the Hamiltonian is $\tilde{H}$ which acts on gauge ($\mathscr{S}$-) invariant functions which we write in the form $F = O^{(\mathscr{D})}(f)$ on the constraint surface of the first class system $\overline{M}$, that is

$$\dot{F}_{|\mathscr{S}=0} = \{\tilde{H},F\}_{|\mathscr{S}=0} = \{O^{(\mathscr{D})}(H), O^{(\mathscr{D})}(f)\}_{|\mathscr{S}=0} = O^{(\mathscr{D})}(\{H,f\}^*_{\mathscr{S},\mathscr{D}})_{|\mathscr{S}=0}. \tag{8.25}$$

Comparing (8.24) and (8.25) we see that the time evolutions are isomorphic when mapping $f_{|\mathscr{S}=\mathscr{D}=0}$ to $O^{(\mathscr{D})}(f)_{|\mathscr{S}=0}$.
Now we consider the case that $H_0 = \mathscr{C}$ itself is constrained to vanish. Then also the Hamiltonian $\tilde{H}$ is constrained to vanish from the point of view of the second class system since is a linear combination of the three constraints $\mathscr{C},\mathscr{S},\mathscr{D}$. Now the following subtlety arises: from the point of view of the first class system, the Hamiltonian $\tilde{H}$ is *not* constrained to vanish because the first class system is only subject to the constraints $\mathscr{C},\mathscr{S}$. But this would clearly be wrong: the first class system would only have the

constraint $\mathscr{S}$ and this would lead to a different dimensionality of the reduced phase space than in the second class system. The correct point of view is the following: the second class system is equivalently described by the three types of constraints $\tilde{H}, \mathscr{S}, \mathscr{D}$ of which $\tilde{H}$ constitutes a first class set of constraints while $(\mathscr{S}, \mathscr{D})$ constitute a second class system of constraints. From the point of view of the first class system we just forget about the $\mathscr{D}$ constraints and instead consider the first class constraint system $\tilde{H}, \mathscr{S}$. The counting of physical number of degrees of freedom is now correct again because both first class constraints $\tilde{H}, \mathscr{S}$ count twice in the first class system while in the second class system $\tilde{H}, \mathscr{S}, \mathscr{D}$ only $\tilde{H}$ counts twice and $\mathscr{S}, \mathscr{D}$ only count once. This also makes sure that there is no true Hamiltonian in both schemes. To compare the observables from both points of view, let $\mathscr{S}_1 := \mathscr{S}$, $\mathscr{S}_2 := \tilde{H}$, $\mathscr{D}_1 := \mathscr{D}$, $\mathscr{D}_2 := \mathscr{G}$ where the gauge fixing condition $\mathscr{G}$ is chosen in such a way that the matrix with entries $F_{IJ} = \{\mathscr{S}_I, \mathscr{D}_J\}$ is non singular. It is easy to see that the second class system $(\mathscr{S}_I, \mathscr{D}_I)$ is of the type to which gauge unfixing applies and the discussion proceeds from here just as in the general case.

## 8.2 Application of gauge unfixing to gravity

We now want to apply the ideas of gauge unfixing to higher dimensional general relativity and start with the Hamiltonian system derived in section 5.2. The second class constraints are given by $\mathscr{S}^{ab}_{\overline{M}} \approx \mathscr{D}^{ab}_{\overline{M}} \approx 0$. As we pointed out in section 5.2.2, the constraints are not independent and the Dirac matrix

$$\{\mathscr{S}^{ab}_{\overline{M}}[c^{\overline{M}}_{ab}], \mathscr{D}^{cd}_{\overline{N}}[d^{\overline{N}}_{cd}]\} =: \int_\sigma d^D x \; c^{\overline{M}}_{ab} \, F^{ab}_{\overline{M}}{}^{cd}_{\overline{N}} \, d^{\overline{N}}_{cd} \tag{8.26}$$

is not invertible. We will neglect this fact for the moment and will see shortly that we can deal with it using the independent sets of constraints of section 5.2.2. We remark that gauge unfixing has been applied previously to $2+1$-dimensional linearised massive gravity [208].

The general discussion of the previous section suggests that the simplicity invariant extension of the Hamiltonian constraint involves an infinite series which is beyond any analytical control already at the classical level. Luckily, the Dirac matrix depends only on $\pi^{aIJ}$ and therefore commutes with the BF-simplicity constraint. Hence repeated commutators acting on functions that depend polynomially on $A$ vanish beyond the order of the polynomial. We calculate explicitly

$$\tilde{\mathscr{H}} = \mathscr{H} - \frac{1}{2}\mathscr{D}^{ab}_{\overline{M}} \left(F^{-1}\right)^{\overline{M}\,\overline{N}}_{ab\;cd} \mathscr{D}^{cd}_{\overline{N}}, \tag{8.27}$$

where terms up to the second order contributed, since $\mathscr{H}$ is quadratic in the connection. The effect of the extra term in the Hamiltonian can be seen when solving the simplicity constraint and reducing the theory to the ADM variables. When doing the calculation (4.65), we have to use $\mathscr{D} \sim (F\bar{K}^{\mathrm{tf}})^{aIJ} = 0$ to eliminate a term proportional to $\bar{K}^{\mathrm{tf}}{}_{aIJ}F^{aIJ,bKL}\bar{K}^{\mathrm{tf}}{}_{bKL}$. This is not necessary any more because the additional $-1/2\mathscr{D}F^{-1}\mathscr{D}$ precisely counters this term.

The Gauß and diffeomorphism constraints only obtain extra terms proportional to the BF-simplicity constraints which can be neglected in the first class theory. We can use the projector identities to calculate the new constraint algebra

$$\left\{\tilde{\mathscr{G}},\tilde{\mathscr{G}}\right\} = \tilde{\mathscr{G}} + \mathscr{S}, \tag{8.28}$$

$$\left\{\tilde{\mathscr{G}},\tilde{\mathscr{H}}^{a}\right\} = \mathscr{S}, \tag{8.29}$$

$$\left\{\tilde{\mathscr{G}},\tilde{\mathscr{H}}\right\} = \mathscr{S}, \tag{8.30}$$

$$\left\{\tilde{\mathscr{H}_a},\tilde{\mathscr{H}_b}\right\} = \tilde{\mathscr{H}_a} + \tilde{\mathscr{G}} + \mathscr{S}, \tag{8.31}$$

$$\left\{\tilde{\mathscr{H}_a},\tilde{\mathscr{H}}\right\} = \tilde{\mathscr{H}} + \tilde{\mathscr{G}} + \mathscr{S}, \tag{8.32}$$

$$\left\{\tilde{\mathscr{G}},\mathscr{S}\right\} = \mathscr{S}, \tag{8.33}$$

$$\left\{\tilde{\mathscr{H}_a},\mathscr{S}\right\} = \mathscr{S}, \tag{8.34}$$

$$\left\{\tilde{\mathscr{H}},\mathscr{S}\right\} = 0. \tag{8.35}$$

By construction it closes without the $\mathscr{D}$ constraint and displays a first class structure.

Concerning gauge invariant phase space functions, it is clear that a vanishing commutator with the BF-simplicity constraint does not constrain the dependence on $\pi^{aIJ}$. Additionally, these functions may only depend on the simplicity invariant extension of $A_{aIJ}$ which is given explicitly by

$$\tilde{A}_{aIJ} = A_{aIJ} + \mathscr{D}^{cd}_{\overline{N}} \left(F^{-1}\right)^{\overline{N}\,\overline{M}}_{cd,ab}\, \epsilon_{\overline{M}IJKL}\pi^{bKL}, \tag{8.36}$$

since $A_{aIJ}$ changes under simplicity gauge transformations as

$$\delta^{\mathscr{S}} A_{aIJ} := \left\{ A_{aIJ}, \mathscr{S}^{bc}_{\overline{M}}[c^{\overline{M}}_{bc}] \right\} = c^{\overline{M}}_{ab} \epsilon_{IJKL\overline{M}} \pi^{bKL}. \tag{8.37}$$

We still have to give a sense to $\left(F^{-1}\right)^{\overline{N}\ \overline{M}}_{cd,ab}$. As we have shown in section 5.2.2, it is enough to consider only a subspace of Lagrange multipliers for the BF-simplicity and $\mathscr{D}^{ab}_{\overline{M}}$ constraints parametrised by the projected test functions

$$d^{\overline{M}}_{ab} = \bar{d}_{(a|IJ} \pi_{b)KL} \epsilon^{IJKL\overline{M}}. \tag{8.38}$$

On this subspace, $F^{aIJ,bKL}$ was shown to be invertible. We therefore make the ansatz

$$\left(F^{-1}\right)^{\overline{N}\ \overline{M}}_{cd,ab} = \alpha \epsilon^{EFGH\overline{N}} \pi_{(c|EF} \left(F^{-1}\right)_{d)GH,(a|AB} \pi_{b)CD} \epsilon^{ABCD\overline{M}} \tag{8.39}$$

for some constant $\alpha$, where

$$\left(F^{-1}\right)_{aIJ,bKL} := \frac{-s}{(D-1)} \pi_{aAC} \pi_{bBD} \left(\pi^{cEC} \pi_{cE}{}^{D} - s\eta^{CD}\right) \left(\eta^{AB} \eta_{K[I} \eta_{J]L} - 2\eta^{A}_{[L} \eta_{K][J} \eta^{B}_{I]}\right) \tag{8.40}$$

only depends on $\pi^{aIJ}$ and reduces to the correct expression on the simplicity constraint surface when contracted in the above equation. Insertion into $\tilde{A}$ yields

$$\alpha = \frac{1}{16(D-1)^2((D-3)!)^2} \tag{8.41}$$

when demanding that $\tilde{A}$ is independent of $\mathscr{D}$, i.e. that the $\bar{K}^{\text{tf}}{}_{aIJ}$ term is cancelled. Since all simplicity invariant phase space functions are arbitrary functions of $\tilde{A}_{aIJ}$ and $\pi^{aIJ}$, we have shown that the proposed expression for $\left(F^{-1}\right)^{\overline{N}\ \overline{M}}_{cd,ab}$ yields the desired results. This can of course also be obtained by direct inversion of the projected version of the matrix $F$. This way we obtain a connection formulation for gravity in $D+1>3$ without second class constraints. Notice however that the observables (with respect to the simplicity constraint) $(\tilde{A}, \pi)$ have complicated Poisson brackets, only the brackets of the canonical pair $(A, \pi)$ are simple, therefore suggesting a Dirac quantisation approach (quantisation at the kinematical level).

Let us summarise and compare with the connection formulation in $D+1=4$:

1.

On the surface where the simplicity constraint vanishes, $\pi^{aIJ} = 2n^{[I}E^{a|J]}$, we can describe the situation more explicitly. From the above formula it is obvious that both

$n^I \delta^{\mathscr{S}} A_{aIJ} = 0$ and $E^{aI} \delta^{\mathscr{S}} A_{aIJ} = 0$, since we always may choose $c^{\overline{M}}_{[ab]} = 0$. Thus, when decomposing the connection $A_{aIJ} = \Gamma_{aIJ} + \bar{K}_{aIJ} + 2n_{[I} K_{a|J]}$ into hybrid spin connection and rotational (i.e. transversal) and boost (i.e longitudinal) components of hybrid contorsion $K_{aIJ}$, we find that the simplicity constraint generates on-shell gauge transformations of the trace free part of the rotational (transversal) components of the SO$(1, D)$ (or SO$(D+1)$ in the Hamiltonian framework of the previous chapter) hybrid contorsion $\bar{K}^{\text{tf}}{}_{aIJ}$. As we have seen in equation (4.60), the remaining trace component of the rotational part $\bar{K}^{\text{tr}}_J$ is proportional to the boost part of the Gauß constraint and vanishes if $n_I \mathscr{G}^{IJ} = 0$ holds. In total, we find that observables in this connection theory may not depend on the value of the rotational components of the SO$(1, D)$ (or SO$(D+1)$) hybrid contorsion at all. The whole physical information contained in the connection is encoded in the boost components of the contorsion, which becomes conjugate to the vielbein after solving the simplicity constraint. Therefore, when removing the boost gauge freedom by choosing the time gauge, there is no physical information left in the SO$(D)$ connection.

2.

In $D + 1 = 4$, this formulation therefore differs from the formulation in terms of real Ashtekar variables considered in [146], which remains a connection formulation also after imposing the time gauge. This is achieved by mixing boost and rotational components of the connection using the total antisymmetric tensor, i.e. ${}^{(\gamma)}A_{ajk} = A_{ajk} - \gamma \epsilon_{0ijk} A_{a0i}$, to "rotate" physical degrees of freedom into the rotational components of the connection, and $\gamma$ now is the Barbero Immirzi parameter. Thus, this procedure exploits a peculiarity of dimension $D + 1 = 4$, and therefore is not possible in any other dimension. As we have seen, it is possible to arrive at the new connection formulation also by enlarging the ADM phase space. Following this route allowed for the introduction of a free parameter $\beta$ similar to the Barbero-Immirzi parameter, but the transformation made to obtain the connection is very different in nature since there is no mixing of boost and rotational parts. This will become even clearer in section 9.3, when we will restrict to $D+1 = 4$ and then introduce two free parameters, the one being $\beta$ of chapter 7 and the other one corresponding to the Barbero Immirzi parameter $\gamma$ (cf. chapter 6).

# 9

# Extensions and related material

## 9.1 Linear simplicity constraint

In [6], an $\mathrm{SO}(1,D)$ or $\mathrm{SO}(D+1)$ connection formulation was introduced which, instead of the quadratic simplicity constraint we considered so far, involves a linear simplicity constraint similar to the one used in the new spin foam models [186–191]. While in these approaches, discrete versions of the constraints appear, continuum versions of the linear simplicity constraints already appeared in [209]. However, their treatment is rather different than the one displayed here, having a focus on the Lagrangian formulation and constructing the linear constraint using a three form field. In our case instead, an additional unit length scalar field $N^I$ is introduced, which upon solving the linear simplicity constraint will coincide with the hybrid vielbein normal $n^I[E]$. Moreover, as we will see later in section 15.1, the way to couple the Rarita Schwinger field ("gravitino") of supergravity theories as shown in [6] uses these normal fields $N^I$ (and therefore the use of the linear simplicity constraints) in an intricate way, and so far it is unknown if this can also be achieved with the quadratic constraint. In the following, we will shortly outline the construction of this first class constrained system in any dimensions, which upon symplectic reduction again is equivalent to the ADM formulation. The quantisation of the additional field $N^I$ [6] and the implementation of both, the linear and quadratic simplicity constraints at the quantum level [3, 5], will be discussed in section 11.2. Our exposition in this section will follow [6].

### 9.1.1 Introducing linear simplicity constraints

We want to remind the reader of the solution[1] to the quadratic simplicity constraint $\mathscr{S}^{ab}{}_{\overline{M}} = 0 \Leftrightarrow \overset{(\beta)}{\pi}{}^{aIJ} = \frac{2}{\beta} n^{[I} E^{a|J]}$, where $n^I[E]$ the unique (up to sign) unit normal to the hybrid vielbein $E^{aI}$. While the quadratic simplicity constraint is defined solely in terms of $\overset{(\beta)}{\pi}{}^{aIJ}$, the linear constraint usually demands that there is a vector field $N^I$ such that

$$\mathscr{S}^{a}_{I\overline{M}} := \epsilon_{IJKL\overline{M}} \; N^J \; \overset{(\beta)}{\pi}{}^{aKL} \tag{9.1}$$

vanishes. This equation defines the linear constraint we want to consider in the following. We do not want to fix the vector field $N^I$ by hand, so we have to postulate it as new phase space degrees of freedom together with its conjugate momentum $P_J$. For any dimension $D \geq 3$, this constraint demands that $\overset{(\beta)}{\pi}{}^{aIJ}$ is a simple bivector with one of its factors being $N^I$, $\overset{(\beta)}{\pi}{}^{aIJ} = \frac{2}{\beta} N^{[I} E^{a|J]}$, where we can choose w.l.o.g. $N^I E^a{}_I = 0$. The hybrid vielbein, however, which now encodes the physical information in $\overset{(\beta)}{\pi}{}^{aIJ}$, already fixes the direction of $N^I$ (up to sign). To get rid of the unphysical information about $N$'s length, we add the normalisation constraints

$$\mathscr{N} := N^I N_I - \zeta, \tag{9.2}$$

and reobtain the solution of the quadratic constraints. Of course, these constraints up to now only reduce $\overset{(\beta)}{\pi} \to E$ and $N \to n(E)$. To account for the momenta $P_J$ being non-physical, we could introduce additional constraints. However, if we manage to implement $\mathscr{S}$, $\mathscr{N}$ being first class, we expect that the gauge transformations of these constraints get rid of the additional degrees of freedom in $P_J$.

Apart from the change from quadratic to linear simplicity and normalisation constraints, we want to construct the theory similar to what we did in chapter 7. The phase space is coordinatised by $\{A_{aIJ}, \overset{(\beta)}{\pi}{}^{bKL}, N^I, P_J\}$ with the non-vanishing Poisson brackets (7.58) and

$$\left\{N^I(x), P_J(y)\right\} = \delta^I_J \delta^{(D)}(x-y). \tag{9.3}$$

[1] For $D = 3$, the quadratic constraint allows for an additional topological solution sector, which we excluded by hand. The linear constraint will not allow for these solutions.

This phase space is subject to the already introduced linear simplicity and normalisation constraints, but of course, to reproduce general relativity, we have to add more constraints. In order that the constraint algebra closes, we change the definition of $\mathscr{G}^{IJ}$ and $\mathscr{H}_a$ such that they generate $\mathrm{SO}(D+1)$ or $\mathrm{SO}(1,D)$ gauge transformations and spatial diffeomorphisms respectively on all phase space variables, in particular as well on $N^I$, $P_J$,

$$\mathscr{G}^{IJ} = D^A{}_a \overset{(\beta)}{\pi}{}^{aIJ} + 2P^{[I}N^{J]}, \tag{9.4}$$

$$\mathscr{H}_a = \frac{1}{2}\overset{(\beta)}{\pi}{}^{bIJ}\partial_a A_{bIJ} - \frac{1}{2}\partial_b\left(\overset{(\beta)}{\pi}{}^{bIJ}A_{aIJ}\right) + P_I\partial_a N^I. \tag{9.5}$$

This already makes sure that the algebra of all the constraints we introduced so far is closing. For the Hamiltonian constraint, we will work with the original version which is obtained by simply replacing the ADM variables in the ADM Hamiltonian constraint by (7.60, 7.61),

$$\mathscr{H} = -\frac{1}{4\sqrt{q}}\left[\overset{(\beta)}{\pi}{}^{[a|IJ}\overset{(\beta)}{\pi}{}^{b]KL}\left(A-\Gamma[\pi]\right)_{bIJ}\left(A-\Gamma[\pi]\right)_{aKL}\right] - \frac{s}{2}\sqrt{q}R[\pi]. \tag{9.6}$$

In particular, we cannot add to the Hamiltonian constraint the terms one would expect for gravity coupled the space time scalar fields $N^I$, since $\{\mathscr{H},\mathscr{S}^a_{\overline{IM}}\}$ and $\{\mathscr{H},\mathscr{N}\}$ would not vanish weakly, spoiling the constraint algebra.

Like in the case of the quadratic constraint, to prove that this constrained system is indeed equivalent to the ADM formulation, we first will define (weak) Dirac observables with respect to Gauß, simplicity and normalisation constraints corresponding to $q_{ab}$, $P^{cd}$, show that their Poisson brackets at least weakly reproduce the ADM canonical brackets and furthermore show that the constraint algebra is closing and, in particular, the Poisson brackets between $\mathscr{H}$ and $\mathscr{H}_a$ weakly reduces to (2.31). Since we already gave calculational details when dealing with the quadratic constraints in chapter 7 and the analysis in this case is analogous, we will be rather brief here.

The map to the ADM phase space is of course again given by (7.60, 7.61). $q_{ab}$, $P^{cd}$ are obviously still Gauß invariant and trivially Poisson commute with the normalisation constraint, and since we still have $\{P,\mathscr{S}\}\propto\mathscr{S}$, both are also (weak) simplicity observables. That the ADM brackets $\{q,q\}$, $\{q,P\}$ are reproduced follows directly from the treatment in chapter 7. To conclude that $\{P, P\}$ still is weakly zero needs some more work, since we changed both, the simplicity and the Gauß constraint which were needed in this calculation. However, since the solutions to the quadratic simplicity constraint and the linear simplicity and normalisation constraints coincide, we conclude that whatever terms weakly vanished due to $\mathscr{S}^{ab}{}_{\overline{M}} = 0$ will now also weakly vanish when $\mathscr{S}^{a}_{I\overline{M}} = 0 = \mathscr{N}$. Furthermore, we did not need the whole Gauß constraint, but rather $\mathscr{G}_{ab} := \overset{(\beta)}{\pi}{}^{cIJ} q_{c[a}(A - \Gamma[\pi])_{b]IJ}$, which simplicity on shell already vanishes modulo the rotational components $\bar{\mathscr{G}}^{IJ}$ of the Gauß constraints. The rotational components of the Gauß constraint are, however, still unchanged, since the additional term $2P^{[I}N^{J]} \approx 2P^{[I}n^{J]}[E]$ weakly is pure "boost" modulo the linear simplicity and normalisation constraints. Therefore, the ADM canonical brackets are weakly reproduced like in the case of the quadratic constraint. We are left with studying the constraint algebra, or more precisely, the Hamiltonian constraint, since we have already seen that all the other constraints weakly Poisson commute. Since $\mathscr{H}$ is constructed using the maps to the ADM phase space, we know that it weakly Poisson commutes with $\mathscr{G}^{IJ}$, $\mathscr{S}^{a}_{I\overline{M}}$, $\mathscr{N}$, and of course also with the generator of spatial diffeomorphisms $\mathscr{H}_a$. Left with the bracket between two Hamiltonian constraint, we invoke the previous result that the ADM brackets are reproduced to conclude that

$$\{\mathscr{H}[N], \mathscr{H}[M]\} \approx -s\mathscr{H}'_a[q^{ab}\,(N\partial_b M - M\partial_b N)], \tag{9.7}$$

where $\mathscr{H}'_a = -2q_{ac}D_b P^{bc}$ now denotes the ADM diffeomorphism constraint. It is straightforward to show that $\mathscr{H}_a$ and $\mathscr{H}'_a$ are weakly equivalent and therefore, not only does the constraint algebra close, but moreover the hypersurface deformation algebra is weakly reproduced. Concerning the counting of the number of degrees of freedom, note that the constraints $\mathscr{S}^{a}_{I\overline{M}}$ again are not irreducible, but we know that they reduce $\overset{(\beta)}{\pi}{}^{aIJ} \to E^{aI}$ and $N^I \to ||N||$, therefore removing $\frac{D(D+1)(D-2)}{2} + D = \frac{D^2(D-1)}{2}$ degrees of freedom (without modding out by their gauge orbits). We obtain the familiar $(D+1)(D-2)$ phase space degrees of freedom of general relativity.

Finally, we want to remark that related formulations of general relativity with (timelike) normal as independent dynamical field, already exist in the literature [146, 194, 196]. However, while our formulation features both the simplicity constraint and the timelike normal vector field at the same time, in the other approaches this field only appears in

| Variable | Dof | 1st cl. constraints | Dof (count twice!) |
|---|---|---|---|
| $A_{aIJ}$ | $\frac{D^2(D+1)}{2}$ | $\mathscr{H}$ | 1 |
| $\overset{(\beta)}{\pi}{}^{bKL}$ | $\frac{D^2(D+1)}{2}$ | $\mathscr{H}_a$ | $D$ |
| $N^I$ | $D+1$ | $\mathscr{G}^{IJ}$ | $\frac{D(D+1)}{2}$ |
| $P_J$ | $D+1$ | $\mathscr{S}^a_{\overline{IM}}$ | $\frac{D^2(D-1)}{2}$ |
| | | $\mathscr{N}$ | 1 |
| Sum: | $D^3+D^2+2D+2$ | Sum: | $D^3+3D+4$ |

**Table 9.1:** The new variables with linear simplicity constraint: counting of degrees of freedom

the process of solving the simplicity constraint while keeping the whole Gauß constraint, i.e. not choosing time gauge. In other words, the time normal is an integral part of the simplicity constraint in our approach, not a concept emerging after its solution.

### 9.1.2 Solution of the linear simplicity constraints

Symplectic reduction with respect to the linear simplicity and normalisation constraint of course is analogous to the treatment in section 4.2.4. Using the solution $\overset{(\beta)}{\pi}{}^{aIJ} = \frac{2}{\beta} n^{[I} E^{a|J]}$ and the ansatz $A_{aIJ} = \Gamma_{aIJ}[\pi] + \beta K_{aIJ}$, we find

$$\begin{aligned}\frac{1}{2}\overset{(\beta)}{\pi}{}^{aIJ}\dot{A}_{aIJ} + P_I\dot{N}^I &\approx -\zeta\bar{K}_{aJ}\dot{E}^{aJ} - \bar{K}_{aIJ}E^{aJ}\dot{n}^I + \bar{P}_I\dot{n}^I \\ &\approx \left[-\zeta\bar{K}_{aJ} - n_J E_{aI}\left(\bar{K}^{\mathrm{tr}I} + \bar{P}^I\right)\right]\dot{E}^{aJ} \\ &=: E^{aJ}\dot{K}_{aJ}, \end{aligned} \tag{9.8}$$

where we have dropped total time derivatives and divergences. Compared with $K_{aI}$ in (4.62), $K_{aI}$ here is defined with an additional term $\propto \bar{P}^I$. In terms of these variables, we find for the constraints, like after the reduction in section 4.2.4, the eADM expressions (3.27, 3.28, 3.29).

## 9.2 SO$(p, D+q)$ formulation

This section will be once again inspired by Peldán's programme. With Ashtekar's new variables, general relativity was formulated on a Yang Mills phase space, which suggested to consider the unification of these two theories. After work of Peldán in 2+1 dimensions [210], in [105, 106] Chakraborty and Peldán studied such unified models inspired by Ashtekar's new formulation also in 3+1 dimensions, which, with certain choice of gauge group would reproduce general relativity, while in a weak field expansion around de Sitter spacetime, would give conventional Yang-Mills theory to the lowest order. The way the authors proceeded was to generalise Ashtekar's formulation to arbitrary gauge group (without adding new constraints), i.e. a phase space coordinatised by a connection $A$ valued in the corresponding Lie algebra and a conjugate (Lie algebra valued) generalised vielbein, subject to Gauß, Hamiltonian and spatial diffeomorphism constraint. As we have seen in chapter 7, this phase space will in general have more physical degrees of freedom than general relativity, allowing for extra Yang Mills degrees of freedom. However, since these models are based on the original complex Ashtekar variables, no mathematically rigorous quantisation thereof can be carried out, and, more severely, already classically their construction in Lorentzian signature is incomplete.

Here, we will study a different possibility of obtaining unified theories, which is however in the same spirit as Peldán's treatment: Having constructed a $\mathrm{SO}(D+1)$ or $\mathrm{SO}(1,D)$ Hamiltonian connection formulation for general relativity, it is rather not surprising that one can extend the gauge group further to $\mathrm{SO}(p, D+q)$, with $p \geq 0$, $q \geq 0$, $p+q =: k \geq 1$. The idea of how to obtain unified models is then to start with this pure gravity formulation, to drop first class constraints, which for sure will enlarge the number of physical degrees of freedom, and to study what kinds of matter coupled theories can be obtained in that way. Note that these groups are of particular interest for unification, since in particular they include SO(10), which one of the GUT models is based on [211, 212].

We will build up this formulation step by step, first studying the ingredients necessary to extend the usual vielbein formulation to the gauge groups $\mathrm{SO}(D+p,q)$, and in a second step turning it into a real connection formulation.

We will show that this programme probably fails: While we are able to give an explicit construction of an $\mathrm{SO}(D+p,q)$ Yang Mills formulation of pure general relativity, all of the additionally introduced first class constraints are needed in order to obtain a first class Hamiltonian constraint in this extended theory, and dropping constraints has to come, if at all possible, with non-trivial alterations of the Hamiltonian constraint.

### 9.2.1 Extension of the eADM phase space

Let us start by extending the eADM phase space of section 3.2.3 further. We will still use $K_{aI}$, $E^{bJ}$ as phase space coordinates, but now understand that they are $\mathbb{R}^{p,D+q}$ valued, i.e. $I, J, ... \in \{0, 1, ..., D+k-1\}$. Internal indices are now moved with $\eta_{IJ} = \text{diag}(\underbrace{-,...,-}_{p}, \underbrace{+,...,+}_{D+q})_{IJ}$. We will call this vielbein a $k$-hybrid vielbein, the internal space having $k$ dimensions more than the spacetime. This makes the hybrid vielbein considered before a 1-hybrid vielbein. When compared to the theory with 1-hybrid vielbein, the enlarged vielbein now has $D(D+k)-D(D+1) = D(k-1)$ additional degrees of freedom, while the Gauß constraint obtains $\frac{(D+k)(D+k-1)}{2} - \frac{D(D+1)}{2} = \frac{(2D+k)(k-1)}{2}$ additional components. Therefore, there now are $(2D+k)(k-1)-2D(k-1) = k(k-1)$ *less* physical phase space degrees of freedom, and we are forced to introduce additional fields.

A possible way how to proceed is to introduce $x$ space time scalars / internal (unit) vector fields, $n_i{}^I$, $i, j, ... \in \{1, ..., x\}$, vielbein- and mutually orthogonal,

$$\mathscr{C}^a_i := n_i{}^I E^a{}_I = 0, \tag{9.9}$$

$$\mathscr{C}_{ij} := n_i{}^I n_{jI} - \eta_{ij} = 0, \tag{9.10}$$

where $\eta_{ij} = \text{diag}(\underbrace{-,...,-}_{p}, \underbrace{+,...,+}_{q})_{ij}$, together with their conjugate momenta $p^i{}_J$. These are $2x(D+k)$ additional phase space degrees of freedom, subject to $Dx + \frac{x(x+1)}{2}$ constraints (note that their mutual orthogonality constitutes a symmetric constraint). Demanding that these fields account for the missing degrees of freedom results in a quadratic equation for $x$, $k(k-1) = 2x(D+k) - 2Dx - x(x+1)$, which has the solutions $x = k$ and $x = k-1$. I.e., we can either introduce a "completion" of the vielbein $n_1{}^I$, ... , $n_k{}^I$ like in appendix C.3, or we can drop one of these normals, say $n_k{}^I$. This was expected, since the the $k$-hybrid vielbein plus the $k-1$ mutually orthogonal unit normals already fix $n_k{}^I$ (up to sign). Both options can be worked out, here we proceed with introducing all $k$ normals.

The non-vanishing Poisson brackets are given by

$$\{K_{aI}(x), E^{bJ}(y)\} = \delta_a^b \eta_I^J \delta^{(D)}(x-y), \tag{9.11}$$

$$\{n_{iI}(x), p^{jJ}(y)\} = \eta_i^j \eta_I^J \delta^{(D)}(x-y), \tag{9.12}$$

where $i, j, ... \in \{1, .., k\}$. In order that the constraints (9.9, 9.10) transform nicely under $\mathrm{SO}(p, D+q)$ transformations and spatial diffeomorphisms, we will define the generators such that they act also on the newly introduced fields,

$$\mathscr{G}^{IJ} := 2E^{a[I}K_a{}^{J]} + 2p^{i[I}n_i{}^{J]}, \tag{9.13}$$

$$\mathscr{H}_a := E^{bJ}\partial_a K_{bJ} - \partial_b(E^{bJ}K_{aJ}) + p^{iI}\partial_a n_{iI}, \tag{9.14}$$

$$\mathscr{H} := \frac{1}{\sqrt{q}} E^{a[I|}E^{b|J]}K_{aI}K_{bJ} - \frac{s}{2}\sqrt{q}{}^{(D)}R, \tag{9.15}$$

where summation convention is also used for $i, j$, etc. Note that we kept the Hamiltonian constraint in the form it appeared in (3.29), in particular did not add the terms one would expect when minimally coupling these scalar fields to general relativity. The reason is the same as when using the linear simplicity constraint in section 9.1: The Hamiltonian constraint would pick up terms $\propto p^2$, which would spoil the constraint algebra since $\mathscr{H}$ would no longer (weakly) Poisson commute with the $\mathscr{C}$ constraints. In its current form, $\mathscr{H}$ depends only on $K_{aI}E^{bI}$, and it is easy to convince oneself that this combination weakly Poisson commutes with $\mathscr{C}_i^a$. All other Poisson brackets vanish trivially (at least weakly), except the one between two Hamiltonian constraints. For this, we find explicitly

$$\{\mathscr{H}[N], \mathscr{H}[M]\} = \mathscr{H}'_a[q^{ab}(N\partial_b M - M\partial_b N)] + \frac{1}{2}\mathscr{G}^{IJ}[\frac{1}{q}E^a{}_I E^b{}_J D_b(N\partial_a M - M\partial_a N)], \tag{9.16}$$

where in the Gauß constraint term we used that $K^{[a}{}_I E^{b]I} = \frac{1}{2q}\mathscr{G}^{IJ}E^a{}_I E^b{}_J$ and defined

$$\mathscr{H}'{}_a := 2D_{[a}(K_{b]I}E^{bI}), \tag{9.17}$$

which coincides with the original spatial diffeomorphism constraint (3.28) of the eADM formulation. The above calculation is greatly simplified using (A.8) and noting that due to the antisymmetry in $M$, $N$, only terms with derivatives on the multipliers can survive.

To furnish the proof that the constraint algebra closes, we have to show that $\mathscr{H}'_a \approx \mathscr{H}_a$. What saves the day at this point is the result of appendix C.3, namely that there exists a $k$-hybrid spin connection

$$\Gamma^{\mathrm{H}}_{aIJ} := e^b{}_{[I|} D_a e_{b|J]} + n^i{}_{[I|} \partial_a n_{i|J]} \tag{9.18}$$

annihilating $E^{aI}$ and all of the $n_i$s. Using this, we have

$$\begin{aligned}
\mathscr{H}'_a &= 2D_{[a}(K_{b]I}E^{bI}) \\
&= K_{bI} D_a E^{bI} + E^{bI} \partial_a K_{bI} - \partial_b (K_{aI} E^{bI}) + 2\Gamma_{[ab]}{}^c K_{cI} E^{bI} \\
&= -K_{bI} \Gamma^{\mathrm{H}}{}_a{}^{IJ} E^b{}_J + \mathscr{H}_a - p^{iI} \partial_a n_{iI} \\
&= \mathscr{H}_a + \frac{1}{2} \Gamma^{\mathrm{H}}{}_{aIJ} \mathscr{G}^{IJ} - p^{iI} (\partial_a n_{iI} + \Gamma^{\mathrm{H}}{}_{aIJ} n_i{}^J) \\
&= \mathscr{H}_a + \frac{1}{2} \Gamma^{\mathrm{H}}{}_a{}^{IJ} \mathscr{G}_{IJ},
\end{aligned} \tag{9.19}$$

where in the second line, we only wrote out the terms of the first line explicitly, then dropped the last summand due to torsion freeness, in the first summand used that $\Gamma^{\mathrm{H}}$ annihilates $E^{aI}$ and replaced the two summands in the middle of the second line by $\mathscr{H}_a$ minus the terms involving the normals. From line three to four, we replaced the first summand by the Gauß constraint minus the terms involving the normals, and in the last step used that $\Gamma^{\mathrm{H}}$ also annihilates all $n_i$s.

With this, we already want to end our proof that the constrained system (9.11, 9.12; 9.9, 9.10, 9.13, 9.14, 9.15) gives general relativity, since in the gauge $n_i{}^I = \eta^I_i$, it obviously reduces to the SO($D$) eADM formulation.

### 9.2.2 Connection formulation

The transition from the $k$-hybrid vielbein formulation to a connection formulation now is completely analogous to chapter 7. We introduce the canonical pair of variables $A_{aIJ}, \overset{(\beta)}{\pi}{}^{bKL}$ and $n_{iI}$, $p^{jJ}$, $i, j \in \{1, ..., k-1\}$, together with

1.) an additional canonical pair $N_I$, $P^J$ (the $k$th unit vector), subject to the constraints

$$\mathscr{S}^a_{I\overline{M}} := \epsilon_{IJKL\overline{M}} N^J \overset{(\beta)}{\pi}{}^{aKL}, \tag{9.20}$$

$$\mathscr{C}^a_i := \overset{(\beta)}{\pi}{}^{aIJ} N_I n_{iJ}, \qquad i \in \{1,..,k-1\}, \tag{9.21}$$

$$\mathscr{C}_{ij} := n_{iI} n_j{}^I - \eta_{ij}, \qquad i,j \in \{1,...,k\}, \tag{9.22}$$

where in the last constraint we understand that $n_{kI} = N_I$. The linear simplicity constraint demands that $\overset{(\beta)}{\pi}{}^{aIJ} = \frac{2}{\beta} N^{[I|} E^{a|J]}$ and the remaining ones then give the constraints of the $k$-hybrid vielbein formulation,

or

2.) the constraints

$$\mathscr{S}^{ab}{}_{\overline{M}} := \frac{1}{4} \epsilon_{IJKL\overline{M}} \overset{(\beta)}{\pi}{}^{aIJ} \overset{(\beta)}{\pi}{}^{bKL}, \tag{9.23}$$

$$\mathscr{C}^{aI}{}_i := \pi^{aIJ} n_{iJ}, \qquad i \in \{1,...,k-1\}, \tag{9.24}$$

$$\mathscr{C}_{ij} := n_{iI} n_j{}^I - \eta_{ij}, \qquad i,j \in \{1,...,k-1\}. \tag{9.25}$$

The quadratic simplicity constraint enforces $\overset{(\beta)}{\pi}{}^{aIJ} = \frac{2}{\beta} \tilde{N}^{[I|} E^{a|J]}$, and $\mathscr{C}^{aI}{}_i$ demands that both, $\tilde{N}^I$ and $E^{aI}$ are orthogonal to all $n_{iI}$, $i \in \{1,...,k-1\}$.

Again both possibilities can be worked out. We will continue with case 1.). Additional to the constraints we introduced so far, we of course again need

$$\mathscr{G}^{IJ} := D^A{}_a \overset{(\beta)}{\pi}{}^{aIJ} + 2p^{i[I} n_i{}^{J]} + 2P^{[I} N^{J]}, \tag{9.26}$$

$$\mathscr{H}_a := \frac{1}{2} \overset{(\beta)}{\pi}{}^{bIJ} \partial_a A_{bIJ} - \frac{1}{2} \partial_b ( \overset{(\beta)}{\pi}{}^{bIJ} A_{aIJ}) + p^{iI} \partial_a n_{iI} + P^I \partial_a N_I, \tag{9.27}$$

$$\mathscr{H} := -\frac{1}{4\sqrt{q}} \left[ \overset{(\beta)}{\pi}{}^{[a|IJ} \overset{(\beta)}{\pi}{}^{b]KL} \left(A - \Gamma[\pi, n_i, N]\right)_{bIJ} \left(A - \Gamma[\pi, n_i, N]\right)_{aKL} \right] - \frac{s}{2} \sqrt{q} R[\pi], \tag{9.28}$$

where the sums over $i$ here and on the following run from $1,...,k-1$. Note the change in the Hamiltonian constraint when compared to (9.6): The extensions off the simplicity constraint surface of the $k$-hybrid spin connection now necessarily also depends on the unit vectors[1] $n_i$, $N$, e.g.

$$\Gamma_{aIJ}[\pi, n_i, N] = \zeta \pi^b{}_{[I|K} D_a \pi_{b|J]}{}^K - 2\zeta N_{[I} n^i{}_{J]} n_{iM} \partial_a N^M + n^i_{[I|} \partial_a n_{i|J]} + \zeta(3-D) N_{[I|} \partial_a N_{|J]}, \tag{9.29}$$

where $\zeta = ||N||$. One can check that (7.60, 7.61), where again $\Gamma[\pi]$ has to be replaced by $\Gamma[\pi, n_i, N]$, define (weak) Dirac observables with respect to the kinematical constraints and that the ADM canonical Poisson brackets are reproduced. From this, it again easily follows that the constraint algebra is first class and the system indeed is equivalent to the ADM formulation. Moreover, symplectic reduction with respect to the linear simplicity constraint immediately leads back to the $k$-hybrid vielbein formulation of the previous section. We leave it to the interested reader to work out the details.

| Variable | Dof | $1^{\text{st}}$ cl. constraints | Dof (count twice!) |
|---|---|---|---|
| $A_{aIJ}$ | $\frac{D(D+k)(D+k-1)}{2}$ | $\mathscr{H}$ | $1$ |
| $\overset{(\beta)}{\pi}{}^{bKL}$ | $\frac{D(D+k)(D+k-1)}{2}$ | $\mathscr{H}_a$ | $D$ |
| $n_{iI}$ | $k(D+k)$ | $\mathscr{G}^{IJ}$ | $\frac{(D+k)(D+k-1)}{2}$ |
| $p^{jJ}$ | $k(D+k)$ | $\mathscr{S}^a_{\overline{IM}}$ | $\frac{D(D+k-1)(D+k-2)}{2}$ |
| | | $\mathscr{C}^a_i$ | $D(k-1)$ |
| | | $\mathscr{C}_{ij}$ | $\frac{k(k+1)}{2}$ |
| Sum: | $D^3 + D^2(2k-1)$ $+D(k^2+k) + 2k^2$ | Sum: | $D^3 + D^2(2k-2)$ $+D(k^2+k+1) + 2k^2 + 2$ |

**Table 9.2:** SO($p, D+q$) formulation with linear simplicity constraint: counting of degrees of freedom ($k = p+q$)

Finally, let us comment on increasing the phase space degrees of freedom by dropping constraints of the set $\{\mathscr{S}, \mathscr{C}\}$ while retaining a first class constraint algebra: That this is non-trivial can already be seen from (9.19): To conclude that $\{\mathscr{H}, \mathscr{H}\} \approx 0$, we needed that the $k$-hybrid spin connection annihilates $E^{aI}$ as well as all unit vectors. Now, for the existence of the $k$-hybrid spin connection and to show that $\Gamma[\pi, n_i, N]$

[1] Actually, the dependence on one of the unit vectors can be removed by expressing it as function of the other $(k-1)$ unit vectors and $\pi$. However, to simplify the final expression, we refrain from doing so.

in (9.29) reduces to this connection, we needed the mutual orthogonality properties of $E^{aI}$, $n_{iI}$ and $N_I$, which only hold if all constraints $\{\mathscr{S}, \mathscr{C}\}$ are imposed.

However, dropping of constraints comes with the possibility of changing existing constraints: E.g., the previously forbidden scalar field terms $p^2$ one would expect in the Hamiltonian constraint become in principle allowed as soon as none of the constraints in the reduced set $\{\mathscr{S}, \mathscr{C}\}$ depends on the corresponding unit vector $n$ anymore. That these terms miraculously cure the problems caused by the spin connection terms is, however, rather unlikely. We leave the study of this issue for further research.

## 9.3 $D = 3$: Revival of the Barbero-Immirzi parameter

This section is taken from [2] with minor modifications.

In the special case of $3+1$ dimensions, it is straightforward to reintroduce the Barbero Immirzi parameter $\gamma$: Use the method of gauge unfixing to the result of the canonical analysis of the Holst action in section 6.2. The Dirac matrix $F$, which has to be inverted for gauge unfixing, is very simple in this case, given by $F^{ab\,cd} = -\frac{4s\gamma^2}{\gamma^2 - s} q^2 G^{ab\,cd}$ as we have seen. We would obtain a connection formulation with (first class) quadratic simplicity constraints and gauge group SO$(3,1)$, which reduces to the Ashtekar Barbero formulation after solving the simplicity and boost Gauß constraints. (Another straightforward calculations shows that the same procedure gives a possible solution to the open issue $(i)$ in [213]). For quantisation purposes, it again would be nice to be able to work with the compact gauge group SO$(4)$ instead of the Lorentz group. Moreover, the linear simplicity constraint, which was introduced in section 9.1, is favoured in $3+1$ dimensions, since the quadratic simplicity constraint allows for unphysical solutions, usually called the topological sector. Last but not least, a formulation with Barbero Immirzi parameter probably is as near as we can get to the Ashtekar Barbero variables, and with linear simplicity constraints it also maximally mimics the new spin foam models. In this appendix, we will show by extending ADM phase space that both, the formulations with flipped internal signature and with quadratic or linear simplicity constraints, exist.

### 9.3.1 Quadratic simplicity constraints

We start with the formulation given in section 7.2 with variables $\left\{K_{aIJ}, \pi^{bKL}\right\}$. Restricting to $3+1$ dimensions, we can perform a canonical transformation to the pair of variables defined by

$$\underset{(\gamma,\beta)}{K}{}_{aIJ} := \beta(\overset{(\gamma)}{\mathscr{M}}{}^{-1})_{IJ}{}^{KL}K_{aKL} := \beta\overset{(\gamma)}{\mathscr{M}}{}^{-1}K_{aIJ}, \tag{9.30}$$

$$\overset{(\gamma,\beta)}{\pi}{}^{aIJ} := \frac{1}{\beta}(\overset{(\gamma)}{\mathscr{M}})^{IJ}{}_{KL}\pi^{aKL} := \frac{1}{\beta}\overset{(\gamma)}{\mathscr{M}}\pi^{aIJ}, \tag{9.31}$$

where the matrices $\mathscr{M}$, $\mathscr{M}^{-1}$ are given in (6.2) and (6.3), and $\gamma \in \mathbb{R}/\{0\}$, $\gamma^2 \neq \zeta$, is the Barbero Immirzi parameter. This transformation is, of course, inspired of our canonical treatment of the Holst action in section 6.2. Note that we introduced a second free parameter $\beta$ coming from a constant rescaling, which already appeared in section 7.2. To obtain a connection formulation, we the would like to use the canonical pair of variables given by $A_{aIJ} := (\Gamma[\pi] + \underset{(\gamma,\beta)}{K})_{aIJ}$ and $\overset{(\gamma,\beta)}{\pi}{}^{aIJ}$,

$$\left\{A_{aIJ}(x), \overset{(\gamma,\beta)}{\pi}{}^{bKL}(y)\right\} := 2\delta_a^b\delta_{[I}^K\delta_{J]}^L\delta^3(x-y), \tag{9.32}$$

while all other Poisson brackets vanish. We will prove in the following that these variables are indeed a valid extension of the ADM phase space.

For later convenience, we introduce the notations

$$\begin{aligned}
\overset{(\gamma,\beta)}{\pi}{}_{aIJ} &:= \frac{1}{q}q_{ab}\overset{(\gamma,\beta)}{\pi}{}^{b}{}_{IJ}, \\
\pi^{aIJ} &:= \beta\cdot(\overset{(\gamma)}{\mathscr{M}}{}^{-1})^{IJ}{}_{KL}\overset{(\gamma,\beta)}{\pi}{}^{aKL}, \\
\underset{(\gamma,\beta)}{\pi}{}^{aIJ} &:= \beta^2\cdot(\overset{(\gamma)}{\mathscr{M}}{}^{-1})^{IJ}{}_{KL}(\overset{(\gamma)}{\mathscr{M}}{}^{-1})^{KL}{}_{MN}\overset{(\gamma,\beta)}{\pi}{}^{aMN},
\end{aligned} \tag{9.33}$$

where in the first line $\frac{1}{q}q_{ab}$ has to be understood as a function of $\overset{(\gamma,\beta)}{\pi}{}^{aIJ}$ as given in (9.35). Moreover, note that in $3+1$ dimensions the expression for the hybrid spin connection given in section 7.1.3 can be simplified to

$$\Gamma_{aIJ}[\pi] := \zeta\pi_{b[I|K}D_a\pi^b{}_{J]}{}^K := \zeta\left(\pi_{b[I|K}\partial_a\pi^b{}_{J]}{}^K + \pi_{b[I|K}\Gamma_{ac}^b\pi^c{}_{J]}{}^K\right), \tag{9.34}$$

where $\Gamma^b_{ac}$ again denotes the Christoffel symbols and $D_a$ is the covariant derivative annihilating $q_{ab}$. The ADM variables, expressed in terms of $A_{aIJ}$ and $\overset{(\gamma,\beta)}{\pi}{}^{aIJ}$, are given by

$$qq^{ab} := \frac{\zeta}{2}\pi^{aIJ}\pi^b_{IJ} = \frac{\zeta}{2}\overset{(\gamma,\beta)}{\pi}{}^{aIJ}\underset{(\gamma,\beta)}{\pi}{}^b{}_{IJ}, \tag{9.35}$$

$$K^{ab} := -\frac{1}{2\sqrt{q}}\overset{(\gamma,\beta)}{\pi}{}^{(b|IJ}q^{a)c}\left(A - \Gamma[\pi]\right)_{cIJ}, \tag{9.36}$$

$$P^{ab} := -\frac{1}{2}\sqrt{q}G^{ab}{}_{cd}K^{cd} = \frac{1}{4}G^{ab\,c}{}_d[A_{cIJ} - \Gamma_{cIJ}[\pi]]\overset{(\gamma,\beta)}{\pi}{}^{dIJ}, \tag{9.37}$$

which immediately follows from (7.60, 7.61).

Rewriting the constraints in terms of the new variables, we find the Gauß and quadratic simplicity constraints

$$\mathscr{G}^{IJ} := D^A{}_a\overset{(\gamma,\beta)}{\pi}{}^{aIJ} \approx 2\left(A - \Gamma[\pi]\right)_a{}^{[I}{}_K\overset{(\gamma,\beta)}{\pi}{}^{aK|J]}, \tag{9.38}$$

$$\mathscr{S}^{ab} := \frac{1}{4}\epsilon^{IJKL}\pi^a_{IJ}\pi^b_{KL}. \tag{9.39}$$

Note that $\Gamma[\pi]$ weakly annihilates $\overset{(\gamma,\beta)}{\pi}{}^{aIJ}$ as well. In $3+1$ dimensions, the quadratic simplicity constraint has additional solutions which lead to a theory not corresponding to general relativity. We will exclude this sector by hand. In section 9.3.2, we will introduce the linear version of the simplicity constraint, which does not have this additional solution sector.

Using the equations (9.35, 9.37), we find for the ADM constraints

$$\mathscr{H}_a = -\frac{1}{2}D_b\left(\left(A - \Gamma[\pi]\right)_{aIJ}\overset{(\gamma,\beta)}{\pi}{}^{bIJ} - \delta^b_a\left(A - \Gamma[\pi]\right)_{cIJ}\overset{(\gamma,\beta)}{\pi}{}^{cIJ}\right), \tag{9.40}$$

$$\mathscr{H} = -\frac{1}{4\sqrt{q}}\left(\overset{(\gamma,\beta)}{\pi}{}^{[a|IJ}\overset{(\gamma,\beta)}{\pi}{}^{b]KL}\left(A - \Gamma[\pi]\right)_{bIJ}\left(A - \Gamma[\pi]\right)_{aKL}\right) - \frac{s}{2}\sqrt{q}R(\pi), \tag{9.41}$$

where in both equations we dropped terms proportional to $\underset{(\gamma,\beta)}{K}{}_{[a}{}^{IJ}\overset{(\gamma,\beta)}{\pi}{}_{|b]IJ}$, which, as we already have seen several times, vanishes modulo the Simplicity and (rotational components of) the Gauß constraint.

Since $\Gamma_{aIJ}$ given in (9.34) transforms as a connection under the action of the Gauß constraint, (9.36, 9.37, 9.40, 9.41) are invariant under gauge transformations. Since the matrix $\overset{(\gamma)}{\mathscr{M}}$ is built from intertwiners, (9.35) and (9.39) are gauge invariant by inspection. Simplicity invariance of (9.37, 9.40, 9.41) follows from

$$\begin{aligned}\left\{K^{ab}(x), \mathscr{S}^{cd}(y)\right\} &= -\frac{1}{2\sqrt{q}} \overset{(\gamma,\beta)}{\pi}{}^{(b|IJ} q^{a)e}(x) \left\{A_{eIJ}(x), \frac{1}{4}\epsilon^{KLMN} \pi^c_{KL}(y) \pi^d_{MN}(y)\right\} \\ &= -\frac{1}{2\sqrt{q}} \epsilon^{IJKL} \beta \overset{(\gamma)}{\mathscr{M}}{}^{-1} \overset{(\gamma,\beta)}{\pi}{}^{(b|IJ} q^{a)(c} \pi^{d)}_{KL} \; \delta^3(x-y) \\ &= -\frac{1}{\sqrt{q}} \left(\mathscr{S}^{b(d} q^{c)a} + \mathscr{S}^{a(d} q^{c)b}\right) \; \delta^3(x-y) \approx 0. \end{aligned} \tag{9.42}$$

What remains to be checked is if the ADM Poisson brackets are reproduced on the new phase space, which will by construction imply that the constraint algebra closes. The following Poisson brackets will be helpful in the sequel, which are straightforward generalisations of the corresponding ones in appendix A,

$$\delta q = \frac{\zeta}{D-1} q \underset{(\gamma,\beta)}{\pi}{}_{aIJ} \delta \overset{(\gamma,\beta)}{\pi}{}^{aIJ}, \tag{9.43}$$

$$\delta q^{ab} = \frac{\zeta}{q} G^{-1\,ab}{}_{cd} \underset{(\gamma,\beta)}{\pi}{}^{cIJ} \delta \overset{(\gamma,\beta)}{\pi}{}^{d}{}_{IJ}, \tag{9.44}$$

$$\delta \pi_{aIJ} = \left[\frac{1}{q} q_{ab} \mathbb{P}_{IJKL} - \frac{\zeta}{2} \pi_{aKL} \pi_{bIJ}\right] \beta (\overset{(\gamma)}{\mathscr{M}}{}^{-1})^{KL}{}_{MN} \delta \overset{(\gamma,\beta)}{\pi}{}^{bMN}, \tag{9.45}$$

and from the last line follows

$$\overset{(\gamma,\beta)}{\pi}{}^{cIJ}(x) \left\{A_{aIJ}(x), \pi_b{}^{KL}(y)\right\} \approx -2\delta^3(x-y) \pi_a{}^{KL} \delta^c_b. \tag{9.46}$$

The brackets

$$\{q_{ab}[\overset{(\gamma,\beta)}{\pi}], q_{cd}[\overset{(\gamma,\beta)}{\pi}]\} = 0 \quad \text{and} \quad \{q_{ab}[\overset{(\gamma,\beta)}{\pi}], P^{cd}[A, \overset{(\gamma,\beta)}{\pi}]\} = \delta^c_{(a} \delta^d_{b)} \tag{9.47}$$

are easily verified. The remaining Poisson bracket

$$\begin{aligned} &\{P^{ab}[A, \overset{(\gamma,\beta)}{\pi}][A_{ab}], P^{cd}[A, \overset{(\gamma,\beta)}{\pi}][B_{cd}]\} \\ &= \int_\sigma d^3x \int_\sigma d^3y \left[\left(\frac{1}{2} A_{ab} q^{a[e} \overset{(\gamma,\beta)}{\pi}{}^{b]IJ}\right)(x) \left\{A_{eIJ}(x), \left(q^{c[f} \overset{(\gamma,\beta)}{\pi}{}^{d]KL}\right)(y)\right\}\right. \\ &\qquad \left.\left(\frac{1}{2} B_{cd} (A-\Gamma)_{fKL}\right)(y)\right] - [A \leftrightarrow B] \end{aligned} \tag{9.48}$$

$$\begin{aligned} &+ \int_\sigma d^3x \int_\sigma d^3y \left[\left(\frac{1}{2} A_{ab} q^{a[e} \overset{(\gamma,\beta)}{\pi}{}^{b]IJ}\right)(x) \left\{A_{eIJ}(x), (-\Gamma_{fKL})(y)\right\}\right. \\ &\qquad \left.\left(\frac{1}{2} B_{cd} q^{c[f} \overset{(\gamma,\beta)}{\pi}{}^{d]KL}\right)(y)\right] - [A \leftrightarrow B] \end{aligned} \tag{9.49}$$

is much harder and therefore will be discussed in more detail. Here, $A_{ab}$ and $B_{cd}$ are test fields of compact support, which we can choose symmetric w.l.o.g., since $P^{ab}$ is symmetric by definition. The second line (9.48) and third line (9.49) of the above equation vanish independently. For (9.48), we find using (9.44)

$$(9.48) = \ldots = \frac{1}{4}\int_\sigma d^3x\; A_{ab}B_{cd}q^{ac}\overset{(\gamma,\beta)}{\pi}{}^{[d|IJ}q^{b]e}(A-\Gamma)_{eIJ} \propto \bar{\mathscr{G}}^{IJ}[\ldots],$$

which vanishes if the (rotational part of the) Gauß constraint holds. Before we proceed, we define $\alpha_f{}^e := \frac{1}{4}A_{ab}G^{ab\,e}{}_f$ and $\beta_h{}^g := \frac{1}{4}B_{cd}G^{cd\,g}{}_h$ and check that $\alpha_{[ef]} = 0 = \beta_{[gh]}$. Then, we find for the third line (skipping "$-[A \leftrightarrow B]$" for a moment)

$$\begin{aligned}
&(9.49) = \\
&= \int_\sigma d^3x \int_\sigma d^3y\; \alpha_f{}^e \overset{(\gamma,\beta)}{\pi}{}^{fIJ}(x)\left\{A_{eIJ}(x), (-\zeta)\pi_{bKM}\left(D_g\pi^b{}_L{}^M\right)(y)\right\}\beta_h{}^g\overset{(\gamma,\beta)}{\pi}{}^{hKL}(y) \\
&= -\zeta\int_\sigma d^3x \int_\sigma d^3y\; \alpha_f{}^e \overset{(\gamma,\beta)}{\pi}{}^{fIJ}(x)\Big[\left\{A_{eIJ}(x), \pi_{bKM}(y)\right\}\left(D_g\pi^b{}_L{}^M\right)\beta_h{}^g\overset{(\gamma,\beta)}{\pi}{}^{hKL}(y) \\
&\qquad\qquad - \left\{A_{eIJ}(x), \pi^b{}_L{}^M(y)\right\}D_g\left(\pi_{bKM}\beta_h{}^g\overset{(\gamma,\beta)}{\pi}{}^{hKL}\right)(y)\Big] \qquad (9.50) \\
&\quad -\zeta\int_\sigma d^3x \int_\sigma d^3y\; \alpha_f{}^e \overset{(\gamma,\beta)}{\pi}{}^{fIJ}(x)\left\{A_{eIJ}(x), \Gamma^b_{ga}(y)\right\}\pi_{bKM}\pi^a{}_L{}^M\beta_h{}^g\overset{(\gamma,\beta)}{\pi}{}^{hKL}(y). \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (9.51)
\end{aligned}$$

Again, (9.50) and (9.51) vanish separately. For (9.50), we find after a few steps using (9.46)

$$\begin{aligned}
&(9.50) = \ldots = \\
&= 2\zeta\int_\sigma d^3x\; \alpha_f{}^e\; D_g\left(\beta_h{}^g \cdot \mathrm{Tr}\left(\pi_e\; \pi^f\overset{(\gamma,\beta)}{\pi}{}^h\right)\right) \\
&= 2\zeta\int_\sigma d^3x\; \alpha_{fe}\left[D_g\left(\frac{1}{q}\beta_h{}^g \cdot \mathrm{Tr}\left(\pi^{[e}\;\pi^{f]}\overset{(\gamma,\beta)}{\pi}{}^h\right)\right) - (D_g q^{ea})\frac{1}{q}\beta_h{}^g\cdot\mathrm{Tr}\left(\pi_a\;\pi^f\overset{(\gamma,\beta)}{\pi}{}^h\right)\right] \\
&\approx 0,
\end{aligned}$$

which vanishes since the trace $\mathrm{Tr}(abc) := a^I{}_J b^J{}_K c^K{}_I$ of antisymmetric matrices $a$, $b$, $c$ is antisymmetric when exchanging two matrices while $\alpha_{ab}$ is symmetric and $D_a q_{bc} = 0$

by construction. The remaining part (9.51) can be rewritten as

$$
\begin{aligned}
&(9.51) = \\
&= -\zeta \int_\sigma d^3y \ \left\{P^{ab}[A_{ab}], \Gamma^d_{gc}(y)\right\} \beta_h{}^g \ \mathrm{Tr}\left(\pi_d \ \pi^c \overset{(\gamma,\beta)}{\pi}{}^h\right)(y) \\
&= -\zeta \int_\sigma d^3y \ \left[\left\{P^{ab}[A_{ab}], q^{de}(y)\right\} q_{di}\Gamma_{gc\ e} + \left\{P^{ab}[A_{ab}], \Gamma_{gc\ i}(y)\right\}\right] \frac{1}{q}\beta_h{}^g \ \mathrm{Tr}\left(\pi^i \ \pi^c \overset{(\gamma,\beta)}{\pi}{}^h\right) \\
&= -\zeta \int_\sigma d^3y \ \left[-\left\{P^{ab}[A_{ab}], q_{jk}(y)\right\} \delta_i^{(j}\Gamma^{k)}_{gc} + \left\{P^{ab}[A_{ab}], \partial_c q_{ig}(y)\right\}\right] \frac{1}{q}\beta_h{}^g \ \mathrm{Tr}\left(\pi^i \ \pi^c \overset{(\gamma,\beta)}{\pi}{}^h\right) \\
&\approx -\zeta \int_\sigma d^3y \ \left[A_{ab}\Gamma^b_{gc} + A_{ag}\vec{\partial}_c\right] \frac{1}{q}\beta_h{}^g \ \mathrm{Tr}\left(\pi^a \ \pi^c \overset{(\gamma,\beta)}{\pi}{}^h\right) \\
&= -\zeta \int_\sigma d^3y \ A_{ab} D_c \left(\frac{1}{q}\beta_h{}^b \ \mathrm{Tr}\left(\pi^a \ \pi^c \overset{(\gamma,\beta)}{\pi}{}^h\right)\right) \\
&= \zeta \int_\sigma d^3y \ \left(D_c A_{ab}\right) \frac{1}{4q} B_{ed} \left(q^{b(e}\delta^{d)}_h - q^{cd}\delta^b_h\right) \ \mathrm{Tr}\left(\pi^a \overset{(\gamma,\beta)}{\pi}{}^c \ \pi^h\right) \\
&= \zeta \int_\sigma d^3y \ \left(D_c A_{ab}\right) \frac{1}{4q} B^b{}_d \ \mathrm{Tr}\left(\pi^a \overset{(\gamma,\beta)}{\pi}{}^c \ \pi^d\right).
\end{aligned}
$$

In the first step, we just reassembled the terms on the left hand side of the Poisson bracket, in the second we used the definition of the Christoffel symbol, in the third the formula for the derivative of the inverse matrix and antisymmetry of the trace in $(i \leftrightarrow c)$. In the fourth line we used the already known brackets of the metric $q_{ab}$ and its conjugate momentum $P^{cd}$. Note that the density weight and index structure is such that the terms in the fourth line can be reassembled in a covariant derivate. In the sixth line the definition of $\beta_h{}^g$ is inserted, we integrated by parts and we used that $\mathrm{Tr}(ab\overset{(\gamma,\beta)}{c}) = \mathrm{Tr}(a\overset{(\gamma,\beta)}{b}c)$ (this trace property can be shown using the definition of the matrices $\overset{(\gamma)}{\mathscr{M}}$). Thus we find that the second summand appearing in the definition of $\beta_h{}^g$ vanishes due to antisymmetry of the trace in the indices $(a \leftrightarrow b)$. If we now restore the antisymmetry in the test fields $(A \leftrightarrow B)$, we obtain

$$
\begin{aligned}
(9.51) &= \frac{\zeta}{4} \int_\sigma d^3y \ \left[\left(D_c A_{ab}\right) B^b{}_d - \left(D_c B_{ab}\right) A^b{}_d\right] \frac{1}{q}\mathrm{Tr}\left(\pi^a \overset{(\gamma,\beta)}{\pi}{}^c \ \pi^d\right) \\
&\approx \frac{1}{4\beta\gamma} \int_\sigma d^3y \ \epsilon^{cda} \ D_c\left(A_{ab}B^b{}_d\right) \\
&= \frac{1}{4\beta\gamma} \int_\sigma d^3y \ \partial_c\left(\epsilon^{cda} A_{ab}B^b{}_d\right) = 0,
\end{aligned}
$$

where we used the simplicity constraint in the second line and then dropped a surface term. We leave the case where $\sigma$ has a boundary for further research. This furnishes the proof of the validity of the formulation.

### 9.3.2 Barbero Immirzi parameter and linear simplicity constraints

As was demonstrated in section 9.1, instead of the quadratic simplicity constraints we may as well work with linear simplicity (and normalisation) constraints when introducing additional phase space degrees of freedom $\{N^I, P_J\}$. This result extends to the $D = 3$ case with Barbero Immirzi parameter as follows: The theory with linear simplicity and normalisation constraint has the non-vanishing Poisson brackets (9.32) as well as

$$\left\{N^I(x), P_J(y)\right\} = \delta^I_J \delta^3(x-y), \tag{9.52}$$

and the constraints are given by

$$\mathscr{H}_a := \frac{1}{2} \overset{(\gamma,\beta)}{\pi}{}^{bIJ} \partial_a A_{bIJ} - \frac{1}{2}\partial_b \left( \overset{(\gamma,\beta)}{\pi}{}^{bIJ} A_{aIJ} \right) + P_I \partial_a N^I, \tag{9.53}$$

$$\mathscr{H} := -\frac{1}{4\sqrt{q}} \left( \overset{(\gamma,\beta)}{\pi}{}^{[a|IJ} \overset{(\gamma,\beta)}{\pi}{}^{b]KL} \left(A - \Gamma[\pi]\right)_{bIJ} \left(A - \Gamma[\pi]\right)_{aKL} \right) - \frac{s}{2}\sqrt{q} R(\pi), \tag{9.54}$$

$$\mathscr{G}^{IJ} := D^A{}_a \overset{(\gamma,\beta)}{\pi}{}^{aIJ} + 2P^{[I} N^{J]}, \tag{9.55}$$

$$\mathscr{S}^{aI} := \epsilon^{IJKL} N_J \pi^a_{KL}, \tag{9.56}$$

$$\mathscr{N} := N^I N_I - \zeta. \tag{9.57}$$

The proof that this constrained system actually describes general relativity is the same as we gave in section 9.1 for general $D \geq 3$ without Barbero Immirzi parameter. We want to point out that, while we now are mixing boost and rotational components due to the matrices $\mathscr{M}$, $\mathscr{M}^{-1}$, for the proof that the ADM Poisson brackets are reproduced on the extended phase space in section 9.3.1, we again only needed $\bar{\mathscr{G}}^{IJ} \approx 0$ and $\mathscr{S}^{ab} \approx 0$, and therefore, the same argumentation as in section 9.1 goes through. We leave it to the interested reader to work out the details.

### 9.3.3 Solving the linear simplicity and normalisation constraints

While we were rather brief in the last subsection, here we want to be more explicit, since a thorough understanding of the relation between this formulation and the usual Ashtekar Barbero variables will be important when one wants to compare the resulting quantum theories in $D = 3$.

It is instructive to solve first the simplicity (and normalisation) constraints and then the boost Gauß constraint ("time gauge"), which will in the first step lead to a formulation similar to the one given in section 6.2.3 (cf. also [146]), and then to the formulation in Ashtekar-Barbero variables. We will treat the case with linear simplicity constraints, since in $3+1$ dimensions, the linear constraint has the additional advantage that its only solution is general relativity, while the quadratic simplicity constraint also allows for the topological solution.

The solution to the linear simplicity and normalisation constraint is given by $\overset{(\beta)}{\pi}{}^{aIJ} = \frac{2}{\beta} n^{[I} E^{a|J]}$ and therefore

$$\overset{(\gamma,\beta)}{\pi}{}^{aIJ} = \frac{1}{\beta}\left(2n^{[I}E^{a|J]} + \frac{1}{\gamma}\epsilon^{IJKL} n_K E_{aL}\right). \tag{9.58}$$

For the connection, we make the Ansatz $A_{aIJ} = \Gamma_{aIJ} + \underset{(\gamma,\beta)}{K}{}_{aIJ}$. The symplectic potential becomes

$$\begin{aligned}
\frac{1}{2}\overset{(\gamma,\beta)}{\pi}{}^{aIJ}\dot{A}_{aIJ} + P^I\dot{N}_I &= \frac{1}{2}\overset{(\gamma,\beta)}{\pi}{}^{aIJ}\dot{\Gamma}_{aIJ} + \frac{1}{2}\overset{(\gamma,\beta)}{\pi}{}^{aIJ}\underset{(\gamma,\beta)}{\dot{K}}{}_{aIJ} + P^I\dot{N}_I \\
&= \frac{1}{2\beta\gamma}\epsilon^{IJKL} n_K E^a{}_L \dot{\Gamma}_{aIJ} + n^{[I}E^{a|J]}\dot{K}_{aIJ} + P^I\dot{N}_I \\
&= -(\dot{n^{[I}E^{a|J]}})\left(K_{aIJ} + \frac{1}{2\beta\gamma}\epsilon_{IJ}{}^{KL}\Gamma_{aKL}\right) + P^I\dot{N}_I \\
&\approx -\dot{E^{aJ}}\left(\zeta\bar{K}_{aJ} + \frac{1}{2\beta\gamma}\epsilon_{IJ}{}^{KL} n^I \Gamma_{aKL} + n_J \bar{P}^I E_{aI}\right. \\
&\qquad \left. - n_J E^I_a E^{bK}\left(\bar{K}_{bIK} + \frac{1}{2\beta\gamma}\epsilon_{IK}{}^{LM}\Gamma_{bLM}\right)\right) \\
&=: E^{aJ}\dot{A}_{aJ}.
\end{aligned} \tag{9.59}$$

In the next step we express the remaining constraints in terms of the new canonical variables. The reduction of the Gauß constraint yields

$$\begin{aligned}
\frac{1}{2}\Lambda_{IJ}\mathscr{G}^{IJ} &= \frac{1}{2}\Lambda_{IJ} D^A{}_a \overset{(\gamma,\beta)}{\pi}{}^{aIJ} + \Lambda_{IJ} P^I N^J \\
&\approx \Lambda_{IJ} \underset{(\gamma,\beta)}{K}{}_{aK}^{\;I} \overset{(\gamma,\beta)}{\pi}{}^{aKJ} + \Lambda_{IJ} P^I N^J \\
&= \Lambda_{IJ} K^I_{aK} \pi^{aKJ} + \Lambda_{IJ} P^I N^J \\
&= \Lambda_{IJ} K^I_{aK}\left(n^K E^{aJ} - n^J E^{aK}\right) + \Lambda_{IJ} P^I N^J \\
&\approx -\Lambda_{IJ} E^{aJ}\left(\zeta \bar{K}^I_a - n^I E_{aL} E^{bK} \bar{K}^L_{b\;K} + n^I P^K E_{aK}\right)
\end{aligned}$$

$$= -\Lambda_{IJ} E^{aJ} \left( A_a^I - \frac{1}{2\beta\gamma} \left( \epsilon_M{}^{IKL} n^M \Gamma_{aKL} - n^I E_a^N E^{bK} \epsilon_{NK}{}^{LM} \Gamma_{bLM} \right) \right)$$
$$= \frac{1}{2} \Lambda_{IJ} \left( 2E^{a[I} A_a^{J]} + \frac{1}{\beta\gamma} \epsilon^{IJKL} \partial_a \left( n_K E_L^a \right) \right), \tag{9.60}$$

which after time gauge $n^I = \delta_0^I$ and solution of the boost part of the Gauß constraint obviously reproduces the SU(2) Gauß constraint of the Ashtekar-Barbero formulation. The diffeomorphism constraint becomes

$$\mathscr{H}_a = \frac{1}{2} \overset{(\gamma,\beta)}{\pi}{}^{bIJ} \partial_a A_{bIJ} - \frac{1}{2} \partial_b \left( \overset{(\gamma,\beta)}{\pi}{}^{bIJ} A_{aIJ} \right) + P_I \partial_a N^I$$
$$\approx E^{bI} \partial_a A_{bI} - \partial_b \left( E^{bI} A_{aI} \right), \tag{9.61}$$

which coincides on-shell with the spatial diffeomorphism constraint of section 9.3.1,

$$\mathscr{H}_a = -\frac{1}{2} D_b \left( \left( A - \Gamma[\pi] \right)_{aIJ} \overset{(\gamma,\beta)}{\pi}{}^{bIJ} - \delta_a^b \left( A - \Gamma[\pi] \right)_{cIJ} \overset{(\gamma,\beta)}{\pi}{}^{cIJ} \right)$$
$$\approx E^{bI} \partial_a A_{bI} - \partial_b \left( E^{bI} A_{aI} \right) + \frac{1}{2} \Gamma_{aKL}[\pi] \mathscr{G}^{KL}. \tag{9.62}$$

Here, we used $R^{\mathrm{H}}_{abIJ} = E_I^c E_{dJ} R_{abc}{}^d$, $R^{\mathrm{H}}_{abIJ} n^I = 0$ (cf. appendix C) and $\epsilon^{IJKL} R^{\mathrm{H}}_{abIJ} n_K E_L^b = 0$ which follows from the algebraic Bianchi identity. Finally, the Hamiltonian constraint gives

$$\mathscr{H} = -\frac{1}{4\sqrt{q}} \left( \overset{(\gamma,\beta)}{\pi}{}^{[a|IJ} \overset{(\gamma,\beta)}{\pi}{}^{b]KL} \left( A - \Gamma \right)_{bIJ} \left( A - \Gamma \right)_{aKL} \right) - \frac{s}{2} \sqrt{q} R(\overset{(\gamma,\beta)}{\pi})$$
$$\approx -\frac{1}{\sqrt{q}} E^{[a|I} E^{b]J} \left( A_{bI} - \frac{1}{2\beta\gamma} \epsilon_{MI}{}^{KL} n^M \Gamma^H{}_{bKL} \right) \left( A_{aJ} - \frac{1}{2\beta\gamma} \epsilon_{NJ}{}^{AB} n^N \Gamma^H{}_{aAB} \right)$$
$$- \frac{s}{2} \sqrt{q} R(E). \tag{9.63}$$

### 9.3.4 Time gauge

We choose time gauge $n^I = \delta_0^I \Leftrightarrow E^{a0} = 0$ and solve its second class partner, the boost part $\mathscr{G}^{0i} = -E^{ai} A_a^0$ of the Gauß constraint $(i, j, \ldots \in \{1, 2, 3\})$. It is convenient to introduce the rescaled variables $A_{ai} \to A'_{ai} := -\zeta\tilde{\gamma} A_{ai}$ and $E^{bj} \to E'^{bj} := -\frac{\zeta}{\tilde{\gamma}} E^{bj}$, where $\tilde{\gamma} := \beta\gamma$, in terms of which we have

$$E^{aI}\dot{A}_{aI} \to E'^{ai}\dot{A}'_{ai}, \tag{9.64}$$

$$G^{IJ} \to -\frac{1}{2}\epsilon^{kij}G_{ij} = \partial_a E'^{ak} + \epsilon^{kij}A'_{ai}E'^{a}_{\ j}, \tag{9.65}$$

$$\mathscr{H}_a \to \mathscr{H}_a = E'^{bi}\partial_a A'_{bi} - \partial_b\left(E'^{bi}A'_{ai}\right), \tag{9.66}$$

$$\begin{aligned}\mathscr{H} \to \mathscr{H} &= -\frac{1}{\sqrt{q}}E^{[a|i}E^{b]j}\left(A_{bi} - \frac{\zeta}{2\tilde{\gamma}}\epsilon_i^{\ kl}\Gamma_{bkl}\right)\left(A_{aj} - \frac{\zeta}{2\tilde{\gamma}}\epsilon_j^{\ mn}\Gamma_{amn}\right) - \frac{s}{2}\sqrt{q}R(E) \\ &\approx \frac{1}{2\sqrt{q}}\epsilon_{ijk}F'^{\ k}_{ab}E'^{ai}E'^{bj} - \frac{1}{2\sqrt{q}}\left(1 - s\tilde{\gamma}^2\right)\epsilon_{ijk}R_{ab}^{\ \ k}E'^{ai}E'^{bj}.\end{aligned} \tag{9.67}$$

Here, terms proportional to the Gauß constraint have been dropped in the expression for the Hamiltonian constraint. At this stage, only the combination of the parameters $\tilde{\gamma} = \gamma\beta$ is left and plays the role of the Barbero Immirzi parameter in usual Ashtekar Barbero variables, cf. chapter 6. One could ask if one should have worked with one parameter from the beginning. To give a tentative answer to this question, note that the (quadratic) simplicity constraint implies $\frac{1}{2}\epsilon_{IJKL}\overset{(\gamma,\beta)}{\pi}{}^{aIJ}\overset{(\gamma,\beta)}{\pi}{}^{bKL} = \frac{2\zeta\gamma}{\gamma^2+\zeta}\overset{(\gamma,\beta)}{\pi}{}^{aIJ}\overset{(\gamma,\beta)}{\pi}{}^{b}_{\ IJ}$ and therefore

$$\begin{aligned}2\zeta q q^{ab} &= \pi^{aIJ}\pi^{b}_{\ IJ} = \left(\frac{\gamma^2}{\gamma^2-\zeta}\right)^2\left[\left(1+\frac{\zeta}{\gamma^2}\right)\overset{(\gamma,\beta)}{\pi}{}^{aIJ}\overset{(\gamma,\beta)}{\pi}{}^{b}_{\ IJ} - \frac{1}{\gamma}\epsilon_{IJKL}\overset{(\gamma,\beta)}{\pi}{}^{aIJ}\overset{(\gamma,\beta)}{\pi}{}^{bKL}\right] \\ &\approx \frac{\gamma^2\beta^2}{\gamma^2+\zeta}\overset{(\gamma,\beta)}{\pi}{}^{aIJ}\overset{(\gamma,\beta)}{\pi}{}^{b}_{\ IJ}.\end{aligned} \tag{9.68}$$

We expect that the square root of this factor, $\frac{|\gamma\beta|}{\sqrt{\gamma^2+\zeta}}$, will appear in the spectrum of the area operator. It seems improbable that the two parameters $\gamma$, $\beta$ appear just in this peculiar combination in the spectra of operators and therefore, at the quantum level one probably will be able to distinguish between $\gamma$ and $\beta$.

### 9.3.5 Formulation with two commuting SU(2) connections

Note that we could have chosen time gauge before solving the simplicity and normalisation constraints by setting $N^I = \delta^I_0$ and solving the boost part of the Gauß constraint $G^{0i} = D^A{}_a\overset{(\gamma,\beta)}{\pi}{}^{ai} - P^i$, where we used the notation $\overset{(\gamma,\beta)}{\pi}{}^{ai} := \overset{(\gamma,\beta)}{\pi}{}^{a0i}$. Furthermore, we define $A_{ai} := A_{a0i}$. We find

$$\frac{1}{2}\overset{(\gamma,\beta)}{\pi}{}^{aIJ}\dot{A}_{aIJ} + P^I\dot{N}_I \quad\to\quad \frac{1}{2}\overset{(\gamma,\beta)}{\pi}{}^{aij}\dot{A}_{aij} + \overset{(\gamma,\beta)}{\pi}{}^{ai}\dot{A}_{ai} \tag{9.69}$$

and

$$\mathscr{G}^{ij} = \partial_a \overset{(\gamma,\beta)}{\pi}{}^{aij} + 2A_a{}^{[i|}{}_k \overset{(\gamma,\beta)}{\pi}{}^{ak|j]} + 2\overset{(\gamma,\beta)}{\pi}{}^{a[i} A_a^{j]}, \tag{9.70}$$

$$\mathscr{S}^{ai} = \epsilon^{ijk} \overset{(\gamma,\beta)}{\pi}{}^a{}_{jk} - \frac{2\zeta}{\gamma} \overset{(\gamma,\beta)}{\pi}{}^{ai}, \tag{9.71}$$

$$\mathscr{H}_a = \frac{1}{2} \overset{(\gamma,\beta)}{\pi}{}^{bij} \partial_a A_{bij} - \frac{1}{2}\partial_b \left( \overset{(\gamma,\beta)}{\pi}{}^{bij} A_{aij} \right) + \overset{(\gamma,\beta)}{\pi}{}^{bi} \partial_a A_{bi} - \partial_b \left( \overset{(\gamma,\beta)}{\pi}{}^{bi} A_{ai} \right), \tag{9.72}$$

$$\begin{aligned} \mathscr{H} = & -\frac{1}{4\sqrt{q}} \left( \overset{(\gamma,\beta)}{\pi}{}^{[a|ij} \overset{(\gamma,\beta)}{\pi}{}^{b]kl} (A-\Gamma)_{bij} (A-\Gamma)_{akl} \right) \\ & - \frac{1}{\sqrt{q}} \left( \overset{(\gamma,\beta)}{\pi}{}^{[a|i} \overset{(\gamma,\beta)}{\pi}{}^{b]jk} A_{bi} (A-\Gamma)_{ajk} \right) - \frac{1}{\sqrt{q}} \left( \overset{(\gamma,\beta)}{\pi}{}^{[a|i} \overset{(\gamma,\beta)}{\pi}{}^{b]j} A_{bi} A_{aj} \right) - \frac{s}{2}\sqrt{q} R(\pi), \end{aligned} \tag{9.73}$$

where we dropped constants in front of the simplicity constraint and in the Hamiltonian constraint we neglected terms proportional to the simplicity constraint ($\Gamma_{a0i} \approx 0$). Note that in the case without Barbero Immirzi parameter, the simplicity constraint $\mathscr{S}^{ai} = \epsilon^{ijk}\pi^a_{jk}$ demands the vanishing of $\pi^{aij}$ and therefore there is no physical information left in the conjugate SU(2) connection $A_{aij}$. Here, this is not the case and we obtain a genuine connection formulation of general relativity. Moreover, the other canonical pair $\left\{ A_{ai}, \overset{(\gamma,\beta)}{\pi}{}^{bj} \right\}$ has the same structure as $\{K_{ai}, E^{bj}\}$. Then, it follows from the know results when extending the ADM phase space to Ashtekar-Barbero variables (cf. section 6.3) that there exists a spin connection $\Gamma'_{aij}$ which annihilates $\overset{(\gamma,\beta)}{\pi}{}^{ai}$ and that the transformation $\left\{ A_{ai}, \overset{(\gamma,\beta)}{\pi}{}^{bj} \right\} \to \left\{ A^-_{aij} := \Gamma'_{aij} + \alpha\epsilon_{ikj} A^k_a,\ E_-{}^{bkl} := \frac{1}{\alpha}\epsilon^{kml} \overset{(\gamma,\beta)}{\pi}{}^b{}_m \right\}$ for $\alpha \in \mathbb{R}/\{0\}$ is canonical. Defining $A^+_{aij} := A_{aij}$ and $E_+{}^{aij} := \overset{(\gamma,\beta)}{\pi}{}^{aij}$, we obtain the symplectic potential

$$\frac{1}{2} E_+{}^{aij} \dot{A}^+_{aij} + \frac{1}{2} E_-{}^{aij} \dot{A}^-_{aij} \tag{9.74}$$

and constraints

$$\mathscr{G}^{ij} = D^+_a E_+{}^{aij} + D^-_a E_-{}^{aij}, \tag{9.75}$$

$$\frac{1}{2}\epsilon^{ijk} \mathscr{S}^a_k = E_+{}^{aij} + \frac{\zeta\alpha}{\gamma} E_-{}^{aij}, \tag{9.76}$$

$$\begin{aligned} \mathscr{H}_a = & \frac{1}{2} E_+{}^{bij} \partial_a A^+_{bij} - \frac{1}{2}\partial_b \left( E_+{}^{bij} A^+_{aij} \right) \\ & + \frac{1}{2} E_-{}^{bij} \partial_a A^-_{bij} - \frac{1}{2}\partial_b \left( E_-{}^{bij} A^-_{aij} \right), \end{aligned} \tag{9.77}$$

$$\mathscr{H} = -\frac{1}{4\sqrt{q}} \left( E_+{}^{[a|ij} E_+{}^{b]kl} \left( A^+ - \Gamma(E_+, E_-) \right)_{bij} \left( A^+ - \Gamma(E_+, E_-) \right)_{akl} \right)$$

$$
\begin{aligned}
&- \frac{1}{2\sqrt{q}} \left( E_-{}^{[a|ij} E_+{}^{b]kl} \left(A^- - \Gamma'(E_-)\right)_{bij} \left(A^+ - \Gamma(E_+, E_-)\right)_{akl} \right) \\
&- \frac{1}{4\sqrt{q}} \left( E_-{}^{[a|ij} E_-{}^{b]kl} \left(A^- - \Gamma'(E_-)\right)_{bij} \left(A^- - \Gamma'(E_-)\right)_{akl} \right) \\
&- \frac{s}{2}\sqrt{q} R(E_+, E_-),
\end{aligned} \tag{9.78}
$$

where we dropped the term $-\frac{1}{2}E_-{}^{bij}R'_{abij}(E_-)$ in the spatial diffeomorphism constraint, since it vanishes due to the Bianchi identity. We made explicit that the spin connection $\Gamma_{aij}$ in the Hamiltonian constraint does not annihilate $E_+{}^{aij}$ but the physical combination of $E_+{}^{aij}$ and $E_-{}^{aij}$ (i.e. the combination which remains when solving the simplicity constraint). Note however, that on the simplicity constraint surface, we have that $E_+$, $E_-$ and therefore also the physical $\hat{E}$ are multiples of each other with constant coefficients, $\hat{E} \approx \alpha\beta E_- \approx -\gamma\beta\zeta E_+$, and since the spin connection is unchanged by constant rescalings of the vielbein which it annihilates, $\Gamma(E_+, E_-)$ and $\Gamma'(E_-)$ coincide on the simplicity constraint surface. In this formulation we now have two commuting SU(2) connections $A^+_{aij}$ and $A^-_{aij}$, which can be interpreted as the two parts $\text{SU}(2)^+$ and $\text{SU}(2)^-$ of SO(4). They are, however, not uncorrelated and their momenta are multiples of each other (9.76), in complete analogy to the relation $K + \gamma L = 0$ of boost- and rotation generators in the new spin foam models.

Of course, with a suitable choice of variables, one of the two connection carries no physical information. Explicitly, $\hat{E}^{ai} = \frac{1}{2}\epsilon^{kil}\hat{E}^a{}_{kl} := \frac{\gamma\beta}{2(\gamma^2-\zeta)}\epsilon^{kil}(E_+ + \gamma\alpha E_-)^a{}_{kl}$ and $\hat{A}_a{}^i := -\frac{\zeta}{2\beta\gamma}\epsilon^{kil}(A^+ - \frac{\gamma\zeta}{\alpha}A^-)_{akl}$ carry the physical information and can be shown to be a canonical pair. In terms of these, we have

$$
\mathscr{H} \approx -\frac{1}{\sqrt{q}} \hat{E}^{[a|i} \hat{E}^{|b]j} \left( \hat{A}_{bi} - \frac{\zeta}{2\hat{\gamma}} \epsilon_{ikl} \Gamma[\hat{E}]_b{}^{kl} \right) \left( \hat{A}_{aj} - \frac{\zeta}{2\hat{\gamma}} \epsilon_{jmn} \Gamma[\hat{E}]_a{}^{mn} \right) - \frac{s}{2}\sqrt{q} R(\hat{E}), \tag{9.79}
$$

where $\hat{\gamma} := \frac{\alpha\beta\gamma}{(\alpha-\zeta\gamma)}$. This form of the Hamiltonian constraint again coincides with our result of section 9.3.4.

## 9.4 On the $\mathscr{D}$ constraints

We will see later on that the fact that the simplicity constraint is quadratic in the momenta $\pi^{aIJ}$ leads to problems when quantising, in particular, group averaging techniques are not available. Therefore, one might want to substitute this constraint at the classical level by a different one which simplifies quantisation. Of course, a possible way of how to get rid of $\mathscr{S}$ is to gauge fix the simplicity constraint and then try to gauge unfix the gauge fixing condition. The most natural gauge fixing condition to introduce, the $\mathscr{D}$ constraint from section 5.2, is alas much more complicated than the simplicity constraint and even if we succeeded in exchanging their role, we would make the problems in the quantum theory only worse. The question thus arises if we could introduce different gauge fixings with nicer properties, which we will briefly study here.

Let us start our considerations in the $\{K_{aIJ}, \pi^{bKL}\}$-theory from section 7.2 for convenience. Since our gauge fixing should be easier to handle than the simplicity constraint at the quantum level, we are lead to consider gauge fixings which are at most linear in the momenta $\pi^{aIJ}$. Of course, in order to be second class with the simplicity constraints, it necessarily has to be at least linear in $K_{aIJ}$. For general $D \geq 3$, the probably simplest constraint which one can come up with,

$$\mathscr{D}_{ab}^{\overline{M}\prime\prime} := (*^{\overline{M}}\pi^a)^{KL} K_{bKL} = \frac{1}{2}\epsilon^{IJKL\overline{M}} \pi^a{}_{IJ} K_{bKL}, \tag{9.80}$$

meets these needs. If $\mathscr{S} = 0$, it obviously demands that $\bar{K}_{aIJ} = 0$, and therefore is a good first guess. Like the simplicity constraints, the $\mathscr{D}''$ constraints are for $D > 3$ not all independent. However, in $D = 3$ where the multiindex $\overline{M}$ is absent, $\mathscr{D}''$ has 9 components, but we would only expect 6 independent constraints from our experience with the Palatini theory. Therefore, the constraints $\mathscr{D}''$ must have some further source of non-independence, apart from the one stemming from the labelling by a multiindex. Indeed, only the symmetric part

$$\mathscr{D}_{ab}^{\overline{M}\prime} := (*^{\overline{M}}\pi_{(a|})_{KL} K_{|b)}{}^{KL} \tag{9.81}$$

is independent, and one easily finds that if $\mathscr{S} = 0$, it demands that the transverse tracefree part $\bar{K}^{\mathrm{tf}}_{aIJ}$ of $K_{aIJ}$ vanishes. The antisymmetric part then is already constrained to vanish, since it can be shown to be proportional to the boost part of the Gauß constraint $\bar{K}^{\mathrm{tr.}}_I$ and $\bar{K}^{\mathrm{tf}}_{aIJ}$, i.e. the symmetric part, again if $\mathscr{S} = 0$. We could of course argue that, classically, it does not matter if we impose $\mathscr{D}''$ or $\mathscr{D}'$, but this only holds if $\mathscr{S} = 0$ and since we want to get rid of this constraint, we are lead to continue with $\mathscr{D}'$. Note that here we already lost the linearity in the momenta, since $\pi_{aIJ}$ is a complicated function of $\pi^{aIJ}$. We will continue anyway to see if we can obtain a connection formulation with first class albeit complicated $\mathscr{D}'$ constraint. In the following, we will restrict to $D = 3$.

That $\mathscr{D}'$ is equivalent to (5.24) already follows from the fact that both demand the vanishing of $\bar{K}^{\text{tf}}_{aIJ}$. Explicitly, we have

$$\begin{aligned}
\mathscr{D}^{ab} &= 2(*\pi^c)_{KL}\pi^{(a|KN}D^A_c\pi^{b)L}{}_N \\
&= -f^{IJKL}{}_{MN}(*\pi^c)_{IJ}\pi^{(a|}{}_{KL}D^A_c\pi^{b)MN} \\
&\approx -f^{IJKL}{}_{MN}(*\pi^c)_{IJ}\pi^{(a|}{}_{KL}[K_c,\pi^{b)}]^{MN} \\
&= -f^{IJKL}{}_{MN}f_{OPQR}{}^{MN}(*\pi^c)_{IJ}\pi^{(a|}{}_{KL}K_c{}^{OP}\pi^{b)QR} \\
&= \zeta q^2 G^{ab\,cd}\mathscr{D}'_{cd} + \mathscr{S}^{c(b}\pi^{a)IJ}K_{cIJ} - \mathscr{S}^{ab}\pi^{cIJ}K_{cIJ}.
\end{aligned} \tag{9.82}$$

Here, in the second line, we used that the trace over three generators gives the structure constants (cf. appendix D for notation), in the third we used $A = \Gamma[\pi] + K$ and that $\Gamma[\pi]$ weakly annihilates $\pi$ (note that the Christoffel symbols drop out). In the fourth line, we rewrote the commutator using the structure functions and in the fifth line use the relation (D.7). We see that the constraints $\mathscr{D}$ and $\mathscr{D}'$ are (weakly) related by the invertible matrix $G^{ab\,cd}$ (cf. (2.15)).

Using (A.25) we easily deduce

$$\delta\mathscr{D}'_{ab} = (*\pi_{(a})^{IJ}\delta K_{b)IJ} + (*K_{(b})^{IJ}\left[\frac{1}{q}q_{a)c}\mathbb{P}_{IJKL} - \frac{\zeta}{2}\pi_{a)KL}\pi_{cIJ}\right]\delta\pi^{cKL}, \tag{9.83}$$

which will be used repeatedly in calculating Poisson brackets in the following. Note that $\mathbb{P}_{IJ}{}^{KL} := \eta^K_{[I}\eta^L_{J]} - \frac{\zeta}{2}\pi_{aIJ}\pi^{aKL}$ projects orthogonal to $\pi^{aIJ}$, $\mathbb{P}_{IJKL}\pi^{aKL} = 0$. First of all, note that $\mathscr{D}'$ obviously will Poisson commute with the Gauß and spatial diffeomorphism constraint. We find that the Dirac matrix between $\mathscr{S}$ and $\mathscr{D}'$

$$\{\mathscr{S}^{ab}(x), \mathscr{D}'_{cd}(y)\} = -4\delta^{(a}_c\delta^{b)}_d\delta^3(x-y) \tag{9.84}$$

is trivial, in particular phase space independent. Note that in higher dimensions, this cannot be true due to the non-independence of the constraints.

Furthermore, the maps to the ADM phase space (7.44, 7.45) actually are $\mathscr{D}'$ observables,

$$
\begin{aligned}
&\{q_{ab}[\pi](x), \mathscr{D}'{}_{cd}(y)\} = -\frac{\zeta}{q} G^{-1}_{ab\,ef} \pi^{eIJ} \{\pi^f{}_{IJ}(x), K_{(d}{}^{KL}(y)\} (*\pi_{c)KL}) \\
&= \frac{4\zeta}{q^2} G^{-1}_{ab\,e(d} q_{c)f} \mathscr{S}^{ef} \delta^{(3)}(x-y), \qquad (9.85) \\
&\{P^{ab}[\pi, K](x), \mathscr{D}'{}_{cd}(y)\} = \\
&= \frac{1}{2} G^{ab\,e}{}_f \pi^{fIJ} (*K_{(c|})_{KL} \left( \frac{1}{q} q_{d)x} \mathbb{P}^{KL}{}_{IJ} - \frac{\zeta}{2} \pi_{d)}{}^{KL} \pi_{eIJ} \right) \delta^{(3)}(x-y) \\
&\quad - \frac{1}{2} K_{eKL} \delta^f_{(c} (*\pi_{d)})^{IJ} \left( G^{ab\,e}{}_f \eta^K_{[I} \eta_{J]} L + \frac{\zeta}{q} \pi^{(b|KL} G^{a)e}{}_{gf} \pi^g{}_{IJ} \right. \\
&\qquad\qquad \left. - \frac{\zeta}{q} G^{ab}{}_{gf} \pi^{eKL} \pi^g{}_{IJ} \right) \delta^{(3)}(x-y) \\
&= \frac{D}{2} q^{ab} \mathscr{D}'_{cd} \delta^{(3)}(x-y) - \frac{1}{2} q_{(c}{}^{(b} q_{d)e} q^{a)f} \mathscr{D}''^e{}_f \delta^{(3)}(x-y) \\
&\quad - \frac{\zeta}{q^2} K_{fIJ} \left( \pi^{(b|IJ} G^{a)f}{}_{e(c|} - \pi^{fIJ} G^{ab}{}_{e(c|} \right) q_{d)f} \mathscr{S}^{ef} \delta^{(3)}(x-y), \qquad (9.86)
\end{aligned}
$$

where in the first line we used (A.21) and in the third line (7.53). As we already stated, the $\mathscr{D}''$ terms in the second to last line vanish weakly modulo $\mathscr{D}'$, $\mathscr{G}$ and $\mathscr{S}$. This proves that $\mathscr{D}'$ weakly Poisson commutes with the Hamilton constraint when expressed via (7.44, 7.45). Finally, due to the very simple form of $\mathscr{D}'$, we find that it is actually first class with itself,

$$
\begin{aligned}
\{\mathscr{D}'{}_{ab}(x), \mathscr{D}'_{cd}(y)\} &= \frac{2}{q} (*K_{(d})^{IJ} q_{c)(b} (*\pi_{a)})_{IJ} \delta^{(3)}(x-y) \\
&\quad - \frac{\zeta}{q^3} \mathscr{S}^{gh} q_{g(a} \left[ q_{b)e} q_{(c|x} + q_{b)(c|} q_{ex} \right] \mathscr{D}''^e{}_{|d)} \delta^{(3)}(x-y) \\
&\quad - ab \leftrightarrow cd \\
&\approx \frac{2}{q} (*K_{(d})^{IJ} q_{c)(b} (*\pi_{a)})_{IJ} \delta^{(3)}(x-y) - ab \leftrightarrow cd \\
&= \frac{8\zeta}{q} K_{(d}{}^{IJ} q_{c)(b} \pi_{a)IJ} \delta^{(3)}(x-y) - ab \leftrightarrow cd \\
&= \frac{4\zeta}{q} \left( K_{[d}{}^{IJ} \pi_{a]IJ} q_{bc} + K_{[c}{}^{IJ} \pi_{a]IJ} q_{bd} \right. \\
&\qquad \left. + K_{[d}{}^{IJ} \pi_{b]IJ} q_{ac} + K_{[c}{}^{IJ} \pi_{b]IJ} q_{ad} \right) \delta^{(3)}(x-y) \approx 0, \qquad (9.87)
\end{aligned}
$$

where the terms in the last line vanish since $K_{[a|IJ} \pi_{b]}{}^{IJ} \approx 0$ up to Gauß and simplicity constraint.

We see that, at least in $D = 3$ and in variables $K$, $\pi$, there is a gauge fixing $\mathscr{D}'$ of the simplicity constraint with nicer properties: The Dirac matrix becomes trivial and $\mathscr{D}'$ is first class with respect to all other constraints. But gauge unfixing and obtaining a connection formulation still fail because of two reasons.

Firstly, $\mathscr{D}'$ is weakly Poisson commuting with itself and the Hamiltonian constraint only up to the simplicity constraint. Therefore, we cannot simply drop the simplicity constraint unless we change $\mathscr{D}'$ and $\mathscr{H}$ such that their algebra closes without $\mathscr{S}$.

Secondly, we still do not yet have a connection formulation. Of course, we can express $\mathscr{D}'$ in terms of new connection variables, $\mathscr{D}' = (*\overset{(\beta)}{\pi}{}_{(a})^{IJ}(A - \Gamma[\pi])_{b)IJ}$. But since we were not able to prove that the transformation $\{K_{aIJ}, \pi^{bKL}\} \to \{A_{aIJ}, \overset{(\beta)}{\pi}{}^{bKL}\}$ is canonical, we do not know if the constraint algebra is reproduced. In particular, when trying to redo the calculation $\{\mathscr{D}', \mathscr{D}'\}$ in connection variables, one cannot reproduce the result of (9.87) because of exactly the same problems which appeared when trying to prove then canonicity of the transformation to connection variables in section 7.2. Note that also the original $\mathscr{D}$ constraint of section 5.2 was not poisson self commuting.

While we will show in the next section that the first of these two problems can be overcome, i.e. we will construct a formulation with first class $\mathscr{D}$ constraints, but we do not know how to solve the second.

## 9.5 First class Hamiltonian formulation with arbitrary internal space

When introducing additional fields in the Lagrangian, one usually would as well introduce constraints which take care of the superfluous degrees of freedom. This is what happens when e.g. going from the Einstein Hilbert action (3.2) to the Plebański action (5.4). But when comparing the Einstein Hilbert actions with metric (2.1) and vielbein (3.7) as fundamental degree of freedom, what catches the eye is that, while working with more fields ($e_\mu{}^I$ having $(D+1)^2$ components while $g_{\mu\nu}$, being symmetric, only has $\frac{(D+1)(D+2)}{2}$ independent components), we do not need to change the action by additional constraints. The extra degrees of freedom are pure gauge, which is reflected in a new invariance of the action: $e_\mu{}^I \to e'_\mu{}^I := g^I{}_J e_\mu{}^J$ for $g \in \mathrm{SO}(D+1)$ or $\mathrm{SO}(1,D)$ in the Euclidean or Lorentzian case respectively. On the Hamiltonian side, as we have seen in section 3.2, additional *primary* constraints arise corresponding to these internal gauge transformations. I.e., the action by construction only depends on $\dot{q}_{ab}$ and surely we cannot solve the defining equations of the momenta for any other velocities. The velocities which cannot be solved for turn out to multiply exactly the Gauß constraints in the Hamiltonian corresponding to the additional internal symmetry.

This leads to the question: what happens if we choose not to work with a vielbein, but decompose the metric differently? E.g., suppose we consider the Einstein Hilbert action as a functional of a Lie algebra valued co vielbein field $\pi_\mu{}^{IJ}$ using $g_{\mu\nu} = \pi_{\mu IJ}\pi_\nu{}^{IJ}$, similar to what is used in part II. Do we need a simplicity constraint? The answer we will give in the following to this question is no, for the Lagrangian picture no new constraints are needed. This rises the question if they will somehow reappear in the Hamiltonian formulation. Performing the canonical analysis, we will find that the answer again is no. Actually, the simplicity constraints cannot appear as primary constraints, since primary constraints are at least linear in the momenta canonically conjugate to the $\pi$s which we will call $K$ in analogy with section 7.2. Instead, apart from a "Gauß like" constraint, constraints similar to the $\mathscr{D}$ constraints will appear as primary first class constraints, and the algebra of all constraints will be shown to be of the first class, in particular, no secondary constraints will appear. Our discussion will be independent of choice of internal space (except that its dimension should allow for incorporating the metric degrees of freedom) in the beginning, only later we will restrict to the case when $\pi_\mu$ is valued in some Lie algebra $\mathfrak{g}$.

### 9.5.1 Lagrangian viewpoint

Specificly, we will work with $\pi_\mu{}^\alpha$, $\alpha, \beta, \gamma \in \{1, ..., A\}$, $A \geq D+1$. We demand that in the internal space, there exists a constant metric tensor $\delta_{\alpha\beta}$ with inverse $\delta^{\alpha\beta}$. We will decompose the spacetime metric according to

$$g_{\mu\nu} = \pi_\mu{}^\alpha \pi_\nu{}^\beta \delta_{\alpha\beta}. \tag{9.88}$$

Variation of the Einstein Hilbert action (2.1) with respect to $\pi^\mu{}_\alpha$ yields, using $\delta g^{\mu\nu} = 2\pi^{(\mu}{}_\alpha \delta\pi^{\nu)\alpha}$ like in section 3.1,

$$\delta S = \frac{s}{2}\int_{\mathscr{M}} d^{D+1}x \; 2\sqrt{|g|}(\pi) G_{\mu\nu}(\pi)\pi^\nu{}_\alpha \delta\pi^{\mu\alpha}, \tag{9.89}$$

up to a boundary term. Contraction the field equations

$$G_{\mu\nu}(\pi)\pi^\nu{}_\alpha = 0 \tag{9.90}$$

with $\pi_\rho{}^\alpha$, we obtain $G_{\mu\rho} = 0$ as necessary condition, which then solves all the field equations. We thus are still dealing with general relativity, but with a possibly huge gauge symmetry. To find the generators of these transformations (zero eigenvectors of the matrix relating momenta and velocities) later in the Hamiltonian picture, it is helpful to study the new symmetries already at the level of the action. The infinitesimal transformations $\pi_\mu{}^\alpha \to \pi'_\mu{}^\alpha = \pi_\mu{}^\alpha + \epsilon_\mu{}^\alpha$ leaving the action invariant have to satisfy

$$\epsilon_{(\mu}{}^\alpha \pi_{\nu)\alpha} = 0, \tag{9.91}$$

in order to leave $g_{\mu\nu}$ invariant. Since $\pi_\mu{}^\alpha$ are $D+1$ vectors in a $A$-dimensional space, there are $A-(D+1)$ linearly independent internal vectors orthogonal to it which for sure satsify (9.91). These orthogonal directions can be accessed using the projectors

$$\hat{\mathbb{P}}_{\alpha\beta} := \delta_{\alpha\beta} - \pi_{\mu\alpha}\pi_{\nu\beta}g^{\mu\nu}, \qquad \hat{\mathbb{Q}}_{\alpha\beta} := \pi_{\mu\alpha}\pi_{\nu\beta}g^{\mu\nu}, \qquad \delta_{\alpha\beta} = \hat{\mathbb{P}}_{\alpha\beta} + \hat{\mathbb{Q}}_{\alpha\beta},$$
$$\hat{\mathbb{P}}_{\alpha\beta}\hat{\mathbb{P}}^\beta\gamma = \hat{\mathbb{P}}_{\alpha\gamma}, \qquad \hat{\mathbb{Q}}_{\alpha\beta}\hat{\mathbb{Q}}^\beta\gamma = \hat{\mathbb{Q}}_{\alpha\gamma}, \qquad \hat{\mathbb{P}}_{\alpha\beta}\hat{\mathbb{Q}}^\beta\gamma = 0, \tag{9.92}$$

and in particular, $\hat{\mathbb{P}}_{\alpha\beta}\pi_\mu{}^\beta = 0$, $\hat{\mathbb{Q}}_{\alpha\beta}\pi_\mu{}^\beta = \pi_{\mu\alpha}$. Thus, there exist $(D+1)[A-(D+1)]$ independent vectors $\epsilon_\mu{}^\alpha$ satisfying $\epsilon_\mu{}^\alpha := \mathbb{P}^{\alpha\beta}\epsilon_{\mu\beta}$ which constitute solutions to (9.91). Futhermore, exploiting the symmetry of (9.91) in the index pair $\mu$, $\nu$, we obtain $\frac{D(D+1)}{2}$ solutions $\epsilon_\mu{}^\alpha = \Lambda^\rho{}_\mu \pi_\rho{}^\alpha$ with $\Lambda^{\mu\nu} = -\Lambda^{\nu\mu}$. Since $\pi_\mu{}^\alpha$ has $A(D+1)$ components and the symmetries remove $(D+1)(A - \frac{(D+2)}{2})$, we are left with $\frac{(D+2)(D+1)}{2}$, corresponding to the metric degrees of freedom.

### 9.5.2 Canonical analysis

Performing the $D+1$ split, we use in analogy to the vielbein case $\pi_\mu{}^\alpha = {}^{||}\pi_\mu{}^\alpha + sn_\mu n^\alpha$, where ${}^{||}\pi_\mu{}^\alpha := q_\mu{}^\nu \pi_\nu{}^\alpha$, $n^\alpha = \pi_\mu{}^\alpha n^\mu$, and we have ${}^{||}\pi_\mu{}^\alpha n_\alpha = 0$, $n^\alpha n_\alpha = s$. Trivially, the split form of the action is given by (2.11), where the spatial metric now is considered as constructed from the pullback of ${}^{||}\pi_\mu{}^\alpha$, $q_{ab} = \pi_a{}^\alpha \pi_{b\alpha}$, and (2.13) immediately translates into $K_{ab} = \frac{1}{N}\pi_{(a}{}^\alpha \left(\dot{\pi}_{b)\alpha} + (\mathscr{L}_N\pi)_{b)\alpha}\right)$.

For the momenta, we find

$$\begin{aligned} K^c{}_\alpha(t,x) &= \frac{\delta S}{\delta \dot{\pi}_c{}^\alpha(t,x)} \\ &= \int_\sigma d^D y \frac{\delta \dot{q}_{ab}(t,y)}{\delta \dot{\pi}_c{}^\alpha(t,x)} \frac{\delta S}{\delta \dot{q}_{ab}(t,y)} \\ &= -\sqrt{q}(t,x) G^{cd\,ab}(t,x)\pi_{d\alpha}(t,x) K_{ab}(t,x) \\ &= G^{cd}_{\alpha\beta}\left(\dot{\pi}_d{}^\beta(t,x) - (\mathscr{L}_N\pi)_d{}^\beta(t,x)\right), \end{aligned} \tag{9.93}$$

where we introduced $G^{ac}_{\alpha\beta} := -\frac{\sqrt{q}}{N}\pi_{b\alpha}G^{ab\,cd}\pi_{d\beta}$ and $G^{ab\,cd}$ was defined in (2.15). From the zero eigenvectors of this matrix, we again can deduce the primary constraints, and indeed, we find zero eigenvectors corresponding to the invariances of the action. Using similar projectors

$$\mathbb{P}_{\alpha\beta} := \delta_{\alpha\beta} - \pi_{a\alpha}\pi_{b\beta}q^{ab}, \qquad\qquad \mathbb{Q}_{\alpha\beta} := \pi_{a\alpha}\pi_{b\beta}q^{ab}, \tag{9.94}$$

these result in the primary constraints

$$\mathscr{G}^{ab} := \pi^{[a}{}_\alpha K^{b]\alpha} = 0, \tag{9.95}$$

$$\mathscr{D}^a{}_\alpha := \mathbb{P}_{\alpha\beta}K^{a\beta} = 0. \tag{9.96}$$

Counting shows that, if these constraints are first class as we will prove later on, they are sufficient: We have in total $\frac{D(D-1)}{2} + D(A-D) = D(A-\frac{D+1}{2})$ constraints to go from $\pi_a{}^\alpha$ to $q_{ab}$, which differ by the same number of components, $DA - \frac{D(D+1)}{2} = D(A - \frac{D+1}{2})$. As before, (9.93) can only be solved for the velocities corresponding to $\dot{q}_{ab}$,

$$\pi_{(a}{}^\beta \dot{\pi}_{b)\beta} = -\frac{N}{\sqrt{q}} G^{(-1)}_{ab\,cd}\pi^{(c|\alpha}K^{d)}_\alpha + \pi_{(a}{}^\beta(\mathscr{L}_N\pi)_{b)\beta}, \tag{9.97}$$

and expressing the kinetic term we expect in the Hamiltonian in terms of $K, \pi, N, \vec{N}$, we obtain using (9.97) and elementary algebra

$$
\begin{aligned}
K^a{}_\alpha \dot{\pi}_a{}^\alpha &= K^{a\alpha}\left(\mathbb{P}_{\alpha\beta} + \mathbb{Q}_{\alpha\beta}\right)\dot{\pi}_a{}^\beta \\
&= \left(K^{a\alpha}\mathbb{P}_{\alpha\beta} + K^{[a\alpha}\pi^{b]}{}_\alpha\pi_{b\beta} + K^{(a\alpha}\pi^{b)}{}_\alpha\pi_{b\beta}\right)\dot{\pi}_a{}^\beta \\
&= \mathscr{D}^{a\alpha}\mathbb{P}_{\alpha\beta}\dot{\pi}_a{}^\beta + \mathscr{G}^{ba}\pi_{[b|\beta}\dot{\pi}_{a]}{}^\beta \\
&\quad + K^{(a\alpha}\pi^{b)}{}_\alpha\left(-\frac{N}{\sqrt{q}}G^{(-1)}_{ab\,cd}\pi^{(c|\alpha}K^{d)}_\alpha + \pi_{(a}{}^\beta(\mathscr{L}_N\pi)_{b)\beta}\right) \\
&= \mathscr{D}^{a\alpha}\mathbb{P}_{\alpha\beta}\dot{\pi}_a{}^\beta + \mathscr{G}^{ba}\pi_{[b|\beta}\dot{\pi}_{a]}{}^\beta - \frac{N}{\sqrt{q}}K^{(a\alpha}\pi^{b)}{}_\alpha G^{(-1)}_{ab\,cd}\pi^{(c|\alpha}K^{d)}_\alpha \\
&\quad + K^{[a\alpha}\pi^{b]}{}_\alpha\pi_a{}^\beta(\mathscr{L}_N\pi)_{b\beta} + K^{b\alpha}\pi^a{}_\alpha\pi_a{}^\beta(\mathscr{L}_N\pi)_{b\beta} \\
&= \mathscr{D}^{a\alpha}\left[\mathbb{P}_{\alpha\beta}\dot{\pi}_a{}^\beta - \mathbb{P}_{\alpha\beta}(\mathscr{L}_N\pi)_a{}^\beta\right] + \mathscr{G}^{ba}\left[\pi_{[b|\beta}\dot{\pi}_{a]}{}^\beta - \pi_{[b}{}^\beta(\mathscr{L}_N\pi)_{a]\beta}\right] \\
&\quad - \frac{N}{\sqrt{q}}K^{(a\alpha}\pi^{b)}{}_\alpha G^{(-1)}_{ab\,cd}\pi^{(c|\alpha}K^{d)}_\alpha + K^{a\alpha}(\mathscr{L}_N\pi)_{a\alpha}.
\end{aligned}
\tag{9.98}
$$

and therefore for the action in Hamiltonian form

$$
\begin{aligned}
S = \int dt \int_\sigma d^Dx \quad &\Big[P^{(N)}\dot{N} + P^{(\vec{N})}_a\dot{N}^a + K^a{}_\alpha\dot{\pi}_a{}^\alpha - \lambda\mathscr{C} - \lambda^a\mathscr{C}_a \\
&\quad - \left(K^a{}_\alpha\dot{\pi}_a{}^\alpha - \frac{s}{2}N\sqrt{q}\left({}^{(D)}R - sG^{ab\,cd}K_{ab}K_{cd}\right)\right)(K,\pi,N,\vec{N})\Big] \\
= \int dt \int_\sigma d^Dx \quad &\Big[P^{(N)}\dot{N} + P^{(\vec{N})}_a\dot{N}^a + K^a{}_\alpha\dot{\pi}_a{}^\alpha - \lambda\mathscr{C} - \lambda^a\mathscr{C}_a \\
&\quad - \left(\frac{1}{2}K^a{}_\alpha\dot{\pi}_a{}^\alpha + \frac{1}{2}K^a{}_\alpha(\mathscr{L}_N\pi)_a{}^\alpha - \frac{s}{2}N\sqrt{q}{}^{(D)}R\right)(K,\pi,N,\vec{N})\Big] \\
= \int dt \int_\sigma d^Dx \quad &\Big[P^{(N)}\dot{N} + P^{(\vec{N})}_a\dot{N}^a + K^a{}_\alpha\dot{\pi}_a{}^\alpha - \lambda\mathscr{C} - \lambda^a\mathscr{C}_a \\
&\quad - d_{a\alpha}\mathscr{D}^{a\alpha} - \Lambda_{ab}\mathscr{G}^{ab} - N^a\mathscr{H}_a - N\mathscr{H}\Big],
\end{aligned}
\tag{9.99}
$$

where in the last step, we used (9.98), integrated by parts and defined

$$
\mathscr{H}_a := K^b{}_\alpha\partial_a\pi_b{}^\alpha - \partial_b(K^b{}_\alpha\pi_a{}^\alpha), \tag{9.100}
$$

$$
\mathscr{H} := -\frac{1}{2\sqrt{q}}K^{(a\alpha}\pi^{b)}{}_\alpha G^{(-1)}_{ab\,cd}\pi^{(c|\alpha}K^{d)}_\alpha - \frac{s}{2}\sqrt{q}{}^{(D)}R. \tag{9.101}
$$

We furthermore replaced the velocities we could not solve for by Lagrange multipliers $d_{a\alpha}$ and $\Lambda_{ab}$. Eliminating as before $N$, $N^a$, we obtain the Hamiltonian

$$
H = \int_\sigma d^Dx \;\left[d_{a\alpha}\mathscr{D}^{a\alpha} + \Lambda_{ab}\mathscr{G}^{ab} + N^a\mathscr{H}_a + N\mathscr{H}\right], \tag{9.102}
$$

and non-vanishing Poisson brackets

$$\{\pi_a{}^\alpha(t,x), K^b{}_\beta(t,y)\} = \delta^b_a \delta^\alpha_\beta \delta^{(D)}(x-y). \tag{9.103}$$

Let us study the stability of the constraints under time evolution. First, since $\mathscr{H}_a$ again is obviously the generator of spatial diffeomorphisms, it is clearly first class and its Poisson brackets with all constraints are clear. $\mathscr{H}$ only depends on $q_{ab}(\pi) = \pi_a{}^\alpha \pi_{b\alpha}$ (and derivatives thereof) and $P^{cd}(K,\pi) = \frac{1}{2} K^{(c|\beta} \pi^{d)}{}_\beta$, and they satisfy (weakly) the ADM Poisson brackets,

$$\begin{aligned}
\{q_{ab}(\pi)(t,x), q_{cd}(\pi)(t,y)\} &= 0, \\
\{q_{ab}(\pi)(t,x), P^{cd}(K,\pi)(t,y)\} &= \delta^c_{(a}\delta^d_{b)} \delta^{(D)}(x-y), \\
\{P^{ab}(K,\pi)(t,x), P^{cd}(K,\pi)(t,y)\} &= -\frac{1}{4}\left(q^{a(c}\mathscr{G}^{d)b} + q^{b(c}\mathscr{G}^{d)a}\right)(x)\ \delta^{(D)}(x-y),
\end{aligned} \tag{9.104}$$

which in turn tells us that the hypersurface deformation algebra 2.31 is (weakly) reproduced. What is left to check is if the Poisson brackets between the new constraints weakly vanish, and whether $q_{ab}(\pi)$ and $P^{cd}(K,\pi)$ are Dirac observables with respect to the new constraints. Straightforward calculation shows

$$\{\mathscr{D}^a{}_\alpha(t,x), \mathscr{D}^b{}_\beta(t,y)\} = \left(2\mathbb{P}_{\alpha\beta}\mathscr{G}^{ab} + \pi^a{}_\beta \mathscr{D}^b{}_\alpha - \pi^b{}_\alpha \mathscr{D}^a{}_\beta\right)\delta^{(D)}(x-y), \tag{9.105}$$

$$\{\mathscr{G}^{ab}(t,x), \mathscr{D}^c{}_\alpha(t,y)\} = q^{c[a}\mathscr{D}^{b]}{}_\alpha \delta^{(D)}(x-y), \tag{9.106}$$

$$\{\mathscr{G}^{ab}(t,x), \mathscr{G}^{cd}(t,y)\} = \left(q^{c[a}\mathscr{G}^{b]d} - q^{d[a}\mathscr{G}^{b]c}\right)\delta^{(D)}(x-y), \tag{9.107}$$

$$\{\mathscr{D}^a{}_\alpha(t,x), q_{cd}(\pi)(t,y)\} = 0, \tag{9.108}$$

$$\{\mathscr{G}^{ab}(t,x), q_{cd}(\pi)(t,y)\} = 0, \tag{9.109}$$

$$\{\mathscr{D}^a{}_\alpha(t,x), P^{cd}(K,\pi)(t,y)\} = -\frac{1}{2} q^{a(c}\mathscr{D}^{d)}{}_\alpha \delta^{(D)}(x-y), \tag{9.110}$$

$$\{\mathscr{G}^{ab}(t,x), P^{cd}(K,\pi)(t,y)\} = \frac{1}{2}\left(q^{a(c}\mathscr{G}^{d)b} - q^{b(c}\mathscr{G}^{d)a}\right)\delta^{(D)}(x-y). \tag{9.111}$$

proving that the constraints are all first class and no secondary constraints appear. To interpret the additional constraints $\mathscr{D}^a{}_\alpha$ and $\mathscr{G}^{ab}$, note that $\mathscr{D}^a{}_\alpha$ demands that $K$ lies in the same $D$-dimensional subspace in the internal space as the $\pi$s (in the notation used in part II, say, the $\pi$s are given by $\pi_a{}^{IJ} = 2n^{[I}e_a{}^{J]}$, then only the component $K^a{}_{IJ} = 2n_{[I}K^a{}_{J]}$ survives) and then $\mathscr{G}^{ab}$ reduces to the Gauß like constraint $e^{[a}{}_I K^{b]I}$, which is related with the symmetry of the extrinsic curvature.

| Variable | Dof | $1^{\text{st}}$ cl. constraints | Dof (count twice!) |
|---|---|---|---|
| ${\pi_a}^\alpha$ | $DA$ | $\mathscr{H}$ | $1$ |
| ${K^b}_\beta$ | $DA$ | $\mathscr{H}_a$ | $D$ |
| | | $\mathscr{G}^{ab}$ | $\frac{D(D-1)}{2}$ |
| | | $\mathscr{D}^{a\alpha}$ | $D(A-D)$ |
| Sum: | $2DA$ | Sum: | $2DA-(D^2-D-2)$ |

**Table 9.3:** First class formulation with arbitrary internal space: counting of degrees of freedom

### 9.5.3 Connection formulation?

Here, we end our general analysis, and in the following will specify the internal space. Since we want to discuss the possibility of obtaining a connection formulation, the case of interest is that $\pi_a$ be $\mathfrak{g}$-valued for some compact Lie algebra $\mathfrak{g}$. As usual, we can assume the Killing metric to be given by $\delta_{\alpha\beta}$ and the structure constants $f_{\alpha\beta\gamma}$ to be totally antisymmetric and to satisfy ${f_\alpha}^{\gamma\delta}f_{\beta\gamma\delta}=\delta_{\alpha\beta}$. In this case, we of course expect the appearance of a Gauß constraint $\mathscr{G}_\alpha := f_{\alpha\beta\gamma}{\pi_a}^\beta K^{a\gamma}$. Indeed, this constraint corresponds to the zero eigenvectors $(V_{a\alpha})_\beta := f_{\alpha\beta\gamma}{\pi_a}^\gamma$ of $G^{ab}_{\alpha\beta}$. However, it has to be and, of course, is already included in the constraints we introduced before. To see this, it is convenient to first recombine these constraints to the equivalent set of constraints

$$\hat{\mathscr{D}}_{\alpha\beta} := \pi_{a[\alpha}{K^a}_{\beta]}. \tag{9.112}$$

Their equivalence can be seen as follows: First of all, $\hat{\mathscr{D}}_{\alpha\beta}$ has $\frac{A(A-1)}{2}$ components, of which $\frac{(A-D)(A-D-1)}{2}$ trivially vanish (project both indices on the $(A-D)$-dimensional subspace accessed via $\mathbb{P}^{\alpha\beta}$), i.e. we have with $\frac{A(A-1)}{2}-\frac{(A-D)(A-D-1)}{2}=D(A-\frac{D+1}{2})$ the right number of constraints. Furthermore, contraction with $\pi^{a\alpha}\pi^{b\beta}$ yields $\mathscr{G}^{ab}$, and a single contraction with $\pi^{a\alpha}$ and using $\mathscr{G}^{ab}$, we obtain $\mathscr{D}^{a\alpha}$.

In a second step, we decompose $\hat{\mathscr{D}}_{\alpha\beta}$ according to

$$\begin{aligned}\hat{\mathscr{D}}^{\alpha\beta} &= \left(\delta^\alpha_\gamma\delta^\beta_\delta - {f^{\alpha\beta}}_\epsilon {f_{\gamma\delta}}^\epsilon + {f^{\alpha\beta}}_\epsilon {f_{\gamma\delta}}^\epsilon\right)\hat{\mathscr{D}}^{\alpha\beta} \\ &=: \mathscr{D}^{\alpha\beta} + {f^{\alpha\beta}}_\gamma \mathscr{G}^\gamma,\end{aligned} \tag{9.113}$$

where $\mathscr{D}^{\alpha\beta} := \left(\delta^\alpha_\gamma \delta^\beta_\delta - f^{\alpha\beta}{}_\epsilon f_{\gamma\delta}{}^\epsilon\right) \pi_a{}^{[\gamma} K^{a\delta]} =: M^{\alpha\beta}{}_{\gamma\delta} \pi_a{}^{[\gamma} K^{a\delta]}$ has the property $f_{\alpha\beta\gamma}\mathscr{D}^{\beta\gamma} = 0$ due to $f_{\alpha\gamma\delta} f_\beta{}^{\gamma\delta} = \delta_{\alpha\beta}$. The geometrical interpretation of this $\mathscr{D}$-constraint is that it generates all transformations which leave $q_{ab} = \pi_{a\alpha}\pi_b{}^\alpha$ invariant and which are not Gauß transformations. Its form crucially depends on the Lie algebra under consideration: For so(3), it is easy to see that $\mathscr{D}^{\alpha\beta} = 0$ identically since $M^{\alpha\beta}{}_{\gamma\delta} = 0$. For $\mathfrak{g} = \mathrm{so}(D+1)$ or $\mathrm{so}(1,D)$ with $D > 2$, we have $M^{[IJ][KL]}{}_{[MN][OP]} \propto \left(\epsilon^{IJ}{}_{MN}{}^{\overline{M}} \epsilon^{KL}{}_{OP\overline{M}} - \epsilon^{KL}{}_{MN}{}^{\overline{M}} \epsilon^{IJ}{}_{OP\overline{M}}\right)$ (cf. appendix D), which should be compared with (5.28), where a very similar contraction of two epsilons appeared when obtaining the independent set of $\mathscr{D}$- and simplicity constraints such that the Dirac matrix is invertible.

Let us calculate the constraint algebra of the newly introduced constraints. Using $\{\hat{\mathscr{D}}_{\alpha\beta}, \hat{\mathscr{D}}_{\gamma\delta}\} = \delta_{\beta[\gamma}\hat{\mathscr{D}}_{\delta]\alpha} - \delta_{\alpha[\gamma}\hat{\mathscr{D}}_{\delta]\beta}$ and the Jacobi identity satisfied by the structure constants, it is straightforward to obtain

$$\{\mathscr{G}_\alpha, \mathscr{G}_\beta\} = -2 f_\alpha{}^{\gamma\epsilon} f_\beta{}^\delta{}_\epsilon \hat{\mathscr{D}}_{\gamma\delta} = -f_{\alpha\beta}{}^\epsilon \mathscr{G}_\epsilon, \tag{9.114}$$

$$\{\mathscr{D}_{\alpha\beta}, \mathscr{G}_\gamma\} = -2 M_{\alpha\beta}{}^{\delta\epsilon} f_{\gamma\epsilon}{}^\zeta \hat{\mathscr{D}}_{\delta\zeta} = 2 f_{\gamma[\beta|}{}^\delta \mathscr{D}_{\delta|\alpha]}, \tag{9.115}$$

$$\{\mathscr{D}_{\alpha\beta}, \mathscr{D}_{\gamma\delta}\} = -2 M_{\alpha\beta}{}^{\epsilon\zeta} M_{\gamma\delta}{}^{\zeta\eta} \hat{\mathscr{D}}_{\epsilon\eta} = -2 M_{\alpha\beta}{}^{\epsilon\zeta} M_{\gamma\delta}{}^{\zeta\eta} \mathscr{D}_{\epsilon\eta} + 2 M_{\alpha\beta}{}^\epsilon{}_{[\gamma} f_{\delta]\epsilon}{}^\zeta \mathscr{G}_\zeta. \tag{9.116}$$

Now we have separated the Gauß constraint from the $\mathscr{D}$-part, we can give a tentative definition of what one could call the simplicity constraint $\mathscr{S}$ for any Lie algebra: A gauge fixing for $\mathscr{D}$, i.e. a constraint such that $\{\mathscr{S}, \mathscr{D}\}$ yields an invertible matrix on a suitably chosen space of Lagrange multipliers, while $\{\mathscr{S}, \mathscr{C}\} \approx 0$ for any other constraint $\mathscr{C}$. Equivalently, $\mathscr{S}$ should annihilate all components of $\pi$ which do not contribute to the metric up to the Gauß constraint. Ideally, we would like $\mathscr{S}$ to be constructed solely from $\pi$ and in the following restrict attention to this case. Since it is very easy to write down constraints which for sure will weakly Poisson commute with the Gauß and spatial diffeomorphism constraint and since $\{\mathscr{S}, q_{ab}[\pi]\} = 0$ by construction, the only thing which needs to be checked is if $\{\mathscr{S}, P^{ab}[K, \pi]\} \propto \mathscr{S} \approx 0$. Being independent of $K^a{}_\alpha$, no other constraint except $\mathscr{S}$ can appear on the right hand side of this Poisson bracket. Finding $\mathscr{S}$ with the above properties seems to be the core problem, and probably is impossible in most cases.

Still, assuming we have accomplished this, the second step in order to obtain a connection formulation is to gauge unfix $\mathscr{S}$. This again is non-trivial, the problem being that for the Poisson bracket $\{\mathscr{H}, \mathscr{H}\}$, we need that

$$\begin{aligned} &\{P^{ab}(K,\pi)[f_{ab}], P^{cd}(K,\pi)[f'_{cd}]\} = \\ &= \int_\sigma d^D x \left[ -\frac{1}{2} f^{a[b} f'^{\,c]}_a \pi_{b\alpha}\pi_{c\beta} \left( M^{\alpha\beta}{}_{\gamma\delta}\mathscr{D}^{\gamma\delta} + f^{\alpha\beta}{}_{\gamma}\mathscr{G}^{\gamma} \right) \right] \overset{!}{\approx} 0, \end{aligned} \tag{9.117}$$

has to vanish weakly *without* using $\mathscr{D}$. This gives us a hint towards what the simplicity constraints have to be, namely a necessary condition is that the matrix multiplying $K^a{}_\alpha$ in the $\mathscr{D}$-term satisfies $\pi_{a\alpha}\pi_{b\beta}M^{\alpha\beta}{}_{\gamma\delta}\pi_c{}^\gamma =: \mathscr{S}'_{abc\,\delta} \overset{!}{\propto} \mathscr{S}$, and one could conjecture that this matrix actually constitutes the simplicity constraint. In particular, one finds $\{\mathscr{S}', P^{cd}[K,\pi]\} \propto \mathscr{S}'$. However, $\mathscr{S}'$ are algebraically by far too many constraints, and have to be hugely redundant in order to be correct. Again comparing[1] to the case $\mathrm{SO}(D+1)$ or $\mathrm{SO}(1,D)$ of part II, we have that $\mathscr{S}'$ indeed is proportional to the simplicity constraint $\mathscr{S}'_{abc\,IJ} \propto (^{\overline{M}} * \pi_{[a|})_{IJ}\mathscr{S}_{|b]c\overline{M}}$. Using the solution of the simplicity constraint $\pi^{aIJ} = 2n^{[I}E^{a|J]}$ after calculating the Poisson bracket, one can show that with the constraint

$$\mathscr{S}_a{}^{IJ} = \mathbb{P}^{IJKL} M_{KL\,MN}{}^{OPQR} \mathbb{Q}^{MN}{}_{OP}\pi_{aQR}, \tag{9.118}$$

being a certain contraction and projection of the full $\mathscr{S}'_{abc\,IJ}$, the Poisson bracket $\{\mathscr{S}, \mathscr{D}\}$ is invertible on transversal trace free Lagrange multipliers as introduced in part II, but apart from this indirect proof of the validity of the chosen set, finding an independent set of simplicity constraints is rather complicated even in this case. We want to point out that this independent set of constraints is again very similar to the independent set of simplicity constraints obtained in section 5.2.

Pushing further, we need to find a connection which, at least weakly, solves

$$D_a \pi_{b\alpha} + f_\alpha{}^{\beta\gamma}\Gamma_{a\beta}(\pi)\pi_{b\gamma} \approx 0. \tag{9.119}$$

At first sight, this seems hopeless, since these are $D^2 A$ equations for $DA$ unknowns $\Gamma_{a\alpha}(\pi)$. However, we only need that $\Gamma$ annihilates $\pi$ weakly, i.e. only those parts of

[1] Note that for direct comparison with the $\mathrm{SO}(D+1)$ or $\mathrm{SO}(1,D)$ case, we again should perform the canonical transformation $\{\pi_a{}^\alpha, K^b{}_\beta\} \to \{K'_{a\alpha} := \frac{1}{\sqrt{q}}(\pi_{b\alpha}\pi_{a\beta} - \frac{1}{D-1}\pi_{a\alpha}\pi_{b\beta} - q_{ab}\mathbb{P}_{\alpha\beta})K^{b\beta}, \pi'^{a\alpha} := \sqrt{q}q^{ab}\pi_b{}^\alpha\}$.

$\pi$ which are left when solving $\mathscr{S}$, and this makes a solution - at least algebraically - possible: Those parts constitute $DA - [D(A - \frac{D+1}{2}) - A] = \frac{D(D+1)}{2} + A$ degrees of freedom. The requirements that they be annihilated are $D$ times that number, $D[\frac{D(D+1)}{2} + A]$. However, they are not all independent, the $\frac{D^2(D+1)}{2}$ equations corresponding to $2\pi_{(c|\alpha}D_a\pi_{b)}{}^{\alpha} = D_a q_{bc} = 0$ are identically satisfied, leaving $DA$ equations for $DA$ unknowns. However, the number of degrees of freedom in the freedom in the "vielbein" resulting when solving both, $\mathscr{D}$ and $\mathscr{S}$, being $\frac{D(D+1)}{2} + A$, is rather odd (except for e.g. $A = \frac{D(D+1)}{2}$), which makes it unlikely that a corresponding simplicity constraint can be found which removes the unnecessary degrees of freedom in $\pi_a{}^{\alpha}$ in a gauge invariant way.

Finally, even if $\mathscr{S}$ and $\Gamma_a{}^{\alpha}[\pi]$ can be found, we do not know of a general argument indicating that the corresponding transformation to connection variables is canonical (or at least that we can construct a corresponding extension of the ADM phase space).

Summarising, it seems doubtful that other connection formulations can be obtained due to the problems mentioned, although we did not prove that it is impossible. At this point one cannot proceed any further without making a specific choice of the gauge group and study in detail the corresponding matrix $M_{\alpha\beta}{}^{\gamma\delta}$. We leave the study of specific groups for further research.

# Part III

# Quantisation

This part is taken from [3]. The presentation has been changed slightly, and chapter 11.2 has been enlarged to incorporate also a summary of the findings of [5] and parts of [6].

We provide a loop quantisation of the new connection formulation of $D+1$ dimensional general relativity ($D+1 \geq 3$) introduced in part II, namely an SO($D+1$) gauge theory subject to SO($D+1$) Gauß constraint, simplicity constraint, spatial diffeomorphism constraint and Hamiltonian constraint. Apart from the different gauge group which however is compact and the additional simplicity constraint, the situation is precisely the same as for LQG and the quantisation of our connection formulation is therefore in complete analogy with LQG. We can therefore simply follow any standard text on LQG such as [61, 62] and follow all the quantisation steps. This way we arrive at the holonomy-flux algebra, its unique spatially diffeomorphism invariant state whose GNS data are the analogue for SO($D+1$) of the Ashtekar-Isham-Lewandowski Hilbert space, the analogue of spin network functions, kinematical geometrical operators such as the volume operator which is pivotal for the quantisation of the Hamiltonian constraint, the SO($D+1$) Gauß constraint, the spatial diffeomorphism constraint, the Hamiltonian constraint and a corresponding master constraint.

The only structurally new ingredient is the simplicity constraint which constrains the type of allowed SO($D+1$) representations, and therefore the corresponding section 11.2 will be considerably longer than the ones treating the other kinematical constraints. The simplicity constraints have been intensely studied in the spin foam literature, but here we want to take an unbiased look at them in the canonical picture and work with methods independent of the spacetime dimension. We want to stress that we will not present a completely satisfactory solution to the simplicity constraint puzzle, but rather suggest for both, the quadratic and the linear constraint, some new starting points for further research which will hopefully help finding such a solution in the future.

This part is organised as follows: in the first chapter, we define the SO($D+1$) holonomy-flux algebra and the corresponding Hilbert space representation. In chapter two we implement the kinematical constraints, that is Gauß, simplicity and spatial diffeomorphism constraints. This chapter will come with its own introduction, mainly sketching the spin foam results and outlining our own findings on the simplicity constraints. In chapter three we develop kinematical geometrical operators, specifically $D$-dimensional area and volume operators. Lower dimensional operators such as length operators etc. can be constructed similarly but are left for future publication. Finally, in chapter four we quantise the Hamiltonian constraint. Most of the presentation will be brief since all the constructions literally parallel those of LQG. We therefore refer the interested reader to [62], the exposition of which we follow, for all the missing details.

# 10

# Introduction to the holonomy-flux-algebra

The construction of the kinematical Hilbert has been performed in [20, 21, 24–27] for four and higher space-time dimension and arbitrary compact gauge group. These results apply for the case considered here, since we are using the compact group $\mathrm{SO}(D+1)$ irrespective of the signature of the space-time metric. We therefore only cite the main results in this section and introduce notation needed later on.

Since the Poisson brackets between $A_{aIJ}$ and $\pi^{bKL}$ are distributional, we have to smear them with test functions. In order to obtain non-distributional Poisson brackets, smearing has to be done at least $D$-dimensional in total. $A_{aIJ}$ is a one-form, thus naturally smeared along a one-dimensional curve. From $\pi^{aIJ}$, being a vector density of weight one, we can construct the $\mathrm{so}(D+1)$ - valued pseudo $(D-1)$-form $(*\pi)_{a_1 \ldots a_{D-1}} := \pi^{aIJ} \epsilon_{a a_1 \ldots a_{D-1}} \tau_{IJ}$ which is integrated over a $(D-1)$-dimensional surface in a background-independent way. These considerations lead to the definitions of holonomies and fluxes, which yield a natural starting point for a background independent quantisation. In the following, we choose $(\tau_{IJ})^K{}_L = \frac{1}{2}\left(\delta^K_I \delta_{JL} - \delta^K_J \delta_{IL}\right)$ as a basis of the Lie algebra $\mathrm{so}(D+1)$.

## 10.1 Holonomies, distributional connections, cylindrical functions, kinematical Hilbert space and spin network states

Denote by $\mathscr{A}$ the space of smooth connections over $\sigma$. We define the holonomy $h_c(A) \in \mathrm{SO}(D+1)$ of the connection $A \in \mathscr{A}$ along a curve $c : [0,1] \to \sigma$ as the unique solution to the differential equation

$$\frac{d}{ds} h_{c_s}(A) = h_{c_s}(A) A(c(s)), \quad h_{c_0} = 1_{D+1}, \quad h_c(A) = h_{c_1}(A), \tag{10.1}$$

where $c_s(t) := c(st)$, $s \in [0,1]$, $A(c(s)) := A_a^{IJ}(c(s)) \tau_{IJ} \dot{c}^a(s)$. The solution is explicitly given by

$$h_c(A) = \mathscr{P} \exp\left( \int_c A \right) = 1_{D+1} + \sum_{n=1}^{\infty} \int_0^1 dt_1 \int_{t_1}^1 dt_2 \ldots \int_{t_{n-1}}^1 dt_n A(c(t_1)) \ldots A(c(t_n)), \tag{10.2}$$

where $\mathscr{P}$ denotes the path ordering symbol which orders the smallest path parameter to the left. Like in $3+1$ dimensional LQG, we will restrict ourselves to piecewise analytic and compactly supported curves.

The holonomies coordinatise the classical configuration space. In quantum field theory it is generic that the measure underlying the scalar product of the theory is supported on a distributional extension of the classical configuration space. For gravity, this enlargement of the configuration space is done by generalising the idea of a holonomy. Since the equations

$$h_{c \circ c'}(A) = h_c(A) h_{c'}(A) \qquad h_{c^{-1}}(A) = h_c(A)^{-1} \tag{10.3}$$

hold, we see that an element $A \in \mathscr{A}$ is a homomorphism from the set of piecewise analytic paths with compact support $\mathscr{P}$ into the gauge group. We now introduce the set $\overline{\mathscr{A}} := \mathrm{Hom}(\mathscr{P}, \mathrm{SO}(D+1))$ of all algebraic homomorphisms (without continuity assumptions) from $\mathscr{P}$ into the gauge group. This space $\overline{\mathscr{A}}$ is called the space of distributional connections over $\sigma$ and constitutes the quantum configuration space. The algebra of cylindrical functions $\mathrm{Cyl}(\overline{\mathscr{A}})$ on the space of distributional $\mathrm{SO}(D+1)$ connections is chosen as the algebra of kinematical observables. The former algebra can be written as the union of the set of functions of distributional connections defined on piecewise analytic graphs $\gamma$, $\mathrm{Cyl}(\overline{\mathscr{A}}) = \cup_\gamma \mathrm{Cyl}_\gamma(\overline{\mathscr{A}})/\sim$. $\mathrm{Cyl}_\gamma(\overline{\mathscr{A}})$ is defined as follows. A piecewise analytic graph $\gamma \in \sigma$ consists of analytic edges $e_1$,...,$e_n$, which meet at most at their endpoints, and vertices $v_1$,...,$v_m$. We denote the edge and vertex set of $\gamma$ by $E(\gamma)$ ($|E(\gamma)| = n$) and $V(\gamma)$ ($|V(\gamma)| = m$), respectively. A function $f_\gamma \in \mathrm{Cyl}_\gamma(\overline{\mathscr{A}})$ is labelled by the graph $\gamma$ and typically looks like $f_\gamma(A) = F_\gamma\left(h_{e_1}(A), ..., h_{e_{|E|}}(A)\right)$, where $F_\gamma : \mathrm{SO}(D+1)^{|E|} \to \mathbb{C}$. One and the same cylindrical function $f \in \mathrm{Cyl}(\overline{\mathscr{A}})$ can be represented on different graphs leading to cylindrically equivalent representations of that function. It is understood in the above union that such functions are identified. We will denote the pullback of a function $f_\gamma$ defined on $\gamma$ on the bigger[1] graph $\gamma' \succ \gamma$ via the cylindrical projections by $p^*_{\gamma'\gamma}$. Then, the equivalence relation just mentioned can be made more explicit, $f_\gamma \sim f'_{\gamma'}$ iff $p^*_{\gamma''\gamma} f_\gamma = p^*_{\gamma''\gamma'} f'_{\gamma'}$ $\forall \gamma, \gamma' \prec \gamma''$. The pullback on the projective limit function space will be denoted by $p^*_\gamma$. The functions cylindrical with respect to a graph that are $N$ times differentiable with respect to the standard differentiable structure on $\mathrm{SO}(D+1)$ will be denoted by $\mathrm{Cyl}^N_\gamma(\overline{\mathscr{A}})$ and $\mathrm{Cyl}^N(\overline{\mathscr{A}}) := \cup_\gamma \mathrm{Cyl}^N_\gamma(\overline{\mathscr{A}})/\sim$.

The action of gauge transformations $g$ and piecewise analytic diffeomorphisms $\phi$ on a cylindrical function $f = p^*_l f_l$ are given by

$$\delta_g(f) := p^*_l f_l(\{g(b(e))h_e(A)g(f(e))^{-1}\}_{e\in E(\gamma)}), \tag{10.4}$$

$$\delta_\phi(f) := p^*_l f_l(\{h_{\phi(e)}(A)\}_{e\in E(\gamma)}). \tag{10.5}$$

Since in the end we are interested only in gauge invariant quantities, after solving the Gauß constraint (classically oder quantum mechanically) we have to consider the algebra of cylindrical functions on the space of distributional connections modulo gauge transformations $\mathrm{Cyl}(\overline{\mathscr{A}/\mathscr{G}})$. For representatives $f_\gamma$ of elements $f$ of this space, the complex-valued function $F_\gamma$ on $\mathrm{SO}(D+1)^{|E|}$ has to be such that $f_\gamma(A)$ is gauge invariant. We will slightly abuse notation and use the same notation for the new projectors $p_{\gamma'\gamma} : \mathscr{A}_{\gamma'}/\mathscr{G}_{\gamma'} \to \mathscr{A}_\gamma/\mathscr{G}_\gamma$. There is a unique [28, 29] choice of a diffeomorphism invariant, faithful measure $\mu_0$ on $\overline{\mathscr{A}/\mathscr{G}}$ which equips us with a kinematical, gauge invariant Hilbert space $\mathscr{H}^0 := L_2\left(\overline{\mathscr{A}/\mathscr{G}}, d\mu_0\right)$ appropriate for a representation in which $A$ is

[1]The graph $\gamma$ can be enlarged by e.g. adding or subdividing edges. See e.g. [62] for a precise definition of the partial order on tame subgroupoids defined by graphs.

diagonal. This measure is entirely characterised by its cylindrical projections defined by

$$\begin{aligned}\int_{\overline{\mathscr{A}/\mathscr{G}}} d\mu_0(A)f(A) &= \int_{\overline{\mathscr{A}/\mathscr{G}}} d\mu_{0,\gamma}(A)f_\gamma(A) \\ &= \int_{\mathrm{SO}(D+1)^{|E(\gamma)|}} \left[\prod_{e\in E(\gamma)} d\mu_H(h_e)\right] F_\gamma\left(h_1, ..., h_{|E|}\right), \qquad (10.6)\end{aligned}$$

where $\mu_H$ is the Haar probability measure on $\mathrm{SO}(D+1)$.

An orthonormal basis on $\mathscr{H}^0$ is given by spin-network states [214–216], which are defined as follows. Given a graph $\gamma$, label its edges $e \in E(\gamma)$ with non-trivial irreducible representations $\pi_{\Lambda_e}$ of $\mathrm{SO}(D+1)$, i.e. $\Lambda_e$ is the highest weight vector associated with $e$, and its vertices $v \in V(\gamma)$ with intertwiners $c_v$, i.e. matrices which contract all the matrices $\pi_{\Lambda_e}(h_e)$ for $e$ incident at $v$ in a gauge invariant way. A spin-network state is simply a $C^\infty$ cylindrical function on $\overline{\mathscr{A}/\mathscr{G}}$ constructed on the above defined so-called spin-net, $T_{\gamma,\vec{\Lambda},\vec{c}}[A] := \mathrm{tr}\left[\otimes_{i=1}^{|E|}\pi_{\Lambda_{e_i}}(h_{e_i}(A)) \cdot \otimes_{j=1}^{|V|} c_j\right]$, where $\vec{\Lambda} = (\Lambda_e)$, $\vec{c} = (c_v)$ have indices corresponding to the edges and vertices of $\gamma$ respectively.

## 10.2 (Electric) fluxes and flux vector fields

Since the momenta $\pi^{aIJ}$ are Lie algebra-valued vector densities of density weight one, $(*\pi)_{a_1...a_{D-1}} := \pi^{aIJ}\epsilon_{aa_1...a_{D-1}}\tau_{IJ}$ is a pseudo $(D-1)$-form and is naturally integrated over a $(D-1)$-dimensional face $S$. We therefore define the (electric) fluxes

$$\pi^n(S) := \int_S n_{IJ}(*\pi)^{IJ} = \int_S n_{IJ}\pi^{aIJ}\epsilon_{aa_1...a_{D-1}}dx^{a_1}\wedge\ldots\wedge dx^{a_{D-1}}, \qquad (10.7)$$

where $n = n^{IJ}\tau_{IJ}$ denotes a Lie algebra-valued scalar function of compact support. We again restrict to piecewise analytic surfaces $S$, to ensure finiteness of the number of isolated intersection points of $S$ with a piecewise analytic path. In order to compute Poisson brackets, we have to suitably regularise the holonomies and fluxes to objects smeared in $D$ spatial dimensions. A possible regularisation in any dimension is given in [62]. Removal of the regulator leads to the following action of the Hamiltonian vector fields $Y_n(S)$ corresponding to $\pi_n(S)$ on adapted representatives $f_{\gamma_S}$

$$Y^n_{\gamma_S}(S)\,[f_{\gamma_S}] = \sum_{e\in E(\gamma_S)} \epsilon(e,S)\ [n(b(e))\, h_e(A)]_{AB}\ \frac{\partial F_{\gamma_S}}{\partial h_e(A)_{AB}}\Big(h_{e_1}(A),...,h_{e_{|E(\gamma_S)|}(A)}\Big)$$
$$= \sum_{e\in E(\gamma_S)} \epsilon(e,S)\, n^{IJ}(e\cap S)\, R^e_{IJ} f_{\gamma_S}. \qquad (10.8)$$

$f_{\gamma_S}$ is an adapted representative of the cylindrical function $f \in Cyl^1(\overline{\mathscr{A}})$ in the sense that all intersection points of $S$ and $\gamma_S$ are beginning points $b(e)$ of edges $e \in E(\gamma_S)$ (this can always be achieved by suitably splitting and inverting edges). In the above equation, $\epsilon(e,S)$ is a type-indicator function, which is $+(-)1$ if the beginning segment of the edge $e$ lies above (below) the surface $S$ and zero otherwise. $R^e_{IJ}$ ($L^e_{IJ}$) is the right (left) invariant vector field on the copy of SO$(D+1)$ labelled by $e$,

$$(R_{IJ}f)\,(h) := \left(\frac{d}{dt}\right)_{t=0} f(e^{t\tau_{IJ}}h) \quad \text{and} \quad (L_{IJ}f)\,(h) := \left(\frac{d}{dt}\right)_{t=0} f(he^{t\tau_{IJ}}). \quad (10.9)$$

The algebra of right (left) invariant vector fields is given by

$$\left[R^e_{IJ}, R^{e'}_{KL}\right] = \frac{1}{2}\delta_{e,e'}\left(\eta_{JK}R^e_{IL} + \eta_{IL}R^e_{JK} - \eta_{IK}R^e_{JL} - \eta_{JL}R^e_{IK}\right),$$
$$\left[R^e_{IJ}, L^{e'}_{KL}\right] = 0, \qquad (10.10)$$

and analogously for $L^e_{IJ}$. We remark that, in order to calculate functional derivatives, we had to restrict $f$ to $\mathscr{A}$ in the beginning. The end result (10.8), however, can be extended to all of $\overline{\mathscr{A}}$. Following the standard treatment, these vector fields are generalised from adapted to non-adapted graphs and shown to yield a cylindrically consistent family of vector fields, thus they define a vector field $Y_n(S)$ on $\overline{\mathscr{A}}$. The $Y_n(S)$ are called flux vector fields.

On the Hilbert space defined in section 10.1, the elements of the classical holonomy-flux algebra become operators which act by

$$\hat{f}\cdot\psi := f\ \psi,$$
$$\hat{Y}_n(S)\cdot\psi := i\hbar\kappa\beta Y_n(S)[\psi], \qquad (10.11)$$

where the right hand side is the action of the vector field $Y_n(S)$ on the cylindrical function $\psi$. The appearance of $\beta$ is due to the fact that we defined the fluxes using $\pi$, whereas the momenta conjugate to the connection is given by $\overset{(\beta)}{\pi} = \frac{1}{\beta}\pi$. The momentum operators $\hat{Y}_n(S)$, with dense domain $Cyl^1$, can be shown to be essentially self-adjoint operators on $\mathscr{H}^0$ analogously to the $(3+1)$-dimensional case [25].

# 11

# Implementation and solution of the kinematical constraints

In this chapter, implementation of the kinematical constraints will be discussed. While the Gauß and spatial diffeomorphism constraint can be treated as in usual LQG, the simplicity constraint is new in the canonical theory and we will discuss it in much more detail than the afore mentioned ones.

As we have seen in chapter 5, the simplicity constraint already appeared in Plebański's constrained BF theory formulation of $D = 3$ general relativity and its higher dimensions generalisations [169]. We have seen in part II that it comes in two variants, the quadratic [1, 2] and the linear version [6]. Discrete versions of these constraint have quite a history in quantum gravity research, they appear in spin foam models [185, 188, 189], in group field theory [217–219] and also in the canonical lattice models [220, 221] as well as in the construction of phase spaces for simplicial geometries [222, 223].

The quadratic constraint used in the original Barrett Crane model [185] is anomalous and the strong imposition of it at the quantum level leads to a one-dimensional intertwiner space. It was shown that this is too restrictive and problems with the asymptotic behaviour of the vertex amplitude were traced back to this fact in [224, 225]. This led to an intense study of the quantum simplicity constraints and the development of the new spin foam models [186–191], in which the quadratic constraints are replaced by the linear version. The linear constraint is still anomalous in general[1], but in the new spin foam models, at least parts of the constraints are imposed weakly which allows for intertwiner spaces mimicking the canonical theory. While the new models pass the tests which led to changing the original Barrett Crane model [226–228], the correct implementation of the simplicity constraints is still a highly debated issue also in the spin foam community (cf. [229] for recent criticism on the implementation in the new models), and new proposals for its correct implementation continue to appear (e.g. [230–232]).

Recently, the quadratic simplicity constraint has also been found to be anomalous in the canonical theory (cf. [3, 213, 222, 223]). To deal with this anomaly, in [3, 5] we of course were inspired by but did not closely follow the proposals made so far in the spin foam literature, the main reason being that many of them make use of special properties of SO(4) (in the Euclidean theory) which simply are not shared by higher rotation groups, or use procedures which are incompatible with the in the canonical picture mandatory cylindrical consistency.

Tentative requirements we could impose on the implementation of both, the linear and the quadratic constraints, apart from mathematical consistency are the following: First of all, to avoid overconstraining the system and erroneous removal of physical degrees of freedom, we would like the constraint operators to be non-anomalous. Secondly, in $D = 3$, we are in the very convenient situation of having two quantisations of the same theories at our disposal, namely the one SU(2) gauge theory obtained when using the Ashtekar Barbero variables, and the SO(4) theory when using the variables introduced here. Classically, both theories reduce to the ADM formulation if we solve the SU(2) Gauß constraint or the SO(4) Gauß and simplicity constraints, respectively. It would be desirable to have a quantum analogon of this classical equivalence, i.e. there should exists a natural unitary map from the joint kernel of the SO(4) Gauß and simplicity constraint to the kernel of the SU(2) Gauß constraint, spanned by gauge invariant SU(2) spin network states[2]. Our considerations will be lead by these two

[1] The anomaly is only absent for the cases $\gamma = \pm\sqrt{\zeta}$, where $\gamma$ denotes the Barbero Immirzi parameter and $\zeta$ again is the signature of the internal space, or $\gamma = \infty$.

[2] Actually, when using the linear simplicity constraint and introducing the Barbero Immirzi parameter in the SO(4) theory like in section 9.3, the classical equivalence is even stronger, since already

requirements. Note that the latter actually is no necessary criterion, both theories only have to have as classical limit general relativity to be considered genuine quantisations thereof. However, conjecturing that the emerging quantum theory should be reasonably unique, this requirement is a tentative guideline of how to implement the simplicity constraints for $D = 3$. If successful, this implementation then can be mimicked when generalising to higher dimensions.

Out presentation will be as follows: in section 11.1, we will very briefly review the implementation of the Gauß constraint. The simplicity constraint will be studied in section 11.2. The quadratic constraint will be studied in detail, while the findings concerning the linear constraint of [5] will only be sketched.

Starting with the quadratic constraint in 11.2.1.1, we will show how it can be represented as constraint operator on the kinematical Hilbert space. It can be easily solved on edges and leads to the well-known simple representations of SO$(D+1)$ [169], which allow for a natural mapping to SU(2) representations (section 11.2.1.2). However, due to the singular smearing of the fluxes, it is “anomalous” when acting on vertices, like in spin foams, and the unique solution is the Barrett Crane intertwiner (or its higher dimensional analogon [169]). We will study the anomaly in detail in section 11.2.1.3, and introduce necessary and sufficient “building blocks” of the quadratic simplicity constraint, which are easier to handle. Based on these building blocks, we will comment on possible remedies, namely a master constraint treatment (section 11.2.1.4) or the imposition of a maximally commuting subset of vertex simplicity constraints corresponding to a recoupling scheme (section 11.2.1.5). While the second option does not come without problems which have to be further studied (e.g. one needs to choose a recoupling scheme for each vertex), it has the advantage that it leads to a natural unitary map to the usual SU(2) based kinematical Hilbert space of LQG.

In section 11.2.2 we will turn to the linear constraint. After constructing a kinematical Hilbert space for the additional field $N^I$ in section 11.2.2.1, we will show how the linear simplicity constraint, being linear in the fluxes, can be quantised in analogy to

when solving the simplicity constraint and the boost part $N_I \mathscr{G}^{IJ}$ of the SO(4) Gauß constraint, the theory reduces to the Ashtekar Barbero formulation.

the fluxes themselves and actually has a closing quantum algebra (section 11.2.2.2). Note that this is different from the linear constraint in spin foams and related with the missing "mixing" of rotational and boost components, which is caused in usual Ashtekar Barbero variables due to the presence of the Barbero Immirzi parameter[1]. However, we will find that the unique solution to these non-anomalous constraints is a certain, $N^I$-dependent intertwiner, and inserting this intertwiner at all points of a given spin network is in conflict with cylindrical consistency. While there might be a chance to make this infinite placing of this $N^I$-dependent intertwiners cylindrically consistent using a rigging map, in [5] we did not succeed in its construction. From this perspective, the quadratic constraints seem to be favoured. But as we will see in section 15.1, introduction of the additional field $N^I$ becomes necessary when dealing with supergravity. Therefore, we will briefly sketch the proposal of the mixed quantisation (section 11.2.2.3), where the linear constraint is replaced by the quadratic constraint while $N^I$ is kept as phase space degree of freedom, at the cost of an additional constraint demanding the equality of the unit vectors $N^I$ and $n^I(\pi)$. This new set of constraints does not share the problems with cylindrical consistency we encountered when solving the linear constraints. However, the solution to the additional constraint is unknown.

While several other ideas of how to possibly deal with this issue were discussed in [5], we will stick to the ones outlined, as they give a mathematically consistent proposal for both, the theory with and without the extra field $N^I$. We refer the interested reader to the original literature for further information. In any case, we do not claim to give a "final answer" to the simplicity constraint problem and further research has to be conducted to derive an entirely satisfactory treatment of these constraints.

Finally, in section 11.3, we sketch the implementation of the diffeomorphism constraint already known from $D = 3$.

[1]Therefore, if using the new variables but additionally introducing the Barbero Immirzi parameter in $D = 3$ like described in section 9.3, it is easy to show that linear constraint becomes anomalous.

## 11.1 Gauß constraint

Working with the gauge invariant Hilbert space from the beginning, the Gauß constraint is already solved. Yet we want to summarise its implementation on the gauge variant Hilbert space $\mathscr{H} = L_2\left(\overline{\mathscr{A}}, d\mu_0'\right)$, since we want to compute quantum commutators of the constraint with the simplicity constraint in the next section. The implementation (as well as the solution) of the Gauß constraint can be copied from the $(3+1)$-dimensional case without modification.

According to the RAQ programme, we choose the dense subspace $\Phi = Cyl^\infty(\overline{\mathscr{A}})$ in the Hilbert space. Then, we are looking for an algebraic distribution $L \in \Phi'$ such that the following equation holds

$$L\left(p_\gamma^* \left[\sum_{e\in E(\gamma);\ v=b(e)} R^e_{IJ} - \sum_{e\in E(\gamma);\ v=f(e)} L^e_{IJ}\right] f_\gamma\right) = 0 \tag{11.1}$$

for any $v \in V(\gamma)$, any graph $\gamma$ and $f_\gamma \in Cyl^\infty_\gamma(\overline{\mathscr{A}})$. The general solution for $L$ is given by a linear combination of $\langle\psi, .\rangle$, where $\psi \in \mathscr{H}^0$ is gauge invariant. Thus, for an adapted graph $\gamma'$ (all edges outgoing from the vertex $v$ in question), gauge invariance amounts to vanishing sum of all right invariant vector fields at a vertex,

$$\sum_{e\in E(\gamma');\ v=b(e)} R^e_{IJ} f_{\gamma'} = 0. \tag{11.2}$$

## 11.2 Simplicity constraints

### 11.2.1 Quadratic simplicity constraints

#### 11.2.1.1 From classical to quantum

Classically, vanishing of the simplicity constraints $\mathscr{S}^{ab}_{\overline{M}}(x) = \frac{1}{4}\epsilon_{IJKL\overline{M}}\pi^{aIJ}(x)\pi^{bKL}(x)$ at all points $x \in \sigma$ is completely equivalent to the vanishing of

$$C_{\overline{M}}(S^x, S'^x) := \lim_{\epsilon,\epsilon'\to 0} \frac{1}{\epsilon^{(D-1)}\epsilon'^{(D-1)}} \epsilon_{IJKL\overline{M}}\pi^{IJ}(S^x_\epsilon)\pi^{KL}(S'^x_{\epsilon'}) \tag{11.3}$$

for all points $x \in \sigma$ and all surfaces $S^x_\epsilon, S'^x_{\epsilon'} \subset \sigma$ containing $x$ and shrinking to $x$ as $\epsilon$, $\epsilon'$ tend to zero. More precisely, we use faces of the form $S^x : (-1/2, 1/2)^{D-1} \rightarrow$

$\sigma$; $(u_1, ..., u_{D-1}) \mapsto S^x(u_1, ..., u_{D-1})$ with semi-analytic but at least once differentiable functions $S^x(u_1, ..., u_{D-1})$ and $S^x(0, ..., 0) = x$, and define $S^x_\epsilon(u_1, ..., u_{D-1}) := S^x(\epsilon u_1, ..., \epsilon u_{D-1})$. We find that (10.7) becomes (with the choice $n_{IJ} = \delta^K_{[I}\delta^L_{J]}$)

$$\begin{aligned} \frac{1}{\epsilon^{(D-1)}}\pi^{IJ}(S^x_\epsilon) = & \frac{1}{\epsilon^{(D-1)}} \int_{(-\epsilon/2,\epsilon/2)^{D-1}} du_1 ... du_{D-1} \epsilon_{aa_1...a_{D-1}} (\partial S^{xa_1}/\partial u_1)(u_1, ..., u_{D-1}) \\ & \times (\partial S^{xa_{D-1}}/\partial u_{D-1})(u_1, ..., u_{D-1}) \;\; \pi^{aIJ}(S^x(u_1, ..., u_{D-1})) \\ = & \, n_a(S)\pi^{aIJ}(x) + O(\epsilon) \end{aligned} \tag{11.4}$$

with $n_a(S) = \epsilon_{aa_1...a_{D-1}}(\partial S^{xa_1}/\partial u_1)(0, ..., 0) \times ... \times (\partial S^{xa_{D-1}}/\partial u_{D-1})(0, ..., 0)$, from which the claim follows. Now, similar to the treatment of the area operator in section 12.1, we just plug in the known quantisation of the electric fluxes and hope to get a well-defined constraint operator in the end. Using the regularised action of the flux vector fields on cylindrical functions (10.8), we find for a representative $f_{\gamma_{SS'}}$ of $f \in \text{Cyl}^2(\overline{\mathscr{A}})$ on a graph $\gamma_{SS'}$ adapted to both $S^x$ and $S'^x$,

$$\begin{aligned} \hat{C}_{\overline{M}}(S^x, S'^x)_{\gamma_{SS'}} \left[f_{\gamma_{SS'}}\right] := & \lim_{\epsilon,\epsilon'\to 0} \frac{1}{\epsilon^{(D-1)}\epsilon'^{(D-1)}} \epsilon_{IJKL\overline{M}} \hat{Y}^{IJ}_{\gamma_{SS'}}(S^x_\epsilon) \hat{Y}^{KL}_{\gamma_{SS'}}(S'^x_{\epsilon'})[f_{\gamma_{SS'}}] \\ = & \lim_{\epsilon,\epsilon'\to 0} \frac{1}{\epsilon^{(D-1)}\epsilon'^{(D-1)}} \epsilon_{IJKL\overline{M}} \sum_{e\in E(\gamma_{SS'});b(e)=x} \; \sum_{e'\in E(\gamma_{SS'});b(e')=x} \\ & \epsilon(e, S^x)\epsilon(e', S'^x) R^{IJ}_e R^{KL}_{e'} f_{\gamma_{SS'}} \\ =: & \lim_{\epsilon,\epsilon'\to 0} \frac{1}{\epsilon^{(D-1)}\epsilon'^{(D-1)}} \hat{\tilde{C}}_{\overline{M}}(S^x, S'^x)_{\gamma_{SS'}}[f_{\gamma_{SS'}}]. \end{aligned} \tag{11.5}$$

The flux vector fields only act locally on the intersection points $e \cap S$, $e \in E(\gamma_{SS'})$. Therefore, in the second line we used that for small surfaces $S^x_\epsilon$, $S'^x_{\epsilon'}$, the action of the constraint will be trivial expect for $x$ (and of course only non-trivial if $x$ is in the range of $\gamma_{SS'}$), thus independent of $\epsilon$. In the limit $\epsilon, \epsilon' \to 0$ the expression in the last line of the above calculation clearly diverges except for $\hat{\tilde{C}}f = 0$, where the whole expression vanishes identically. Since the kernels of the constraint operators $\hat{C}$ and $\hat{\tilde{C}}$ coincide, we can work with the latter and propose the constraint (omitting the $\sim$ again)

$$\begin{aligned} \hat{C}^{\overline{M}}(S, S', x)_\gamma p^*_\gamma f_\gamma = & \, p^*_{\gamma_{SS'}} \epsilon^{IJKL\overline{M}} \sum_{e,e'\in\{e''\in E(\gamma_{SS'}),b(e'')=x\}} \epsilon(e, S^v)\epsilon(e', S'^v) R^e_{IJ} R^{e'}_{KL} p^*_{\gamma_{SS'}\gamma} f_\gamma \\ = & \, p^*_{\gamma_{SS'}} \epsilon^{IJKL\overline{M}} \left(R^{up}_{IJ} - R^{down}_{IJ}\right)\left(R^{up'}_{KL} - R^{down'}_{KL}\right) p^*_{\gamma_{SS'}\gamma} f_\gamma, \end{aligned} \tag{11.6}$$

where $R_{IJ}^{up(')} := \sum_{e \in E(\gamma_{SS'}), b(e)=x, \epsilon(e,S('))=1} R_{IJ}^e$ and similar for $R_{IJ}^{down(')}$. In the following, will drop the superscript $^x$ for the surfaces for simplicity.

The proof that the family $\hat{C}_\gamma^{\overline{M}}(S, S', x)$ is consistent and defines a vector field $\hat{C}^{\overline{M}}(S, S', x)$ on $\overline{\mathscr{A}}$ follows from the consistency of $\hat{Y}_n(S)$. To see that the operator is essentially self-adjoint, let $\mathscr{H}^0_{\gamma,\vec{\pi}}$ be the finite-dimensional Hilbert subspace of $\mathscr{H}^0$ given by the closed linear span of spin network functions over $\gamma$ where all edges are labelled with the same irreducible representations given by $\vec{\pi}$, $\mathscr{H}^0 = \overline{\oplus_{\gamma,\vec{\pi}} \mathscr{H}^0_{\gamma,\vec{\pi}}}$. Given any surfaces $S$, $S'$ we can restrict the sum over graphs to adapted ones since we have $\mathscr{H}^0_{\gamma,\vec{\pi}} \subset \mathscr{H}^0_{\gamma_{SS'},\vec{\pi}'}$ for the choice $\pi'_{e'} = \pi_e$ with $E(\gamma_{SS'}) \ni e' \subset e \in E(\gamma)$. Since $\hat{C}^{\overline{M}}(S, S', x)$ preserves each $\mathscr{H}^0_{\gamma,\vec{\pi}}$, its restriction is a symmetric operator on a finite-dimensional Hilbert space, therefore self-adjoint. To see that it is symmetric, note that the right hand side of the first line of (11.6) consists of right-invariant vector fields which commute. This is obvious for the summands with vector fields acting on distinct edges $e \neq e'$, and for $e = e'$ note that $[R^e_{IJ}, R^e_{KL}]$ is antisymmetric in $(IJ) \leftrightarrow (KL)$ and thus vanishes if contracted with $\epsilon^{IJKL\overline{M}}$. Now it is straightforward to see that $\hat{C}^{\overline{M}}(S, S', x)$ itself is essentially self-adjoint.

Note that we did not follow the standard route to quantise operators, which would be to adjust the density weight of the simplicity constraint to be +1 (in its current form it is +2) and quantise it using the methods in [46]. Rather, the quantisation displayed above parallels the quantisation of the (square of the) area operator in 3+1 dimensions and indeed we could have considered $\int d^{D-1}u \sqrt{|n_a^S n_b^S \mathscr{S}^{ab}_{\overline{M}}|}$ for arbitrary surfaces $S$ and would have arrived at the above expression in the limit that $S$ shrinks to a point without having to take away the regulator $\epsilon$ (the dependence on two rather than one surface can be achieved, to some extent, by an appeal to the polarisation identity). If we would have quantised it using the standard route then it would be necessary to have access to the volume operator. We will see in section 12.2 that for the derivation of the volume operator in certain dimensions in the form we propose, which is a generalisation of the $3+1$ dimensional treatment, we need the above simplicity constraint operator to cancel some unwanted terms. Of course, there might be other proposals for volume operators which can be defined in any dimension without using the simplicity constraint. Still, the quantisation of the simplicity constraint presented here will (1) give contact to the simplicity constraints used in spin foam models and (2) enable us to solve the constraint in any dimension when acting on edges. Its action on the vertices, i.e. the requirements on the intertwiners, is more subtle. We will first present the action on edges and afterwards derive a suitable set of necessary and sufficient "building blocks" for the vertex simplicity constraints, which will help us to prove its anomalous nature and to propose possible routes of how to proceed, namely the master constraint method or the choice of a maximally commuting subset. For following calculations, note that we always can adapt a graph to a finite number of surfaces. Furthermore, it is understood that all surfaces intersect $\gamma'$ in one point only (we may always shrink the surfaces until this is true).

#### 11.2.1.2 Edge constraints and their solution

The action of the quantum simplicity constraint at an interior point $x$ of an analytic edge $e = e_1 \circ (e_2)^{-1}$ for both surfaces $S$, $S'$ not containing $e$ (otherwise the action is trivial) is given by

$$\begin{aligned}
\hat{C}^{\overline{M}}(S,S',x)p^*_\gamma f_\gamma &= \pm p^*_{\gamma_{SS'}} \epsilon^{IJKL\overline{M}} \left(R^{e_1}_{IJ} - R^{e_2}_{IJ}\right)\left(R^{e_1}_{KL} - R^{e_2}_{KL}\right) p^*_{\gamma_{SS'}\gamma} f_\gamma \\
&= \pm p^*_{\gamma_{SS'}} 2\epsilon^{IJKL\overline{M}} \left(R^{e_1}_{IJ} - R^{e_2}_{IJ}\right) R^{e_1}_{KL} p^*_{\gamma_{SS'}\gamma} f_\gamma \\
&= \pm p^*_{\gamma_{SS'}} 2\epsilon^{IJKL\overline{M}} R^{e_1}_{KL} \left(R^{e_1}_{IJ} - R^{e_2}_{IJ}\right) p^*_{\gamma_{SS'}\gamma} f_\gamma \\
&= \pm p^*_{\gamma_{SS'}} 4\epsilon^{IJKL\overline{M}} R^{e_1}_{IJ} R^{e_1}_{KL} p^*_{\gamma_{SS'}\gamma} f_\gamma,
\end{aligned} \tag{11.7}$$

where the sign is + if the orientation of the two surface $S$, $S'$ with respect to $e$ coincides and − otherwise. In the second and fourth step we used gauge invariance at the vertex $v$ of an adapted graph, $\left[\sum_{e\in E(\gamma);\, v=b(e)} R^e_{IJ}\right] f_{\gamma_{SS'}} = 0$, and in the third step we used that $[R^{e_1}, R^{e_2}] = 0$. This leads to the requirement on the generators of $\mathrm{SO}(D+1)$ for all edges

$$\tau_{[IJ}\tau_{KL]} = 0. \tag{11.8}$$

It was found in [169] that this constraint is satisfied by so-called simple representations of $\mathrm{SO}(D+1)$. These representations have been studied in the mathematical literature in quite some detail, where they are called most degenerate representations [233–235], (completely) symmetric representations [234, 236–238] or representations of class one (with respect to a $\mathrm{SO}(D)$ subgroup) [239]. Irreducible simple representations are given by homogeneous harmonic polynomials $\mathscr{H}_N^{(D+1)}$ of degree $N$. While the highest weight vector of irreps of SO$(D+1)$ usually are of the form $\Lambda = (n_1, ..., n_n)$, $n_i \in \mathbb{N}_0$ and $n = \lfloor \frac{D+1}{2} \rfloor$, simple irreps are in any dimension labelled by one positive integer $N$, $\Lambda = (N, 0, ..., 0)$. In this sense, there is a similarity between the simple representations of SO$(D+1)$ and the representations of SO(3) (which all can be thought of as being simple). In particular, for $D+1 = 4$ we obtain the well-known simple representations of SO(4) used in spin foams labelled by $j^+ = j = j^-$.

The commutator with gauge transformations at an interior point $x$ of an analytic edge $e = e_1 \circ (e_2)^{-1}$ ($e_1$, $e_2$ outgoing at $x$) yields, analogously to the classical calculation,

$$\begin{aligned}
&\left[\hat{G}_{\gamma_{SS'}}[\Lambda], \hat{C}^{\overline{M}}(S, S', x)_{\gamma_{SS'}}\right] \\
&= \pm\, \Lambda^{AB}(x)\epsilon^{IJKL\overline{M}} \left[\left(R^{e_1}_{AB} + R^{e_2}_{AB}\right), \left(R^{e_1}_{IJ} - R^{e_2}_{IJ}\right)\left(R^{e_1}_{KL} - R^{e_2}_{KL}\right)\right] \\
&= \pm \left\{\Lambda^{AB}(x)\epsilon^{IJKL\overline{M}} \left[R^{e_1}_{AB}, R^{e_1}_{IJ}R^{e_1}_{KL} - 2R^{e_1}_{IJ}R^{e_2}_{KL}\right] + (e_1 \leftrightarrow e_2)\right\} \\
&= \pm \sum_{i=1}^{D-3} \Lambda^{M_i}{}_{M_i'}(x)\epsilon^{IJKLM_1...M_{i-1}M_i'M_{i+1}...M_{D-3}} \left(R^{e_1}_{IJ}R^{e_1}_{KL} - 2R^{e_1}_{IJ}R^{e_2}_{KL} + R^{e_2}_{IJ}R^{e_2}_{KL}\right) \\
&= \sum_{i=1}^{D-3} \Lambda^{M_i}{}_{M_i'}(x)\, \hat{C}^{M_1...M_{i-1}M_i'M_{i+1}...M_{D-3}}(S, S', x).
\end{aligned} \tag{11.9}$$

Two constraints acting at the same interior point $x$ of an edge $e = e_1 \circ (e_2)^{-1}$ commute weakly. Using the gauge invariance of $Cf$ if $f$ is gauge invariant, we find

$$\begin{aligned}
&\left[\hat{C}^{\overline{M}}(S, S', x), \hat{C}^{\overline{N}}(S'', S''', x')\right] p^*_\gamma f_\gamma \\
&\approx \pm\, 16 p^*_\gamma \delta_{x,x'} \epsilon^{IJKL\overline{M}} \epsilon^{OPQR\overline{N}} \left[R^{e_1}_{IJ}R^{e_1}_{KL}, R^{e_1}_{OP}R^{e_1}_{QR}\right] f_\gamma + \mathscr{O}(\hat{C}f_\gamma) + \mathscr{O}(\hat{G}f_\gamma) \\
&\sim p^*_\gamma \delta_{x,x'} \left(\epsilon R^{e_1} \cdot \hat{C}^{e_1,rot} + \hat{C}^{e_1,rot} \cdot \epsilon R^{e_1}\right) f_\gamma \\
&\sim p^*_\gamma \delta_{x,x'} \left(\epsilon R^{e_1} \cdot \hat{C}^{e_1,rot} + [\hat{C}^{e_1,rot}, \epsilon R^{e_1}] + \epsilon R^{e_1} \cdot \hat{C}^{e_1,rot}\right) f_\gamma \\
&\sim p^*_\gamma \delta_{x,x'} \left(2\epsilon R^{e_1} \cdot \hat{C}^{e_1,rot} + \epsilon \cdot \hat{C}^{e_1,rot,rot}\right) f_\gamma \approx 0,
\end{aligned} \tag{11.10}$$

which can be seen by the fact that the simplicity on an edge is quadratic in the rotation generator $R^{e_1}$ on that edge, and we used the notation

$$\sum_{i=1}^{D-3} \Lambda^{M_i}{}_{M_i'}\, \epsilon^{ABCDM_1...M_{i-1}M_i'M_{i+1}...M_{D-3}} R^e_{AB} R^e_{CD} =: \Lambda \cdot \hat{C}^{e,rot} \tag{11.11}$$

for a simplicity with a infinitesimal rotation acting on the multi-index $\overline{M}$ (cf. (11.9)). Here, we chose a graph $\gamma$ adapted to all four surfaces $S$, $S'$, $S''$, $S'''$. Note that classically, before introducing singular smearing, the Poisson bracket of two simplicity constraints vanished strongly. In the quantum theory, we see that this is only true in a weak sense. However, this already is the case at the level of the classical holonomies and fluxes, i.e. can be traced back to the singular smearing which is used. In this sense, the simplicity constraints acting on an edge are non-anomalous and can be solved by labelling all edges by simple representations of $\mathrm{SO}(D+1)$.

#### 11.2.1.3 Vertex simplicity constraints: Anomaly

When acting on a node then, like the off-diagonal constraints in spin foam models, we will find that the simplicity constraints do not (weakly) commute anymore. To analyse the anomaly in detail, here we will first introduce a both necessary and sufficient set of simple "building blocks" of the simplicity constraint at the node, and then calculate the commutator of these building blocks. Having them at hand will also be convenient later on when giving tentative proposals of how to treat these vertex constraints which, due to their second class nature, should not be imposed strongly anymore.

Considering (11.6), an obviously sufficient set of building blocks at the vertex $v$ is given by

$$R^e_{[IJ}R^{e'}_{KL]}f_\gamma = 0 \quad \forall e, e' \in \{e'' \in E(\gamma); v = b(e'')\}. \tag{11.12}$$

Note that they exactly coincide with the off-diagonal simplicity constraints which appear in spin foam models, see e.g. [169, 190]. For necessity, we have to prove that we can choose surfaces in such a way that these building blocks follow. Note that it has already been shown in [240] that all right invariant vector fields $R^e$ for single edges $e$ can be generated by the $Y(S)$, but the construction involves commutators of the fluxes. Since we want to explore if the simplicity constraints acting on vertices are anomalous, we cannot use commutators in our argument. Instead, we will construct the right invariant vector fields $R^e$ by using linear combinations of fluxes only. To this end, we will prove the following lemma:

**Lemma 1.**
*For each edge $e \in E(v)$ at the vertex $v$ we can always choose two surfaces $S$, $\tilde{S}$, such that the orientations with respect to $S$, $\tilde{S}$ of all edges but $e$ coincide.*

The intuitive idea of how to find these surfaces is to start with a surface containing the edge $e$ while intersecting all other edges $e' \in E(v), e' \neq e$ transversally, and then slightly distort this surface in the two directions "above" and "below" defined by the surface, such that the edge $e$ in consideration is once above and once below the surface, while the orientations of all other edges with respect to the surfaces remain unchanged, in particular none of them lies inside the surfaces. When subtracting the flux vector fields corresponding to the two distorted surfaces, all terms will cancel except the terms involving $R^e$.

*Proof.* To prove the statement above, two cases have to be distinguished: (a) the case where no $e' \in E(v)$ is (a segment of) the analytic extension through $v$ of the edge $e$ and (b) the case where $e$ has a partner $\tilde{e}$ which is a analytic extension of $e$ through $v$.
*Case (a):* The construction of the surface $S_{v,e}$ with the following properties

1. $s_e \subset S_{v,e}$ for some beginning segment $s_e$ of $e$, and the other edges $e' \in E(v), e' \neq e$ intersect $S_{v,e}$ transversally in $v$.

2. For $e' \in E(v), e' \neq e$: $e' \cap S_{v,e} = v$, and for $e' \notin E(v)$, $e' \cap S_{v,e} = \emptyset$.

is given in [240] and we summarise the result shortly. An analytic surface (edge) is completely determined by its germ $[S]_v$ ($[e]_v$)

$$S(u_1, ..., u_{D-1}) = \sum_{m_1,...,m_{D-1}=0}^{\infty} \frac{u_1^{m_1}...u_{D-1}^{m_{D-1}}}{m_1!...m_{D-1}!} S^{(m_1,...,m_{D-1})}(0, ..., 0),$$
$$e(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} e^{(n)}(0). \tag{11.13}$$

To ensure that $s_e \subset S_{v,e}$, we just need to choose a parametrisation of $S$ such that $S(t, 0, ..., 0) = e(t)$ which fixes the Taylor coefficients $S^{(m,0,...,0)}(0, ..., 0) = e^{(m)}(0)$. For the finite number $k = |E(v)| - 1$ of remaining edges at $v$, we can now use the freedom in choosing the other Taylor coefficients to assure that there are no (beginning segments of) other edges contained in $S_{v,e}$ [240]. In particular, only a finite number of Taylor coefficients is involved.

Now we state that the intersection properties of a finite number of transversal edges at $v$ with any (sufficiently small) surface $S$ are already fixed by a finite number of Taylor coefficients of $S$. We will discuss the case $D = 3$ for simplicity, higher dimensions are treated analogously. Locally around $v$ we may always choose coordinates such that the surface is given by $z = 0$, $S(x, y) = (x, y, 0)$. The edge $e$ contained in the surface is given by $e(t) = (x(t), y(t), 0)$ and for any transversal edge at $v$ we find $e'(t) = (x'(t), y'(t), z'(t))$ where $z'(t) = \frac{t^{n-1}}{(n-1)!} z'^{(n-1)}(0) + \mathscr{O}(t^n)$, and $n < \infty$ since otherwise $e'$ would be contained in $S$. The sign of the lowest non-vanishing Taylor coefficient $z'^{(n-1)}(0)$ determines if the edge is "up"- or "down"-type locally. Set $N = \max_{e' \in E(v), e' \neq e}(n)$, and obviously $N < \infty$. Thus, we can e.g. by modifying $S^{(N,0)}(0,0)$ choose the surface $\tilde{S}(x, y) = (x, y, \pm x^N)$, which locally has the same intersection properties with the edges $e' \in E(v), e' \neq e$ and certainly does not contain $e$ anymore.

Coming back to the general case considered before, there always exists $N < \infty$ such that we can change $S^{(N,0,...,0)}(0, ..., 0)$ without modifying the intersection properties of any of the edges $e' \in E(v), e' \neq e$, in particular the "up"- or "down"-type properties are unaffected. However, the edge $e$ no longer is of the inside type, but becomes either "up" or "down" (depending on whether $S^{(N,0,...,0)}(0, ..., 0)$ is scaled up or down and on the orientation of $S$). In general, new intersection points $v' \in E(v) \cap S, v' \neq v$ may occur when modifying the surface in the above described way, but we may always make $S$ smaller to avoid them.

Now choose a pair of surfaces $S$, $\tilde{S}$ for the edge $e$ such that it is once "up"- and once "down"-type to obtain the desired result

$$\left[\hat{Y}_{IJ}(S) - \hat{Y}_{IJ}(\tilde{S})\right] p_\gamma^* f_\gamma = 2 p_\gamma^* R_{IJ}^e f_\gamma. \tag{11.14}$$

*Case (b):* In the case that there is a partner $\tilde{e}$ which is a analytic continuation of $e$ through $v$, we cannot construct an analytic surface (without boundary) $S_{v,e}$ containing a beginning segment of $e$ and not containing a segment of $\tilde{e}$. However, we can construct an analytic surface $S_{v,\{e,\tilde{e}\}}$ containing (beginning segments of) $e$, $\tilde{e}$ and sharing the remaining properties with $S_{v,e}$ above. The method is the same as in case (a) [240]. Again, there always exists $N < \infty$ such that we can change $S^{(N,0,...,0)}(0, ..., 0)$ without modifying the intersection properties of any of the edges $e' \in E(v), e' \neq \{e, \tilde{e}\}$, and such that both edges $e$, $\tilde{e}$ become either "up" or "down"-type. Moreover, if we choose $N$ even, then $e$, $\tilde{e}$ will be of the same type with respect to the modified surface, while for $N$ odd one edge will be "up" and its partner will be "down". Calling the modified surface $S$ for $N$ even and $\tilde{S}$ for $N$ odd, we find with the same calculation (11.14) as in case (a) the desired result.

This furnishes the proof of the above lemma[1]. □

Choosing the surfaces as described above, we find that the following linear combination

$$\begin{aligned}&\frac{1}{4}\left(\hat{C}^{\overline{M}}(S,S',x)-\hat{C}^{\overline{M}}(\tilde{S},S',x)-\hat{C}^{\overline{M}}(S,\tilde{S}',x)+\hat{C}^{\overline{M}}(\tilde{S},\tilde{S}',x)\right)p_\gamma^* f_\gamma\\ &=p_\gamma^*\epsilon^{IJKL\overline{M}}R^e_{IJ}R^{e'}_{KL}f_\gamma \end{aligned} \tag{11.15}$$

proves the necessity of the building blocks. Using the fact that the edge representations are already simple, we can rewrite the building blocks as

$$\begin{aligned}R^e_{[IJ}R^{e'}_{KL]}f_\gamma &= \frac{1}{2}\left[(R^e_{[IJ}+R^{e'}_{[IJ})(R^e_{KL]}+R^{e'}_{KL]})-R^e_{[IJ}R^e_{KL]}-R^{e'}_{[IJ}R^{e'}_{KL]}\right]f_\gamma\\ &=\frac{1}{2}(R^e_{[IJ}+R^{e'}_{[IJ})(R^e_{KL]}+R^{e'}_{KL]})f_\gamma =: \frac{1}{2}\Delta^{ee'}_{IJKL}f_\gamma.\end{aligned} \tag{11.16}$$

We proceed by showing that the building blocks are anomalous, starting with the case $D=3$. We calculate for $e\neq e'\neq e''\neq e$

$$\left[\epsilon^{IJKL}\Delta^{ee'}_{IJKL},\epsilon^{ABCD}\Delta^{e'e''}_{ABCD}\right]\sim\delta^{ABC}_{IJK}(R_{e''})_{AB}(R_e)^{IJ}(R_{e'})^K{}_C, \tag{11.17}$$

where we used the notation $\delta^{I_1...I_n}_{J_1...J_n}:=n!\ \delta^{I_1}_{[J_1}\delta^{I_2}_{J_2}...\delta^{I_n}_{J_n]}$. To show that this expression can not be rewritten as a linear combination of the of building blocks (11.16), we antisymmetrise the indices $[ABIJ]$, $[ABKC]$ and $[IJKC]$ and find in each case that the result is zero. Therefore, a simplicity building block can not be contained in any linear combination of terms of the type (11.17). For $D>3$, we have

$$\left[\epsilon^{IJKL\overline{M}}\Delta^{ee'}_{IJKL},\epsilon^{ABCD\overline{E}}\Delta^{e'e''}_{ABCD}\right]\sim\delta^{ABC\overline{E}}_{IJK\overline{M}}(R_{e''})_{AB}(R_e)^{IJ}(R_{e'})^K{}_C. \tag{11.18}$$

Choosing $\overline{M}=\overline{E}$ fixed, the anomaly is the same as above.

A short remark concerning the terminology "anomaly" here and in the title of this section is in order at this place. Normally, the term anomaly denotes that a certain classical structure, e.g. the constraint algebra, is not preserved at the quantum

[1]This also establishes that the right invariant vector fields $R^e_{IJ}$ are not only contained in the Lie algebra generated by the flux vector fields $\hat{Y}(S)$, but are already contained in the flux vector space, which to the best of our knowledge has not been shown.

level, e.g. by factor ordering ambiguities. The non-commutativity of the simplicity constraints, however, is a classical effect, since it already arises when introducing holonomies and fluxes as basic variables. Thus, one could argue that it would be more precise to talk of a quantisation of (partly) second class constraints. On the other hand, since the holonomy-flux algebra is an integral part of the quantum theory and at the classical level it would be perfectly fine to use a non-singular smearing and thus first class simplicity constraints, we will nevertheless use the term anomaly to describe this phenomenon, since its consequence, the erroneous removal of degrees of freedom at the quantum level, is the same.

Independently of the terminology chosen, we cannot quantise the simplicity constraints acting on vertices using the Dirac procedure since this will lead to the additional constraints (11.18) being imposed. The unique solution to these constraints has been worked out in [169] and is given by the Barrett-Crane intertwiner in four dimensions and a higher-dimensional analogue thereof. Several options of how to proceed are at our disposal at this point. We will first discuss the introduction a vertex master constraint as given in [3], and then the choice of a maximally commuting subset of vertex simplicity constraints as introduced in [5].

#### 11.2.1.4 Quadratic vertex simplicity master constraint

While equivalent at the classical level, the master constraint introduced in [60] allows to quantise also second class constraints by a strong operator equation. Due to the second class nature, one expects the master constraint operator to have an empty kernel or at least a kernel which is too small to describe the physical Hilbert space. Since we know that the Barrett-Crane intertwiner is a solution to the strong imposition of all vertex simplicity constraints, we are in the second case. In order to find a larger kernel of the master constraint, one modifies it by adding terms to it which vanish in the classical limit, i.e. performs $\hbar$-corrections. The merits of this procedure are exemplified by the construction of the EPRL intertwiner [190] in four dimensions, which results from a master constraint for the linear simplicity constraint upon $\hbar$-corrections. Since we are not aware of a suitable solution for the quadratic vertex master simplicity constraint, we will contend ourselves by giving a definition of this constraint operator. The task remaining for solving the vertex simplicity master constraint operator is thus to find a proper $\hbar$-correction which results in a physical Hilbert space with the desired properties, e.g. that there exists a unitary map to SU(2) spin networks in four dimensions.

A general simplicity master constraint is given by

$$\hat{\boldsymbol{M}}_v p^*_\gamma f_\gamma = p^*_\gamma \sum_{e,e',e'',e'''\in E(v)} c^{e''e'''}_{ee'}{}^{MNOP}_{IJKL} \Delta^{ee'}_{IJKL} \Delta^{e''e'''}_{MNOP} f_\gamma \tag{11.19}$$

with a positive matrix $c^{e''e'''}_{ee'}{}^{MNOP}_{IJKL}$, which we will choose diagonal for simplicity, i.e. $c^{e''e'''}_{ee'}{}^{MNOP}_{IJKL} = \frac{1}{4!} c_{ee'} \delta^{e''}_{(e} \delta^{e'''}_{e')} \delta^{MNOP}_{IJKL}$. The diagonal elements $c_{ee'}$ can be chosen symmetric because of the symmetry of the building blocks. We choose $c_{ee'} = 1\,\forall\, e, e', e \neq e'$ and $c_{ee} = 0$ since the edge representations are already simple, leading to the final version of the master constraint we propose,

$$\hat{\boldsymbol{M}}_v p^*_\gamma f_\gamma = p^*_\gamma \sum_{e,e'\in E(v), e\neq e'} \Delta^{ee'}_{IJKL} \Delta^{ee'}_{IJKL} f_\gamma. \tag{11.20}$$

Cylindrical consistency and essential self-adjointness follows analogously to the case of $C(S, S', x)$ in section 11.2.1.1.

For the case of SO(4), we can use the decomposition in self-dual and anti-selfdual generators to find that $\epsilon^{IJKL} R^e_{IJ} R^{e'}_{KL} = \vec{J}^e_+ \cdot \vec{J}^{e'}_+ - \vec{J}^e_- \cdot \vec{J}^{e'}_-$, which implies

$$\epsilon^{IJKL} \Delta^{ee'}_{IJKL} = \left(\vec{J}^e_+ + \vec{J}^{e'}_+\right) \cdot \left(\vec{J}^e_+ + \vec{J}^{e'}_+\right) - \left(\vec{J}^e_- + \vec{J}^{e'}_-\right) \cdot \left(\vec{J}^e_- + \vec{J}^{e'}_-\right) =: \Delta^{ee'}_+ - \Delta^{ee'}_-. \tag{11.21}$$

This leads to the master constraint

$$\hat{\boldsymbol{M}}_v p^*_\gamma f_\gamma = p^*_\gamma \sum_{e,e'\in E(v), e\neq e'} \left(\Delta^{ee'}_+ \Delta^{ee'}_+ - 2\Delta^{ee'}_+ \Delta^{ee'}_- + \Delta^{ee'}_- \Delta^{ee'}_-\right) f_\gamma, \tag{11.22}$$

where + and − now label independent copies of SO(3). Thus, we can calculate the matrix elements of this constraint in a recoupling basis analogously to the standard LQG volume operator matrix elements [241].

#### 11.2.1.5 Choice of maximally commuting subset of vertex simplicity constraints

Looking back at chapter 8, one could alternatively try to gauge unfix the second class vertex simplicity constraints which result after classically introducing holonomy and flux variables, to obtain a first class system subject to only a subset of the vertex simplicity constraints. In this process, one would have to pick out a first class subset of the simplicity constraints which has a closing algebra with the remaining constraints. The construction of a possible choice of such a subset was discussed in [5] and we will briefly summarise these findings.

At the heart of the construction lies the fact that a basis in space of intertwiners can be given by specifying a recoupling scheme and labelling the "internal lines" by internal irreducible representations. We have seen in (11.16) that, using that the edge representation already are constrained to be simple, the simplicity building blocks can be rewritten as $R^e_{[IJ}R^{e'}_{KL]}f_\gamma = \frac{1}{2}(R^e_{[IJ}+R^{e'}_{[IJ})(R^e_{KL]}+R^{e'}_{KL]})f_\gamma$, which now demands that not only the edge representations, but also the representation to which they couple, be simple. Demanding all simplicity building blocks thus means that, no matter which recoupling scheme is chosen for the intertwiner, all internal representations have to be simple. As we already commented, this requirement is very restrictive and only allows for one solution, the Barrett Crane intertwiner (or its higher dimensional version) [169]. The non-commutativity of the building blocks thus can be understood as the fact that the property of one internal representation being simple in one recoupling scheme in general is not preserved under a change of recoupling scheme. However, it is proven in [5] that in one fixed recoupling scheme, we may demand that all internal lines be simple. Moreover, it is shown (under a certain assumption, cf. [5]) that this set of commuting vertex simplicity constraints is maximal, i.e. adding any other building block spoils the closure of the algebra.

A intertwiner of $N$ edges which satisfies such a maximally commuting subset of constraints can thus be labelled by the $N-3$ simple representations, i.e. "spins", on its internal lines in the given recoupling scheme. We will call such an intertwiner a simple $\mathrm{SO}(D+1)$ intertwiner. Choosing the same recoupling scheme for an SU(2) intertwiner, we can construct a unitary (with respect the scalar products induced by the respective Ashtekar-Lewandowski measures) map from the set of simple $\mathrm{SO}(D+1)$ intertwiners to the SU(2) intertwiners by simply identifying the spins on the internal lines.

Of course, this also makes apparent the problem of this proposal: We have to make a choice of maximally commuting set of vertex simplicity constraints, i.e. a recoupling scheme, for each vertex. While at the level of SU(2), a change of recoupling scheme only is a change of basis in the intertwiner space, the corresponding change of recoupling scheme at the level of SO($D+1$) does not preserve the property of the intertwiners being simple. It also is questionable if the Hamiltonian constraint leaves the space of simple intertwiners in a certain recoupling scheme invariant, and it probably has to be modified accordingly. Another puzzle is that the "size" of the kinematical Hilbert space after solving the simplicity constraint in the above described manner is *the same* for any dimension $D \geq 3$ (neglecting subtleties related with the solution of the diffeomorphism constraint), and the dimension of spacetime could become an emerging concept stemming form the choice of semiclassical states. For an extended discussion on these issues, we refer the interested reader to the original work [5].

### 11.2.2 Linear simplicity constraint

As we have seen in section 9.1, classically it is equivalent to use the linear simplicity constraints instead of the quadratic constraints we treated so far. We will see shortly in section 15.1 that this option even seems favoured if (in particular, Majorana) fermions are coupled.

To study this constraint in the quantum theory, it is firstly necessary to construct a kinematical Hilbert space for the additional field $N^I$ appearing, and secondly one has to represent and try to solve the constraint in the quantum theory. The kinematical Hilbert space for $N^I$ was given in [6] and the linear constraint was further studied in [5]. We will shortly summarise the findings, and refer the interested reader to the original articles for more detailed display.

#### 11.2.2.1 Kinematical Hilbert space for $N^I$

From a spacetime point of view, the fields $N^I$ are scalars. In [45, 62], already two routes of how to obtain kinematical Hilbert spaces for scalar fields in a background independent way were given.

The first route [45] is based on point holonomies, the construction of which works fine if the scalar field is valued in the Lie algebra of some compact gauge group. However, in the case at hand $N^I$ transforms in the defining representation of SO$(D+1)$ and it is at least not obvious if the point holonomies also can be constructed in this case.

The second possibility [62], which actually could be applied straightforwardly here, leads to a diffeomorphism invariant Fock representation. However, the field $N^I$ we are dealing with has one crucial property which usual scalar fields do not share and which led the authors in [5] to construct the Hilbert space differently: it is itself (weakly) valued in a compact set, namely the $D$-sphere $S^D$. This is exactly what is ensured by the normalisation constraint $\mathscr{N} = N^I N_I - 1$.
To make $N^I$ strongly valued in $S^D$, the normalisation constraint is gauge fixed by introducing the additional constraint $\tilde{\mathscr{N}} = N^I P_I$. The resulting second class pair $\mathscr{N}$, $\tilde{\mathscr{N}}$ is in a second step strongly solved by going over to the corresponding Dirac bracket. The remaining fields $N^I$, $\bar{P}_I$ (where $||N|| = 1$ now holds strongly) do not have a closing Dirac bracket algebra any longer. However, the rotation generators $L_{IJ} := 2N_{[I}\bar{P}_{J]}$ together with $N^I$ do have a closing algebra, and moreover, by $L_{IJ}N^J = -\bar{P}_I$, we see that $\{N^I, L_{JK}\}$ surely separate the points of the phase space (of course, neglecting $A, \pi$).

The Hilbert space $\mathscr{H}_N$ now is constructed in analogy to the one usually used in LQG: Wave functions are cylindrical functions over finite point sets $F[N]$ of the form $F[N] = F_{p_1,..,p_n}(N(p_1),..,N(p_n))$ where $F_{p_1..p_n}$ is a polynomial with complex coefficients of the $N^I(p_k)$, $k = 1,..,n$, $I = 0,..,D+1$. The cylindrical measure is constructed using that there exists an SO$(D+1)$ invariant probability measure $d\nu$ on $S^D$, and the operator $\hat{N}_I(x)$ acts by multiplication by $N_I(x)$ on this space. An orthonormal basis in this Hilbert space is given by spherical harmonic vertex functions $F_{\vec{v},\vec{l},\vec{M}}(N) := \prod_{v\in\vec{v}} \Xi_{l_v}^{\vec{M}_v}(N)$, where $\Xi_l^{\vec{M}}(N)$ are generalisations of spherical harmonics $Y_l^m(\theta,\phi)$ to higher dimensions and constitute an orthonormal basis for the Hilbert space $\mathscr{H}_p = L_2(S^D, d\nu)$ of square integrable functions on $S^D$. The label $l$ here stands for the highest weight of the representation $\Lambda = (l, 0, ..., 0)$, $l \in \mathbb{N}$, and $\vec{M}$ denotes an integer sequence $\vec{M} := (M_1, \ldots, M_{D-2}, \pm M_{D-1})$ satisfying $l \geq M_1 \geq \ldots \geq M_{D-1} \geq 0$. For more details on these functions, we refer the interested reader to our original article [5] or to [239] for a comprehensive treatment.

The combined Hilbert space for the scalar field and the gravitational so$(D+1)$ connection is simply given by the tensor product, $\mathscr{H}_T = \mathscr{H}_{\text{grav}} \otimes \mathscr{H}_N$. An orthonormal basis thereof is given by a slight generalisation of the usual gauge-variant spin network states (cf., e.g., [45]), where each vertex is labelled by an additional simple SO$(D+1)$ irreducible representation coming from the field $N^I$, and the intertwiners of course have to be altered accordingly to contract also the indices coming from this additional representation.

#### 11.2.2.2 Regularisation and anomaly freedom

The regularisation of the linear simplicity constraint, being a vector density of weight one, is similar to the regularisation of the fluxes. $\mathscr{S}^a_{I\overline{M}}$ is most naturally smeared over $(D-1)$-dimensional surfaces,

$$\mathscr{S}^b(S) := \int_S b^{L\overline{M}}(x)\epsilon_{IJKL\overline{M}} N^I(x)\pi^{aJK}(x)\epsilon_{ab_1\ldots b_{D-1}} dx^{b_1} \wedge \ldots \wedge dx^{b_{D-1}}, \qquad (11.23)$$

where $S$ again is a $D-1$-surface, and $b^{L\overline{M}}$ an arbitrary semianalytic smearing function of compact support, and the corresponding quantum operator is given by

$$\begin{aligned}\hat{S}^b(S)f &= \hat{Y}^{\epsilon b \hat{N}}(S)f = p^*_{\gamma_S}\hat{Y}^{\epsilon b\hat{N}}_{\gamma_S}(S)f_{\gamma_S}\\ &= p^*_{\gamma_S}\sum_{e\in\gamma_S}\epsilon(e,S)\epsilon_{IJKL\overline{M}}b^{L\overline{M}}(b(e))\hat{N}^I(b(e))R^{JK}_e f_{\gamma_S}.\end{aligned} \qquad (11.24)$$

Using that the right invariant vector fields actually are in the linear span of the flux vector fields as we have seen in section 11.2.1.3, it is found in [5] that is necessary and sufficient to demand that

$$\bar{R}^{IJ}_e \cdot f_\gamma = 0 \qquad (11.25)$$

for all points of $\gamma$, i.e. the generators of the SO$(D)_N$ subgroup of rotations stabilising $N^I$ have to annihilate physical states. While we have found an anomaly in the case of the quadratic simplicity constraint, the linear constraint actually is non-anomalous, since the generators of rotations stabilising $N^I$ form a closed subalgebra, i.e. commute weakly[1]. Consulting a standard textbook on representation theory [239], we find that

[1]Note that the constraint is "non-anomalous" in the same sense as the quadratic constraint is "anomalous": while classically strongly Poisson commuting, the linear simplicity constraint lose this

by definition, the only irreducible representations of SO($D+1$) which have in their representation space non-zero vectors which are invariant under an SO($D$) subgroup are irreps of class one, and they exactly coincide with what has been termed simple representations in the spin foam literature. This already tells us that the above requirement (11.25) can only be met if all edge representations are simple. Moreover, one finds that SO($D$) is a massive subgroup of SO($D+1$), which means that the unit length vector $\xi$ invariant under this subgroup is unique if it exists [239]. The constraint thus is satisfied if the ends of all edges meeting at the point where the constraint operator acts are each individually contracted with (possibly a multiple of) this unit length invariant vector $\xi_{\pi_e}(N)$ in the irrep $\pi_e$ of the edge (or, depending on the orientation, its dual; note that dual representations of simple representations are simple again).

This of course poses an immediate problem: First of all, the intertwiner space at any vertex becomes one-dimensional when solving both, the linear simplicity and Gauß constraint (all endpoints of the edges have to be contracted with invariant vectors to fulfil the simplicity constraints, and any further non-trivial $N$-dependence would make the vertex non-gauge invariant), which seems too restrictive. Moreover, since the constraint has to be satisfied for all surfaces $S$, it in particular has to hold for all points of a given graph. However, to insert this $N$-dependent intertwiner at all points of $\gamma$ is in conflict with the definition of cylindrical functions. In [5], the possibility of bringing the proposal in agreement with cylindrical consistency using a rigging map construction [242–244] is discussed. However, no rigging map with satisfactory properties is found and we have to leave this issue for further studies.

#### 11.2.2.3 Mixed quantisation

We have seen that, while the linear constraint has the nice property of being non-anomalous if quantised as outlined, solving the constraint causes problems. While one would expect from the experience with the quadratic constraint that, acting on edges, it only demands the representations carried to be simple, this is not the case and it is hard to give mathematical sense to the solution space. Therefore it seems that for pure

property upon singular smearing and therefore also in the quantum theory, but at least it remains weakly commuting or "non-anomalous". Note, however, that for $D = 3$, this property is in general lost when introducing the Barbero Immirzi parameter as we did in section 9.3, cf. also [5].

gravity, the quadratic constraint is favourable. However, the formulation we will give for supergravity (cf. section 15.1) forces us to introduce the field $N^I$. Therefore, in [5] a third possibility is discussed, namely replacing the linear constraint classically by the quadratic constraint plus an additional constraint of the form $\sqrt{q}(n^I(\pi) - N^I) \approx 0$. While classically completely equivalent, at the quantum level this avoids the above problems: The quadratic constraints can be implemented as before, in particular restricts the edge representations to be simple, while the additional constraint has to be quantised using the master constraint method [245], since otherwise it most probably does not commute with the Hamiltonian constraint operator. Choosing a suitable factor ordering, we can make sure that the additional constraint vanishes when acting on edges, but the restrictions it imposes on the intertwiner spaces when acting on vertices cannot be easily deduced and have not been studied so far.

For an extended discussion of the above briefly raised problems and the proposal of several tentative remedies, as well as a comparison with the approaches used in spin foams to deal with the simplicity constraints, we refer the interested reader to [5]. We will also revisit the simplicity problem in the discussion 18.2 at the end of this work.

## 11.3 Diffeomorphism constraint

The diffeomorphism constraint can again be treated in exact agreement with the (3+1)-dimensional case. Consider the set of smooth cylindrical functions $\Phi := \mathrm{Cyl}^\infty(\overline{\mathscr{A}/\mathscr{G}})$ which can be shown to be dense in $\mathscr{H}^0$. By a distribution $\psi \in \Phi'$ on $\Phi$ we simply mean a linear functional on $\Phi$. The group average of a spin-network state $T_{\gamma,\vec{\Lambda},\vec{c}}$ is defined by the following well-defined distribution on $\Phi$

$$T_{[\gamma],\vec{\Lambda},\vec{c}} := \sum_{\gamma' \in [\gamma]} < T_{\gamma',\vec{\Lambda},\vec{c}}, . > , \tag{11.26}$$

where $[\gamma]$ denotes the orbit of $\gamma$ under smooth diffeomorphisms of $\sigma$ which preserve the analyticity of $\gamma$ including an average over the graph symmetry group (see, e.g., [56] for technical details). Since we already solved the simplicity constraint on single edges, we can restrict attention to spin network states with edges labelled by simple $\mathrm{SO}(D+1)$ representations, $\Lambda_e = (N_e, 0, ...)$. The group average $[f]$ of a general cylindrical function $f$ is defined by demanding linearity of the averaging procedure, i.e. first decompose $f$

into spin-network states and then average each of the spin-network states separately. An inner product for the diffeomorphism invariant Hilbert space can be constructed. We will not give details and refer the reader to [27, 56].

# 12

# Geometrical operators

## 12.1 The $D-1$ area operator

The area operator was first considered in [32] and defined mathematically rigorously in the LQG representation in [34]. In [62], the results of [34] are generalised for arbitrary dimension $D$. Using the classical identity $\pi^{aIJ}\pi^{b}{}_{IJ} = 2qq^{ab}$, we can basically copy the treatment found there. Let S be a surface and $X : U_0 \to S$ the associated embedding, where $U_0$ is an open submanifold of $\mathbb{R}^{D-1}$. Then the area functional is given by

$$\mathrm{Ar}[S] := \int_{U_0} d^{D-1}u \sqrt{det\left(\left[X^{*}q\right](u)\right)}. \tag{12.1}$$

Introduce $U_0 = \cup_{U\in\mathscr{U}} U$, a partition of $U_0$ by closed sets $U$ with open interior, $\mathscr{U}$ being the collection of these sets. Then the area functional can be written as the limit as $|U| \to \infty$ of the Riemann sum

$$\mathrm{Ar}[S] := \sum_{U\in\mathscr{U}} \sqrt{\frac{1}{2}\pi_{IJ}(S_U)\pi^{IJ}(S_U)}, \tag{12.2}$$

where $S_U = X(U)$ and $\pi_{IJ}(S_U)$ is the electric flux with choice $n^{IJ} = \delta^{I}_{[K}\delta^{J}_{L]}$, which has been quantised already. Let $f \in \mathrm{Cyl}^2(\overline{\mathscr{A}})$, choose a representative $f_\gamma$ and, using the known action of the quantised electric fluxes, obtain as in the $(3+1)$-dimensional case

$$\widehat{\mathrm{Ar}_\gamma}[S]p^{*}_{\gamma}f_\gamma = \kappa\hbar\beta p^{*}_{\gamma_S} \sum_{x\in\{e\cap S; e\in E(\gamma_S)\}} \sqrt{-\frac{1}{2}\left\{\sum_{e\in E(\gamma_S), x\in\partial e} \epsilon(e,S)R^{e}_{IJ}\right\}^2} \, p^{*}_{\gamma_S\gamma}f_\gamma, \tag{12.3}$$

where $\gamma_S \succ \gamma$ is an adapted graph. The family of operators $\widehat{\mathrm{Ar}_\gamma[S]}$ has dense domain $\mathrm{Cyl}^2(\overline{\mathscr{A}})$. Its independence of the adapted graph follows from that of the electric fluxes.

Moreover, the properties of the area operator like cylindrical consistency, essential self-adjointness and discreteness of the spectrum can be shown analogously to [62].

The complete spectrum can be derived using the standard methods. We use

$$\left\{ \sum_{e\in E(\gamma_S), x\in\partial e} \epsilon(e,S) R^e_{IJ} \right\}^2 = 2\left(R^{x,up}_{IJ}\right)^2 + 2\left(R^{x,down}_{IJ}\right)^2 - \left(R^{x,up}_{IJ} + R^{x,down}_{IJ}\right)^2$$

$$=: -\Delta^{up} - \Delta^{down} + \frac{1}{2}\Delta^{up+down}, \tag{12.4}$$

where the $\Delta$s are mutually commuting primitive Casimir operators of $\mathrm{SO}(D+1)$. Thus their spectrum is given by the Eigenvalues $\lambda_\pi > 0$. We have to distinguish the cases $D+1 = 2n$ even, $\mathbb{N} \ni n \geq 2$ and $D+1 = 2n+1$ odd, $n \in \mathbb{N}$. In a representation of $\mathrm{SO}(D+1)$ with highest weight $\Lambda = (n_1, ..., n_n)$, $n_i \in \mathbb{N}_0$, we find for the eigenvalues of the Casimir[1] $\Delta := -\frac{1}{2} X_{IJ} X^{IJ}$

$$\Delta v_\Lambda := \lambda_{\pi_\Lambda} v_\Lambda = \left[\sum_{i=1}^n f_i^2 + 2\sum_{j=2}^n \sum_{i<j} f_i\right] v_\Lambda \quad \text{for } \mathrm{SO}(2n),$$

$$\Delta v_\Lambda := \lambda_{\pi_\Lambda} v_\Lambda = \left[\sum_{i=1}^n f_i^2 + 2\sum_{j=2}^n \sum_{i<j} f_i + \sum_{i=1}^n f_i\right] v_\Lambda \quad \text{for } \mathrm{SO}(2n+1), \tag{12.5}$$

where we used the following notation

$$f_i = \sum_{j=i}^{n-2} n_j + \frac{n_{n-1}+n_n}{2},\ i \leq (n-2);\ f_{n-1} = \frac{n_{n-1}+n_n}{2};\ f_n = \frac{n_n - n_{n-1}}{2} \ \text{ for } \mathrm{SO}(2n),$$

$$f_i = \sum_{j=i}^{n-1} n_j + \frac{n_n}{2},\ i \leq (n-1);\ f_n = \frac{n_n}{2} \ \text{ for } \mathrm{SO}(2n+1), \tag{12.6}$$

such that $f_1 \geq f_2 \geq ... \geq f_n$. Note that the above formulas hold for general irreducible $\mathrm{Spin}(D+1)$ representations. Irreducible representations of $\mathrm{SO}(D+1)$ are found by the restriction that all $f_i$ be integers. Denoting by $\Pi$ a collection of representatives of irreducible representations of $\mathrm{SO}(D+1)$, one for each equivalence class, we find for the area spectrum

$$\mathrm{Spec}(\widehat{\mathrm{Ar}}[S]) = \left\{ \frac{\kappa\hbar\beta}{2} \sum_{n=1}^N \sqrt{2\lambda_{\pi^1_n} + 2\lambda_{\pi^2_n} - \lambda_{\pi^{12}_n}};\ N\in\mathbb{N}, \pi^1_n, \pi^2_n, \pi^{12}_n \in \Pi, \pi^{12}_n \in \pi^1_n \otimes \pi^2_n \right\}. \tag{12.7}$$

[1] Note that $R^{IJ} = 1/2 X^{IJ}$, such that $X^{IJ}$ fulfil the standard Lie algebra relations without the factor 1/2 appearing in (10.10).

Note that the above formulas (12.5) significantly simplify if we restrict to simple representations, $\Lambda_0 = (N, 0, 0, ...)$,

$$\begin{aligned}\Delta v_{\Lambda_0} &= N(N+2n-2)v_{\Lambda_0} &&= N(N+D-1)v_{\Lambda_0} \quad \text{for SO}(2n),\\ \Delta v_{\Lambda_0} &= N(N+2n+1-2)v_{\Lambda_0} &&= N(N+D-1)v_{\Lambda_0} \quad \text{for SO}(2n+1).\end{aligned} \tag{12.8}$$

We cannot use this simplified expression for the SO$(D+1)$ Casimir operator in the general case (12.7), since in the decomposition of a tensor product of irreducible simple representations usually non-simple representations will appear[1], but we can use it for a single edge. When acting on a single edge $e = e_1 \circ (e_2)^{-1}$ intersecting $S$ transversally, we know that due to gauge invariance

$$\left\{R^{e_1}_{IJ} - R^{e_2}_{IJ}\right\}^2 h_e = 4\left(R^{e_1}_{IJ}\right)^2 h_e = -2N(N+D-1)h_e. \tag{12.9}$$

The action of the area operator on a single edge $e$, $e \cap S \neq \emptyset$ is thus given by

$$\widehat{\mathrm{Ar}_e}[S]p_e^* h_e = \kappa\hbar\beta\sqrt{N(N+D-1)}p_e^* h_e = 16\pi\beta\left(l_p^{(D+1)}\right)^{D-1} \times \sqrt{N(N+D-1)}p_e^* h_e, \tag{12.10}$$

where $l_p^{(D+1)} := \sqrt[D-1]{\frac{\hbar G^{(D+1)}}{c^3}}$ is the unique length in $D+1$ dimensions, and $\kappa = 16\pi G^{(D+1)}/c^3$ in any dimension, where $G^{(D+1)}$ denotes the gravitational constant. Note that for $D = 3$, we find the factor $\sqrt{N(N+2)}$ in the area spectrum of an edge stemming from irreducible simple representations of SO(4). Replace the non-negative integer $N$ labelling the weight by $N = 2j$, $j$ half integer, to find the factor $2\sqrt{j(j+1)}$ of SO(4) spin foam models, which coincides with the usual spacing in $(3+1)$-dimensional LQG,

$$\widehat{\mathrm{Ar}_e}[S]p_e^* h_e = 2\kappa\hbar\beta\sqrt{j(j+1)}p_e^* h_e = 32\pi\beta\left(l_p^{(D+1)}\right)^{D-1} \times \sqrt{j(j+1)}p_e^* h_e. \tag{12.11}$$

In standard LQG, instead of the gauge group SO(3) one extends to the double cover Spin(3) $\cong$ SU(2) and allows also for half integer representations. Note that in our case, we cannot allow for general Spin$(D+1)$ representations at the edges, since the edge simplicity constraint is not satisfied in representations of Spin$(D+1)$ which are not as well representations of SO$(D+1)$, $D \geq 3$ [169].

[1] For the tensor product of two irreducible simple representations of SO$(n)$ holds [237, 238] (w.l.o.g. $M \geq N$) $[M, 0, .., 0] \otimes [N, 0, .., 0] = \sum_{K=0}^{N} \sum_{L=0}^{N-K} [M+N-2K-L, L, 0, .., 0]$.

## 12.2 The volume operator

The derivation of the volume operator is analogous to the treatment in [62] and requires only a slight adjustment. The exposure is geared towards using the quadratic simplicity constraint. For an alternative construction of the volume operator when using the linear constraint, cf. [5].

The volume of a region $R$ is classically measured by

$$V(R) := \int_R d^D x \sqrt{q}, \tag{12.12}$$

where $\sqrt{q}$ has to be expressed in terms of the canonical variables. The derivation is performed for $\beta = 1$, the general result is obtained by multiplying the resulting operator by $\beta^{D/(D-1)}$.

### 12.2.1 $D + 1$ even

Let $n = (D-1)/2$. Let $\chi_\Delta(p, x)$ be the characteristic function in the coordinate $x$ of a hypercube with centre $p$ spanned by the $D$ vectors $\vec{\Delta}^i := \Delta^i \vec{n}^i$, $i = 1, \ldots, D$, where $\vec{n}^i$ is a normal vector in the frame under consideration and which has coordinate volume $\text{vol} = \Delta^1 \ldots\ \Delta^D \det(\vec{n}^1, \ldots, \vec{n}^D)$ (we assume the vectors to be right-oriented). In other words,

$$\chi_\Delta(p, x) = \prod_{i=1}^{D} \Theta\left(\frac{\Delta^i}{2} - \left|< n^i, x - p >\right|\right) \tag{12.13}$$

where $< \cdot, \cdot >$ is the standard Euclidean inner product and $\Theta(y) = 1$ for $y > 0$ and zero otherwise. We will use lower indices $(\Delta^1_I, \ldots, \Delta^D_I)$ to label different hypercubes. It will turn out to be convenient to label the $D$ edges appearing in the following formulae by $e, e_1, \ldots, e_n, e'_1, \ldots, e'_n$.

We consider the smeared quantity

$$\begin{aligned}
&\pi(p, \Delta_1, \ldots, \Delta_D) \\
&= \frac{1}{\text{vol}(\Delta_1) \ldots \text{vol}(\Delta_D)} \int_\sigma d^D x_1 \ldots \int_\sigma d^D x_D \\
&\quad \chi_{\Delta_1}(p, x_1) \chi_{\Delta_2}(2p, x_1 + x_2) \ldots \chi_{\Delta_D}(Dp, x_1 + \ldots + x_D) \\
&\quad \frac{1}{2D!} \epsilon_{a a_1 b_1 \ldots a_n b_n} \epsilon_{IJI_1J_1I_2J_2 \ldots I_nJ_n} \pi^{aIJ} \pi^{a_1 I_1 K_1} \pi^{b_1 J_1}{}_{K_1} \ldots \pi^{a_n I_n K_n} \pi^{b_n J_n}{}_{K_n}.
\end{aligned} \tag{12.14}$$

Then it is easy to see that the classical identity

$$V(R) = \lim_{\Delta_1 \to 0} \ldots \lim_{\Delta_D \to 0} \int_R d^D p \; |\pi(p, \Delta_1, \ldots, \Delta_D)|^{\frac{1}{D-1}} \tag{12.15}$$

holds. The canonical brackets

$$\left\{ A_{aIJ}(x), \pi^{bKL}(y) \right\} = 2\delta^D(x-y)\delta_a^b \delta_I^{[K} \delta_J^{L]} \tag{12.16}$$

give rise to the operator representation

$$\hat{\pi}^{bKL} = -\frac{\hbar}{i} \frac{\delta}{\delta A_{bKL}} \tag{12.17}$$

while the connection acts by multiplication.

Let a graph $\gamma$ be given. In order to simplify the notation, we subdivide each edge $e$ with endpoints $v, v'$ which are vertices of $\gamma$ into two segments $s, s'$ where $e = s \circ (s')^{-1}$ and $s$ has an orientation such that it is outgoing at $v'$. This introduces new vertices $s \cap s'$ which we will call pseudo-vertices because they are not points of non-semianalyticity of the graph. Let $E(\gamma)$ be the set of these segments of $\gamma$ but $V(\gamma)$ the set of true (as opposed to pseudo) vertices of $\gamma$. Let us now evaluate the action of

$$\hat{\pi}^{aIJ}(p, \Delta) := \frac{1}{\mathrm{vol}(\Delta)} \int_\Sigma d^D x \, \chi(p, x) \hat{\pi}^{aIJ} \tag{12.18}$$

on a function $f = p_\gamma^* f_\gamma$ cylindrical with respect to $\gamma$. We find ($e : [0,1] \to \sigma, t \to e(t)$ being a parametrisation of the edge $e$)

$$\hat{\pi}^{aIJ}(p, \Delta) f = \frac{i\hbar}{\mathrm{vol}(\Delta)} \sum_{e \in E(\gamma)} \int_0^1 \chi_\Delta(p, e(t)) \dot{e}^a(t) \mathrm{tr} \left( \left[ h_e(0,t) \tau^{IJ} h_e(t,1) \right]^T \frac{\partial}{\partial h_e(0,1)} \right) f_\gamma. \tag{12.19}$$

Here we have used (1) the fact that a cylindrical function is already determined by its values on $\mathscr{A}/\mathscr{G}$ rather than $\overline{\mathscr{A}/\mathscr{G}}$ so that it makes sense to take the functional derivative, (2) the definition of the holonomy as the path-ordered exponential of $\int_e A$ with the smallest parameter value to the left, (3) $A = dx^a A_{aIJ} \tau^{IJ}$ where $\tau^{IJ} \in \mathrm{so}(D+1)$ and we have defined (4) $\mathrm{tr}(h^T \partial / \partial g) = h_{AB} \partial / \partial_{AB}$, $A, B, C, \ldots$ being $\mathrm{SO}(D+1)$ indices. The state that appears on the right-hand side of (12.19) is actually well-defined, in the sense of functions of connections, only when $A$ is smooth for otherwise the integral over $t$ does not exist, see [246] for details. However, as announced, we will be interested only in quantities constructed from operators of the form (12.19) and for which the limit of shrinking $\Delta \to 0$ to a point has a meaning in the sense of $\mathscr{H} = L_2(\overline{\mathscr{A}/\mathscr{G}}, d\mu_0)$ and therefore will not be concerned with the actual range of the operator (12.19) for the moment.

We now wish to evaluate the whole operator $\hat{\pi}(p, \Delta^1, \ldots, \Delta^D)$ on $f$. It is clear that we obtain $D$ types of terms, the first type comes from all three functional derivatives acting on $f$ only, the second type comes from $D-1$ functional derivatives acting on $f$ and the remaining one acting on the trace appearing in (12.19), and so forth.

The first term (type) is explicitly given by

$$
\begin{aligned}
&\hat{\pi}(p, \Delta_1, .., \Delta_D) f \\
= &\frac{1}{2D!} \frac{(i\hbar)^D}{\mathrm{vol}(\Delta_1)..\mathrm{vol}(\Delta_D)} \epsilon_{a a_1 b_1 .. a_n b_n} \epsilon_{IJI_1J_1I_2J_2..I_nJ_n} \int_{[0,1]^D} dt\, dt_1..dt_n\, dt'_1..dt'_n \sum_{e_1,..,e_D \in E(\gamma)} \\
&\chi_{\Delta_1}(p, x_1)\chi_{\Delta_2}(2p, x_1 + x_2)..\chi_{\Delta_D}(Dp, x_1 + .. + x_D) \\
&\dot{e}^a(t)\dot{e}_1^{a_1}(t_1)..\dot{e}_n^{a_n}(t_n)\dot{e}'^{\,b_1}_1(t'_1)..\dot{e}'^{\,b_n}_n(t'_n) \mathrm{tr}\left( \left[h_e(0,t)\tau^{IJ}h_e(t,1)\right]^T \frac{\partial}{\partial h_e(0,1)} \right) \\
&\mathrm{tr}\left( \left[h_{e_1}(0,t_1)\tau^{I_1K_1}h_{e_1}(t_1,1)\right]^T \frac{\partial}{\partial h_{e_1}(0,1)} \right) \\
&\mathrm{tr}\left( \left[h_{e'_1}(0,t'_1)\tau^{J_1}{}_{K_1}h_{e'_1}(t'_1,1)\right]^T \frac{\partial}{\partial h_{e'_1}(0,1)} \right) ... \\
&\mathrm{tr}\left( \left[h_{e_n}(0,t_n)\tau^{I_nK_n}h_{e_n}(t_n,1)\right]^T \frac{\partial}{\partial h_{e_n}(0,1)} \right) \\
&\mathrm{tr}\left( \left[h_{e'_n}(0,t'_n)\tau^{J_n}{}_{K_n}h_{e'_n}(t'_n,1)\right]^T \frac{\partial}{\partial h_{e'_n}(0,1)} \right) f_\gamma. \qquad (12.20)
\end{aligned}
$$

The other terms are vanishing due to either the same symmetry / anti-symmetry properties as in the usual treatment or the simplicity constraint in case the first derivative is involved.

Given a $D$-tuple $e_1 \ldots e_D$ of (not necessarily distinct) edges of $\gamma$, consider the functions

$$x_{e_1,\ldots,e_D}(t_1, \ldots, t_D) := e_1(t_1) + \ldots + e_D(t_D). \qquad (12.21)$$

This function has the interesting property that the Jacobian is given by

$$\det\left(\frac{\partial(x^1_{e_1,\ldots,e_D},\ldots x^D_{e_1,\ldots,e_D})(t_1,\ldots,t_D)}{\partial(t_1,\ldots,t_D)}\right)=\epsilon_{a_1\ldots a_D}\dot{e}_1(t_1)^{a_1}\ldots\dot{e}_D(t_D)^{a_D} \tag{12.22}$$

which is precisely the form of the factor which enters the integral (12.20).

We now consider the limit $\Delta^1,\ldots,\Delta^D\to 0$. The idea is that all quantities in (12.20) are meaningful in the sense of functions on smooth connections and thus limits of functions as $\Delta\to 0$ are to be understood with respect to any Sobolev topology. The miracle is that the final function is again cylindrical and thus the operator that results in the limit has an extension to all of $\overline{\mathscr{A}/\mathscr{G}}$.

**Lemma 2.**
*For each D-tuple of edges $e_1,\ldots,e_D$ there exists a choice of vectors $\vec{n}^1_1,\ldots,\vec{n}^1_D,\vec{n}^2_1,\ldots,\vec{n}^D_D$ and a way to guide the limit $\Delta^1_1,\Delta^1_2,\ldots,\Delta^D_D\to 0$ such that*

$$\int_{[0,1]^D}\det\left(\frac{\partial x^a_{e_1,\ldots,e_D}}{\partial(t_1,\ldots,t_D)}\right)\chi_{\Delta_1}(p,e_1)\ldots\chi_{\Delta_D}(Dp,e_1+\ldots e_D)\hat{O}_{e_1,\ldots,e_D} \tag{12.23}$$

*vanishes if*

*(a) if $e_1,\ldots,e_D$ do not all intersect p or*

*(b) $\det\left(\frac{\partial x^a_{e_1,\ldots,e_D}}{\partial(t_1,\ldots,t_D)}\right)_p=0$ (which is a diffeomorphism invariant statement).*

*Otherwise it tends to*

$$\frac{1}{2^D}sgn\left(\det\left(\frac{\partial x^a_{e_1,\ldots,e_D}}{\partial(t_1,\ldots,t_D)}\right)\right)_p\hat{O}_{e_1,\ldots,e_D}(p)\prod_{i=1}^{D}\Delta^i_D. \tag{12.24}$$

*Here we have denoted by $\hat{O}_{e_1,\ldots,e_D}(p)$ the trace(s) involved in the various terms of (12.20).*

We conclude that (12.20) reduces to

$$\begin{aligned}&\lim_{\Delta_D\to 0}\hat{\pi}(p,\Delta_1,\ldots,\Delta_D)f\\&=\sum_{e_1,\ldots,e_D}\frac{(i\hbar)^D s(e_1,\ldots,e_D)}{2^D D!\mathrm{vol}(\Delta_1)\ldots\mathrm{vol}(\Delta_{D-1})}\chi_{\Delta_1}(p,v)\ldots\chi_{\Delta_{D-1}}(p,v)\hat{O}_{e_1,\ldots,e_D}(0,\ldots,0),\end{aligned} \tag{12.25}$$

where $v$ on the right-hand side is the intersection point of the $D$-tuple of edges and it is understood that we only sum over such $D$-tuples of edges which are incident at a common vertex and $s(e_1, \ldots, e_D) := \mathrm{sgn}(\det(\dot{e}_1(0), \ldots, \dot{e}_D(0)))$. Moreover,

$$\hat{O}_{e_1,\ldots,e_D}(0,\ldots,0) = \frac{1}{2}\epsilon_{IJI_1J_1I_2J_2\ldots I_nJ_n} R_e^{IJ} R_{e_1}^{I_1K_1} R^{J_1}_{e'_1}{}_{K_1} \ldots R_{e_n}^{I_nK_n} R^{J_n}_{e'_n}{}_{K_n} \tag{12.26}$$

and

$$R_e^{IJ} := R^{IJ}(h_e(0,1)) := \mathrm{tr}\left((\tau^{IJ} h_e(0,1))^T \frac{\partial}{\partial h_e(0,1)}\right) \tag{12.27}$$

is a right-invariant vector field in the $\tau^{IJ}$ direction of $\mathrm{SO}(D+1)$, that is, $R(hg) = R(h)$. We have also extended the values of the sign function to include 0, which takes care of the possibility that one has $D$-tuples of edges with linearly dependent tangents.

The final step is choosing $\Delta_1 = \ldots = \Delta_{D-1}$ and exponentiating the modulus by $1/(D-1)$. We replace the sum over all $D$-tuples incident at a common vertex $\sum_{e_1,\ldots,e_D}$ by a sum over all vertices followed by a sum over all $D$-tuples incident at the same vertex $\sum_{v\in V(\gamma)} \sum_{e_1\cap\ldots\cap e_D = v}$. Now, for small enough $\Delta$ and given $p$, at most one vertex contributes, that is, at most one of $\chi_\Delta(v,p) \neq 0$ because all vertices have finite separation. Then we can take the relevant $\chi_\Delta(p,v) = \chi_\Delta(p,v)^2$ out of the exponential and take the limit, which results in

$$\hat{V}(R) = \int_R d^Dp\, |\widehat{\det(q)(p)}|_\gamma = \int_R d^Dp \hat{V}(p)_\gamma, \tag{12.28}$$

$$\hat{V}(p) = \left(\frac{\hbar}{2}\right)^{\frac{D}{D-1}} \sum_{v\in V(\gamma)} \delta^D(p,v) \hat{V}_{v,\gamma}, \tag{12.29}$$

$$\hat{V}_{v,\gamma} = \left| \frac{i^D}{D!} \sum_{e_1,\ldots,e_D \in E(\gamma),\, e_1\cap\ldots\cap e_D = v} s(e_1,\ldots,e_D) q_{e_1,\ldots,e_D} \right|^{\frac{1}{D-1}}, \tag{12.30}$$

$$q_{e_1,\ldots,e_D} = \frac{1}{2}\epsilon_{IJI_1J_1I_2J_2\ldots I_nJ_n} R_e^{IJ} R_{e_1}^{I_1K_1} R^{J_1}_{e'_1}{}_{K_1} \ldots R_{e_n}^{I_nK_n} R^{J_n}_{e'_n}{}_{K_n}. \tag{12.31}$$

### 12.2.2 $D+1$ odd

The case $D+1$ uneven works analogously, except that the expression for $\det(q)$ is changed a bit. With $n = D/2$, the result is

$$\hat{V}(R) = \int_R d^D p \, |\widehat{\det(q)(p)}|_{\gamma} = \int_R d^D p \hat{V}(p)_{\gamma}, \tag{12.32}$$

$$\hat{V}(p) = \left(\frac{\hbar}{2}\right)^{\frac{D}{D-1}} \sum_{v \in V(\gamma)} \delta^D(p,v) \hat{V}_{v,\gamma}, \tag{12.33}$$

$$\hat{V}^I_{v,\gamma} = \frac{i^D}{D!} \sum_{e_1,\ldots,e_D \in E(\gamma),\, e_1 \cap \ldots \cap e_D = v} s(e_1,\ldots,e_D) q^I_{e_1,\ldots,e_D}, \tag{12.34}$$

$$\hat{V}_{v,\gamma} = \left| \hat{V}^I_{v,\gamma} \hat{V}_{I\,v,\gamma} \right|^{\frac{1}{2D-2}}, \tag{12.35}$$

$$q^I_{e_1,\ldots,e_D} = \epsilon_{I I_1 J_1 I_2 J_2 \ldots I_n J_n} R^{I_1 K_1}_{e_1} R^{J_1}_{e'_1\;K_1} \ldots R^{I_n K_n}_{e_n} R^{J_n}_{e'_n\;K_n}. \tag{12.36}$$

### 12.2.3 More results and open questions

The derivations of cylindrical consistency, symmetry, positivity, self-adjointness and anomaly-freeness given in [62] generalise immediately to the higher dimensional volume operator. The question of uniqueness of the prefactor [247, 248] in front of the expression under the square root of the volume operator or the computation of the matrix elements [249–252] have not been addressed so far, however these are not necessary steps in order to use the volume operator for a consistent quantisation of the Hamiltonian constraint in what follows. We leave these open questions for future research.

# 13

# Implementation of the Hamiltonian constraint

The implementation of the Hamiltonian constraint will follow along the lines of [62], see [30] for original literature and details. In section 7.3 (see also [1, 2]), we derived the classical expression

$$\begin{aligned}\mathscr{H} &= -\frac{1}{2\sqrt{q}}\Big(F_{abIJ}\pi^{aIK}\pi^{b}{}_{K}{}^{J} + \bar{D}_{\mathrm{tf}}{}^{aIJ}\,(F^{-1})_{aIJ,bKL}\,\bar{D}_{\mathrm{tf}}{}^{bKL} \\ &\qquad + \frac{1}{\sqrt{q}(D-1)^2}[D_b{}^a\,D_a{}^b - (D_c{}^c)^2]\Big) - \frac{1}{2\beta^2(D-1)^2}[D_b{}^a\,D_a{}^b - (D_c{}^c)^2] \\ &= -\frac{1}{\sqrt{q}}\mathscr{H}_E + \frac{1}{2\sqrt{q}}\mathscr{D}^{ab}_{\overline{M}}\,\left(F^{-1}\right)^{\overline{M}\,\overline{N}}_{ab\;cd}\,\mathscr{D}^{cd}_{\overline{N}} - \frac{1}{2\sqrt{q}}(\beta^2+1)(K_a{}^bK_b{}^a - (K_c{}^c)^2), \quad (13.1)\end{aligned}$$

where we specified $s=-1$, $\zeta=1$ in (7.89) and in the second step we introduced the notation $D_a{}^b =: \beta(D-1)K_a{}^b$, where $K_a{}^b$ now actually is weakly given by the (densitised) extrinsic curvature (cf. (7.79)). These correction terms changing the extrinsic curvature contribution to the constraint also appear when using Ashtekar Barbero variables (except for $\gamma^2 = s$), and can be quantised, as we will see, in analogy to the treatment in 3+1 dimensions. We furthermore defined the analogon of the (density weight two) Euclidean Hamiltonian constraint in $D=3$ (although here, this object does not reduce to the Euclidean Hamiltonian constraint) $\mathscr{H}_E := -\frac{1}{2}\pi^{aIK}\pi^{bJ}{}_{K}F_{abIJ}$, and rewrote the terms removing the $\bar{K}^{\mathrm{tf}}_{aIJ}$ terms in the form they appear after gauge unfixing (this is only to keep the notation used in [3]).

In order to have a well defined quantum version of this constraint, we have to express it in terms of holonomy and flux variables. As in the $3+1$-dimensional case, the volume operator turns out to be a cornerstone of the quantisation.

At first, we will introduce a graph adapted triangulation of $\sigma$ in order to regularise the Hamiltonian constraint. Next, classical identities to express the Hamiltonian constraint in terms of holonomies and fluxes are derived. Since the complete expression for the Hamiltonian constraint will turn out to be rather laborious to write down, we will derive the regularisation piece by piece. Next, we show how to assemble the regularised pieces to the complete constraint and describe the quantisation. Finally, we construct a Hamiltonian master constraint in order to avoid some of the usual difficulties associated with quantisation.

## 13.1 Triangulation

A natural choice for a triangulation turns out to be the following (we simplify the presentation drastically, the details can be found in [30]): given a graph $\gamma$ one constructs a triangulation $T(\gamma,\epsilon)$ of $\sigma$ *adapted* to $\gamma$ which satisfies the following basic requirements.

(a) The graph $\gamma$ is embedded in $T(\gamma,\epsilon)$ for all $\epsilon > 0$.

(b) The valence of each vertex $v$ of $\gamma$, viewed as a vertex of the infinite graph $T(\gamma,\epsilon)$, remains constant and is equal to the valence of $v$, viewed as a vertex of $\gamma$, for each $\epsilon > 0$.

(c) Choose a system of semianalytic[1] arcs $a^{\epsilon}_{\gamma,v,e,e'}$, one for each pair of edges $e,e'$ of $\gamma$ incident at a vertex $v$ of $\gamma$, which do not intersect $\gamma$ except in its endpoints where they intersect transversally. These endpoints are interior points of $e,e'$ and are those vertices of $T(\gamma,\epsilon)$ contained in $e,e'$ closest to $v$ for each $\epsilon > 0$ (i.e., no others are in between). For each $\epsilon,\epsilon' > 0$ the arcs $a^{\epsilon}_{\gamma,v,e,e'}$, $a^{\epsilon'}_{\gamma,v,e,e'}$ are diffeomorphic with respect to semianalytic diffeomorphisms. The segments $e,e'$ incident at $v$ with outgoing orientation that are determined by the endpoints of the arc $a^{\epsilon}_{\gamma,v,e,e'}$ will be denoted by $s^{\epsilon}_{\gamma,v,e}$, $s^{\epsilon}_{\gamma,v,e'}$ respectively. Finally, if $\phi$ is a semianalytic diffeomorphism then $s^{\epsilon}_{\phi(\gamma),\phi(v),\phi(e)}$, $a^{\epsilon}_{\phi(\gamma),\phi(v),\phi(e),\phi(e')}$ and $\phi(s^{\epsilon}_{\gamma,v,e})$, $\phi(a^{\epsilon}_{\gamma,v,e,e'})$ are semianalytically diffeomorphic.

[1] Semianalyticity is a more precise version of piecewise analytic. See [28] for complete definitions.

(d) Choose a system of mutually disjoint neighbourhoods $U^\epsilon_{\gamma,v}$, one for each vertex $v$ of $\gamma$, and require that for each $\epsilon > 0$ the $a^\epsilon_{\gamma,v,e,e'}$ are contained in $U^\epsilon_{\gamma,v}$. These neighbourhoods are nested in the sense that $U^\epsilon_{\gamma,v} \subset U^{\epsilon'}_{\gamma,v}$ if $\epsilon < \epsilon'$. and $\lim_{\epsilon\to 0} U^\epsilon_{\gamma,v} = \{v\}$.

(e) Triangulate $U^\epsilon_{\gamma,v}$ by $D$-simplices $\Delta(\gamma, v, e_1, \ldots, e_D)$, one for each ordered $D$-tuple of distinct edges $e_1, \ldots, e_D$ incident at $v$, bounded by the segments $s^\epsilon_{\gamma,v,e_1}, \ldots, s^\epsilon_{\gamma,v,e_D}$ and the arcs $a^\epsilon_{\gamma,v,e_1,e_2}, a^\epsilon_{\gamma,v,e_1,e_3}, \ldots, a^\epsilon_{\gamma,v,e_{D-1},e_D}$ ($D(D-1)/2$ arcs) from which loops $\alpha^\epsilon_{\gamma;v;e_1,e_2}$, etc. are built and triangulate the rest of $\sigma$ arbitrarily. The ordered $D$-tuple $e_1, \ldots, e_D$ is such that their tangents at $v$, in this sequence, form a matrix of positive determinant.

Requirement (a) prevents the action of the Hamiltonian constraint operator from being trivial. Requirement (b) guarantees that the regulated operator $\hat{H}^\epsilon(N)$ is densely defined for each $\epsilon$. Requirements (c), (d) and (e) specify the triangulation in the neighbourhood of each vertex of $\gamma$ and leave it unspecified outside of them.

The reason why those $D$-simplices lying outside the neighbourhoods of the vertices described above are irrelevant will rest crucially on the choice of ordering with $[\hat{h}_s^{-1}, \hat{V}]$ on the rightmost: if $f$ is a cylindrical function over $\gamma$ and $s$ has support outside the neighbourhood of any vertex of $\gamma$, then $V(\gamma \cup s) - V(\gamma)$ consists of planar at most four-valent vertices only so that $[\hat{h}_s^{-1}, \hat{V}]f = 0$.

We will define our operator on functions cylindrical over coloured graphs, that is, we define it on spin network functions. The domain for the operator that we will choose is a finite linear combination of spin-network functions, hence this defines the operator uniquely as a linear operator. Any operator automatically becomes consistent if one defines it on a basis, the consistency condition simply drops out.

The volume operator will appear in every term of the regulated Hamiltonian constraint. We will choose a factor ordering such that the Hamiltonian constraint acts only on vertices. It is therefore sufficient to regularise the constraint at vertices. As in the usual treatment, we use the tangents to the edges at a vertex as tangent vectors spanning the tangent space of the spatial coordinates. To emphasise this, we will abuse the notation in the following way: Let $e_a(\Delta)$ denote the $D$ edges incident at the vertex $v$ of an analytic $D$-simplex $\Delta \in T(\gamma, \epsilon)$. The matrix consisting of the tangents of the edges $e_1(\Delta), \ldots, e_D(\Delta)$ at $v$ (in that sequence) has non-negative determinant, which induces an orientation of $\Delta$. Furthermore, let $\alpha_{ab}$ be the arc on the boundary of $\Delta$ connecting the endpoints of $e_a(\Delta)$, $e_b(\Delta)$ such that the loop $\alpha_{ab}(\Delta) = e_a(\Delta) \circ a_{ab}(\Delta) \circ e_b(\Delta)^{-1}$ has positive orientation in the induced orientation of the boundary for $a < b$ (modulo cyclic permutation) and negative in the remaining cases.

## 13.2 Key classical identities

The following classical identities are key for the rest of the discussion.

### 13.2.1 $D + 1 \geq 3$ arbitrary

We observe that

$$\sqrt{q}\pi_{aIJ}(x) := -(D-1)\{A_{aIJ}, V(x,\epsilon)\}, \tag{13.2}$$

where $V(x,\epsilon) := \int d^D y \, \chi_\epsilon(x,y)\sqrt{q}$ is the volume of the region defined by $\chi_\epsilon(x,y) = 1$ measured by $q_{ab}$ and $\chi_\epsilon(x,y) = \prod_{a=1}^{D} \Theta(\epsilon/2 - |x^a - y^a|)$ is the characteristic function of a cube of coordinate volume $\epsilon^D$ with centre $x$. Also,

$$n^I(x) n_J(x) \approx \frac{1}{D-1}\left(\pi^{aKI}(x)\pi_{aKJ}(x) - \eta^I{}_J\right). \tag{13.3}$$

We can write the extrinsic curvature terms in the same way as in the usual 3 + 1-dimensional case (“KKEE” terms in this case) , using

$$K(x) := K_a{}^a(x) \approx \frac{D-1}{D}\left\{\mathscr{H}_E(x), V(x,\epsilon)\right\}. \tag{13.4}$$

Further,

$$K_a{}^b(x) \approx \frac{(D-1)}{2D}\pi^{bKL}(x)\left\{A_{aKL}(x), \{\mathscr{H}_E[1](x,\epsilon), V(x,\epsilon)\}\right\} \tag{13.5}$$

gives us access to all the needed terms.

### 13.2.2 $D + 1$ even

Let $n = (D-1)/2$. It is easy to see that

$$\pi^{aIJ}(x) \approx \frac{1}{(D-1)!}\epsilon^{ab_1c_1\ldots b_nc_n}\epsilon^{IJI_1J_1\ldots I_nJ_n}\mathrm{sgn}(\det e)(x)$$
$$\pi_{b_1I_1K_1}(x)\pi_{c_1J_1}{}^{K_1}(x)\ldots\pi_{b_nI_nK_n}(x)\pi_{c_nJ_n}{}^{K_n}(x)\sqrt{q}^{D-1}(x). \tag{13.6}$$

The sign of the determinant of $e_a^I$ where the internal space is the subspace perpendicular to $n^I$ is accessible through

$$\mathrm{sgn}(\det(e_a^I))(x) \approx \frac{1}{2D!}\epsilon^{IJI_1J_1\dots I_nJ_n}\epsilon^{aa_1b_1\dots a_nb_n}\sqrt{q}^{D-1}\pi_{aIJ}(x)$$
$$\pi_{a_1I_1K_1}(x)\pi_{b_1J_1}{}^{K_1}(x)\dots\pi_{a_nI_nK_n}(x)\pi_{b_nJ_n}{}^{K_n}(x). \tag{13.7}$$

For the Euclidean part of the Hamiltonian constraint, we need

$$\frac{\pi^{[a|IK}\pi^{b]J}{}_K}{\sqrt{q}}(x)$$
$$\approx \frac{1}{4(D-2)!}\epsilon^{abca_1b_1\dots a_{n-1}b_{n-1}}\epsilon^{IJKLI_1J_1\dots I_{n-1}J_{n-1}}\mathrm{sgn}(\det e)(x)$$
$$\pi_{cKL}(x)\pi_{a_1I_1K_1}(x)\pi_{b_1J_1}{}^{K_1}(x)\dots\pi_{a_{n-1}I_{n-1}K_{n-1}}(x)\pi_{b_{n-1}J_{n-1}}{}^{K_{n-1}}(x)\sqrt{q}^{D-2}(x). \tag{13.8}$$

Regarding quantisation, we have to choose a classical expression for $\frac{\pi^{[a|IK}\pi^{b]J}{}_K}{\sqrt{q}}(x)$. The above expression would be favourable by arguments of simplicity if it would not contain the additional factor of $\mathrm{sgn}(\det(e_a^I))(x)$ which has to be accounted for. Therefore, we can equally well express the two factors of $\pi^{aIJ}$ separately and absorb the inverse square root into volume operators.

### 13.2.3 $D+1$ odd

Let $n = (D-2)/2$. With only minor modifications of the $D+1$ even case, we get

$$\pi^{aIJ}(x) \approx \frac{1}{(D-1)!}\epsilon^{abb_1c_1\dots b_nc_n}\epsilon^{IJKI_1J_1\dots I_nJ_n}\mathrm{sgn}(\det e)(x)\pi_{bLK}(x)n^L(x)$$
$$\pi_{b_1I_1K_1}(x)\pi_{c_1J_1}{}^{K_1}(x)\dots\pi_{b_nI_nK_n}(x)\pi_{c_nJ_n}{}^{K_n}(x)\sqrt{q}^{D-1}(x) \tag{13.9}$$

with

$$n^I(x) \approx \frac{1}{D!}\epsilon^{a_1b_1\dots a_{n+1}b_{n+1}}\epsilon^{II_1J_1\dots I_{n+1}J_{n+1}}\mathrm{sgn}(\det e)(x)\sqrt{q}^{D-1}(x)$$
$$\pi_{a_1I_1K_1}(x)\pi_{b_1J_1}{}^{K_1}(x)\dots\pi_{a_{n+1}I_{n+1}K_{n+1}}(x)\pi_{b_{n+1}J_{n+1}}{}^{K_{n+1}}(x). \tag{13.10}$$

For the Euclidean part of the Hamiltonian constraint, we need

$$\frac{\pi^{[a|IK}\pi^{b]J}{}_K}{\sqrt{q}} \approx \frac{1}{2(D-2)!}\epsilon^{aba_1b_1\dots a_nb_n}\epsilon^{IJKI_1J_1\dots I_nJ_n}\mathrm{sgn}(\det e)$$
$$n_K\pi_{a_1I_1K_1}\pi_{b_1J_1}{}^{K_1}\dots\pi_{a_nI_nK_n}\pi_{b_nJ_n}{}^{K_n}\sqrt{q}^{D-2} \tag{13.11}$$

and observe that the factor of $\mathrm{sgn}(\det(e_a^I))(x)$ is canceled by another such factor coming from $n^I$. The Euclidean part of the Hamiltonian constraint therefore has the same amount of complexity, measured by the "number of involved operators", in even and odd dimensions.

## 13.3 General scheme

The basic idea of the regularisation of the Hamiltonian constraint operator is to approximate the constraint operator on the graph adapted triangulation and then to take the limit of an infinitely refined triangulation. For this procedure to work, it is mandatory that the constraint operator has a density weight of +1. A typical term of the classical Hamiltonian constraint (or any other operator one wants to regulate) will, after using the above classical identities, consist of

- an integral $\int_\sigma d^D x$,
- $n \in \mathbb{N}_0$ spatial $\epsilon$ symbols,
- factors of $A_{aIJ}(x)$,
- Poisson brackets involving a factor of $A_{aIJ}(x)$ as one of its two arguments as well as either the volume of a neighbourhood of $x$, the Euclidean part of the Hamiltonian constraint smeared with unit lapse over a region containing $x$, or the Poisson bracket of the Euclidean part of the Hamiltonian constraint with the volume, smeared as before, as the other argument,
- field strength tensors,
- a factor of $\sqrt{q}^{1-n}$,
- (covariant) derivatives.

Operators that are well defined on the kinematical Hilbert space are holonomies and the volume operator. We will show in the following that we can construct the Euclidean part of the Hamiltonian constraint operator, which gives us access to the remaining part of the constraint operator. As a start, it is therefore mandatory to write the Euclidean part of the Hamiltonian constraint in terms of holonomies and volume operators. We stress that we do not quantise the $\pi^{aIJ}$ as flux operators, which would also be possible. The reason is that the Hamiltonian constraint operator would not simplify significantly by using fluxes instead of derived flux operators. On the other hand, the appearance of fluxes only through volume operators can be seen as a certain simplification. Anyhow, different regularisations are possible and the discrimination between different regularisations has to be considered in the semiclassical limit.

We begin with rewriting the integral. Given a $D$-tuple of edges $(e_1, \ldots, e_D)$ incident at $v$ with outgoing orientation consider the $D$-simplex $\Delta^\epsilon(\gamma, e_1, \ldots, e_D)$ bounded by the $D$ segments $s^\epsilon_{\gamma,v,e_1}$, $\ldots$, $s^\epsilon_{\gamma,v,e_D}$ incident at $v$ and the $D(D-1)/2$ arcs $a^\epsilon_{\gamma,v,e_a,e_b}$, $1 \leq a < b \leq D$. We now define the "mirror images"

$$\begin{aligned}
s^\epsilon_{\gamma,v,\bar{p}}(t) &:= 2v - s^\epsilon_{\gamma,v,p}(t), \\
a^\epsilon_{\gamma,v,\bar{p},\bar{p}'}(t) &:= 2v - a^\epsilon_{\gamma,v,p,p'}(t), \\
a^\epsilon_{\gamma,v,\bar{p},p'}(t) &:= a^\epsilon_{\gamma,v,\bar{p},\bar{p}'}(t) - 2t[v - s^\epsilon_{\gamma,v,p'}(1)], \\
a^\epsilon_{\gamma,v,p,\bar{p}'}(t) &:= a^\epsilon_{\gamma,v,p,p'}(t) + 2t[v - s^\epsilon_{\gamma,v,p'}(1)],
\end{aligned} \tag{13.12}$$

where $p \neq p' \in e_1, \ldots, e_D$ and we have chosen some parametrisation of segments and arcs. Using the data (13.12) we build $2^D - 1$ more "virtual" $D$-simplices bounded by these quantities so that we obtain altogether $2^D$ $D$-simplices that saturate $v$ and triangulate a neighbourhood $U^\epsilon_{\gamma,v,e_1,\ldots,e_D}$ of $v$. Let $U^\epsilon_{\gamma,v}$ be the union of these neighbourhoods as we vary the ordered $D$-tuple of edges of $\gamma$ incident at $v$. The $U^\epsilon_{\gamma,v}$, $v \in V(\gamma)$ were chosen to be mutually disjoint in point (d) above. Let now

$$\begin{aligned}
\bar{U}^\epsilon_{\gamma,v,e_1,\ldots,e_D} &:= U^\epsilon_{\gamma,v} - U^\epsilon_{\gamma,v,e_1,\ldots,e_D}, \\
\bar{U}^\epsilon_\gamma &:= \sigma - \bigcup_{v \in V(\gamma)} U^\epsilon_{\gamma,v},
\end{aligned} \tag{13.13}$$

then we may write any classical integral (symbolically) as

$$\begin{aligned}
\int_\sigma &= \int_{\bar{U}^\epsilon_\gamma} + \sum_{v \in V(\gamma)} \int_{U^\epsilon_{\gamma,v}} \\
&= \int_{\bar{U}^\epsilon_\gamma} + \sum_{v \in V(\gamma)} \frac{1}{E(v)} \sum_{v = b(e_1)\cap\ldots\cap b(e_D)} \left( \int_{U^\epsilon_{\gamma,v,e_1,\ldots,e_D}} + \int_{\bar{U}^\epsilon_{\gamma,v,e_1,\ldots,e_D}} \right) \\
&\approx \int_{\bar{U}^\epsilon_\gamma} + \sum_{v \in V(\gamma)} \frac{1}{E(v)} \left[ \sum_{v = b(e_1)\cap\ldots\cap b(e_D)} 2^D \int_{\Delta^\epsilon_{\gamma,v,e_1,\ldots,e_D}} + \int_{\bar{U}^\epsilon_{\gamma,v,e_1,\ldots,e_D}} \right],
\end{aligned} \tag{13.14}$$

where in the last step we have noticed that classically the integral over $U^\epsilon_{\gamma,v,e_1,\ldots,e_D}$ converges to $2^D$ times the integral over $\Delta^\epsilon_{\gamma,v,e_1,\ldots,e_D}$, $\approx$ means approximately and $E(v) = \binom{n(v)}{D}$ with $n(v)$ being the valence of the vertex. Now when triangulating the regions of the integrals over $\bar{U}^\epsilon_{\gamma,v,e_1,\ldots,e_D}$ and $\bar{U}^\epsilon_\gamma$ in (13.14), regularisation and quantisation gives operators that vanish on $f_\gamma$ because the corresponding regions do not contain a non-planar vertex of $\gamma$.

As a next step, we approximate the integral

$$\int_{\Delta^\epsilon_{\gamma,v,e_1,\ldots,e_D}} d^D x \, g(x) \approx \frac{1}{D!} \epsilon^D g(v) \tag{13.15}$$

for some function $g(x)$. Here we assumed the coordinate length of each segment $s^{\epsilon}_{\gamma,v,e_a}$ to be $\epsilon$. The general case of arbitrary coordinate length works analogously, since the factors of $\epsilon$ will be hidden in holonomies and derivatives contracted with an epsilon symbol which addresses each segment exactly once. The factor $1/D!$ accounts for the volume of a $D$-simplex. We now multiply the nominator and the denominator by $\epsilon^{D(n-1)}$. Together with the factors $\sqrt{q}^{1-n}(v)$ and the factor $\epsilon^D$ from the integral, we get $\epsilon^{Dn}/V(v,\epsilon)^{n-1}$. The volumes in the denominator are absorbed into the Poisson brackets by the standard technique. The factors of $A_{aIJ}$ are turned into holonomies $(h_{s_a})_{KL} = \delta_{KL} + \epsilon \dot{e}^a(0) A_{aIJ} \left(\tau^{IJ}\right)_{KL} + \mathscr{O}(\epsilon^2)$ using the same amount of factors of $\epsilon$ since we note that the zeroth order of the expansion of the holonomies vanishes when inserted into the Poisson brackets. We abbreviated $s_a = s^{\epsilon}_{\gamma,v,e_a}$ to simplify notation.

The field strength tensors can be dealt with as follows. Let $e, e'$ be arbitrary paths which are images of the interval $[0,1]$ under the corresponding embeddings, which we also denote by $e, e'$ such that $v = e(0) = e'(0)$. For any $0 < \epsilon < 1$ set $e_\epsilon(t) := e(\epsilon t)$ for $t \in [0,1]$ and likewise for $e'$. Then we expand $h_{e_\epsilon}(A)$ in powers of $\epsilon$. Consider the loop $\alpha_{e_\epsilon, e'_\epsilon}$ where in a coordinate neighbourhood

$$\alpha_{e_\epsilon,e'_\epsilon}(t) = \begin{cases} e_\epsilon(4t) & 0 \le t \le 1/4 \\ e_\epsilon(1) + e'_\epsilon(4t-1) - v & 1/4 \le t \le 1/2 \\ e'_\epsilon(1) + e_\epsilon(3-4t) - v & 1/2 \le t \le 3/4 \\ e'_\epsilon(4-4t) & 3/4 \le t \le 1. \end{cases} \tag{13.16}$$

Now expanding again in powers of $\epsilon$ we easily find $h_{\alpha_{e_\epsilon,e'_\epsilon}} = 1_{D+1} + \epsilon^2 F_{abIJ} \tau^{IJ} \dot{e}^a(0) \dot{e}'^b(0) + \mathscr{O}(\epsilon^3)$. Since the indices of the field strength tensors are contracted only with other antisymmetric index pairs, the zeroth order of the expansion vanishes as well as the orders beyond $\epsilon^2$ in the limit $\epsilon \to 0$. The remaining factors of $\epsilon$ are absorbed into covariant derivatives using the approximation

$$\begin{aligned} &\left(h_e(0,\epsilon)\pi^a(e(\epsilon))h_e(0,\epsilon)^{-1} - \pi^a(v)\right)^{AB} \\ =& \Big((1+\epsilon\dot{e}^b(0)A_b)(\pi^b(v) + \epsilon\dot{e}^c(0)\partial_c\pi^b(v))(1-\epsilon\dot{e}^d(0)A_d) - \pi^b(v)\Big)^{AB} + \mathscr{O}(\epsilon^2) \\ =& \epsilon\dot{e}^c(0) D^A{}_c \pi^{aAB}(v) + \mathscr{O}(\epsilon^2). \end{aligned} \tag{13.17}$$

We note that partial derivatives can be dealt with in the same way.

At this point, all factors of $\epsilon$ have been absorbed into holonomies and derivatives. It is key that the volume operators are ordered to the right in the quantum theory since then, the Hamiltonian constraint evaluated on a cylindrical function $f_\gamma$ will only act on the vertices of $\gamma$. The action at vertices however does not depend on the value of $\epsilon > 0$ and we can take the limit $\epsilon \to 0$, thus removing the regulator.

In order to quantise the Hamiltonian constraint, we have to replace the holonomies by multiplication operators, the volumes by volume operators, and the Poisson brackets by $i/\hbar$ times the commutator.

## 13.4 Regularised quantities

In order to construct a well defined Hamiltonian constraint operator, we have to express it in terms of operators well defined on the kinematical Hilbert space. Instead of writing down the explicit regularisation for the proposed Hamiltonian constraint, we want to provide a toolkit for a general class of operators. In the following, we will propose "regulated" versions of the phase space variables, marked by an upper $^\epsilon$ in front. The idea will be to replace all phase space variables in the classical Hamiltonian constraint by their corresponding regulated versions, do some additional minor modifications and directly arrive at the Hamiltonian constraint operator, without explicitly dealing with the triangulation and the correct powers of $\epsilon$. Since the final constraint operator will only act on vertices of $\gamma$, it is sufficient to regularise the phase space variables at vertices $v$.

In what follows, we use a graph adapted coordinate system, meaning that the spatial coordinates $a, b, \ldots = 1, \ldots, D$ enumerate the $D$ edges incident at $v$ of a $D$-simplex.

### 13.4.1 $D + 1 \geq 3$ arbitrary

We will express all the basic variables in terms of holonomies living on the edges of the adapted triangulation and volume operators acting on it. First, we notice that

$$^\epsilon(\sqrt{q}^{x+1}\pi_{aIJ}(v)) := \frac{(D-1)}{(x+1)}(h_{s_a})_I{}^K\{(h_{s_a})^{-1}_{KJ}, (V(v,\epsilon))^{x+1}\} \tag{13.18}$$

is gauge covariant and reduces to $\epsilon\sqrt{q}^{x+1}\pi_{aIJ}(v)$ in the limit $\epsilon \to 0$. The factor of $\epsilon$ is expected as the regulated quantity has a lower spatial index. In the end, when the complete constraint operator will be assembled, all factors of $\epsilon$ will cancel out. We restrict $x > -1$ because powers of the volume operator will be defined by the spectral theorem in the quantum theory.

For the extrinsic curvature terms, we propose

$$
\begin{aligned}
&{}^{\epsilon}\!\left(\frac{1}{2\sqrt{q}}(K_a{}^b K_b{}^a - (K_c{}^c)^2)\right)\\
&\approx \frac{(D-1)^2}{4D^2}\,{}^{\epsilon}(\sqrt[4]{q}^{-1}\pi^{[a|KL}(v))(h_{s_a})_K{}^O\left\{(h_{e_a})^{-1}_{OL}, \{\mathscr{H}_E[1](v,\epsilon), V(v,\epsilon)\}\right\}\\
&\qquad\times {}^{\epsilon}(\sqrt[4]{q}^{-1}\pi^{b]MN}(v))(h_{s_b})_M{}^P\left\{(h_{e_b})^{-1}_{PN}, \{\mathscr{H}_E[1](v,\epsilon), V(v,\epsilon)\}\right\},
\end{aligned}
\tag{13.19}
$$

where the ${}^{\epsilon}\pi^{aIJ}$ will be defined below.

Next, we regulate the gauge unfixing term $\mathscr{D}F^{-1}\mathscr{D}$ with density weight 1. We will place zero density into $F^{-1}$ and a density weight of 1/2 into each $\mathscr{D}$. Accordingly,

$$
\sqrt{q}^4\left(F^{-1}\right)^{\overline{N}\,\overline{M}}_{cd,ab} = \alpha\sqrt{q}^4\epsilon^{EFGH\overline{N}}\pi_{(c|EF}\left(F^{-1}\right)_{d)GH,(a|AB}\pi_{b)CD}\epsilon^{ABCD\overline{M}} \tag{13.20}
$$

becomes

$$
{}^{\epsilon}\left(\sqrt{q}^4F^{-1}\right)^{\overline{N}\,\overline{M}}_{cd,ab} = \alpha\epsilon^{EFGH\overline{N}}\,{}^{\epsilon}(\sqrt{q}\pi_{(c|EF})\,{}^{\epsilon}\left(\sqrt{q}^2F^{-1}\right)_{d)GH,(a|AB}\,{}^{\epsilon}(\sqrt{q}\pi_{b)CD})\epsilon^{ABCD\overline{M}} \tag{13.21}
$$

with

$$
\begin{aligned}
{}^{\epsilon}\left(\sqrt{q}^2F^{-1}\right)_{aIJ,bKL} :=& \frac{1}{(D-1)}\,{}^{\epsilon}(\sqrt{q}\pi_{aAC})\,{}^{\epsilon}(\sqrt{q}\pi_{bBD})\left({}^{\epsilon}(\sqrt{q}^{-1}\pi^{cEC})\,{}^{\epsilon}(\sqrt{q}\pi_{cE}{}^{D}) + \eta^{CD}\right)\\
&\left(\eta^{AB}\eta^{K[I}\eta^{J]L} - 2\eta^{A[L}\eta^{K][J}\eta^{I]B}\right),
\end{aligned}
\tag{13.22}
$$

cf. (8.40). The $\mathscr{D}$ constraint contains a covariant derivative which we regularise as

$$
{}^{\epsilon}(\sqrt{q}^{-1}D^A{}_a\pi^{bAB}) := \left(h_{s_a}{}^{\epsilon}(\sqrt{q}^{-1}\pi^b(s_a))h^{-1}_{s_a} - {}^{\epsilon}(\sqrt{q}^{-1}\pi^b(v))\right)^{AB}. \tag{13.23}
$$

The full $\mathscr{D}$ constraint

$$
\mathscr{D}^{ab}_{\overline{M}} = -\epsilon_{IJKL\overline{M}}\pi^{cIJ}\left(\pi^{(a|KN}D^A{}_c\pi^{b)L}{}_N\right) \tag{13.24}
$$

can thus be regularised as

$$
{}^{\epsilon}(\sqrt{q}^{-3/2}\mathscr{D}^{ab}_{\overline{M}}) = -\epsilon_{IJKL\overline{M}}\,{}^{\epsilon}(\sqrt{q}^{-1/2}\pi^{cIJ})\left({}^{\epsilon}(\sqrt{q}^{-1}\pi^{(a|KN})\,{}^{\epsilon}(\sqrt{q}^{-1}D^A_c\pi^{b)L}{}_N)\right). \tag{13.25}
$$

In the paper [3], a second regularisation of the $\mathscr{D}F^{-1}\mathscr{D}$ part of the Hamiltonian constraint is given, which rest on the classic relation[1]

$$
2D_{[a}\sqrt{q}\pi_{b]IJ}(x) = -(D-1)\{F_{abIJ}(x), V(x,\epsilon)\}. \tag{13.26}
$$

[1] Using this was suggested by Wieland [213].

The resulting part in the Hamiltonian constraint is quadratic in the field strength and therefore this procedure results in a more non-local operation of the Hamiltonian constraint. We refer the interested reader to the original literature.

In general, a generic power of $1/\sqrt{q}$ needed to turn the individual terms with densities $> 1$ into densities of weight 1 can be constructed as

$$^{\epsilon}\left(\frac{1}{\sqrt{q}^{(-2xD-2)}}\right) \approx \left(\frac{1}{2}\right)^{D} \det\left({}^{\epsilon}(\sqrt{q}^{x+1}\pi_{aIJ})^{\epsilon}(\sqrt{q}^{x+1}\pi_{b}{}^{IJ})\right) \tag{13.27}$$

with the usual $x > -1$.

The field strength tensors are regularised as

$$^{\epsilon}F_{abIJ} = \left(h_{\alpha_{s_a,s_b}}\right)_{KL} \delta^{K}_{[I}\delta^{L}_{J]} \tag{13.28}$$

while we set

$$^{\epsilon}\{A_{aIJ}(v), \cdot\} = -(h_{s_a})_{I}{}^{K}\{(h_{s_a}{}^{-1})_{KJ}, \cdot\}. \tag{13.29}$$

### 13.4.2 $D + 1$ even

Let $n = (D-1)/2$. We "regulate"

$$\begin{aligned} {}^{\epsilon}(\sqrt{q}^{(D-1)x}\pi^{aIJ}(v)) \approx \frac{1}{(D-1)!}&\epsilon^{ab_1c_1\ldots b_nc_n}\epsilon^{IJI_1J_1\ldots I_nJ_n}\mathrm{sgn}(\det e)(v) \\ &{}^{\epsilon}(\sqrt{q}^{(1+x)}\pi_{b_1I_1K_1}(v))^{\epsilon}(\sqrt{q}^{(1+x)}\pi_{c_1J_1}{}^{K_1}(v))\ldots \\ &{}^{\epsilon}(\sqrt{q}^{(1+x)}\pi_{b_nI_nK_n}(v))^{\epsilon}(\sqrt{q}^{(1+x)}\pi_{c_nJ_n}{}^{K_n}(v)) \end{aligned} \tag{13.30}$$

and

$$\begin{aligned} {}^{\epsilon}(\mathrm{sgn}(\det(e^I_a))) \approx \frac{1}{2D!}&\epsilon^{IJI_1J_1\ldots I_nJ_n}\epsilon^{aa_1b_1\ldots a_nb_n}{}^{\epsilon}(\sqrt{q}^{(D-1)/D}\pi_{aIJ}) \\ &{}^{\epsilon}(\sqrt{q}^{(D-1)/D}\pi_{a_1I_1K_1})^{\epsilon}(\sqrt{q}^{(D-1)/D}\pi_{b_1J_1}{}^{K_1})\ldots \\ &{}^{\epsilon}(\sqrt{q}^{(D-1)/D}\pi_{a_nI_nK_n})^{\epsilon}(\sqrt{q}^{(D-1)/D}\pi_{b_nJ_n}{}^{K_n}). \end{aligned} \tag{13.31}$$

For the Euclidean part of the Hamiltonian constraint, we need

$$\begin{aligned} {}^{\epsilon}\left(\frac{\pi^{[a|IK}\pi^{b]J}{}_{K}}{\sqrt{q}}\right) \approx &\frac{1}{4(D-2)!}\epsilon^{abca_1b_1\ldots a_{n-1}b_{n-1}}\epsilon^{IJKLI_1J_1\ldots I_{n-1}J_{n-1}}\mathrm{sgn}(\det e) \\ &{}^{\epsilon}(\sqrt{q}\pi_{cKL})^{\epsilon}(\sqrt{q}\pi_{a_1I_1K_1})^{\epsilon}(\sqrt{q}\pi_{b_1J_1}{}^{K_1})\ldots \\ &{}^{\epsilon}(\sqrt{q}\pi_{a_{n-1}I_{n-1}K_{n-1}})^{\epsilon}(\sqrt{q}\pi_{b_{n-1}J_{n-1}}{}^{K_{n-1}}). \end{aligned} \tag{13.32}$$

As stressed before, the two possibilities to express the Euclidean part of the Hamiltonian constraint are equally complicated.

### 13.4.3 $D+1$ odd

Let $n = (D-2)/2$. We "regulate"

$$
{}^{\epsilon}(\sqrt{q}^{(D-1)x}\pi^{aIJ}(v)) \approx \frac{1}{(D-1)!}\epsilon^{abb_1c_1\ldots b_nc_n}\epsilon^{IJKI_1J_1\ldots I_nJ_n}\mathrm{sgn}(\det e)(v)\,{}^{\epsilon}(\sqrt{q}^{(1+x)}\pi_{bLK}(v))
\\ {}^{\epsilon}n^L(v)\,{}^{\epsilon}(\sqrt{q}^{(1+x)}\pi_{b_1I_1K_1}(v))\,{}^{\epsilon}(\sqrt{q}^{(1+x)}\pi_{c_1J_1})^{K_1}(v)\ldots
\\ {}^{\epsilon}(\sqrt{q}^{(1+x)}\pi_{b_nI_nK_n}(v))\,{}^{\epsilon}(\sqrt{q}^{(1+x)}\pi_{c_nJ_n}{}^{K_n}(v)) \tag{13.33}
$$

and

$$
{}^{\epsilon}n^I(v) \approx \frac{1}{D!}\epsilon^{a_1b_1\ldots a_{n+1}b_{n+1}}\epsilon^{II_1J_1\ldots I_{n+1}J_{n+1}}\mathrm{sgn}(\det e)(v)
\\ {}^{\epsilon}(\sqrt{q}^{(D-1)/D}\pi_{a_1I_1K_1}(v))\,{}^{\epsilon}(\sqrt{q}^{(D-1)/D}\pi_{b_1J_1}{}^{K_1}(v))\ldots
\\ {}^{\epsilon}(\sqrt{q}^{(D-1)/D}\pi_{a_{n+1}I_{n+1}K_{n+1}}(v))\,{}^{\epsilon}(\sqrt{q}^{(D-1)/D}\pi_{b_{n+1}J_{n+1}}{}^{K_{n+1}}(v)). \tag{13.34}
$$

For the Euclidean part of the Hamiltonian constraint, we need

$$
{}^{\epsilon}\left(\frac{\pi^{[a|IK}\pi^{b]J}{}_K}{\sqrt{q}}\right) \approx \frac{1}{2(D-2)!}\epsilon^{aba_1b_1\ldots a_nb_n}\epsilon^{IJKI_1J_1\ldots I_nJ_n}\mathrm{sgn}(\det e) \tag{13.35}
\\ {}^{\epsilon}(n_K)\,{}^{\epsilon}(\sqrt{q}\pi_{a_1I_1K_1})\,{}^{\epsilon}(\sqrt{q}\pi_{b_1J_1}{}^{K_1})\ldots{}^{\epsilon}(\sqrt{q}\pi_{a_nI_nK_n})\,{}^{\epsilon}(\sqrt{q}\pi_{b_nJ_n}{}^{K_n}).
$$

## 13.5 The Hamiltonian constraint operator

At this point, we are ready to assemble the Hamiltonian constraint operator. The general idea of the regularisation has been described in section 13.3. Here, we provide a toolkit in order to assemble the constraint operator.

(1) The "Euclidean part" $\frac{1}{\sqrt{q}}\mathscr{H}_E = -\frac{1}{2\sqrt{q}}\pi^{aIK}\pi^{bJ}{}_KF_{abIJ}$ of the Hamiltonian constraint can be quantised with the methods described above and using the following recipe. The corresponding operator can then be used in commutators to express additional parts of the full Hamiltonian constraint operator.

(2) Use classical identities in order to express the Hamiltonian constraint in terms of connections $A_{aIJ}$, volumes $V(x,\epsilon)$ and Euclidean Hamiltonian constraints $\mathscr{H}_E(x,\epsilon)$.

(3) Replace all phase space variables by their corresponding regulated quantities.

(4) Instead of the integration $\int_\sigma d^Dx$, put a sum $\frac{1}{D!}\sum_{v\in V(\gamma)}$ over all the vertices $v$ of the graph $\gamma$.

(5) For every spatial $\epsilon$-symbol, put a sum $\frac{2^D}{E(v)} \sum_{v(\Delta)=v}$ over all $D$-simplices having $v$ as a vertex. The holonomies associated with the $\epsilon$-symbol are evaluated along the edges spanning $\Delta$.

(6) Substitute the Poisson brackets by $\frac{i}{\hbar}$ times the commutator of the corresponding operators, i.e. the multiplication operator $\hat{h}_e$ and the volume operator $\hat{V}$.

In order to understand the double sum over $D$-simplices appearing in the $KKEE$ and the gauge unfixing term, consider the following argument given in a similar form in [46]: Since $\lim_{\epsilon\to 0}(1/\epsilon^D)\chi_\epsilon(x,y) = \delta^D(x,y)$ we have $\lim_{\epsilon\to 0}(1/\epsilon^D)V(x,\epsilon) = \sqrt{q}(x)$. It is also easy to see that for each $\epsilon > 0$ we have that $\delta V/\delta\pi^{aIJ}(x) = \delta V(x,\epsilon)/\delta\pi^{aIJ}(x)$. The terms under consideration are of the form

$$\int d^D x \, \frac{\sqrt{q}(x)\pi_{aIJ}(x)Z^{aIJ}(x)\sqrt{q}(x)\pi_{bKL}(x)Z^{bKL}(x)}{\sqrt{q}(x)}, \tag{13.36}$$

where $Z^{aIJ}$ is a density of weight $+1$ and stands symbolically for the remaining terms, including a spatial $\epsilon$-symbol with upper indices, one of which is $a$. We rewrite this expression as

$$\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{\epsilon^D}4(D-1)^2\int d^D x\,\frac{\{A_{aIJ}(x),V\}Z^{aIJ}(x)}{2\sqrt[4]{q}(x)}\int d^D y\,\chi_\epsilon(x,y)\frac{\{A_{bKL}(y),V\}Z^{bKL}(y)}{2\sqrt[4]{q}(y)}\\
=&\lim_{\epsilon\to 0}\frac{1}{\epsilon^D}4(D-1)^2\int d^D x\,\frac{\{A_{aIJ}(x),V(x,\epsilon)\}Z^{aIJ}(x)}{2\sqrt[4]{q}(x)}\\
&\qquad\qquad\int d^D y\,\chi_\epsilon(x,y)\frac{\{A_{bKL}(y),V(y,\epsilon)\}Z^{bKL}(y)}{2\sqrt[4]{q}(y)}\\
=&\lim_{\epsilon\to 0}\frac{1}{\epsilon^D}4(D-1)^2\int d^D x\,\frac{\{A_{aIJ}(x),V(x,\epsilon)\}Z^{aIJ}(x)}{2\sqrt{V(y,\epsilon)/\epsilon^D}}\\
&\qquad\qquad\int d^D y\,\chi_\epsilon(x,y)\frac{\{A_{bKL}(y),V(y,\epsilon)\}Z^{bKL}(y)}{2\sqrt{V(y,\epsilon)/\epsilon^D}}\\
=&\lim_{\epsilon\to 0}4(D-1)^2\int d^D x\,\frac{\{A_{aIJ}(x),V(x,\epsilon)\}Z^{aIJ}(x)}{2\sqrt{V(y,\epsilon)}}\\
&\qquad\qquad\int d^D y\,\chi_\epsilon(x,y)\frac{\{A_{bKL}(y),V(y,\epsilon)\}Z^{bKL}(y)}{2\sqrt{V(y,\epsilon)}}\\
=&\lim_{\epsilon\to 0}4(D-1)^2\int d^D x\,\{A_{aIJ}(x),\sqrt{V(x,\epsilon)}\}Z^{aIJ}(x)\\
&\qquad\qquad\int d^D y\,\chi_\epsilon(x,y)\{A_{bKL}(y),\sqrt{V(y,\epsilon)}\}Z^{bKL}(y).
\end{aligned}\tag{13.37}$$

Triangulation leads to two sums over vertices and two sums over $D$-simplices containing the individual vertices. In the limit $\epsilon \to 0$ however the two sums over vertices collapse to a single sum over vertices due to the $\chi_\epsilon$ term and we have the desired result.

## 13.6 Solution of the Hamiltonian constraint

As in the $3+1$-dimensional treatment, we realise that the only spin changing operation of the Hamiltonian constraint is performed by its Euclidean part. The construction of a set of rigorously defined solutions to the diffeomorphism and the Hamiltonian constraint described in [31] thus immediately generalises to our case.

## 13.7 Master constraint

The implementation of the master constraint

$$\boldsymbol{M} = \frac{1}{2}\int_\sigma d^D x \, \frac{\mathscr{H}(x)^2}{\sqrt{q}(x)} \tag{13.38}$$

works analogously to the $3+1$-dimensional case described in [245]. The inverse square root is split up between the two Hamiltonian constraints and hidden by adjusting the power of the volume operators as before. The result of the derivation is the master constraint operator

$$\hat{\boldsymbol{M}} T_{[s]} := \sum_{[s_1]} Q_{\boldsymbol{M}}(T_{[s_1]}, T_{[s]}) T_{[s_1]} \tag{13.39}$$

with

$$Q_{\boldsymbol{M}}(l, l') = \sum_{[s]} \eta_{[s]} \sum_{v \in V(\gamma(s_0[s]))} \overline{l(\hat{C}_v^\dagger T_{s_0([s])})} l'(\hat{C}_v^\dagger T_{s_0([s])}) \tag{13.40}$$

and $l(\hat{C}_v^\dagger T_{s_0([s])})$ being the evaluation of $l$ on the Hamiltonian constraint operator with the additional $1/\sqrt[4]{q}$ hidden in the volume operator(s). The proof of the following theorem generalises with obvious modifications from the treatment in [62].

**Theorem 6.**

*(i) The positive quadratic form $Q_{\boldsymbol{M}}$ is closable and induces a unique, positive self-adjoint operator $\hat{\boldsymbol{M}}$ on $\mathscr{H}_{diff}$.*

*(ii) Moreover, the point zero is contained in the point spectrum of $\hat{\boldsymbol{M}}$.*

We deal with the problem of $\mathscr{H}_{\text{diff}}$ not being separable by using $\theta$-equivalence classes of spin-networks, see [245]. Now, a direct integral decomposition of $\mathscr{H}^\theta_{\text{diff}}$ is available:

**Theorem 7.**
*There is a unitary operator $V$ such that $V\mathscr{H}^{\theta}_{diff}$ is the direct integral Hilbert space*

$$\mathscr{H}^{\theta}_{diff} \propto \int_{\mathbb{R}+}^{\oplus} d\mu(\lambda)\, \mathscr{H}^{\theta}_{diff}(\lambda) \tag{13.41}$$

*where the measure class of $\mu$ and the Hilbert space $\mathscr{H}^{\theta}_{diff}(\lambda)$, in which $V\hat{\boldsymbol{M}}V^{-1}$ acts by multiplication by $\lambda$, are uniquely determined.*
*The physical Hilbert space is given by $\mathscr{H}^{\theta}_{phys} = \mathscr{H}^{\theta}_{diff}(0)$.*

We notice that we could define an extended master constraint that also involves the simplicity constraint.

## 13.8 Factor ordering

In [247, 248], it has been shown that there is a unique factor ordering which results in a non-vanishing flux operator expressed through the volume operator and holonomies in the usual $3+1$ dimensional LQG. The idea, translated to our case, is that the volume operator in the expression for ${}^{\epsilon}\pi^{aIJ}$ has to act on an at least $D$-valent non-planar vertex and the holonomies in the expression have to be ordered to the right for this to be ensured. Apart from ordering individual terms of the sums appearing differently (which would be highly unnatural), this leaves only one possible factor ordering. We remark that the proof of the equivalence of the "normal" and "derived" flux operator given in [247, 248] does not generalise trivially to our case since it is explicitly based on SU(2) as the internal gauge group. We leave this point open for further research.

In order to ensure that the Hamiltonian constraint only acts on vertices, we order in all three terms either a commutator $[\hat{h}_e^{-1}, \hat{V}]$ or a double-commutator $[\hat{h}_e^{-1}, [\mathscr{H}_E, \hat{V}]$ to the right.

We leave the remaining details of the factor ordering open, as here we only intend to show that a quantisation is possible in principle.

## 13.9 Outlook on consistency checks

At this point, one might ask if there are good indications whether the proposed theory is physically viable. In case of the usual formulation of LQG in terms of Ashtekar-Barbero variables, it was shown in [69] that a quantisation of Euclidean general relativity in three dimensions with methods very similar to the ones used in LQG recovers the known solutions of three-dimensional general relativity familiar from other approaches. The reason why these theories match is that they both use the gauge groups SU(2) and that a suitable redefinition of the Lagrange multipliers of Euclidean three-dimensional general relativity leads to a Hamiltonian constraint with the same algebraic structure as the Euclidean part of the constraint familiar from LQG. A similar check is conceivable for the presented theory in that we can describe Lorentzian three-dimensional general relativity using SU(2) as a gauge group, which would result in a different Hamiltonian constraint. One could now check if the solution space of Lorentzian three-dimensional general relativity is reproduced when using SU(2) as a gauge group and thus mimicking the internal signature switch which is also done in this formulation. As for the simplicity constraint, we cannot use three-dimensional general relativity as a testbed since the simplicity constraints only appear in four and higher dimensions.

Another approach to consistency checks is to compare our formulation in four dimensions to the usual LQG formulation. In section 12.1, the area operator was shown to have the same spectrum as in standard LQG, which however does not come as a surprise regarding similar results from spin foam models. As for the volume operator, we do not know whether the spectrum matches the one of standard LQG. This is also tied to the fact that we are only interested in the spectrum on the solution space to the vertex simplicity constraint operators, for which we do not have a completely satisfactory proposal. We remark that a matching spectrum of the volume operator can be obtained by using a weak implementation of the linear vertex simplicity constraints [253], but as we have seen, the linear constraint comes with its own problems in the canonical theory.

# Part IV

# Inclusion of matter and extension to supersymmetry

In the previous chapters, we introduced a new connection formulation of vacuum general relativity with compact gauge group in any spacetime dimension $D + 1 \geq 3$ and its loop quantisation. Now we consider coupling of this theory to matter. Concerning standard matter, it will suffice to consider Dirac fermions because gauge bosons and scalar fields can be coupled in the same way as it has been done in $3 + 1$ dimensions already [46]: Scanning through the details of [46] one realises that nothing depends substantially on $D = 3$ and we can consider the gauge boson and scalar sector as treated already. However, for supergravity theories, also non-standard matter fields, most prominently, the "gravitino" in fermionic sector, which has spin 3/2 and usually is a Majorana fermion, and Abelian higher $p$-form gauge fields in the bosonic sector, appear (and more).

Therefore, in the first chapter 14 of this part, we will derive a connection formulation of Lorentzian general relativity coupled to Dirac fermions in dimensions $D+1 \geq 3$ with compact gauge group. The technique that accomplishes that is similar to the one that has been introduced in $3 + 1$ dimensions already: First one performs a canonical analysis of Lorentzian general relativity coupled to the Dirac field using the time gauge and then introduces an extension of the phase space analogous to the one employed in chapter 7 to obtain a connection theory with $\mathrm{SO}(D + 1)$ as the internal gauge group subject to additional constraints. The success of this method rests heavily on the strong similarity of the Lorentzian and Euclidean Clifford algebras. A quantisation of the Hamiltonian constraint is provided. The presentation is taken from [4] with only minor modifications.

In chapter 15, we will finally turn to non-standard matter fields needed for the extension to supersymmetric theories. Since the focus of this thesis is on the higher dimensional extension of LQG, we will only briefly summarise the findings of our original articles [6, 7] with results towards this goal: In section 15.1, we will follow [6] in performing an analysis of the spin 3/2 Rarita Schwinger field ("gravitino"). This field usually is a Majorana fermion, i.e. belongs to real representation spaces of $\mathrm{SO}(1, D)$. The obstacle that there is no action of $\mathrm{SO}(D + 1)$ on these representation spaces is circumvented by introducing an auxiliary unit vector field $N^I$ and to define an action of $\mathrm{SO}(D + 1)$ on a combined objected formed by this field and the Majorana fermion. The additionally introduced degrees of freedom introduced with this field are naturally removed by using the linear simplicity constraint. We construct a background independent Hilbert space representation for the real valued Majorana spinor fields that implements its canonical Dirac anti-bracket and $^*$-relations. To this end, a new method needed to be developed since the treatment of the Dirac field does not carry over because the $^*$-algebra is different.

Afterwards in section 15.2, we will sketch the study of [7] of the three-form gauge field of $d = 11$ $N = 1$ supergravity as an example of an Abelian higher $p$-form field. Due to an additional Chern Simons term in the supergravity action, a straightforward generalisation to higher form degree of the usual loop quantisation procedure fails. We propose a reduced phase space quantisation instead: We compute the algebra of the Weyl elements corresponding to a full set of Dirac observables with respect to the (generalisation of the) Gauß constraint and show that it allows for a state of the Narnhofer-Thirring type.

While the fields we study allow for a loop quantisation of a large class of Lorentzian supergravity theories in diverse dimensions, including the $d = 4$ $N = 8$, $d = 10$ $N = 1$, and $d = 11$ $N = 1$ supergravities, the study is far from complete. We refer the reader to section 18.2 for open problems and suggestions for further research.

# 14

# Inclusion of Dirac fermions

Our starting point is the standard canonical treatment of Dirac fermions coupled to general relativity. To the best of our knowledge, Kibble [254] was the first to consider the canonical formulation of fermions coupled to vierbein gravity. The classical coupling of fermions to the Ashtekar's new variables [13] was provided in [255]. Since then, several papers appeared debating issues arising when including fermions. Among others, the role of the Immirzi parameter [256], the appearance of torsion [257, 258] and the correct form of the Holst modification [259] are ongoing debates. Here we will consider the simplest possibility, namely the standard coupling of Dirac fermions to vielbein gravity.

In $3+1$ dimensions the quantisation of this theory was carried out for the first time in the context of Ashtekar's new variables in [45, 46]. The new ingredient was the passage to Graßmann valued half densities and a representation in terms of holomorphic wave functions of the fermionic variables. Technically, in $3+1$ dimensions one works in the time gauge and with the Ashtekar Barbero connection which can be obtained by an extension of the ADM phase space subject to an SU(2) Gauß constraint.

In higher dimensions, an Ashtekar Barbero like connection is not available and therefore a new idea is needed in order to arrive at a connection formulation with *compact gauge group* although we are considering *Lorentzian* gravity. We start from the usual Dirac - Palatini Lagrangian for Lorentzian general relativity and introduce the time gauge. This results in a formulation in terms of a canonical pair $(K_a^i, E_i^a)$, $a, b, c, .. = 1, .., D$; $i, j, k, .. = 1, .., D$ which is subject to an SO($D$) Gauß constraint (cf. section 3.2.4 for the corresponding vacuum formulation). We now extend this phase space by a canonical pair $(A_{aIJ}, \pi^{aIJ})$ subject to the simplicity and SO($D+1$) Gauß constraint like in part II. This way we arrive at a connection formulation in terms of the compact gauge group $\mathrm{SO}(D+1)$ although we are considering Lorentzian gravity. Of course, the fermionic contribution to the Hamiltonian constraint of Lorentzian gravity, just as in $3+1$ dimensions, acquires correction terms as compared to its Euclidean counterpart which in part is due to switching from Lorentzian to Euclidean $\gamma$ matrices. Yet, these corrections are not as cumbersome as one might expect because the Lorentzian Clifford algebra differs from the Euclidean one just by a factor of $i$ in front of $\gamma^0$.

After having obtained the fermionic contributions to the classical constraints we quantise them using standard methods [46] and using the representation [45].

## 14.1 Canonical analysis of Lorentzian gravity coupled to Dirac fermions

As opposed to pure gravity where, in the end, it does not matter whether one starts with a first or a second order formulation of the theory, this choice results in inequivalent theories when dealing with fermions. The reason for this is that the torsion freeness condition which one derives when starting with first order general relativity is modified by a term quadratic in the fermions, thus resulting in a non-vanishing torsion. At the end of the canonical analysis, one arrives at the same set of variables, but, after solving the equations of motion for the torsion part of the connection, one obtains more interaction terms, most prominently four-fermion interactions, which are not present in the theory when starting with a second order formulation. To the best of the author's knowledge, it is unclear which type of action should be preferred on physical grounds. The second order variant leads to less interaction terms and could thus be preferred by demanding simplicity. On the other hand, when deriving the Ashtekar Barbero variables from the Holst action, one deals in a first order framework and one could thus consider it more natural to choose this route. Here, we will choose the first order approach since the results of the canonical analysis in part II can be nicely used in order to deal with the torsion terms. For further literature on this topic, we refer to [46, 255, 258].

We start with the first order action

$$S_{\mathrm{G+F}} = -\int_{\mathscr{M}} d^{D+1}x \left( \frac{1}{2} e e^{\mu I} e^{\nu J} F_{\mu\nu IJ}(A) + \frac{i}{2} \overline{\Psi} e^{\mu}_{I} \gamma^{I} \nabla^{A}{}_{\mu} \Psi - \frac{i}{2} \overline{\nabla^{A}{}_{\mu} \Psi} e^{\mu}_{I} \gamma^{I} \Psi \right). \quad (14.1)$$

$\Psi$ denotes a Graßmann valued Dirac spinor, $\overline{\Psi} = \Psi^\dagger \gamma^0$ and $\nabla^A{}_\mu \Psi = \partial_\mu \Psi + \frac{i}{2} A_{aIJ} \Sigma^{IJ} \Psi$, $\Sigma^{IJ} = -\frac{i}{4}[\gamma^I, \gamma^J]$. Spinor indices will be mostly suppressed. The properties of the $\gamma$ matrices are summarised in appendix E. The remaining notation is as before, $e^{\mu I}$ denotes the vielbein and $F_{\mu\nu IJ} := \partial_\mu A_{\nu IJ} - \partial_\nu A_{\mu IJ} + [A_\mu, A_\nu]_{IJ}$ is the field strength of the $\mathrm{SO}(1,D)$ connection $A_{\mu IJ}$. The gravitational part of this action has already been analysed in part II, we will therefore concentrate on the fermionic part.

The split in space and time is performed analogously to the $D = 3$ case (cf. section 3.2.1) and we additionally choose the time gauge prior to the canonical analysis (the time gauge is a canonical gauge, see, for instance, [161]) by setting $n^I = \delta^I_0$. The split form of the action is found to be

$$S_{\mathrm{G+F}} = \int dt \int_\sigma d^D x \left( \dot{E}^a_i K^i_a + i(\sqrt[4]{q}\Psi^\dagger)(\sqrt[4]{q}\Psi)^{\cdot} - N\mathscr{H} - N^a \mathscr{H}_a - \lambda_{ij}\mathscr{G}^{ij} - (\lambda_i + *_i)\mathscr{G}^i \right), \tag{14.2}$$

where

$$\begin{aligned}
\mathscr{H} = {} & \frac{1}{2}\sqrt{q}R + \frac{1}{\sqrt{q}} E^{[a|i} E^{b]j} K_{ai} K_{bj} + \frac{1}{\sqrt{q}} \frac{1}{8} \bar{K}^{\mathrm{tf}}_{aij} F^{aij,bkl} \bar{K}^{\mathrm{tf}}_{bkl} \\
& + \frac{i}{2}\frac{1}{\sqrt{q}} \overline{\sqrt[4]{q}\Psi} E^a_i \gamma^i D^\Gamma{}_a(\sqrt[4]{q}\Psi) - \frac{i}{2}\frac{1}{\sqrt{q}} \overline{D^\Gamma{}_a(\sqrt[4]{q}\Psi)} E^a_i \gamma^i (\sqrt[4]{q}\Psi) \\
& - \frac{1}{\sqrt{q}} (\sqrt[4]{q}\Psi)^\dagger \Sigma^{ij} (\sqrt[4]{q}\Psi) K_{ai} E^a_j - \frac{1}{4}\frac{1}{\sqrt{q}} \overline{\sqrt[4]{q}\Psi} E^a_k \left\{ \gamma^k, \Sigma^{ij} \right\} (\sqrt[4]{q}\Psi) \bar{K}^{\mathrm{tf}}_{aij},
\end{aligned} \tag{14.3}$$

$$\mathscr{H}_a = -2E^{bj} D^\Gamma{}_{[a} K_{b]j} + \frac{i}{2}(\sqrt[4]{q}\Psi)^\dagger D^\Gamma{}_a(\sqrt[4]{q}\Psi) - \frac{i}{2}(D^\Gamma{}_a(\sqrt[4]{q}\Psi))^\dagger(\sqrt[4]{q}\Psi) + \frac{1}{2}\bar{K}^{\mathrm{tf}}_{aij}\mathscr{G}^{ij}, \tag{14.4}$$

$$\mathscr{G}^{ij} = 2K^{[i}_a E^{a|j]} - (\sqrt[4]{q}\Psi)^\dagger \Sigma^{ij} (\sqrt[4]{q}\Psi), \tag{14.5}$$

$$\mathscr{G}_i = \bar{K}^{\mathrm{tr}}_i, \tag{14.6}$$

and small Latin indices $i, j, k, \ldots = 1, \ldots, D$ are internal indices in the time gauge. $F^{aij,bkl}$ and the derivation of the symplectic structure have been described in chapter 4. We have decomposed $A_{aIJ} = \Gamma_{aIJ} + 2n_{[I}\bar{K}_{a|J]} + \bar{K}_{aIJ}$, where the bar notation $\bar{K}_{aIJ}$ as before means that the internal indices are orthogonal on $n^I$. In the time gauge, this is equivalent of having only small latin indices running from 1 to $D$. We have $E^{ai} = \sqrt{q}e^{ai}$ and $D^\Gamma{}_a$ is the covariant derivative associated to the spin connection $\Gamma_{aij}$ annihilating the vielbein. The splits in trace and trace free parts are done with respect to the vielbein. The Gauß constraint has been split into its rotational part $\mathscr{G}^{ij}$ and its boost part $\mathscr{G}^i$. $\lambda_{ij} := -T^\mu A_{\mu ij}$ and $\lambda_i = \lambda_{i0}$. All terms proportional to $\bar{K}^{\mathrm{tr}}_i := \bar{K}_{aij}E^{aj}$ not belonging to the boost part of the Gauß constraint have been written as $*_i\mathscr{G}^i$.

The boost part of the Gauß constraint does not acquire a fermionic part because of the cancellation $\Sigma^{0i} + (\Sigma^{0i})^\dagger = 0$. The Dirac spinors in the above equations appear only as half-densities, i.e. $\sqrt[4]{q}\Psi$. Since the symplectic structure tells us that these half-densities are the natural canonical variables, we will abuse notation and denote by $\Psi$ from now on a half-density. The importance of using half densities stems from the simple form of the symplectic structure. Otherwise, the connection would acquire a complex part [260] and the techniques introduced in [21, 24–27] would not be accessible.

In order to facilitate the canonical analysis, we will employ the equations of motion for $\lambda_i$ and $\bar{K}^{\text{tf}}_{aij}$ at the Lagrangian level. Their solutions translate directly to a purely canonical treatment as one can check. Variation of the Lagrangian with respect to $\lambda_i$ sets the boost part of the Gauß constraint to zero. Variation with respect to $\bar{K}^{\text{tf}}_{aij}$ yields

$$\bar{K}^{\text{tf}}_{aij} = F^{-1}_{aij,bkl}\overline{\Psi}E^b_m \left\{\gamma^m, \Sigma^{ij}\right\}\Psi, \tag{14.7}$$

which we use to eliminate $\bar{K}^{\text{tf}}_{aij}$ in $\mathscr{H}$.

Next, we perform the Legendre transform, yielding the constraints

$$\begin{aligned}
\mathscr{H} &= \frac{1}{2}\sqrt{q}R + \frac{1}{\sqrt{q}}E^{[a|i}E^{b]j}K_{ai}K_{bj} \\
&\quad + \frac{i}{2}\frac{1}{\sqrt{q}}\overline{\Psi}E^a_i\gamma^i D^\Gamma{}_a\Psi - \frac{i}{2}\frac{1}{\sqrt{q}}\overline{D^\Gamma{}_a\Psi}E^a_i\gamma^i\Psi \\
&\quad - \frac{1}{2}\Psi^\dagger\Sigma^{ij}\Psi\Psi^\dagger\Sigma_{ij}\Psi + \frac{1}{32}\overline{\Psi}\left\{\gamma^k, \Sigma^{ij}\right\}\Psi\overline{\Psi}\left\{\gamma_k, \Sigma_{ij}\right\}\Psi, \qquad (14.8)\\
\mathscr{H}_a &= -2E^{bj}D^\Gamma{}_{[a}K_{b]j} + \frac{i}{2}\Psi^\dagger D^\Gamma{}_a\Psi - \frac{i}{2}(D^\Gamma{}_a\Psi)^\dagger\Psi, \qquad (14.9)\\
\mathscr{G}^{ij} &= 2K_a^{[i}E^{a|j]} - \Psi^\dagger\Sigma^{ij}\Psi, \qquad (14.10)
\end{aligned}$$

as well as the non-vanishing (generalised) Poisson (anti-) brackets [161]

$$\{E^{ai}(x), K_{bj}(y)\} = \delta^D(x-y)\delta^a_b\delta^i_j \quad \text{and} \quad \{\Psi^\alpha(x), -i\Psi^\dagger_\beta(y)\} = -\delta^D(x-y)\delta^\alpha_\beta. \tag{14.11}$$

A term proportional to the Gauß constraint has been omitted in $\mathscr{H}$ and $\mathscr{H}_a$.

We define the generator of spatial diffeomorphisms

$$\tilde{\mathscr{H}}_a := \mathscr{H}_a - \frac{1}{2}\Gamma_{aij}\mathscr{G}^{ij} = -E^{bj}\partial_a K_{bj} + \partial_b(E^{bj}K_{aj}) + \frac{i}{2}\Psi^\dagger\partial_a\Psi - \frac{i}{2}(\partial_a\Psi^\dagger)\Psi, \tag{14.12}$$

which acts as

$$\Big\{E^{ai}, \tilde{\mathscr{H}}_b[N^b]\Big\} = N^b\partial_b E^{ai} + (\partial_b N^b)E^{ai} - (\partial_b N^a)E^{bi}, \tag{14.13}$$

$$\Big\{K_{ai}, \tilde{\mathscr{H}}_b[N^b]\Big\} = N^b\partial_b K_{ai} + (\partial_a N^b)K_{bi}, \tag{14.14}$$

$$\Big\{\Psi, \tilde{\mathscr{H}}_b[N^b]\Big\} = N^b\partial_b \Psi + \frac{1}{2}(\partial_a N^a)\Psi, \tag{14.15}$$

$$\Big\{\Psi^\dagger, \tilde{\mathscr{H}}_b[N^b]\Big\} = N^b\partial_b \Psi^\dagger + \frac{1}{2}(\partial_a N^a)\Psi^\dagger, \tag{14.16}$$

$$\tag{14.17}$$

by Lie derivatives. The Gauß constraint acts as

$$\left\{E^{ai}, \frac{1}{2}\mathscr{G}^{ij}[\lambda_{ij}]\right\} = \lambda^i{}_j E^{aj}, \tag{14.18}$$

$$\left\{K_{ai}, \frac{1}{2}\mathscr{G}^{ij}[\lambda_{ij}]\right\} = \lambda_i{}^j K_{aj}, \tag{14.19}$$

$$\left\{\Psi, \frac{1}{2}\mathscr{G}^{ij}[\lambda_{ij}]\right\} = \frac{1}{2}i\lambda_{ij}\Sigma^{ij}\Psi, \tag{14.20}$$

$$\left\{\Psi^\dagger, \frac{1}{2}\mathscr{G}^{ij}[\lambda_{ij}]\right\} = -\frac{1}{2}i\Psi^\dagger\lambda_{ij}\Sigma^{ij}, \tag{14.21}$$

$$\left\{D^\Gamma{}_a\Psi, \frac{1}{2}\mathscr{G}^{ij}[\lambda_{ij}]\right\} = \frac{1}{2}i\lambda_{ij}\Sigma^{ij}D^\Gamma{}_a\Psi, \tag{14.22}$$

$$\left\{(D^\Gamma{}_a\Psi)^\dagger, \frac{1}{2}\mathscr{G}^{ij}[\lambda_{ij}]\right\} = -\frac{1}{2}i(D^\Gamma{}_a\Psi)^\dagger\lambda_{ij}\Sigma^{ij}, \tag{14.23}$$

$$\left\{\Psi^\dagger\Sigma^{ij}\Psi, \frac{1}{2}\mathscr{G}^{ij}[\lambda_{ij}]\right\} = \Psi^\dagger[\lambda,\Sigma]^{ij}\Psi, \tag{14.24}$$

$$\left\{\overline{\Psi}\{\gamma^k,\Sigma^{ij}\}\Psi, \frac{1}{2}\mathscr{G}^{ij}[\lambda_{ij}]\right\} = \overline{\Psi}\left(\{\gamma^k,[\lambda,\Sigma]^{ij}\} + \{\lambda_{km}\gamma^m,\Sigma^{ij}\}\right)\Psi. \tag{14.25}$$

We therefore conclude that the algebra of the diffeomorphism and Gauß constraints closes and that they both Poisson-commute with the Hamiltonian constraint, at least weakly.

Thus we are left with checking the Poisson bracket of two Hamiltonian constraints. We split $\mathscr{H} = \mathscr{H}_{\text{grav}} + \mathscr{H}_{2\text{F}} + \mathscr{H}_{4\text{F}}$ into the purely gravitational part, a part containing two fermions and a part containing the four-fermion terms and define $V_a := M\partial_a N - N\partial_a M$ as well as $V_{ab} := (\partial_a M)(\partial_b N) - (\partial_b M)(\partial_a N)$. The non-vanishing Poisson brackets are given as

$$\{\mathscr{H}_{\mathrm{grav}}[M],\mathscr{H}_{\mathrm{grav}}[N]\} = \int_\sigma d^Dx\left(V_a q^{ab}\left(-2E^{cj}D^\Gamma{}_{[b}K_{c]j}\right) + V_{ab}\frac{E^{ai}E^{bj}}{q}K_{a[i}E^a_{j]}\right),$$

$$\{\mathscr{H}_{2\mathrm{F}}[M],\mathscr{H}_{2\mathrm{F}}[N]\} = \int_\sigma d^Dx\left(V_a q^{ab}\left(\frac{i}{2}\Psi^\dagger D^\Gamma{}_a\Psi - \frac{i}{2}(D^\Gamma{}_a\Psi)^\dagger\Psi\right) - V_{ab}\frac{E^{ai}E^{bj}}{2q}\Psi^\dagger\Sigma^{ij}\Psi\right),$$

$$\{\mathscr{H}_{\mathrm{grav}}[M],\mathscr{H}_{2\mathrm{F}}[N]\} + \{\mathscr{H}_{2\mathrm{F}}[M],\mathscr{H}_{\mathrm{grav}}[N]\} = \int_\sigma d^Dx\left(\frac{1}{8q}V_a\overline{\Psi}\{E^a_k\gamma^k,\Sigma^{ij}\}\Psi\Psi^\dagger\Sigma_{ij}\Psi\right),$$

$$\{\mathscr{H}_{2\mathrm{F}}[M],\mathscr{H}_{4\mathrm{F}}[N]\} + \{\mathscr{H}_{4\mathrm{F}}[M],\mathscr{H}_{2\mathrm{F}}[N]\} = \int_\sigma d^Dx\left(-\frac{1}{8q}V_a\overline{\Psi}\{E^a_k\gamma^k,\Sigma^{ij}\}\Psi\Psi^\dagger\Sigma_{ij}\Psi\right),\tag{14.26}$$

and sum up to

$$\{\mathscr{H}[M],\mathscr{H}[N]\} = \int_\sigma d^Dx\left(V_a q^{ab}\mathscr{H}_b + V_{ab}\frac{E^a_iE^b_j}{2q}\mathscr{G}^{ij}\right).\tag{14.27}$$

The constraints are therefore consistent and the canonical analysis ends here.

## 14.2 Phase space extension

In part II, we have seen that the extension of the ADM phase space $(q_{ab}, P^{ab})$ to the extended phase space $(A_{aIJ}, \pi^{aIJ})$ subject to Gauß and simplicity constraint is equivalent to the ADM phase space. Moreover, this is possible using $\mathrm{SO}(D+1)$ as the structure group while considering Lorentzian gravity. Since spinors can only be coupled to vielbeins, we have to construct a transformation from $(E^{ai}, K_{ai})$ to $(A_{aIJ}, \pi^{aIJ})$. The calculation turns out to be very similar to the one described in part II, we therefore only give the result and comment on some peculiarities.

The explicit construction is given by

$$\bar{E}^{aI} = \zeta\bar{\eta}^I{}_J\pi^{aKJ}n_K, \quad \bar{K}_{aI} = \zeta\bar{\eta}_I{}^J(A-\Gamma)_{aKJ}n^K,\tag{14.28}$$

where, as before, $\bar{\eta}^{IJ} = \eta^{IJ} - \zeta n^I n^J \approx \eta^{IJ} - \frac{\zeta}{D-1}\left(\pi^{aKI}\pi_{aK}{}^J - \zeta\eta^{IJ}\right)$, and $\Gamma_{aIJ}$ is the hybrid spin connection of $E^{aI}$ (see appendix C for details). The peculiarity of these expressions is the appearance of $n^I$, which can only be directly (that is, without non-polynomial terms except for $\sqrt{q}$) expressed in terms of $\pi^{aIJ}$ for $D$+1 odd. For general $D$, we only have access to $n^I n^J$ and then can define $\pm n^I$ through $\pm n^I = \sqrt{n^I n^I}\mathrm{sgn}(n^0 n^I)$ (no summation understood here and one substitutes for $n^I n^I$ under the square root the expression for $n^I n^J$ at $I = J$). Fortunately, we can avoid to make use of this explicit square root expression by invoking the following trick: Ultimately the non-vanishing Poisson bracket involving $n^I$ is of the form $n^J\{A_{aIJ}, n^K\}$. Since $n^K n_K \approx \zeta$ modulo simplicity constraint we have $n^J\{A_{aIJ}, n^K\}n_K \approx 0$. To see this, notice that the simplicity constraint reads $\mathscr{S}^{cd}_{\overline{M}} = \frac{1}{4}\epsilon_{IJKL\overline{M}}\pi^{cIJ}\pi^{dKL}$. It follows

$$n^J\{A_{aIJ}, \mathscr{S}^{cd}_{\overline{M}}\} = \frac{1}{2}n^J\epsilon_{IJKL\overline{M}}\delta^{(c}_a\pi^{d)KL} \approx 0$$

on the constraint surface $\pi^{aIJ} = 2n^{[I}E^{aJ]}$. It follows $n^J\{A_{aIJ}, n^K\} \approx n^J\{A_{aIJ}, n^L\}\bar{\eta}^K_L = -n^J\{A_{aIJ}, \bar{\eta}^K_L\}n^L$. However, $\{A_{aIJ}, \bar{\eta}^K_L\} = -\zeta\{A_{aIJ}, n^K n_L\}$ and $n^K n_L$ can be expressed unambiguously as above in terms of $\pi^{aIJ}$. In order to compute the brackets between $\bar{E}^{aI}, \bar{K}_{aI}$ one then just hast to carefully insert the definition of $n_I n_J$ in terms of $\pi^{aIJ}$. The only term which cannot easily be seen to vanish by algebraic manipulations alone occurs in the bracket $\{K_{aI}, K_{aJ}\}$ and is of the form

$$\bar{\eta}^K_I n^L \bar{\eta}^M_J n^N\{[A-\Gamma]_{aIJ}, [A-\Gamma]_{bKL}\} = -\bar{\eta}^K_I n^L \bar{\eta}^M_J n^N[\{[A_{aIJ}, \Gamma_{bKL}\} - \{A_{bKL}, \Gamma_{aIJ}\}].$$

This term vanishes due to the weak integrability (modulo simplicity constraint) of the hybrid connection $\Gamma_{aIJ}$ and by using the trick mentioned above, see section 7.1.3 for more details.

After a tedious calculation, the Poisson brackets of $\bar{E}^{aI}$ and $\bar{K}_{aI}$ expressed as functions of $A_{aIJ}$ and $\pi^{aIJ}$ are given by

$$\{\bar{E}^{aI}(x), \bar{E}^{bJ}(y)\} = 0, \quad \{\bar{K}_{aI}(x), \bar{K}_{bJ}(y)\} \approx 0, \quad \{\bar{E}^{aI}(x), \bar{K}_{bJ}(y)\} \approx -\zeta\delta^D(x-y)\delta^a_b\bar{\delta}^I_J. \tag{14.29}$$

modulo simplicity constraint.

The only task left to do is to write down a Hamiltonian theory in the variables $A_{aIJ}$ and $\pi^{bKL}$ with internal gauge group SO$(D+1)$ which reduces to the theory derived in the previous section on the constraint surface $\mathscr{S}^{ab}_{\overline{M}} = n_I\mathscr{G}^{ij} = 0$. The basic idea is to first derive a Hamiltonian formulation of Euclidean gravity coupled to fermions and then to adjust the Hamiltonian constraint to mimic Lorentzian gravity. The reason why the procedure already used in the vacuum case in part II generalises nicely to Dirac fermions is the strong resemblance of the Clifford algebras, which differ only by factors of $i$ for different signatures and the Euclidean signature of the internal gauge group which ensures that $\Sigma^{IJ}$ is a Hermitian matrix the Euclidean case.

This requires care at several places, e.g. the cancellation $\Sigma^{0i} + (\Sigma^{0i})^\dagger = 0$ from the boost part of the Lorentzian Gauß constraint is no longer present. In order to derive the Euclidean constraints, we start as in the previous section with the action (14.1) and perform a $D+1$ decomposition. We replace $\gamma^0$ with $n_I\gamma^I$, which reduces to $\gamma^0$ in the time gauge. We note that the object $\overline{\Psi}\Psi$ is not a Lorentz scalar any more when using Euclidean signature, because the $\gamma^0$ inherent in $\overline{\Psi}\Psi$ is needed in order to maintain invariance under boosts which are generated by the anti-Hermitian $\Sigma^{0i}$. In Euclidean signature the boost generator is also Hermitian and thus $\Psi^\dagger\Psi$ rather than $\overline{\Psi}\Psi$ is now the appropriate Euclidean scalar to be used while $\Psi^\dagger\gamma^I\Psi$ is a Euclidean covariant vector with index $I$. The substitution $\gamma^0 \to n_I\gamma^I$ is therefore natural for Euclidean signature and allows for the construction of a manifestly $\mathrm{SO}(D+1)$ gauge invariant theory. We use the additional $n^I$ in the action to form $\pi'^{aIJ} = 2n^{[I}E^{a|J]}$ and introduce the simplicity constraint in order to replace $\pi'^{aIJ}$ by $\pi^{aIJ}$. The Euclidean Hamiltonian theory is then given by the constraints

$$\mathscr{H}^E = \frac{1}{2}\pi^{aIK}\pi^{bJ}{}_K F_{abIJ} + \left(\frac{1}{2}\Psi^\dagger\pi^{aIJ}\Sigma_{IJ}D^A{}_a\Psi + CC\right), \tag{14.30}$$

$$\mathscr{H}_a^E = \frac{1}{2}\pi^{bIJ}F_{abIJ} + \frac{i}{2}\Psi^\dagger D^A{}_a\Psi - \frac{i}{2}(D^A{}_a\Psi)^\dagger\Psi, \tag{14.31}$$

$$\mathscr{G}_E^{IJ} = D^A{}_a\pi^{aIJ} - \Psi^\dagger\Sigma^{IJ}\Psi, \tag{14.32}$$

$$\mathscr{S}^{ab}_{\overline{M}} = \frac{1}{4}\epsilon_{IJKL\overline{M}}\pi^{aIJ}\pi^{bKL}, \tag{14.33}$$

and the (non-vanishing) Poisson brackets

$$\begin{aligned}\{A_{aIJ}(x), \pi^{bKL}(y)\} &= \delta^D(x-y)\delta_a^b(\delta_K^I\delta_L^J - \delta_L^I\delta_K^J),\\ \{\Psi^\alpha(x), -i\Psi^\dagger_\beta(y)\} &= -\delta^D(x-y)\delta^\alpha_\beta.\end{aligned} \tag{14.34}$$

The task of "Lorentzifying" the gravitational part of $\mathscr{H}^E$ has already been addressed. For the fermionic part, we observe that we should add a factor of $i$ in front of the fermionic term in order to compensate for $\gamma_E^0 = i\gamma_L^0$ and denote the changed constraint by $\mathscr{H}^E_{(i)}$. The Hamiltonian constraint now reduces to

$$\begin{aligned}\mathscr{H}^E_{(i)} \to{}& \frac{1}{2}\sqrt{q}R - \frac{1}{\sqrt{q}}E^{[a|i}E^{b]j}K_{ai}K_{bj} + \left(\frac{i}{2\sqrt{q}}\Psi^\dagger\gamma_L^0E_i^a\gamma^iD^\Gamma{}_a\Psi + CC\right)\\ &- \frac{1}{2\sqrt{q}}\Psi^\dagger\Psi E^{ai}K_{ai} - \frac{1}{2\sqrt{q}}\frac{D-2}{D-1}\Psi^\dagger\Sigma_E^{0i}\Psi\Psi^\dagger\Sigma^E_{0i}\Psi + \partial_a\left(\frac{E_i^a}{\sqrt{q}}\Psi^\dagger\Sigma_E^{0i}\Psi\right) + \mathscr{O}(K^{\mathrm{tf}}_{aij}).\end{aligned} \tag{14.35}$$

The terms proportional to $K^{\text{tf}}_{aij}$ can be dealt with using ideas from gauge unfixing. We calculate

$$\{\mathscr{S}^{ab}_{\overline{M}}[c^{\overline{M}}_{ab}], \mathscr{H}^E_{(i)}[N]\} = {}_G\mathscr{D}^{ab}_{\overline{M}}[Nc^{\overline{M}}_{ab}] + \int_\sigma d^Dx \frac{N}{4} c^{\overline{M}}_{ab} \pi^{aIJ} \pi^{bMN} \epsilon_{MNKL\overline{M}} \Psi^\dagger \{\Sigma_{IJ}, \Sigma_{KL}\} \Psi$$
$$:= {}_G\mathscr{D}^{ab}_{\overline{M}}[Nc^{\overline{M}}_{ab}] + {}_F\mathscr{D}^{ab}_{\overline{M}}[Nc^{\overline{M}}_{ab}] \tag{14.36}$$

and see that the gravitational constraint ${}_G\mathscr{D}^{ab}_{\overline{M}} = -\epsilon_{IJKL\overline{M}} \pi^{cIJ} \left(\pi^{(a|KN} D^A{}_c \pi^{b)L}{}_N\right) \approx 0$ now receives a fermionic contribution ${}_F\mathscr{D}$. $F^{abcd}_{\overline{M}\,\overline{N}}$, the Dirac matrix introduced in section 8.2, however remains unchanged since ${}_F\mathscr{D}$ Poisson-commutes with the simplicity constraint and gauge unfixing works as before. Next to compensating the terms proportional to $K^{\text{tf}}_{aij}$, gauge unfixing also produces a four-fermion term, which we have to subtract again in order to build the correct Lorentzian Hamiltonian constraint.

Comparison with the previous section leads to the following correction terms:

$$\begin{aligned}
\mathscr{H}^L = {} & \mathscr{H}^E_{(i)} + \frac{2}{\sqrt{q}} E^{[a|I} E^{b]J} K_{aI} K_{bJ} - \frac{1}{2} {}_G\mathscr{D}^{ab}_{\overline{M}} \left(F^{-1}\right)^{\overline{M}\,\overline{N}}_{ab\,cd} {}_G\mathscr{D}^{cd}_{\overline{N}} \\
& - \frac{1}{2} {}_G\mathscr{D}^{ab}_{\overline{M}} \left(F^{-1}\right)^{\overline{M}\,\overline{N}}_{ab\,cd} {}_F\mathscr{D}^{cd}_{\overline{N}} - \frac{1}{2} {}_F\mathscr{D}^{ab}_{\overline{M}} \left(F^{-1}\right)^{\overline{M}\,\overline{N}}_{ab\,cd} {}_G\mathscr{D}^{cd}_{\overline{N}} \\
& - \frac{1}{2\sqrt{q}} \Psi^\dagger \Sigma^{ij} \Psi \Psi^\dagger \Sigma_{ij} \Psi + \frac{1}{2\sqrt{q}} \frac{D-2}{D-1} \Psi^\dagger \Sigma^{0i}_E \Psi \Psi^\dagger \Sigma^E_{0i} \Psi \\
& - \partial_a \left( \frac{E^a_i}{\sqrt{q}} \Psi^\dagger \Sigma^{0i}_E \Psi \right) + \frac{1}{2\sqrt{q}} \Psi^\dagger \Psi E^{aI} K_{aI} + \frac{1}{32} \overline{\Psi} \{\gamma^k, \Sigma^{ij}\} \Psi \overline{\Psi} \{\gamma_k, \Sigma_{ij}\} \Psi.
\end{aligned} \tag{14.37}$$

This Hamiltonian has to be rewritten in terms of $A_{aIJ}$ and $\pi^{bKL}$ only, desirably as simple as possible regarding the quantisation. We propose the Hamiltonian

$$\begin{aligned}
\mathscr{H}^L = {} & \frac{1}{2} \pi^{aIK} \pi^{bJ}{}_K F_{abIJ} + \left( i \frac{1}{2} \Psi^\dagger \pi^{aIJ} \Sigma_{IJ} D^A{}_a \Psi + CC \right) \\
& + \frac{2}{\sqrt{q}} E^{[a|I} E^{b]J} K_{aI} K_{bJ} - \frac{1}{2} {}_G\mathscr{D}^{ab}_{\overline{M}} \left(F^{-1}\right)^{\overline{M}\,\overline{N}}_{ab\,cd} {}_G\mathscr{D}^{cd}_{\overline{N}} \\
& - \frac{1}{2} {}_G\mathscr{D}^{ab}_{\overline{M}} \left(F^{-1}\right)^{\overline{M}\,\overline{N}}_{ab\,cd} {}_F\mathscr{D}^{cd}_{\overline{N}} - \frac{1}{2} {}_F\mathscr{D}^{ab}_{\overline{M}} \left(F^{-1}\right)^{\overline{M}\,\overline{N}}_{ab\,cd} {}_G\mathscr{D}^{cd}_{\overline{N}} \\
& - \frac{1}{2\sqrt{q}} \Psi^\dagger \Sigma^{IJ} \Psi \Psi^\dagger \Sigma_{IJ} \Psi + \frac{1}{2\sqrt{q}} \frac{3D-4}{D-1} \Psi^\dagger \Sigma^{IK} \Psi \Psi^\dagger \Sigma_{JK} \Psi n_I n^J \\
& - \partial_a \left( \frac{\pi^a{}_{IJ}}{\sqrt{q}} \Psi^\dagger \Sigma^{IJ} \Psi \right) + \frac{1}{2\sqrt{q}} \Psi^\dagger \Psi E^{ai} K_{ai} \\
& - \frac{1}{32} \Psi^\dagger \gamma^{[I} \gamma^J \gamma^K \gamma^{L]} \Psi n_L n^M \Psi^\dagger \gamma_{[I} \gamma_J \gamma_K \gamma_{M]} \Psi
\end{aligned} \tag{14.38}$$

for quantisation, although we are well aware of the fact that other choices might lead to equally reasonable classical starting points. We shift the problem of choosing the "correct" Hamiltonian constraint to the semiclassical analysis. The expressions for $n^I n_J$ and $E^{aI} K_{bI}$ were already given in part II and all $\gamma$-matrices appearing are those for Euclidean signature.

A last remark concerning the use of the linear simplicity constraint (cf. [6] and section 9.1) instead of the quadratic version above is in order. Since, using the linear simplicity constraint, we have direct access to the internal unit vector $N^I$, the above construction is, in fact, simpler in that case. In analogy to (14.28), we define the map to the eADM phase space by

$$\bar{E}^{aI} = \zeta \bar{\eta}^I{}_J \pi^{aKJ} N_K, \quad \bar{K}_{aI} = \zeta \bar{\eta}_I{}^J (A - \Gamma)_{aKJ} N^K, \tag{14.39}$$

where $\Gamma_{aIJ}$ is understood as functions of $\pi^{aIJ}$ but $\bar{\eta}_{IJ}$ now is understood as a function of $N^I$. The proof that the extension of $\left(K_{ai}, E^{bj}\right)$ with SO($D$) Gauß constraint to $\left(A_{aIJ}, \pi^{bKL}, N^I, P_J\right)$ with SO($D$+1) (or SO(1, $D$)) Gauß, linear simplicity and normalisation constraint then is analogous to the one above and therefore will not be detailed here.

## 14.3 Kinematical Hilbert space for fermions

The construction of the kinematical Hilbert space for fermions was discussed in [45]. Results obtained there apply for the case at hand, so we only give a short summary. It is crucial to work with half-densitised fermionic variables $\Psi$ for what follows, as was stressed in [45].

Faithful implementation of the reality conditions enforces the use of a representation in which the objects

$$\theta_\alpha(x) := \int_\sigma d^D y \sqrt{\delta(x,y)}\ \Psi_\alpha(y) := \lim_{\epsilon \to 0} \int_\sigma d^D y \frac{\chi_\epsilon(x,y)}{\sqrt{\epsilon^D}} \Psi_\alpha(y) \tag{14.40}$$

become densely defined multiplication operators. Their adjoints $\overline{\theta}^\alpha$ become derivative operators. Here, $\alpha = 1, ..., n := 2^{\lfloor (D+1)/2 \rfloor}$ ($\lfloor . \rfloor$ denotes the integer part of .) and $\chi_\epsilon(x,y)$ denotes the characteristic function of a box of Lebesgue measure $\epsilon^D$ centered at $x$. In the above equation, the half-densities $\Psi$ are "dedensitised" using the $\delta$-distribution, which is a scalar in one of its arguments and a density in the other. Thus, the variables $\theta$ are Graßmann-valued scalar quantities, which is important for diffeomorphism invariance [45]. In calculations it is understood that the $\epsilon \to 0$ limit is performed after the manipulation under consideration is performed.

The variables $\theta_\alpha(v)$ coordinatise together with their conjugates the superspace $S_v$ at the point $v$. The quantum configuration space is the uncountable direct product $\overline{\mathscr{S}} := \prod_{v\in\sigma} S_v$. In order to define an inner product on $\overline{\mathscr{S}}$, it turns out to be sufficient to define an inner product on each $S_v$ coming from a probability measure. The "measure" on $S_v$ is a modified form of the Berezin symbolic integral [261]

$$dm\left(\overline{\theta},\theta\right) = d\overline{\theta}d\theta e^{\overline{\theta}\theta} \text{ and } dm_v = \otimes_{\alpha=1}^{n} dm\left(\overline{\theta}_\alpha(v),\theta_\alpha(v)\right), \tag{14.41}$$

which has the additional property of being positive on holomorphic functions (those which only depend on the $\theta_\alpha$ and not on $\overline{\theta}_\alpha$). Since the $\theta$ are Graßmann variables and thus anti-commute, any product of more than $n$ of these variables will vanish. The vector space of monomials of order $k$ is $n!/k!(n-k)!$-dimensional $(0 \leq k \leq n)$ and the full vector space $Q_v$ built from all monomials has dimension $2^n$. The full fermionic Hilbert space is a space of holomorphic square integrable functions on $\overline{\mathscr{S}}$ with respect to $d\mu_F$

$$\mathscr{H}_F = L_2\left(\overline{\mathscr{S}}, d\mu_F\right) = \otimes_{v\in\sigma} L_2\left(S_v, dm_v\right). \tag{14.42}$$

When restricted to a point $v$, the inner product can be seen to coincide with the standard inner product on $Q_v$ when viewed as a vector space of exterior forms of maximal degree $D+1$. For a more complete treatment, the reader is referred to [45] where it is shown that the fermion measure $d\mu_F$ is gauge and diffeomorphism invariant and that the reality conditions $\overline{\theta}_\alpha = -i\pi_\alpha$ are faithfully implemented in the inner product.

## 14.4 Implementation of the Hamiltonian constraint operator

The quantisation of the purely gravitational Hamiltonian constraint in dimensions $D+1 \geq 3$ has already been discussed in part III. The quantisation of fermionic degrees of freedom was described in detail in [45, 46], which we assume the reader to be familiar with. Next to an explicit example, we will only provide a toolkit to quantise the fermionic part of the Hamiltonian constraint operator as writing down the explicit terms is rather laborious.

Quantisation of the $\theta$ variables is performed by promoting $\theta_\alpha$ to a multiplication operator and $\widehat{(\theta^\dagger)^\beta} = -\hbar\frac{\partial^L}{\partial\theta_\beta}$, where $L$ indicates the left derivative. The explicit quantisation follows the (extended) toolkit of [3]:

(1) Choose a triangulation $T(\gamma,\epsilon)$ of the spatial slice $\sigma$ adapted to the graph $\gamma$.

(2) Use classical identities in order to express the Hamiltonian constraint in terms of connections $A_{aIJ}$, volumes $V(x,\epsilon)$ and Euclidean Hamiltonian constraints $\mathscr{H}_E(x,\epsilon)$.

(3) Replace all phase space variables by their corresponding regulated quantities.

(4) Instead of the the integration $\int_\sigma d^Dx$, put a sum $\frac{1}{D!}\sum_{v\in V(\gamma)}$ over all the vertices $v$ of the graph $\gamma$.

(5) For every spatial $\epsilon$-symbol, put a sum $\frac{2^D}{E(v)}\sum_{v(\Delta)=v}$ over all $D$-simplices having $v$ as a vertex. The holonomies associated with the $\epsilon$-symbol are evaluated along the edges spanning $\Delta$.

(6) Substitute the generalised Poisson (anti-)brackets by $\frac{i}{\hbar}$ times the (anti-)commutator of the corresponding operators, i.e. the multiplication operator $\hat{h}_e$, the volume operator $\hat{V}$, the multiplication operator $\hat{\theta}_\alpha$ and the derivation operator $-\hbar\frac{\partial^L}{\partial\theta_\alpha}$.

The kinetic fermionic part of the Hamiltonian constraint operator is a bit more involved since it contains a derivative. Following [45], we explicitly get

$$\begin{aligned}&\hat{\mathscr{H}}^\epsilon_{\text{Dirac, kin}}(N)f_\gamma\\ =&\Bigg(\frac{i\hbar}{2}\sum_{v\in V(\gamma)}\frac{2^D}{D!}\frac{N_v}{E(v)}\sum_{v(\Delta)=v}{}^{\hat{\epsilon}}\left(\frac{\pi^a{}_{IJ}}{\sqrt{q}}(v)\right)\times\\ &\left(\Sigma^{IJ}\left((h_{s_a(\Delta)}\theta(s_a(\Delta)(\epsilon)\right)-\theta(v)\right)_\alpha\frac{\partial^L}{\partial\theta_\alpha(v)}+H.C.\Bigg)f_\gamma,\end{aligned} \tag{14.43}$$

where by ${}^{\hat{\epsilon}}(\ldots)$ we mean the regulated quantity with the Poisson brackets substituted by $i/\hbar$ times the commutator of the corresponding operators. The Hermitian conjugation operation $H.C.$ is meant with respect to the inner product on the Hilbert space.

Due to its length, we refrain from writing down the complete Hamiltonian constraint operator which can be easily done when following the quantisation recipe. We remark that we could split the Dirac fermions for $D+1$ even into left- and right-handed parts, however, the presentation does not benefit from this. Details are supplied in [6]. The quantisation ambiguities from LQG are also present when considering fermions and, as usual, we shift this problem to the semiclassical limit.

# 15

# Extension to supersymmetric theories

*"Maybe the way we now interpret Kaluza-Klein ideas is totally wrong. Perhaps one should instead consider field theories with a variable number of dimensions [...], maybe even continuous dimensions [...], maybe even a new Schrödinger equation in which one of the canonical variables is a variable dimension. The fact that we live in $d = 4$ would then simply be a Bohr-quantization rule. Perhaps certain integer dimensions are singled out in a path integral approach because they have more differentiable structures than other integer dimensions [...]. In these lectures, however, we will stick to the "conservative" viewpoint that our world is eleven dimensional."*

- Peter van Nieuwenhuizen [133]

In [6, 7], non-standard matter fields which appear generically in supergravity theories were included in the loop quantisation framework, namely the spin 3/2 Rarita Schwinger field ("gravitino") and the three-form gauge field from the $d = 11$ $N = 1$ supergravity as a specific example of a higher $p$-form field.

In the fermionic part 15.1, we will exclusively study the case of Majorana fermions. The Rarita Schwinger action actually is not tied to the use of Majorana fermions, there exist also Dirac and Weyl versions of this field. Also, it is a well-known fact that Majorana fermions do not exist in any dimension, but in those dimensions instead one can define anti- or symplectic Majorana fermions with slightly more complicated

Majorana conditions (cf. e.g. [133]). Scanning the literature on supergravity theories in various dimensions (cf. e.g. [134] for a collection of important original articles which additional explanations, historical remarks and extensive reference to further literature), it transpires that there are supergravities in which the role of the gravitino actually is played by any of these possibilities: Weyl, Majorana-Weyl, anti- or symplectic Majorana Rarita Schwinger fields appear. However, for the $d = 4$ $N = 1$, and $d = 11$ $N = 1$ supergravities, the gravitino actually is a Majorana fermion [134], and therefore, although the Majorana case cannot be called generic, we already cover these arguably interesting theories. Still, in [6] it is shown exemplarily that, without further complications, we are also able to include spin 3/2 Majorana-Weyl fermions both, at the classical and quantum level (important for e.g. $d = 10$ $N = 1$ supergravity), as well as spin 1/2 Majorana and spin 1/2 Dirac-Weyl fermions which also appear in some supergravities. This makes us confident that the methods developed actually allow for the loop quantisation of the fermionic sector of a large class of supergravity theories. In the bosonic sector (section 15.2), while only treating the three-form field from $d = 11$ $N = 1$ supergravity explicitly, we also expect that our methods carry over to more general $p$-form fields. However, here we make do with only presenting the mentioned examples and leave the generalisations to the interested reader.

## 15.1 Rarita Schwinger field

### 15.1.1 Classical extension to SO$(D+1)$ gauge supergravity

We start from an action of the form

$$S_{\mathrm{SUGRA}}[e, A, \psi, \mathrm{more}] = -\int_{\mathscr{M}} d^d x \, \left( \frac{1}{2} e \, e^{\mu}{}_{I} e^{\nu}{}_{J} F_{\mu\nu}{}^{IJ}(A) + ie \, \bar{\psi}_{\mu} \gamma^{\mu\rho\sigma} \nabla^{A}{}_{\rho} \psi_{\sigma} + \mathrm{more} \right), \tag{15.1}$$

which is quite generic for (first order) supergravity theories[1]. The action consists of three parts: The first term is given by the Palatini action for gravity known from chapter 4, the second is the action of the Rarita Schwinger field $\psi_\mu$ (spinor indices will be mostly suppressed; note that in general there might be several Rarita Schwinger fields appearing), and "more" stands for all other terms which are demanded by supersymmetry. We defined $\gamma^{\mu\nu\rho} := \gamma^{IJK} e^{\mu}{}_{I} e^{\nu}{}_{J} e^{\rho}{}_{K}$, and the covariant derivative acting on the spinor

[1]Here, we want to make explicit the transition from the Lagrangian to the Hamiltonian formulation in time gauge once and choose exemplarily the first order framework. We could, however, as well start with a second order action. The subsequent considerations in the Hamiltonian theory actually do not depend on this choice of starting point.

field is defined like before for Dirac spinors, $\nabla^A{}_\mu\psi_\nu = \partial_\mu\psi_\nu + \frac{i}{2}A_{\mu IJ}\Sigma^{IJ}\psi_\nu$. Being a Majorana fermion, $\psi$ has to satisfy the Majorana reality condition $\bar{\psi}_\mu := \psi_\mu^\dagger\gamma^0 = \psi_\mu^{\mathrm{T}}C$, where $C$ is called charge conjugation matrix. We will work in a Majorana representation, in which the spinors are real and we have $C = \gamma^0$. Note that this implies a restriction to those dimensions in which a Majorana representation of the Lorentzian Clifford algebra exists, which however includes the particularly interesting cases $d = 4$, 10, 11. Like in the case of Dirac fermions, we again want to first impose time gauge $n^I = \delta_0^I \Leftrightarrow E^{a0} = 0$ to reduce the internal SO$(1, D)$ symmetry to SO$(D)$ and then extend it again to SO$(D + 1)$. Performing the $D + 1$ split like in the pure gravity case then leads to

$$S_{\mathrm{SUGRA}} = \int dt \int_\sigma d^Dx \left( E^{ai}\dot{K}_{ai} - i^{(D)}e\ \psi_a^{\mathrm{T}}\gamma^{ab}\dot{\psi}_b \right.$$
$$\left. - N\mathscr{H} - N^a\mathscr{H}_a - \frac{1}{2}\lambda_{ij}\mathscr{G}^{ij} - \bar{\psi}_t\mathfrak{S} + \text{more}\right). \quad (15.2)$$

Here, "more" stands for the kinetic terms of all other present fields and for further constraints which might appear. Of course, we obtain the usual Hamiltonian, spatial diffeomorphism and SO$(D)$ Gauß constraint, but also $\bar{\psi}_t$ plays the role of a Lagrange multiplier field and therefore, one further constraint, the supersymmetry constraints $\mathfrak{S}$, arises. The form of $\mathscr{H}$ and $\mathfrak{S}$ depends strongly on the theory under consideration. With a suitable choice of Lagrange multipliers, $\mathscr{H}_a$ is the generator of spatial diffeomorphisms on all phase space variables and therefore has a generic form[1], as has $\mathscr{G}^{ij}$, generating internal SO$(D)$ transformations on all phase space variables in the corresponding representation. Reading off the momenta conjugate to $\psi_a$

$$\pi^a := i^{(D)}e\ \psi_b^{\mathrm{T}}\gamma^{ba}, \quad (15.3)$$

we find

$$\mathscr{G}^{ij} = 2K_a^{[i}E^{a|j]} + \pi^a[i\Sigma^{ij}]\psi_a + \text{more}, \quad (15.4)$$
$$\mathscr{H}_a = E^{bj}\partial_a K_{bj} - \partial_b(E^{bj}K_{aj}) - \pi^b\partial_a\psi_b + \partial_b(\pi^b\psi_a) + \text{more}, \quad (15.5)$$

where "more" stands for the corresponding terms of any additional fields, as well as the non-vanishing canonical anti-bracket relations (CAR)

$$\{\psi_a^\alpha(x), \pi_\beta^b(y)\} = -\delta_\beta^\alpha\delta_a^b\delta^{(D)}(x - y). \quad (15.6)$$

[1]Suitable choice of Lagrange multiplier here is equivalent to choosing a certain linear combination of constraints. The "natural" vector constraint which appears, namely the one whose corresponding Lagrange multiplier is the shift vector, actually generates a mixture of spatial diffeomorphisms, internal rotations and local supersymmetry transformations.

Since the Gauß and spatial diffeomorphism constraint are expected to be treatable in the same way as in the non-supersymmetric case, our strategy will be to focus on the CAR and reality conditions to obtain a kinematical quantisation of this sector of the theory.

The three main manipulations we have to perform in order to arrive at CAR and reality conditions which are amenable at the quantum level (for details, see [6]):

(1) It was shown in [45] (and applied in chapter 14) that for Dirac fermions, it is mandatory to use half densities as fundamental fermionic variables, since otherwise one has a complex valued gravitational connection and problems with implementing both, the reality conditions and the canonical Poisson anti-bracket at the quantum level. Interestingly, in the case at hand, we again will be forced to use half-densitised fermionic variables. Due to the reality of the fermionic field, the usual defining equation of the fermionic momenta 15.3 actually are *second class with themselves*, a complication which is absent in the case of the Dirac field,

$$\Omega^a := \pi^a - i^{(D)}e \; \psi_b^{\mathrm{T}} \gamma^{ba} \approx 0, \tag{15.7}$$

$$\{\Omega^a(x), \Omega^b(y)\} = 2i^{(D)}e \; \gamma^{ab} \delta^{(D)}(x-y). \tag{15.8}$$

One might at this point again want to use gauge unfixing, but to this end we had to split the constraints $\Omega^a$ in a covariant way such that we can drop half of them. A natural splitting would be a chiral one, but the concept of chirality only is defined in even spacetime dimensions and we do not want to impose further restrictions on the number of dimensions. Therefore, we are lead to using the Dirac bracket. For the Dirac anti-bracket between two fermionic fields, we find

$$\{\psi_a(x), \psi_b(y)\}_{\mathrm{DB}} = \frac{i}{2(D-1)^{(D)}e}((2-D)q_{ab} + \gamma_{ab})\delta^{(D)}(x-y), \tag{15.9}$$

and furthermore, one finds $\{K_a{}^i(x), K_b{}^j(y)\}_{\mathrm{DB}} \neq 0$, $\{\psi_a(x), K_c{}^i(y)\}_{\mathrm{DB}} \neq 0$, and $\{\psi_a(x), E^b{}_j(y)\}_{\mathrm{DB}} = 0$. The latter Dirac brackets are rather disastrous in view of later quantisation: From the first one, we expect that the $\mathrm{SO}(D+1)$ connection will not be Poisson self - commuting, and the latter two indicate that the Rarita Schwinger field will have a very complicated action on the connection, being constructed from both, $E^{bj}$ and $K_{ck}$.

These obstacles can be circumvented using half densitised and vielbein contracted fermionic variable[1] $\phi_i := \sqrt{^{(D)}e}e^{ai}\psi_a$. From the kinetic term in the action, it is easy to read off that, when using this fermionic variable, the variable conjugate to the vielbein changes as $K_a{}^i \to K'_a{}^i := K_a{}^i - iE_a{}^k\phi_j\gamma^{ji}\phi_k$. After this change of variables, the non-vanishing Dirac (anti-)brackets are (this was already observed in [120] in $D = 3$)

$$\{K'_{ai}(x), E^{bj}(y)\}_{\text{DB}} = \delta_a^b\delta_i^j\delta^{(D)}(x-y), \tag{15.10}$$

$$\{\phi_i(x), \phi_j(y)\}_{\text{DB}} = \frac{i}{2(D-1)}((2-D)\eta_{ij} + \gamma_{ij})\delta^{(D)}(x-y), \tag{15.11}$$

and all fields are real. To simplify notation, we will drop the subscript DB in the following.

(2) Having sidestepped this first major problem, we can start thinking of extending the internal gauge group to SO$(D+1)$. Since we "decoupled" the fermionic and gravitational degrees of freedom at the level of the Dirac bracket in the last step, we can treat the bosonic degrees of freedom like in the vacuum case to obtain a SO$(D+1)$ connection formulation. In the fermionic sector, we have to either get rid of the matrix appearing on the right hand side of (15.11) before the extension to SO$(D+1)$ or give an SO$(D+1)$ version thereof. The latter option is problematic, since the naive extension (just adding one internal direction, $\eta_{ij} \to \eta_{IJ}$, $\gamma_{ij} \to \gamma_{IJ}$) does not lead to a symmetric matrix (under the exchange of both, $I$, $J$ and the spinor indices), which, however, is demanded by the symmetry of the anti-bracket. Therefore, we stick to the former route, and simplify the bracket by decomposing $\phi_i$ into trace and trace free components with respect to $\gamma^i$. To this end, we define

$$\mathbb{P}^{ij}_{\alpha\beta} := \eta^{ij}\delta_{\alpha\beta} - \frac{1}{D}(\gamma^i\gamma^j)_{\alpha\beta} = \frac{D-1}{D}\eta^{ij}\delta_{\alpha\beta} - \frac{2i}{D}\Sigma^{ij}_{\alpha\beta}, \tag{15.12}$$

$$\mathbb{Q}^{ij}_{\alpha\beta} := \frac{1}{D}(\gamma^i\gamma^j)_{\alpha\beta} = \frac{1}{D}\eta^{ij}\delta_{\alpha\beta} + \frac{2i}{D}\Sigma^{ij}_{\alpha\beta}, \tag{15.13}$$

where $\alpha, \beta, ... \in \{1, ..., 2^{\lfloor D+1/2 \rfloor}\}$ denote spinor indices. It is easy to check that this actually defines projectors, $\mathbb{P}^{ij}_{\alpha\beta}\mathbb{Q}^{\beta\gamma}_{jk} = 0$, $\mathbb{P}^{ij}_{\alpha\beta}\mathbb{P}^{\beta\gamma}_{jk} = \mathbb{P}^{i\gamma}_{\alpha k}$, $\mathbb{Q}^{ij}_{\alpha\beta}\mathbb{Q}^{\beta\gamma}_{jk} = \mathbb{Q}^{i\gamma}_{\alpha k}$, $\mathbb{P} + \mathbb{Q} = \mathbb{1}\eta$, and using them, we can decompose the Rarita-Schwinger field as follows

$$\phi_i = \mathbb{P}_{ij}\phi^j + \mathbb{Q}_{ij}\phi^j =: \rho_i + \frac{1}{D}\gamma_i\sigma, \tag{15.14}$$

[1] This choice of variables actually appeared much earlier in the literature on Hamiltonian supergravity [262] when requiring that the kinetic term of the Rarita Schwinger field be explicitly vielbein independent.

with $\rho_i := \mathbb{P}_{ij}\phi^j$ and $\sigma := \gamma^i\phi_i$[1]. At the cost of introducing a new constraint

$$\Lambda := \gamma^i \rho_i, \tag{15.15}$$

which demands that $\rho_i$ is trace free with respect to $\gamma^i$, we then find very convenient non-vanishing fermionic anti-brackets

$$\{\rho_i(x), \rho_j(y)\} = -\frac{i}{2}\mathbb{1}\eta_{ij}\ \delta^{(D)}(x-y), \tag{15.16}$$

$$\{\sigma(x), \sigma(y)\} = i\frac{D}{2(D-1)}\mathbb{1}\ \delta^{(D)}(x-y), \tag{15.17}$$

and again all fields are real. A final comment is in order: Note that the constraint (15.15) is again second class with itself. If we calculate the corresponding Dirac bracket, (15.16) changes to $\{\rho_i(x), \rho_j(y)\}_{\mathrm{DB}} = -\frac{i}{2}\mathbb{P}_{ij}\ \delta^{(D)}(x-y)$, and it seams that not much has been gained when compared with (15.11). However, as we will see later when quantising, it will be central that the right hand side of the Dirac bracket gives a projector and this is only true for $\rho_i$, not for $\phi_i$.

(3) Finally, we have to extend the internal space, which actually poses the most intricate problem: Since we started with real valued Lorentzian Dirac matrices, the corresponding generators ($\propto \Sigma_{IJ}$) of $\mathrm{SO}(D+1)$ in the spinor representation are necessarily complex (more concretely, in our conventions $\gamma_0$ for Euclidean and Lorentzian signature differs by a factor of $i$ and therefore becomes imaginary). This implies that the real vector space $V$ of Majorana fermions is not preserved under the action of the extended, $\mathrm{SO}(D+1)$ Gauß constraint, and it seems that the Majorana reality condition and the internal signature switch are incompatible.

However, note that any element $g \in \mathrm{SO}(D+1)$ can be written as a $g = b \cdot r$, where $b$ is an "Euclidean boost" in the $0j$ - plane and $r$ a rotation stabilizing $n_0^I := \delta_0^I$, and only the "Euclidean boosts" spoil the action of $\mathrm{SO}(D+1)$ on $V$ (all $\gamma_i$, $i \in \{1, ..., D\}$ are real valued). If we started with a real spinor $\in V$ and kept track of all "Euclidean boosts", we could still impose sensible reality conditions, namely that the spinor when "boosted" back to $V$ is real. A natural way how to keep track of these boosts lies in the use of the linear simplicity constraint: The unit vector field $N^I$ encodes the $D$ boost parameters, $N^I =: b^I{}_J(N) n_0^J$.

[1] Also this decomposition already appeared earlier in treatments of the free Rarita-Schwinger field to isolate the physical degrees of freedom. The trace part $\sigma$ is found to be unphysical in this case, cf. e.g. [262].

The inverse matrix $b^{-1}$ rotates $N^I$ back to its "time gauge" value $n_0^J$. Starting from time gauge and spinors in $V$, we then impose as reality condition that the spinors when rotated back to "time gauge" are always real valued, i.e. e.g. $b^{-1}(N)\sigma = (b^{-1}(N)\sigma)^*$, where $b(N)$ here is in the spinor representation. Based on this idea, one can indeed obtain a faithful extension of the supergravity phase space to the gauge group $\mathrm{SO}(D+1)$, and we refer the reader to the original work [6] for details. For what follows, we only need that after solving all second class constraints, the final non-vanishing anti-brackets read

$$\{\rho_r^i(x), \rho_r^j(y)\} = -\frac{i}{2}\mathbb{P}^{ij}\ \delta^{(D)}(x-y), \tag{15.18}$$

$$\{\rho_r^0(x), \rho_r^j(y)\} = 0, \tag{15.19}$$

$$\{\rho_r^0(x), \rho_r^0(y)\} = 0, \tag{15.20}$$

$$\{\sigma_r(x), \sigma_r(y)\} = i\frac{D}{2(D-1)}\mathbb{1}\ \delta^{(D)}(x-y), \tag{15.21}$$

together with the usual canonical brackets we encountered in part II in the bosonic sector, and all fields are real valued (The subscript $r$ is to remind that these fields are not the same as above). In terms of these fields, the spatial diffeomorphism constraint and the linear simplicity and normalisation constraints read as one would expect, but $\mathscr{H}$ and $\mathfrak{S}$ are intricate. More surprisingly, also the Gauß constraint is very complicated, which is related with the non-covariant split in $\rho^0$ and $\rho^j$- components in the anti-brackets. Still one can prove that all of these remaining constraints, $\mathscr{H}$, $\mathscr{H}_a$, $\mathscr{G}^{IJ}$, $\mathscr{S}^a_{\overline{IM}}$, $\mathscr{N}$ and $\mathfrak{S}$, are first class, and we again refer the interested reader to [6] for the details.

### 15.1.2 Kinematical Hilbert space for the Rarita Schwinger field

Consider the finite dimensional complex vector space $V$ of polynomials of $N$ real valued Graßmann variables $\theta_A$, $A \in \{1, ..., N\}$, with complex coefficients. A polynomial $f \in V$ can be written as

$$f = \sum_{n=0}^{N} \sum_{1 \leq A_1 < .. < A_n \leq N} f^{(n)}_{A_1..A_n}\ \theta^{A_1}..\theta^{A_n}, \tag{15.22}$$

where $f^{(n)}_{A_1..A_n}$ are a complex $n$ - forms. An obviously positive definite sesqui-linear form on $V$ is given by

$$\langle f, f' \rangle := \sum_{n=0}^{N} \sum_{A_1 < .. < A_n} \overline{f^{(n)}_{A_1..A_N}}\ f^{(n)\prime}_{A_1..A_N}, \tag{15.23}$$

which is invariant under U($N$) acting on $V$ by

$$f \mapsto U \cdot f; \quad [U \cdot f]^{(n)}_{A_1..A_N} = f^{(n)}_{B_1..B_N} U_{B_1 A_1}..U_{B_N A_N}. \tag{15.24}$$

On the Graßmann variables, this corresponds to $\theta_A \mapsto U_{AB}\theta_B$. Note that this is not an action on real Graßmann variables unless $U$ is real valued. Therefore we restrict U($N$) to O($N$) (more precisely, to a subgroup thereof), and to real valued coefficients in the polynomials $f$ in what follows.

One can check that with the above inner product the operators

$$[\theta_A \cdot f](\theta) := \theta_A\, f(\theta), \quad [\partial_A \cdot f](\theta) := \partial^l f(\theta)/\partial\theta_A \tag{15.25}$$

[1]satisfy the adjointness relations $\theta_A^\dagger = \partial_A$. Therefore, we can define the operators $\hat{\theta}_A := \sqrt{\hbar}[\theta_A + \partial_A]$, which obviously are self-adjoint and can be checked to satisfy the anticommutation relations

$$[\hat{\theta}_A, \hat{\theta}_B]_+ = 2\hbar\delta_{AB}. \tag{15.26}$$

This already gives (up to a constant factor) a faithful representation of the abstract CAR *-algebra for $\sigma_r^\alpha$ if we interpret $A$ as spinor index,

$$\hat{\sigma}_\alpha := \frac{1}{2}\sqrt{\frac{D\hbar}{D-1}}[\theta_\alpha + \partial_\alpha]. \tag{15.27}$$

For $\rho_r^i$, $A$ is compound index $(j, \alpha)$, $j \in \{1, ..., D\}$, $\alpha \in \{1, ..., 2^{\lfloor (D+1)/2 \rfloor}\}$. Using that $\mathbb{P}^{\alpha\beta}_{ij}$ is a real valued projector (in particular symmetric and positive semidefinite), we can define the self adjoint operators

$$\hat{\rho}_i^\alpha := \frac{\sqrt{\hbar}}{2}\mathbb{P}^{\alpha\beta}_{ij}[\theta_j^\beta + \partial_j^\beta] \tag{15.28}$$

satisfying

$$[\hat{\rho}_j^\alpha, \hat{\rho}_k^\beta]_+ = \frac{\hbar}{2}\mathbb{P}^{\alpha\beta}_{jk}. \tag{15.29}$$

The Hilbert space $\mathscr{H}_v$ for each point $v$ on the spatial slice then is just given by the tensor product of the Hilbert spaces we just constructed for both $\rho$ and $\sigma$, and the field theoretic generalisation thereof is constructed as in the case of Dirac fermions, either using an inductive limit of the finite tensor products of the point Hilbert spaces $\mathscr{H}_v$ or the infinite tensor product of these Hilbert spaces over all points $v$, cf. [62, 263] for details.

[1] $\frac{\partial^l}{\partial\theta_A}$ here denotes the left derivative, see, e.g., [161] for more details.

## 15.2 Three-form field of $d = 11$ $N = 1$ supergravity

In this section, we will study the quantisation of new bosonic fields in supergravity theories using the example of the three-form field ("three index photon") of $d = 11$ $N = 1$ supergravity. We will start by shortly reviewing the classical canonical theory and afterwards study its quantisation. Again, our exposition will be rather brief and we refer the interested reader to [7] for more details.

### 15.2.1 Canonical formulation

The Hamiltonian analysis of the full $d = 11$ $N = 1$ supergravity Lagrangian[135] was studied in [264, 265]. Here, we will restrict to the contributions to the action stemming form the three-form $A_{\mu\nu\rho} = A_{[\mu\nu\rho]}$, since the remaining parts (graviton- and gravitino-part in $d = 11$) already were in included in the loop quantisation programme. This part of the Lagrangian is not solely the (generalisation of) the Maxwell term coupled to a current, but due to the presence of a Chern Simons term, the field actually becomes self-interacting. It is given up to a numerical constant by

$$\begin{aligned} L = &-\frac{1}{2}\sqrt{|g|}F_{\mu_1..\mu_4}\ F^{\mu_1..\mu_4} - \alpha\sqrt{|g|}F_{\mu_1..\mu_4}\ J^{\mu_1..\mu_4} \\ &- \frac{c}{2}\sqrt{|g|}F_{\mu_1..\mu_4}\ F_{\nu_1..\nu_4}\ A_{\rho_1..\rho_3}\epsilon^{\mu_1..\mu_4\nu_1..\nu_4\rho_1..\rho_3}, \end{aligned} \tag{15.30}$$

where $F = dA$, $F_{\mu_1..\mu_4} = \partial_{[\mu_1}A_{\mu_2..\mu_3]}$ is the curvature of $A$, $J$ is a totally skew tensor current bilinear in the Rarita Schwinger field and furthermore depending on the vielbein, not containing derivatives. The specific form will not be important in what follows. Furthermore, $c$ and $\alpha$ are positive constants fixed by the requirement of local supersymmetry. We will call $c$ the level of the Chern Simons theory in analogy to the three dimensional case.

We only want to highlight the main results of the canonical analysis, which is straight forward but tedious. Performing the $10 + 1$ split as in section 2.2.1, we find that the Lagrangian is singular in Dirac's [157] terminology: While the momenta $\pi^{abc}$ to the spatial components $A_{abc}$ of the three-form field can be solved for the corresponding velocities, the temporal components $A_{tab}$ act as Lagrange multipliers fields and give rise to the primary constraint

$$\mathscr{G}^{a_1a_2} := \partial_{a_3}\pi^{a_1..a_3} - \frac{c}{2}\epsilon^{a_1a_2b_1..b_4c_1..c_4}F_{b_1..b_4}F_{c_1..c_4}. \tag{15.31}$$

This is the analogue of the Gauß constraint in the Maxwell case, however, due to the presence of the Chern Simons term in the action, it gets an additional contribution corresponding to the second term in (15.31). This leads to the following action on the phase space variables

$$\{A, \mathscr{G}[\lambda]\} = -d\lambda, \tag{15.32}$$

$$\{*\pi, \mathscr{G}[\lambda]\} = c\ (d\lambda) \wedge F, \tag{15.33}$$

where we introduced the dual seven-pseudo form $(*\pi)_{a_1..a_7} := \frac{1}{3!\,7!}\epsilon_{b_1..b_3a_1..a_7}\pi^{b_1..b_3}$, smeared versions $\mathscr{G}[\lambda]$ of the constraint as in parts I, II, where $\lambda$ is an arbitrary two form field on $\sigma$, and used the canonical Poisson brackets

$$\{A_{b_1..b_3}(x), \pi^{a_1..a_3}(y)\} = \delta^{a_1}_{[b_1}\delta^{a_2}_{b_2}\delta^{a_3}_{b_3]}\ \delta^{(10)}(x-y). \tag{15.34}$$

Note that, unlike in the Maxwell case, $\pi$ (the analogue of the electric field) is not invariant under the action of the "twisted" Gauß constraint $\mathscr{G}$, which has tremendous consequences for quantisation as we will see.

Apart from the appearance of the three-form Gauß constraint, of course, the canonical analysis will lead to corresponding three-form contributions to the Hamiltonian, spatial diffeomorphism and supersymmetry constraint, but their explicit form will not be important for what follows. We only want to point out that $\mathscr{G}$ can be checked to be an Abelian ideal in the constraint algebra, i.e. Poisson commutes strongly with all constraints including itself.

### 15.2.2 Reduced phase space quantisation

Trying to quantise the theory on a kinematical Hilbert space of the type usually used in LQG immediately leads to several problems. Integrating $A$, $\pi$ over oriented three-dimensional and seven-dimensional submanifolds, respectively, to write down the generalisation of the holonomy flux algebra and the LQG type positive linear functional thereon which then gives a Hilbert space representation by the GNS construction, can be done analogously. However, the extra term in the twisted Gauß constraint $F \wedge F$ does not exists in this representation, being discontinuous in the holonomies. Even if we would give a procedure of how to regularise this term, the usual solution space to the untwisted Gauß constraint (spanned by a generalisation of gauge invariant spin networks) would not solve these constraints, since its elements would be annihilated by $\partial_a\pi^{abc}$ but not by the second term $F \wedge F$.

Therefore, in [7] we propose a reduced phase space quantisation. This suggests itself since $\mathscr{G}$ is an ideal in the constraint algebra. Indeed, one finds that

$$P^{abc} := \pi^{abc} + c\epsilon^{abcd_1..d_4e_1..e_3} F_{d_1..d_4} A_{e_1..e_3} \quad \Leftrightarrow \quad *P = *\pi + c\, A \wedge F, \tag{15.35}$$

and $F = dA$ are Dirac observables with respect to $\mathscr{G}$ and all other constraints can be expressed in terms of $F$, $P$ and terms independent of $A$, $\pi$.

Computing the observable algebra, we find

$$\{F[h], F[h']\} = 0, \quad \{P[f], F[h]\} = \int h \wedge df, \quad \{P[f], P[f']\} = -3c\, F[f \wedge f'], \tag{15.36}$$

where we introduced $P[f] := \int_\sigma f \wedge *P$ and $F[h] := \int_\sigma h \wedge F$ for a three-form $f$ and a six-form $h$. While the algebra closes, $P$ and $F$ are not conjugate. In particular, if we in analogy to LQG choose a discontinuous representation in which only the corresponding Weyl elements are defined but $F[h]$ itself does not exists, (15.36) shows that $P[f]$ also cannot be defined. We therefore are looking for a representation in which only the Weyl elements $W[h, f] := \exp\left(i(F[h] + P[f])\right)$ corresponding to both, $F$ and $P$ exists.

For the $^*$-relations and Weyl relations, we find[1]

$$W[h, f]^* = W[-h, -f], \tag{15.37}$$

$$W[h, f]\, W[h', f'] = W[h + h' + \frac{3c}{2} f \wedge f', f + f'] \times \exp\left(\frac{i}{4} \int \left[2(h \wedge df' - h' \wedge df) - cf \wedge f' \wedge d(f - f')\right]\right). \tag{15.38}$$

Note that also the Weyl relations get twisted due to the presence of the Chern Simons term ($c \neq 0$).

The Narnhofer-Thirring type functional [136], which also was applied in the context of loop quantisation of the closed bosonic string [93],

$$\omega(W(h, f)) = \begin{cases} 1 & h = f = 0 \\ 0 & \text{else} \end{cases}, \tag{15.39}$$

can be shown to give a positive linear functional on the $^*$-algebra $\mathfrak{A}$ generated by the Weyl elements, and therefore a Hilbert space representation thereof by means of the GNS construction. This representation is strongly discontinuous in both, $h$ and $f$ and

[1] To compute the latter, one needs to generalise the Baker-Campbell-Hausdorff formula [266–271] to higher commutators [272].

while cyclic, not irreducible.

Finally, it was studied in [7] if the Weyl algebra and the state $\omega$ continue to be well-defined if we introduce singular smearing in the spirit of holonomies and fluxes in usual LQG, i.e. when restricting the smearing functions $h$, $f$ to the form factors of four- and seven-surfaces respectively. The answer turns out to be in the affirmative, and we refer the reader to the original literature [7] for details. This implies that terms in the Hamiltonian and supersymmetry constraint depending on $F$, $*P$ can be regularised in the spirit of [30].

# Part V

# Approaching black holes in higher dimensional LQG: Isolated horizon boundary degrees of freedom

The first articles on black hole entropy in LQG by Krasnov [39] and Rovelli [40] appeared almost simultaneously in 1996. Roughly, employing the discreteness of area in LQG, by counting the microstates compatible with a certain macroscopic area of a two-surface, an entropy proportional to its area is derived. This result was significantly strengthened by Ashtekar and collaborators [273–276]. Invoking the newly introduced isolated horizon framework, which gives a quasi-local notion of black holes, it could be shown at the classical level that, when using the Ashtekar's variables and imposing the boundary conditions corresponding to a spherically symmetric isolated horizon at the inner boundary of a given spacetime, a U(1) Chern Simons theory arises on the horizon in order to render the variational principle well defined. In fact, the Chern Simons connection turns out to be nothing else than the pull back to the horizon of the Ashtekar connection in the bulk. Smolin in a seminal work [38] already anticipated the role of this topological field theory on inner boundaries in spacetime. Quantisation of the three dimensional Chern Simons theory is well studied [184] and subsequent state counting lead to a rigorous derivation of $S \propto A$ within LQG. Moreover, the constant of proportionality can be chosen to coincide with Bekenstein's and Hawking's result when fixing the Barbero Immirzi parameter $\gamma$ appropriately. The methods of counting were subsequently corrected and refined in [277, 278]. Sophisticated number theoretical and combinatorial methods introduced by Barbero and collaborators (cf. e.g. [279] and references therein) finally allowed for an exact computation of the entropy.

Central to the early derivations was the spherical symmetry of the horizon and the related constancy of the Ricci curvature scalar of the horizon two-sphere cross sections, and only later was extended to axisymmetric horizons [280] and finally to arbitrary horizon shape [281].

Quite recently, it was argued by Perez and collaborators that the U(1) Chern Simons theory arises only due to a certain unnecessary gauge fixing, and that one should work with an SU(2) Chern Simons boundary theory instead [282]. The full bulk group SU(2) entered the picture already earlier when deriving logarithmic corrections to the entropy formula [283]. To the best of the author's knowledge, it is still debated which of the two should be preferred [284]. While conceptually the same, the use of SU(2) leads to a different prediction for the value of $\gamma$ and different logarithmic corrections. Furthermore, the SU(2) analysis suggests that one actually can allow for more general connections on the boundary, not necessarily equal with the (pullback of the) bulk connection, which in turn allows to obtain the right prefactor in the entropy formula without fixing the

Barbero Immirzi parameter [43]. Extension of the SU(2) theory to horizons without spherical symmetry turns out to be possible, but is more complicated than in the U(1) case [285] and leads to new challenges at the quantum level. For a recent review comparing the U(1) and SU(2) framework and introducing the state counting, we refer the interested reader to [43]. First results that the entropy formula can also be recovered with spin foams are given in [286, 287].

In this part, we generalise the isolated horizon treatment of usual $D = 3$ LQG to higher dimensions $D = 2n + 1$, resulting in an $\mathrm{SO}(2(n + 1))$ Chern Simons symplectic structure on the intersections of the black hole horizon with the spatial slice. We will also derive higher dimensional analogues of the boundary condition $F \propto \Sigma$. We have to restrict to even spacetime dimensions $D+1 = 2(n+1)$, since $a$) otherwise there does not exist a higher dimensional Chern Simons theory on the odd dimensional horizon and $b$) the Euler topological density of the $(D-1)$ dimensional intersection of the horizon and the spatial slice, which plays a central role in our considerations, is only defined in even dimensions. We comment briefly on a possible quantisation of the horizon theory and argue that the local degrees of freedom naturally arising in higher-dimensional non-Abelian Chern Simons theory could be erased at the quantum level by quantising the boundary conditions. The exposition follows [10] and several parts are taken from there.

The part is organised as follows: in the section on preliminaries 16, we will firstly introduce in section 16.1 some new notation which was so far not necessary. Then, in section 16.2 we will briefly discuss the Hamiltonian formulation of Chern Simons theory in higher dimensions, in particular the derivation of its symplectic structure. This of course is well-known (cf. e.g. [137]) and only added for completeness. Thereafter, we will introduce the notion of higher dimensional isolated horizons in section 16.3 and derive their consequences. They already have been studied in [288–291] and our definition of higher dimensional, undistorted, non-rotating isolated horizons (UDNRIH) does not differ significantly from the definitions given there.

Thereafter, we will turn to the derivation of the boundary degrees of freedom (chapter 17). Firstly, we will give a comparison of the results obtained in this part with the ones from the usual treatment in $3+1$ dimensions (section 17.1). This section (partly) summarises the results obtained in the following sections, in which lengthy derivations are provided.

Afterwards we turn to the case of structure group $\mathrm{SO}(1, D)$, where we can start our considerations from the Palatini action principle. We will derive suitable boundary conditions connecting the (pullback to the horizon cross sections of the) Palatini connection and an its conjugate momenta at the boundary (section 17.2.1). Moreover, we find that the symplectic structure of an $\mathrm{SO}(1, D)$ Chern Simons arises at as boundary contribution to the symplectic structure at the UDNRIH, and moreover, that the Chern Simons connection coincides with the (pullback of the) Palatini connection (section 17.2.2).

For the structure group $\mathrm{SO}(D+1)$, which we have to prefer when quantising, we cannot start from an action principle and we have to work purely in the Hamiltonian picture. What we will show is that one can formulate boundary conditions whose form is similar to those in the $\mathrm{SO}(1, D)$ case, but which now connect the momenta conjugate to the bulk with the hybrid connection $\Gamma^0$ of appendix C on the horizon cross sections. Furthermore, the boundary term to the symplectic structure obtained when extending the phase space from ADM to the new $\mathrm{SO}(D+1)$ formulation can be reformulated in terms of an $\mathrm{SO}(D+1)$ Chern Simons symplectic structure of exactly this connection (section 17.3). Changing to $\mathrm{SO}(D+1)$ seems to force us into the extended paradigm of Perez and collaborators (the bulk and the Chern Simons connection need not be directly related), but apart from that no conceptual novelties show up. Like in [292], the boundary connection is not uniquely determined and we shortly comment on its non-uniqueness in $D = 3$.

Up to this point, we considered undistorted horizons only. In section 17.4, we will discuss the generalisation of both, the Engle-Bettle [281] method as well as the Perez-Pranzetti [285] method of how to incorporate distortion. The former generalises to the $\mathrm{SO}(4)$ theory, but it is unclear if it works also in higher dimensions. The latter employed two $\mathrm{SU}(2)$ Chern Simons connections and few additional, more or less manageable constraints already in $D = 3$. In higher dimensions, a straight forward generalisation is possible but invokes $\lceil \frac{D+1}{4} \rceil + 1$ SO(D+1) Chern Simons connections and many more constraints, which make it doubtful if this route can be continued to the quantum level.

We will close with some comments on quantisation in section 17.5, which is far from straight forward since non-Abelian Chern Simons theory in higher dimensions becomes non-topological. Further comments and tentative research directions can be found in the general discussion section 18.2. Some additional material for this part is provided in appendix F, where an overview over the higher-dimensional Newman-Penrose formalism is given, as well as in appendix G where further calculational details to derivations of the main text are provided.

# 16

# Preliminaries

## 16.1 Further notation and conventions

This part will need some additional notation not used in the remainder of this thesis, which we want to introduce briefly.

Apart from $D$-dimensional Cauchy surfaces $\Sigma$, we will now also have to deal with $D$-dimensional null surface within the spacetime manifold $\mathscr{M}$, which we will denote by $\Delta$. We will restrict the topology of $\Delta$ to be $S \times \mathbb{R}$, where $S$ is a $(D-1)$-dimensional compact Riemannian manifolds which has non-zero Euler characteristic. Examples are the $(D-1)$-spheres $S^{D-1}$ or hyperbolic spaces $H^{D-1}$ divided by a freely acting discrete subgroup $\Gamma$, e.g. handle bodies with genus $g > 1$ for $D = 3$ (at the level of topology) and the corresponding black hole solutions, given e.g. in [293]. For notational simplicity, we will refer to all these manifolds as spheres in this work but keep in mind that more general topologies are allowed. We mostly restrict attention to even spacetime dimensions $D + 1 =: 2(n + 1)$, having the advantages that (a) there can exist a Chern Simons theory on the odd $(2n + 1)$-dimensional $\Delta$ and (b) the Euler density [294] is defined for the even $(2n)$-dimensional intersections $S \cong S^{D-1}$ of $\Sigma$ and $\Delta$. In addition to the index conventions of the remainder of this thesis, we will use:

- tensorial indices on $\Delta$ will be denoted by the $\underleftarrow{\mu}, \underleftarrow{\nu}, \underleftarrow{\rho}$ (the pullback arrow will sometimes be omitted if there should be no confusion whether the equation is referring to $\mathscr{M}$ or $\Delta$).

- tensorial indices in $(D-1)$-dimensional subspaces $S$ will be denoted by lower Greek letters from the beginning of the alphabet: $\alpha, \beta, \gamma, \ldots \in \{1, \ldots, D-1\}$ or by $\underset{\Leftarrow}{\mu}$.

- Lie algebra indices of some gauge group $G$ will be denoted by capital Latin letters from the beginning of the alphabet: $A, B, C \in \{1, \ldots, \dim(\mathfrak{g})\}$. Note that this differs from our conventions in part II.

Apart from the spacetime metric $g_{\mu\nu}$ and the spatial metric $q_{ab}$, the (degenerate) metric on $\Delta$ denoted by $h_{\underset{\longleftarrow}{\mu\nu}}$ and the Riemannian metric on $(D-1)$ dimensional subspaces denoted by $h_{\alpha\beta}$ need to be introduced. The corresponding Levi-Civita connections will be denoted by $\nabla_\mu$, $D_a$, $D_{\underset{\leftarrow}{\mu}}$ and $D_\alpha$. We denote by $E^{(D+1)} := \epsilon^{\mu_1\nu_1\ldots\mu_{n+1}\nu_{n+1}}\epsilon^{I_1J_1\ldots I_{n+1}J_{n+1}}R_{\mu_1\nu_1I_1J_1}\ldots R_{\mu_{n+1}\nu_{n+1}I_{n+1}J_{n+1}}$ the Euler topological density [294] and remark that it coincides with other definitions in the literature only up to normalisation, i.e. the integral of this density over a closed compact manifold, in our case $S^{D-1}$, denoted by $\langle E^{(2n)} \rangle$, gives a only a *multiple* of the Euler characteristic $\chi_S$ of $S$. We choose this definition since it simplifies many formulas. Explicitly, we have

$$\chi_S = \frac{1}{(8\pi)^n n!}\int_S E^{(2n)}, \tag{16.1}$$

which in our case, i.e. spheres $S^{2n}$, results in $\chi_{S^{2n}} = 2$.

The null normal to $\Delta$ will be denoted by $l$ and the vector field normal to the $(D-1)$ – sphere cross-sections by[1] $k$, normalised to $l \cdot k = -1$ (cf. section 16.3). $k$ can be extended uniquely to a spacetime 1-form at points of $\Delta$ by requiring it to be null. Then, at points of $\Delta$, we can decompose the metric according to $g_{\mu\nu} = h_{\mu\nu} - 2l_{(\mu}k_{\nu)}$. We will denote the $h$-projected vielbein by $m$, $m_{\mu I} = h^\nu_\mu e_{\nu I}$, and furthermore use the notation $l^I = l^\mu e_\mu{}^I$, $k^I = k^\mu e_\mu{}^I$, and, since $l, k$ are null and normalised, $k^I k^J \eta_{IJ} = 0 = l^I l^J \eta_{IJ}$, $k^I l^J \eta_{IJ} = -1$. We will call $\{l, k, \{m_I\}\}$ a generalised null frame. Elements of higher dimensional Newman-Penrose formalism in this frame will be introduced in appendix F.

We will denote with $s$ the spacelike normal to the $(D-1)$ - dimensional cross-sections $\Sigma \cap \Delta$, $s^2 = 1, s \cdot n = 0$, pointing outward of $\sigma$ ($n$ of course is again the future pointing timelike unit normal to a spatial slice $\Sigma$). When dealing with the Hamiltonian formulation, we will choose the foliation such that $l = \frac{1}{\sqrt{2}}(n-s)$, $k = \frac{1}{\sqrt{2}}(n+s)$ holds, where $l$ and $k$ are the (representatives of the equivalence class of the) null normals to

[1] We refrain from using the usual notation $n$ for this normal here, to avoid confusion with the normal to spatial slices, and also to make clear the difference between the hybrid vielbein normal $n^I$ and $k^I = k^\mu e_\mu{}^I$.

a given isolated horizon as specified in section 16.3. Furthermore, we will use the notation $s^I := s^\mu e_\mu{}^I$ and introduce $\bar{\bar{\eta}}_{IJ} := \eta_{IJ} + n_I n_J - s_I s_J = \eta_{IJ} + 2l_{(I}k_{J)} = m_{\mu I} m^\mu{}_J$, $\bar{\bar{\eta}}_{IJ} n^J = \bar{\bar{\eta}}_{IJ} s^J = \bar{\bar{\eta}}_{IJ} l^J = \bar{\bar{\eta}}_{IJ} k^J = 0$. An upper twiddle indicates the density weight of one, e.g. $\tilde{s}^I := \sqrt{\det h}\; s^I$.

Finally, a word of caution: If using the structure group $\mathrm{SO}(D+1)$, which implies that the internal and external signature do not match, several of the above formulas get changed by signs ($n^I$ becomes spacelike, and the $n\ n$ - terms in the definitions of $\bar{\bar{\eta}}$) or even become obsolete (since, to perform the signature switch, we already are in the Hamiltonian framework, $l^I$ and $k^I$ are not null anymore).

## 16.2 Higher dimensional Chern-Simons theory

We will review some facts about Chern-Simons theory in higher dimensions relevant for this work, with focus on the canonical formulation. In particular, we will derive the symplectic structure of the theory. We want to stress that these results are not new, but we state them here for completeness. For a more elaborate canonical treatment of higher dimensional Chern-Simons theory, we refer the reader to [138].

The Chern-Simons action is defined for all odd dimensions $2n+1$ and gauge groups $G$ by the equation

$$dL_{CS}^{2n+1} = i_{A_1 A_2 \ldots A_{n+1}} F^{A_1} \wedge \ldots \wedge F^{A_{n+1}}, \tag{16.2}$$

where $F^A = dA^A + 1/2\ [A,A]^A = dA^A + 1/2\ f^A{}_{BC}\ A^B \wedge A^C$ is the field strength of the connection one form $A^B$ valued in the Lie algebra of $G$, $f^A{}_{BC}$ are the structure constants of $G$, $A_j, B, C \in \{1, ..., \dim(\mathfrak{g})\}$ are Lie algebra indices and $i_{A_1 \ldots A_n}$ is a rank $(n+1)$ symmetric tensor invariant under the adjoint action of the group. Explicitly,

$$\begin{aligned} L_{CS}^{2n+1} = i_{A_1 \ldots A_{n+1}} \sum_{p=0}^{n} (-1)^p \frac{\binom{2n+1}{n-p}}{\binom{2n+1}{n}} \times \\ \underbrace{F^{A_1} \wedge \ldots \wedge F^{A_{n-p}}}_{n-p} \wedge \underbrace{\left(1/2\ [A,A]^{A_{n-p+1}}\right) \wedge \ldots \wedge \left(1/2\ [A,A]^{A_n}\right)}_{p} \wedge A^{A_{n+1}} \\ =: i \cdot \sum_{p=0}^{n} (-1)^p \frac{\binom{2n+1}{n-p}}{\binom{2n+1}{n}} F^{n-p} \wedge (1/2\ [A,A])^p \wedge A \quad , \end{aligned} \tag{16.3}$$

where the second line defines the short hand notation we will use in the following. For our purposes, it will be sufficient to restrict attention to the groups $\mathrm{SO}(1,D)$ or

SO$(D+1)$ where $D = 2n+1$. It is convenient to label the $\frac{D(D+1)}{2}$ generators of the corresponding Lie algebras by an anti-symmetric combination of two indices in the fundamental representation $I, J = 0, ..., D$ (e.g. the connection one form will be denoted by $A^{IJ}$ with $A^{(IJ)} = 0$). We will furthermore restrict the invariant tensor to be the epsilon tensor $\epsilon^{I_1 J_1 \ldots I_{n+1} J_{n+1}}$, which is the one relevant for our application. However, we want to point out that all results of this section are independent of the choice of gauge group and invariant tensor.

In order to obtain the (pre-)symplectic structure, we invoke the covariant canonical formalism [295–297], according to which the presymplectic potential is given by the boundary term of the first variation of the action, while the presymplectic structure is the exterior derivative of the potential.

Using the relation

$$\begin{aligned}\delta\left(\epsilon\ F^{n-p} \wedge \frac{1}{2}[A,A]^p \wedge A\right) = \epsilon\Big\{&(n+p+1)\ \delta A \wedge F^{n-p} \wedge \frac{1}{2}[A,A]^p + \\ &(n-p)\ \delta A \wedge F^{n-p-1} \wedge \frac{1}{2}[A,A]^{p+1} + \\ &(n-p)\ d\left[\delta A \wedge F^{n-p-1} \wedge \frac{1}{2}[A,A]^p \wedge A\right]\Big\}, \end{aligned} \tag{16.4}$$

the first variation of the Chern-Simons action yields

$$\begin{aligned}\delta S_{CS}^{2n+1} &= \delta \int_{\mathscr{M}} L_{CS}^{2n+1} \\ &= \int_{\mathscr{M}} \left[\epsilon \cdot \sum_{p=0}^{n} (-1)^p \frac{\binom{2n+1}{n-p}}{\binom{2n+1}{n}} (n+p+1) F^{n-p} \wedge \frac{1}{2}[A,A]^p\right] \wedge \delta A \\ &\quad + \int_{\mathscr{M}} \left[\epsilon \cdot \sum_{p=0}^{n-1} (-1)^p \frac{\binom{2n+1}{n-p}}{\binom{2n+1}{n}} (n-p) F^{n-p-1} \wedge \frac{1}{2}[A,A]^{p+1}\right] \wedge \delta A \\ &\quad + \int_{\mathscr{M}} d\left\{\delta A \wedge \left[\epsilon \cdot \sum_{p=0}^{n-1} (-1)^p \frac{\binom{2n+1}{n-p}}{\binom{2n+1}{n}} (n-p) F^{n-p-1} \wedge \frac{1}{2}[A,A]^p \wedge A\right]\right\} \\ &= \int_{\mathscr{M}} (n+1)\ \epsilon \cdot F^n \wedge \delta A \\ &\quad + \int_{\mathscr{M}} d\left\{\delta A \wedge \left[\epsilon \cdot \sum_{p=0}^{n-1} (-1)^p \frac{\binom{2n+1}{n-p}}{\binom{2n+1}{n}} (n-p) F^{n-p-1} \wedge \frac{1}{2}[A,A]^p \wedge A\right]\right\}. \end{aligned} \tag{16.5}$$

Note that the two sums of the bulk contribution cancel each other term by term, and the only term surviving is the $(p = 0)$ – term of the first sum. We obtain the Chern-Simons equations of motion[1]

$$\epsilon \cdot \underbrace{F \wedge ... \wedge F}_{n \text{ times}} = 0 \ , \tag{16.6}$$

which in 2+1 dimensions (which corresponds to $n = 1$) reduces to $F = 0$. Let $\sigma$ be a $2n$-dimensional Cauchy slice. The presymplectic potential can be read off the boundary term of the first variation and is given by

$$\theta_\sigma(\delta) = \int_\sigma \delta A \wedge \left[ \epsilon \sum_{p=0}^{n-1} \frac{\binom{2n+1}{n-p}}{\binom{2n+1}{n}} (-1)^p (n-p) F^{n-p-1} \wedge \frac{1}{2} [A, A]^p \wedge A \right] \ . \tag{16.7}$$

For its variation, the equation

$$\begin{aligned} &\delta_{[2} \left[ \epsilon \cdot \delta_{1]} A \wedge F^{n-p-1} \wedge \frac{1}{2} [A, A]^p \wedge A \right] = \\ &\qquad \epsilon \cdot \left[ \frac{1}{2}(n+p+1)\ \delta_{[1} A \wedge \delta_{2]} A \wedge F^{n-p-1} \wedge \frac{1}{2} [A, A]^p \right. \\ &\qquad\qquad \left. + \frac{1}{2}(n-p-1)\ \delta_{[1} A \wedge \delta_{2]} A \wedge F^{n-p-2} \wedge \frac{1}{2} [A, A]^{p+1} \right] \ , \end{aligned} \tag{16.8}$$

is useful. Actually, in the above result a boundary term was dropped, but in defining the symplectic current, we are allowed to drop this term since we will integrate the symplectic current we want to derive in this step over the boundary of the spacetime region we are interested in. We find for the symplectic current

$$\begin{aligned} &d\theta_\sigma(\delta_2, \delta_1) \\ &= \frac{1}{2\binom{2n+1}{n}}\ \epsilon \cdot \delta_{[1} A \wedge \delta_{2]} A \wedge \left[ \sum_{p=0}^{n-1} \binom{2n+1}{n-p} (-1)^p (n-p)(n+p+1)\ F^{n-p-1} \wedge \frac{1}{2} [A, A]^p \right. \\ &\qquad \left. + \sum_{p=0}^{n-2} \binom{2n+1}{n-p} (-1)^p (n-p)(n-p-1)\ F^{n-p-2} \wedge \frac{1}{2} [A, A]^{p+1} \right] \\ &= \frac{n(n+1)}{2}\ \epsilon \cdot \delta A \wedge \delta A \wedge F^{n-1} \ , \end{aligned} \tag{16.9}$$

where again the terms in the two sums cancel each other out, with only the $(p = 0)$ – term in the first sum remaining. Therefore, the presymplectic structure is given by

$$\Omega_\sigma(\delta_2, \delta_1) = \frac{n(n+1)}{2} \int_\sigma \epsilon \cdot \delta_{[1} A \wedge \delta_{2]} A \wedge F^{n-1} \ . \tag{16.10}$$

[1]Note that the bulk term of the variation can be obtained within two lines by varying 16.2.

Usually, in order to have a meaningful phase space description, one now imposes suitable boundary conditions and checks if the presymplectic structure is independent of the choice of the Cauchy slice $\sigma$ and, for noncompact $\sigma$, if the integral is finite. However, in this thesis we are only interested in a spacetime with internal isolated horizon boundary on which the Chern-Simons symplectic structure arises and we only have to answer this questions for the full spacetime.

From 16.9, we can also read off that the Dirac matrix of Chern-Simons theory is given, up to numerical factors, by $\epsilon \cdot F^{n-1}$, which coincides with the result in [138, eq. (2.7)].

## 16.3 Higher dimensional isolated horizons

The isolated horizon framework was introduced in a series of seminal papers [273–275, 298] and extended to higher dimensions in [288–291]. We will therefore only briefly state the definition of undistorted, non-rotating horizons in higher dimensions which we will be using, and discuss its consequences. The definition is geared towards the goal of the next section, namely to obtain the boundary condition which will lead to a higher-dimensional Chern-Simons theory on the boundary. We will start by giving the weaker definitions of near expanding and weakly isolated horizons and a brief discussion of their consequences in a manner very similar to [275]:

**Definition 1.** *A sub-manifold $\Delta$ of $(M, g)$ is said to be a non-expanding horizon (NEH) if*

*(1) $\Delta$ is topologically $\mathbb{R} \times S^{D-1}$ and null.*

*(2) Any null normal $l$ of $\Delta$ has vanishing expansion $\theta_l := h^{\mu\nu}\nabla_\mu l_\nu$[1].*

*(3) All field equations hold at $\Delta$ and $-T^\mu_\nu l^\nu$ is a future-causal vector for any future directed null normal $l$.*

We will state the consequences of definition 1. For more details on the derivations, we refer the interested reader to the standard literature cited above:

(a) *Properties of $l$*: Being a null normal to $\Delta$, $l$ is automatically twist free and geodesic. Moreover, using the vanishing of $\theta_l$, the Raychaudhuri equation and the condition on the stress energy tensor, one can show it is additionally shear free and $R_{\mu\nu}l^\mu l^\nu \mathrel{\widehat{=}} 0$.

[1]On $\Delta$, $h^{\mu\nu}$ is any tensor such that $h_{\mu\nu} = h_{\mu\mu'}h^{\mu'\nu'}h_{\nu\nu'}$

(b) *Conditions on the Ricci tensor:* From the condition on $T_{\mu\nu}$, the field equations and the relation for $R_{\mu\nu}$ in (a) it follows that $R_{\underleftarrow{\mu}\nu}l^\nu \mathrel{\widehat{=}} 0$, or, in Newman-Penrose formalism,

$$\Phi_{00} = R_{\mu\nu}l^\mu l^\nu \mathrel{\widehat{=}} 0 \quad \text{and} \quad \Phi_{0J} = R_{\mu\nu}l^\mu m^\nu_J \mathrel{\widehat{=}} 0. \tag{16.11}$$

(c) *Induced Connection on* $\Delta$: Due to (a), there exists a unique intrinsic derivative operator $D$ on $\Delta$. Its action on vector fields $X \in T\Delta$ and on 1-forms $\eta \in T^*\Delta$ are given by

$$D_\mu X^\nu \mathrel{\widehat{=}} \nabla_{\underleftarrow{\mu}} \tilde{X}^\nu \quad \text{and} \quad D_\mu \eta_\nu \mathrel{\widehat{=}} \nabla_{\underleftarrow{\mu}} \tilde{\eta}_{\underleftarrow{\nu}}\ , \tag{16.12}$$

where $\tilde{X}$ and $\tilde{\eta}$ are arbitrary extensions of $X$, $\eta$ to $M$.

(d) *Natural connection 1-form on* $\Delta$: From the properties of $l$, it follows that there exists a one-form $\omega^l_\mu$ such that

$$\nabla_{\underleftarrow{\mu}} l^\nu \mathrel{\widehat{=}} \omega^l_\mu l^\nu, \tag{16.13}$$

which implies

$$\mathscr{L}_l h_{\underleftarrow{\mu\nu}} \mathrel{\widehat{=}} 0. \tag{16.14}$$

We define the acceleration of $l$ by $l^\mu \nabla_\mu l^\nu = \kappa^l l^\nu$. We infer $\kappa^l = i_l \omega^l$.

(e) *Conditions on the Weyl tensor*: From the defining equation of the Riemann tensor, it follows that

$$2(D_{[\mu}\omega^l_{\nu]})l^\rho \mathrel{\widehat{=}} -R_{\underleftarrow{\mu\nu\sigma}}{}^\rho l^\sigma \mathrel{\widehat{=}} -C_{\underleftarrow{\mu\nu\sigma}}{}^\rho l^\sigma, \tag{16.15}$$

where in the last step we used (b). Contracting (16.15) with $m_{\rho J}$, we find

$$\Psi_{0I0J} \mathrel{\widehat{=}} 0 \quad \text{and} \quad \Psi_{0IJK} \mathrel{\widehat{=}} 0, \tag{16.16}$$

and therefore also

$$0 \mathrel{\widehat{=}} \Psi_{010J} = \Psi_{0IJ}{}^I. \tag{16.17}$$

Using this and (b), we find

$$0 \mathrel{\widehat{=}} C_{\underleftarrow{\mu\nu\rho\sigma}} l^\nu l^\rho k^\sigma \mathrel{\widehat{=}} R_{\underleftarrow{\mu\nu\rho\sigma}} l^\nu l^\rho k^\sigma \mathrel{\widehat{=}} -\mathscr{L}_l \omega^l_{\underleftarrow{\mu}} + D_{\underleftarrow{\mu}} \kappa^l. \tag{16.18}$$

**Definition 2.** *A pair $(\Delta, [l])$, where $\Delta$ is a NEH and $[l]$ an equivalence class*[1] *of null normals, is said to be a weakly isolated horizon (WIH) if*

*4.* $\mathscr{L}_l \omega \mathrel{\hat{=}} 0$

*for any $l \in [l]$.*

Note that, while $\omega^l$ in general depends on the choice of null normal $l$, it is invariant under constant rescalings of $l$ and therefore depends only on the equivalence class $[l]$ we fixed. Therefore, we will drop the superscript $^l$ in the following. We immediately infer from (16.18) that the $0^{\text{th}}$ law holds for WIH,

$$\underleftarrow{d\kappa^l} \mathrel{\hat{=}} 0. \tag{16.19}$$

In the following, we will slightly strengthen this usual definition of WIHs in a way which is very similar to the definitions given in [274] by introducing some extra structure. Fix a foliation of $\Delta$ by $(D-1)$ - spheres. Denote by $[k]$ an equivalence class of 1-form fields normal to the foliation of $\Delta$ by $(D-1)$ - spheres[2]. We require that any $k \in [k]$ is closed on $\Delta$. We extend them uniquely to spacetime 1-forms on $\mathscr{M}$ by requiring that they be null. Now, we introduce the equivalence class of pairs $[l, k]$ where each pair $(l^\mu, k_\nu)$ satisfies $i_l k = -1$, i.e. we fix $l$ and $k$ up to mutually inverse and constant rescaling. Since $k$ is closed and $\Delta \mathrel{\hat{\cong}} S^{D-1} \times \mathbb{R}$ is simply connected, $k = -dv$ for some function $v$ on $\Delta$, and each leaf $S_v \mathrel{\hat{\cong}} S^{D-1}$ of the fixed foliation is characterised by $v = \text{const}$. By *spherically symmetric*, we will in the following mean *constant on the leaves $S_v$*, e.g. for a spherically symmetric function $f = f(v)$.

**Definition 3.** *A undistorted non-rotating isolated horizon (UDNRIH) is a WIH where to each $l \in [l]$ there is a $k$ like above, such that*

5. *$k$ is shear-free with nowhere vanishing spherically symmetric expansion and vanishing Newman - Penrose coefficients $\pi_J \mathrel{\hat{=}} l^\mu m^\nu_J \nabla_\mu k_\nu$ on $\Delta$.*

6. *The Euler density $E^{(D-1)}$ of the $(D-1)$ – sphere cross sections obeys $E^{(D-1)}/\sqrt{h} = f(v)$ for some function $f$, i.e. the given ratio is constant on each leaf $S_v$.*

Two remarks are in order: Firstly, in $D = 3$, one finds for undistorted non-rotating isolated horizons [274], instead of the last condition,

[1] Two null normals $l$ and $l'$ are said to belong to the same equivalence class $[l]$ if $l = cl'$ for some positive constant $c$.

[2] Again, two 1-forms $k, k'$ are called equivalent if $k = ck'$ for some constant $c$.

*6'. $T_{\mu\nu}l^\mu k^\nu$ is spherically symmetric at $\Delta$.*

It is only for $D = 3$ that *6.* and *6'.* are equivalent. *6'.* can be shown to be equivalent to demanding that the curvature scalar $R^{(2)}$ of the 2-sphere cross sections be constant. In two dimensions, we have $E^{(2)} = \text{const.} \times R^{(2)}\sqrt{h} = f(v)\sqrt{h}$ for some scalar function $f$. In higher dimensions, condition *6'.* still is equivalent to demanding that $R^{(D-1)}$ is constant on $S_v$. However, we will see that for our purposes, this condition is unnecessary, but has to be replaced by *6.* This will be discussed explicitly in section 17.2.1. Apart from that, compared with [274], our definition 3 is slightly stronger (more restrictive) in that [274] does not demand *4.* Furthermore, whereas we only allow for constant rescaling of $l, k$, in [274] they are fixed up to spherically symmetric and mutually inverse rescaling, but later in that paper, the gauge freedom of rescaling is fixed completely.

Secondly, the definition given above is tied to a foliation. The standard definitions of (W)IH are usually foliation independent, though some results rely on the existence of a so called good cuts foliation. Moreover, when going to the Hamiltonian formulation, one usually demands that the spacetime foliation is such that at the boundary, the foliation coincides with this preferred foliation. Note that our fixed foliation is a good cuts foliation. We leave the question if all results obtained here hold in the more general context of weaker definitions of (W)IH or ones without reference to a fixed foliation for further research and continue by stating the consequences of definition 3:

(f) *Properties of $k$, $\omega$ and its curvature*: By the above requirements, we find for vectors $u$ tangential to $\Delta$ using $k^\mu \nabla_u k_\mu = 0$

$$\begin{aligned} \nabla_u k_\nu &= u^\mu \left( h_\nu^{\nu'} h_\mu^{\mu'} \nabla_{\mu'} k_{\nu'} - k_\nu \omega_\mu \right) \\ &= u^\mu \left( \frac{1}{D-1} \theta_k h_{\mu\nu} - k_\nu \omega_\mu \right). \end{aligned} \tag{16.20}$$

Furthermore, we have for tangential vectors $u$ and $v$

$$0 = u^\mu v^\nu \nabla_{[\mu} k_{\nu]} = -u^\mu v^\nu k_{[\nu}\omega_{\mu]}, \tag{16.21}$$

from which we conclude that $\omega = \hat{f}\, k$ for some function $\hat{f}$. Since $i_l \omega = \kappa^l$, we have $\hat{f} = -\kappa^l$ or

$$\omega = -\kappa^l k. \tag{16.22}$$

Contraction of (16.15) with $k_\rho$ yields

$$2D_{[\mu}\omega_{\nu]} \mathrel{\hat{=}} C_{\underleftarrow{\mu\nu\sigma}}{}^{\rho} l^\sigma k_\rho \mathrel{\hat{=}} m^I_{\underleftarrow{\mu}} m^J_{\underleftarrow{\nu}} \Psi_{01IJ}, \tag{16.23}$$

where in the last step we used the trace freeness of the Weyl tensor and (16.16). We can furthermore conclude that $d\omega \mathrel{\widehat{=}} 0$ and $\Psi_{01IJ} \mathrel{\widehat{=}} 0$, since $\omega \mathrel{\widehat{=}} -\kappa^l \underleftarrow{k}$ and $\underleftarrow{d\kappa^l} \mathrel{\widehat{=}} 0 \mathrel{\widehat{=}} \underleftarrow{dk}$. This can be traced back to the requirement $\pi_J \mathrel{\widehat{=}} 0$ in the definition of UDNRIHs, and in analogy to the $D = 3$ case, this is why we refer to these horizons as non-rotating (note that $\Psi_{01IJ}$ is the analog of $\mathfrak{Im}\Psi_2$ in $D = 3$).

# 17

# The boundary degrees of freedom

## 17.1 Undistorted case: Comparison with $D = 3$

Let us very briefly review the main steps of the the classical part of the black hole treatment in LQG (we will follow [62]) and compare them with what we expect to encounter in higher dimensions, which already partly summarises our results. The following sections then will give rather lengthy derivations thereof.

### 17.1.1 Boundary condition and role of topological invariants:

Usually in $D = 3$, the derivation of the boundary condition goes as follows: Due to the isolated horizon boundary conditions (IHBC), the field equations have to be satisfied at the horizon. In particular, starting with the Palatini theory, we have $F^{(4)}_{\mu\nu IJ} = R^{(4)}_{\mu\nu\rho\sigma}\Sigma^{\rho\sigma}_{IJ}$ where $\Sigma^{\rho\sigma}_{IJ} = e^{[\rho}{}_I e^{\sigma]}{}_J$, due to the equation of motion demanding torsion freeness of the Palatini connection. Pulling back to the horizon cross sections, we find using again the IHBC that $\underset{\Leftarrow}{R^{(4)}_{\mu\nu}}{}_{\rho\sigma}\Sigma^{\rho\sigma}_{IJ} = \underset{\Leftarrow}{R}{}^{(2)}_{\mu\nu\rho\sigma}\Sigma^{\rho\sigma}_{IJ}$. In two dimensions, the Riemann tensor is already determined by the curvature scalar, $R^{(2)}_{\mu\nu\rho\sigma} \propto R^{(2)} g_{[\mu|\rho} g_{|\nu]\sigma}$. Combining these findings and choosing time gauge to obtain the structure group SU(2), we have $\underset{\Leftarrow}{F}{}^{(4)}_{\mu\nu}{}^{i} \propto R^{(2)} \underset{\Leftarrow}{\Sigma}{}_{\mu\nu}{}^{i}$, where $\underset{\Leftarrow}{\Sigma}{}_{\mu\nu i} = \epsilon_{jik} \underset{\Leftarrow}{e}{}_{[\mu}{}^{j} \underset{\Leftarrow}{e}{}_{\nu]}{}^{k}$.

To continue, we have to invoke that in two dimensions the integral over the Ricci scalar is a topological invariant by the Gauß-Bonnet theorem. Due to the spherical symmetry of the horizon cross section, it follows that the Ricci curvature actually is a constant given by $\frac{-2\pi\chi_S}{A_S}$, where $\chi_S$ denotes the Euler characteristic of $S$ (which equals 2 in our case of spheres). $A_S$ here denotes the area of the two-sphere cross sections, which also is a constant in time due to the IHBC. Therefore, we actually have the boundary condition $\underset{\Leftarrow}{F}^{(4)i} \propto \underset{\Leftarrow}{\Sigma}^i \propto \underset{\longleftarrow}{*E}^i$. In the last step, we used that, when expressing in canonical fields, the middle term coincides with the pullback to $S$ of the two form dual to the densitised spatial triad. Classically, the surface degrees of freedom are determined by the bulk fields by continuity. At the quantum level, this ceases to be true and it is this equation which relates the bulk (triad) and surface (Chern Simons connection) degrees of freedom.

In higher dimensions, the first steps towards a derivation of a similar boundary condition can be literally copied and we also find $\underset{\Leftarrow}{F}^{(D+1)}_{\mu\nu IJ} = \underset{\Leftarrow}{R}^{(D-1)}_{\mu\nu\rho\sigma}\Sigma^{\rho\sigma}_{IJ}$. However, in higher dimensions the Riemann tensor of course has more than one independent component and also the Ricci scalar ceases to play the topological role he had for $D = 3$. The idea of how to generalise this aspect of the boundary condition to higher dimensions comes from the observation that in two dimensions we have $\sqrt{h}R^{(2)} \propto \epsilon^{\alpha\beta}\epsilon^{IJ}R^{(2)}_{\alpha\beta IJ}$, i.e. the (densitised) Ricci scalar coincides (up to constant factors) with the Euler topological density [294] which generalises to even dimensions,

$$E^{(2n)} := \epsilon^{\mu_1\nu_1\ldots\mu_n\nu_n}\epsilon^{I_1J_1\ldots I_nJ_n}R_{\mu_1\nu_1I_1J_1}\ldots R_{\mu_n\nu_nI_nJ_n}. \tag{17.1}$$

This motivates that a boundary condition in higher dimensions should read

$$\epsilon^{K_1L_1\ldots K_nL_nIJ}\underset{\Leftarrow}{\epsilon}^{\mu_1\nu_1\ldots\mu_n\nu_n}\underset{\Leftarrow}{F}_{\mu_1\nu_1K_1L_1}\ldots\underset{\Leftarrow}{F}_{\mu_n\nu_nK_nL_n} = \frac{2E^{(2n)}}{\sqrt{h}}n^{[I}\tilde{s}^{J]} \approx \frac{E^{(2n)}}{\sqrt{h}}\pi^{aIJ}s_a, \tag{17.2}$$

where $s_a \in T^*\sigma$ denotes the unit conormal vector to $S$ pointing outward of $\sigma$, $s^I := s_a e^{aI}$ and the twiddle indicates the density weight of one, $\tilde{s}^I := \sqrt{h}\ s^I$. This indeed will be verified in section 17.2.1. By the same arguments as above, we have that $\frac{E^{(2n)}}{\sqrt{h}} = \frac{(8\pi)^n n! \chi_S}{A_S}$ is constant on a history. However, we will see that the condition (17.2) actually is not sufficient in higher dimensions and additional boundary conditions have to be imposed in order to determine the boundary connection in terms of the bulk fields.

In the case of structure group $\mathrm{SO}(D+1)$, the form of the boundary condition turns out to be the same, but since we have no action principle to start with, the derivations will be different. In particular, the connection on the left hand side of (17.2) will in this case simply be given by the $\mathrm{SO}(D+1)$ spin connection $\Gamma^0$ on $S$ annihilating $n^I$, $s^J$ and ${m_\alpha}^K$ (cf. appendix C) and not coincide with the (pullback of the) Palatini connection. The connection to the Palatini connection turns out to be irrelevant at this point, the most important role of the boundary condition being to relate the *boundary* connection with the bulk degrees of freedom (cf. also [292]).

### 17.1.2 Boundary contribution to the symplectic structure:

Within the Hamiltonian framework, when extending the ADM to the Ashtekar Barbero phase space, we added an exact one form to the canonical action. However, when looking closely at (6.31), we see that this is no longer true in the presence of inner boundaries, where we obtain a contribution

$$\begin{aligned}\int_\sigma d^3x \; {}^{(\gamma)}E^{ai}\delta\Gamma_{ai} &= \frac{\gamma}{2}\int_\sigma d^3x \; \epsilon^{abc}\partial_a\left(\operatorname{sgn} e \; e_{bk}\delta e_c{}^k\right) \\ &= \frac{\gamma}{2}\int_S d^2x \; \epsilon^{\alpha\beta} \; m_{\alpha k}\delta m_\beta{}^k, \end{aligned} \tag{17.3}$$

where we assumed $\operatorname{sgn} e = 1$. After another gauge fixing, it is shown again making use of the IHBC and ruthermore restricting the horizon area to be constant throughout the histories we are considering, $\delta A_S = 0$, that the corresponding symplectic structure can be rewritten as the symplectic structure of a U(1) Chern Simons theory.

Similarly, from (7.39) we see that a similar transformation when going over to the new variables leads to the boundary contribution to the symplectic potential

$$\begin{aligned}\int_\sigma d^Dx \; {}^{(\beta)}\pi^{aIJ}\delta\Gamma_{aIJ} &\approx \frac{1}{\beta}\int_\sigma d^Dx \; \partial_a\left(2E^{aI}\delta n_I\right) \\ &= \frac{1}{\beta}\int_S d^{D-1}x \; 2\tilde{s}^I\delta n_I. \end{aligned} \tag{17.4}$$

Of course, the structure of (17.3) and (17.4) is necessarily different, since it is unclear how to generalise (17.3) to higher dimensions. This again underlines the difference between Ashtekar's and the new connection variables. In $3+1$ dimensions, we have the possibility to introduce a Holst - like modification (cf. section 9.3). Repeating the above calculation then yields the modified boundary term

$$\begin{aligned}\int_\sigma d^3x \; {}^{(\beta,\gamma)}\!\!\stackrel{}{\pi}{}^{aIJ}\delta\Gamma_{aIJ} &\approx \frac{1}{\beta}\int_\sigma d^3x \; \partial_a\left(2E^{aI}\delta n_I - \frac{1}{\gamma}\epsilon^{abc}e_{bM}\delta e_c^M\right) \\ &= \frac{1}{\beta}\int_S d^2x \left(2\tilde{s}^I\delta n_I - \frac{1}{\gamma}\epsilon^{\alpha\beta}m_{\alpha I}\delta m_\beta^I\right), \end{aligned} \tag{17.5}$$

the new term appearing corresponding to the boundary term (17.3) for Ashtekar-Barbero variables (Note that $\gamma$ in (17.3) and in (17.5) do *not* coincide, as is explained also in section 9.3).

From (17.4), it is easy to obtain the boundary contribution to the symplectic structure

$$\Omega^S(\delta_1, \delta_2) = \int_S \frac{2}{\beta}(\delta_{[1}\tilde{s}^I)(\delta_{2]}n_I), \tag{17.6}$$

and it will be shown that, again restricting to $\delta A_S = 0$, this symplectic structure can be rewritten as the symplectic structure of an SO$(D+1)$ (or SO$(1,D)$, depending on the structure group in the bulk) Chern Simons theory of the hybrid spin connection $\Gamma^0$ (cf. appendix C) on $S$

$$\Omega^S_{\text{CS}}(\delta_1,\delta_2) = \frac{nA_S}{\beta\left\langle E^{(2n)}\right\rangle}\int_S \epsilon^{IJKLM_1N_1...M_{n-1}N_{n-1}}\epsilon^{\alpha\beta\alpha_1\beta_1...\alpha_{n-1}\beta_{n-1}} \left(\delta_{[1}\Gamma^0_{\alpha IJ}\right)\left(\delta_{2]}\Gamma^0_{\beta KL}\right)R^0_{\alpha_1\beta_1M_1N_1}...R^0_{\alpha_{n-1}\beta_{n-1}M_{n-1}N_{n-1}} \quad . \tag{17.7}$$

This connection is not uniquely determined and exemplarily, we point out possible modifications of the connection for $D=3$. Actually, in the case of structure group SO$(1,D)$ and for any (even) spacetime dimension, there is a modification which allows to interpret the Chern Simons connection as the pullback of the bulk connection, as will be shown in detail in section 17.2.2, but like in the boundary condition, this is not necessary.

The Euler density and its topological nature play a central role again in this derivation, both in the $D=3$ and the higher dimensional case.

## 17.2 SO(1,D) as internal gauge group

### 17.2.1 Boundary condition

In this section, we will derive the boundary condition relating the bulk with the horizon degrees of freedom starting from the Palatini action. This forces us to use SO$(1,D)$ as the internal gauge group as opposed to SO$(D+1)$, which can be used in the Hamiltonian formalism even for Lorentzian signature. In a later chapter, we will rederive the boundary condition independently of the internal signature, thus allowing us to use the loop quantisation based on SO$(D+1)$ connection variables for the bulk degrees of freedom.

Due to *3.* of definition 1, we have at points of $\Delta$

$$F_{\mu\nu}{}^{IJ} \mathrel{\hat{=}} R_{\mu\nu}{}^{IJ} = R^{(D+1)}_{\mu\nu\rho\sigma}\; e^{\rho I}e^{\sigma J}. \tag{17.8}$$

In the following, we will use the notation introduced in appendix F for the Weyl tensor also for the Riemann tensor, e.g. $R_{01IJ} = R^{(D+1)}_{\mu\nu\rho\sigma}l^\mu k^\nu m^\rho{}_I m^\sigma{}_J$. Note that therefore, the internal indices appearing on $R$ and $\Psi$ are perpendicular to $l^I$ and $k^I$, which will be used in several calculations in this section. Pulling back to $\Delta$, we obtain

$$\begin{aligned}
&F_{\underleftarrow{\mu\nu}}{}^{IJ} = R_{\underleftarrow{\mu\nu}}{}^{IJ} = R^{(D+1)}_{\underleftarrow{\mu\nu}\rho\sigma} e^{\rho I} e^{\sigma J} \\
&= \left( h_{\underleftarrow{\mu}}{}^{\mu'} h_{\underleftarrow{\nu}}{}^{\nu'} R^{(D+1)}_{\mu'\nu'\rho\sigma} - 2 k_{[\underleftarrow{\mu}} h_{\underleftarrow{\nu}]}{}^{\nu'} l^{\mu'} R^{(D+1)}_{\mu'\nu'\rho\sigma} \right) \Big( m^{\rho I} m^{\sigma J} - 2 m^{\rho[I} l^{\sigma} k^{J]} - 2 m^{\rho[I} k^{\sigma} l^{J]} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + 2 l^{[\rho} k^{\sigma]} k^{[I} l^{J]} \Big) \\
&= h_{\underleftarrow{\mu}}{}^{\mu'} h_{\underleftarrow{\nu}}{}^{\nu'} R^{(D+1)}_{\mu'\nu'\rho\sigma} m^{\rho I} m^{\sigma J} + m_{\underleftarrow{\mu}}{}^{K} m_{\underleftarrow{\nu}}{}^{L} \left( -2 R_{KL}{}^{[I}{}_{0} k^{J]} - 2 R_{KL}{}^{[I}{}_{1} l^{J]} + 2 R_{KL01} k^{[I} l^{J]} \right) \\
&\quad - 2 k_{[\underleftarrow{\mu}} m_{\underleftarrow{\nu}]}{}^{K} \left( R_{0K}{}^{IJ} - 2 R_{0K}{}^{[I}{}_{0} k^{J]} - 2 R_{0K}{}^{[I}{}_{1} l^{J]} + 2 R_{0K01} k^{[I} l^{J]} \right) \\
&= h_{\underleftarrow{\mu}}{}^{\mu'} h_{\underleftarrow{\nu}}{}^{\nu'} R^{(D-1)}_{\mu'\nu'\rho\sigma} m^{\rho I} m^{\sigma J} + m_{\underleftarrow{\mu}}{}^{K} m_{\underleftarrow{\nu}}{}^{L} \left( -2 \Psi_{KL}{}^{[I}{}_{0} k^{J]} - 2 R_{KL}{}^{[I}{}_{1} l^{J]} + 2 \Psi_{KL01} k^{[I} l^{J]} \right) \\
&\quad - 2 k_{[\underleftarrow{\mu}} m_{\underleftarrow{\nu}]}{}^{K} \left( \Psi_{0K}{}^{IJ} - 2 \Psi_{0K}{}^{[I}{}_{0} k^{J]} - 2 R_{0K}{}^{[I}{}_{1} l^{J]} + 2 \Psi_{0K01} k^{[I} l^{J]} \right) \\
&= h_{\underleftarrow{\mu}}{}^{\mu'} h_{\underleftarrow{\nu}}{}^{\nu'} R^{(D-1)}_{\mu'\nu'\rho\sigma} m^{\rho I} m^{\sigma J} + 4 k_{[\underleftarrow{\mu}} m_{\underleftarrow{\nu}]}{}^{K} R_{0K}{}^{[I}{}_{1} l^{J]} \\
&= h_{\underleftarrow{\mu}}{}^{\mu'} h_{\underleftarrow{\nu}}{}^{\nu'} R^{(D-1)}_{\mu'\nu'\rho\sigma} m^{\rho I} m^{\sigma J} + \frac{4}{D-1} k_{[\underleftarrow{\mu}} m_{\underleftarrow{\nu}]}{}^{[I} l^{J]} \left[ \nabla_l \theta_k + \kappa^l \theta_k \right],
\end{aligned} \tag{17.9}$$

where in the fourth line, we used that $\Phi_{0J} \mathrel{\widehat{=}} 0$, $\Phi_{00} \mathrel{\widehat{=}} 0$ to replace some Riemann tensor components by the corresponding Weyl tensor components, and in the fifth line we used $0 \mathrel{\widehat{=}} \Psi_{0IJK} \mathrel{\widehat{=}} \Psi_{01JK} \mathrel{\widehat{=}} \Psi_{0I0J} \mathrel{\widehat{=}} \Psi_{010J}$ and furthermore for $u_\sigma$ such that $u \cdot l = 0 = u \cdot k$,

$$\begin{aligned}
R^{(D-1)\sigma}_{\mu\nu\rho} u_\sigma &= [D_\mu D_\nu] u_\rho \\
&= 2 h^{\mu'}_{[\mu} h^{\nu'}_{\nu]} h^{\rho'}_{\rho} \nabla_{\mu'} h^{\nu''}_{\nu'} h^{\rho''}_{\rho'} \nabla_{\nu''} u_{\rho''} \\
&= h^{\mu'}_{\mu} h^{\nu'}_{\nu} h^{\rho'}_{\rho} h^{\sigma}_{\sigma'} R^{(D+1)\sigma'}_{\mu'\nu'\rho'} u_\sigma + 2 h^{\mu'}_{[\mu} h^{\nu'}_{\nu]} h^{\rho'}_{\rho} (\nabla_{[\mu'} h^{\nu''}_{\nu']} h^{\rho''}_{\rho'}) \nabla_{\nu''} u_{\rho''} \\
&\mathrel{\widehat{=}} h^{\mu'}_{\mu} h^{\nu'}_{\nu} h^{\rho'}_{\rho} h^{\sigma}_{\sigma'} R^{(D+1)\sigma'}_{\mu'\nu'\rho'} u_\sigma.
\end{aligned} \tag{17.10}$$

The second term in the second to last line vanishes due to

$$\begin{aligned}
h^{\mu'}_{[\mu} h^{\nu'}_{\nu]} h^{\rho'}_{\rho} (\nabla_{[\mu'} h^{\nu''}_{\nu']} h^{\rho''}_{\rho'}) \nabla_{\nu''} u_{\rho''} &= h^{\mu'}_{[\mu} h^{\nu'}_{\nu]} h^{\rho''}_{\rho} \nabla_{[\mu'} (l_{\nu']} k^{\nu''} + k_{\nu']} l^{\nu''}) \nabla_{\nu''} u_{\rho''} \\
&\quad + h^{\mu'}_{[\mu} h^{\nu''}_{\nu]} h^{\rho'}_{\rho} \nabla_{\mu'} (l_{\rho'} k^{\rho''} + k_{\rho'} l^{\rho''}) \nabla_{\nu''} u_{\rho''} \\
&\mathrel{\widehat{=}} h^{\mu'}_{[\mu} h^{\nu'}_{\nu]} h^{\rho''}_{\rho} ((\nabla_{[\mu'} l_{\nu']}) k^{\nu''} + (\nabla_{[\mu'} k_{\nu']}) l^{\nu''}) \nabla_{\nu''} u_{\rho''} \\
&\quad + h^{\mu'}_{[\mu} h^{\nu''}_{\nu]} h^{\rho'}_{\rho} ((\nabla_{\mu'} l_{\rho'}) k^{\rho''} + (\nabla_{\mu'} k_{\rho'}) l^{\rho''}) \nabla_{\nu''} u_{\rho''} \\
&\mathrel{\widehat{=}} h^{\mu'}_{[\mu} h^{\nu'}_{\nu]} h^{\rho''}_{\rho} (\nabla_{[\mu'} k_{\nu']}) l^{\nu''} \nabla_{\nu''} u_{\rho''} \\
&\quad - h^{\mu'}_{[\mu} h^{\nu''}_{\nu]} h^{\rho'}_{\rho} (\nabla_{\mu'} k_{\rho'}) u^{\rho''} \nabla_{\nu''} l_{\rho''}
\end{aligned}$$

$$\hat{=} 0 \quad , \tag{17.11}$$

where in the first line we used $\nabla g = 0$, in the second line that $h(l,.) = 0 = h(k,.)$, in the third that $\underleftarrow{\nabla_\mu l_\nu} = 0$ and $l^\mu \nabla_\rho u_\mu = -u^\mu \nabla_\rho l_\mu$, and in the fourth line and $\underleftarrow{dk} = 0$.

Finally, we have to account for the vanishing of $R_{IJK1}$ in (17.9), which follows from

$$\begin{aligned}
R_{IJK1} &= \Psi_{IJK1} + \frac{2}{D-1}\bar{\bar{\eta}}_{K[I}\Phi_{J]1} \\
&= m^\mu_I m^\nu_J m^\rho_K R^{(D+1)}_{\mu\nu\rho\sigma} k^\sigma = m^\mu_I m^\nu_J m^\rho_K \left[\nabla_\mu, \nabla_\nu\right] k_\rho \\
&= 2\ m^\mu_I m^\nu_J m^\rho_K \nabla_{[\mu} \left( \left( h^{\nu'}_{\nu]} - l_{\nu]} k^{\nu'} - k_{\nu]} l^{\nu'} \right) \left( h^{\rho'}_\rho - l_\rho k^{\rho'} - k_\rho l^{\rho'} \right) \nabla_{\nu'} k_{\rho'} \right) \\
&\hat{=} 2\ m^\mu_I m^\nu_J m^\rho_K \nabla_{[\mu} \left( h^{\nu'}_{\nu]} \left( h^{\rho'}_\rho - k_\rho l^{\rho'} \right) \nabla_{\nu'} k_{\rho'} \right) \\
&\hat{=} 2\ m^\mu_{[I} m^\nu_{J]} m^\rho_K \nabla_\mu \left( \frac{1}{D-1} h^{\nu'}_\nu h^{\rho'}_\rho h_{\nu'\rho'} \theta_k - h^{\nu'}_\nu k_\rho \omega^l_{\nu'} \right) \\
&\hat{=} \frac{2}{D-1}\ m^\mu_{[I} m^\nu_{J]} m^\rho_K \left( h_{\nu\rho} \nabla_\mu \theta_k - h_{\mu\rho} \theta_k \omega^l_\nu \right) \\
&\hat{=} \frac{2}{D-1}\ m^\mu{}_{[I} \bar{\bar{\eta}}_{J]K} \left( \nabla_\mu \theta_k + \theta_k \omega^l_\mu \right) \\
&\hat{=} \frac{2}{D-1}\ m^\mu{}_{[I} \bar{\bar{\eta}}_{J]K} \left( -(\nabla_l \theta_k) k_\mu - \theta_k \kappa^l k_\mu \right)\ \hat{=}\ 0.
\end{aligned} \tag{17.12}$$

From the third to the fourth line, we dropped the second two summands in the first round bracket because $l$ and $k$ are twist free, and the second summand in the second bracket since $k^\mu \underleftarrow{\nabla} k_\mu = 0$. In the fifth line, we used that $k$ is twist and shear free and that $l^\mu \underleftarrow{\nabla} k_\mu = \omega^l$. In line 6, we again invoke the twist and shear freeness of $k$. In the last line, we used that $\underleftarrow{d\theta_k} = -k\nabla_l \theta_k$ since it is spherical symmetric by definition 3 and that $\omega^l = -\kappa^l k$.[1]

In the last line of (17.9), we furthermore used

$$\begin{aligned}
R_{0I1J} &= C_{0I1J} + \frac{1}{D-1}\left(\bar{\bar{\eta}}_{IJ}\Phi_{01} - \Phi_{IJ}\right) - \frac{1}{D(D+1)}\bar{\bar{\eta}}_{IJ} R^{(D+1)} \\
&= -\frac{1}{D-1}\bar{\bar{\eta}}_{IJ}\left[\nabla_l \theta_k + \kappa^l \theta_k\right],
\end{aligned} \tag{17.13}$$

which can be shown analogously.

[1] Comparing with the $3+1$ dimensional case, we find $R_{IJK1} = \Psi_{IJK1} + \frac{2}{D-1}\bar{\bar{\eta}}_{K[I}\Phi_{J]1} = 0$ corresponds to $\Psi_3 - \Phi_{21} = 0$, $\Psi_{KLJ0} = 0$ to $\Psi_0 = 0$ and $\Psi_1 = 0$, and $\Psi_{KL01} = 0$ to the non-rotating condition $\mathfrak{Im}\Psi_2 = 0$.

Since the pullback to $H$ of the second summand in (17.9) is zero ($\underset{\Leftarrow}{k} = 0$), we finally obtain when pulling back once more

$$\underset{\Leftarrow}{F}_{\mu\nu IJ} = \underset{\Leftarrow}{R}^{(D+1)}_{\mu\nu IJ} = \underset{\Longleftarrow}{h_\mu}{}^{\mu'} \underset{\Longleftarrow}{h_\nu}{}^{\nu'} R^{(D+1)}_{\mu'\nu'\rho\sigma} e^{\rho I} e^{\sigma J} = \underset{\Longleftarrow}{h_\mu}{}^{\mu'} \underset{\Longleftarrow}{h_\nu}{}^{\nu'} R^{(D-1)}_{\mu'\nu'\rho\sigma} m^{\rho I} m^{\sigma J} \tag{17.14}$$

and therefore, for $D-1 = 2n$ even,

$$\begin{aligned} &\epsilon^{K_1 L_1 \ldots K_n L_n IJ} \underset{\Leftarrow}{\epsilon}{}^{\mu_1\nu_1\ldots\mu_n\nu_n} \underset{\Leftarrow}{F}_{\mu_1\nu_1 K_1 L_1} \ldots \underset{\Leftarrow}{F}_{\mu_n\nu_n K_n L_n} \\ &= \epsilon^{K_1 L_1 \ldots K_n L_n IJ} \underset{\Leftarrow}{\epsilon}{}^{\mu_1\nu_1\ldots\mu_n\nu_n} R^{(D-1)}_{\mu_1\nu_1\rho_1\sigma_1} \ldots R^{(D-1)}_{\mu_n\nu_n\rho_n\sigma_n} m^{\rho_1 K_1} m^{\sigma_1 L_1} \ldots m^{\rho_n K_n} m^{\sigma_n L_n} \\ &= \frac{1}{\sqrt{h}} \underset{\Leftarrow}{\epsilon}{}^{\rho_1\sigma_1\ldots\rho_n\sigma_n} \underset{\Leftarrow}{\epsilon}{}^{\mu_1\nu_1\ldots\mu_n\nu_n} R^{(D-1)}_{\mu_1\nu_1\rho_1\sigma_1} \ldots R^{(D-1)}_{\mu_n\nu_n\rho_n\sigma_n} 2n^{[I} s^{J]} \approx \frac{E^{(2n)}}{\sqrt{q}} \pi^{aIJ} s_a \quad , \end{aligned} \tag{17.15}$$

where $E^{(2n)}$ denotes the Euler density of the $(D-1)$ – sphere cross sections and $\approx$ means equal up to the simplicity constraint. Finally, by *6.* of definition 3, $E^{(2n)} = f(v)\sqrt{h}$. Some comment on the role of the equations (17.14, 17.15) is in order.

Firstly, notice that both of these equations are generalisations of the 3+1 dimensional boundary conditions $\underset{\Leftarrow}{F}{}^4_{\mu\nu IJ} \propto R^{(2)} \underset{\Leftarrow}{\Sigma}_{\mu\nu IJ}$ known from the U(1) and SU(2) treatments. (17.14) has the same left hand side, but further manipulation of the right hand side as in the 3+1 dimensional case is not possible, since the Riemann tensor is in general not completely determined by the Ricci scalar in higher dimensions and the Ricci scalar also ceases to play a topological role. (17.15) generalises the right hand side, the topological role now being played by the Euler density, while the left hand side is more complicated than in the 3+1 dimensional case.

Secondly, at the quantum level, we want to work with an independent Chern-Simons connection on the horizon from the outset and demand by constraint that the boundary connection actually is determined by the bulk fields. This constraint is in 3+1 dimensions precisely given by the boundary condition $\underset{\Leftarrow}{F}{}^4_{\mu\nu IJ} \propto \underset{\Leftarrow}{\Sigma}_{\mu\nu IJ}$. In higher dimensions, one can easily convince oneself that (17.15) is insufficient to determine the boundary connection and one has to impose (17.14) at the quantum level. However, (17.15) connects the momenta conjugate to the bulk connection with Chern-Simons excitations and therefore is a direct generalisation of what is imposed at the quantum level in the 3+1 dimensional case. It therefore could serve as a consistency requirement additionally to (17.14), see the discussion in section 17.5. One last comment concerning *6'.*: Assuming this condition to hold, one easily obtains that

$$G_{\mu\nu} l^\mu k^\nu = \Phi_{01} + \frac{D-1}{2(D+1)} R^{(D+1)} \tag{17.16}$$

is spherically symmetric. Moreover, taking the trace of (17.13), we infer that

$$C_{0I1}{}^{I} + \frac{D-3}{D-1}\Phi_{01} - \frac{D-1}{D(D+1)}R^{(D+1)} = -\nabla_l \theta_k - \kappa^l \theta_k \tag{17.17}$$

is spherically symmetric since the right hand side is. Finally, from (17.10),

$$\begin{aligned} R^{(D-1)} = R_{IJ}{}^{IJ} &= 2C_{0I1}{}^{I} + \frac{4(D-2)}{D-1}\Phi_{01} + \frac{(D-2)(D-1)}{D(D+1)}R^{(D+1)} \\ &= 2\left(C_{0I1}{}^{I} + \frac{D-3}{D-1}\Phi_{01} - \frac{D-1}{D(D+1)}R^{(D+1)}\right) + 2\left(\Phi_{01} + \frac{D-1}{2(D+1)}R^{(D+1)}\right), \end{aligned} \tag{17.18}$$

where Weyl tensor component identities from appendix F were used. Since both summands in round brackets are spherically symmetric, we find that $R^{(D-1)}$ is also spherically symmetric. As we already remarked at the beginning of section 16.3, this property will not be needed in higher dimensions, but instead *6.* will be crucial in the next section.

### 17.2.2 Hamiltonian framework

In this section, we will show, starting from the Palatini action in $(D+1) = 2(n+1)$ dimensions, how the symplectic structure of $(2n+1)$ - dimensional Chern-Simons theory arises as boundary contribution to the symplectic structure for an internal boundary with UDNRIH conditions. We restrict to a vanishing cosmological constant. Note that the mechanics of higher dimensional isolated horizons has already been studied in the quasi-local, the asymptotically flat [289] as well as the asymptotically anti-de Sitter [290] case. However, in all these treatments, the internal $\mathrm{SO}(1,D)$ transformations were (partially) gauge fixed. In view of the boundary term (eq. (7.39)) of the generating functional for the canonical transformation to $\mathrm{SO}(1,D)$ connection variables which we found in part II and which we expect to be related to the boundary symplectic structure, we are not allowed to fix the internal gauge freedom completely. In particular, in the usual time gauge $n^I = \delta^I_0$, this boundary term vanishes since it is proportional to $\delta n^I$. Therefore, we will rederive the Hamiltonian framework for IH in higher dimensions for our specific definition of UDNRIH and without using any internal gauge fixing[1]. Indeed, the derivation deviates from the usual treatment and we obtain the same boundary contribution to the symplectic structure we found in 17.1, which a) vanishes in time gauge and b) can be reexpressed as $\mathrm{SO}(1,D)$ Chern-Simons symplectic

[1]Note, however, that there are interesting allowed gauge fixings, e. g. $n^I = g^{0i}\eta^I_i$, $s^I = g^{1i}\eta^I_i$ for $g \in \mathrm{SO}(2)$ $(i \in \{0,1\})$.

structure.

Consider a region $\mathscr{M}$ in a $(D+1)$ - dimensional Lorentzian spacetime $(\mathscr{M}', g)$ bounded by two (partial) Cauchy slices $\Sigma_1$ and $\Sigma_2$, $\Delta$, and possibly an outer boundary $\mathscr{T}$. On $\Delta$, we impose the UDNRIH boundary conditions and furthermore require that $\Sigma_1, \Sigma_2$ intersect $\Delta$ in leaves ($(D-1)$ - spheres) of the preferred foliation $S_1, S_2$, respectively. Moreover, as usual in the IH literature, for a given history $(e, A)$ the horizon area $A_S$ is constant in time as we will show shortly (below (17.27)). We will now furthermore fix the horizon area to be a constant throughout the histories we are considering, $\delta A_S = 0$. We will not specify any boundary conditions on $\mathscr{T}$ and neglect boundary terms related with it which are possibly needed to obtain a well defined variational principle since they are not relevant for the purpose of this thesis. For a discussion of these issues in higher dimensions, we refer the interested reader to e.g. [299] and, specifically in the IH framework, [289]. The Palatini action is given by

$$S[A, e] = \int_{\mathscr{M}} \Sigma_{IJ} \wedge F^{IJ}, \tag{17.19}$$

where $F = 1/2 F_{\mu\nu} dx^\mu \wedge dx^\nu$, $F_{\mu\nu}{}^{IJ} = 2\partial_{[\mu} A_{\nu]}{}^{IJ} + [A_\mu, A_\nu]^{IJ}$, $\Sigma := - * (e \wedge e)$, or in coordinates $- * (e \wedge e)_{\mu_1 \dots \mu_{D-1} IJ} = \frac{1}{(D-1)!} e^{K_1}_{\mu_1} \dots e^{K_{D-1}}_{\mu_{D-1}} \epsilon_{IJK_1 \dots K_{D-1}}$, and as already stated, boundary terms possibly needed for $\mathscr{T}$ are neglected. Variation with respect to $A$ gives rise to a surface term

$$\int_\Delta \underset{\leftarrow}{\Sigma}_{IJ} \wedge \delta \underset{\leftarrow}{A}^{IJ}, \tag{17.20}$$

which, however, vanishes when imposing the UDNRIH boundary conditions, and therefore, the variation only yields the bulk equations of motion. This is a standard result in the IH literature, but will be derived here without any internal gauge fixing. Using $\underset{\leftarrow}{e}_{\mu I} = m_{\mu I} - k_\mu l_I$, we immediately find

$$\underset{\leftarrow}{\Sigma}_{IJ} = -\frac{1}{(D-1)!} \epsilon_{IJK_1 .. K_{D-1}} \left[ m^{K_1} \wedge .. \wedge m^{K_{D-1}} - (D-1) l^{K_1}\ k \wedge m^{K_2} \wedge .. \wedge m^{K_{D-1}} \right]. \tag{17.21}$$

For the pullback of the space time connection $A$ we find analogous to the calculations in section 17.2.1

$$\underset{\leftarrow}{A}_{\mu IJ} = \underset{\leftarrow}{\Gamma}_{\mu IJ} = \Gamma^0_{\mu IJ} + \frac{2}{D-1} l_{[I} m_{\mu|J]} \theta_k - 2\omega_\mu l_{[I} k_{J]}, \tag{17.22}$$

$$\Gamma^0_{\mu IJ} = m^\nu_{[I} \underset{\leftarrow}{\nabla}_\mu m_{\nu|J]} - l_{[I} \underset{\leftarrow}{\nabla}_\mu k_{|J]} - k_{[I} \underset{\leftarrow}{\nabla}_\mu l_{|J]}, \tag{17.23}$$

where $\Gamma^0$ here denotes the connection on $\Delta$ which annihilates $m_{\mu K}$, $l_I$ and $k_J$. Here and in the following, we will understand that $m^{\mu I} := h^{\mu\nu} m_\nu{}^I$ and $h^{\mu\nu} = g^{\mu\mu'} h_{\mu'\nu'} g^{\nu'\nu}$ such that $h^{\mu\nu} k_\nu = 0$.

For the variation of $\underset{\leftarrow}{A}$, we find

$$\begin{aligned} \delta \underset{\leftarrow}{A}_{\mu IJ} &= \delta\Gamma^0{}_{\mu IJ} + \frac{2}{D-1}\left[(\delta l_{[I}) m_{\mu|J]}\theta_k + l_{[I}(\delta m_{\mu|J]})\theta_k + l_{[I} m_{\mu|J]}(\delta\theta_k)\right] \\ &\quad - 2\left[(\delta\omega_\mu) l_{[I} k_{J]} + \omega_\mu \delta(l_{[I} k_{J]})\right], \end{aligned} \tag{17.24}$$

which for the case at hand can be reduced to

$$\begin{aligned} \delta \underset{\leftarrow}{A}_{\mu IJ} &= 2k_\mu l_{[I} k_{J]} l^K k^L l^\nu \delta \underset{\leftarrow}{A}_{\nu KL} - 2k_{[I} \bar{\bar{\eta}}_{J]}{}^L l^K h^\nu_\mu \delta \underset{\leftarrow}{A}_{\nu KL} + \mathscr{R} \\ &= 2k_\mu l_{[I} k_{J]} l^K k^L l^\nu \left[\delta\Gamma^0{}_{\nu KL} - 2 l_{[K} k_{L]} \delta\omega_\nu\right] - 2k_{[I} \bar{\bar{\eta}}_{J]}{}^L l^K h^\nu_\mu \delta\Gamma^0{}_{\nu KL} + \mathscr{R} \\ &= 2k_\mu l_{[I} k_{J]} l^\nu \left[k^L D^{\Gamma^0}_\nu \delta l_L + \delta\omega_\nu\right] - 2k_{[I} \bar{\bar{\eta}}_{J]}{}^L h^\nu_\mu D^{\Gamma^0}_\nu \delta l_L + \mathscr{R}, \end{aligned} \tag{17.25}$$

where in the first line, we made use of the fact that only certain components of $\delta \underset{\leftarrow}{A}$ will appear when contracted with $\underset{\leftarrow}{\Sigma}$ and $\mathscr{R}$ stands for the remaining terms which vanish in this contraction. In the second step, several terms drop out due to $l^\nu \delta m_{\nu I} = -m_{\nu I} \delta l^\nu = -m_{\nu I} c_\delta l^\nu = 0$ since $l$ is fixed up to constant rescaling on $\Delta$, $l^I \delta l_I = 0$ since $l^2 = 0$ on $\Delta$, and $h^\nu_\mu \omega_\nu \mathrel{\widehat{=}} 0$. Finally, we used that $l^K \delta\Gamma^0_{\mu KL} = -\delta D^{\Gamma^0}_\mu l_L + D^{\Gamma^0}_\mu \delta l_L = D^{\Gamma^0}_\mu \delta l_L$ since $\Gamma^0$ annihilates $l^I$. Putting all together, we recover for the definition of an UDNRIH as given in section 16.3 the result that there is no boundary term in symplectic potential for the horizon,

$$\begin{aligned} \int_\Delta \underset{\leftarrow}{\Sigma} \wedge \delta \underset{\leftarrow}{A} &= \int_\Delta \underset{\leftarrow}{\Sigma} \wedge \delta \underset{\leftarrow}{\Gamma} \\ &= -\frac{1}{(D-1)!} \int_\Delta \left(m^{K_1} \wedge \ldots \wedge m^{K_{D-1}} - (D-1) l^{K_1}\, k \wedge m^{K_2} \wedge \ldots \wedge m^{K_{D-1}}\right) \\ &\qquad\qquad \epsilon_{IJK_1 \ldots K_{D-1}} \wedge \left\{-2 l^{[I} k^{J]} \left[d(k_M \delta l^M) + \delta\omega\right] + 2\bar{\bar{\eta}}^I_{I'} k_J d_{\Gamma^0} \delta l^{I'}\right\} \\ &= -\frac{2}{(D-1)!} \int_\Delta \epsilon^{D-1} \wedge k \left(\mathscr{L}_l (k_I \delta l^I)\right) + \frac{2}{(D-1)!} \int_\Delta \epsilon^{D-1} \wedge \delta\omega \\ &\quad + \frac{2}{(D-2)!} \int_\Delta k \wedge m^{K_2} \wedge \ldots \wedge m^{K_{D-1}} l^{K_1} \epsilon_{IJK_1 \ldots K_{D-1}} k^J d_{\Gamma^0} \delta l^I \\ &= 0, \end{aligned} \tag{17.26}$$

where in the second step, we used (17.21) and (17.25), which results in three terms in the third step, each of which vanishes separately. The first one since we can partially integrate the Lie derivative (boundary terms drop since $\delta l^I = 0$ on $S_1$, $S_2$) and we have $\underleftarrow{\mathscr{L}_l \epsilon^{D-1}} \mathrel{\widehat{=}} 0$ and $\underleftarrow{\mathscr{L}_l k} \mathrel{\widehat{=}} 0$. Note that here, we defined

$$\epsilon^{D-1} = \epsilon_{IJK_1 \ldots K_{D-1}} l^I k^J m^{K_1} \wedge \ldots \wedge m^{K_{D-1}}. \tag{17.27}$$

To see that it is Lie dragged, note that

$$\begin{aligned} \underleftarrow{\mathscr{L}_l m_{\mu I}} &= l^\nu \underleftarrow{\nabla}_\nu m_{\mu I} + m_{\nu I} \underleftarrow{\nabla}_\mu l^\nu = l^\nu \underleftarrow{\nabla}_\nu m_{\mu I} = -l^\nu {\Gamma^0}_{\nu I}{}^J m_{\mu J}, & (17.28)\\ \underleftarrow{\mathscr{L}_l l^I} &= l^\nu \underleftarrow{\nabla}_\nu l^I = -l^\nu {\Gamma^0}_{\nu I}{}^J l_J, & (17.29)\\ \underleftarrow{\mathscr{L}_l k^I} &= l^\nu \underleftarrow{\nabla}_\nu k^I = -l^\nu {\Gamma^0}_{\nu I}{}^J k_J. & (17.30) \end{aligned}$$

Using this, to prove that $\underleftarrow{\mathscr{L}_l \epsilon^{D-1}} = 0$ we only need to use the invariance of $\epsilon^{I_1 \ldots I_{D+1}}$ under (infinitesimal) SO$(1, D)$ transformations. A similar argument shows that

$$\underleftarrow{d\epsilon}^{D-1} = 0. \tag{17.31}$$

The second term in (17.26) is zero since $\delta\omega$ is fixed on $S_1$, $S_2$ and also Lie dragged along $l$, so the whole integrand is Lie dragged an vanishes at the boundary, which implies that the integral vanishes (This argument is e.g. given in [275]). The last term vanishes since the derivative $d_{\Gamma^0}$ annihilates the whole expression (note that $\underleftarrow{dk} \mathrel{\widehat{=}} 0$) and therefore leads only to a boundary contribution which vanishes again due to $\delta l|_{S_1,S_2} = 0$.

The second variation of the action yields the symplectic current $\delta_{[1}\Sigma^{IJ}\delta_{2]}A_{IJ}$ which is closed by standard arguments,

$$\left(\int_{\Sigma_2} - \int_{\Sigma_1} + \int_\Delta\right)\delta_{[1}\Sigma^{IJ}\delta_{2]}A_{IJ} = 0. \tag{17.32}$$

Moreover, the contribution at $\Delta$ is a pure surface term, and we will show in the following that

$$\int_\Delta \delta_{[1}\underleftarrow{\Sigma}^{IJ}\delta_{2]}\underleftarrow{A}_{IJ} = \Omega^{S_2}_{\text{CS}}(\delta_1, \delta_2) - \Omega^{S_1}_{\text{CS}}(\delta_1, \delta_2), \tag{17.33}$$

where

$$\Omega^S_{\text{CS}} = \frac{nA_S}{\langle E^{(2n)}\rangle} \int_S \epsilon^{IJKLM_2N_2 \ldots M_nN_n} \left(\delta_{[1}\underleftarrow{A}_{IJ}\right) \wedge \left(\delta_{2]}\underleftarrow{A}_{KL}\right) \wedge \underleftarrow{F}_{M_2N_2} \wedge \ldots \wedge \underleftarrow{F}_{M_nN_n} \tag{17.34}$$

denotes the Chern-Simons symplectic structure (cf. appendix 16.2), and therefore, the symplectic structure is given by

$$\Omega(\delta_1,\delta_2) = \int_\Sigma \delta_{[1}\Sigma^{IJ}\delta_{2]}A_{IJ} \\ + \frac{nA_S}{\left\langle E^{(2n)}\right\rangle}\int_S \epsilon^{IJKLM_2N_2...M_nN_n}\left(\delta_{[1}\underset{\Leftarrow}{A}_{IJ}\right)\wedge\left(\delta_{2]}\underset{\Leftarrow}{A}_{KL}\right)\wedge\underset{\Leftarrow}{F}_{M_2N_2}\wedge...\wedge\underset{\Leftarrow}{F}_{M_nN_n}, \tag{17.35}$$

and is independent of the choice of $\Sigma$.

To prove (17.35), we will first show that the contribution to the symplectic structure at $\Delta$ is given by the boundary term we already found in section 17.1,

$$\int_\Delta \delta_{[1}\underset{\leftarrow}{\Sigma}^{IJ}\delta_{2]}\underset{\leftarrow}{A}_{IJ} = \int_{S_2} 2(\delta_{[1}\tilde{s}^I)(\delta_{2]}n_I) - \int_{S_1} 2(\delta_{[1}\tilde{s}^I)(\delta_{2]}n_I), \tag{17.36}$$

where $\tilde{s}_I = \sqrt{h}s_I$, and in a second step that the boundary contribution can be rewritten as

$$\int_S 2(\delta_{[1}\tilde{s}^I)(\delta_{2]}n_I) = \frac{A_S}{\left\langle E^{(2n)}\right\rangle}\int_S 2\frac{E^{(2n)}}{\sqrt{h}}(\delta_{[1}\tilde{s}^I)(\delta_{2]}n_I) \\ = \frac{nA_S}{\left\langle E^{(2n)}\right\rangle}\int_S \epsilon^{IJKLM_2N_2...M_nN_n}\left(\delta_{[1}\underset{\Leftarrow}{A}_{IJ}\right)\wedge\left(\delta_{2]}\underset{\Leftarrow}{A}_{KL}\right)\wedge\underset{\Leftarrow}{F}_{M_2N_2}\wedge...\wedge\underset{\Leftarrow}{F}_{M_nN_n}. \tag{17.37}$$

For the variation of $\underset{\leftarrow}{\Sigma}$, we find using (17.21)

$$\begin{aligned} &-(D-1)!\,\delta\underset{\leftarrow}{\Sigma}_{IJ} \\ &= \epsilon_{IJK_1...K_{D-1}}\left[(D-1)(\delta m^{K_1})\wedge m^{K_2}\wedge...\wedge m^{K_{D-1}}\right. \\ &\qquad -(D-1)(D-2)l^{K_1}\ k\wedge(\delta m^{K_2})\wedge m^{K_3}\wedge...\wedge m^{K_{D-1}} \\ &\qquad \left.-(D-1)\left(l^{K_1}(\delta k)+(\delta l^{K_1})k\right)\wedge m^{K_2}\wedge...\wedge m^{K_{D-1}}\right] \\ &= \epsilon_{IJK_1...K_{D-1}}\left[(D-1)m_L\wedge m^{K_2}\wedge...\wedge m^{K_{D-1}}(i_{m^L}\delta m^{K_1})\right. \\ &\qquad -(D-1)(D-2)l^{K_1}\ k\wedge m_L\wedge m^{K_3}\wedge...\wedge m^{K_{D-1}}(i_{m^L}\delta m^{K_2}) \\ &\qquad \left.-(D-1)\left(-l^{K_1}(i_l\delta k)+(\delta l^{K_1})\right)k\wedge m^{K_2}\wedge...\wedge m^{K_{D-1}}\right], \end{aligned} \tag{17.38}$$

where we used

$$\begin{aligned} \delta\underset{\leftarrow}{m}_I &= \underset{\leftarrow}{m}_J(i_{m^J}\delta m_I) - \underset{\leftarrow}{k}(i_l\delta m_I) = \underset{\leftarrow}{m}_J(i_{m^J}\delta m_I) + \underset{\leftarrow}{k}(i_{m_I}\delta l) \\ &= \underset{\leftarrow}{m}_J(i_{m^J}\delta m_I) + \underset{\leftarrow}{k}c_\delta(i_{m_I}l) = \underset{\leftarrow}{m}_J(i_{m^J}\delta m_I), \end{aligned} \tag{17.39}$$

$$\delta\underset{\leftarrow}{k} = -k(i_l\delta k). \tag{17.40}$$

In total, after a long calculation explained in appendix G.1, one finds for (17.36)

$$
\begin{aligned}
&\int_\Delta \delta_{[1} \underset{\leftarrow}{\Sigma}^{IJ} \delta_{2]} \underset{\leftarrow}{A}_{IJ} = \\
&= \frac{2}{(D-1)!}\int_\Delta \Big\{ d\left[\delta_{[1}(\epsilon^{D-1}k_I)\delta_{2]}l^I\right] + \delta_{[1}\epsilon^{D-1}\wedge\delta_{2]}\omega^l \\
&\qquad +(D-1)d\left[\left(c_\delta + (k_M\delta_{[1}l^M)\right) k\wedge m^{K_2}\wedge ...\wedge m^{K_{D-1}} l^I k^J \epsilon_{IJK_1...K_{D-1}}\delta_{2]}l^{K_1}\right] \\
&\qquad +(D-2)d\left[k\wedge m^M \wedge m^{K_3}\wedge ...\wedge m^{K_{D-1}} l^I k^J \epsilon_{IJK_1...K_{D-1}} (i_{m_M}\delta_{[1}m^{K_2})\delta_{2]}l^{K_1}\right]\Big\} \\
&= \frac{2}{(D-1)!}\int_\Delta \Big\{ d\left[\delta_{[1}(\epsilon^{D-1}k_I)\delta_{2]}l^I\right] + \delta_{[1}\epsilon^{D-1}\wedge\delta_{2]}\omega^l \Big\} \\
&= 2\int_\Delta \left\{ d\left[\delta_{[1}\tilde{s}^I\delta_{2]}n_I\right] + \frac{1}{(D-1)!}\delta_{[1}\epsilon^{D-1}\wedge\delta_{2]}\omega^l \right\}.
\end{aligned} \tag{17.41}
$$

We used $\delta k = -c_\delta k$ and $\underset{\Leftarrow}{k} = 0$. Since we also restricted to constant area $A_S$ throughout the phase space region we are considering ($\delta A_S = 0$), we furthermore find

$$
\begin{aligned}
\int_\Delta \delta_{[1}\epsilon^{D-1}\wedge\delta_{2]}\omega^l &= -\int_\Delta \delta_{[1}\epsilon^{D-1}\wedge\delta_{2]}(\kappa^l k) = +\int_\Delta \delta_{[1}\epsilon^{D-1}\wedge d\delta_{2]}(\kappa^l v) \\
&= +\left[\delta_{[2}(\kappa^l v)|_{S_2}\int_{S_2}\delta_{1]}\epsilon^{D-1} - \delta_{[2}(\kappa^l v)|_{S_1}\int_{S_1}\delta_{1]}\epsilon^{D-1}\right] \\
&= +\left[\delta_{[2}(\kappa^l v)|_{S_2}\delta_{1]}A_{S_2} - \delta_{[2}(\kappa^l v)|_{S_1}\delta_{1]}A_{S_1}\right] = 0.
\end{aligned} \tag{17.42}
$$

Now, since we have $E^{(D-1)} = f(v)\epsilon^{D-1}/(D-1)!$ for a spherically symmetric function $f$ by the conditions for an UDNRIH, and since

$$\int_S E^{(2n)} = (8\pi)^n n!\, \chi_S = 2(8\pi)^n n! =: \langle E^{(2n)}\rangle, \tag{17.43}$$

$$\int_S \epsilon^{D-1} = (D-1)!\, A_S, \tag{17.44}$$

are both constant in time, we have $f = \frac{\langle E^{(2n)}\rangle}{A_S}$ where $2n = D-1$. Here, since in our case $S$ has spherical topology, we used that the Euler number is $\chi_S = 2$. The first line of (17.37) easily follows. In fact, this also shows that $f(v)$ is independent of $v$.

For the second pullback of $A$, we find since $\underset{\Leftarrow}{\omega} = 0$,

$$\underset{\Leftarrow}{A}_{IJ} = \Gamma^0{}_{IJ} + \frac{2}{D-1} l_{[I}m_{J]}\theta_k =: \Gamma^0{}_{IJ} + \underset{\Leftarrow}{K}_{IJ}. \tag{17.45}$$

Since $\theta_k$ is constant on the $(D-1)$ - sphere cross sections of the chosen foliation, we have $\underset{\Longleftarrow}{d_{\Gamma^0}K} = 0$. Since also $[\underset{\Leftarrow}{K}, \underset{\Leftarrow}{K}] = 0$, we obtain $\underset{\Leftarrow}{F} = \underset{\Leftarrow}{R}^0$ which was already derived in section 17.2.1. We now want to show that (17.37) holds, which is shown to be true in (G.5) if the connection would be given by $\Gamma^0$. Therefore, what needs to be checked is if

$$\epsilon^{IJKLM_2N_2..M_nN_n}\left(2\delta_{[1}\Gamma^0_{IJ}\wedge\delta_{2]}\underset{\Leftarrow}{K}_{KL}+\delta_{[1}\underset{\Leftarrow}{K}_{IJ}\wedge\delta_{2]}\underset{\Leftarrow}{K}_{KL}\right)\wedge R^0{}_{M_2N_2}\wedge..R^0{}_{M_nN_n}=0. \tag{17.46}$$

Using

$$\delta\underset{\Leftarrow}{K}_{IJ}=\frac{2}{D-1}\left[-l_{[I}k_{J]}l^K\theta_k\delta m_K+l_{[I}\bar{\bar{\eta}}_{J]K}\left(\theta_k\delta m^K-m^K\theta_k k^L\delta l_L+m^K\delta\theta_k\right)\right.$$
$$\left.+\bar{\bar{\eta}}_{[I}{}^K m_{J]}\theta_k\delta l_K\right], \tag{17.47}$$

we find in a first step

$$\begin{aligned}E_\perp^{IJKL}\wedge\delta_{[1}\underset{\Leftarrow}{K}_{IJ}\wedge\delta_{2]}\underset{\Leftarrow}{K}_{KL}&=-\frac{8}{(D-1)^2}E_\perp^{IJKL}\wedge l_Ik_J\bar{\bar{\eta}}^N_K\theta_k^2l^M\delta_{[1}m_M\wedge m_L\delta_{2]}l_N\\&=\frac{8}{(D-1)^2}E_\perp^{IJ[N|L}\wedge l_Ik_J\theta_k^2m^{M]}\wedge m_L\delta_{[1}l_M\delta_{2]}l_N=0.\end{aligned} \tag{17.48}$$

$E_\perp^{IJKL}=\epsilon^{IJKLM_2N_2...M_nN_n}R^0{}_{M_2N_2}\wedge...\wedge R^0{}_{M_nN_n}$ in the above formula stands for the terms in (17.46) contracted with $\delta K\wedge\delta K$. $\perp$ indicates that fact that $E_\perp$ needs to be contracted with $k^I, l^J$ since it vanishes otherwise, therefore only one combination of terms survives when we use (17.47) in the first step. In the second line, we made use of $l^I\delta m_I=-m_I\delta l^I$ and therefore, the expression is antisymmetric in the index pair $M, N$. Adding terms until all indices of the epsilon symbol in $E_\perp$ plus the index $M$ are totally antisymmetric and subtracting the therefore needed terms again, we find that the whole expression vanishes: The total antisymmetrisation since there is no nontrivial rank $D+2$ antisymmetric tensor in $D+1$ dimensions, and the subtracted terms since they are either of the form $l^Im_I=0$ or $k^Im_I=0$, or $R^0{}_{MN}\wedge m^N$ which vanishes due to the Bianci identity, or $m^L\wedge m_L=0$.

Furthermore, we have

$$\begin{aligned}&E_\perp^{IJKL}\wedge\delta_{[1}\Gamma^0{}_{IJ}\wedge\delta_{2]}\underset{\Leftarrow}{K}_{KL}\\&=\frac{2}{(D-1)}E_\perp^{IJKL}\wedge\Big[-\bar{\bar{\eta}}^{I'}_{[I}\bar{\bar{\eta}}^{J'}_{J]}\delta_{[1}\Gamma^0{}_{I'J'}\wedge l_{[K}k_{L]}\theta_kl^M\delta_{2]}m_M\\&\quad-2k_{[I}l^{I'}\bar{\bar{\eta}}^{J'}_{J]}\delta_{[1}\Gamma^0{}_{I'J'}\wedge l_{[K}\bar{\bar{\eta}}_{L]M}\left(\theta_k\delta_{2]}m^M-m^M\theta_kk^N\delta_{2]}l_N+m^M\delta_{2]}\theta_k\right)\\&\quad+2l_{[I}k_{J]}k^{I'}l^{J'}\delta_{[1}\Gamma^0{}_{I'J'}\wedge\bar{\bar{\eta}}^M_{[K}m_{L]}\theta_k\delta_{2]}l_M\Big]\end{aligned}$$

$$
\begin{aligned}
= \frac{2}{(D-1)} E_{\perp}^{IJKL} \wedge \Big[ &- \bar{\bar{\eta}}_{[I}^{I'} m_{\alpha|J]} \left( -d_{\Gamma^0} \delta_{[1} m^{\alpha}_{I'} - m^{\beta}_{I'} \delta_{[1} \Gamma^{\alpha}_{\bullet\beta} \right) \wedge l_{[K} k_{L]} \theta_k l^M \delta_{2]} m_M \\
&- 2k_{[I} \bar{\bar{\eta}}_{J]}^{J'} (d_{\Gamma^0} \delta_{[1} l_{J'}) \wedge l_{[K} \bar{\bar{\eta}}_{L]M} \left( \theta_k \delta_{2]} m^M - m^M \theta_k k^N \delta_{2]} l_N + m^M \delta_{2]} \theta_k \right) \\
&- 2 l_{[I} k_{J]} [d_{\Gamma^0} (k^{I'} \delta_{[1} l_{I'})] \wedge \bar{\bar{\eta}}^M_{[K} m_{L]} \theta_k \delta_{2]} l_M \Big], \qquad (17.49)
\end{aligned}
$$

where we used $\bar{\bar{\eta}}_{IJ} = m^{\alpha}{}_I m_{\alpha J}$ in the last step as well as the fact that $\Gamma^0$ annihilates $m^K, l^I, k^J$ and therefore, e.g. $l^J \delta \Gamma^0{}_{IJ} = \delta(d_{\Gamma^0} l_I) - d_{\Gamma^0} \delta l_I = -d_{\Gamma^0} \delta l_I$. In the last expression, the second summand in the second to last line and the term in the last line together just give a surface term which vanishes since the $(D-1)$ sphere cross sections have no boundary. To see this, one needs to make use of the fact that $d_{\Gamma^0} R^0 = d_{\Gamma^0} m = d_{\Gamma^0} l^I = d_{\Gamma^0} k^J = d_{\Gamma^0} \bar{\bar{\eta}} = d\theta_K = 0$. Moreover, we also have $d\delta\theta_K = 0$ since $\delta\theta_K$ has to be constant on the $(D-1)$ - sphere cross sections, and therefore also the last term in the second to last line is a surface term. Using the notation $\delta\Gamma^{\alpha}_{\bullet\beta}$ to indicate that $\delta\Gamma$ is considered as a form in the index $\bullet$, the terms in the first line of (17.49) give

$$
\begin{aligned}
& \frac{2\theta_k}{(D-1)} l^K k^L E^{\perp}_{IJKL} \wedge m_N \, m_{\alpha}{}^{[J} \wedge \left[ \left( d_{\Gamma^0} \delta_{[1} m^{\alpha|I]} + m^{\beta|I]} \delta_{[1} \Gamma^{\alpha}_{\bullet\beta} \right) \right] \delta_{2]} l^N \\
= & \frac{2\theta_k}{(D-1)} l^K k^L E^{\perp}_{IJKL} \wedge m_M \wedge m_N \times \\
& \left[ m^{\beta M} m^{\alpha[I} D^{\Gamma^0}_{\beta} \delta_{[1} m_{\alpha}{}^{|J]} + m^{\beta[J} m^{\alpha|I]} D^{\Gamma^0}_{\beta} \delta_{[1} m_{\alpha}{}^{M} - m^{\beta[I|} m^{\alpha M} D^{\Gamma^0}_{\beta} \delta_{[1} m_{\alpha}{}^{|J]} \right] \delta_{2]} l^N \\
= & \frac{2\theta_k}{(D-1)} l^K k^L E^{\perp}_{IJKL} \wedge m_M \wedge m_N \times \\
& \left[ \frac{1}{3} m^{\beta[M} m^{\alpha I} D^{\Gamma^0}_{\beta} \delta_{[1} m_{\alpha}{}^{J]} + 2 m^{\beta[J} m^{\alpha|I]} D^{\Gamma^0}_{\beta} \delta_{[1} m_{\alpha}{}^{M} \right] \delta_{2]} l^N \\
= & \frac{4\theta_k}{(D-1)} l^K k^L E^{\perp}_{IJKL} \wedge m_M \wedge m_N \left[ -m^{\beta[I} m^{\alpha|J]} D^{\Gamma^0}_{\beta} \delta_{[1} m_{\alpha}{}^{M} \right] \delta_{2]} l^N. \qquad (17.50)
\end{aligned}
$$

In the third step, the term totally antisymmetric in the indices $M, J, I$ vanishes since

$$
\begin{aligned}
& l^K k^L E^{\perp}_{[IJ|KL} \wedge m_{|M]} \wedge m_N \\
= & \epsilon_{[IJ|KLM_2N_2 \ldots M_nN_n} l^K k^L R_0{}^{M_2N_2} \wedge \ldots \wedge R_0{}^{M_nN_n} \wedge m_{|M]} \wedge m_N \\
= & \frac{(D+2)}{3} \epsilon_{[IJKLM_2N_2 \ldots M_nN_n|} l^K k^L R_0{}^{M_2N_2} \wedge \ldots \wedge R_0{}^{M_nN_n} \wedge m_{|M]} \wedge m_N \\
= & 0, \qquad (17.51)
\end{aligned}
$$

since $R_0^{KL} \wedge m_L = 0$ due to the Bianci identity and $m_I l^I = 0 = m_I k^I$, and the antisymmetrisation of $D+2$ indices vanishes. Finally, the first term in the second to last line of (17.49) gives

$$\begin{aligned} &\frac{4\theta_k}{(D-1)} l^K k^L E^{\perp}_{IJKL} \wedge \left[ (d_{\Gamma^0} \delta_{[1} l^{[J}) \wedge \delta_{2]} m^{I]} \right] \\ &= \frac{4\theta_k}{(D-1)} l^K k^L E^{\perp}_{IJKL} \wedge ((d_{\Gamma^0} \delta_{[1} m^{[I}) \delta_{2]} l^{J]}) + d(\ldots) \\ &= \frac{4\theta_k}{(D-1)} l^K k^L E^{\perp}_{IJKL} \wedge m_M \wedge m_N \left[ m^{\beta[M} m^{\alpha|N]} D^{\Gamma^0}_{\beta} \delta_{[1} {m_\alpha}^{[I} \right] \delta_{2]} l^{J]} + d(\ldots), \end{aligned} \tag{17.52}$$

up to a boundary term $d(\ldots)$ that vanishes, as above, after integration over $H$, which means that (17.50) and (17.52) together are of the form

$$\begin{aligned} & l^K k^L E^{\perp}_{IJKL} \wedge m_M \wedge m_N \left[ \alpha^{IJ} \beta^{MN} - \alpha^{MN} \beta^{IJ} \right] \\ &= l^K k^L \epsilon_{IJKLM_2N_2..M_nN_n} {R_0}^{M_2N_2} \wedge .. \wedge {R_0}^{M_nN_n} \wedge m_M \wedge m_N \left[ \alpha^{IJ} \beta^{MN} - \alpha^{MN} \beta^{IJ} \right] \\ &= \big[ (D+2) l^K k^L \epsilon_{[IJKLM_2N_2..M_nN_n|} {R_0}^{M_2N_2} \wedge .. \wedge {R_0}^{M_nN_n} \wedge m_{|M]} \wedge m_N \\ &\quad - 2 l^K k^L \epsilon_{JMKLM_2N_2..M_nN_n} {R_0}^{M_2N_2} \wedge .. \wedge {R_0}^{M_nN_n} \wedge m_I \wedge m_N \big] \left[ \alpha^{IJ} \beta^{MN} - \alpha^{MN} \beta^{IJ} \right] \\ &= -2 l^K k^L \epsilon_{JMKLM_2N_2..M_nN_n} {R_0}^{M_2N_2} \wedge .. \wedge {R_0}^{M_nN_n} \wedge m_I \wedge m_N \left[ \alpha^{NM} \beta^{JI} - \alpha^{MN} \beta^{IJ} \right] \\ &= 0, \end{aligned} \tag{17.53}$$

where $\alpha^{IJ}$ and $\beta^{KL}$ are antisymmetric matrices. This furnishes the proof of (17.35).

## 17.3 SO($D+1$) as internal gauge group

In the previous sections, we have derived the isolated horizon boundary condition relating the connection on the horizon with the bulk degrees of freedom, as well as the symplectic structure on the horizon, which coincides with the one of higher dimensional Chern-Simons theory. Since we started from the space-time covariant Palatini action, the internal gauge group was fixed to SO$(1, D)$. In the light of quantising the bulk degrees of freedom however, it was pointed out in [1] that one can change the internal gauge group to SO$(D+1)$ by a canonical transformation from the ADM phase space. After this reformulation, the quantisation of the bulk degrees of freedom can be performed with standard LQG methods as spelled out in [3]. Thus, we are interested in reformulating the horizon boundary condition and the horizon symplectic structure so that it fits in the SO$(D+1)$ scheme.

As for the boundary condition, the generalisation to the Euclidean internal group is straight forward, since the construction of the connection $\Gamma^0$ in appendix C works independently of the internal signature. Thus, constructing $\Gamma^0$ such that it annihilates both $n^K$ and $s^K = s_a e^{aK}$ additionally to $m^K_\alpha = \underset{\leftarrow}{e}{}_\alpha^{\ K}$, the horizon boundary conditions

$$R^{0,\text{horizon}}_{\alpha\beta IJ} = R^{0,\text{bulk}}_{\alpha\beta IJ} \tag{17.54}$$

$$\epsilon^{K_1L_1...K_nL_nIJ}\epsilon^{\alpha_1\beta_1...\alpha_n\beta_n} R^{0,\text{horizon}}_{\alpha_1\beta_1K_1L_1}...R^{0,\text{horizon}}_{\alpha_n\beta_nK_nL_n} = \frac{E^{(2n)}}{\sqrt{q}}\pi^{aIJ}s_a \quad . \tag{17.55}$$

follow immediately from the fact that ${R^0}_{\alpha\beta KL}n^K = {R^0}_{\alpha\beta KL}s^K = 0$. We will drop the superscripts "bulk" and "horizon" in what follows.

In order to derive the new symplectic structure, we first perform a symplectic reduction of the theory derived in the previous chapters by solving the Gauß and simplicity constraint. This leads us to the ADM phase space, from which we can perform further canonical transformations. This step is important since it tells us that using an isolated horizon as a boundary of our manifold, we will have a vanishing horizon symplectic structure when using ADM variables. We remark that this does not follow trivially for any boundary if one starts with the Einstein-Hilbert action and performs the Legendre transform, since one is picking up boundary terms in the Gauß-Codazzi equation which are neglected in order to arrive at the standard ADM symplectic structure.

In section 17.1 we found that the canonical transformation to SO($D+1$) connection variables leads to the boundary symplectic structure

$$\Omega^S(\delta_1,\delta_2) = \frac{2}{\beta}\int_S d^{D-1}x\ \delta_{[1}\tilde{s}_I\delta_{2]}n^I \quad . \tag{17.56}$$

Furthermore, under the non-distortion condition $\delta\frac{E^{(2n)}}{\sqrt{h}} = 0$, i.e. restricting to the part of phase space where $\frac{E^{(2n)}}{\sqrt{h}} = \frac{\langle E^{(2n)}\rangle}{A_S}$ is constant, it is shown in appendix G.2 that

$$2\frac{E^{(2n)}}{\sqrt{h}}(\delta_{[1}\tilde{s}^I)(\delta_{2]}n_I) = n\epsilon^{IJKLM_1N_1...M_{n-1}N_{n-1}}\epsilon^{\alpha\beta\alpha_1\beta_1...\alpha_{n-1}\beta_{n-1}} \left(\delta_{[1}\Gamma^0_{\alpha IJ}\right)\left(\delta_{2]}\Gamma^0_{\beta KL}\right)R^0_{\alpha_1\beta_1M_1N_1}...R^0_{\alpha_{n-1}\beta_{n-1}M_{n-1}N_{n-1}}, \tag{17.57}$$

which results in the Chern-Simons type boundary symplectic structure

$$\Omega^S_{\text{CS}}(\delta_1,\delta_2) = \frac{nA_S}{\beta\langle E^{(2n)}\rangle}\int_S \epsilon^{IJKLM_1N_1...M_{n-1}N_{n-1}}\epsilon^{\alpha\beta\alpha_1\beta_1...\alpha_{n-1}\beta_{n-1}} \left(\delta_{[1}\Gamma^0_{\alpha IJ}\right)\left(\delta_{2]}\Gamma^0_{\beta KL}\right)R^0_{\alpha_1\beta_1M_1N_1}...R^0_{\alpha_{n-1}\beta_{n-1}M_{n-1}N_{n-1}}. \tag{17.58}$$

Concluding, we have shown that also for the case of SO($D+1$) as an internal gauge group, one arrives at a higher dimensional Chern-Simons symplectic structure at the isolated horizon boundary of $\sigma$.

A remark concerning the uniqueness of $\Gamma^0$ is in order. In $D = 3$, one easily finds that there are more connections which allow for carrying out the whole programme. Exemplarily, we can introduce a constant parameter $\Phi \in \mathbb{R}$ and choose ${\Gamma^{\phi}}_{\alpha}{}^{IJ} = {\Gamma^0}_{\alpha}{}^{IJ} + 2\Phi n^{[I} {m_\alpha}^{J]}$ as connections for the Chern-Simons theory on the boundary. We then find

$$ {R^{\Phi}}_{\alpha\beta}{}^{IJ} = {R^0}_{\alpha\beta}{}^{IJ} - 2\Phi^2 {m_a}^{[I} {m_b}^{J]}, \qquad (17.59)$$

$$ \epsilon^{IJKL}\epsilon^{\alpha\beta} {R^{\Phi}}_{\alpha\beta KL} = \left( \frac{E^{(2)}}{\sqrt{h}} - 4\Phi^2 \right) \pi^{aIJ} s_a, \qquad (17.60)$$

$$ \frac{A_S}{\langle E^{(2)} \rangle - 4\Phi^2 A_S} \epsilon^{IJKL}\epsilon^{\alpha\beta} \delta_{[1} {\Gamma^{\Phi}}_{\alpha IJ} \delta_{2]} {\Gamma^{\Phi}}_{\beta KL} = 2\delta_{[1} \tilde{s}^I \delta_{2]} n_I. \qquad (17.61)$$

A further modification of $\Gamma^0$, which in particular allows for generalisation to distorted horizons, will be introduced in section 17.4.1, where a non-constant field $\Psi$ is added to the connection. The introduction of $\Psi$ and $\Phi$ cannot be combined non-trivially, since otherwise there will be terms $\propto n^{[I} {m_\alpha}^{J]}$ contributing to $R^{\Phi,\Psi}_{\alpha\beta IJ}$.

A third possibility to change the connection in $D = 3$, which can be combined with both of the above methods, is as follows. As we have already seen in 17.1, if we introduce the Barbero Immirzi parameter $\gamma$ in $D = 3$ [2], it will appear in the boundary symplectic structure. The boundary condition in this case reads

$$ \epsilon^{\alpha\beta} \left( \epsilon^{IJKL} {R^0}_{\alpha\beta KL} + \frac{1}{\gamma} {R^0}_{\alpha\beta}{}^{IJ} \right) = \frac{E^{(2)}}{\sqrt{h}} \overset{(\gamma)}{\pi}{}^{aIJ} s_a, \qquad (17.62)$$

where

$$ \overset{(\gamma)}{\pi}{}^{aIJ} = \pi^{aIJ} + \frac{1}{2\gamma} \epsilon^{IJ}{}_{KL} \pi^{aKL}. \qquad (17.63)$$

To show that the boundary symplectic structure can be rewritten according to

$$ \frac{2}{\beta} \int_S d^2x \left( \delta_{[1} \tilde{s}^I \delta_{2]} n_I - \frac{1}{2\gamma} \epsilon^{\alpha\beta} \delta_{[1} m_{\alpha I} \delta_{2]} m^I_\beta \right) = \frac{A_S}{\beta \langle E^{(2)} \rangle} \int_S \epsilon^{\alpha\beta} \left( \epsilon^{IJKL} \delta_{[1} \Gamma^0_{\alpha IJ} \delta_{2]} \Gamma^0_{\beta KL} + \frac{2}{\gamma} \delta_{[1} {\Gamma^0}_{\alpha IJ} \delta_{2]} {\Gamma^0}_{\beta}{}^{IJ} \right), \qquad (17.64)$$

it remains to verify that

$$ \frac{E^{(2)}}{\sqrt{h}} \delta m^I \wedge \delta m_I = -2 \delta \Gamma^{0\,IJ} \wedge \delta {\Gamma^0}_{IJ}. \qquad (17.65)$$

Since the scalar curvature $R = \frac{E^{(2)}}{2\sqrt{h}}$ is constant on the 2-spheres, the metric $h$ is fixed up to diffeomorphism. Therefore, $m_I, \Gamma^0{}_{IJ}$ are fixed up to diffeomorphism and SO$(D+1)$ rotations, i.e. $\delta m_I = \Lambda_I{}^J \delta m_J + \mathscr{L}_\xi m_I$ and $\delta \Gamma^0{}_{IJ} = -d_{\Gamma^0}\Lambda_{IJ} + \mathscr{L}_\xi \Gamma^0{}_{IJ}$. Using this for the variations, (17.65) can be proven straight forwardly using $0 = d_{\Gamma^0} m_I = dm_I + \Gamma^0{}_{IJ} \wedge m^J$, $d\Gamma^0{}_{IJ} + \frac{1}{2}[\Gamma^0, \Gamma^0]_{IJ} = R^0{}_{IJ} = \frac{1}{2} R m_I \wedge m_J$ and the properties of the exterior and Lie derivative.

In higher dimensions, it is less trivial to modify the connection $\Gamma^0$. In particular, the above constructions can at least not be applied trivially. While (17.59) continues to hold, in (17.60) mixed terms of the form $R^0 \wedge ... \wedge (\Phi m \wedge m)$ will appear which spoil the construction, and also the introduction of $\gamma$ is tied to $D = 3$.

## 17.4 Inclusion of distortion

So far, we have treated undistorted horizons exclusively. Note that our definition of undistorted only poses a restriction on the Euler density and therefore in higher dimensions already is a rather weak requirement. Now we want to turn to completely distorted horizons. In the $D = 3$ case, this extension was studied in the U(1) framework for the first time in [300], where a generalisation to axi-symmetric horizons was achieved. This result was considerably extended by Engle and Beetle [281], who with a beautiful idea managed to generalise the treatment to arbitrarily shaped spherical horizons. The same was achieved within the more recent SU(2) framework by Perez and Pranzetti [285], although their method is more complicated. We will test both proposals for a possible generalisation to higher dimensions.

### 17.4.1 Beetle-Engle method

A key ingredient in the derivation of the symplectic structure on the spatial two-sphere cross section $S$ of the horizon is the "undistortedness" of $S$, i.e. the constancy of $E^{(2n)}/\sqrt{h}$ (or, equivalently $R^{(2)}$) on $S$. Beetle and Engle showed within the U(1) framework that also for distorted $S$ one can construct a U(1) connection such that the corresponding curvature scalar is constant on $S$. They start with the ansatz

$$\overset{\circ}{V}_\alpha := \frac{1}{2}\theta_\alpha - \epsilon_{\alpha\beta} h^{\beta\gamma} D_\gamma \Psi. \tag{17.66}$$

For $\Psi = 0$, this reduces to the connection used in spherical symmetry. The additional freedom to choose the "curvature potential" $\Psi$ now is used to have the following equation satisfied:

$$d\overset{\circ}{V}\overset{!}{=}-\frac{\left\langle R^{(2)}\right\rangle}{4}\epsilon=-\frac{2\pi}{A_S}\Sigma_i s^i. \tag{17.67}$$

This leads to the following condition on $\Psi$

$$\Delta\Psi = R^{(2)} - \left\langle R^{(2)}\right\rangle, \tag{17.68}$$

which with the additional condition $\langle\Psi\rangle = 0$ has a unique solution.

In four spacetime dimensions, this idea can be easily generalised to the gauge group SO(4) or SO(1,3). Using the ansatz

$$A_{\alpha IJ} = \Gamma^0_{\alpha IJ} + 2m_{\alpha[I}m_{\beta|J]}h^{\beta\gamma}(D_\gamma\psi), \tag{17.69}$$

for the corresponding connection, and demanding the boundary condition

$$\epsilon^{\alpha\beta}\epsilon^{IJKL}F_{\alpha\beta KL}(A) = 2\langle E^{(2)}\rangle n^{[I}\tilde{s}^{J]} \tag{17.70}$$

leads to the requirement

$$\Delta\psi = \frac{1}{4}\left(\frac{E^{(2)}}{\sqrt{h}} - \langle E^{(2)}\rangle\right). \tag{17.71}$$

A lenghty calculation in G.3 furthermore shows that for this connection, it holds that

$$2\langle E^{(2)}\rangle(\delta_{[1}\tilde{s}^I)(\delta_{2]}n_I) = \epsilon^{IJKL}\epsilon^{\alpha\beta}\left(\delta_{[1}A_{\alpha IJ}\right)\left(\delta_{2]}A_{\beta KL}\right). \tag{17.72}$$

A generalisation of this procedure to higher dimensions, however, is far from straight forward. The main problem is that the boundary condition in higher dimensions becomes non-linear in the curvature. With the same ansatz for the connection, we obtain a non-linear partial differential equation for $\Psi$ for which a mathematical solution theory to the best of the author's knowledge has not been developed.

### 17.4.2 Perez-Pranzetti method

The extendibility of the Beetle-Engle method actually suggests that their method should be applicable also in the case of SU(2). However, Perez and Pranzetti [285] proceed rather differently. To include distortion, they propose to use two SU(2) Chern Simons connections

$$A^i_\gamma = \Gamma^i + \gamma e^i, \qquad A^i_\sigma = \Gamma^i + \sigma e^i, \tag{17.73}$$

and find for the corresponding curvatures by demanding the IHBC

$$F^i(A_\gamma) = \Psi_2\Sigma^i + \frac{1}{2}(\gamma^2 + c)\Sigma^i, \quad F^i(A_\sigma) = \Psi_2\Sigma^i + \frac{1}{2}(\sigma^2 + c)\Sigma^i. \tag{17.74}$$

Here $\Psi_2$ it a Newman-Penrose coefficient (related to $R^{(2)}$) and $c$ an extrinsic curvature scalar. It follows that their difference satisfies an equation of the sought form,

$$F^i(A_\gamma) - F^i(A_\sigma) = \frac{1}{2}(\gamma^2 - \sigma^2)\Sigma^i, \tag{17.75}$$

with just a constant appearing in front of $\Sigma$ on the right hand side. This allows to rewrite the boundary symplectic structure in the arbitrarily distorted case in terms of two Chern Simons theories. The downside is that the equations (17.74) (or equivalent constraints) have to be imposed at the quantum level in order to account for superfluous boundary degrees of freedom. A proposal how this is to be done is given in [285].

Let us mimic the procedure in higher dimensions. We start naively by introducing $N$ Chern-Simons connections

$$A^{(a_i)}_{\alpha IJ} = \Gamma^0_{\alpha IJ} + 2\sqrt{a_i} s_{[I} m_{\alpha|J]}, \; i \in \{1, ..., N\}. \tag{17.76}$$

For their field strengths, we find

$$F^{(a_i)}_{\alpha\beta IJ} = R^0_{\alpha\beta IJ} - 2m_{\alpha[I} m_{\beta|J]} a_i. \tag{17.77}$$

When we insert this in the formula needed for the higher dimensional boundary condition, we find

$$\begin{aligned} E^{IJ}_{(a_i)}(A^{(a_i)}) &:= \epsilon^{\beta_1\gamma_1...\beta_n\gamma_n} \epsilon^{IJK_1L_1...K_nL_n} F^{(a_i)}_{\beta_1\gamma_1K_1L_1} \dots F^{(a_i)}_{\beta_n\gamma_nK_nL_n} \\ &= \sum_{k=0}^{n} a_i^k X_k, \end{aligned} \tag{17.78}$$

where, schematically, $X_k \propto (R^0)^{n-k} \wedge (m \wedge m)^k$. Only the $k = 0$ term, being exactly of the form "$n^{[I}\tilde{s}^{J]} \times$ const." we need, is allowed to survive when linear combining the $E^{IJ}_{(a_i)}$ with coefficients $b_i \in \mathbb{R}$, $i \in \{1, ..., N\}$,

$$\sum_{i=1}^{N} b_i \; E^{IJ}_{(a_i)}(A^{(a_i)}) \overset{!}{\propto} n^{[I}\tilde{s}^{J]}, \tag{17.79}$$

which leads to the system of equations

$$\begin{aligned} &\sum_{i=1}^{N} b_i (a_i)^k = 0, \quad k \in \{0, ..., n-1\}, \\ &\sum_{i=1}^{N} b_i (a_i)^n = d, \end{aligned} \tag{17.80}$$

for some constant $d \neq 0$. Suppose w.l.o.g. that $a_1 \neq 0$, $b_1 \neq 0$. Introducing a new $\tilde{d} = \frac{d}{b_1(a_1)^n}$, we find that the above $n+1$ equations for fixed $\tilde{d}$, actually only depend on the $2(N-1)$ unknowns $(a_i/a_1)$, $(b_i/b_1)$. Since $N$ is integer and $2n = D-1$, we find that we need at least $N = \lceil \frac{n+1}{2} \rceil + 1 = \lceil \frac{D+1}{4} \rceil + 1$ Chern Simons theories on the boundary, which for $D = 3$ reproduces $N = 2$. However, we now have to implement many additional constraints corresponding to (17.77) consistently, which makes a success of this route at the quantum level rather doubtful (see, however, our comments on quantisation in section 17.5).

## 17.5 Comments on quantisation

In a seminal paper, Witten [184] studied the quantisation of Chern Simons theory in three dimensions, making heavily use of the fact that it is a topological field theory (see also [301] for an exhaustive treatment): The field equations read $F = 0$, and we obtain as solution space the finite dimensional moduli space of flat connections modulo gauge transformations. The quantisation of the boundary degrees of freedom is based on this work: A key result in the isolated horizon framework is that the field strength vanishes almost everywhere due to the isolated horizon boundary condition, except at points where the bulk spin network punctures the isolated horizon. Only at these points, the flux operator, which determines the field strength on $S$ via the isolated horizon boundary condition (17.55), is non-vanishing. The resulting quantum theory on the horizon is a Chern-Simons theory with topological defects induced by these spin network punctures, which result in a finite-dimensional Hilbert space.

In higher dimensions, Chern Simons theory admits local degrees of freedom in general [137, 138]. This can be easily understood looking at the field equations (16.6), which now are more complicated and do not restrict the connection to be flat in general. To treat black holes in higher dimensions at the quantum level, a full quantisation of the non-topological boundary field theory seems a rather ambitious goal. Here, we will briefly discuss two proposals for alternative routes for quantisation. Firstly, we will point out that one of the boundary conditions we derived might actually lead to flat connections except at the punctures in section 17.5.1. We want to stress that this proposal is incomplete and definitely deserves further study. In section 17.5.2, we will discuss the possibility of gauge fixing from SO(D+1) to U(1). The U(1) Chern Simons theory is exceptional and suggests itself for quantisation, since it lacks of local degrees of freedom in any dimension [302]. However, we we did not succeed in performing this reduction.

### 17.5.1 SO$(D+1)$ as gauge group

Since the symplectic structure on the isolated horizon is exactly of Chern Simons type, one would expect to obtain a higher-dimensional Chern-Simons theory on the boundary. Due to the distributional nature of the space of generalised connections in LQG, see e.g. [62], one promotes the connection on the isolated horizon to an independent degree of freedom in the quantum theory, here called $A_{IJ}$ with field strength $F_{IJ} = F(A)_{IJ}$. Furthermore, a quantisation of the boundary condition (17.55) (neglecting for a moment the stronger condition (17.54) and thus the fact that the connection on the isolated horizon is given by $\Gamma^0$) yields the quantum first class constraints of a higher-dimensional Chern-Simons theory with punctures,

$$E^{I_1 J_1}(x) := \epsilon^{I_1 J_1 \ldots I_n J_n} F_{I_2 J_2}(x) \wedge \ldots \wedge F_{I_n J_n}(x) \propto s_a \widehat{\pi}^{a I_1 J_1}(x). \tag{17.81}$$

The quantum interpretation of this equation is that the punctures of bulk spin networks act as "particle excitations" for the Chern Simons theory, exactly as in the $3+1$-dimensional case [292]. The immediate problem with this approach of course are the local degrees of freedom of higher dimensional Chern Simons theories. As a direct consequence, one would expect to obtain an infinite entropy by counting the allowed states in the Hilbert space.

Still, it seems that the functions $\epsilon_{I_1 J_1 \ldots I_n J_n} F^{I_2 J_2} \wedge \ldots \wedge F^{I_n J_n}$ entering the first class constraints of higher dimensional Chern Simons theory [137] constitute an important sub-sector of the theory which one should consider for entropy calculations, as we will argue in the following. The algebra of these excitations can be explicitly shown to reproduce the lie algebra relations of so$(D+1)$,

$$\left\{E^{IJ}(x), E^{KL}(y)\right\} \propto \delta^{(D-1)}(x-y) f^{IJ,KL,}{}_{MN} E^{MN}(x), \tag{17.82}$$

where $f$ are the corresponding structure constants like given in appendix D. Preliminary calculations also indicate that the a straight forward generalisation of the quantisation prescription in [292] leads to boundary excitations which automatically carry simple representations. This is appealing since, on the one hand, the use of the variables $n^I$ and $s^J$ inherently implies that this constraint is also solved classically at the horizon, and on the other hand (17.81) requires these representations to be simple since the SO$(D+1)$ representations in the bulk are simple. But from $D=3$ we know that the boundary Hilbert space typically turns out not to be simply the tensor product of the individual representation spaces corresponding to the punctures, but rather a subspace thereof, since there are additional global constraints resulting form the horizon topology. The global constraints which need to be imposed in higher dimensions remain to be studied. Another open question is the role played by the vertex simplicity constraints at the boundary.

Despite these attractive features, we still have to deal with the local degrees of freedom. One point that we overlooked up to now is that the classical analogue of the boundary condition (17.81) does not constrain the Chern-Simons connection $A_{\alpha IJ}$ to be ${\Gamma^0}_{\alpha IJ}$. In section 17.3, it was shown that some modifications of the boundary connection parametrised by constants are allowed. Furthermore, the idea of Beetle and Engle introduced in section 17.4.1 suggests that further modifications are conceivable, possibly an infinite set. Thus, we should introduce a constraint which restricts the degrees of freedom of the higher-dimensional Chern-Simons theory as if the horizon connection would be given by $\Gamma^0$. Since the gauge invariant (local) information of a connection is contained in its field strength, we should introduce the boundary condition (17.54) in the form

$$F(A)^{\text{horizon}}_{\alpha\beta IJ} = F(\Gamma^0)^{\text{bulk}}_{\alpha\beta IJ} \tag{17.83}$$

on $S$. Note that although this condition seems physically sensible, it cannot be strictly derived due to the non-uniqueness of the boundary connection. In analogy to the $3+1$ dimensional treatment, we would quantise this boundary condition by promoting the left hand side to an operator in the higher-dimensional Chern-Simons theory and act with a proper quantisation of the right hand side on the bulk spin network (as with a flux operator). Since we would regularise the right hand side by fluxes and commutators involving volume operators as in [3, 30], it would vanish everywhere, except at punctures[1]. This mechanism could thus get rid of the local degrees of freedom and result in a finite entropy much in the same way as in $3 + 1$ dimensions. Still, there are many missing and imprecise steps in this argument, e.g. that one would first need an actual quantisation of higher-dimensional Chern-Simons theory before a quantum boundary condition as (17.83) could be even imposed.

To conclude, we don't have a satisfactory quantisation of the resulting boundary theory

[1] We would expect that the corresponding operator would even vanish at punctures, since the volume operator annihilates edges. On the other hand, we would demand consistency with (17.81), i.e. we would rather use (17.81) at punctures. This underlines again that the discussion here does not provide a satisfactory answer.

and thus also no direct access to entropy calculations at the moment. The biggest uncertainty certainly is that no quantisation of higher-dimensional Chern-Simons theory with a non-Abelian gauge group is known.

### 17.5.2 Reduction to U(1)

In view of quantisation, the U(1) Chern Simons theory of course is distinguished by its lack of local degrees of freedom also in higher dimensions. The natural question arises why not to reformulate the boundary degrees of freedom accordingly. This question will be pursued in this section, but as we will see, we did not succeed in giving a satisfactory description of the boundary degrees of freedom with this structure group.

Two routes suggest themselves: 1) Gauge fix the SO($D+1$) Chern Simons theory we obtained in the course of this thesis down to SO(2), or 2) impose the gauge fixing directly at the level of the boundary symplectic structure and rewrite it in terms of an SO(2) Chern Simons theory.

Concerning the first route, note that gauge fixing cannot change the number of physical degrees of freedom. Naturally, one would expect the SO($D+1$) Chern Simons theory on the boundary to have local degrees of freedom. If this turns out to be true, gauge fixing to SO(2) cannot be possible, simply because this would imply a change of number of degrees of freedom. But as we commented on in the previous section 17.5.1, the boundary condition might render the boundary degrees of freedom finite even for the structure group SO($D+1$), so there would be at least no immediate contradiction. However, it is easy to see that the SO($D+1$) invariant tensor used to construct the Chern Simons theory, namely $\epsilon^{I_1 \ldots I_{D+1}}$, does not admit a gauge fixing to SO(2) and therefore, the first route fails. We will follow route 2) in what follows.

We introduce the gauge fixing $n^I = g^{0i}\delta^I_i$, $s^J = g^{1j}\delta^J_j$, where $i, j \in \{0, 1\}$ and $g \in \mathrm{SO}(2)$. Let us use the usual parametrisation of rotations by an angle $\phi$, $g_{00} = g_{11} = \cos\phi$, $g_{01} = -g_{10} = \sin\phi$. The boundary contribution to the symplectic structure reads in this gauge

$$\delta_{[1}\tilde{s}^I \delta_{2]} n_I = \delta_{[1}\sqrt{h}\ \delta_{2]}\phi. \tag{17.84}$$

In the SO($D+1$) case, to show that a Chern Simons symplectic structure arises on the horizon cross sections, it was important that $\sqrt{h}$ and the Euler density are essentially the same. Introducing an SO(2) connection $A_\alpha$, the analogue of this requirement would read

$$\sqrt{h} = \epsilon^{\alpha_1 \ldots \alpha_{2n}} F_{\alpha_1 \alpha_2} \ldots F_{\alpha_{2n-1} \alpha_{2n}}, \tag{17.85}$$

where $F_{\alpha\beta} = 2\partial_{[\alpha} A_{\beta]}$. It follows that $\delta\sqrt{h} = 2n\epsilon^{\alpha_1 \ldots \alpha_{2n}} \left(\partial_{[\alpha_1} \delta A_{\alpha_2]}\right) F_{\alpha_3 \alpha_4} \ldots F_{\alpha_{2n-1} \alpha_{2n}}$ and therefore (upon partial integration)

$$\delta_{[1} \tilde{s}^I \delta_{2]} n_I = 2n\epsilon^{\alpha_1 \ldots \alpha_{2n}} \left(\delta_{[1} A_{\alpha_1}\right) \left(\delta_{2]} \partial_{\alpha_2} \phi\right) F_{\alpha_3 \alpha_4} \ldots F_{\alpha_{2n-1} \alpha_{2n}}. \tag{17.86}$$

With the additional requirement that $A = d\phi$, this would become the symplectic structure of an SO(2) Chern Simons theory on the boundary. However, from this requirement we also conclude that $F = 0$, which is in contradiction with (17.85), and therefore also our second route fails. It thus seems that we have to stick to the SO($D+1$) theory on the boundary and one should try to make progress with its quantisation as outlined above.

# 18

# Conclusions and outlook

## 18.1 Summary

In this thesis, we succeeded in constructing a canonical connection formulation of general relativity in any spacetime dimension $D + 1 \geq 3$, based on the gauge group $\mathrm{SO}(D+1)$ or $\mathrm{SO}(1, D)$. The choice of gauge group, being $\mathrm{SO}(4)$ or $\mathrm{SO}(1, 3)$ in four spacetime dimensions, already makes explicit that the theory is genuinely different from the $\mathrm{SU}(2)$ Ashtekar Barbero formulation. Our presentation interrelates this new formulation with several other, well-known Hamiltonian formulations of general relativity and shows how it arises rather naturally from them. To this end, we derived the formulation both using Hamiltonian methods, i.e. extending the ADM phase space appropriately, as well as by performing a detailed canonical analysis of the Palatini action and applying the procedure of gauge unfixing to get rid of the appearing second class constraints.

The theory of course is subject to the usual spatial diffeomorphism and Hamiltonian constraint, but the latter is necessarily more complicated than the one of the $\mathrm{SU}(2)$ Ashtekar Barbero theory. This more complicated form is needed in order to allow for first class simplicity constraints, which enter the Hamiltonian picture as a new ingredient and play a central role in both derivations. These three sets of constraints together with the Gauß constraint familiar from usual LQG constitute all first class constraints of the system.

The Hamiltonian route towards the new variables is more general in that it allows for the introduction of a free parameter $\beta$, similar to but different from the Barbero

Immirzi parameter $\gamma$. Moreover, the internal ($\zeta$) and spacetime ($s$) signature are not necessarily tied to each other, but rather all four possible combinations thereof can be chosen. This is particularly important regarding quantisation, enabling us to work with the compact gauge group $\mathrm{SO}(D+1)$ for both, Euclidean and Lorentzian general relativity in $D+1$ dimensions. It is this $\mathrm{SO}(D+1)$ formulation which ultimately features all the properties needed for loop quantisation. The background independent quantisation techniques developed in LQG for spatially diffeomorphism invariant theories of connections are formulated independently of the number of dimensions and choice of compact structure group, and further results like the implementation of Gauß and spatial diffeomorphism constraint as well as results on the Hamiltonian constraint and the uniqueness of the representation generalise to arbitrary dimensions (cf. e.g. [62] and references therein).

The simplicity constraints constitute a novelty in canonical LQG research, but actually have a long history in its covariant cousin, the spin foam models [185, 188, 189]. Being constructed from discretised $B$-fields (spin foams) or singularly smeared fluxes (canonical approach) which are non-commutative, they lead to anomalies at the quantum level. Various proposal are available in the literature on how to deal with this issue, but some use special properties of the groups $\mathrm{SU}(2)$, $\mathrm{SO}(4)$ and $\mathrm{SO}(1,3)$ which do not hold in higher dimensions and some are simply not applicable in the canonical picture. We proposed some new but still incomplete ideas towards a satisfactory solution to this issue in the canonical picture, and will comment on open problems in section 18.2.

We furthermore showed that the framework can be extended further to all standard model matter fields and also to various kinds of other fields appearing in (higher dimensional) supergravity theories. The coupling of gauge bosons (for arbitrary compact structure groups) as well as scalar fields can literally be copied from the 3+1 dimensional treatment in [45, 46]. Dirac fermions need a special treatment at the classical level in the Lorentzian case, since we have to exchange the Lorentzian by the Euclidean Clifford algebra in order to obtain a compact structure group for general relativity. We showed that this can be accomplished, and after this classical manipulation, the quantisation known from 3+1 dimensions can be applied.

Turning to supergravity theories, many new fields arise to complete the super multiplets, most prominently, the spin 3/2 Rarita Schwinger field ("gravitino"). Compared to Dirac fermions, a new technical challenge arises: supersymmetry usually demands this field to be a Majorana fermion. The corresponding Majorana condition in the Hamiltonian picture leads to a non-trivial Dirac anti-bracket, which on the one hand hugely complicates the switch of the structure group $\mathrm{SO}(1, D)$ to $\mathrm{SO}(D+1)$ and on the other hand calls for a quantisation different from that for the Dirac field. We showed that, using an auxiliary field known from the linear simplicity constraint, the internal signature switch is still possible and a background independent Hilbert space representation for the Rarita Schwinger field is provided. Our methods also extend to spin 3/2 Majorana Weyl, spin 1/2 Dirac Weyl and spin 1/2 Majorana fermions. On the bosonic side, typical new fields are e.g. Abelian higher $p$-form fields and, exemplarily, we studied the quantisation of the three form gauge field ("three index photon") of $d = 11$, $N = 1$ supergravity. Due to a Chern Simons term in the corresponding supergravity action, this field becomes self-interacting and a non-standard $^*$-algebra of observables with respect to the (equivalent of the) Gauß constraint arises. The resulting Weyl algebra allows a state of the Narnhofer-Thirring type. These findings allow for the LQG type quantisation of at least a subset of supergravity theories, including the arguably interesting cases of $d = 11$ $N = 1$, $d = 10$ $N = 1$ and $d = 4$ $N = 8$ supergravity.

Finally, as a first application of the developed framework, we took a first step in direction of a quantum gravity derivation of the famous Bekenstein Hawking formula for the black hole entropy also in higher dimensional LQG. Concretely, we derived a suitable boundary condition as well as the boundary symplectic structure for undistorted non-rotating isolated horizons in $2(n+1)$ dimensional spacetimes and showed that it yields an $\mathrm{SO}(2(n+1))$ Chern Simons theory.

## 18.2 Discussion of open problems and directions for further research

Finally, we want to give a (non-exhaustive) list of open problems and interesting directions for further research.

**1. Implementation of the simplicity constraints and connection to spin foams**

Regarding vacuum general relativity, the simplicity constraint is the most unsettled point in our analysis. Classically, it is equivalent to work with the linear or the quadratic version of the constraint, or even use a mixing of both. At the quantum level, conceptual differences appear (the quadratic constraint operators not forming a closed algebra while the linear do) and each case has to be studied individually and, in particular, dynamical stability of a tentative solution has to be checked. Each implementation has advantages and disadvantages: the quadratic constraint has the appealing feature that it leads to simple irreps of $\mathrm{SO}(D+1)$ on the edges [169], which in any dimension are labelled by only one integer and therefore allow for a natural map to SU(2) irreps. For $D = 3$, this in particular fits nicely with the SU(2) edge labels of usual canonical LQG. However, the quadratic constraint is anomalous at vertices and, to avoid the Barrett Crane solution, we either need to use a master constraint or implement only a maximally commuting subset of the constraints. The latter option, while leading to a (natural) unitary map to the Ashtekar Lewandowski Hilbert space, needs further study (Can it be shown rigorously that we are allowed to drop the non-commuting constraints? Are those which are dropped possibly solved weakly? Can the chosen subset be made dynamically stable? Why are the Hilbert spaces for all dimensions of the "same size"?). These questions are discussed in [5] in more detail than we do in this thesis, but no final answers are provided. The master constraint on the other hand is rather complicated and since the results from spin foams suggest that there are possible easier solutions, those should be preferred if they can be rigorously implemented.

The linear constraint is actually non-anomalous (except for additional introduction of the Barbero Immirzi parameter in $D = 3$), but strong implementation leads to a one-dimensional intertwiner space and is troublesome at the edges.

Mixing both proposals, one obtains simple representations at the edges, but again a complicated master constraint has to be implemented at the vertices (still, its advantage when compared to the quadratic simplicity master constraint is the access to the unit vector field $N^I$ needed for supergravity).

For further research, we think that it is interesting to study in detail the implementation of the simplicity constraint in the spin foam literature regarding their applicability in the Hamiltonian picture (a first comparison of the results above to spin foam methods has already been given in [5]) and generalisability to higher dimensions. Eventually, this will lead to new developments in the both fields and build new bridges between them. In particular the classical formulation presented in section 9.3, when reintroducing the Barbero Immirzi parameter $\gamma$ with both, the linear and the quadratic simplicity constraint in $D = 3$, mimics the classical starting point of the new spin foam models as much as a Hamiltonian formulation possibly can. This suggests that one can also make stronger contact also at the quantum level (although an implementation of the simplicity constraint found following this route might have the disadvantage of being not generalisable to higher dimensions due to the peculiar role played by $\gamma$).

Some open points concerning a contact to spin foams are: in the Euclidean theory, when strongly implementing the quadratic simplicity constraints, we actually recover the boundary Hilbert space of the original Barrett-Crane model: SO(4) spin networks with simple representations at the edges and the unique Barrett-Crane intertwiner at the vertices. This would be very appealing if we did not know about the problems with the Barrett Crane model, and if the intertwiner spaces would not be too small when compared with the kinematical Hilbert space of standard LQG.

Turning to the new models [186–190], the Immirzi parameter and the linear simplicity constraints enter the picture, and with them the $\gamma$-simple SO(4) representations, where still one SU(2) label suffices to label the SO(4) irreps, but left- and right handed spins are no longer equal, and, most prominently, the EPRL intertwiner space are introduced. It is argued in [5] how the EPRL intertwiner space could arise also in the canonical picture, but these results remain to be made rigorous, and an equivalent of the $\gamma$ simple representation was not shown to arise in the canonical picture for neither the quadratic nor the linear simplicity constraints.

In the Lorentzian theory, of course the apparent difference between the SO(1,3) based EPRL model and the canonical SO(4) theory of section 9.3 is the gauge group. From a canonical point of view, this difference is necessary since background independent quantisation methods have not been developed for non-compact gauge groups so far. Actually, Alexandrov started a line of research studying SO(1,3) canonical LQG and introduced so called projected spin networks [303, 304] in which the non-compact gauge group is projected down to SU(2). Although definitely a challenging quest, perhaps contact can be made by making his proposals mathematically precise.

Finally, it is interesting to study if the gauge unfixing terms in the Hamiltonian play any role for spin foam models, where the second class partner of the simplicity constraints usually is neglected (see, however, [305, 306]). The reason is that this second class partner is a secondary constraint (cf. 5.2), i.e. needed in order that the primary simplicity constraint is preserved by the dynamics. However, in spin foams the simplicity constraints are implemented at every time step and it is generally argued that the secondary constraints are therefore not needed. Recently the time evolution operator in spin foams has been shown to be of the form $\mathbb{P}T\mathbb{P}$ [307], where $\mathbb{P}$ projects onto the solutions of the primary simplicity constraints. This leads to conjecture that in the continuum limit one should actually recover the gauge unfixed Hamiltonian introduced in this work.

**2. Supergravity theories - limitations of the presented treatment**
Our considerations for supergravity theories are not completely general since for our treatment of the Majorana Rarita Schwinger field, we used a real representation of the Lorentzian Clifford algebra, which does not exist in any dimension.

Furthermore, the list of fields we studied is not exhaustive: in some supergravity theories, anti- or symplectic Majorana fermions appear, and some feature non-Abelian higher $p$-forms or non-compact gauge groups (cf. e.g. [134]). While the different Majorana fermions probably only need a minor generalisation of the framework we outlined, the latter two pose genuine barriers. The last hinders the application of the rich machinery developed for background independent quantisation of gauge theories with compact structure groups, if one is not able to exchange the non-compact group by a compact one like we did in the case of the gravitational field. Higher non-Abelian $p$-form fields probably call for further development in the field of higher gauge theory [308].

In some supergravity theories, the algebra of local supersymmetry generators closes only when using the equations of motion, otherwise being second class. We do not know how to deal with these on-shell formulations, and can only speculate that again gauge unfixing might offer a way to construct a corresponding off-shell formulation.

Finally, our treatment of the Rarita Schwinger field probably is also not the most elegant one. In particular, the attractive feature of former treatments of loop supergravity [112], employing an Osp(1|2) connection combining both, bosonic and fermionic degrees of freedom, is lost. Maybe a formulation in terms of superfields would be more appealing.

**3. Supergravity theories - quantum constraint algebra**
For supergravity theories, it would be highly desirable to have a faithful representation of the super Dirac algebra at the quantum level. Forgetting for a moment the

additional problems posed by the non-existence of an operator corresponding to the diffeomorphism constraint and the presence of the other constraints, this implies the reproduction of the additional Poisson brackets $\{\mathfrak{S}, \mathfrak{S}\} \propto \mathfrak{S} + \mathscr{H}$, $\{\mathfrak{S}, \mathscr{H}\} \propto \mathfrak{S}$ and $\{\mathscr{H}, \mathscr{H}\} \propto \mathfrak{S}$ (cf. e.g. [120, 265]).

This, in a sense, is both, a blessing and a curse: while it is probably tremendously complicated to study the quantum algebra, these requirements might be that strong that they actually reduce the quantisation ambiguities in both, the Hamiltonian and the supersymmetry constraint. As testbed, three dimensional supergravity suggests itself, coming with huge simplifications both, in the bosonic as well as in the fermionic sector (cf. [309, 310] for previous approaches to loop quantisation of $d = 3$ supergravity), and in particular allowing for a study of this issue independent of the simplicity constraint problem, which does not exist in $d = 3$.

**4. Cosmology**
Loop quantum cosmology, the quantisation of various cosmological models with LQG methods, has been extraordinary successful. Not only does it lead to a rather generic resolution of singularities present in classical and the older Wheeler deWitt quantum cosmology, e.g. the big bang singularity which instead is replaced by a quantum bounce at minimal finite volume of the universe, but also its effective dynamics have been shown to be in favor of inflation compatible with the 7 years WMAP data (see e.g. [41, 42] and references therein). Revisiting these cosmological models within the new higher dimensional, possibly supersymmetric approach, allows for a study of several open question for the full theory in much simpler model systems, like how to obtain an effective four dimensional theory from higher dimensions or how to break supersymmetry.

On the other hand, the cosmological sector of string theory has been studied extensively (cf. e.g. [96, 97]) and thus could be a first point of contact of string theory with the framework of higher dimensional LQG proposed here.

**5. Black holes**
As another early point of contact to string theory, black holes in higher dimensions suggest themselves. While they are studied in string theory (in particular, the first derivation of the black hole entropy formula from string theory was performed for supersymmetric black holes in five dimensions [98]), this was so far not possible in LQG due to the restriction to $D = 3$. While we made first steps in that direction, there

are several open problems, most prominently, the quantisation of the (non-topological) higher dimensional SO($D+1$) Chern Simons theory on the boundary and role of the simplicity constraint we commented on in section 17.5. These have to be settled before rigorous state counting and a derivation of the entropy formula, possibly with logarithmic corrections, come into reach.

Experience from the $D = 3$ case suggests that, to reproduce the right prefactor 1/4 of the leading order term in the entropy formula, one has to fix the new parameter $\beta$, which most probably will depend on the dimensionality of spacetime. In more recent work, there have been found ways to reproduce 1/4 without fixing of the Barbero Immirzi parameter [285, 311]. This could possibly be recovered in higher dimensions using the freedom in the boundary connection we commented on in section 17.3.

The logarithmic corrections in $D = 3$ are independent of $\gamma$ and therefore seem to give a more stringent benchmark which might be used as a cross check if the quantisation of the boundary degrees of freedom is correct. However, there seems to be less consensus on what the prefactor of these logarithmic corrections should be: there are general arguments (cf. [312, 313]), that the logarithmic corrections should to be $-3/2 \log(A/4)$ independent of the spacetime dimension, which is supported by calculations in different models. This prefactor $-3/2$ was also found in the LQG derivation using SU(2) as gauge group, the U(1) case, instead, leads to $-1/2$ (cf. e.g.[43] and references therein). However, this does not indicate that the SU(2) treatment of Perez and collaborators is favoured. In fact, there are also many (non-loop) derivations of the factor $-1/2$ (see e.g. [314] and references therein). Finally, very recent calculations by Sen [315] using Euclidean gravity methods lead to an again different value of the prefactor. The issue is, to the best of the author's knowledge, unsettled.

Furthermore, there is an extension of the isolated horizon framework to supersymmetry [316, 317]. The study of supersymmetric black holes and the role played by supersymmetry in the subsequent entropy derivation suggest itself for further research. When treating non-supersymmetric isolated horizons, the Hamiltonian constraint usually does not need to be taken into account since the lapse function vanishes at the horizon. The constraint algebra displayed in **3.** leads one to the conjecture that either also the supersymmetry constraint needs not to be taken into account or both have to.

Finally, since no hair theorems for four dimensions generally fail in higher dimensions, there is a whole "zoo" of black hole solution to be explored (cf. e.g. [318]). In particular, it is argued that the black hole entropy actually should depend on the horizon topology, more precisely on the Euler characteristic $\chi$, which seems to nicely fit with the results obtained here. However, there seems to be no consensus on how topology enters the entropy formula: when studying exotic topologies, there are results from LQG [319] as well as results not employing loop techniques [320–322] indicating that only the sub-leading terms should depend on topology. In contrary, [323] finds that the leading order term depends on $\chi$. To hopefully give answers to these questions in the future, of course we first have to make progress on the quantisation of the boundary degrees of freedom.

**6. Recovering "every day life" physics**

Of course, starting from a higher dimensional, possibly supersymmetric theory of quantum gravity poses the immediate question of how to recover an effective, four dimensional and non-supersymmetric theory, and in particular, if problems similar to the "landscape" in string theory emerge. So far, we only have the observation that the implementation of a maximally commuting set of simplicity constraints in chapter 11.2 suggests that the dimensionality of spacetime might be irrelevant at the kinematical quantum level, only reemerging at the semiclassical level (or possibly through dynamics).

The study of this issue definitely is an ambitious project: before it can be attacked, one probably needs to make progress on **1.** and **3.**, and furthermore, sufficient control on the semiclassical sector of the theory needs to be gained.

**7. Connection to string theory**

While certainly not the least interesting, the connection to string theory is definitely a hard and long term goal. Even with a future, further developed loop quantum supergravity (LQSG) at hand, the comparison at the level of supergravity is only indirect. In particular, there is still no "string in higher dimensional LQSG", and the quantisation methods in string theory are still (at least a priori) background dependent. Therefore, we think that, parallel to LQSG, the research direction started in [93] of a loop quantisation of string theory should be further developed.

Apart from the already mentioned contact points, cosmology and black holes, an interesting but more speculative application we have in mind is a test of the conjectured

AdS/CFT correspondence by e.g. loop quantising type IIB supergravity on a 10d manifold ($\mathrm{AdS}^5 \times \mathrm{S}^5$?) and compare it with a loop quantisation of $N = 4$ super Yang-Mills theory. Progress in this direction would probe the non-perturbative limit of the conjectured equivalence [85–87] of type IIB string theory on $\mathrm{AdS}^5 \times \mathrm{S}^5$ and $N = 4$ super Yang-Mills theory on the four-dimensional boundary of $\mathrm{AdS}^5$.

In conclusion, many technical problems remain so far unsettled by our work and definitely deserve further studies, but also many interesting new research directions are opened up. We hope that this work leads to a stronger bridge between canonical LQG and spin foam models, stimulates a further development of LQSG in any dimensions and ultimately contributes to an enhanced exchange between strings and loops in the future.

# A

# Variational formulae

In this appendix, we will collect various variational formulae which will be helpful for calculations in the main text.

**Inverse metric and determinant:** Since $\delta(g_{\mu\nu}g^{\nu\rho}) = (\delta g_{\mu\nu})g^{\nu\rho} + g_{\mu\nu}(\delta g^{\nu\rho}) = 0$, we have

$$\delta g^{\mu\nu} = -g^{\mu\rho}g^{\nu\sigma}\delta g_{\rho\sigma}. \tag{A.1}$$

For the determinant of the metric, we find using Jacobi's formula

$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu}, \tag{A.2}$$

$$\delta\sqrt{|g|} = \frac{1}{2\sqrt{|g|}}\delta|g| = \frac{1}{2}\sqrt{|g|}g^{\mu\nu}\delta g_{\mu\nu}, \tag{A.3}$$

where in the last step, it was important that the metric is non-degenerate.

**Christoffel symbols:** Two affine connections differ by a tensor field of rank (1,2). Therefore, one expects that also the variation of the Christoffel symbols yields a tensor field, which is indeed the case and was, to the best of the author's knowledge, first observed in [147].

$$\begin{aligned}
\delta\Gamma^{\rho}_{\mu\nu} &= \frac{1}{2}\delta\left[g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)\right] \\
&= \frac{1}{2}g^{\rho\sigma}\left(\partial_\mu \delta g_{\nu\sigma} + \partial_\nu \delta g_{\mu\sigma} - \partial_\sigma \delta g_{\mu\nu}\right) - \frac{1}{2}g^{\rho\alpha}g^{\sigma\beta}\delta g_{\alpha\beta}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \\
&= \frac{1}{2}g^{\rho\sigma}\left(\nabla_\mu \delta g_{\nu\sigma} + \nabla_\nu \delta g_{\mu\sigma} - \nabla_\sigma \delta g_{\mu\nu}\right).
\end{aligned} \tag{A.4}$$

The final expression shows that the variation actually is a tensor field. The easiest way to comprehend the last step in the above calculation is to work backwards, writing out the covariant derivatives explicitly and simplifying the resulting expression reproduces the second line.

**Riemann tensor, Ricci tensor, Ricci scalar:** For the variation of the Riemann tensor, we find

$$\begin{aligned}\delta R_{\mu\nu\rho}{}^{\sigma} &= -2\delta\left(\partial_{[\mu}\Gamma^{\sigma}_{\nu]\rho} - \Gamma^{\lambda}_{[\mu|\rho}\Gamma^{\sigma}_{\nu]\lambda}\right)\\ &= -2\nabla_{[\mu}\delta\Gamma^{\sigma}_{\nu]\rho}.\end{aligned} \tag{A.5}$$

Contracting this equation with $\delta^{\nu}_{\sigma}$, we obtain for the variation of the Ricci tensor

$$\delta R_{\mu\rho} = -2\nabla_{[\mu}\delta\Gamma^{\nu}_{\nu]\rho}, \tag{A.6}$$

Contraction of (A.6) with the inverse metric and using (A.4), we find

$$g^{\mu\rho}\delta R_{\mu\rho} = \nabla^{\mu}\left(\nabla^{\rho}\delta g_{\mu\rho} - \nabla_{\mu}g^{\rho\sigma}\delta g_{\rho\sigma}\right), \tag{A.7}$$

which yields the surface term in the variation of the Einstein Hilbert action in section 2.1. Finally, we find for the Ricci scalar

$$\delta R = \delta(g^{\mu\rho}R_{\mu\rho}) = \nabla^{\mu}\nabla^{\rho}\delta g_{\mu\rho} - \nabla^{\mu}\nabla_{\mu}g^{\rho\sigma}\delta g_{\rho\sigma} - R^{\mu\rho}\delta g_{\mu\rho}. \tag{A.8}$$

**Laplacian:** For the variation of the Laplacian $\Delta = g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}$ of a scalar field $\phi$, we find using (A.1, A.2, A.4)

$$\begin{aligned}\delta\Delta\phi &= \delta\left(g^{\mu\nu}\partial_{\mu}\partial_{\nu}\psi - g^{\mu\nu}\Gamma^{\rho}_{\mu\nu}\partial_{\rho}\phi\right)\\ &= \Delta\delta\phi - \left(\delta g_{\mu\nu}\right)\nabla^{\mu}\nabla^{\nu}\phi - \left(\nabla^{\mu}\delta g_{\mu\nu} - \frac{1}{\sqrt{|g|}}\nabla_{\nu}\delta\sqrt{|g|}\right)\nabla^{\nu}\phi.\end{aligned} \tag{A.9}$$

**Vielbein and related variations:** If, instead of a metric, one works with a (co)-vielbein, the following formulas might be helpful. Their derivation is straight forward.

$$\delta e^{\mu I} = -e^{\mu J}e^{\nu I}\delta e_{\nu J}, \tag{A.10}$$

$$\delta e = e e^{\mu I}\delta e_{\mu I}, \tag{A.11}$$

$$\delta g_{\mu\nu} = 2e_{(\mu}{}^{I}\delta e_{\nu)I}, \tag{A.12}$$

$$\delta|g| = 2e^{2}e^{\mu I}\delta e_{\mu I}, \tag{A.13}$$

$$g^{\mu\nu}\delta g_{\mu\nu} = 2e^{\mu I}\delta e_{\mu I}. \tag{A.14}$$

Therefore, if we work with a densitised (co)-vielbein $E^{\mu I} := ee^{\mu I}$, we have

$$\delta E^{\mu I} = 2ee^{\mu[I|}e^{\nu|J]}\delta e_{\nu J}, \tag{A.15}$$

$$\delta e_{\mu I} = e\left(\frac{1}{d-1}E_{\mu I}E_{\nu J} - E_{\nu I}E_{\mu J}\right)\delta E^{\nu J}, \tag{A.16}$$

where $E_{\mu I} = \frac{1}{e}e_{\mu I}$. Note that $e = (\det E)^{\frac{1}{(d-1)}}$ holds. The first line follows easily from the above equations and for the second we merely have to invert the matrix appearing in the first line. More importantly for this work, if we work with a hybrid vielbein (or its densitised version $E^{aI} := \sqrt{q}e^{aI}$, cf. section 3.2.3 for notation), we have

$$\delta e^{aI} = \left(\zeta n^I n^J q^{ab} - e^{aJ}e^{bI}\right)\delta e_{bJ}, \tag{A.17}$$

$$\delta\sqrt{q} = \sqrt{q}e^{aK}\delta e_{aK}, \tag{A.18}$$

$$\delta E^{aI} = \sqrt{q}\left(\zeta n^I n^J q^{ab} + 2e^{a[I|}e^{b|J]}\right)\delta e_{bJ}, \tag{A.19}$$

$$\delta e_{aI} = \sqrt{q}\left(\zeta(E^{aK}E^b{}_K)^{-1}n_I n_J + \frac{1}{D-1}E_{aI}E_{bJ} - E_{bI}E_{aJ}\right)\delta E^{bJ}, \tag{A.20}$$

$$\delta q_{ab} = 2e_{(a|I}\delta e_{b)}{}^I = -\frac{1}{q}G^{-1}_{ab\,cd}(\delta q q^{cd}) = -\frac{2}{q}G^{-1}_{ab\,cd}E^{cI}\delta E^d{}_I, \tag{A.21}$$

where $\bar{\eta}_{IJ} = E_{cI}E^c{}_J$ and $\zeta n^I n^J = \eta^{IJ} - \bar{\eta}^{IJ}$, and furthermore $\sqrt{q} = (\det(E^{cL}E^d{}_L))^{\frac{1}{2(D-1)}}$ holds. $G^{-1}_{ab\,cd}$ here and in the following is the same matrix that appeared in (2.16).

Finally, when working with $\pi^{aIJ} = 2n^{[I}E^{a|J]}$, $2\zeta q q^{ab} = \pi^{aIJ}\pi^b{}_{IJ}$, as in sections 4.2, 5.2 and part II, we have

$$\delta q q^{ab} = \zeta\pi^{(a|IJ}\delta\pi^{|b)}{}_{IJ}, \tag{A.22}$$

$$\delta q = \frac{\zeta}{D-1}q\pi_{aIJ}\delta\pi^{aIJ}, \tag{A.23}$$

$$\delta q_{ab} = -\frac{\zeta}{q}G^{-1}_{ab\,cd}\pi^{cIJ}\delta\pi^d{}_{IJ}, \tag{A.24}$$

$$\delta\pi_{aIJ} = \left[\frac{1}{q}q_{ab}\mathbb{P}_{IJ\,KL} - \frac{\zeta}{2}\pi_{aKL}\pi_{bIJ}\right]\delta\pi^{bKL}. \tag{A.25}$$

To derive these, the previous formulae are helpful. We used the notation $\pi_{aIJ} = \frac{1}{q}q_{ab}\pi^b{}_{IJ}$ and in the last equation, we introduced the projector $\mathbb{P}_{IJ}{}^{KL} := \eta^K_{[I}\eta^L_{J]} - \frac{\zeta}{2}\pi_{aIJ}\pi^{aKL}$, which projects orthogonal to $\pi^{aIJ}$, $\mathbb{P}_{IJ\,KL}\pi^{aKL} = 0$.

**Vielbein compatible spin connection:** As for the Christoffel symbol, the variation of $\Gamma_{\mu IJ}$ should yield a tensor. Indeed, we find after some simple algebra using (A.4, A.10)

$$\begin{aligned}\delta\Gamma_{\mu IJ} &= \delta\left(e^{\nu}_{[I|}\nabla_{\mu}e_{\nu|J]}\right)\\ &= e^{\nu}_{[I|}\nabla^{\Gamma}_{\mu}\delta e_{\nu|J]} - e^{\nu}_{[I|}\nabla^{\Gamma}_{\nu}\delta e_{\mu|J]} - e^{\nu}{}_{[I}e^{\rho}_{J]}e_{\mu}{}^{K}\nabla^{\Gamma}_{[\nu}\delta e_{\rho]K}.\end{aligned} \tag{A.26}$$

**Curvature tensors:** In general, we find for SO$(D+1)$ or SO$(1, D)$ curvature tensors $F_{\mu\nu IJ}$

$$\begin{aligned}\delta F_{\mu\nu IJ} &= 2\delta\left(\partial_{[\mu}A_{\nu]IJ} + A_{\mu[I|}{}^{K}A_{\nu K|J]}\right)\\ &= 2\nabla^{A}_{[\mu}\delta A_{\nu]IJ}.\end{aligned} \tag{A.27}$$

It is instructive for the unfamiliar reader to rederive (A.8) from the variation of $e^{\mu I}e^{\nu J}$ $R_{\mu\nu IJ}$ using (A.26, A.27) and various other formulas above.

# B

# Spatial - temporal decompositions

In the following, we collect and derive formulas helpful for $D+1$ decompositions of spacetime tensors.

**Gauß Codacci equations:** In section 2.2.1, we used the famous Gauß Codacci equations to express ${}^{(D+1)}R$ in terms of ${}^{(D)}R$ and the extrinsic curvature $K_{\mu\nu}$, which we will derive in the following. We follow [62] and start with noting that for a spatial covector $u_\mu$, we have, using the definition of the covariant spatial derivative in 2.2.1, the fact that $\nabla$ annihilates $g_{\mu\nu}$ and that $q_{\mu\nu} = g_{\mu\nu} - s n_\mu n_\nu$,

$$\begin{aligned}
&{}^{(D)}R_{\mu\nu\rho}{}^{\sigma} u_\sigma = [D_\mu, D_\nu] u_\rho = 2 q^{\mu'}_{[\mu} q^{\nu'}_{\nu]} q^{\rho'}_{\rho} \left(\nabla_{\mu'} q^{\nu''}_{\nu'} q^{\rho''}_{\rho'} \nabla_{\nu''} u_{\rho''}\right) \\
&= 2 q^{\mu'}_{[\mu} q^{\nu'}_{\nu]} q^{\rho'}_{\rho} \left[ -s(\nabla_{\mu'} n_{\nu'} n^{\nu''}) q^{\rho''}_{\rho'} \nabla_{\nu''} u_{\rho''} - s q^{\nu''}_{\nu'} (\nabla_{\mu'} n_{\rho'} n^{\rho''}) \nabla_{\nu''} u_{\rho''} + \nabla_{\mu'} \nabla_{\nu'} u_{\rho'} \right] \\
&= 2 q^{\mu'}_{[\mu} q^{\nu'}_{\nu]} q^{\rho'}_{\rho} \left[ -s q^{\nu''}_{\nu'} (\nabla_{\mu'} n_{\rho'}) n^{\rho''} \nabla_{\nu''} u_{\rho''} + \nabla_{\mu'} \nabla_{\nu'} u_{\rho'} \right] \\
&= 2 q^{\mu'}_{[\mu} q^{\nu'}_{\nu]} q^{\rho'}_{\rho} \left[ s q^{\nu''}_{\nu'} (\nabla_{\mu'} n_{\rho'}) u_{\rho''} \nabla_{\nu''} n^{\rho''} + \nabla_{\mu'} \nabla_{\nu'} u_{\rho'} \right] \\
&= \left[ 2s K_{[\mu|\rho} K_{\nu]}{}^{\sigma} + q^{\mu'}_{\mu} q^{\nu'}_{\nu} q^{\rho'}_{\rho} {}^{(D+1)}R_{\mu'\nu'\rho'}{}^{\sigma'} q^{\sigma}_{\sigma'} \right] u_\sigma,
\end{aligned} \tag{B.1}$$

where from line 3 to line 4 we used that $q^{\mu'}_{\mu} q^{\nu'}_{\nu} \nabla_{[\mu'} n_{\nu']} = 0$ due to Frobenius' theorem, and from line 4 to 5 that $u_\mu$ is spatial and therefore $n^\mu \nabla_\nu u_\mu = -u_\mu \nabla_\nu n^\mu$. This is the famous Gauß equation. Contracting it, we obtain

$$\begin{aligned}
{}^{(D)}R &= q^{\mu\rho} q^{\nu\sigma} {}^{(D)}R_{\mu\nu\rho\sigma} \\
&= s\left[K^2 - K_{\mu\nu} K^{\mu\nu}\right] + q^{\mu\rho} q^{\nu\sigma} {}^{(D+1)}R_{\mu\nu\rho\sigma}.
\end{aligned} \tag{B.2}$$

On the other hand, we have

$$\begin{aligned}
{}^{(D+1)}R &= g^{\mu\rho} g^{\nu\sigma\,(D+1)} R_{\mu\nu\rho\sigma} \\
&= (q^{\mu\rho} + s n^\mu n^\rho)(q^{\nu\sigma} + s n^\nu n^\sigma)^{(D+1)} R_{\mu\nu\rho\sigma} \\
&= (q^{\mu\rho} q^{\nu\sigma} + 2s n^\mu n^\rho q^{\nu\sigma})^{(D+1)} R_{\mu\nu\rho\sigma} \\
&= q^{\mu\rho} q^{\nu\sigma\,(D+1)} R_{\mu\nu\rho\sigma} - 2s n^\mu q^{\nu\sigma} [\nabla_\mu, \nabla_\nu] n_\sigma \\
&= q^{\mu\rho} q^{\nu\sigma\,(D+1)} R_{\mu\nu\rho\sigma} - 2s n^\mu [\nabla_\mu, \nabla_\nu] n^\nu, \qquad \text{(B.3)}
\end{aligned}$$

and for the last term we have

$$\begin{aligned}
2n^\mu \nabla_{[\mu} \nabla_{\nu]} n^\nu &= 2\nabla_{[\mu}(n^\mu \nabla_{\nu]} n^\nu) - 2(\nabla_{[\mu} n^\mu)(\nabla_{\nu]} n^\nu) \\
&= 2\nabla_{[\mu}(n^\mu \nabla_{\nu]} n^\nu) - K^2 + K_{\mu\nu} K^{\mu\nu}, \qquad \text{(B.4)}
\end{aligned}$$

where in the last step we used

$$\begin{aligned}
\nabla_\mu n^\mu &= g^{\mu\nu} \nabla_\mu n_\nu = (q^{\mu\nu} + s n^\mu n^\nu) \nabla_\mu n_\nu \\
&= q^{\mu\nu} \nabla_\mu n_\nu = K, \qquad \text{(B.5)} \\
(\nabla_\mu n^\nu)(\nabla_\nu n^\mu) &= (\nabla_\mu n_\rho)(\nabla_\nu n_\sigma) g^{\mu\sigma} g^{\nu\rho} \\
&= (\nabla_\mu n_\rho)(\nabla_\nu n_\sigma)(q^{\mu\sigma} + s n^\mu n^\sigma)(q^{\nu\rho} + s n^\nu n^\sigma) \\
&= (\nabla_\mu n_\rho)(\nabla_\nu n_\sigma) q^{\mu\sigma} q^{\nu\rho} \\
&= K_{\mu\nu} K^{\mu\nu}. \qquad \text{(B.6)}
\end{aligned}$$

Here, we repeatedly used that $n^\mu \nabla_\nu n_\mu = \frac{1}{2}(\nabla_\nu n^\mu n_\mu) = 0$. Combining (B.2, B.3, B.4), one arrives at the Codacci equation

$$ {}^{(D+1)}R = {}^{(D)}R - s[K_{\mu\nu} K^{\mu\nu} - K^2] - 4s \nabla_{[\mu}(n^\mu \nabla_{\nu]} n^\nu). \qquad \text{(B.7)}$$

**Spin connection and its curvature:** For the spatial components of the spin connection, we find

$$\begin{aligned}
q_\mu^{\mu'} \Gamma_{\mu'}{}^{IJ} &= q_\mu^{\mu'} e^{\nu[I} \nabla_{\mu'} e_\nu{}^{J]} \\
&= q_\mu^{\mu'} ({}^{||}e^{\nu[I} + s n^\nu n^{[I}) \nabla_{\mu'} ({}^{||}e_\nu{}^{J]} + s n_\nu n^{J]}) \\
&= {}^{||}e^{\nu[I} D_\mu {}^{||}e_\nu{}^{J]} + s n^{[I} D_\mu n^{J]} - 2s n^{[I||} e^{\nu|J]} K_{\mu\nu} \\
&= \Gamma^{\mathrm{H}}{}_\mu{}^{IJ} - 2s n^{[I||} e^{\nu|J]} K_{\mu\nu}, \qquad \text{(B.8)}
\end{aligned}$$

where we defined

$$\Gamma^{\mathrm{H}}{}_\mu{}^{IJ} := {}^{||}e^{\nu[I} D_\mu {}^{||}e_\nu{}^{J]} + s n^{[I} D_\mu n^{J]}, \qquad \text{(B.9)}$$

which upon pulling back to the spatial manifold $\sigma$ becomes the hybrid spin connection [103] introduced in appendix C.

For the projection of the spin connection on the timelike unit normal $n^\mu$, we similarly find

$$\begin{aligned} n^\mu \Gamma_\mu{}^{IJ} &= n^\mu e^{\nu[I} \nabla_\mu e_\nu{}^{J]} \\ &= n^\mu ({}^{||}e^{\nu[I} + s n^\nu n^{[I}) \nabla_\mu ({}^{||}e_\nu{}^{J]} + s n_\nu n^{J]}) \\ &= {}^{||}e^{\nu[I} \nabla_n {}^{||}e_\nu{}^{J]} + s n^{[I} \nabla_n n^{J]} - 2 s n^{[I||} e^{\nu|J]} \nabla_n n_\nu \\ &= {}^{||}e^{\nu[I} \nabla_n {}^{||}e_\nu{}^{J]} + s n^{[I} \nabla_n n^{J]} + 2 n^{[I||} e^{\nu|J]} D_\nu \log N, \end{aligned} \tag{B.10}$$

where in the last line, we used that (see e.g. [62, page 55])

$$\nabla_n n_\mu = -\frac{s}{N} D_\mu N. \tag{B.11}$$

For its curvature, we immediately find from (C.10)

$$\begin{aligned} e^{\mu I} e^{\nu J} R_{\mu\nu IJ} &= e^{\mu I (D+1)} R_{\mu\nu\rho}{}^\nu e^\rho{}_I \\ &= {}^{(D+1)}R \\ &= {}^{(D)}R - s[K_{\mu\nu} K^{\mu\nu} - K^2] - 4s \nabla_{[\mu} (n^\mu \nabla_{\nu]} n^\nu) \\ &= {}^{||}e^{\mu I ||} e^{\nu J} R^{\mathrm{H}}{}_{\mu\nu IJ} - s[K_{\mu\nu} K^{\mu\nu} - K^2] - 4s \nabla_{[\mu} (n^\mu \nabla_{\nu]} n^\nu), \end{aligned} \tag{B.12}$$

where we used (B.7) in the third step and $R^{\mathrm{H}}{}_{\mu\nu}{}^{IJ} = {}^{(D)}R_{\mu\nu\rho\sigma} {}^{||}e^{\rho I ||| } e^{\sigma J}$ and $R^{\mathrm{H}}{}_{\mu\nu}{}^{IJ}$ denotes the curvature of $\Gamma^{\mathrm{H}}{}_\mu{}^{IJ}$.

# C

# (Hybrid) spin connection and generalisations

This appendix is taken from [10]. We will introduce several connections relevant for the main text, namely the spin connection compatible with the vielbein, Peldán's "hybrid" spin connection [103] and extensions thereof to higher dimensional internal space.

## C.1 Vielbein compatible spin connection

It is a well-known fact that, given an SO($D$) vielbein ${e_a}^i$ in $D$ dimensions (or, equivalently, an SO($D+1$) or SO($1,D$) vielbein in $D+1$ dimensions), there exists a unique spin connection $\Gamma_{aij}[e]$ compatible with it, which is obtained by solving

$$0 \stackrel{!}{=} {D^\Gamma}_a {e_b}^i = D_a {e_b}^i + \Gamma[e]_a{}^i{}_j {e_b}^j \tag{C.1}$$

for $\Gamma[e]_{aij}$, where $D_a$ denotes the torsion free metric compatible covariant derivative. These are $D^3$ equations for $\frac{D^2(D-1)}{2}$ unknowns $\Gamma[e]_{aij}$, but $\frac{D^2(D+1)}{2}$ of these, namely $2e_{(b|}{}^i {D^\Gamma}_a e_{|c)i} = D_a q_{bc} = 0$, are identically satisfied (or, if we do not fix the affine connection, can be solved for the $\frac{D^2(D+1)}{2}$ components of $\Gamma^a_{bc}$). Therefore, the number of equations equals the number of unknowns, and we can solve for

$$\Gamma[e]_{aij} = e^b{}_{[i} D_a e_{b|j]}. \tag{C.2}$$

Note that an equivalent requirement in this case is the torsion freeness condition

$$0 \stackrel{!}{=} {D^\Gamma}_{[a} e_{b]}{}^i, \tag{C.3}$$

which constitutes $\frac{D^2(D-1)}{2}$ independent equations for the $\frac{D^2(D-1)}{2}$ unknowns $\Gamma[e]_{aij}$. Its curvature $R_{ab}{}^{ij}$ satisfies

$$R_{abcd} = R_{ab}{}^{ij} e_{ci} e_{dj}, \tag{C.4}$$

$$R_{ab}{}^{ij} = R_{abcd} e^{ci} e^{dj}, \tag{C.5}$$

which will be needed in the main text. These equations can be easily derived from

$$0 = [D^\Gamma{}_a, D^\Gamma{}_b] e_c{}^i = R_{abc}{}^d e_d{}^i + R_{ab}{}^i{}_j e_c{}^j. \tag{C.6}$$

## C.2 Peldán's hybrid connection

Starting from a Lagrangian formulation of general relativity on a $D+1$ dimensional space time manifold, the natural gauge group is $\mathrm{SO}(1,D)$ or $\mathrm{SO}(D{+}1)$ for the Lorentzian or Euclidean theory, respectively. When passing to the corresponding Hamiltonian system, a $D+1$ split is performed and we are naturally led to consider a $\mathrm{SO}(1,D)$ or $\mathrm{SO}(D+1)$ vielbein $e_a{}^J$ on the $D$ dimensional spatial manifold, which we call hybrid vielbein (cf. also chapter 3). However, from the Hamiltonian perspective, the signature of the internal space $\zeta$ is not necessarily tied to the space time signature $s$, since we can always start with an $\mathrm{SO}(D)$ vielbein on the spatial slice and introduce gauge degrees of freedom corresponding either to $\mathrm{SO}(1,D)$ or $\mathrm{SO}(D+1)$. In the following, we will therefore treat internal and space time signature independently. Peldán [103] investigated if one could define a compatible connection also for this hybrid vielbein. We have

$$0 \stackrel{!}{=} D^{\mathrm{H}}_a e_b{}^J = D_a e_b{}^J + \Gamma^{\mathrm{H}}[e]_a{}^J{}_K e_b{}^K, \tag{C.7}$$

which constitutes $D^2(D{+}1)$ equations for $\frac{D^2(D+1)}{2}$ unknowns $\Gamma^{\mathrm{H}}[e]_a{}^{IJ}$. However, again the $\frac{D^2(D+1)}{2}$ equations $2e_{(b|}{}^I D^{\mathrm{H}}{}_a e_{|c)I} = D_a q_{bc} = 0$ are identically satisfied, and again, the number of equations matches the number of unknowns. We actually can solve for the unique "hybrid" spin connection,

$$\Gamma^{\mathrm{H}}[e]_{aIJ} = e^b{}_{[I} D_a e_{b|J]} + \zeta n_{[I} D_a n_{J]}, \tag{C.8}$$

where $n^I$ is the unique (up to sign) unit normal to the hybrid vielbein, $n^I e_{aI} = 0$, $n^I n^J \eta_{IJ} = \zeta$, and $\zeta$ again denotes the internal signature, $\zeta = -1$ for $\mathrm{SO}(1,D)$ and $+1$ for $\mathrm{SO}(D+1)$. Note that the sign ambiguity is absent in $\Gamma^{\mathrm{H}}[e]_{aIJ}$ since $n^I$ appears quadratically.

In this case, the conditions

$$0 \stackrel{!}{=} D^{\mathrm{H}}_{[a} e_{b]}{}^J \tag{C.9}$$

are insufficient [103], being only $\frac{D^2(D-1))}{2}$ independent equations. Again, since

$$0 = [D_a^{\rm H}, D_b^{\rm H}] e_c{}^I = R_{abc}{}^d e_d{}^I + R^{{\rm H}\,I}_{ab}{}_J e_c{}^J, \tag{C.10}$$

$$0 = [D_a^{\rm H}, D_b^{\rm H}] n^I = R^{{\rm H}\,I}_{ab}{}_J n^J, \tag{C.11}$$

we have

$$R_{abcd} = R^{{\rm H}\,IJ}_{ab} e_{cI} e_{dJ}, \tag{C.12}$$

$$R^{{\rm H}\,IJ}_{ab} = R_{abcd} e^{cI} e^{dJ}. \tag{C.13}$$

The superscript "H" on $\Gamma^{\rm H}$ and $R^{\rm H}$ will be skipped in several formulae throughout this thesis, since already the index structure distinguishes the hybrid from the usual spin connection.

## C.3 Extensions to higher dimensional internal space

Now we want to extend this result to a higher dimensional internal space, which is necessary for black hole applications in part V, since we have to deal with the vielbein on the $D-1$ dimensional inner boundaries of the spatial slice, and also allows for the construction of $\mathrm{SO}(p, D+q)$ gauge theories ($p \geq 0$, $q \geq 0$, $p+q \neq 0$) of gravity in section 9.2.

We will start quite general by introducing an $\mathbb{R}^{p,D+q}$ – valued vielbein $e_a{}^J$ in $D$ dimensions (note that in this section, we will have $I, J, K... = 1, ..., D+k$), $e_a{}^I e_b{}^J \eta_{IJ} = q_{ab}$ where $\eta_{IJ} = \mathrm{diag}(\underbrace{-, ..., -}_{p}, \underbrace{+, ..., +}_{D+q})$ and $p+q = k$, and ask for a $\mathrm{so}(p, D+q)$ connection $\Gamma^{\rm H}_{aIJ}$ annihilating $e^J_a$. We have

$$0 \stackrel{!}{=} D^{\rm H}_a e_b{}^J = D_a e_b{}^J + \Gamma^{{\rm H}\,J}_{a}{}_K e_b{}^K, \tag{C.14}$$

corresponding to $D^2(D+k)$ equations to determine $\Gamma^{\rm H}_{aIJ}$. However, these equations are not all independent, since

$$0 = e_{(c|}{}^I D^{\rm H}_a e_{|b)I} \tag{C.15}$$

are identically satisfied due to the antisymmetry of the $\mathrm{so}(p, D+q)$ connection and the metric compatibility of $D_a$. The result are

$$D^2(D+k) - D^2(D+1)/2 = D^2((D-1)/2 + k) \tag{C.16}$$

independent equations for the

$$D(D+k)(D+k-1)/2 \tag{C.17}$$

unknowns $\Gamma^{\rm H}_{aIJ}$. It is clear that $\Gamma^{\rm H}_{aIJ}$ cannot be determined uniquely for any $k$, since the number of equations grows, for fixed $D$, linearly with $k$, while the connection components grow quadratically. More precisely, equating both, we obtain $(C.16) = (C.17) \Leftrightarrow Dk(k-1)/2 = 0$, i.e. the connection is only uniquely determined for the gauge groups $SO(D)$, corresponding to $k=0$, and $SO(1,D)$ or $SO(D+1)$ for $k=1$. Let us study the indeterminacy for $k>1$ in more detail. First we "complete" the vielbein by choosing an orthonormal set of $k$ unit vectors $n_i{}^I$, $i=1,...,k$, normal to the vielbein, i.e. $n_i{}^I e_{aI} = 0\ \forall i = 1,...,k$ and $n_i{}^I n_j{}^J \eta_{IJ} = \eta_{ij}\ \forall i,j = 1,...,k$ where $\eta_{ij} = \text{diag}(\underbrace{-,...,-}_{p},\underbrace{+,...,+}_{q})$[1]. The indices $i,j,...$ will be raised and lowered using this metric and its inverse $\eta^{ij}$. Then we can decompose $\Gamma^{\rm H}_{aIJ}$ according to

$$\Gamma^{\rm H}_{aIJ} = \overline{\Gamma}_{aIJ} + 2n_{i[I|}\overline{\Gamma}^i{}_{a|J]} + n_{i[I|}n_{j|J]}\Gamma^{ij}{}_a, \tag{C.18}$$

where summation over repeated indices $i,j$ is understood and $\overline{\Gamma}_{aIJ}n_i{}^J = 0\ \forall i = 1,...,k$, $\overline{\Gamma}^i{}_{aJ}n_j{}^J = 0\ \forall i,j = 1,...,k$. Inserting this decomposition of $\Gamma^{\rm H}_{aIJ}$ into (C.14), we find that $\Gamma^{ij}{}_a$ simply drops out and therefore cannot be solved for, and the number of its components, $Dk(k-1)/2$ since it is antisymmetric in $i,j$, precisely matches the indeterminacy. For the other components, one obtains

$$\overline{\Gamma}_{aIJ} = e^b{}_{[I|}\bar{\eta}_{J]K} D_a e_b{}^K, \tag{C.19}$$

$$\overline{\Gamma}^i{}_{aJ} = \bar{\eta}_{JK} D_a n^{iK}, \tag{C.20}$$

where $\bar{\eta}_{IJ} := e_{aI}e^a{}_J$. Inserting back into (C.18), we find

$$\Gamma^{\rm H}_{aIJ} = 2e^b{}_{[I|}D_a e_{b|J]} - e^b{}_{[I}\bar{\eta}_{J]K}D_a e_b{}^K + n_{i[I|}n_{j|J]}\Gamma^{ij}{}_a \tag{C.21}$$

and therefore a $Dk(k-1)/2$ – parameter family of connections annihilating $e_a{}^I$. To obtain a unique connection, we have to add additional requirements, e.g. we could demand that $\Gamma^{ij}{}_a = 0\ \forall\ i,j = 1,...,k$ (these requirements are independent of the choice of "completion" for the vielbein $\{n_i{}^I\}_{i=1}^k$). This connection $\Gamma^1_{aIJ}$ would be special in that it would only depend on $e_a{}^I$,

$$\Gamma^1_{aIJ} = 2e^b{}_{[I|}D_a e_{b|J]} - e^b{}_{[I}\bar{\eta}_{J]K}D_a e_b{}^K. \tag{C.22}$$

[1]Actually, we can as well specify $k-1$ vectors, since the last one, $n_k{}^I$, is already determined (up to sign) by the mentioned requirements.

Having in mind the application to black holes, we will proceed differently. For a fixed extension, the extra conditions we impose are $D^{\mathrm{H}}_a n_i{}^I = 0 \ \forall i = 1, ..., k-1$[1] (these requirements are sensitive to the choice of completion). Again, these conditions are not all independent. We have $e_b{}^I D^{\mathrm{H}}_a n_{iI} = 0$ and $n_{(i}{}^I D^{\mathrm{H}}_a n_{j)I} = 0$ already satisfied, which results in $D(k-1)(D+k) - (D^2(k-1) + Dk(k-1)/2) = Dk(k-1)/2$ independent equations. This equals the number of undetermined components $\Gamma^{ij}{}_a$. Solving for these, we find

$$\Gamma^{ij}{}_a = -n^{[i}{}_I D_a n^{j]I} \tag{C.23}$$

and

$$\Gamma^0_{aIJ}[e,n] := e^b{}_{[I|} D_a e_{b|J]} + n^i{}_{[I|} D_a n_{i|J]} \tag{C.24}$$

as the unique connection annihilating the chosen completion of $e_a{}^J$. This connection has several nice properties. For all connection of the family, we have

$$R^H_{abIJ} e_c{}^I e^{dJ} = R_{abc}{}^d, \tag{C.25}$$

$$R^H_{abIK} n_i{}^I \bar{\eta}^{KJ} = 0, \tag{C.26}$$

which follows from contraction of

$$0 = [D^{\mathrm{H}}_a, D^{\mathrm{H}}_b] e_c{}^I = R^{H\,I}_{ab}{}_J e_c{}^J + R_{abc}{}^d e_d{}^I. \tag{C.27}$$

But for this connection $\Gamma^0$, we additionally have

$$R^0_{ab}{}^I{}_J n_i{}^J = [D^0_a, D^0_b] n_i{}^I = 0 \tag{C.28}$$

and therefore

$$R^0_{abIJ} = R_{abc}{}^d \; e^c{}_I e_{dJ}. \tag{C.29}$$

From the right hand side of (C.29), we see that, while $\Gamma^0_{aIJ}$ depends on the choice of $\{n_i{}^I\}_{i=1}^k$, $R^0_{abIJ}$ is independent of $n$, determined completely by $e_a{}^I$ and its first and second derivatives. Explicitly, choosing a different completion $\{\tilde{n}_i{}^I\}_{i=1}^k$ of $e_a{}^I$, which is related to $\{n_i{}^I\}_{i=1}^k$ by a $\mathrm{SO}(p,q)$ transformation $g$ via $\tilde{n}_i = g_i{}^j n_j$, we find

$$\Gamma^0_{aIJ}[e,\tilde{n}] = \Gamma^0_{aIJ}[e,n] + K_{aIJ}, \tag{C.30}$$

$$K_{aIJ} := g^i{}_k n^k{}_{[I|} n_{l|J]} D_a g_i{}^l, \tag{C.31}$$

[1] Note that, since $n_k{}^I$ is given by $e_a{}^I$, $n_i{}^J$, $i = 1, ..., k-1$, up to sign, it is automatically annihilated by $D_a$ if the latter are.

and

$$R^0_{abIJ}[\Gamma^0[e,\tilde{n}]] = R^0_{abIJ}[\Gamma^0[e,n]] + 2D^0[e,n]_{[a}K_{b]IJ} + [K_a, K_b]_{IJ} = ... = R^0_{abIJ}[\Gamma^0[e,n]]. \tag{C.32}$$

For even dimensions $D = 2n$, it follows from (C.29)

$$\epsilon^{K_1...K_kI_1J_1...I_nJ_n} \; \epsilon^{a_1b_1...a_nb_n} \; R^0_{a_1b_1I_1J_1}...R^0_{a_nb_nI_nJ_n} = E^{(D)}\epsilon^{i_1...i_k}n_{i_1}{}^{[K_1}...n_{i_k}{}^{K_k]}, \tag{C.33}$$

the right hand side of which is also manifestly invariant under SO$(p,q)$ rotations and where $E^{(D)}$ denotes the $D$ - dimensional Euler density

$$E^{(D)} := \frac{1}{\sqrt{q}}\epsilon^{a_1b_1...a_nb_n}\epsilon^{c_1d_1...c_nd_n}R_{a_1b_1c_1d_1}...R_{a_nb_nc_nd_n}. \tag{C.34}$$

Note that $R^0_{abIJ}$ is not the only curvature tensor constructed from $e_a{}^I$ only. Of course, the connection $\Gamma^1_{aIJ}$ we considered earlier, obtained by choosing $\Gamma^{ij}_a = 0$, is constructed solely from $e_a{}^I$ and so is the corresponding curvature tensor, but it fails to satisfy (C.29). More precisely, we find

$$R^1_{abIJ} = R^0_{abIJ} + 2(\eta - \bar{\eta})_{K[I}(\eta - \bar{\eta})_{J]L}q^{cd}(D_{[a|}e_c{}^K)(D_{|b]}e_d{}^L). \tag{C.35}$$

# D

# The Lie algebras so$(1, D)$ and so$(D+1)$

This appendix is mostly taken from [2]. We generalise a so$(1,3)$ structure constant identity given in [103] to so$(1, D)$ or so$(D+1)$. In our notation,

$$(T_{AB})^I{}_J = \eta^I{}_{[A}\eta_{B]J} \tag{D.1}$$

denotes the generators of so$(1, D)$ or so$(D+1)$ in the fundamental representation. The *antisymmetric* index pair $AB$ labels the $D(D+1)/2$ generators, $I$ and $J$ are matrix indices, also antisymmetric. In the following, a generator $T_{AB}$ will always have a label, but the matrix indices will be mostly suppressed. Insertion of the definitions shows that the generators satisfy the usual Lorentz algebra

$$[T_{AB}, T_{CD}]^I{}_J = 2\eta_{A][C}\left(T_{D][B}\right)^I{}_J =: f_{AB,CD,}{}^{EF}\left(T_{EF}\right)^I{}_J \tag{D.2}$$

with

$$f_{AB,CD,EF} = -2\eta_{B][C}\eta_{D][E}\eta_{F][A} = -2\mathrm{Tr}\left(T_{AB}T_{CD}T_{EF}\right). \tag{D.3}$$

We further define the Cartan-Killing metric

$$q_{IJ,KL} = \eta_{I[K}\eta_{L]J} \;\Leftrightarrow\; -\mathrm{Tr}\left(T_{AB}T_{CD}\right) = \left(T_{AB}\right)^{IJ} q_{IJ,KL}\left(T_{CD}\right)^{KL} \tag{D.4}$$

and the object

$$(q^{*\overline{M}})_{IJ,KL} = \frac{1}{2}\epsilon_{IJKL}{}^{\overline{M}} \tag{D.5}$$

defining the dual

$$T^{*\overline{M}}_{AB} = (q^{*\overline{M}})_{AB,}{}^{CD}T_{CD} \tag{D.6}$$

generators. We note that self-duality is a concept reserved for $3+1$ dimensions.
These definitions lead us to the main result of this appendix:

$$f_{AB,CD,IJ} f_{EF,GH,}{}^{IJ} = \frac{1}{2} q_{AB,EF} q_{GH,CD} + \frac{\zeta \eta_{\overline{MN}}}{2(D-3)!} (q^{*\overline{M}})_{AB,EF} (q^{*\overline{N}})_{GH,CD} - ({}_{EF} \leftrightarrow {}_{GH}), \quad \text{(D.7)}$$

where $\eta_{\overline{MN}} := \eta_{M_1[N_1|} \eta_{M_2|N_2|} ... \eta_{M_{D-3}|N_{D-3}]}$ is defined with total weight one and $\zeta = -1$ $(+1)$ for so$(1, D)$ (so$(D+1)$) as before. It can be proven by carefully inserting the definitions and writing out explicitly each term.
Using these definitions, we can rewrite

$$[\Lambda, \Omega]^I{}_J = \Lambda^{AB} \Omega^{CD} f_{AB,CD,}{}^I{}_J, \quad \text{(D.8)}$$

$$\text{Tr}\,(\Lambda\Omega\Xi) = -\frac{1}{2} \Lambda^{AB} \Omega^{CD} \Xi^{EF} f_{AB,CD,EF}, \quad \text{(D.9)}$$

and use (D.7) to simplify certain calculations in the main text.

# E

# Gamma matrices

This appendix is taken from [4]. The properties of the gamma matrices can be found in most textbooks on quantum field theory, see, for instance, [324]. Their basic property is the Clifford algebra

$$\{\gamma^I, \gamma^J\} = 2\eta^{IJ}, \tag{E.1}$$

where $\eta^{IJ}$ is the flat Minkowski metric of a spacetime with signature $(p, q)$. From this relation alone, one deduces,

$$[\Sigma^{IJ}, \gamma^K] = -i\gamma^I \eta^{JK} + i\gamma^J \eta^{IK} \tag{E.2}$$

and

$$i[\Sigma^{IJ}, \Sigma^{KL}] = \eta^{LJ}\Sigma^{KI} - \eta^{LI}\Sigma^{KJ} + \eta^{JK}\Sigma^{IL} - \eta^{IK}\Sigma^{JL}, \tag{E.3}$$

where $\Sigma^{IJ} := -\frac{i}{4}[\gamma^I, \gamma^J]$. $\Sigma^{IJ}$ thus constitutes a representation of the Lie algebra $\mathrm{so}(p, q)$ on spinor space.
Furthermore, the expression $\{\gamma^K, \Sigma^{IJ}\} = -i\gamma^{[K}\gamma^I\gamma^{J]}$ is completely antisymmetric in $I, J, K$.
It is noteworthy that $\Sigma^{IJ}$ is a Hermitian matrix for Euclidean signature. In general, $\left(\Sigma^{IJ}\right)^\dagger = \eta^{II}\eta^{JJ}\Sigma^{IJ}$, which becomes important when dealing with Lorentzian signature, i.e. the boost part of the Gauß constraint is purely rotational as $\Sigma^{0i} + (\Sigma^{0i})^\dagger = 0$.
Explicit representations of the gamma matrices exist for all dimensions $D + 1 \geq 2$, see, for instance, [325], or [326]. A generalisation of left- and right-handed spinors exists for $D + 1$ even and is spelled out e.g. in [6].

# F

# Higher dimensional Newman Penrose formalism

In this appendix, which is taken from [10], we will very briefly introduce the higher dimensional Newman Penrose formalism as far as it is needed for the purpose of this thesis. Firstly, the Riemann tensor can be decomposed as follows

$$\begin{aligned} R^{(D+1)}_{\mu\nu\rho\sigma} &= C^{(D+1)}_{\mu\nu\rho\sigma} + \frac{2}{D-1}\left(R^{(D+1)}_{[\mu|\rho}g_{|\nu]\sigma} - R^{(D+1)}_{[\mu|\sigma}g_{|\nu]\rho}\right) - \frac{2}{D(D-1)}g_{[\mu|\rho}g_{|\nu]\sigma}R^{(D+1)} \\ &= C^{(D+1)}_{\mu\nu\rho\sigma} + \frac{2}{D-1}\left(J^{(D+1)}_{[\mu|\rho}g_{|\nu]\sigma} - J^{(D+1)}_{[\mu|\sigma}g_{|\nu]\rho}\right) + \frac{2}{D(D+1)}g_{[\mu|\rho}g_{|\nu]\sigma}R^{(D+1)}, \end{aligned} \tag{F.1}$$

where $C^{(D+1)}_{\mu\nu\rho\sigma}$ denotes the $(D+1)$ Weyl tensor and $J^{(D+1)}_{\mu\nu} := R^{(D+1)}_{\mu\nu} - \frac{1}{D+1}g_{\mu\nu}R^{(D+1)}$ the tracefree Ricci tensor. In a given null frame $\{l, k, \{m_I\}\}$, $l^2 = k^2 = l \cdot m_I = k \cdot m_I = 0$, $l \cdot k = -1$, $m_I \cdot m_J = \bar{\eta}_{IJ}$, we will use the following notation (cf. [327]) for the components of the Weyl tensor

$$\begin{aligned} \Psi_{0101} &:= C^{(D+1)}_{\mu\nu\rho\sigma} l^\mu k^\nu l^\rho k^\sigma, & \Psi_{010I} &:= C^{(D+1)}_{\mu\nu\rho\sigma} l^\mu k^\nu l^\rho m^\sigma_I, \\ \Psi_{011I} &:= C^{(D+1)}_{\mu\nu\rho\sigma} l^\mu k^\nu k^\rho m^\sigma_I, & \Psi_{01IJ} &:= C^{(D+1)}_{\mu\nu\rho\sigma} l^\mu k^\nu m^\rho_I m^\sigma_J, \\ \Psi_{0I0J} &:= C^{(D+1)}_{\mu\nu\rho\sigma} l^\mu m^\nu_I l^\rho m^\sigma_J, & \Psi_{0I1J} &:= C^{(D+1)}_{\mu\nu\rho\sigma} l^\mu m^\nu_I k^\rho m^\sigma_J, \\ \Psi_{0IJK} &:= C^{(D+1)}_{\mu\nu\rho\sigma} l^\mu m^\nu_I m^\rho_J m^\sigma_K, & \Psi_{1I1J} &:= C^{(D+1)}_{\mu\nu\rho\sigma} k^\mu m^\nu_I k^\rho m^\sigma_J, \\ \Psi_{1IJK} &:= C^{(D+1)}_{\mu\nu\rho\sigma} k^\mu m^\nu_I m^\rho_J m^\sigma_K, & \Psi_{IJKL} &:= C^{(D+1)}_{\mu\nu\rho\sigma} m^\mu_I m^\nu_J m^\rho_K m^\sigma_L. \end{aligned} \tag{F.2}$$

We will use analogous notation for the $(D+1)$ Riemann tensor if convenient. From curvature tensor symmetries and tracelessness, the relations

$$\Psi_{0I0}{}^I = \Psi_{1I1}{}^I = 0,\ \Psi_{0[IJK]} = \Psi_{1[IJK]} = \Psi_{I[JKL]} = 0,\ \Psi_{0101} = -\Psi_{0I1}{}^I,$$

$$\Psi_{010J} = -\Psi_{0IJ}{}^{I},\ \Psi_{011J} = \Psi_{1IJ}{}^{I},\ \Psi_{0I1J} = \frac{1}{2}\left(\Psi_{01IJ} + \Psi_{IKJ}{}^{K}\right) \tag{F.3}$$

can be derived [327]. For the components of the tracefree Ricci tensor $J_{\mu\nu}^{(D+1)}$, we introduce the notation

$$\begin{aligned} \Phi_{00} &= J_{\mu\nu}^{(D+1)} l^{\mu} l^{\nu}, & \Phi_{01} &= J_{\mu\nu}^{(D+1)} l^{\mu} k^{\nu}, & \Phi_{0I} &= J_{\mu\nu}^{(D+1)} l^{\mu} m_I^{\nu}, \\ \Phi_{11} &= J_{\mu\nu}^{(D+1)} k^{\mu} k^{\nu}, & \Phi_{1I} &= J_{\mu\nu}^{(D+1)} k^{\mu} m_I^{\nu}, & \Phi_{IJ} &= J_{\mu\nu}^{(D+1)} m_I^{\mu} m_J^{\nu}, \end{aligned} \tag{F.4}$$

and, because of tracelessness, it holds that

$$2\Phi_{01} = \Phi_I{}^{I}. \tag{F.5}$$

# G

# Details on calculations for part V

This appendix is taken from [10] and provides calculational details for several derivations of part V.

## G.1 Symplectic structure via the Palatini action

In this appendix, we provide calculational details for showing (17.41),

$$\int_{\Delta} \delta_{[1} \underset{\leftarrow}{\Sigma}^{IJ} \delta_{2]} \underset{\leftarrow}{A}_{IJ} = 2 \int_{\Delta} \left\{ d \left[ \delta_{[1} \tilde{s}^{I} \delta_{2]} n_I \right] + \frac{1}{(D-1)!} \delta_{[1} \epsilon^{D-1} \wedge \delta_{2]} \omega^{l} \right\}. \tag{G.1}$$

We will contract any of the three lines of (17.38) separately with (17.24) and multiply them by $\frac{-1}{(D-1)!}$. For the first line, we find

$$\begin{aligned}
&(D-1)\epsilon_{IJK_1...K_{D-1}} \left[ m_L \wedge m^{K_2} \wedge ... \wedge m^{K_{D-1}} (i_{m^L} \delta_{[1} m^{K_1}) \right] \\
&\quad \wedge \left[ \delta_{2]} \Gamma^{0IJ} - 2(\delta_{2]}\omega) l^{[I} k^{J]} - 2\omega \delta_{2]} (l^{[I} k^{J]}) \right] \\
= &(D-1)\epsilon_{IJK_1...K_{D-1}} l^I k^J \left[ m_L \wedge m^{K_2} \wedge ... \wedge m^{K_{D-1}} (i_{m^L} \bar{\bar{\eta}}^{K_1 M} \delta_{[1} m_M) \right] \\
&\quad \wedge \left[ -2 k_{I'} d_{\Gamma^0} \delta_{2]} l^{I'} - 2\delta_{2]}\omega \right] \\
&+ (D-1)\epsilon_{IJK_1...K_{D-1}} \left[ m_L \wedge m^{K_2} \wedge ... \wedge m^{K_{D-1}} (-i_{m^L} l^{K_1} k^M \delta_{[1} m_M) \right] \\
&\quad \wedge \left[ 2k^J d_{\Gamma^0} \delta_{2]} l^I - 2k^J \omega (\delta_{2]} l^I) \right] \\
&+ (D-1)\epsilon_{IJK_1...K_{D-1}} \left[ m_L \wedge m^{K_2} \wedge ... \wedge m^{K_{D-1}} (-i_{m^L} k^{K_1} l^M \delta_{[1} m_M) \right] \\
&\quad \wedge \left[ 2l^J d_{\Gamma^0} \delta_{2]} k^I + 2l^J \omega (\delta_{2]} k^I) \right] \\
= &-2(D-1)\epsilon_{IJK_1...K_{D-1}} l^I k^J \delta_{[1} m^{K_1} \wedge m^{K_2} \wedge ... \wedge m^{K_{D-1}} \wedge \left[ k_{I'} d_{\Gamma^0} \delta_{2]} l^{I'} + \delta_{2]}\omega \right] \\
&-2(D-1)\epsilon_{IJK_1...K_{D-1}} l^I k^J m_M \wedge m^{K_2} \wedge ... \wedge m^{K_{D-1}} \\
&\quad \wedge \left[ \delta_{[1} k^M d_{\Gamma^0} \delta_{2]} l^{K_1} - \omega (\delta_{[1} k^M)(\delta_{2]} l^{K_1}) \right]
\end{aligned}$$

$$
\begin{aligned}
& -2(D-1)\epsilon_{IJK_1\ldots K_{D-1}} l^I k^J m_M \wedge m^{K_2} \wedge \ldots \wedge m^{K_{D-1}} \\
& \quad \wedge \left[\delta_{[2} l^M d_{\Gamma^0} \delta_{1]} k^{K_1} + \omega(\delta_{[2} l^M)(\delta_{1]} k^{K_1})\right] \\
= & -2\delta_{[1}\epsilon^{D-1} \wedge \left[d(k_{I'}\delta_{2]} l^{I'}) + \delta_{2]}\omega\right] \\
& -2(D-1)^2 \epsilon_{IJK_1[K_2\ldots K_{D-1}|} l^I k^J m_{|M]} \wedge m^{K_2} \wedge \ldots \wedge m^{K_{D-1}} \\
& \quad \wedge \left[\delta_{[1} k^M d_{\Gamma^0} \delta_{2]} l^{K_1} - \omega(\delta_{[1} k^M)(\delta_{2]} l^{K_1})\right] \\
& -2(D-1)\epsilon_{IJK_1\ldots K_{D-1}} l^I k^J m_M \wedge m^{K_2} \wedge \ldots \wedge m^{K_{D-1}} \\
& \quad \wedge \left[\delta_{[2} l^M d_{\Gamma^0} \delta_{1]} k^{K_1} + \omega(\delta_{[2} l^M)(\delta_{1]} k^{K_1})\right] \\
= & -2\delta_{[1}\epsilon^{D-1} \wedge \left[d(k_{I'}\delta_{2]} l^{I'}) + \delta_{2]}\omega\right] \\
& -2(D-1)\epsilon_{IJMK_2\ldots K_{D-1}} l^I k^J m_{K_1} \wedge m^{K_2} \wedge \ldots \wedge m^{K_{D-1}} \\
& \quad \wedge \left[\delta_{[1} k^M d_{\Gamma^0} \delta_{2]} l^{K_1} - \omega(\delta_{[1} k^M)(\delta_{2]} l^{K_1})\right] \\
& -2(D-1)\epsilon_{IJK_1\ldots K_{D-1}} l^I k^J m_M \wedge m^{K_2} \wedge \ldots \wedge m^{K_{D-1}} \\
& \quad \wedge \left[\delta_{[2} l^M d_{\Gamma^0} \delta_{1]} k^{K_1} + \omega(\delta_{[2} l^M)(\delta_{1]} k^{K_1})\right] \\
= & -2\delta_{[1}\epsilon^{D-1} \wedge \left[d(k_{I'}\delta_{2]} l^{I'}) + \delta_{2]}\omega\right] \\
& -2(D-1)\epsilon_{IJK_1K_2\ldots K_{D-1}} l^I k^J m_M \wedge m^{K_2} \wedge \ldots \wedge m^{K_{D-1}} \\
& \qquad \wedge \left[\delta_{[1} k^{K_1} d_{\Gamma^0} \delta_{2]} l^M + \delta_{[2} l^M d_{\Gamma^0} \delta_{1]} k^{K_1} - \omega(\delta_{[1} k^{K_1})(\delta_{2]} l^M) + \omega(\delta_{[2} l^M)(\delta_{1]} k^{K_1})\right] \\
= & -2\delta_{[1}\epsilon^{D-1} \wedge \left[d(k_{I'}\delta_{2]} l^{I'}) + \delta_{2]}\omega\right] \\
& -2(D-1)\epsilon_{IJK_1[K_2\ldots K_{D-1}|} l^I k^J m^M \wedge m^{K_2} \wedge \ldots \wedge m^{K_{D-1}} \wedge d_{\Gamma^0}(\delta_{[1} k^{K_1} \delta_{2]} l_{|M]}) \\
= & -2\delta_{[1}\epsilon^{D-1} \wedge \left[d(k_{I'}\delta_{2]} l^{I'}) + \delta_{2]}\omega\right] \\
& +4\epsilon_{JK_1K_2\ldots K_{D-1}M} l^{[I} k^{J]} m^M \wedge m^{K_2} \wedge \ldots \wedge m^{K_{D-1}} \wedge d_{\Gamma^0}(\delta_{[1} k^{K_1} \delta_{2]} l_I) \\
& -2\epsilon_{IJMK_2\ldots K_{D-1}} l^I k^J m^M \wedge m^{K_2} \wedge \ldots \wedge m^{K_{D-1}} \wedge d(\delta_{[1} k^{K_1} \delta_{2]} l_{K_1}) \\
= & -2\delta_{[1}\epsilon^{D-1} \wedge \left[d(k_{I'}\delta_{2]} l^{I'}) + \delta_{2]}\omega\right] \\
& -2\epsilon_{JK_1MK_2\ldots K_{D-1}} l^J k^{K_1} m^M \wedge m^{K_2} \wedge \ldots \wedge m^{K_{D-1}} \wedge d_{\Gamma^0}(l_N \delta_{[1} k^N k^I \delta_{2]} l_I) \\
& -2\epsilon^{D-1} \wedge d(\delta_{[1} k^{K_1} \delta_{2]} l_{K_1}) \\
= & -2\delta_{[1}\epsilon^{D-1} \wedge \left[d(k_{I'}\delta_{2]} l^{I'}) + \delta_{2]}\omega\right] + 2\epsilon^{D-1} \wedge d_{\Gamma^0}((k^N \delta_{[1} l_N)(k^I \delta_{2]} l_I)) \\
& -2\epsilon^{D-1} \wedge d(\delta_{[1} k^{K_1} \delta_{2]} l_{K_1}) \\
= & -2d\left[(\delta_{[1}\epsilon^{D-1} k_{I'})(\delta_{2]} l^{I'})\right] - 2\delta_{[1}\epsilon^{D-1} \wedge \delta_{2]}\omega. \qquad \text{(G.2)}
\end{aligned}
$$

Similar calculations of the same length show that for the second and third line of (17.38) contracted with (17.24), we obtain

$$\begin{aligned}
&-(D-1)(D-2)\epsilon_{IJK_1\dots K_{D-1}} l^{K_1}\; k\wedge m_L \wedge m^{K_3}\wedge \dots \wedge m^{K_{D-1}} (i_{m^L}\delta_{[1} m^{K_2})\\
&\quad \wedge \left[\delta_{2]}\Gamma^{0IJ} + \frac{2}{D-1}(\delta_{2]} l^{[I}) m^{|J]}\theta_k\right]\\
&= -2(D-2)\epsilon_{IJK_1\dots K_{D-1}} d\left[l^I k^J k\wedge m^M \wedge m^{K_3}\wedge\dots\wedge m^{K_{D-1}}\left((i_{m_M}\delta_{[1}m^{K_2})\delta_{2]}l^{K_1}\right)\right],
\end{aligned} \tag{G.3}$$

and

$$\begin{aligned}
&-(D-1)\epsilon_{IJK_1\dots K_{D-1}}\left(-l^{K_1}(i_l\delta_{[1}k) + (\delta_{[1}l^{K_1})\right) k\wedge m^{K_2}\wedge\dots\wedge m^{K_{D-1}}\\
&\quad \wedge\left[\delta\Gamma^{0IJ} + \frac{2}{D-1}\left((\delta l^{[I})m^{|J]}\theta_k + l^{[I}(\delta m^{|J]})\theta_k\right)\right]\\
&= -2(D-1)\epsilon_{IJK_1\dots K_{D-1}} d\left[l^I k^J k\wedge m^{K_2}\wedge\dots\wedge m^{K_{D-1}}\left(i_l\delta_{[1}k + k^M\delta_{[1}l_M\right)\delta_{2]}l^{K_1}\right],
\end{aligned} \tag{G.4}$$

respectively. Summing up the three lines, we arrive at (17.41) rescaled by the factor $\frac{-1}{(D-1)!}$ introduced before.

## G.2 Symplectic structure independent of the internal signature

In this appendix, we provide calculational details for showing that under the assumption[1] $\delta\frac{E^{(2n)}}{\sqrt{h}} = 0$ $(2n = D-1)$, we have

$$\begin{aligned}
2\frac{E^{(2n)}}{\sqrt{h}}(\delta_{[1}\tilde{s}^I)(\delta_{2]}n_I) = n\epsilon^{IJKLM_1N_1\dots M_{n-1}N_{n-1}}\epsilon^{\alpha\beta\alpha_1\beta_1\dots\alpha_{n-1}\beta_{n-1}}\left(\delta_{[1}\Gamma^0_{\alpha IJ}\right)\left(\delta_{2]}\Gamma^0_{\beta KL}\right)\times\\
R^0_{\alpha_1\beta_1M_1N_1}\dots R^0_{\alpha_{n-1}\beta_{n-1}M_{n-1}N_{n-1}},
\end{aligned} \tag{G.5}$$

where $\Gamma^0_{\alpha IJ}$ is the generalised hybrid connection and $R^0_{\alpha\beta IJ}$ the corresponding curvature tensor which are given in appendix C.3.

[1]Note that this requirement for an UDNRIH is equivalent to restricting to histories with a fixed value of the horizon area, $\delta A_S = 0$, which can be seen as follows: Since $E^{(2n)} = f(v)\sqrt{h}$, by integrating both sides over $S$ we obtain $f(v) = f = \frac{\langle E^{(2n)}\rangle}{A_S}$ actually is independent of $v$ since both, $A_S$ and $\langle E^{(2n)}\rangle$ are. Therefore, we have $\delta\frac{E^{(2n)}}{\sqrt{h}} = \delta\frac{\langle E^{(2n)}\rangle}{A_S} = -\frac{\langle E^{(2n)}\rangle}{A_S^2}\delta A_S$, where we used that the topology of S is fixed.

Starting with (G.5), we first calculate

$$\begin{aligned}
\delta\left(\frac{E^{(2n)}}{\sqrt{h}}\right) &= \delta\left(\frac{1}{h}\epsilon^{\alpha_1\beta_1\ldots\alpha_n\beta_n}\epsilon^{\gamma_1\delta_1\ldots\gamma_n\delta_n}R_{\alpha_1\beta_1\gamma_1\delta_1}\ldots R_{\alpha_n\beta_n\gamma_n\delta_n}\right)\\
&= -(\delta\log h)\frac{E^{(2n)}}{\sqrt{h}}\\
&\quad+\frac{n}{h}\epsilon^{\alpha_1\beta_1\ldots\alpha_n\beta_n}\epsilon^{\gamma_1\delta_1\ldots\gamma_n\delta_n}\left(-2h_{\delta_1\epsilon_1}D_{\alpha_1}\delta\Gamma^{\epsilon_1}_{\beta_1\gamma_1}+R_{\alpha_1\beta_1\gamma_1}{}^{\epsilon_1}\delta h_{\delta_1\epsilon_1}\right)\times\\
&\quad R_{\alpha_2\beta_2\gamma_2\delta_2}\ldots R_{\alpha_n\beta_n\gamma_n\delta_n}\\
&= -(\delta\log h)\frac{E^{(2n)}}{\sqrt{h}}\\
&\quad-\frac{2n}{h}\epsilon^{\alpha_1\beta_1\ldots\alpha_n\beta_n}\epsilon^{\gamma_1\delta_1\ldots\gamma_n\delta_n}\left(D_{\alpha_1}D_{\gamma_1}\delta h_{\beta_1\delta_1}\right)R_{\alpha_2\beta_2\gamma_2\delta_2}\ldots R_{\alpha_n\beta_n\gamma_n\delta_n}\\
&\quad+\frac{n}{h}\epsilon^{\alpha_1\beta_1\ldots\alpha_n\beta_n}\epsilon^{\gamma_1\delta_1\ldots\gamma_n\delta_n}R_{\alpha_1\beta_1\gamma_1}{}^{\epsilon_1}\left(\delta h_{\delta_1\epsilon_1}\right)R_{\alpha_2\beta_2\gamma_2\delta_2}\ldots R_{\alpha_n\beta_n\gamma_n\delta_n}\\
&= -(\delta\log h)\frac{E^{(2n)}}{2\sqrt{h}}\\
&\quad-\frac{2n}{h}\epsilon^{\alpha_1\beta_1\ldots\alpha_n\beta_n}\epsilon^{\gamma_1\delta_1\ldots\gamma_n\delta_n}\left(D_{\alpha_1}D_{\gamma_1}\delta h_{\beta_1\delta_1}\right)R_{\alpha_2\beta_2\gamma_2\delta_2}\ldots R_{\alpha_n\beta_n\gamma_n\delta_n}.
\end{aligned}\tag{G.6}$$

In the second line, we just explicitly wrote down all variations appearing using (A.5). In the third, we used (A.4) and in the last step, we used

$$\frac{n}{h}\epsilon^{\alpha_1\beta_1\ldots\alpha_n\beta_n}\epsilon^{\gamma_1\delta_1\ldots\gamma_n\delta_n}R_{\alpha_1\beta_1\gamma_1}{}^{\epsilon_1}\left(\delta h_{\delta_1\epsilon_1}\right)R_{\alpha_2\beta_2\gamma_2\delta_2}\ldots R_{\alpha_n\beta_n\gamma_n\delta_n}=\frac{E^{(2n)}}{2\sqrt{h}}\left(\delta\log h\right).\tag{G.7}$$

This last identity can be verified as follows:

$$\begin{aligned}
&\frac{n}{h}\epsilon^{\alpha_1\beta_1..\alpha_n\beta_n}\epsilon^{\gamma_1\delta_1..\gamma_n\delta_n}R_{\alpha_1\beta_1\gamma_1}{}^{\epsilon_1}\left(\delta h_{\delta_1\epsilon_1}\right)R_{\alpha_2\beta_2\gamma_2\delta_2}..R_{\alpha_n\beta_n\gamma_n\delta_n}\\
&=-\frac{n}{h}\epsilon^{\alpha_1\beta_1..\alpha_n\beta_n}\epsilon^{[\gamma_1|\delta_1..\gamma_n\delta_n}\left(\delta h^{\epsilon_1|\zeta_1]}\right)h_{\delta_1\epsilon_1}R_{\alpha_1\beta_1\gamma_1\zeta_1}R_{\alpha_2\beta_2\gamma_2\delta_2}..R_{\alpha_n\beta_n\gamma_n\delta_n}\\
&=\frac{n}{2h}\epsilon^{\alpha_1\beta_1..\alpha_n\beta_n}\left[\left(\delta h^{\epsilon_1\delta_1}\right)\epsilon^{\gamma_2\delta_2..\delta_n\zeta_1\gamma_1}+2(n-1)\left(\delta h^{\epsilon_1\gamma_2}\right)\epsilon^{\delta_2..\delta_n\zeta_1\gamma_1\delta_1}\right]\\
&\qquad\qquad\times h_{\delta_1\epsilon_1}R_{\alpha_1\beta_1\gamma_1\zeta_1}R_{\alpha_2\beta_2\gamma_2\delta_2}..R_{\alpha_n\beta_n\gamma_n\delta_n}\\
&=\frac{nE^{(2n)}}{2\sqrt{h}}\left(\delta\log h\right)-\frac{n(n-1)}{h}\epsilon^{\alpha_1\beta_1..\alpha_n\beta_n}\epsilon^{\gamma_1\delta_1..\gamma_n\delta_n}R_{\alpha_1\beta_1\gamma_1}{}^{\epsilon_1}(\delta h_{\delta_1\epsilon_1})R_{\alpha_2\beta_2\gamma_2\delta_2}..R_{\alpha_n\beta_n\gamma_n\delta_n},
\end{aligned}\tag{G.8}$$

where in the first step, we used $h^{\epsilon\zeta}\delta h_{\delta\epsilon}=-h_{\delta\epsilon}\delta h^{\epsilon\zeta}$, then we added zero by adding all terms necessary that the expression in the second line becomes antisymmetric in $\gamma_1,\delta_1,...,\gamma_n,\delta_n,\zeta_1$ and immediately subtracting them again. Since these are $D$ indices in dimension $D-1$, the antisymmetrisation vanishes and we are left with the subtracted terms. The first of these gives, using $h_{\delta\epsilon}\delta h^{\delta\epsilon} = -\delta \log h$, the first term in the fourth line, while the remaining ones, after renaming indices, reproduce up to numerical factors the expression we started with. Comparing the first and the last line of (G.8), one easily infers (G.7).

Next, we will calculate $\delta\Gamma^0_{\alpha IJ}$:

$$\begin{aligned}
\delta\Gamma^0_{\alpha IJ} &= \left(\delta\Gamma^0_{\alpha KL}\right)\eta^K{}_I\eta^L{}_J \\
&= \left(\delta\Gamma^0_{\alpha KL}\right)\left(\bar{\bar{\eta}}^K{}_I + \zeta n^K n_I + s^K s_I\right)\left(m^{\beta L}m_{\beta J} + \zeta n^L n_J + s^L s_J\right) \\
&= \bar{\bar{\eta}}^K{}_{[I}\Big[\left(\left(\delta D^0_\alpha m^\beta{}_K\right) - \left(D^0_\alpha \delta m^\beta{}_K\right) - \left(\delta\Gamma^\beta_{\alpha\gamma}\right)m^\gamma_K\right)m_{\beta|J]} \\
&\qquad + 2\zeta\left(\left(\delta D^0_\alpha n_K\right) - \left(D^0_\alpha\delta n_K\right)\right)n_{|J]} + 2\left(\left(\delta D^0_\alpha s_K\right) - \left(D^0_\alpha \delta s_K\right)\right)s_{|J]}\Big] \\
&\quad + \zeta n_{[I}s_{J]}\left[n^K\left(\left(\delta D^0_\alpha s_K\right) - \left(D^0_\alpha\delta s_K\right)\right) - s^K\left(\left(\delta D^0_\alpha n_K\right) - \left(D^0_\alpha \delta n_K\right)\right)\right] \\
&= \bar{\bar{\eta}}^K{}_{[I}m_{\beta|J]}\left[-\left(D^0_\alpha\delta m^\beta{}_K\right) - \left(\delta\Gamma^\beta_{\alpha\gamma}\right)m^\gamma_K\right] - 2\zeta\bar{\bar{\eta}}^K{}_{[I}n_{J]}\left(D^0_\alpha \delta n_K\right) \\
&\quad - 2\bar{\bar{\eta}}^K{}_{[I}s_{J]}\left(D^0_\alpha\delta s_K\right) - 2\zeta n_{[I}s_{J]}\left(D^0_\alpha n^K \delta s_K\right) \qquad \text{(G.9)}
\end{aligned}$$

where in the second step we used $\eta_{IJ} = \bar{\bar{\eta}}_{IJ} + \zeta n_I n_J + s_I s_J$ and $\bar{\bar{\eta}}_{IJ} = m^\beta{}_I m_{\beta J}$, in the third that $(\delta\Gamma^0_{\alpha IJ})m^{\beta J} = \left(\delta D^0_\alpha m^\beta{}_I\right) - \left(D^0_\alpha\delta m^\beta{}_I\right) - \left(\delta\Gamma^\beta_{\alpha\gamma}\right)m^\gamma_I$ and corresponding equations for $n, s$, and finally in the fourth step we used that $\Gamma^0_{\alpha IJ}$ annihilates the hybrid vielbein and $n, s$. This way of expressing $\delta\Gamma^0_{\alpha IJ}$ is convenient for several reasons. First of all, we explicitly separated the *(bar bar)*, *(bar n)*, *(bar s)* and *(n s)* terms. Since the two variations of $\Gamma^0_{\alpha IJ}$ in (G.5) are contracted with an $\epsilon$, which is *bar* projected on all other indices (remember $R^0_{\alpha\beta IJ} = \bar{R}^0_{\alpha\beta IJ}$, cf. C), the only contributions will come from *(bar bar)* $\cdot$ *(n s)* and *(bar n)* $\cdot$ *(bar s)* terms. Secondly, many of the terms are such that covariant derivates $D^0_\alpha$ appear explicitly. This simplifies further manipulations like partial integrations, since almost all appearing objects are annihilated by $D^0_\alpha$. Furthermore, since $S$ already is a boundary, no boundary terms appear when partially integrating. Using (G.9), we thus find

$$\begin{aligned}
& n\epsilon^{IJKLM_2N_2...M_nN_n}\epsilon^{\alpha\beta\alpha_2\beta_2...\alpha_n\beta_n}\left(\delta_{[1}\Gamma^0_{\alpha IJ}\right)\left(\delta_{2]}\Gamma^0_{\beta KL}\right)R^0_{\alpha_2\beta_2M_2N_2}...R^0_{\alpha_n\beta_nM_nN_n} \\
&= n\epsilon^{IJKLM_2N_2...M_nN_n}\epsilon^{\alpha\beta\alpha_2\beta_2...\alpha_n\beta_n}R^0_{\alpha_2\beta_2M_2N_2}...R^0_{\alpha_n\beta_nM_nN_n} \\
&\quad\times\Big[8\zeta n_I\bar{\bar{\eta}}_{JJ'}\left(D^0_\alpha\delta_{[1}n^{J'}\right)s_K\bar{\bar{\eta}}_{LL'}\left(D^0_\beta\delta_{2]}s^{L'}\right) \\
&\qquad +4\zeta\bar{\bar{\eta}}^{I'}{}_I m_{\delta J}\left(\left(D^0_\alpha\delta_{[1}m^\delta{}_{I'}\right) + \left(\delta_{[1}\Gamma^\delta_{\alpha\gamma}\right)m^\gamma_{I'}\right)n_K s_L\left(D^0_\beta n^P\delta_{2]}s_P\right)\Big] \\
&= -\frac{4n}{\sqrt{h}}\epsilon^{\gamma\delta\gamma_2\delta_2...\gamma_n\delta_n}\epsilon^{\alpha\beta\alpha_2\beta_2...\alpha_n\beta_n}R_{\alpha_2\beta_2\gamma_2\delta_2}...R_{\alpha_n\beta_n\gamma_n\delta_n}
\end{aligned}$$

$$
\begin{aligned}
&\times \left[2m_{\gamma J}\left(D^0_\alpha \delta_{[1} n^J\right) m_{\delta L}\left(D^0_\beta \delta_{2]} s^L\right) - \left(D^0_\gamma \delta_{[1} h_{\alpha\delta}\right)\left(D_\beta n^P \delta_{2]} s_P\right)\right] \\
=& -\frac{4n}{h}\epsilon^{\gamma\delta\gamma_2\delta_2\ldots\gamma_n\delta_n}\epsilon^{\alpha\beta\alpha_2\beta_2\ldots\alpha_n\beta_n} R_{\alpha_2\beta_2\gamma_2\delta_2}\ldots R_{\alpha_n\beta_n\gamma_n\delta_n} \\
&\times \left[2m_{\gamma J}\left(D^0_\alpha \delta_{[1} n^J\right) m_{\delta L}\left(D^0_\beta \delta_{2]} \tilde{s}^L\right) - \left(D^0_\alpha D^0_\gamma \delta_{[1} h_{\beta\delta}\right) n^P\left(\delta_{2]} \tilde{s}_P\right)\right] \\
=& -\frac{8n}{h}\epsilon^{\gamma\delta\gamma_2\delta_2\ldots\gamma_n\delta_n}\epsilon^{\alpha\beta\alpha_2\beta_2\ldots\alpha_n\beta_n} R_{\alpha_2\beta_2\gamma_2\delta_2}\ldots R_{\alpha_n\beta_n\gamma_n\delta_n} m_{\gamma J}\left(D^0_\alpha \delta_{[1} n^J\right) m_{\delta L}\left(D^0_\beta \delta_{2]} \tilde{s}^L\right) \\
&- \left[2\left(\delta_{[1}\frac{E^{(2n)}}{\sqrt{h}}\right) + \frac{E^{(2n)}}{\sqrt{h}}\left(\delta_{[1}\log h\right)\right] n^P\left(\delta_{2]}\tilde{s}_P\right). \qquad \text{(G.10)}
\end{aligned}
$$

In the third line, note that the term containing $D^0_\alpha \delta m^\beta_I$ vanishes, since when partially integrating, we obtain a term of the form $\left(D^0_{[\alpha} D^0_{\beta]} n^P \delta s_P\right)$, which vanishes due to torsion freeness. In the second step, we used

$$
\epsilon^{IJM_1N_1\ldots M_nN_n} n_I s_J m^{\gamma_1}{}_{M_1} m^{\delta_1}{}_{N_1}\ldots m^{\gamma_n}{}_{M_n} m^{\delta_n}{}_{N_n} = \frac{\zeta}{\sqrt{h}}\epsilon^{\gamma_1\delta_1\ldots\gamma_n\delta_n} \qquad \text{(G.11)}
$$

and again (A.4). In the third step, we densitised $s^I$ (note that $s^I$ is always contracted such that variations on the density $\sqrt{h}$ drop out), partially integrated in the last summand and interchanged the indices $\alpha$ and $\beta$. In the fourth step, we replaced the second summand in square brackets using (G.6).

Now we will have a closer look at the left hand side of (G.5).

$$
\begin{aligned}
2\frac{E^{(2n)}}{\sqrt{h}}(\delta_{[1}\tilde{s}^I)(\delta_{2]} n_I) &= 2E^{(2n)}(\delta_{[1} s^I)(\delta_{2]} n_I) + \frac{E^{(2n)}}{\sqrt{h}}\tilde{s}_I(\delta_{[1}\log h)(\delta_{2]} n^I) \\
&= 2E^{(2n)}(\delta_{[1} s^I)(\delta_{2]} n_I) + \frac{E^{(2n)}}{\sqrt{h}} n^I(\delta_{[1}\tilde{s}_I)(\delta_{2]}\log h). \qquad \text{(G.12)}
\end{aligned}
$$

Here, in the first step we varied $s^I$ and the density $\sqrt{h}$ independently. In the second step, we interchanged the variations and used $s_I \delta n^I = -n^I \delta s_I$ in the second summand. For the first summand, we find

$$
\begin{aligned}
&2E^{(2n)}(\delta_{[1} s^I)(\delta_{2]} n_I) \\
=& -\frac{2}{\sqrt{h}}\epsilon^{\alpha_1\beta_1\ldots\alpha_n\beta_n}\epsilon^{\gamma_1\delta_1\ldots\gamma_n\delta_n} R_{\alpha_1\beta_1\gamma_1\delta_1}\ldots R_{\alpha_n\beta_n\gamma_n\delta_n}(\delta_{[1} n^I)(\delta_{2]} s_I) \\
=& -2\zeta\epsilon^{\alpha_1\beta_1\ldots\alpha_n\beta_n}\epsilon^{IJK_1L_1\ldots K_nL_n} R^0_{\alpha_1\beta_1K_1L_1}\ldots R^0_{\alpha_n\beta_nK_nL_n} n_I s_J(\delta_{[1} n^M)(\delta_{2]} s_M) \\
=& -4\zeta\epsilon^{\alpha_1\beta_1\ldots\alpha_n\beta_n}\epsilon^{IJK_1L_1\ldots K_nL_n} R^0_{\alpha_1\beta_1K_1L_1}\ldots R^0_{\alpha_n\beta_nK_nL_n} n_I s_{[J}(\delta_{[1} n^M)(\delta_{2]} s_{|M]}) \\
=& -4\zeta\epsilon^{\alpha_1\beta_1\ldots\alpha_n\beta_n}(\delta_{[1} n^{[M})\epsilon^{I|J]K_1L_1\ldots K_nL_n} R^0_{\alpha_1\beta_1K_1L_1}\ldots R^0_{\alpha_n\beta_nK_nL_n} n_I s_J(\delta_{2]} s_M) \\
=& -2\zeta\epsilon^{\alpha_1\beta_1\ldots\alpha_n\beta_n}\left((\delta_{[1} n^I)\epsilon^{K_1L_1\ldots K_nL_nMJ} + 2n(\delta_{[1} n^{K_1})\epsilon^{L_1\ldots K_nL_nMJI}\right) \\
&\times R^0_{\alpha_1\beta_1K_1L_1}\ldots R^0_{\alpha_n\beta_nK_nL_n} n_I s_J(\delta_{2]} s_M)
\end{aligned}
$$

$$
\begin{aligned}
&= -4\zeta n \epsilon^{\alpha_1\beta_1...\alpha_n\beta_n}\epsilon^{L_1...K_nL_nMJI} R^0_{\alpha_1\beta_1K_1L_1}(\delta_{[1}n^{K_1})R^0_{\alpha_2\beta_2K_2L_2}...R^0_{\alpha_n\beta_nK_nL_n} n_I s_J(\delta_{2]}s_M) \\
&= 8\zeta n \epsilon^{\alpha_1\beta_1...\alpha_n\beta_n}\epsilon^{L_1...K_nL_nMJI}(D^0_{\alpha_1}D^0_{\beta_1}\delta_{[1}n_{L_1})R^0_{\alpha_2\beta_2K_2L_2}...R^0_{\alpha_n\beta_nK_nL_n} n_I s_J(\delta_{2]}s_M) \\
&= -\frac{8n}{h}\epsilon^{\alpha_1\beta_1...\alpha_n\beta_n}\epsilon^{\gamma_1\delta_1...\gamma_n\delta_n} R^0_{\alpha_2\beta_2\gamma_2\delta_2}...R^0_{\alpha_n\beta_n\gamma_n\delta_n} m_{\delta_1}{}^J(D^0_{\beta_1}\delta_{[1}n_J)m_{\gamma_1}{}^L(D^0_{\alpha_1}\delta_{2]}\tilde{s}_L),
\end{aligned}
\tag{G.13}
$$

which shows that (G.12) coincides with (G.10) iff $\delta\left(\frac{E^{(2n)}}{\sqrt{h}}\right) = 0$. Here, in the first step, we used the defining equation for $E^{(2n)}$ and in the second step we used (G.11) and (C.29). In the third step, we antisymmetrise in the lower pair of indices $J$ and $M$. Note that the additional term vanishes since $s^J\delta s_J = 0$ and the epsilon tensor enforces $\delta s_J$ to be projected into that direction. The fifth line is exactly the same as the fourth, we just moved $\delta n^M$ to the front and antisymmetrised the upper indices $J$ and $M$ instead of the lower ones. Now we again antisymmetrise the $D+2$ upper indices $M, I, J, K_1, L_1, ..., K_nL_n$, which gives zero, and subtract the term we added for antisymmetrisation again. The first of these, the first summand in the round brackets in line 6, gives zero due to $n^I\delta n_I = 0$. The others all give the same term of the form $R^0_{\alpha\beta KL}\delta n^L = 2D^0_{[\alpha}D^0_{\beta]}\delta n_K$, which we used in the second to last line. One more integration by parts in the last line, again using (G.11) and densitising $s^I$ gives the final result.

## G.3 Symplectic structure for the SO(4) based Beetle-Engle connection

For $D = 3$, we will show that one can bypass the restriction to spherically symmetric isolated horizons in complete analogy to the treatment of Beetle and Engle [281],

$$2\langle E^{(2)}\rangle(\delta_{[1}\tilde{s}^I)(\delta_{2]}n_I) = \epsilon^{IJKL}\epsilon^{\alpha\beta}\left(\delta_{[1}A_{\alpha IJ}\right)\left(\delta_{2]}A_{\beta KL}\right), \tag{G.14}$$

where $\langle E^{(2)}\rangle := \int_S d^2x E^{(2)}$ coincides, up to constant factors, with the Euler characteristic of the intersection of the Isolated Horizon with the spatial slices, and $A_{\alpha IJ}$ was defined in (17.69). The assumption $\delta\frac{E^{(2)}}{\sqrt{h}} = 0$ is then replaced by $\delta\langle E^{(2)}\rangle = 0$, which however is already enforced by our choice of topology of the horizon.
To prove (G.14), we start by noting that

$$
\begin{aligned}
&\epsilon^{IJKL}\epsilon^{\alpha\beta}\left(\delta_{[1}A_{\alpha IJ}\right)\left(\delta_{2]}A_{\beta KL}\right) \\
&\quad= \epsilon^{IJKL}\epsilon^{\alpha\beta}\left[\left(\delta_{[1}\Gamma^0_{\alpha IJ}\right)\left(\delta_{2]}\Gamma^0_{\beta KL}\right) + 2\left(\delta_{[1}\Gamma^0_{\alpha IJ}\right)\left(\delta_{2]}K_{\beta KL}\right) + \left(\delta_{[1}K_{\alpha IJ}\right)\left(\delta_{2]}K_{\beta KL}\right)\right] \\
&\quad=: A + B + C,
\end{aligned}
\tag{G.15}
$$

where we introduced the abbreviations $A$, $B$, $C$ for the three summands. The first summand in square brackets is, up to factors, the restriction to $D = 3$ of what we just calculated above,

$$\begin{aligned} A &= \epsilon^{IJKL}\epsilon^{\alpha\beta}\left(\delta_{[1}\Gamma^0_{\alpha IJ}\right)\left(\delta_{2]}\Gamma^0_{\beta KL}\right) \\ &= \frac{2E^{(2)}}{\sqrt{h}}(\delta_{[1}\tilde{s}^I)(\delta_{2]}n_I) - 2\left(\delta_{[1}\frac{E^{(2)}}{\sqrt{h}}\right)n^P\left(\delta_{2]}\tilde{s}_P\right). \end{aligned} \tag{G.16}$$

Next, we need to calculate

$$\begin{aligned} \delta K_{\alpha IJ} &= \delta\left(2m_{\alpha[I}m_{\beta|J]}h^{\beta\gamma}(D_\gamma\psi)\right) \\ &= 2m_{\alpha[I}m^\beta{}_{J]}(D_\beta\delta\psi) + 4(\delta m_{[\alpha|K})m_{\beta][J}\bar{\bar{\eta}}^K{}_{I]}h^{\beta\gamma}(D_\gamma\psi) + 2m_{\alpha[I}m_{\beta|J]}(\delta h^{\beta\gamma})(D_\gamma\psi) \\ &\quad + 4\zeta(\delta m_{[\alpha|K})m_{\beta][J}n^K n_{I]}h^{\beta\gamma}(D_\gamma\psi) + 4(\delta m_{[\alpha|K})m_{\beta][J}s^K s_{I]}h^{\beta\gamma}(D_\gamma\psi), \end{aligned} \tag{G.17}$$

where we again split the *(bar bar)* terms (second line) from the *(bar n)*, *(bar s)* terms (third line). Since no *(n s)* terms appear, we find for $C$

$$\begin{aligned} C &= \epsilon^{IJKL}\epsilon^{\alpha\beta}\left(\delta_{[1}K_{\alpha IJ}\right)\left(\delta_{2]}K_{\beta KL}\right) \\ &= 32\zeta\epsilon^{IJKL}\epsilon^{\alpha\beta}(\delta_{[1}m_{[\alpha|M})m_{\gamma]J}n^M n_I h^{\gamma\epsilon}(D_\epsilon\psi)(\delta_{2]}m_{[\beta|N})m_{\delta]L}s^N s_K h^{\delta\zeta}(D_\zeta\psi) \\ &= -32\sqrt{h}\epsilon^{\alpha\beta}(\delta_{[1}m_{[\alpha|M})\epsilon_{\gamma][\delta}(\delta_{2]}m_{\beta]N})n^M h^{\gamma\epsilon}(D_\epsilon\psi)s^N h^{\delta\zeta}(D_\zeta\psi) \\ &= 0, \end{aligned} \tag{G.18}$$

where in the second step we used

$$\epsilon^{IJKL}n_I s_J m_{\alpha K}m_{\beta L} = \zeta\sqrt{h}\epsilon_{\alpha\beta} \tag{G.19}$$

and the last equality is easily obtained when explicitly writing out all antisymmetrisations. For $B$, we find using (G.9) and (G.17)

$$\begin{aligned} B &= 2\epsilon^{IJKL}\epsilon^{\alpha\beta}\left(\delta_{[1}\Gamma^0_{\alpha IJ}\right)\left(\delta_{2]}K_{\beta KL}\right) \\ &= 2\epsilon^{IJKL}\epsilon^{\alpha\beta}\Big\{\left[-2\zeta n_I s_J\left(D_\alpha n^M\delta_{[1}s_M\right)\right]\Big[2m_{\beta K}m^\gamma{}_L D_\gamma\delta_{2]}\psi + 2m_{\beta K}m_{\gamma L}(\delta_{2]}h^{\gamma\delta})D_\delta\psi \\ &\qquad + 4(\delta_{2]}m_{[\beta|N})m_{\gamma]L}\bar{\bar{\eta}}^N{}_K h^{\gamma\delta}D_\delta\psi\Big] \\ &\qquad + \left[-2\zeta\bar{\bar{\eta}}^M{}_{[I}n_{J]}\left(D_\alpha\delta_{[1}n_M\right)\right]\left[4(\delta_{2]}m_{[\beta|N})m_{\gamma]L}s^N s_K h^{\gamma\delta}D_\delta\psi\right] \\ &\qquad + \left[-2\bar{\bar{\eta}}^M{}_{[I}s_{J]}\left(D_\alpha\delta_{[1}s_M\right)\right]\left[4\zeta(\delta_{2]}m_{[\beta|N})m_{\gamma]L}n^N n_K h^{\gamma\delta}D_\delta\psi\right]\Big\} \end{aligned}$$

$$
\begin{aligned}
&= -8\sqrt{h}\epsilon^{\alpha\beta}\,\Big\{\left(D_\alpha n^M\delta_{[1}s_M\right)\Big[\epsilon_{\beta\gamma}h^{\gamma\delta}D_\delta\delta_{2]}\psi + 2(\delta_{2]}m_{[\beta|N})\epsilon_{\epsilon|\gamma]}m^{\epsilon N}h^{\gamma\delta}D_\delta\psi \\
&\qquad\qquad + \epsilon_{\beta\gamma}(\delta_{2]}h^{\gamma\delta})D_\delta\psi\Big] \\
&\qquad + 2m^{\epsilon M}\left(D_\alpha\delta_{[1}n_M\right)(\delta_{2]}m_{[\beta|N})\epsilon_{\epsilon|\gamma]}s^N h^{\gamma\delta}D_\delta\psi \\
&\qquad - 2m^{\epsilon M}\left(D_\alpha\delta_{[1}s_M\right)(\delta_{2]}m_{[\beta|N})\epsilon_{\epsilon|\gamma]}n^N h^{\gamma\delta}D_\delta\psi\Big\} \\
&= -8\sqrt{h}\,\Big\{\left(D_\alpha n^M\delta_{[1}s_M\right)\Big[-D^\alpha\delta_{2]}\psi + 2(\delta_{2]}m_{\beta N})m^{[\alpha|N}D^{\beta]}\psi - (\delta_{2]}m_{\beta N})m^{\alpha N}D^\beta\psi \\
&\qquad\qquad - (\delta_{2]}h^{\alpha\delta})D_\delta\psi\Big] \\
&\qquad + 2\left(D_\alpha\delta_{[1}n_M\right)(\delta_{2]}m_{\beta N})s^N m^{[\alpha|M}D^{\beta]}\psi - \left(D_\alpha\delta_{[1}n_M\right)(\delta_{2]}m_{\beta N})s^N m^{\alpha M}D^\beta\psi \\
&\qquad - 2\left(D_\alpha\delta_{[1}s_M\right)(\delta_{2]}m_{\beta N})n^N m^{[\alpha|M}D^{\beta]}\psi + \left(D_\alpha\delta_{[1}s_M\right)(\delta_{2]}m_{\beta N})n^N m^{\alpha M}D^\beta\psi\Big\} \\
&= -8\sqrt{h}\,\Big\{\left(D_\alpha n^M\delta_{[1}s_M\right)\Big[-D^\alpha\delta_{2]}\psi - (\delta_{2]}m_{\beta N})m^{\beta N}D^\alpha\psi - (\delta_{2]}h^{\alpha\delta})D_\delta\psi\Big] \\
&\qquad + \left(D_\alpha\delta_{[1}n_M\right)(\delta_{2]}s^N)\bar{\bar{\eta}}_N{}^M D^\alpha\psi - \left(D_\alpha\delta_{[1}s_M\right)(\delta_{2]}n^N)\bar{\bar{\eta}}_N{}^M D^\alpha\psi\} \\
&= -8\sqrt{h}\,\Big\{\left(n^M\delta_{[1}s_M\right)\Big[D_\alpha D^\alpha\delta_{2]}\psi + D_\alpha((\delta_{2]}\log\sqrt{h})D^\alpha\psi) + D_\alpha((\delta_{2]}h^{\alpha\delta})D_\delta\psi)\Big] \\
&\qquad + \left(D_\alpha\delta_{[1}n_M\delta_{2]}s^N\right)\bar{\bar{\eta}}_N{}^M D^\alpha\psi\} \\
&= -8\sqrt{h}\,\{\left(n^M\delta_{[1}s_M\right)\ \Big[\Delta\delta_{2]}\psi + (D_\alpha\delta_{2]}\log\sqrt{h})D^\alpha\psi + (\delta_{2]}\log\sqrt{h})\Delta\psi \\
&\qquad\qquad - (\delta_{2]}h_{\alpha\delta})D^\alpha D^\delta\psi - (D^\alpha\delta_{2]}h_{\alpha\delta})D^\delta\psi\Big] - (\delta_{[1}n_M)(\delta_{2]}s^M)\Delta\psi\Big\} \\
&= -8\sqrt{h}\Big\{\left(n^M\delta_{[1}s_M\right)(\delta_{2]}\Delta\psi) - \frac{1}{\sqrt{h}}(\delta_{[1}n_M)\Big[\sqrt{h}(\delta_{2]}s^M) + s^M(\delta_{2]}\sqrt{h})\Big]\Delta\psi\Big\} \\
&= -8\,\{\left(n^M\delta_{[1}\tilde{s}_M\right)(\delta_{2]}\Delta\psi) - (\delta_{[1}n_M)(\delta_{2]}\tilde{s}^M)\Delta\psi\}\,,
\end{aligned}
$$

and since we assumed that $\Delta\psi = \frac{1}{4}\left(\frac{E^{(2)}}{\sqrt{h}} - \langle E^{(2)}\rangle\right)$ and $\delta\langle E^{(2)}\rangle = 0$, we find

$$
= -2\left\{\left(n^M\delta_{[1}\tilde{s}_M\right)\left(\delta_{2]}\frac{E^{(2)}}{\sqrt{h}}\right) + (\delta_{[1}\tilde{s}_M)(\delta_{2]}n^M)\left(\frac{E^{(2)}}{\sqrt{h}} - \langle E^{(2)}\rangle\right)\right\}. \tag{G.20}
$$

Here, in the second line, we inserted the expressions for $\delta\Gamma^0{}_{\alpha IJ}$ and $\delta K_{\alpha IJ}$ (G.9, G.17). Note that since $\delta K_{\alpha IJ}$ does not contain *(n s)* terms, the *(bar bar)* terms of $\delta\Gamma^0{}_{\alpha IJ}$ drop out. In the third step, we used (G.19) and $\bar{\bar{\eta}}_{IJ} = m_{\alpha I}m^\beta{}_J$, and in the fourth step, epsilon identities were used and antisymmetrisations in $(\beta,\gamma)$ were written out explicitly. When furthermore writing out the antisymmetrisations in $(\alpha,\beta)$, we find that several terms cancel (step 5) and additionally used $(\delta m_{\alpha I})n^I = -(\delta n^I)m_{\alpha I}$, $(\delta m_{\alpha I})s^I = -(\delta s^I)m_{\alpha I}$ and $m_{\alpha I}m^\alpha{}_J = \bar{\bar{\eta}}_{IJ}$. In the sixth step, the upper line is partially integrated and we used $(\delta m_{\alpha I})m^{\alpha I} = \frac{1}{2}(\delta h_{\alpha\beta})h^{\alpha\beta} = \frac{1}{\sqrt{h}}\delta\sqrt{h}$, and the two summands of the lower line are combined into one term. The seventh step consists of writing out all individual appearing in the square brackets explicitly and partially integrating the last term. In step 8, we used (A.9) and the remaining steps are straightforward.

Combining (G.16), (G.20) and (G.18), we find immediately

$$\begin{aligned}
&\epsilon^{IJKL}\epsilon^{\alpha\beta}\left(\delta_{[1}A_{\alpha IJ}\right)\left(\delta_{2]}A_{\beta KL}\right)\\
&\quad= -\frac{2E^{(2)}}{\sqrt{h}}(\delta_{[1}n^I)(\delta_{2]}\tilde{s}_I) + 2\left(\delta_{[1}\frac{E^{(2)}}{\sqrt{h}}\right)n^P\left(\delta_{2]}\tilde{s}_P\right)\\
&\quad\quad - 2\left\{\left(n^M\delta_{[1}\tilde{s}_M\right)(\delta_{2]}\frac{E^{(2)}}{\sqrt{h}}) + (\delta_{[1}\tilde{s}_M)(\delta_{2]}n^M)\left(\frac{E^{(2)}}{\sqrt{h}} - \langle E^{(2)}\rangle\right)\right\}\\
&\quad= 2\langle E^{(2)}\rangle(\delta_{[1}\tilde{s}_M)(\delta_{2]}n^M).
\end{aligned} \tag{G.21}$$

# References


[1] Norbert Bodendorfer, Thomas Thiemann, and Andreas Thurn. **New Variables for Classical and Quantum Gravity in all Dimensions: I. Hamiltonian Analysis**. *Classical and Quantum Gravity*, **30**:045001, 2013. xxxvii, 21, 87, 181, 213, 288

[2] Norbert Bodendorfer, Thomas Thiemann, and Andreas Thurn. **New Variables for Classical and Quantum Gravity in all Dimensions: II. Lagrangian Analysis**. *Classical and Quantum Gravity*, **30**:045002, 2013. xxxvii, 20, 21, 22, 25, 46, 68, 87, 92, 99, 101, 146, 181, 213, 290, 323

[3] Norbert Bodendorfer, Thomas Thiemann, and Andreas Thurn. **New Variables for Classical and Quantum Gravity in all Dimensions: III. Quantum Theory**. *Classical and Quantum Gravity*, **30**:045003, 2013. xxxvii, 135, 173, 182, 194, 213, 222, 244, 288, 296

[4] Norbert Bodendorfer, Thomas Thiemann, and Andreas Thurn. **New Variables for Classical and Quantum Gravity in all Dimensions: IV. Matter Coupling**. *Classical and Quantum Gravity*, **30**:045004, 2013. xxxvii, 231, 325

[5] Norbert Bodendorfer, Thomas Thiemann, and Andreas Thurn. **On the Implementation of the Canonical Quantum Simplicity Constraint**. *Classical and Quantum Gravity*, **30**:045005, 2013. xxxvii, 135, 173, 182, 183, 184, 194, 196, 197, 198, 199, 200, 201, 206, 302, 303

[6] Norbert Bodendorfer, Thomas Thiemann, and Andreas Thurn. **Towards Loop Quantum Supergravity (LQSG): I. Rarita-Schwinger Sector**. *Classical and Quantum Gravity*, **30**:045006, 2013. xxxvii, 135, 173, 181, 197, 231, 242, 244, 245, 246, 248, 251, 325

[7] Norbert Bodendorfer, Thomas Thiemann, and Andreas Thurn. **Towards Loop Quantum Supergravity (LQSG): II. p-Form Sector**. *Classical and Quantum Gravity*, **30**:045007, 2013. xxxvii, 231, 232, 245, 253, 254, 256

[8] Norbert Bodendorfer, Thomas Thiemann, and Andreas Thurn. **Towards Loop Quantum Supergravity (LQSG)**. *Physics Letters B*, **711**:205–211, 2012. xxxvii

[9] Norbert Bodendorfer, Thomas Thiemann, and Andreas Thurn. **New Variables for Classical and Quantum (Super)-Gravity in all Dimensions**. *PoS(QGQGS 2011)*, **022**:1–8, 2011. xxxvii

[10] Norbert Bodendorfer, Thomas Thiemann, and Andreas Thurn. **New Variables for Classical and Quantum Gravity in all Dimensions V. Isolated Horizon Boundary Degrees of Freedom**. *to appear soon*, 2013. xxxvii, 260, 317, 327, 329

[11] Norbert Bodendorfer, Alexander Stottmeister, and Andreas Thurn. **Loop quantum gravity without the Hamiltonian constraint**. *Classical and Quantum Gravity*, **30**:082001, 2013. xxxvii

[12] Norbert Bodendorfer, Alexander Stottmeister, and Andreas Thurn. **On a partially reduced phase space quantisation of general relativity conformally coupled to a scalar field**. *arXiv preprint*, pages 1–51, 2012. Accepted for publication in Classical and Quantum Gravity. xxxvii

[13] Abhay Ashtekar. **New Variables for Classical and Quantum Gravity**. *Physical Review Letters*, **57**(18):2244–2247, 1986. 1, 22, 87, 89, 233

[14] Abhay Ashtekar. **New Hamiltonian formulation of general relativity**. *Physical Review D*, **36**(6):1587–1602, 1987. 1, 22, 87, 89

[15] Amitabha Sen. **Gravity as a spin system**. *Physics Letters B*, **119**(1-3):89–91, 1982. 1

[16] J. Fernando Barbero G. **Real Ashtekar variables for Lorentzian signature space-times**. *Physical Review D*, **51**(10):5507–5510, 1995. 1, 71, 87, 88

[17] J. Fernando Barbero G. **Reality conditions and Ashtekar variables: A different perspective**. *Physical Review D*, **51**:5498–5506, 1995. 1, 71, 87, 88

[18] Giorgio Immirzi. **Real and complex connections for canonical gravity**. *Classical and Quantum Gravity*, **14**:L177–L181, 1997. 1, 71, 88, 93

[19] Giorgio Immirzi. **Quantum gravity and Regge calculus**. *Nuclear Physics B - Proceedings Supplements*, **57**:65–72, 1997. 1, 71, 88

[20] Abhay Ashtekar and Chris J. Isham. **Representations of the holonomy algebras of gravity and non-Abelian gauge theories**. *Classical and Quantum Gravity*, **9**:1433–1468, 1992. 2, 88, 175

[21] Abhay Ashtekar and Jerzy Lewandowski. **Representation Theory of Analytic Holonomy C* Algebras**. In John C. Baez, editor, *Knots and Quantum Gravity*. Oxford University Press, 1994. 2, 88, 175, 236

[22] John C. Baez. **Generalized measures in gauge theory**. *Letters in Mathematical Physics*, **31**:213–223, 1994. 10.1007/BF00761713. 2

[23] John C. Baez. **Diffeomorphism-invariant generalized measures on the space of connections modulo gauge transformations** . In David N. Yetter, editor, *Proceedings of the Conference on Quantum Topology*, pages 21–43, Singapore, 1994. World Scientific Press. 2

[24] Donald Marolf and José M. Mourão. **On the support of the Ashtekar-Lewandowski measure**. *Communications in Mathematical Physics*, **170**(3):583–605, 1995. 2, 175, 236

[25] Abhay Ashtekar and Jerzy Lewandowski. **Differential geometry on the space of connections via graphs and projective limits**. *Journal of Geometry and Physics*, **17**(3):191–230, 1995. 2, 175, 180, 236

[26] Abhay Ashtekar and Jerzy Lewandowski. **Projective techniques and functional integration for gauge theories**. *Journal of Mathematical Physics*, **36**:2170–2191, 1995. 2, 175, 236

[27] Abhay Ashtekar, Jerzy Lewandowski, Donaldo Marolf, José M. Mourão, and Thomas Thiemann. **Quantization of diffeomorphism invariant theories of connections with local degrees of freedom**. *Journal of Mathematical Physics*, **36**:6456–6493, 1995. 2, 175, 202, 236

[28] Jerzy Lewandowski, Andrzej Okołów, Hanno Sahlmann, and Thomas Thiemann. **Uniqueness of Diffeomorphism Invariant States on Holonomy-Flux Algebras**. *Communications in Mathematical Physics*, **267**(3):703–733, 2006. 2, 88, 177, 214

[29] Christian Fleischhack. **Representations of the Weyl Algebra in Quantum Geometry**. *Communications in Mathematical Physics*, **285**(1):67–140, 2009. 2, 88, 177

[30] Thomas Thiemann. **Quantum spin dynamics (QSD)**. *Classical and Quantum Gravity*, **15**:839–873, 1998. 2, 77, 88, 213, 214, 256, 296

[31] Thomas Thiemann. **Quantum spin dynamics (QSD): II. The kernel of the Wheeler-DeWitt constraint operator**. *Classical and Quantum Gravity*, **15**:875–905, 1998. 2, 226

[32] Lee Smolin. **Recent developments in nonperturbative quantum gravity**. In Juan Pérez-Mercader, Enric Verdaguer, and Joan Solà, editors, *Quantum Gravity and Cosmology: Proceedings of the XXIIth GIFT International Seminar on Theoretical Physics, Sant Feliu de Guíxols, Catalonia, Spain*, pages 3–84. World Scientific, 1992. 2, 203

[33] Carlo Rovelli and Lee Smolin. **Discreteness of area and volume in quantum gravity**. *Nuclear Physics B*, **442**(3):593–619, 1995. 2

[34] Abhay Ashtekar and Jerzy Lewandowski. **Quantum theory of gravity I: Area operators**. *Classical and Quantum Gravity*, **14**:A55–A81, 1997. 2, 203

[35] Abhay Ashtekar and Jerzy Lewandowski. **Quantum theory of geometry. 2. Volume operators**. *Advances in Theoretical and Mathematical Physics*, **1**:388–429, 1998. 2

[36] Thomas Thiemann. **A length operator for canonical quantum gravity**. *Journal of Mathematical Physics*, **39**:3372–3392, 1998. 2

[37] Martin Bojowald. **Absence of a Singularity in Loop Quantum Cosmology**. *Physical Review Letters*, **86**:5227–5230, 2001. 2

[38] Lee Smolin. **Linking Topological Quantum Field Theory and Nonperturbative Quantum Gravity**. *Journal of Mathematical Physics*, **36**:6417–6455, 1995. 2, 259

[39] Kirill V. Krasnov. **Geometrical entropy from loop quantum gravity**. *Physical Review D*, **55**:3505 – 3513, 1997. 2, 259

[40] Carlo Rovelli. **Black Hole Entropy from Loop Quantum Gravity**. *Physical Review Letters*, **77**:3288–3291, 1996. 2, 259

[41] Martin Bojowald. **Loop Quantum Cosmology**. *Living Reviews in Relativity*, **8**(11. URL (cited on 21.11.2012)), 2005. 2, 7, 305

[42] Abhay Ashtekar and Parampreet Singh. **Loop quantum cosmology: a status report**. *Classical and Quantum Gravity*, **28**(21):213001, 2011. 2, 7, 305

[43] Jacobo Diaz-Polo and Daniele Pranzetti. **Isolated Horizons and Black Hole Entropy In Loop Quantum Gravity**. *SIGMA*, **8**(048):58, 2012. 2, 7, 14, 260, 306

[44] J. Fernando Barbero G., Jerzy Lewandowski, and Eduardo J. S. Villaseñor. **Quantum isolated horizons and black hole entropy**. *PoS(QGQGS 2011)*, **023**:28, 2011. 2, 7, 14

[45] Thomas Thiemann. **Kinematical Hilbert spaces for Fermionic and Higgs quantum field theories**. *Classical and Quantum Gravity*, **15**:1487–1512, 1998. 2, 4, 197, 199, 233, 234, 242, 243, 244, 248, 300

[46] Thomas Thiemann. **Quantum spin dynamics (QSD) V: Quantum Gravity as the Natural Regulator of Matter Quantum Field Theories**. *Classical and Quantum Gravity*, **15**:1281–1314, 1998. 2, 4, 187, 225, 231, 233, 234, 243, 300

[47] Yongge Ma. **Extension of Loop Quantum Gravity to Metric Theories beyond General Relativity**. *Journal of Physics: Conference Series*, **360**(1):012006, 2012. 2

[48] Maïté Dupuis. **Spin Foam Models for Quantum Gravity and semi-classical limit**. *PhD thesis (Ecole Normale Supérieure Lyon)*, page 192, 2011. 3

[49] Alejandro Perez. **The Spin Foam Approach to Quantum Gravity**. *To appear in Living Reviews in Relativity*, page 121, 2012. 3, 22

[50] Carlo Rovelli. **Zakopane lectures on loop gravity**. *arXiv preprint*, page 34, 2011. 3

[51] Kristina Giesel and Thomas Thiemann. **Algebraic quantum gravity (AQG): IV. Reduced phase space quantization of loop quantum gravity**. *Classical and Quantum Gravity*, **27**:175009, 2010. 3, 128

[52] Marcin Domagała, Kristina Giesel, Wojciech Kamiński, and Jerzy Lewandowski. **Gravity quantized: Loop quantum gravity with a scalar field**. *Physical Review D*, **82**:104038, 2010. 3

[53] Viqar Husain and Tomasz Pawłowski. **Time and a physical Hamiltonian for quantum gravity**. *to appear in Physical Review Letters*, page 5, 2011. 3

[54] Kristina Giesel and Thomas Thiemann. **Scalar Material Reference Systems and Loop Quantum Gravity**. *arXiv preprint*, page 28, 2012. 3

[55] Thomas Thiemann. **Gauge field theory coherent states (GCS): I. General properties**. *Classical and Quantum Gravity*, **18**:2025–2064, 2001. 3

[56] Thomas Thiemann. **Quantum spin dynamics (QSD) III: Quantum constraint algebra and physical scalar product in quantum general relativity**. *Classical and Quantum Gravity*, **15**:1207–1247, 1998. 3, 201, 202

[57] Alok Laddha and Madhavan Varadarajan. **Polymer parametrized field theory**. *Physical Review D*, **78**:044008, 2008. 3

[58] Thomas Thiemann. **Lessons for Loop Quantum Gravity from Parametrised Field Theory**. *arXiv preprint*, page 37, 2010. 3

[59] Alok Laddha and Madhavan Varadarajan. **The diffeomorphism constraint operator in loop quantum gravity**. *Classical and Quantum Gravity*, **28**:195010, 2011. 3

[60] Thomas Thiemann. **The Phoenix Project: master constraint programme for loop quantum gravity**. *Classical and Quantum Gravity*, **23**(7):2211, 2006. 3, 194

[61] Carlo Rovelli. *Quantum Gravity.* Cambridge University Press, Cambridge, 2004. 3, 173

[62] Thomas Thiemann. *Modern Canonical Quantum General Relativity.* Cambridge University Press, Cambridge, 2007. 3, 21, 26, 28, 32, 40, 42, 77, 79, 87, 173, 174, 177, 178, 197, 198, 203, 204, 206, 211, 213, 226, 252, 273, 295, 300, 313, 315

[63] Martin Bojowald. *Canonical Gravity and Applications: Cosmology, Black Holes, and Quantum Gravity.* Cambridge University Press, Cambridge, 2011. 3

[64] Rodolfo Gambini and Jorge Pullin. *A First Course in Loop Quantum Gravity.* Oxford University Press, Oxford, 2011. 3

[65] Thomas Thiemann. **Introduction to Modern Canonical Quantum General Relativity**. *arXiv preprint*, page 301, 2001. 3

[66] Abhay Ashtekar and Jerzy Lewandowski. **Background independent quantum gravity: a status report**. *Classical and Quantum Gravity*, **21**(15):R53, 2004. 3

[67] Kristina Giesel and Hanno Sahlmann. **From Classical To Quantum Gravity: Introduction to Loop Quantum Gravity**. *arXiv preprint*, pages 1–55, 2011. 3

[68] Abhay Ashtekar. **Introduction to Loop Quantum Gravity**. *arXiv preprint*, pages 1–30, 2012. 3

[69] Thomas Thiemann. **QSD 4: (2+1) Euclidean quantum gravity as a model to test (3+1) Lorentzian quantum gravity**. *Classical and Quantum Gravity*, **15**:1249–1280, 1998. 4, 227

[70] Víctor Aldaya, Carlos Barceló, and José L. Jaramillo. **Spanish Relativity Meeting (ERE 2010): Gravity as a Crossroad in Physics**. *Journal of Physics: Conference Series*, **314**(1):011001, 2011. 5

[71] **Quantum Geometry and Gravitation**. *Conference Webpage*, (June 2011, ETH Zurich, Switzerland (cited on 03.12.2012)). 5

[72] Peter G. Bergmann. **Concluding Remarks**. In Léopold Infeld, editor, *Conférence internationale sur les théories relativistes de la gravitation*, Paris, 1964. Gauthier-Villar. 5

[73] Marc H. Goroff and Augusto Sagnotti. **Quantum gravity at two loops**. *Physics Letters B*, **160**(1-3):81–86, 1985. 5

[74] Marc H. Goroff and Augusto Sagnotti. **The ultraviolet behavior of Einstein gravity**. *Nuclear Physics B*, **266**(3-4):709–736, 1986. 5

[75] Carlo Rovelli. **Notes for a brief history of quantum gravity**. *arXiv preprint*, 2001. 5

[76] Stanley Deser. **Two outcomes for two old (super)problems**. In Mikhail Shifman, editor, *The many faces of the Superworld, Yuri Gol'fand memorial volume*, page 7. World Publishing, 2000. 5

[77] Stanley Deser. **Infinities in quantum gravities**. *Annalen der Physik*, **9**:299–307, 2000. 5

[78] Stanley Deser. **Nonrenormalizability of (last hope) D= 11 supergravity, with a terse survey of divergences in quantum gravities**. *arXiv preprint*, page 8, 1999. 5

[79] Nima Arkani-Hamed, Freddy Cachazo, and Jared Kaplan. **What is the simplest quantum field theory?** *Journal of High Energy Physics*, **2010**:1–92, 2010. 10.1007/JHEP09(2010)016. 5

[80] Renata Kallosh. **On UV finiteness of the four loop N= 8 supergravity**. *Journal of High Energy Physics*, **2009**(09):116, 2009. 5

[81] Michael B. Green, John H. Schwarz, and Edward Witten. *Superstring Theory, Vol. 1: Introduction*. Cambridge University Press, Cambridge, 1988. 5

[82] Michael B. Green, John H. Schwarz, and Edward Witten. *Superstring Theory, Vol. 2: Loop Amplitudes, Anomalies and Phenomenology*. Cambridge University Press, Cambridge, 1988. 5

[83] Joseph Polchinski. *String Theory, Vol. 1: An Introduction to the bosonic string*. Cambridge University Press, Cambridge, 1998. 5

[84] Joseph Polchinski. *String Theory, Vol. 2: Superstring theory and beyond*. Cambridge University Press, Cambridge, 1998. 5

[85] Juán M. Maldacena. **The Large N limit of superconformal field theories and supergravity**. *Advances in Theoretical and Mathematical Physics*, **2**:231–252, 1998. 6, 308

[86] Steven S. Gubser, Igor R. Klebanov, and Alexander M. Polyakov. **Gauge theory correlators from non-critical string theory**. *Physics Letters B*, **428**(1-2):105–114, 1998. 6, 308

[87] Edward Witten. **Anti-de Sitter space and holography**. *Advances in Theoretical and Mathematical Physics*, **2**:253–291, 1998. 6, 308

[88] Jens L. Petersen. **Introduction to the Maldacena conjecture on AdS / CFT**. *International Journal of Modern Physics A*, **14**:3597–3672, 1999. 6

[89] Joseph Polchinski. **Introduction to Gauge/Gravity Duality**. *arXiv preprint*, page 43, 2010. 6

[90] Niklas Beisert et al. **Review of AdS/CFT Integrability: An Overview**. *Letters in Mathematical Physics*, **99**:3–32, 2012. 10.1007/s11005-011-0529-2. 6

[91] Gary T. Horowitz. **Introduction to Holographic Superconductors**. *arXiv preprint*, page 34, 2010. 6

[92] Ofer Aharony, Steven S. Gubser, Juan Maldacena, Hirosi Ooguri, and Yaron Oz. **Large N field theories, string theory and gravity**. *Physics Reports*, **323**(3-4):183–386, 2000. 6

[93] Thomas Thiemann. **The LQG string - loop quantum gravity quantization of string theory: I. Flat target space**. *Classical and Quantum Gravity*, **23**(6):1923–1970, 2006. 6, 10, 14, 255, 307

[94] Robert C. Helling and Giuseppe Policastro. **String quantization: Fock vs. LQG Representations**. *arXiv preprint*, page 19, 2004. 6, 10

[95] Dieter Lüst. **Seeing through the string landscape – a string hunter's companion in particle physics and cosmology**. *Journal of High Energy Physics*, **2009**(03):149, 2009. 7

[96] Liam McAllister and Eva Silverstein. **String cosmology: a review**. *General Relativity and Gravitation*, **40**:565–605, 2008. 7, 305

[97] Johanna Erdmenger. *String Cosmology: Modern String Theory Concepts from the Big Bang to Cosmic Structure*. Wiley-VCH, New Jersey, 2009. 7, 305

[98] Andrew Strominger and Cumrun Vafa. **Microscopic origin of the Bekenstein-Hawking entropy**. *Physics Letters B*, **379**:99–104, 1996. 7, 305

[99] Juán M. Maldacena. **Black Holes in String Theory**. *PhD thesis (Princeton University)*, page 80, 1996. 7

[100] Florian Girelli, Franz Hinterleitner, and Seth Major. **Loop Quantum Gravity Phenomenology: Linking Loops to Physics**. *to be published in SIGMA*, page 75, 2012. 8

[101] Ivan Agullo, Abhay Ashtekar, and William Nelson. **A quantum gravity extension of the inflationary scenario**. *arXiv preprint*, page 4, 2012. 8

[102] Emanuele Alesci and Francesco Cianfrani. **A new perspective on cosmology in Loop Quantum Gravity**. *arXiv preprint*, page 5, 2012. 8

[103] Peter Peldán. **Actions for gravity, with generalizations: A Review**. *Classical and Quantum Gravity*, **11**:1087–1132, 1994. 8, 9, 19, 20, 21, 22, 37, 56, 58, 91, 94, 315, 317, 318, 319, 323

[104] Peter Peldán. **Ashtekar's variables for arbitrary gauge group**. *Physical Review D*, **46**(6):R2279–R2282, 1992. 9

[105] Subenoy Chakraborty and Peter Peldán. **Gravity and Yang Mills theory: Two faces of the same theory?** *International Journal of Modern Physics D*, **03**(04):695–722, 1994. 9, 140

[106] Subenoy Chakraborty and Peter Peldán. **Towards a Unification of Gravity and Yang-Mills Theory**. *Physical Review Letters*, **73**:1195–1198, 1994. 9, 140

[107] Juan A. Nieto. **Towards an Ashtekar formalism in eight dimensions**. *Classical and Quantum Gravity*, **22**(6):947–955, 2005. 9

[108] Juan A. Nieto. **Oriented matroid theory and loop quantum gravity in (2+2) and eight dimensions**. *Revista mexicana de física*, **57**:400–405, 2011. 9

[109] Juan A. Nieto. **Towards an Ashtekar formalism in 12 dimensions**. *General Relativity and Gravitation*, **39**(8):1109–1119, 2007. 9

[110] Ted Jacobson. **New variables for canonical supergravity**. *Classical and Quantum Gravity*, **5**:923–935, 1988. 9

[111] Hiroshi Kunitomo and Takashi Sano. **The Ashtekar Formulation for Canonical N=2 Supergravity**. *International Journal of Modern Physics D*, **1**:559–570, 1992. 9

[112] Géza Fülöp. **About a super-Ashtekar–Renteln ansatz**. *Classical and Quantum Gravity*, **11**:1–10, 1994. 9, 304

[113] D. Armand-Ugon, Rodolfo Gambini, Octavio Obrégon, and Jorge Pullin. **Towards a loop representation for quantum canonical supergravity**. *Nuclear Physics B*, **460**(3):615–631, 1996. 9

[114] Luis F. Urrutia. **Towards a loop representation of connection theories defined over a super Lie algebra**. *AIP Conference Proceedings*, **359**(1):243–277, 1996. 9

[115] Yi Ling. **Supersymmetric spin networks**. *Journal of Mathematical Physics*, **43**(1):154–169, 2002. 9

[116] Yi Ling and Lee Smolin. **Supersymmetric spin networks and quantum supergravity**. *Physical Review D*, **61**(4):044008, 2000. 9

[117] David Williams and John F. Cornwell. **The Haar integral for Lie supergroups**. *Journal of Mathematical Physics*, **25**(10):2922–2932, 1984. 9

[118] H. C. Paul Lee and Ruibin B. Zhang. **Geometry and representations of the quantum supergroup OSP[sub q](1—2n)**. *Journal of Mathematical Physics*, **40**(6):3175–3190, 1999. 9

[119] Motomu Tsuda. **Generalized Lagrangian of N=1 supergravity and its canonical constraints with the real Ashtekar variable**. *Physical Review D*, **61**(2):024025, 1999. 9

[120] Manabu Sawaguchi. **Canonical formalism of N = 1 supergravity with the real Ashtekar variables**. *Classical and Quantum Gravity*, **18**:1179–1186, 2001. 9, 249, 305

[121] Sandipan Sengupta and Romesh K. Kaul. **Canonical supergravity with Barbero-Immirzi parameter**. *Physical Review D*, **81**:024024, 2010. 9

[122] Hermann Nicolai, Kasper Peeters, and Marija Zamaklar. **Loop quantum gravity: an outside view**. *Classical and Quantum Gravity*, **22**(19):R193, 2005. 10

[123] Thomas Thiemann. **Loop Quantum Gravity: An Inside View**. In Ion-Olimpiu Stamatescu and Erhard Seiler, editors, *Approaches to Fundamental Physics*, **721** of *Lecture Notes in Physics*, pages 185–263. Springer Berlin / Heidelberg, 2007. 10

[124] Carlo Rovelli. **A critical look at strings**. *arXiv preprint*, page 7, 2011. 10

[125] Klaus Pohlmeyer. **A group-theoretical approach to the quantization of the free relativistic closed string**. *Physics Letters B*, **119**:100–104, 1982. 10

[126] Catherine Meusburger und Karl-Henning Rehren. **Algebraic Quantization of the Closed Bosonic String**. *Communications in Mathematical Physics*, **237**:69–85, 2003. 10

[127] Dorothea Bahns, Katarzyna Reizner, and Jochen Zahn. **The effective theory of strings**. *arXiv preprint*, pages 1–46, 2012. 10

[128] Winston Fairbairn, Karim Noui, and Francesco Sardelli. **Canonical analysis of algebraic string actions**. *Journal of High Energy Physics*, **2010**:1–33, 2010. 10.1007/JHEP01(2010)045. 10

[129] Winston Fairbairn, Karim Noui, and Francesco Sardelli. **Equivalence of the self-dual and Nambu-Goto strings**. *Journal of High Energy Physics*, **2011**:1–24, 2011. 10.1007/JHEP08(2011)070. 10

[130] Stephan Melosch and Hermann Nicolai. **New canonical variables for d=11 supergravity**. *Physics Letters B*, **416**(1-2):91–100, 1998. 10

[131] Lee Smolin. **The cubic matrix model and a duality between strings and loops**. *arXiv preprint*, page 32, 2000. 10

[132] Lee Smolin. **A quantization of topological M theory**. *Nuclear Physics B*, **739**(1-2):169–185, 2006. 10

[133] Peter van Nieuwenhuizen. **An Introduction to Simple Supergravity and the Kaluza-Klein Program**. In Bryce S. DeWitt and R. Stora, editors, *Relativity, Groups & Topology, II: Proceedings of the Les Houches Summer School, Session XL*. Elsevier, 1984. 13, 245, 246

[134] Abdus Salam and Ergin Sezgin, editors. *Supergravities in Diverse Dimensions*. North-Holland, World Scientific, 1989. 13, 246, 304

[135] Eugène Cremmer, Bernard Julia, and Jo el Scherk. **Supergravity theory in 11 dimensions**. *Physics Letters B*, **76**:409–412, 1978. 14, 253

[136] Heide Narnhofer and Walter E. Thirring. **Covariant QED Without Indefinite Metric**. *Reviews in Mathematical Physics*, **4**:197–211, 1992. 14, 255

[137] Máximo Bañados, Luis Garay, and Marc Henneaux. **Existence of local degrees of freedom for higher dimensional pure Chern-Simons theories**. *Physical Review D*, **53**:R593–R596, 1996. 14, 260, 294, 295

[138] Máximo Bañados, Luis J. Garay, and Marc Henneaux. **The dynamical structure of higher dimensional Chern-Simons theory**. *Nuclear Physics B*, **476**:635–611, 1996. 14, 265, 268, 294

[139] Tullio Regge and Claudio Teitelboim. **Role of surface integrals in the Hamiltonian formulation of general relativity**. *Annals of Physics*, **88**:286–318, 1974. 19

[140] Vladimir O. Soloviev. **How canonical are Ashtekar's variables?** *Physics Letters B*, **292**:30–34, 1992. 19

[141] Thomas Thiemann. **Generalised boundary conditions for general relativity for the asymptotically flat case in terms of Ashtekar's variables**. *Classical and Quantum Gravity*, **12**:181–198, 1995. 19

[142] Bryce S. DeWitt and Cécile DeWitt-Morette. **The Quantum Theory of Interacting Gravitational and Spinor Fields**. *Physical Review*, **87**:116–122, 1952. 20

[143] Robert M. Wald. *General Relativity*. University Of Chicago Press, 1984. 20, 26

[144] Abhay Ashtekar and Ranjeet S. Tate. *Lectures on Non-Perturbative Canonical Gravity*. World Scientific, 1991. 20

[145] Sören Holst. **Barbero's Hamiltonian derived from a generalized Hilbert-Palatini action**. *Physical Review D*, **53**(10):5966–5969, 1996. 20, 22, 71, 119

[146] Nuno Barros e Sá. **Hamiltonian analysis of general relativity with the Immirzi parameter**. *International Journal of Modern Physics D*, **10**(3):261–272, 2001. 20, 71, 134, 138, 153

[147] Attilio Palatini. **Deduzione invariantiva delle equazioni gravitazionali dal principio di Hamilton**. *Rendiconti del Circolo Matematico di Palermo*, **43**:203–212, 1919. 20, 309

[148] Roberto Hojman and Chandrasekher Mukku. **Invariant Deduction of the Gravitational Equations from the Principle of Hamilton (English translation of Palatini's 1919 article)**. In Peter G. Bergmann and Venzo de Sabbata, editors, *Cosmology and gravitation: spin, torsion, rotation, and supergravity*, NATO advanced study institutes series: Physics, pages 479–488. Plenum Press, New York, 1980. 20

[149] Albert Einstein. **Einheitliche Feldtheorie von Gravitation und Elektrizität**. *Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin*, pages 414–419, 1925. 20

[150] Marco Ferraris, Mauro Francaviglia, and Cesare Reina. **Variational formulation of general relativity from 1915 to 1925, "Palatini's method" discovered by Einstein in 1925**. *General Relativity and Gravitation*, **14**:243–254, 1982. 20

[151] David Hilbert. **Die Grundlagen der Physik. Erste Mittelung**. *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen. Math.-phys. Klasse*, pages 395–407, 1915. 20

[152] F. A. E. Pirani and A. Schild. **On the Quantization of Einstein's Gravitational Field Equations**. *Physical Review*, **79**:986–991, 1950. 21

[153] Peter G. Bergmann, Robert Penfield, Ralph Schiller, and Henry Zatzkis. **The Hamiltonian of the General Theory of Relativity with Electromagnetic Field**. *Physical Review*, **80**:81–88, 1950. 21

[154] F. A. E. Pirani, A. Schild, and R. Skinner. **Quantization of Einstein's Gravitational Field Equations. II**. *Physical Review*, **87**:452–454, 1952. 21

[155] Paul A. M. Dirac. **The Theory of Gravitation in Hamiltonian Form**. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, **246**:333–343, 1958. 21

[156] Paul A. M Dirac. **Generalized Hamiltonian dynamics**. *Canadian Journal of Mathematics*, **2**:129–148, 1950. 21

[157] Paul A. M. Dirac. *Lectures on Quantum Mechanics.* Belfer Graduate School of Science, Yeshiva University Press, New York, 1964. 21, 29, 253

[158] Peter G. Bergmann. **Non-Linear Field Theories**. *Physical Review*, **75**:680–685, 1949. 21

[159] Peter G. Bergmann and Johanna H. M. Brunings. **Non-Linear Field Theories II. Canonical Equations and Quantization**. *Review of Modern Physics*, **21**:480–487, 1949. 21

[160] James L. Anderson and Peter G. Bergmann. **Constraints in Covariant Field Theories**. *Physical Review*, **83**:1018–1025, 1951. 21

[161] Marc Henneaux and Claudio Teitelboim. *Quantization of Gauge Systems.* Princeton University Press, 1994. 21, 42, 121, 235, 236, 252

[162] Richard Arnowitt, Stanley Deser, and Charles W. Misner. **Canonical Variables for General Relativity**. *Physical Review*, **117**:1595–1602, 1960. 21, 32

[163] Richard Arnowitt, Stanley Deser, and Charles W. Misner. **The Dynamics of General Relativity**. In Louis Witten, editor, *Gravitation: An introduction to current research*, pages 227–265. Wiley, New York, 1962. 21, 32

[164] Norbert Bodendorfer. **Canonical analysis of gravity theories without the time gauge**. *Diploma thesis*, (FAU Erlangen - Nuremberg), 2009. 22, 46

[165] Andreas Thurn. **Constraint analysis of the D+1 dimensional Palatini action**. *Diploma thesis*, (FAU Erlangen - Nuremberg), 2009. 22, 46

[166] Muxin Han, Yongge Ma, You Ding, and Li Qin. **Hamiltonian analysis of n-dimensional Palatini gravity with matter**. *Modern Physics Letters A*, **20**:725–732, 2005. 22, 120

[167] Jerzy F. Plebański. **On the separation of Einsteinian substructures**. *Journal of Mathematical Physics*, **18**:2511–2520, 1977. 22, 62

[168] Alejandro Perez. **Spin foam models for quantum gravity**. *Classical and Quantum Gravity*, **20**(6):R43–R104, 2003. 22

[169] Laurent Freidel, Kirill V. Krasnov, and Raymond Puzio. **BF description of higher-dimensional gravity theories**. *Advances in Theoretical and Mathematical Physics*, **3**:1289–1324, 1999. 22, 62, 64, 65, 101, 120, 181, 183, 188, 190, 194, 196, 205, 302

[170] Eric Buffenoir, Marc Henneaux, Marc Noui, and Philippe Roche. **Hamiltonian analysis of Plebanski theory**. *Classical and Quantum Gravity*, **21**:5203–5220, 2004. 22, 63

[171] Riccardo Capovilla, John Dell, and Ted Jacobson. **General relativity without the metric**. *Physical Review Letters*, **63**:2325–2328, 1989. 23, 80

[172] Peter Peldán. **Connection formulation of (2+1)-dimensional Einstein gravity and topologically massive gravity**. *Classical and Quantum Gravity*, **9**:2079–2092, 1992. 23

[173] Peter Peldán. **Legendre transforms in Ashtekar's theory of gravity**. *Classical and Quantum Gravity*, **8**:1765–1783, 1991. 23, 80

[174] Erwin Schrödinger. *Space-Time Structure*, pages 112–115. Cambridge University Press, Cambridge, 1950. 23

[175] Jack Heller and Peter G. Bergmann. **A Canonical Field Theory with Spinors**. *Physical Review*, **84**:665–670, 1951. 23

[176] Gerardus 't Hooft. **A chiral alternative to the vierbein field in general relativity**. *Nuclear Physics B*, **357**:211–221, 1991. 23

[177] Ludvig D. Faddeev. **New action for the Hilbert-Einstein equations**. *arXiv:0906.4639v2 [hep-th]*, 2009. 23

[178] Bryce S. DeWitt. **Quantum Theory of Gravity. I. The Canonical Theory**. *Physical Review*, **160**:1113–1148, 1967. 23, 29

[179] Bryce S. DeWitt. **Quantum Theory of Gravity. II. The Manifestly Covariant Theory**. *Physical Review*, **162**:1195–1239, 1967. 23

[180] Bryce S. DeWitt. **Quantum Theory of Gravity. III. Applications of the Covariant Theory**. *Physical Review*, **162**:1239–1256, 1967. 23

[181] Robert Geroch. **Domain of Dependence**. *Journal of Mathematical Physics*, **11**:437–449, 1970. 25

[182] Antonio N. Bernal and Miguel Sanchez. **On Smooth Cauchy Hypersurfaces and Geroch's Splitting Theorem**. *Communications in Mathematical Physics*, **243**(3):461–470, 2003. 25

[183] Friedrich W. Hehl, Paul von der Heyde, G. David Kerlick, and James M. Nester. **General relativity with spin and torsion: Foundations and prospects**. *Review of Modern Physics*, **48**:393–416, 1976. 26

[184] Edward Witten. **Quantum field theory and the Jones polynomial**. *Communications in Mathematical Physics*, **121**(3):351–399, 1989. 61, 259, 294

[185] John W. Barrett and Louis Crane. **Relativistic spin networks and quantum gravity**. *Journal of Mathematical Physics*, **39**(6):3296–3302, 1998. 62, 181, 300

[186] Jonathan Engle, Roberto Pereira, and Carlo Rovelli. **Loop-Quantum-Gravity Vertex Amplitude**. *Physical Review Letters*, **99**(16):161301, 2007. 62, 63, 135, 181, 303

[187] Etera R. Livine and Simone Speziale. **New spinfoam vertex for quantum gravity**. *Physical Review D*, **76**(8):084028, 2007. 62, 135, 181, 303

[188] Laurent Freidel and Kirill V. Krasnov. **A new spin foam model for 4D gravity**. *Classical and Quantum Gravity*, **25**(12):125018, 2008. 62, 135, 181, 300, 303

[189] Jonathan Engle, Roberto Pereira, and Carlo Rovelli. **Flipped spinfoam vertex and loop gravity**. *Nuclear Physics B*, **798**:251–290, 2008. 62, 63, 135, 181, 300, 303

[190] Jonathan Engle, Etera R. Livine, Roberto Pereira, and Carlo Rovelli. **LQG vertex with finite Immirzi parameter**. *Nuclear Physics B*, **799**(1-2):136–149, 2008. 62, 135, 181, 190, 194, 303

[191] Wojciech Kamiński, Marcin Kisielowski, and Jerzy Lewandowski. **Spin-foams for all loop quantum gravity**. *Classical and Quantum Gravity*, **27**(9):095006, 2010. 62, 135, 181

[192] Jonathan Engle. **A spin-foam vertex amplitude with the correct semiclassical limit**. *arXiv preprint*, page 4, 2012. 63

[193] Sergei Alexandrov. **SO (4, C )-covariant Ashtekar-Barbero gravity and the Immirzi parameter**. *Classical and Quantum Gravity*, **17**(20):4255–4268, 2000. 70, 87

[194] Sergei Alexandrov and Etera R. Livine. **SU(2) loop quantum gravity seen from covariant theory**. *Physical Review D*, **67**(4):044009, 2003. 70, 87, 120, 138

[195] Francesco Cianfrani and Giovanni Montani. **Towards Loop Quantum Gravity without the Time Gauge**. *Physical Review Letters*, **102**:091301, 2009. 70

[196] Marc Geiller, Marc Lachieze-Rey, Karim Noui, and Francesco Sardelli. **A Lorentz-Covariant Connection for Canonical Gravity**. *SIGMA*, **7**:083, 2011. 70, 138

[197] Ted Jacobson and Lee Smolin. **Covariant action for Ashtekar's form of canonical gravity**. *Classical and Quantum Gravity*, **5**(4):583, 1988. 72

[198] Sergei Alexandrov and Philippe Roche. **Critical overview of loops and foams**. *Physics Reports*, **506**(3-4):41–86, 2011. 87

[199] Parthasarathi Mitra and Ramamurti Rajaraman. **Gauge-invariant reformulation of an anomalous gauge theory**. *Physics Letters B*, **225**(3):267–271, 1989. 89, 121

[200] Parthasarathi Mitra and Ramamurti Rajaraman. **Gauge-invariant reformulation of theories with second-class constraints**. *Annals of Physics*, **203**(1):157–172, 1990. 89

[201] Ramesh Anishetty and A. S. Vytheeswaran. **Gauge invariance in second-class constrained systems**. *Journal of Physics A: Mathematical and General*, **26**(20):5613–5619, 1993. 89, 121

[202] A. S. Vytheeswaran. **Gauge unfixing in second-class constrained systems**. *Annals of Physics*, **236**(2):297–324, 1994. 89, 121, 125

[203] Steven Carlip. *Quantum Gravity in 2+1 Dimensions*. Cambridge University Press, Cambridge, 2003. 91

[204] Bianca Dittrich. **Partial and complete observables for Hamiltonian constrained systems**. *General Relativity and Gravitation*, **39**(11):1891–1927, 2007. 121, 127

[205] Carlo Rovelli. **What is observable in classical and quantum gravity?** *Classical and Quantum Gravity*, **8**(2):297–316, 1991. 121

[206] Carlo Rovelli. **Partial observables**. *Physical Review D*, **65**(12):124013, 2002. 121

[207] Thomas Thiemann. **Reduced phase space quantization and Dirac observables**. *Classical and Quantum Gravity*, **23**(4):1163–1180, 2006. 121, 127

[208] Pío José Arias and Jorge Stephany. **Gauge invariance and second class constraints in three-dimensional linearized massive gravity**. *Journal of Mathematical Physics*, **36**:1868–1876, 1995. 131

[209] Steffen Gielen and Daniele Oriti. **Classical general relativity as BF-Plebanski theory with linear constraints**. *Classical and Quantum Gravity*, **27**(18):185017, 2010. 135

[210] Peter Peldán. **Unification of gravity and Yang-Mills theory in (2+1) dimensions**. *Nuclear Physics B*, **395**(1-2):239–262, 1993. 140

[211] Howard Georgi. **The state of the art–gauge theories**. In Carl E. Carlson, editor, *Particles and Fields–1974*, pages 575–582, New York, 1975. American Institute of Physics. 140

[212] Harald Fritzsch and Peter Minkowski. **Unified interactions of leptons and hadrons**. *Annals of Physics*, **93**(1-2):193–266, 1975. 140

[213] WM Wieland. **Complex Ashtekar variables and reality conditions for Holst's action**. *Annales Henri Poincaré*, **13**:425–448, 2011. 146, 182, 222

[214] Carlo Rovelli and Lee Smolin. **Spin networks and quantum gravity**. *Physical Review D*, **52**(10):5743–5759, 1995. 178

[215] John C. Baez. **Spin Networks in Gauge Theory**. *Advances in Mathematics*, **117**(2):253–272, 1996. 178

[216] Thomas Thiemann. **The inverse loop transform**. *Journal of Mathematical Physics*, **39**:1236–1248, 1998. 178

[217] Roberto De Pietri, Laurent Freidel, Kirill V. Krasnov, and Carlo Rovelli. **Barrett - Crane model from a Boulatov - Ooguri field theory over a homogeneous space**. *Nuclear Physics B*, **574**(3):785–806, 2000. 181

[218] Joseph Ben Geloun, Razvan Gurau, and Vincent Rivasseau. **EPRL/FK group field theory**. *EPL (Europhysics Letters)*, **92**(6):60008, 2010. 181

[219] Aristide Baratin and Daniele Oriti. **Group Field Theory with Noncommutative Metric Variables**. *Physical Review Letters*, **105**(22):221302, 2010. 181

[220] Henri Waelbroeck and José A. Zapata. **A Hamiltonian lattice formulation of topological gravity**. *Classical and Quantum Gravity*, **11**(4):989, 1994. 181

[221] José A. Zapata. **Topological lattice gravity using self-dual variables**. *Classical and Quantum Gravity*, **13**(10):2617, 1996. 181

[222] Bianca Dittrich and James P. Ryan. **Phase space descriptions for simplicial 4D geometries**. *Classical and Quantum Gravity*, **28**(6):065006, 2011. 181, 182

[223] Bianca Dittrich and James P. Ryan. **Simplicity in simplicial phase space**. *Physical Review D*, **82**:064026, 2010. 181, 182

[224] Emanuele Alesci and Carlo Rovelli. **Complete LQG propagator: Difficulties with the Barrett-Crane vertex**. *Physical Review D*, **76**:104012, 2007. 181

[225] Emanuele Alesci and Carlo Rovelli. **Complete LQG propagator. II. Asymptotic behavior of the vertex**. *Physical Review D*, **77**:044024, 2008. 181

[226] Emanuele Alesci, Eugenio Bianchi, and Carlo Rovelli. **LQG propagator: III. The new vertex**. *Classical and Quantum Gravity*, **26**(21):215001, 2009. 182

[227] Eugenio Bianchi, Elena Magliaro, and Claudio Perini. **LQG propagator from the new spin foams**. *Nuclear Physics B*, **822**(1-2):245–269, 2009. 182

[228] Carlo Rovelli and Mingyi Zhang. **The Euclidean three-point function in loop and perturbative gravity**. *Classical and Quantum Gravity*, **28**(17):175010, 2011. 182

[229] Frank Hellmann and Wojciech Kamiński. **Geometric asymptotics for spin foam lattice gauge gravity on arbitrary triangulations**. *arXiv preprint*, page 10, 2012. 182

[230] Maïté Dupuis and Etera R. Livine. **Revisiting the simplicity constraints and coherent intertwiners**. *Classical and Quantum Gravity*, **28**(8):085001, 2011. 182

[231] Maïté Dupuis and Etera R. Livine. **Holomorphic simplicity constraints for 4D spinfoam models**. *Classical and Quantum Gravity*, **28**:215022, 2011. 182

[232] Maïté Dupuis, Laurent Freidel, Etera R. Livine, and Simone Speziale. **Holomorphic Lorentzian simplicity constraints**. *Journal of Mathematical Physics*, **53**(3):032502, 2012. 182

[233] Markus Hormessand Georg Junker. **More on coupling coefficients for the most degenerate representations of SO(n)**. *Journal of Physics A: Mathematical and General*, **32**:4249–4256, 1999. 188

[234] Sigitas Alisauskas. **Coupling coefficients of SO(n) and integrals involving Jacobi and Gegenbauer polynomials**. *Journal of Physics A: Mathematical and General*, **35**:7323–7345, August 2002. 188

[235] Georg Junker. **Explicit evaluation of coupling coefficients for the most degenerate representations of SO(n)**. *Journal of Physics A: Mathematical and General*, **26**(7):1649–1661, 1993. 188

[236] Sigitas Alisauskas. **Some coupling and recoupling coefficients for symmetric representations of SO n**. *Journal of Physics A: Mathematical and General*, **20**:35, 1987. 188

[237] Georges Girardi, Antonino Sciarrino, and Paul Sorba. **Generalized Young tableaux and Kronecker products of SO(n) representations**. *Physica A: Statistical Mechanics and its Applications*, **114**:365–369, 1982. 188, 205

[238] Georges Girardi, Antonino Sciarrino, and Paul Sorba. **Kronecker products for SO(2p) representations**. *Journal of Physics A: Mathematical and General*, **15**:1119–1129, 1982. 188, 205

[239] Naum J. Vilenkin. *Special Functions and the Theory of Group Representations*. American Mathematical Society, 1968. 188, 198, 199, 200

[240] Hanno Sahlmann and Thomas Thiemann. **Irreducibility of the Ashtekar-Isham-Lewandowski representation**. *Classical and Quantum Gravity*, **23**(13):4453–4471, 2006. 190, 191, 192

[241] Thomas Thiemann. **Closed formula for the matrix elements of the volume operator in canonical quantum gravity**. *Journal of Mathematical Physics*, **39**(6):3347–3371, 1998. 195

[242] Atsushi Higuchi. **Quantum linearization instabilities of de Sitter spacetime. II**. *Classical and Quantum Gravity*, **8**(11):1983–2004, 1991. 200

[243] Domenico Giulini and Donald Marolf. **On the generality of refined algebraic quantization**. *Classical and Quantum Gravity*, **16**(7):2479–2488, 1999. 200

[244] Domenico Giulini and Donald Marolf. **A uniqueness theorem for constraint quantization**. *Classical and Quantum Gravity*, **16**:2489–2505, 1999. 200

[245] Thomas Thiemann. **Quantum spin dynamics: VIII. The master constraint**. *Classical and Quantum Gravity*, **23**(7):2249–2265, 2006. 201, 226

[246] José M. Mourão, Thomas Thiemann, and José M. Velhinho. **Physical properties of quantum field theory measures**. *Journal of Mathematical Physics*, **40**(5):2337–2353, 1999. 207

[247] Kristina Giesel and Thomas Thiemann. **Consistency check on volume and triad operator quantization in loop quantum gravity: I**. *Classical and Quantum Gravity*, **23**:5667–5691, 2006. 211, 227

[248] Kristina Giesel and Thomas Thiemann. **Consistency check on volume and triad operator quantization in loop quantum gravity: II**. *Classical and Quantum Gravity*, **23**:5693–5771, 2006. 211, 227

[249] Johannes Brunnemann and Thomas Thiemann. **Simplification of the Spectral Analysis of the Volume Operator in Loop Quantum Gravity**. *Classical and Quantum Gravity*, **23**:1289–1346, 2006. 211

[250] Johannes Brunnemann and David Rideout. **Properties of the volume operator in loop quantum gravity: I. Results**. *Classical and Quantum Gravity*, **25**(6):065001, 2008. 211

[251] Johannes Brunnemann and David Rideout. **Properties of the volume operator in loop quantum gravity: II. Detailed presentation**. *Classical and Quantum Gravity*, **25**(6):065002, 2008. 211

[252] Johannes Brunnemann and David Rideout. **Oriented matroids - combinatorial structures underlying loop quantum gravity**. *Classical and Quantum Gravity*, **27**(20):205008, 2010. 211

[253] You Ding and Carlo Rovelli. **The volume operator in covariant quantum gravity**. *arXiv:0911.0543v2 [gr-qc]]*, 2010. 228

[254] Tom W. B. Kibble. **Canonical Variables for the Interacting Gravitational and Dirac Fields**. *Journal of Mathematical Physics*, **4**(11):1433–1437, 1963. 233

[255] Abhay Ashtekar, Joseph Romano, and Ranjeet Tate. **New variables for gravity: Inclusion of matter**. *Physical Review D*, **40**(8):2572–2587, 1989. 233, 234

[256] Alejandro Perez and Carlo Rovelli. **Physical effects of the Immirzi parameter in loop quantum gravity**. *Physical Review D*, **73**(4):044013, 2006. 233

[257] Martin Bojowald and Rupam Das. **Canonical gravity with fermions**. *Physical Review D*, **78**(6):064009, 2008. 233

[258] Simone Mercuri. **Fermions in the Ashtekar-Barbero connection formalism for arbitrary values of the Immirzi parameter**. *Physical Review D*, **73**(8):084016, 2006. 233, 234

[259] Ghanashyam Date, Romesh Kaul, and Sandipan Sengupta. **Topological interpretation of Barbero-Immirzi parameter**. *Physical Review D*, **79**(4):044008, 2009. 233

[260] Ted Jacobson. **Fermions in canonical gravity**. *Classical and Quantum Gravity*, **5**:L143–L148, 1988. 236

[261] Yvonne Choquet-Bruhat and Cécile DeWitt-Morette. *Analysis, Manifolds and Physics. Part II - Revised and Enlarged.* North Holland, Amsterdam, 2000. 243

[262] Stanley Deser, J. H. Kay, and K. S. Stelle. **Hamiltonian formulation of supergravity**. *Physical Review D*, **16**(8):2448–2455, 1977. 249, 250

[263] Thomas Thiemann and Oliver Winkler. **Gauge field theory coherent states (GCS): IV. Infinite tensor product and thermodynamical limit**. *Classical and Quantum Gravity*, **18**(23):4997–5053, 2001. 252

[264] Alvaro Diaz. **Hamiltonian formulation of eleven-dimensional supergravity**. *Physical Review D*, **33**(10):2801–2808, 1986. 253

[265] Alvaro Diaz. **Constraint algebra in eleven-dimensional supergravity**. *Physical Review D*, **33**(10):2809–2812, 1986. 253, 305

[266] John E. Campbell. **On a Law of Combination of Operators bearing on the Theory of Continuous Transformation Groups**. *Proceedings of the London Mathematical Society*, **s1-28**:381–390, 1896. 255

[267] John E. Campbell. **On a Law of Combination of Operators (Second Paper)**. *Proceedings of the London Mathematical Society*, **s1-29**:14–32, 1897. 255

[268] Henry F. Baker. **Further Applications of Metrix Notation to Integration Problems**. *Proceedings of the London Mathematical Society*, **s1-34**:347–360, 1901. 255

[269] Henry F. Baker. **On the Integration of Linear Differential Equations**. *Proceedings of the London Mathematical Society*, **s1-35**:333–378, 1902. 255

[270] Henry F. Baker. **Alternants and Continuous Groups**. *Proceedings of the London Mathematical Society*, **s2-3**:24–47, 1905. 255

[271] Felix Hausdorff. **Die symbolische Exponentialformel in der Gruppentheorie**. *Berichte über die Verhandlungen der Sächsischen Akademie der Wissenschaften*

*zu Leipzig, Mathematisch-Physische Klasse*, **58**:19–48, 1906. 255

[272] WILHELM MAGNUS. **On the exponential solution of differential equations for a linear operator**. *Communications on Pure and Applied Mathematics*, **7**:649–673, 1954. 255

[273] ABHAY ASHTEKAR, ALEJANDRO CORICHI, AND KIRILL V. KRASNOV. **Isolated Horizons: the Classical Phase Space**. *Advances in Theoretical and Mathematical Physics*, **3**:419–478, 2000. 259, 268

[274] ABHAY ASHTEKAR, CHRISTOPHER BEETLE, AND STEPHEN FAIRHURST. **Mechanics of isolated horizons**. *Classical and Quantum Gravity*, **17**:253–298, 2000. 259, 268, 270, 271

[275] ABHAY ASHTEKAR, STEPHEN FAIRHURST, AND BADRI KRISHNAN. **Isolated horizons: Hamiltonian evolution and the first law**. *Physical Review D*, **62**:104025, 2000. 259, 268, 283

[276] ABHAY ASHTEKAR, JOHN C. BAEZ, AND KIRILL V. KRASNOV. **Quantum Geometry of Isolated Horizons and Black Hole Entropy**. *Advances in Theoretical and Mathematical Physics*, **4**:1–94, 2000. 259

[277] MARCIN DOMAGAŁA AND JERZY LEWANDOWSKI. **Black-hole entropy from quantum geometry**. *Classical and Quantum Gravity*, **21**(22):5233, 2004. 259

[278] KRZYSZTOF A. MEISSNER. **Black-hole entropy in loop quantum gravity**. *Classical and Quantum Gravity*, **21**(22):5245, 2004. 259

[279] IVAN AGULLO, J. FERNANDO BARBERO G., ENRIQUE F. BORJA, JACOBO DIAZ-POLO, AND EDUARDO J. S. VILLASENOR. **Detailed black hole state counting in loop quantum gravity**. *Physical Review D*, **82**:084029, 2010. 259

[280] ABHAY ASHTEKAR, JONATHAN ENGLE, AND CHRIS VAN DEN BROECK. **Quantum horizons and black-hole entropy: inclusion of distortion and rotation**. *Classical and Quantum Gravity*, **22**:L27, 2005. 259

[281] CHRISTOPHER BEETLE AND JONATHAN ENGLE. **Generic isolated horizons in loop quantum gravity**. *Classical and Quantum Gravity*, **27**:235024, 2010. 259, 261, 291, 335

[282] JONATHAN ENGLE, KARIM NOUI, AND ALEJANDRO PEREZ. **Black Hole Entropy and SU(2) Chern-Simons Theory**. *Physical Review Letters*, **105**:031302, 2010. 259

[283] ROMESH K. KAUL AND PARTHASARATHI MAJUMDAR. **Logarithmic Correction to the Bekenstein-Hawking Entropy**. *Physical Review Letters*, **84**:5255–5257, 2000. 259

[284] ANDREW DEBENEDICTIS, STEVE KLOSTER, AND JOHAN BRANNLUND. **A note on the symmetry reduction of SU(2) on horizons of various topologies**. *Classical and Quantum Gravity*, **28**(10):105023, 2011. 259

[285] ALEJANDRO PEREZ AND DANIELE PRANZETTI. **Static isolated horizons: SU(2) invariant phase space, quantization, and black hole entropy**. *Entropy*, **13**:744–777, 2011. 260, 261, 291, 292, 293, 306

[286] J. MANUEL GARCÍA-ISLAS. **BTZ black hole entropy: a spin foam model description**. *Classical and Quantum Gravity*, **25**(24):245001, 2008. 260

[287] EUGENIO BIANCHI. **Entropy of Non-Extremal Black Holes from Loop Gravity**. *arXiv preprint*, page 6, 2012. 260

[288] JERZY LEWANDOWSKI AND TOMASZ PAWŁOWSKI. **Quasi-local rotating black holes in higher dimension: geometry**. *Classical and Quantum Gravity*, **22**:1573–1598, May 2005. 260, 268

[289] MIKOŁAJ KORZYŃSKI, JERZY LEWANDOWSKI, AND TOMASZ PAWŁOWSKI. **Mechanics of multidimensional isolated horizons**. *Classical and Quantum Gravity*, **22**:2001–2016, 2005. 260, 268, 280, 281

[290] ABHAY ASHTEKAR, TOMASZ PAWŁOWSKI, AND CHRIS VAN DEN BROECK. **Mechanics of higher dimensional black holes in asymptotically anti-de Sitter spacetimes**. *Classical and Quantum Gravity*, **24**:625–644, 2007. 260, 268, 280

[291] Tomáš Liko and Ivan Booth. **Isolated horizons in higher dimensional Einstein-Gauss-Bonnet gravity**. *Classical and Quantum Gravity*, **24**(14):3769, 2007. 260, 268

[292] Jonathan Engle, Karim Noui, Alejandro Perez, and Daniele Pranzetti. **Black hole entropy from the $SU(2)$-invariant formulation of type I isolated horizons**. *Physical Review D*, **82**:044050, 2010. 261, 274, 295

[293] Cristián Martínez, Ricardo Troncoso, and Jorge Zanelli. **Exact black hole solution with a minimally coupled scalar field**. *Physical Review D*, **70**:084035, 2004. 263

[294] Mikio Nakahara. *Geometry, Topology and Physics*. Taylor & Francis, 2nd edition, 2003. 263, 264, 274

[295] Cedomir Crnkovic and Edward Witten. **Covariant Description of Canonical Formalism in Geometrical Theories**. In Werner Israel and Steven W. Hawking, editors, *Three Hundred Years of Gravitation*, pages 676–684, Cambridge, 1987. Cambridge University Press. 266

[296] Joohan Lee and Robert Wald. **Local symmetries and constraints**. *Journal of Mathematical Physics*, **31**:725–743, 1990. 266

[297] Abhay Ashtekar, Luca Bombelli, and Oscar Reula. **Covariant Phase Space of Asymptotically flat Gravitational Fields**. In Mauro Francaviglia, editor, *Mechanics, Analysis and Geometry: 200 Years after Lagrange*, Amsterdam, 1990. North Holland. 266

[298] Abhay Ashtekar, Christopher Beetle, and Stephen Fairhurst. **Isolated horizons: a generalization of black hole mechanics**. *Classical and Quantum Gravity*, **16**:L1–L7, 1999. 268

[299] Abhay Ashtekar and David Sloan. **Action and Hamiltonians in higher-dimensional general relativity: first-order framework**. *Classical and Quantum Gravity*, **25**:225025, 2008. 281

[300] Abhay Ashtekar, Jonathan Engle, and Chris Van Den Broeck. **Quantum horizons and black-hole entropy: inclusion of distortion and rotation**. *Classical and Quantum Gravity*, **22**(4):L27–L34, 2005. 291

[301] Scott Axelrod, Steve Della Pietra, and Edward Witten. **Geometric quantization of Chern-Simons gauge theory**. *Journal of Differential Geometry*, **33**(3):787–902, 1991. 294

[302] Roberto Floreanini and Roberto Percacci. **Higher dimensional abelian Chern-Simons theory**. *Physics Letters B*, **224**(3):291–294, 1989. 294

[303] Sergei Alexandrov. **Hilbert space structure of covariant loop quantum gravity**. *Physical Review D*, **66**:024028, 2002. 303

[304] Etera R Livine. **Projected spin networks for Lorentz connection: linking spin foams and loop gravity**. *Classical and Quantum Gravity*, **19**(21):5525–5541, 2002. 303

[305] Jonathan Engle, Muxin Han, and Thomas Thiemann. **Canonical path integral measures for Holst and Plebanski gravity: I. Reduced phase space derivation**. *Classical and Quantum Gravity*, **27**(24):245014, 2010. 303

[306] Muxin Han. **Canonical path integral measures for Holst and Plebanski gravity: II. Gauge invariance and physical inner product**. *Classical and Quantum Gravity*, **27**(24):245015, 2010. 303

[307] Wojciech Kamiński Bianca Dittrich, Frank Hellmann. **Holonomy Spin Foam Models: Boundary Hilbert spaces and Time Evolution Operators**. *arXiv preprint*, pages 1–51, 2012. 304

[308] John C. Baez and John Huerta. **An invitation to higher gauge theory**. *General Relativity and Gravitation*, **43**:2335–2392, 2011. 304

[309] Hans-Jürgen Matschull and Hermann Nicolai. **Canonical quantum supergravity in three dimensions**. *Nuclear Physics B*, **411**(2-3):609–646, 1994. 305

[310] Etera R. Livine and Robert Oeckl. **Three-dimensional Quantum Supergravity and Supersymmetric Spin Foam Models**. *Advances in Theoretical and Mathematical Physics*, **7**:951–1001, 2004. 305

[311] Amit Ghosh and Alejandro Perez. **Black Hole Entropy and Isolated Horizons Thermodynamics**. *Physical Review Letters*, **107**:241301, 2011. 306

[312] Steven Carlip. **Logarithmic corrections to black hole entropy, from the Cardy formula**. *Classical and Quantum Gravity*, **17**(20):4175, 2000. 306

[313] Steven Carlip. **Black Hole Entropy from Conformal Field Theory in Any Dimension**. *Physical Review Letters*, **82**:2828–2831, 1999. 306

[314] Amit Ghosh and Parthasarathi Mitra. **Log correction to the black hole area law**. *Physical Review D*, **71**:027502, 2005. 306

[315] Ashoke Sen. **Logarithmic Corrections to Schwarzschild and Other Non-extremal Black Hole Entropy in Different Dimensions**. *arXiv:1205.0971v1 [hep-th]*, pages 1–40, 2012. 306

[316] Tomáš Liko and Ivan Booth. **Supersymmetric isolated horizons**. *Classical and Quantum Gravity*, **25**(10):105020, 2008. 306

[317] Ivan Booth and Tomáš Liko. **Supersymmetric isolated horizons in ADS spacetime**. *Physics Letters B*, **670**(1):61–66, 2008. 306

[318] Roberto Emparan and Harvey S. Reall. **Black Holes in Higher Dimensions**. *Living Reviews in Relativity*, **11**, 2008. 307

[319] Steve Kloster, Johan Brannlund, and Andrew DeBenedictis. **Phase space and black-hole entropy of higher genus horizons in loop quantum gravity**. *Classical and Quantum Gravity*, **25**(6):065008, 2008. 307

[320] Luciano Vanzo. **Black holes with unusual topology**. *Physical Review D*, **56**:6475–6483, 1997. 307

[321] Robert B. Mann and Sergey N. Solodukhin. **Universality of quantum entropy for extreme black holes**. *Nuclear Physics B*, **523**(1-2):293–307, 1998. 307

[322] Tomáš Liko. **Topological deformation of isolated horizons**. *Physical Review D*, **77**:064004, 2008. 307

[323] Stefano Liberati and Giuseppe Pollifrone. **Entropy and topology for gravitational instantons**. *Physical Review D*, **56**:6458–6466, 1997. 307

[324] Steven Weinberg. *The Quantum Theory of Fields, Volume 1: Foundations*. Cambridge University Press, 2005. 325

[325] Roger Penrose and Wolfgang Rindler. *Spinors and Space-Time: Volume 2, Spinor and Twistor Methods in Space-Time Geometry*. Cambridge University Press, 1988. 325

[326] Tomás Ortín. *Gravity and Strings*. Cambridge University Press, 2007. 325

[327] Alan Coley, Robert Milson, Vojtech Pravda, and Alena Pravdová. **Classification of the Weyl tensor in higher dimensions**. *Classical and Quantum Gravity*, **21**:L35, 2004. 327, 328