Universal aspects of perturbative gauge theory

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Abstract

Gravity amplitudes, at tree level, can be computed from gauge theory amplitudes using the (field theory) KLT relation. Underlying the KLT relation are the 'Kleiss-Kuijf' and 'fundamental BCJ' identities satisfied by gauge theory partial amplitudes. These identities, and the KLT relation, are proved in an elementary way using properties of Lie polynomials, and the properties of a Lie bracket we call the 'S bracket', which has appeared in examples in superstring calculations. The S bracket is also closely related to string-like formulas which express partial amplitudes as sums of residues of rational functions on $\mathcal{M}_{0,n}$. In this context, identities amoung rational functions on $\mathcal{M}_{0,n}$ allow us to prove a formula for non-linear sigma model partial amplitudes.

In the last decade, organising the computation of tree and loop amplitudes into the volumes of polytopes has led to new formulas for partial amplitudes in special cases. More generically, the ABHY associahedron is a polytopal realisation of the associahedron that is directly related to the tree level partial amplitudes of all gauge theories. It is known that the ABHY associahedron can be derived by studying A_n quiver representations. Generalised *ABHY polytopes* are defined here, associated to Artinian module categories. The faces of an ABHY polytope are themselves isomorphic to ABHY polytopes of lower dimension. When this generalised definition is applied to categories associated to surfaces, the ABHY polytopes relate to the 'loop level' and 'multi-trace' partial amplitudes of the conventional Feynman perturbation series for biadjoint scalar theory. It is suggested that understanding these partial amplitudes and their ABHY polytopes will be important for stating and proving a correct generalisation of the (field theory) KLT relation to the full amplitudes of gauge theory and gravity.

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Contents

Chapter 1

Introduction

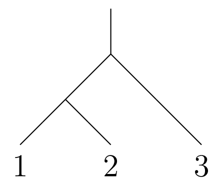
Let

$$A_{\mathrm{YM}}(1, 2, 3, 4) \tag{1.1}$$

be the Yang-Mills scattering amplitude for 4 gluons, with momenta k_i^μ, polarizations ϵ_i^μ, and Lie algebra elements $t_i^a \in \mathrm{ad}_{\mathfrak{g}}$. The leading order, tree contribution to A_{YM} is conventionally expanded as

$$A(123, 4)\mathrm{tr}(t_1t_2t_3t_4) + A(231, 4)\mathrm{tr}(t_2t_3t_1t_4) + A(312, 4)\mathrm{tr}(t_3t_1t_2t_4). \tag{1.2}$$

Let $s_{ij} = (k_i+k_j)^2$ be the Mandelstam variables. Then the partial amplitude $A(123, 4)$ has poles at $s_{12} = 0$ and $s_{23} = 0$, but not at $s_{13} = 0$. This is because the only cubic Feynman diagrams whose colour factors contain $\mathrm{tr}(t_1t_2t_3t_4)$ are



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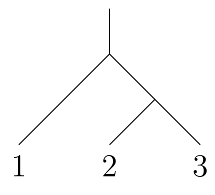
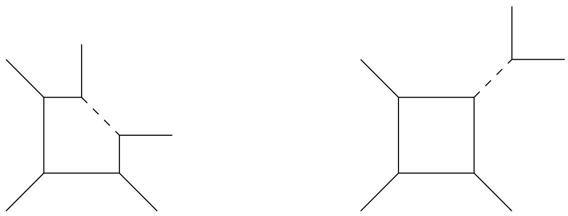


These two diagrams are said to be related to each other by a *mutation* of their internal edge. More generally, all the cubic diagrams that contribute to any given partial amplitude at any order in A_{YM} are related to each other by mutations of this kind. For example, the following one-loop diagrams are related by a mutation of the dashed edge.



Mutations of Feynman diagrams is the common theme of the results in this thesis. In the case of tree diagrams, mutations are closely related to the Jacobi identity. Many known properties of tree amplitudes are shown in the thesis to be direct consequences of elementary facts about Lie monomials, and the Jacobi identity. The appearance of the Jacobi identity in the topology of configuration space Conf_{n-1} explains morally why the string-like 'CHY formulas' for amplitudes are correct, and leads to new results about tree amplitudes in Chapter 6.

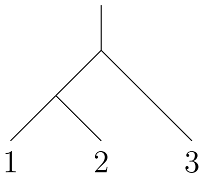
A planer cubic tree can also be regarded as a triangulation of a polygon, and mutations between tree graphs amount to 'flips' of arcs in the triangulation. Likewise, mutations between loop diagrams are described by 'flips' on more complicated surfaces with boundary. Fix such a surface with boundary. There is a graph, whose vertices are labelled by triangulations of this surface, and whose edges are the mutations between them. This graph was studied, e.g., by Penner. [66] These graphs are not regular, in general, but they can be made regular by modifying the notion of a triangulation. Chapters 7 through 10 realize these graphs as the skeletons of polytopes, and relate these polytopes to the conventional gauge theory perturbation series. The method used in the construction of these polytopes is slightly more general than is required for the problem, and the main examples use some known results from representation theory.

The rest of this introduction is a self-contained technical summary of the thesis.

1.1 Tree level

Let α be a cubic tree, with its external edges labelled by $1, 2, ..., n$. Fixing n as the root, α is a rooted tree. The tree then defines an associated bracketing of $1, 2, ..., n-1$.

For instance,



$$= [[1, 2], 3]. \tag{1.3}$$

In most of the thesis, such a bracketing is treated as a Lie monomial. The correspondence between Lie monomials and rooted trees is 2:1, because $[[1, 2], 3]$ and $-[[1, 2], 3] = [[2, 1], 3]$ are different Lie monomials that correspond to the same tree. This sign arises naturally in the computation of an amplitude.

Indeed, the tree amplitude of a gauge theory may be written (not uniquely) as a sum of the form

$$A(1, 2, ..., n) = \sum_{\alpha} \frac{n_\alpha c_\alpha}{s_\alpha}, \tag{1.4}$$

where the sum is over rooted cubic trees, α, labelled as above by $1, ..., n$. The denominator s_α is the product, over the internal edges of α, of the Mandelstam variables, s_I, associated to each edge, I. The colour factor c_α can be written as

$$c_\alpha := \mathrm{tr}(\alpha(t_1, ..., t_{n-1})t_n), \tag{1.5}$$

where the $t_i \in \mathrm{ad}_{\mathfrak{g}}$ are bracketed according to one of the two Lie monomials, $\pm\alpha$, associated to the tree. If the numerators n_α are assigned to Lie monomials in such a way that $n_{-\alpha} = -n_\alpha$, then the signs can be chosen freely for each α. The sum (1.4) can then be regarded as a sum over all pairs $\{\alpha, -\alpha\}$ of Lie monomials on the letters $1, 2, ..., n - 1$.

Kapranov [47] suggested that it should be possible to derive the main properties of tree level amplitudes directly from the properties of Lie monomials. This is carried out in Chapters 3 and 4. Write S for the ring of rational functions of Mandelstam variables. Write $\mathcal{L}$ for the vector space of Lie monomials, with coefficients in S. And write $\mathcal{L}^\vee$ for the vector space of orderings, modulo shuffle relations. $\mathcal{L}^\vee$ and $\mathcal{L}$ are dual vector spaces (see Section 3.2). Abstracting from (1.4) consider the *binary tree map*,

$$T : \mathcal{L}^\vee \to \mathcal{L}, \tag{1.6}$$

which sends a word a to the sum

$$T : a \mapsto \frac{1}{s_a} \sum_{\alpha} \frac{(a,\alpha)\alpha}{s_\alpha}, \tag{1.7}$$

where $s_a = \sum_{i,j} s_{ij}$ is the total momentum, and $(a,\alpha) = \pm 1$ is a sign. The main result is that

Proposition 1.1. *T is an isomorphism.*

Given this, T defines a Lie bracket on $\mathcal{L}^\vee$ by pulling back the Lie bracket on $\mathcal{L}$. Define the Lie bracket $\{\ ,\ \}$ on $\mathcal{L}^\vee$ by

$$T(\{a,b\}) = [T(a), T(b)], \tag{1.8}$$

for $a, b \in \mathcal{L}^\vee$. It is shown in Chapter 4 that this Lie bracket can be inductively defined by

$$\{iaj, b\} = i\{aj, b\} - j\{ia, b\}, \qquad \{a, ibj\} = \{a, ib\}j - \{a, bj\}i, \tag{1.9}$$

for letters i, j, and orderings a, b, with the base case

$$\{i, j\} = s_{ij}\, ij. \tag{1.10}$$

Write

$$S : \mathcal{L} \to \mathcal{L}^\vee, \tag{1.11}$$

for the inverse of T. For a Lie monomial α, written in its bracketed form, $S(\alpha)$ is obtained from α by replacing all brackets $[\ ,\ ]$ with braces $\{\ ,\ \}$. As shown in section 4.2, the map S, when expressed in a basis, is the 'KLT kernel'.

Since $\mathcal{L}$ and $\mathcal{L}^\vee$ are dual vector spaces, $\{\ ,\ \}$ defines a Lie cobracket on $\mathcal{L}$,

$$C : \mathcal{L} \to \mathcal{L} \otimes \mathcal{L}, \tag{1.12}$$

where $C(\alpha)$ is given by

$$(a \otimes b, C(\alpha)) := (\{a, b\}, \alpha). \tag{1.13}$$

The cobracket C has appeared in superstring calculations, and in other recent calculations of super Yang-Mills amplitudes, where it is called the *contact term map*. Properties of $\{\ ,\ \}$ proved in Chapter 4 easily imply properties of the contact term map that have been otherwise difficult to prove. For example, Section 5.4 proves that

$$C([\alpha,\beta]) = [C(\alpha),\beta] + [\alpha, C(\beta)] + s_{A,B}(\alpha\otimes\beta - \beta\otimes\alpha), \tag{1.14}$$

where α is a Lie monomial on a set A, and β is a Lie monomial on a set B, and $s_{A,B}$ is the sum $\sum_{i\in A, j\in B} s_{ij}$. The last term in (1.14) means that C does not make $\mathcal{L}$ into a Lie bialgebra.

The discussion above of the S bracket, $\{\ ,\ \}$, has a natural relation to the configuration space of $n-1$ points, Conf_{n-1}. Write $\mathrm{pt}(123...k)$ for the *broken Parke-Taylor* function

$$\mathrm{pt}(123...k) := \frac{1}{z_{12}z_{23}...z_{k-1k}}. \tag{1.15}$$

If $\mathcal{L}^\vee$ is as above, then pt defines a map from $\mathcal{L}^\vee$ to rational functions on Conf_{n-1}, because it can be shown that

$$\mathrm{pt}(a \sqcup\!\sqcup\, b) = 0. \tag{1.16}$$

The S bracket appears in this context as follows.

Lemma 1.2. Let a and b be orderings of two disjoint sets, A and B, then:

$$\mathrm{pt}(a)\, E_{A,B}\, \mathrm{pt}(b) = \mathrm{pt}(\{a,b\}), \tag{1.17}$$

where

$$E_{A,B} = \sum_{i\in A, j\in B} \frac{s_{ij}}{z_{ij}}, \tag{1.18}$$

are the *scattering equation functions.*

Define a matrix, A, by

$$A_{ij} = \begin{cases} E_{i,j} & i \neq j \\ -\sum_{k\neq i} E_{i,k} & i = j \end{cases} \tag{1.19}$$

Then the matrix tree theorem, combined with some identities that hold in $\mathcal{L}^\vee$, can be used to show that

Lemma 1.3.

$$\det A[1] = \sum_{a \in \mathfrak{S}_{n-2}} \mathrm{pt}(1a) S(1a, 1a), \tag{1.20}$$

where $A[1]$ is the matrix A with column 1 and row 1 removed.

A *CHY formula* expresses the tree amplitudes of a gauge theory as a sum of residues in $\mathcal{M}_{0,n}$. In particular, the NLSM (non-linear sigma model) amplitudes are written as a sum of residues of $\det A[1]$. Lemma 1.3 then implies that

Proposition 1.4. *The NLSM partial amplitudes have an expansion*

$$A(a, n) = \sum_\alpha \frac{(a, \alpha) N_\alpha}{s_\alpha}, \tag{1.21}$$

where the N_α are BCJ numerators given by

$$N_\alpha = \sum_{a \in \mathfrak{S}_{n-2}} (1a, \alpha) S(1a, 1a). \tag{1.22}$$

Equation (1.21) is an example of an expansion of the form (1.4), but the numerators N_α in (1.21) have the following two special properties: (i) the N_α are polynomial functions of the Mandelstam variables; (ii) the N_α define a homomorphism out of the vector space of Lie monomials. Property (ii) means that if α, β, γ are three Lie monomials that sum to zero by the Jacobi identity, then

$$N_\alpha + N_\beta + N_\gamma = 0. \tag{1.23}$$

It has been widely observed that it is always possible to find numerators which satisfy Property (ii).[1] But it is not known in general if it is possible to satisfy *both* properties

[1] This is because the off-shell KLT map, S, can be used to trivially write

$$A(a, n) = \sum_\alpha \frac{(a, \alpha)}{s_\alpha} A(S(\alpha), n). \tag{1.24}$$

The numerators, $N_\alpha = A(S(\alpha), n)$ do not typically satisfy Property (i).

at the same time. Proposition 1.4 shows that this is possible for NLSM. It is known to be possible for Yang-Mills theory, [8] [63] even though an explicit formula like (1.22) is not known (but for explicit computations along these lines see [13] and [42]). Appendix A gives a basic method for expanding amplitudes in the form (1.4) so that Property (i) is satisfied manifestly.

1.2 Loops

In introducing the 'planar limit' [76], 't Hooft pointed out that the conventional perturbative expansion of A_{YM} contains a genus expansion. A more precise version of this statement is that the terms contributing to a gauge theory amplitude can be gathered into partial amplitudes labelled by surfaces with boundary. In the case of the biadjoint scalar theory, the conventional perturbation series can be written as

$$A_{\phi^3}(1,...,n) = \sum_{\Sigma,\Sigma'} c_\Sigma\, \tilde{c}_{\Sigma'} A(\Sigma,\Sigma'), \tag{1.25}$$

where the sum is over all pairs of surfaces with n boundary marked points. The contributions $A(\Sigma,\Sigma')$ are *partial amplitudes*, and the c_Σ and $\tilde{c}_{\Sigma'}$ are *colour factors*, which depend only on the surface. The partial amplitudes are sums of the conventional Feynman integrals, I_D, associated to cubic graphs, D. A cubic ribbon graph, G, is a cubic graph with a cyclic orientation assigned to each vertex. There is a surface, Fat_G, unique up to the choice of orientation, obtained from G by 'fattening it'. Write Thin_G for the cubic graph obtained by forgetting the orientations at the vertices. The choice of orientation of Fat_G can be fixed once and for all by declaring that it is determined by the vertex attached to n, say.

Lemma 1.5.

$$A(\Sigma,\tilde{\Sigma}) = (ig)^k \text{sgn}(\Sigma,\tilde{\Sigma}) \sum_{\substack{G\ s.t.\\ \text{Fat}_G=\Sigma}} \sum_{\substack{G'\ s.t.\\ \text{Fat}_{G'}=\Sigma',\\ \text{Thin}_{G'}=\text{Thin}_G}} I_{\text{Thin}_G}, \tag{1.26}$$

where $\mathrm{sgn}(\Sigma, \tilde{\Sigma})$ *is a sign that is* $+1$ *when* $\Sigma = \tilde{\Sigma}$.[2]

One of the original aims of string theory was to compute $A(\Sigma, \tilde{\Sigma})$ in terms of the surfaces themselves, and not as a sum of Feynman diagrams. We take up this aim here. One basic motivating observation is that $A(\Sigma, \tilde{\Sigma})$ can be written as a sum over triangulations of Σ, because:

Lemma 1.6. *There is a 1:1 correspondence between triangulations of* Σ*, and labelled cubic ribbon graphs that fatten to* Σ*.*

There is again a subtlety involving signs, and the overall orientation of Σ in establishing the correspondence. The second basic observation is that the Feynman integrals appearing in $A(\Sigma, \tilde{\Sigma})$ can be summed without performing the integration. The Feynman integral associated to a cubic graph D (before Feynman parameterization) may be written in the form

$$I_D = \int \mathrm{d(loops)} \prod_{I \in P(D)} \frac{1}{X_I}, \tag{1.27}$$

where the product is over all internal edges of D, and each X_I is a Lorentz invariant expression of the form P^2, for some momentum P^μ. The integral I_D is ill defined for divergent diagrams D (those diagrams with $\#$ loop propogators $\leq 2\#$ loops), and dimensional regularization or a cut-off should be used. Choices are involved when assigning momenta P^μ to internal edges of D. But (Chapter 8):

Lemma 1.7. *Fix a bordered surface* Σ*, with marked points* $\mathbb{P}$*, and triangulation* α*. The arcs of* α*, when oriented, represent classes that span* $H_1(\Sigma, \mathbb{P})$*. A consistent assignment of momenta,* P^μ*, to the edges of* α*, is a homomorphism* $h : H_1(\Sigma, \mathbb{P}) \to \mathcal{V}$*, where* $\mathcal{V}$ *is the free vector space spanned by the external data,* K_i^μ*, for each marked point* i*, and* ℓ *variables of integration* P_i^μ*. For any arc,* I*, representing a class* $[I] \in H_1(\Sigma, \mathbb{P})$*, write*

$$X_I = h([I])^2. \tag{1.28}$$

[2]If $\Sigma \neq \tilde{\Sigma}$, then take any G (which fattens to Σ) and G' (which fattens to Σ') such that $\mathrm{Thin}_G = \mathrm{Thin}_{G'}$. The sign $\mathrm{sgn}(\Sigma, \tilde{\Sigma})$ counts mod 2 the number of vertices where G and G' have different orientations, given that they have the same orientation at the vertex attached to n.

Then, for any triangulation of Σ*,* β*, the associated Feynman integral is*

$$I_\beta = \int \left(\prod_{punctures,\ i} dP_i \right) \prod_{I \in P(\beta)} \frac{1}{X_I}, \tag{1.29}$$

where the product is over all edges, I*, of* β*.*

Postponing the details of the loop variable integration (and regularization), it is thus natural to write the partial amplitude $A(\Sigma, \Sigma)$ as

$$A(\Sigma, \tilde{\Sigma}) = \int \mathrm{d}(\mathrm{loops})\, f(\Sigma, \tilde{\Sigma}), \tag{1.30}$$

where $f(\Sigma, \tilde{\Sigma})$ is a rational function of the X_I variables, and d(loops) is the measure from the Lemma. It is the object of Chapter 10 to compute $f(\Sigma, \Sigma)$ for a family of surfaces, Σ.

At tree level, Σ is a disk, and it is observed in [2] that $f(\Sigma, \Sigma)$ can be computed as the volume of the dual of a polytope, P, the *ABHY polytope*. P is isomorphic as a polytope to the associahedron. But the special property of P is that the normal vectors around each vertex, with specific normalisations, have determinant ± 1: and this property is equivalent to the statement that $f(\Sigma, \Sigma)$ is the volume of the dual of P. [5] shows that P has a natural description in terms of the category of finite dimensional representations of A_n. Chapter 7 generalizes this, and Chapter 10 applies this construction to computing $f(\Sigma, \Sigma)$ when Σ is an annulus or a twice punctured disk.

The appearance here of representation theory warrants explanation. A polytope is the intersection of half-planes $f_i \geq c_i$, for some linear functions f_i. The linear functions f_i of the ABHY associahedron can be interpreted as planes normal to g-vectors, which are defined for any triangulated marked surface (Chapter 9). However, not all choices of the constants c_i give rise to the same polytope, and it is problem to determine all possible c_i which give rise to a polytope with the desired face lattice. The homological algebra approach of Chapter 7 (summarized below) is one natural way to find the constraints on the constants c_i.

For a triangulated marked surface, Σ, there is a quiver, Q, which is 'dual' to the triangulation of Σ (described in Chapter 9), and a quiver path algebra kQ (or kQ/W).

The modules of these quiver path algebras, mod$-kQ/W$, have been well studied as a large class of examples in the representation theory of Artin algebras. [4] In addition to the category mod$-kQ/W$, there is a closely related category $\mathcal{C}_Q$ defined by [18]. In the particular case that Q arises from a marked surface, there is a correspondence between indecomposable objects of $\mathcal{C}_Q$ and arcs on Σ. [1] In this case, the generalized ABHY polytopes of $\mathcal{C}$ contain information about the triangulations of Σ. Proposition 1.9 below, for example, is related to what happens to the surface Σ when an arc cuts it into two disjoint surfaces.

The construction of generalised ABHY polytopes is as follows. Let $\mathcal{C}$ be a category of finite dimensional modules of some finite dimensional Artinian algebra. Let $K_0(\mathcal{C}, 0)$ be the vector space spanned by $[N]$, for $N \in \text{ind}-\mathcal{C}$ the indecomposable objects. The Grothendieck group is the quotient $K_0(\mathcal{C}, 0)/I$, where I is the subspace spanned by the vectors $[A] - [E] + [B]$ for all short exact sequences $A \hookrightarrow E \twoheadrightarrow B$. An object $M \in \mathcal{C}$ defines two vectors in $K_0(\mathcal{C}, 0)$,

$$v_M = \sum_{N \in \text{ind}-\mathcal{C}} \text{hom}(N, M)\,[N], \qquad \text{and} \qquad {}_M v = \sum_{N \in \text{ind}-\mathcal{C}} \text{hom}(M, N)\,[N]. \tag{1.31}$$

The ABHY hyperplane, H_M, is the following affine translation of the subspace I,

$$H_M = I + {}_M v + v_M \subset K_0(\mathcal{C}, 0). \tag{1.32}$$

Write V^+ for the positive orthant of $V = K_0(\mathcal{C}, 0)$, defined by the linear hull of the vectors $+[N]$ for all $N \in \text{ind}-\mathcal{C}$. Then the intersection

$$P_{\mathcal{C},M} = H_M \cap V^+ \tag{1.33}$$

is some (possibly unbounded) generalized ABHY polytope.

For $N \in \text{ind}-\mathcal{C}$, let $[N]^\perp$ be the subspace orthogonal to $[N]$. Write $\mathcal{C}|_N$ for the full subcategory of $\mathcal{C}$ defined by the vanishing of both $\text{Ext}^1(\ , N)$ and $\text{Ext}^1(N,\ )$.

Proposition 1.8. (Faces of P.) *Let $P_{\mathcal{C},M}$ be the polytope associated to $M \in \mathcal{C}$. The face $[A]^\perp \cap P$ is isomorphic to the polytope $P_{\mathcal{C}|_A}$ associated to a certain $M' \in \mathcal{C}|_A$. This determines the face lattice of $P_{\mathcal{C}}$.*

Given a full subcategory $\mathcal{D} \subset \mathcal{C}$, the (left-and-right) orthogonal subcategory, $\mathcal{D}^{\perp}$, is the subcategory with objects $B \in \mathcal{C}$ such that

$$\text{hom}(A, B) = 0, \;\; \text{hom}(B, A) = 0, \;\; \text{ext}^1(A, B) = 0, \;\; \text{ext}^1(B, A) = 0, \tag{1.34}$$

for all $A \in \mathcal{D}$. A full subcategory $\mathcal{D} \subset \mathcal{C}$ is called (left-and-right) *admissible* if

$$\mathcal{D} = (\mathcal{D}^{\perp})^{\perp}. \tag{1.35}$$

Call an admissible subcategory $\mathcal{D} \subset \mathcal{C}$ an *island of* M if $\mathcal{D}$ and $\mathcal{D}^{\perp}$ partition the indecomposable subobjects of M into two parts.

Proposition 1.9. (Factorization property.) *Let $P_{\mathcal{C},M}$ be the polytope associated to $M \in \mathcal{C}$. If $\mathcal{D} \subset \mathcal{C}$ is an island, then $P_{\mathcal{C},M} \simeq P_{\mathcal{D}} \times P_{\mathcal{D}^{\perp}}$, the cross product of the polytopes associated to modules $M_1 \in \mathcal{D}$ and $M_2 \in \mathcal{D}^{\perp}$ (with $M = M_1 \oplus M_2$).*

Example 1.10. (A_n.) *In the case of the n-gon, there is a category $\mathcal{C}$ with one indecomposable object, I, for every diagonal, I, of the n-gon. In $\mathcal{C}$,* $\text{Ext}^1(I, J)$ *or* $\text{Ext}^1(J, I)$ *is non-zero iff the arcs I and J cross. The associated polytope, $P_{\mathcal{C}}$, is an associahedron. The face $H_I \cap P_{\mathcal{C}}$ is isomorphic to $P_{\mathcal{C}[I]}$. But $\mathcal{C}[I]$ contains only those arcs which do not cross I. These form two islands, $\mathcal{D}$ and $\mathcal{D}^{\perp}$, which are the arcs on each of the two sides of I. So the face $H_I \cap P_{\mathcal{C}}$ is isomorphic to the cross product of two associahedra $P_{\mathcal{D}}$ and $P_{\mathcal{D}^{\perp}}$.*

Further families of examples of generalized ABHY polytopes are given in Chapter 10, corresponding to the two-punctured disk, and the annulus. In these cases, the description of the appropriate category $\mathcal{C}$ can be adapted from results in the representation theory of hereditary algebras.

Relation to papers. Some material in this thesis has appeared elsewhere in articles and preprints. Chapters 3 and 4 are partially presented in [40]. Sections 6.2 and 6.3 together elaborate the discussion in [41]. Chapter 5 has been partially published as [39]. Chapters 7 to 10 are subsidiary to an ongoing joint project with Nima Arkani-Hamed, Giulio Salvatori, Hugh Thomas, and Pierre-Guy Plamandon. Chapter 8 is mostly folklore, although some of it is also reviewed in [39].

Chapter 2

Tree partial amplitude identities

The amplitudes of a coloured quantum field theory like Yang-Mills are organised by the Lie algebra structure constants into contributions called 'partial amplitudes'. The leading order contribution comes from tree Feynman diagrams. The YM tree amplitude involving n gluons (in the adjoint representation of $\mathfrak{su}(N)$) is conventionally written as

$$A_{YM}(1,2,...,n) = \sum_{a \in \mathfrak{S}_{n-1}} A(a,n) \operatorname{tr}\left(t^{a(1)} t^{a(2)} ... t^{a(n-1)} t^{n}\right), \tag{2.1}$$

where each partial tree amplitude, $A(a,n)$, is labelled by a permutation, a, of $1,2,...,n-1$. The partial amplitudes $A(a,n)$ satisfy linear identities that are awkward to prove from Feynman diagrams. Some of these identities follow directly from elementary facts about colour structures, as explained in Section 2.1. The BCJ and KLT identities, reviewed in Section 2.2, require more analysis. This chapter concludes by summarizing the observations about tree amplitudes that partially motivate the thesis.

2.1 Lie identities

The $U(1)$ *decoupling identity* is[1]

$$A(i \sqcup\!\sqcup\, a, n) = 0. \tag{2.2}$$

[1]The shuffle product, $\sqcup\!\sqcup$, is defined in Section 3.1.

This identity heuristically follows by replacing an external gluon with a photon. A possible derivation of (2.2) is from string theory. With an appropriate choice of convention for the integration cycles, the open string worldsheet integrals, $\mathcal{A}(a,n)$, satisfy the following monodromy relations,

$$\mathcal{A}(ai,n) + \sum_{a=bc} e^{-2\pi i\alpha' s_{i;c}} \mathcal{A}(bic,n) = 0, \tag{2.3}$$

for appropriate choices of contours and branches. [67] proves (2.3) by constructing a contractible integration contour that, choosing an appropriate branch of the multivalued integrand, reproduces the LHS. The Yang-Mills tree amplitude is recovered from the RNS open string gluon amplitude by evaluating the limit

$$A_{\mathrm{YM}}(a,n) = \lim_{\alpha'\to 0} (\alpha')^{n-3} \mathcal{A}(a,n). \tag{2.4}$$

Applying this limit to (2.3) gives,

$$\lim_{\alpha'\to 0} (\alpha')^{n-3} \mathcal{A}(i \sqcup\!\sqcup a, n) = 0, \tag{2.5}$$

which is the $U(1)$ decoupling identity. The $U(1)$ decoupling identity is a special case of the following relation,

$$A(a \sqcup\!\sqcup b, n) = 0, \tag{2.6}$$

which holds for all non-empty disjoint words a and b. It follows from Lemma 3.5, below, that (2.6) is equivalent to the Kleiss-Kuijf relations,

$$A(aib,n) = (-1)^{|a|} A(i(\overline{a} \sqcup\!\sqcup b), n), \tag{2.7}$$

which were observed to hold in examples in [51]. The rest of this section gives an elementary proof of (2.6) for Yang-Mills, and hence also of (2.7).

The relations (2.6) hold because the *colour factors* appearing in the computation of the amplitude satisfy the Jacobi identities. Equation (2.6) is then a consequence of Ree's theorem. To state Ree's theorem in this context, it is helpful to make the following definitions. *Lie monomials* are total bracketings of distinct letters. For example,

$$[1,[2,3]] = 123 - 132 - 231 + 321 \tag{2.8}$$

is a Lie monomial on the set $\{1,2,3\}$. Let L_{n-1} be the subspace of $\{1,...,n-1\}^*$ given by the linear span of all the Lie monomials. For example, L_3 is the linear span of $[1,[2,3]]$, $[2,[3,1]]$, and $[3,[1,2]]$, which is a 2 dimensional vector space. The vector spaces L_n are discussed in greater detail in Section 3.2. For each Lie monomial $\alpha \in L_{n-1}$, there is a colour factor

$$c_\alpha := ig^{n-2}\mathrm{tr}(\alpha[t^{a_1},...,t^{a_{n-1}}]t^{a_n}), \tag{2.9}$$

where $\alpha[t^1,...,t^{n-1}]$ is the Lie algebra element in $\mathrm{ad}_\mathfrak{g}$ obtained from the map ${\mathrm{ad}_\mathfrak{g}}^{\otimes n-1} \to \mathrm{ad}_\mathfrak{g}$ induced in the obvious way by α. Expanding the commutators in c_α gives a sum of the *trace factors* that appear in the partial amplitude expansion (2.1) above. Write

$$\mathrm{tr}(a,n) := ig^{n-2}\mathrm{tr}(t^{a_1}...t^{a_{n-1}}t^{a_n}), \tag{2.10}$$

for these trace factors.

Lemma 2.1. (Ree's theorem.) *Any sum of the form*

$$\sum_\alpha A_\alpha c_\alpha, \tag{2.11}$$

over some set of Lie monomials $\alpha \in L_{n-1}$, may be expanded as,

$$\sum_{a\in\mathfrak{S}_{n-1}} A(a)\mathrm{tr}(a,n). \tag{2.12}$$

The coefficients in this expansion, $A(a)$, satisfy

$$A(b \sqcup\!\sqcup c) = 0. \tag{2.13}$$

This Lemma can be used to show that the Yang-Mills partial tree amplitudes, $A(a,n)$, satisfy

$$A(b \sqcup\!\sqcup c, n) = 0, \tag{2.14}$$

for all non-trivial shuffle sums $b \sqcup\!\sqcup c$. Indeed, it suffices to show that the tree amplitude may be written in the form

$$A_{\text{YM}}(1, .., n) = \sum_{\alpha} A_\alpha \, c_\alpha, \tag{2.15}$$

for some coefficients A_α. Conventionally, $A_{\text{YM}}(1, ..., n)$ is written as a sum of Feynman diagram contributions,

$$A_{\text{YM}}(1, ..., n) = \sum_{D} A_D. \tag{2.16}$$

The Feynman rules are given as (A.5) and (A.6) in Appendix A. Fix D a cubic Feynman diagram. D can be regarded as a *rooted tree* by fixing leg n as the root. Let $\alpha \in L_{n-1}$ be a Lie monomial corresponding to T. Then the cubic vertex Feynman rule gives that the colour factor associated to D is given by $\pm c_\alpha$. Next, suppose D is a Feynman graph with one or more quartic vertices. The Feynman rule contribution of each quartic vertex is a sum of 3 terms, each proportional to $f^{a_1 a_2 c} f^{a_3 a_4 c}$, where the sum is over cyclic permutations of $1, 2, 3$ (as in equation (A.6)). The term $f^{a_1 a_2 c} f^{a_3 a_4 c}$ is the colour factor of a cubic graph. In this way, a Feynman graph D with k quartic vertices gives a sum of 3^k contributions, each proportional to a colour factor c_α.

2.2 The fundamental BCJ and KLT relations

The subleading terms in the Taylor expansion of (2.3) imply a further relation amoung Yang-Mills partial amplitudes, called the *fundamental BCJ relation*, [12]

$$\sum_{a=bc} s_{i;c} \, A(bic, n) = 0. \tag{2.17}$$

Due to the Lie identities satisfied by $A(a,n)$, there is no unique way to write (2.17). For example, the KK relation implies that the LHS of (2.17) is equal to

$$\sum_{a=bjc} s_{ij} A(ij(\bar{b} \sqcup c), n). \tag{2.18}$$

An invariant definition of the fundamental BCJ relations is studied in Section 4.1. The relations were originally conjectured in a different form in [9], and proved using a Grassmannian integral formula for $A(a,n)$ in [19]. An alternative proof follows Proposition 4.2.

The KLT relation, proposed in [48], expresses closed string integrals as quadratic expressions in open string integrals. It expresses the closed string amplitude $\mathcal{M}(1,...n)$ as a sum,

$$\mathcal{M}(1,...n) = \sum_{a,b\in\mathfrak{S}_{n-2}} \mathcal{A}(1a,n)\mathcal{S}(a,b)_1\mathcal{A}(1b,n), \tag{2.19}$$

for a matrix of coefficients $\mathcal{S}(a,b)_1$ which are periodic functions of the Mandelstam variables s_I. A formula for the components $\mathcal{S}(a,b)_1$ is given in [11]. These components can be computed as the inverse of a matrix of intersection pairings. [17] [62] The field theory limit, as in (2.4), of the KLT relation, (2.19), gives a relation between the gravity tree amplitude, $M(1,...,n)$, and the YM partial tree amplitudes, $A(a,n)$. [6] The explicit formula, given in [11], is

$$M(1,...,n) = \lim_{s_{12...n-1}\to 0} \frac{1}{s_{12...n-1}} \sum_{a,b\in\mathfrak{S}_{n-2}} A(1a,n)S(a,b)_1A(b1,n), \tag{2.20}$$

where the field theory KLT matrix is

$$S(a,b)_1 := \prod_{i=2}^{n-1} \left(\sum_{\substack{k<_{1a}i \\ k>_{1b}i}} s_{ik} \right). \tag{2.21}$$

For fixed i, the sum is over all k that precede i in $1a$, and that are preceded by i in $1b$.

2.3 The BCJ conjectures

As observed in Section 2.1, the YM tree amplitude may be written

$$A_{\mathrm{YM}}(1,...,n) = \sum_{\alpha} A_\alpha \, c_\alpha, \tag{2.22}$$

where the sum is over some set of Lie monomials (or rooted binary trees) $\alpha \in L_{n-1}$. Let α be such a monomial. Let $P(\alpha) \subset \mathcal{P}_{n-1}$ be the set of propagators of α, and let

$$\frac{1}{s_\alpha} := \prod_{I \in P(\alpha)} \frac{1}{s_I}, \tag{2.23}$$

be the product of propagator factors (also defined in Section 3.3). Then (2.22) may be written

$$A_{\mathrm{YM}}(1,...,n) = \sum_{\alpha} \frac{N_\alpha \, c_\alpha}{s_\alpha}, \tag{2.24}$$

where $N_\alpha := s_\alpha A_\alpha$. The N_α are not unique. Given three Lie monomials related by the Jacobi identity, $\alpha + \beta + \gamma = 0$, the corresponding colour factors satisfy

$$c_\alpha + c_\beta + c_\gamma = 0. \tag{2.25}$$

It follows that the replacement

$$A_\alpha \to A_\alpha + C, \qquad A_\beta \to A_\beta + C, \qquad A_\gamma \to A_\gamma + C,$$

does not change the sum, (2.24). For the $n = 4$ amplitude, [30] found numerators N_α such that

$$N_{[[1,2],3]} + N_{[[2,3],1]} + N_{[[3,1],2]} = 0. \tag{2.26}$$

It was conjectured in [9] that, for $n > 4$, a choice of N_α always exists, so that (2.26) holds for all Jacobi triples of Lie monomials, and so that the N_α do not have any poles in the Mandelstam variables. Call such a choice of N_α *local BCJ numerators.* In the case of YM, it is known that local BCJ numerators exist, [8] [63] even if explicit formulas for the numerators have not yet been exhibited (but see [13] and [42] for some explicit computations). Using facts from Chapters 3 and 4, the local BCJ numerators for the nonlinear sigma model are found in Section 6.2.

A key motivation for studying local BCJ numerators is that they can be used to obtain compact formulas for gravity tree amplitudes, $M(1,...,n)$:

Lemma 2.2. *If N_α are BCJ numerators for YM, then the formula*

$$M(1,...,n) = \sum_{\alpha} \frac{N_\alpha N_\alpha}{s_\alpha} \tag{2.27}$$

gives the tree amplitudes for gravity.

Proof. A Lie monomial α has an expansion

$$\alpha = \sum_{a\in\mathfrak{S}_{n-2}} (\alpha, 1a)\ell(1a), \tag{2.28}$$

where $\ell(1a)$ is the left bracketing of $1a$ (see Definition 3.2). If N is a homomorphism out of L_{n-1}, (2.24) may therefore be written as

$$A_{\mathrm{YM}}(1,...,n) = \sum_{a\in\mathfrak{S}_{n-2}} N_{\ell(1a)} \sum_{\alpha\in L_{n-1}} \frac{(\alpha,1a)\, c_\alpha}{s_\alpha}. \tag{2.29}$$

The partial amplitudes, $A(a,n)$, can then be expanded as

$$A(a,n) = \sum_{b\in\mathfrak{S}_{n-2}} N_{\ell(1b)}\, m(1b,n|a,n), \tag{2.30}$$

where $m(1b,n|a,n)$ are the biadjoint scalar partial tree amplitudes. It is known that the field theory KLT matrix, $S_1(a,b)$, inverts the matrix $m(1a,n|1b,n)$, in the sense that

$$\lim_{s_{12...n-1}\to 0} \frac{1}{s_{12...n-1}} \sum_c m(1a,n|1c,n)S_1(c,b) = (a,b). \tag{2.31}$$

A new proof of (2.31) is given in Section 4.2. (2.27) then follows from the KLT relation, (2.20). □

In addition to the conjecture that gauge theory partial amplitudes have a BCJ form, it has been widely conjectured that local BCJ numerators, if they exist, can be expressed as inner products of the form

$$\mathrm{tr}(\alpha[x_1, ..., x_{n-1}], x_n), \tag{2.32}$$

for some fixed elements x_i of a Lie algebra, and [,] some Lie bracket. Such a Lie algebra has not been exhibited in practice, but is referred to as a *kinematic algebra*. Appendix A gives a method for writing partial amplitudes with numerators in the form (2.32), but the bracket the appears in these formulas is not a Lie bracket.

Chapter 3

Orderings and Mandelstam Variables

This section gives the basic results needed for Chapter 4. For a subset $A \subset \mathbb{N}$, an *ordering* of A is a word that uses each letter in A exactly once. Section 3.2 then defines the Lie algebra, $L(A)$, whose elements correspond to binary trees with the leaves labelled by A, and all labels distinct. Section 3.4 defines the ring of Mandelstam variables on A, R_A. This is the ring associated to the Mandelstam variables, which are defined as

$$s_I = \left(\sum_{i \in I} k_i \right)^2, \tag{3.1}$$

for null momentum vectors k_i. Chapter 4 then studies the Lie algebra $\mathcal{L}(A) = L(A) \otimes_k R_A$, over R_A, and proves the identities conjectured in [55] as part of a more general discussion of the properties of $\mathcal{L}(A)$. The properties of $\mathcal{L}(A)$ imply elementary derivations of the field theory KLT and fundamental BCJ relations.

3.1 Free algebras and orderings

Fix a subset $A \subset \mathbb{N}$, and let $k \langle A \rangle$ be the free associative algebra on A. As a vector space, $k \langle A \rangle$ is the free k vector space over the free monoid A^*. Concatenation of words in A^* induces an associative product on $k \langle A \rangle$. For words a and b in A^*, write ab for their concatenation. Write e for the empty word, so that $ae = ea = a$. Denote the pairing between $k \langle A \rangle$ and its dual $k \langle A \rangle^\vee$ by $(\ , \ )$. Write a for the element in

$k\langle A\rangle^\vee$ that is dual to the word $a \in A^*$. That is, for $b \in A^*$,

$$(a,b) = \begin{cases} 1 & a = b, \\ 0 & a \neq b. \end{cases} \tag{3.2}$$

The *shuffle product* is a commutative product, with unit e, defined inductively by

$$(ai) ⧢ (bj) = (a ⧢ bj)i + (ai ⧢ b)j, \tag{3.3}$$

for words $a, b \in A^*$ and letters $i, j \in A$. The shuffle product induces a coproduct, $\delta_⧢$, on $k\langle A\rangle$ defined by

$$(a \otimes b, \delta_⧢ c) := (a ⧢ b, c). \tag{3.4}$$

The dual statement of (3.3) is

$$\delta_⧢(ai) = \delta_⧢(a) \cdot \delta_⧢(i), \tag{3.5}$$

where concatenation is extended to a product on $k\langle A\rangle \otimes k\langle A\rangle$, such that

$$(a \otimes b) \cdot (c \otimes d) = ac \otimes bd. \tag{3.6}$$

Inductively, (3.5) implies that

$$\delta_⧢(ab) = \delta_⧢(a) \cdot \delta_⧢(b). \tag{3.7}$$

This property, together with some further conditions on the unit and counit, means that $k\langle A\rangle$ satisfies the axioms of a bialgebra. It is in fact a Hopf algebra, with antipode

$$a \mapsto \tilde{a} := (-1)^{|a|}\overline{a}, \tag{3.8}$$

where $|a|$ is the length of a, and $\overline{a}$ is the reverse of a.

Definition 3.1. (Orderings.) *An ordering of A is a word that uses each letter $i \in A$ exactly once. Write $\mathfrak{S}(A) \subset A^*$ for the set of orderings of A. Write $W(A) \subset k\langle A\rangle$*

for the vector space

$$W(A) = \bigoplus_{B \subset A} k\mathfrak{S}(B). \tag{3.9}$$

$W(A)$ is not a subalgebra of $k\langle A\rangle$. It can be made into an algebra as follows. Let the product of $a \in W(A)$ and $b \in W(A)$ be their concatenation, ab, unless $ab \notin W(A)$, in which case it is 0. This defines an associative product on $W(A)$, with unit e. The shuffle product on $k\langle A\rangle$ induces a shuffle product on $W(A)^\vee$, and it satisfies (3.7).

3.2 Lie monomials

For $A \subset \mathbb{N}$, let $\mathrm{Lie}\langle A\rangle$ be the free Lie algebra on A, over the field k. There is an inclusion $\mathrm{Lie}\langle A\rangle \hookrightarrow k\langle A\rangle$. Identifying $\mathrm{Lie}\langle A\rangle$ with its image under this inclusion, it can be regarded as a subspace,

$$\mathrm{Lie}\langle A\rangle \subset k\langle A\rangle. \tag{3.10}$$

Define the *restricted Lie algebra* on A as the subspace

$$L(A) := \mathrm{Lie}\langle A\rangle \cap W(A), \tag{3.11}$$

i.e. $L(A)$ restricts to the span of those Lie monomials that contain no repeated letters. $L(A)$ is not a subalgebra of $\mathrm{Lie}\langle A\rangle$, but the product on $W(A)$ induces a Lie bracket on $L(A)$. For Lie monomials $\beta, \gamma \in L(A)$, the Lie bracket of β and γ is given by the commutator, $[\beta, \gamma]$, unless $[\beta, \gamma] \notin L(A)$, in which case the Lie bracket is zero.

The rest of this section proves properties of $L(A)$ and its dual, $L(A)^\vee$. Standard theorems about $\mathrm{Lie}\langle A\rangle$ (as in, e.g. [70]) imply most of these properties, but this section gives simpler self-contained proofs that don't invoke these theorems. The formulas that appear here are used frequently in Chapter 4.

Definition 3.2. *The left-bracketing map $\ell : W(A) \to L(A)$ is the linear map which sends a word $a \in W(A)$ to its full left-bracketing:*

$$\ell(123...n) := [[[1, 2], 3], ..., n]. \tag{3.12}$$

The right-bracketing map, r, is defined in an analogous way, and it is related to ℓ by

$$r(a) = -(-1)^{|a|}\ell(\overline{a}), \tag{3.13}$$

where $\overline{a}$ is the reverse word of a.

Lemma 3.3. (Dynkin-Specht-Wever.) *The map ℓ surjects onto $L(A)$.*

Proof. The identity,

$$\ell(ai) = [\ell(a), i], \tag{3.14}$$

implies that ℓ satisfies (inducting on the length of b, with a fixed)

$$\ell(a\ell(b)) = [\ell(a), \ell(b)]. \tag{3.15}$$

In other words, $[\ell(a), \ell(b)] \in \mathrm{Im}(\ell)$. Let $\alpha = [\beta, \gamma]$. If β and γ are in the image of ℓ, then α is too, by (3.15). □

Corollary 3.4. *For homogeneous $\alpha \in W(A)$, α is Lie iff $\ell(\alpha) = |\alpha|\alpha$.*

Proof. If $\alpha \in L(A)$ is a Lie monomial, then $\alpha = [\beta, \gamma]$, for some Lie monomials β and γ. By (3.15),

$$\ell([\beta, \gamma]) = [\ell(\beta), \gamma] + [\beta, \ell(\gamma)]$$

So $\ell(\alpha) = |\alpha|\alpha$, by induction. Conversely, for $\alpha \in W(A)$, if $\ell(\alpha) = |\alpha|\alpha$, then α is in the image of ℓ. □

Lemma 3.3 implies that $L(A)$ is the linear span of the Lie monomials $\ell(a)$ for $a \in W(A)$. The set of $\ell(a)$ is linearly dependent because of the Jacobi relation. For a letter $i \in A$ and a word $a \in \mathfrak{S}(A \backslash i)$, the left-bracketing of ia is given by the explicit formula (given in, e.g., [71])

$$\ell(ia) = \sum_{b,c} (-1)^{|b|}(b \sqcup c, a)\overline{b}ic. \tag{3.16}$$

This formula can be proved by showing that the RHS satisfies (3.14). Let ℓ^* be the adjoint of ℓ, with respect to the inner product $(\ ,\ )$ on $W(A)$ and its dual. The dual of (3.16) implies that $\ell^*(a)$ is explicitly given by

$$\ell^*(a) = \sum_{a=bic} (a, bic)(-1)^{|b|} i(\bar{b} \sqcup c), \tag{3.17}$$

for some $a \in \mathfrak{S}(A)$. A direct computation gives

$$\ell^*(a \sqcup b) = 0, \tag{3.18}$$

for nonempty and disjoint words $a, b \in W(A)$.

Lemma 3.5. (Ree, Kleiss-Kuijf.) *For homogeneous $\alpha \in W(A)$, the following are equivalent:*

1. *α is Lie*
2. *$\alpha = \ell(a)$ for some $a \in W(A)$*
3. *$(a \sqcup b, \alpha) = 0$, for all nonempty and disjoint words $a, b \in W(A)$*
4. *$(bic, \alpha) = (i(\tilde{b} \sqcup c), \alpha)$, for words $bic \in W(A)$, with $i \in A$ a letter*

Proof. Lemma 3.3 gives $1 \to 2$. Equation (3.18) gives $2 \to 3$. By induction,

$$bic - i(\tilde{b} \sqcup c) = -\sum_{b=xy} \tilde{x} \sqcup (yic),$$

which gives $3 \to 4$. Finally, if α satisfies 4, then, by (3.17),

$$(\ell^*(a), \alpha) = |\alpha|(a, \alpha),$$

for all a. So $\ell(\alpha) = |\alpha|\alpha$ and α is Lie by Corollary 3.4. □

The nontrivial shuffles, $a \sqcup b \in W(A)^\vee$, with a and b nonempety, span a subspace,

$$\mathrm{Sh}(A) \subset W(A). \tag{3.19}$$

Lemma 3.5 implies that $L(A)$ is the orthogonal complement of $\mathrm{Sh}(A)$ with respect to $(\, , \,)$. The dual vector space $L(A)^\vee$ is therefore

$$L(A)^\vee := W(A)/\mathrm{Sh}(A). \tag{3.20}$$

For $a, b \in L(A)^\vee$, write

$$a \sim b$$

if $a - b \in \mathrm{Sh}(A)$. For example, for homogeneous $a \in L(A)^\vee$,

$$a \sim -(-1)^{|a|}\overline{a}, \tag{3.21}$$

which follows from (3.13). This is a special case of

$$xjy \sim j(\tilde{x} \sqcup\!\sqcup y) \sim (x \sqcup\!\sqcup \tilde{y})j, \tag{3.22}$$

which follows from part (4) of Lemma 3.5.

$L(A)$ is graded by word length. Write

$$L^+(A) := L(A) \cap k.\mathfrak{S}(A) \tag{3.23}$$

for the highest weight component of $L(A)$. $L^+(A)$ is linearly spanned by Lie monomials $\alpha \in L(A)$ with length $|A|$. Let A be ordered according to the usual ordering on $\mathbb{N}$. A word $a \in W(A)$ is *Lyndon* if it begins with its smallest letter.[1]

Lemma 3.6. (Basis.) *The Lie monomials $\ell(a)$, for all Lyndon words $a \in W(A)$, are a basis of $L(A)$. The dual basis is given by the $a + \mathrm{Sh}(A) \in L^\vee(A)$, for Lyndon words $a \in W(A)$.*[2] *In particular,*

$$\dim L^+(A) = (|A| - 1)!.$$

[1]A word $a \in A^*$ is called Lyndon iff $a < c$ in the lexicographic ordering for every proper factorization $a = bc$. For a word $a \in W(A)$ that has no repeated letters, this is the same as the condition that a begins with its smallest letter.

[2]This is implied by a less trivial statement about $\mathrm{Lie}(A)$. It is a result of [68] that $k\langle A\rangle$, with the shuffle product, has a multiplicative basis given by the Lyndon words. It follows that a basis of $\mathrm{Lie}\langle A\rangle^\perp$ is given by the Lyndon words.

Proof. Fix A and let $i \in A$ be the smallest letter. The words ia, for $a \in \mathfrak{S}(A \backslash i)$, span the dual of $L^+(A)$, by part (4) of Lemma 3.5. But no linear combination of words ia can belong to $\mathrm{Sh}(A)$, so these words give a basis. Moreover,

$$(ia, \ell(ib)) = (a, b),$$

for $a, b \in \mathfrak{S}(A \backslash i)$. □

3.3 Rooted binary trees

The Lie algebra $L(A)$ is related to binary trees. A *rooted binary tree*, T, is a cubic tree with a marked external leg, the root. The other external legs of T are its *leaves.* A labelling of T by a set A is an assignment of elements of A to the leaves of T. A *plane rooted binary tree* is a rooted binary tree with a planar embedding. Given two plane rooted binary trees, T and T', there is a plane rooted tree,

$$T \vee T', \tag{3.24}$$

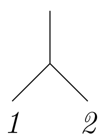
the *grafting* of T and T', that joins the roots of the two trees, with T on the left, and T' on the right.

Definition 3.7. *Given $\alpha \in L(A)$ and a word $a \in A^*$ such that $(a, \alpha) = +1$, there is a plane rooted binary tree, $T(\alpha, a)$, fixed inductively by*

$$T([\alpha, \beta], ab) = T(\alpha, a) \vee T(\beta, b), \tag{3.25}$$

with the base case

$$T_{[1,2],12} =$$



(3.26)

The grafting operation on rooted binary trees (without plane embedding) is commutative.

Definition 3.8. *For* $\alpha \in \mathrm{Lie}\langle A\rangle$*, write* T_α *for the rooted binary tree defined inductively by*

$$T_{[\alpha,\beta]} := T_\alpha \vee T_\beta = T_\beta \vee T_\alpha. \tag{3.27}$$

Note that $T_\alpha = T_{-\alpha}$*. Conversely, each rooted binary tree,* T*, with distinct labels in* A*, arises from two Lie monomials,* $\pm\alpha$*.*

Given $\alpha \in L(A)$, the leaves of T_α are labelled by mutually distinct elements of A. This means that it is possible to uniquely label the internal edges of T_α using subsets of A. Label each edge in T_α by the subset $I \subset A$ of leaves which are below that edge in the tree. Let $P(\alpha) \subset \mathcal{P}A$ be the set of all internal edges of α. For example,

$$P([1,2]) = \emptyset, \qquad \text{and} \qquad P([[1,2],3]) = \{\{1,2\}\}. \tag{3.28}$$

It is clear that

$$P([\alpha,\beta]) = \{A,B\} \cup P(\alpha) \cup P(\beta), \tag{3.29}$$

for $\alpha \in L^+(A)$ and $\beta \in L^+(B)$.

Lemma 3.9. *Two edges* $I, J \in P(\alpha)$ *meet at a vertex in* α *iff* $I \triangle J \in P(\alpha)$*.*

Proof. For any two $I, J \in P(\alpha)$, the two sets are either disjoint, or one set contains the other. If they are disjoint, $I\triangle J = I \cup J$. If $I \subset J$, then $(I\triangle J) \cup I = J$. And if $J \subset I$, then $(I\triangle J)\cup J = I$. □

Given an edge $I \in P(\alpha)$, there is a rooted binary tree T_{α_I} obtained by cutting T_α at I, and declaring I the root of T_{α_I}. Let $\pm\alpha_I \in L(A)$ be the Lie monomial (up to sign) associated to T_{α_I}. The monomial α_I appears nested inside α, if α is written as a nested bracketing.

3.4 The ring of Mandelstam variables

Mandelstam variables are Lorentz invariant functions,

$$s_I := \left(\sum_{i\in I} k_i\right)^2, \tag{3.30}$$

of momentum vectors k_i^μ. As defined, these variables satisfy

$$s_I = \sum_{\{i,j\}\subset I} s_{ij}. \tag{3.31}$$

The s_I are also subject to determinantal relations and inequalities, which follow from their definition as the norm squared of a vector.

Introduce formal Mandelstam variables, s_I, for every subset $I \subset \mathbb{N}$. For $A \subset \mathbb{N}$, write M_A for the set of variables s_I, with $I \subset A$, and form the polynomial ring $k[M_A]$. The summation relations

$$s_I = \sum_{\{i,j\}\subset I} s_{ij} \tag{3.32}$$

generate an ideal, $I \subset k[M_A]$.

Definition 3.10. *The Laurent ring of Mandelstam variables is*

$$R_A := M_A^{-1}\left(k[M_A]/I\right). \tag{3.33}$$

Write

$$s_{A,B} := \sum_{i\in A}\sum_{j\in B} s_{ij}, \tag{3.34}$$

so that

$$s_{AB} - s_A - s_B - 2s_{A,B} \in I, \tag{3.35}$$

where $s_{AB} := s_{A\cup B}$. (3.35) implies that

$$s_{AB} + s_{BC} + s_{CA} - s_A - s_B - s_C - s_{ABC} \in I, \tag{3.36}$$

for any (not necessarily disjoint) subsets A, B, C. And let s_α be the *product of propagators*,

$$s_\alpha := \prod_{I\in P(\alpha)} s_I, \tag{3.37}$$

where $P(\alpha)$ is the set of propagators of α, defined in (3.29). For α in $L^+(A)$, write

$$\tilde{s}_\alpha := s_A\, s_\alpha. \tag{3.38}$$

In particular, for disjoint Lie monomials $\alpha \in L^+(A)$ and $\beta \in L^+(B)$,

$$\tilde{s}_{[\alpha,\beta]} = s_{AB}\, \tilde{s}_\alpha\, \tilde{s}_\beta. \tag{3.39}$$

3.5 Mandelstam forms

The summation relations, (3.32), define a vector subspace

$$\tilde{K}_A \subset k.M_A, \tag{3.40}$$

called *kinematic space.* Write 1-forms in $\bigwedge^1 \tilde{K}_A$ as $\mathrm{d}s_I$. These satisfy

$$\mathrm{d}s_I - \sum_{\{i,j\}\subset I} \mathrm{d}s_{ij} = 0, \tag{3.41}$$

and

$$\mathrm{d}s_{AB} + \mathrm{d}s_{BC} + \mathrm{d}s_{CA} - \mathrm{d}s_A - \mathrm{d}s_B - \mathrm{d}s_C - \mathrm{d}s_{ABC} = 0. \tag{3.42}$$

Definition 3.11. (Tree form.) *Fix a plane rooted binary tree, T, with labels in A. Suppose that T is the grafting $T = T_l \vee T_r$. Then the tree form $\omega_T \in \wedge^\bullet \tilde{K}_A$ is inductively defined by*

$$\omega_T := \omega_{T_l} \wedge ds_A \wedge \omega_{T_r}. \tag{3.43}$$

For the base case, when $T = 1 \vee 2$, define $\omega_T := ds_{12}$.

It follows from (3.43) that the tree forms, $\omega_{\alpha,a} = \omega_{T_{\alpha,a}}$, satisfy the graded-symmetry relation,

$$\omega_{[\alpha,\beta],ab} = -(-1)^{|a||b|}\omega_{[\beta,\alpha],ba}, \tag{3.44}$$

for monomials $\alpha \in L^+(A)$, $\beta \in L^+(B)$, and orderings $a \in \mathfrak{S}(A)$, $b \in \mathfrak{S}(B)$. Moreover, the seven term identity, (3.42), implies that the tree forms satisfy the super-Jacobi relation,

$$(-1)^{|a||c|}\omega_{[\alpha,[\beta,\gamma]],abc} + (-1)^{|b||a|}\omega_{[\beta,[\gamma,\alpha]],bca} + (-1)^{|c||b|}\omega_{[\gamma,[\alpha,\beta]],cab} = 0. \tag{3.45}$$

Remark 3.12. *There is an action of the symmetric group, S_A, on $W(A)$ and $L(A)$. Let ϵ be the sign representation of S_A. Then write $SL(A)$ for the graded Lie algebra which, as a representation of S_A, is $L(A) \otimes \epsilon$. Concretely, this has the effect of introducing signs. The commutator in $SL(A)$ satisfies*

$$[a, b]_{\mathrm{gr}} = -(-1)^{|a||b|}[b, a]_{\mathrm{gr}}, \tag{3.46}$$

and also the super Jacobi identity,

$$(-1)^{|a||c|}[a, [b, c]] + (-1)^{|b||a|}[b, [c, a]] + (-1)^{|c||b|}[c, [a, b]] = 0. \tag{3.47}$$

Dually, there is a signed shuffle product in $SL(A)^\vee$, inductively defined by

$$(ia \sqcup_{\mathrm{gr}} jb) = i(a \sqcup_{\mathrm{gr}} jb) + (-1)^{|ia|} j(ia \sqcup_{\mathrm{gr}} b). \tag{3.48}$$

For example, in $SL(123)$,

$$[[1, 2], 3] = 123 + 213 - 312 - 321,$$

and, in $SL(123)^\vee$,

$$1 \sqcup 23 = 123 - 213 + 231.$$

Equations (3.44) *and* (3.45) *imply that ω defines a homomorphism, $SL(A) \to \wedge^\bullet K_A$, from super Lie monomials to Mandelstam forms. The graded signs occur naturally when studying residues of top forms on Conf_{n-1} in Chapter 5.*

3.6 The binary tree map

For a subset $A \subset \mathbb{N}$, write

$$\mathcal{L}(A) := L(A) \otimes_k R_A,$$

and

$$\mathcal{L}(A)^\vee = L(A)^\vee \otimes_k R_A.$$

Write (,) for the pairing between $\mathcal{L}(A)$ and $\mathcal{L}(A)^\vee$, with values in R_A. The biadjoint scalar tree amplitudes give rise to the following map from $\mathcal{L}(A)^\vee$ to $\mathcal{L}(A)$.

Definition 3.13. (Binary Tree Map.) *The binary tree map, $T : \mathcal{L}(A)^\vee \to \mathcal{L}(A)$, is*

$$T : a \mapsto T(a) = \sum_{\alpha} \frac{(a, \alpha)\alpha}{\tilde{s}_\alpha}, \tag{3.49}$$

where the sum is over all Lie monomials in $L(A)$ (up to sign).

By definition, T is self-adjoint,

$$(a, T(b)) = (b, T(a)). \tag{3.50}$$

Moreover, for $a \in \mathfrak{S}(A)$, $T(a)$ may be expanded as

$$T(a) = \frac{1}{s_A} \sum_{\substack{a=bc \\ b\neq e, c\neq e}} [T(b), T(c)]. \tag{3.51}$$

This follows directly from the identity,

$$(a, [\alpha, \beta]) = (\delta a, \alpha \otimes \beta - \beta \otimes \alpha), \tag{3.52}$$

where δ is the deconcatenation coproduct (but see Remark 3.14 for a technicality). Equation (3.51) is called the Berends-Giele relation for biadjoint scalar theory. The amplitudes of biadjoint scalar theory have a partial amplitude decomposition of the form,

$$m(1, ..., n) = \sum_{a,b \in \mathfrak{S}_{n-1}} m(a, n|b, n)\mathrm{tr}(a, n)\mathrm{tr}(b, n). \tag{3.53}$$

The partial amplitudes are related to T by

$$m(a, n|b, n) := s_A(b, T(a)). \tag{3.54}$$

The Berends-Giele relation for these amplitudes is studied in [54].

Remark 3.14. *Equation* (3.52) *is obvious for* $a, \alpha, \beta \in A^*$*. However, it also holds for* $[\alpha, \beta] \in L(A)$ *and* $a \in L(A)^\vee$*. This is because deconcatenation,* δ*, respects the shuffle product on* $k\langle A\rangle^\vee$*, so that*

$$\delta(a \sqcup\!\sqcup b) = \delta a \sqcup\!\sqcup \delta b, \tag{3.55}$$

where the shuffle product acts on the tensor product as

$$(a \otimes b) \sqcup\!\sqcup (c \otimes d) = (a \sqcup\!\sqcup c) \otimes (b \sqcup\!\sqcup d). \tag{3.56}$$

This is the dual of the bialgebra identity, (3.7)*.*

Chapter 4

The generalised fundamental BCJ relations

As reviewed in Chapter 2, Yang-Mills partial amplitudes satisfy

$$\sum_{a=bc}(-1)^c s_{i;c} A(bi\bar{c}, n) = 0, \tag{4.1}$$

for fixed $a \in \mathfrak{S}([n]\backslash i)$, and where

$$s_{i;c} := \sum_{j\in c} s_{ij}. \tag{4.2}$$

The relations, (4.1), are also satisfied by the partial amplitudes of other coloured theories, such as NLSM, and biadjoint scalar theory. This suggests that there should be a universal explanation for these identities. This chapter studies the binary tree map,

$$T : \mathcal{L}(A)^\vee \to \mathcal{L}(A).$$

The fundamental BCJ relations, (4.1), and the field theory KLT relation, are consequences of a basic property of T. A key role is played by a Lie bracket on $\mathcal{L}(A)^\vee$, which we call the *S bracket* (following, e.g., [57] and [58], where it is called the 'S map'). Section 4.2 uses the S bracket to define the off-shell KLT kernel map

$$S : \mathcal{L}(A) \to \mathcal{L}(A)^\vee,$$

in an arbitrary basis. The definition of S given in Section 4.2 trivializes the fact that the map S inverts the binary tree map,

$$S \circ T = Id, \qquad T \circ S = Id.$$

The on-shell version of this statement is well known, but a conceptually simple proof has been lacking. Finally, the dual of the S bracket is a map that has been called the *contact term map* in studies of superstring correlators and BCJ numerators [13]. Its properties are proved in Section 4.4.

4.1 The S bracket on $\mathcal{L}(A)^\vee$

For words $a, b \in A^*$ and distinct letters $i, j \in A$, define

$$ai \star jb := s_{ij}\, aijb. \tag{4.3}$$

Imposing linearity, this defines a noncommutative associative product on $k\langle A\rangle^\vee \otimes R_A$.

Definition 4.1. *For $a, b \in k\langle A\rangle^\vee \otimes R_A$, the S bracket of a and b is*

$$\{a, b\} := r^*(a) \star \ell^*(b). \tag{4.4}$$

By (3.18), *ℓ^* and r^* vanish on the shuffle ideal,* $\mathrm{Sh}(A) \subset k\langle A\rangle^\vee$. *This means that the formula,* (4.4), *defines a bracket on $\mathcal{L}(A)^\vee$:*

$$\{\ ,\ \} : \mathcal{L}(A)^\vee \otimes \mathcal{L}(A)^\vee \to \mathcal{L}(A)^\vee. \tag{4.5}$$

It can already be verified from the definition that the S-bracket is antisymmetric, using

$$\overline{r^*(a)} = (-1)^{|a|} \ell^*(\overline{a}), \tag{4.6}$$

which is the dual of (3.13). The key interest of the S bracket is its relationship to the Lie bracket $[\ ,\ ]$ on $\mathcal{L}(A)$.

$$
\begin{array}{ccc}
\mathcal{L}(A) \otimes \mathcal{L}(A) & \xrightarrow{[\ ,\ ]} & \mathcal{L}(A) \\
{\scriptstyle T \otimes T}\uparrow & & \uparrow{\scriptstyle T} \\
\mathcal{L}(A)^\vee \otimes \mathcal{L}(A)^\vee & \xrightarrow[\{\ ,\ \}]{} & \mathcal{L}(A)^\vee
\end{array}
$$

Figure 4.1: The maps in Proposition 4.2.

Proposition 4.2. *T intertwines between the S bracket and the Lie bracket, in the sense that*

$$T(\{a, b\}) = [T(a), T(b)], \tag{4.7}$$

for $a, b \in \mathcal{L}(A)^\vee$.

The rest of this section proves Proposition 4.2 directly from the definition, (4.4). Let A denote antisymmetrization,

$$A(x \otimes y) := x \otimes y - y \otimes x. \tag{4.8}$$

Extend the definition of the $\star$ product to $\mathcal{L}(A)^\vee \otimes \mathcal{L}(A)^\vee$ by defining

$$(x \otimes y) \star b := x \otimes (y \star b), \qquad \text{and} \qquad a \star (x \otimes y) := (a \star x) \otimes y. \tag{4.9}$$

Then,

$$\{x \otimes y, b\} := x \otimes \{y, b\}, \qquad \text{and} \qquad \{a, x \otimes y\} := \{a, x\} \otimes y. \tag{4.10}$$

Finally, as a shorthand, write

$$\delta'(a) := \delta(a) - a \otimes e - e \otimes a, \tag{4.11}$$

for the non-trivial part of $\delta(a)$.

Lemma 4.3. *Deconcatenation acts on the S bracket as*

$$\delta'\{a,b\} = \{A\circ\delta(a),b\} + \{a, A\circ\delta(b)\} + s_{A,B}\, a\otimes b, \tag{4.12}$$

for $a \in \mathfrak{S}(A)$ and $b \in \mathfrak{S}(B)$.

Proof. For non-empty words x, y,

$$\delta'(xy) = \delta'(x)\cdot(e\otimes y) + (x\otimes e)\cdot\delta'(y) + x\otimes y.$$

Applying this to the explicit formula, (4.4), for $\{a,b\}$ gives

$$\delta'\{a,b\} = \delta' r^*(a) \star \ell^*(b) + r^*(a)\star\delta'\ell^*(b) + \sum_{\substack{a=xiy\\ b=zjw}} s_{ij}\,(x \sqcup\!\sqcup \tilde{y})i\otimes j(\tilde{z}\sqcup\!\sqcup w). \tag{4.13}$$

But recall that (as in (3.22))

$$xiy \sim i(\tilde{x}\sqcup\!\sqcup y) \sim (x\sqcup\!\sqcup\tilde{y})i.$$

So the sum in (4.13) sums to

$$s_{A,B}\, a\otimes b.$$

Finally, a direct computation using (3.17) shows that

$$\delta'\circ\ell^*(a) = (\ell^*\otimes 1)\circ A\circ\delta'(a), \tag{4.14}$$

and a similar identity holds for $r^*(a)$. $\square$

Proof. (Of proposition 4.2.) The BG relation, (3.51), implies that

$$T(a) = \frac{1}{s_a}\sum_{x,y\neq e}(x\otimes y, \delta(a))[T(x),T(y)], \tag{4.15}$$

for any homogeneous $a \in \mathcal{L}(A)^\vee$. By Lemma 4.3, it follows that

$$\begin{aligned} s_{ab}T(\{a,b\}) = s_{a,b}&[T(a),T(b)] \\ &+ \sum_{a=xy}[T(x),T(\{y,b\})] - (x \leftrightarrow y) \\ &+ \sum_{b=xy}[T(\{a,x\}),T(y)] - (x \leftrightarrow y). \end{aligned}$$

Inductively assuming (4.7), the second line becomes

$$\sum_{a=xy}[T(x),T(\{y,b\})] - (x \leftrightarrow y) = \sum_{a=xy}[[T(x),T(y)],T(b)],$$

after using the Jacobi identity. Likewise for the third line. Proposition 4.2 then follows by using (4.15) again. □

Corollary 4.4. (Off shell fundamental BCJ. [31]) *For a letter i and a word $a \in \mathfrak{S}(A\backslash i)$,*

$$\sum_{a=xjy} s_{ij}\, T(ij(\tilde{x} \sqcup\!\sqcup y)) = [i, T(a)]. \tag{4.16}$$

Recall that the biadjoint scalar amplitudes are related to $T(a)$ by

$$m(a,n|b,n) = \lim_{s_a \to 0} s_a(b, T(a)).$$

Then Corollary 4.4 implies that

$$\sum_{a=xjy} s_{ij}\, m(ij(\tilde{x} \sqcup\!\sqcup y), n|b,n) = 0, \tag{4.17}$$

because $[i,T(a)]$ has no $1/s_{ia}$ pole. The BCJ relation, (4.17), is an alternative form of the more conventional statement of the relation, [19]

$$\sum_{a=xy} s_{i;y}\, m(xiy,n|b,n) = 0. \tag{4.18}$$

The two expressions, (4.17) and (4.18) are equivalent:

$$\begin{aligned}\sum_{a=xy} s_{i;y}\, xiy &\sim \sum_{a=xy} s_{i;y}\, i(\tilde{x} \sqcup y) \\ &= \sum_{a=xjy} s_{i;jy}\, ij(\tilde{x} \sqcup y) - s_{i;y} ij(\tilde{x} \sqcup y) \\ &= \sum_{a=xjy} s_{ij}\, ij(\tilde{x} \sqcup y),\end{aligned}$$

where the second line follows by the defining property of the shuffle product, (3.3).

4.2 The off-shell KLT kernel

This section shows that Proposition 4.2 implies that T is invertible.

For a word $a \in \mathfrak{S}(A)$, write $\ell\{a\} \in \mathcal{L}(A)^\vee$ for the left S-bracketing of a, which is the full left-bracketing of a by S-brackets:

$$\ell\{123...n\} := \{...\{\{1,2\},3\},...,n\}. \tag{4.19}$$

This is analogous to the definition of $\ell(a)$. The right S-bracketing map, $a \mapsto r\{a\}$, is similarly defined, and it is related to the left S-bracketing by

$$r\{a\} = -(-1)^{|a|}\ell\{\overline{a}\}, \tag{4.20}$$

which follows from the fact that the S bracket is antisymmetric.

Lemma 4.5. *For a word $a \in W(A)$,*

$$T(\ell\{a\}) = \ell(a). \tag{4.21}$$

Proof. For any monomial $\alpha \in L^+(A)$, fix some presentation of α as a nested bracketing, and let $\{\alpha\}$ be the expression obtained by replacing each pair of brackets by pairs of braces. Since $T(i) = i$, for $i \in A$, Proposition 4.2 implies that $\alpha = T(\{\alpha\})$. □

Fix $i \in A$. By Lemma 3.6, $\ell\{a\}$ admits a basis expansion of the form,

$$\ell\{a\} = \sum_{b \in \mathfrak{S}(A \setminus i)} (\ell\{a\}, \ell(ib))ib. \tag{4.22}$$

This identity doesn't depend on the choice of basis and dual basis. So write

$$\ell\{a\} = {\sum_{\beta,b}}' (\ell\{a\}, \beta)b, \tag{4.23}$$

where the prime indicates that the sum is restricted to some choice of dual bases.

Definition 4.6. (Off shell KLT map.) *The off-shell KLT map,*

$$S : \mathcal{L}(A) \to \mathcal{L}(A)^\vee,$$

acts on $\alpha \in \mathcal{L}(A)$ *as*

$$S : \alpha \mapsto {\sum_{\beta,b}}' (\{\beta\}, \alpha)b. \tag{4.24}$$

S is clearly independent of the choice of dual bases.

Lemma 4.7. *S is self-adjoint.*

Proof. Since T is self-adjoint, it follows that $S(a, b)$ is a symmetric matrix, because

$$(\ell\{a\}, \ell(b)) = (\ell\{a\}, T(\ell\{b\})) = (\ell(a), \ell\{b\}). \tag{4.25}$$

□

Proposition 4.8. (Off shell KLT.) *The tree map, $T : \mathcal{L}(A)^\vee \to \mathcal{L}(A)$, is invertible, with inverse given by S.*

Proof. (4.21) can be written in the form

$$\ell(a) = \sum_b (\ell\{a\}, \ell(ib))\, T(ib). \tag{4.26}$$

By (4.26),

$$S \circ T(a) = \sum_{b,c}{}' (\ell\{b\}, \ell(c))(T(c), a)b = \sum_{b}{}' (\ell(b), a)b = a,$$

where it has been used that T is self-adjoint. Likewise,

$$T \circ S(\alpha) = \sum_{b,c,d}{}' (d, T(c))(\ell\{c\}, \ell(b))(b, \alpha)\ell(d) = \sum_{b}{}' (b, \alpha)\ell(b) = \alpha,$$

where the symmetry property of S, (4.25), is used. □

Remark 4.9. *Fixing $i \in A$, consider the dual bases, ib and $\ell(ib)$, as in Lemma 3.6. $\alpha \in L^{+}(A)$ is expanded in this basis as*

$$\alpha = \sum_{b} (\alpha, ib)\ell(ib). \tag{4.27}$$

The map S is then, using the same basis,

$$\alpha \mapsto \sum_{a,b} (\alpha, ib)(\ell\{ia\}, \ell(ib))ia, \tag{4.28}$$

so that the components of S in this basis are

$$S(ia, ib) = (\ell\{ia\}, \ell(ib)). \tag{4.29}$$

The components of S in this basis agree with the formula given by [11]. The explicit computation of the components of S from (4.29) is a mechanical exercise, given in Appendix 4.A.

Remark 4.10. *The formulas in this section for S were conjectured in [55] in an attempt to define S for an arbitrary pair of bases. Conventionally, S is given with respect to a dual pair of bases, such as $S(ia, ib) = (\ell\{ia\}, \ell(ib))$ (as in (4.29)), where $i \in A$ is sometimes referred to as the 'pivot'.*

4.3 The S bracket is Lie

Given Proposition 4.8, the S bracket of $a, b \in \mathcal{L}(A)^\vee$ may be written as

$$\{a, b\} = S([T(a), T(b)]). \tag{4.30}$$

In other words, the S bracket $\{\ ,\ \}$ is the pullback of the Lie bracket $[\ ,\ ]$ by the tree map T. It follows immediately that

Lemma 4.11. The S bracket $\{\ ,\ \}$ is a Lie bracket on $\mathcal{L}(A)^\vee$.

This lemma implies properties of $\{\ ,\ \}$ that are not obvious from the original definition of the S bracket. For a Lie monomial $\alpha \in L^+(A)$, write

$$\{\alpha\} \in \mathcal{L}(A)^\vee$$

for the S-bracketing of A according to α. If the sum of three Lie monomials, $\alpha, \beta, \gamma \in L^+(A)$, vanishes in $L^+(A)$, then Lemma 4.11 implies that

$$\{\alpha\} + \{\beta\} + \{\gamma\}$$

vanishes in $\mathcal{L}(A)^\vee$. For example,

$$\{1, \{2, 3\}\} = s_{12}s_{23}123 - s_{13}s_{23}132, \tag{4.31}$$

so that the Jacobi sum is

$$\{1, \{2, 3\}\} + (\text{cyclic } 123) = s_{12}s_{23}(123 - 321) + (\text{cyclic } 123), \tag{4.32}$$

which vanishes in $\mathcal{L}(A)^\vee$ because $123 \sim 321$.

4.4 The tree splitting identity

Given that the S bracket is Lie, it induces a *Lie cobracket* on $\mathcal{L}(A)$ in the following way.

Definition 4.12. *The contact term map*

$$C : \mathcal{L}(A) \to \mathcal{L}(A) \otimes \mathcal{L}(A)$$

is the Lie cobracket dual to the S bracket $\{\ ,\ \}$ *on* $\mathcal{L}(A)^\vee$*. Given* $\alpha \in \mathcal{L}(A)$*,* $C(\alpha)$ *is defined by*

$$(a \otimes b, C(\alpha)) := (\{a, b\}, \alpha). \tag{4.33}$$

$$\begin{array}{ccc} \mathcal{L}^\vee \otimes \mathcal{L}^\vee & \xrightarrow{\{\ ,\ \}} & \mathcal{L}^\vee \\ & \underset{C(\alpha)}{\searrow} & \downarrow \alpha \\ & & k \end{array}$$

By construction, C is antisymmetric and satisfies the co-Jacobi property. The antisymmetry of the S bracket, $\{\ ,\ \}$, implies that

$$S_{12} \circ C = -\delta_{\mathrm{Lie}}, \tag{4.34}$$

where $S_{12}(x \otimes y) = y \otimes x$. It can also be checked that the Jacobi property of the S bracket dualizes to

$$(C \otimes 1) \circ C - (1 \otimes C) \circ C - (1 \otimes S_{12}) \circ (C \otimes 1) \circ C = 0, \tag{4.35}$$

which is the co-Jacobi identity. [61]

Lemma 4.13. *For* $\alpha \in \mathcal{L}(A)$*,*

$$C(\alpha) := A \circ (T \otimes T) \circ \delta \circ S(\alpha). \tag{4.36}$$

Equivalently, for $a \in \mathfrak{S}(A)$,

$$C(T(a)) := \sum_{a=xy} T(x) \otimes T(y) - T(y) \otimes T(x). \tag{4.37}$$

Proof. Proposition 4.2 can be summarizes as the commutativity of the following diagram:

$$\begin{array}{ccc} L \otimes L & \xrightarrow{[\ ,\ ]} & L \\ {\scriptstyle T \otimes T} \uparrow & & \uparrow {\scriptstyle T} \\ L^\vee \otimes L^\vee & \xrightarrow[\{\ ,\ \}]{} & L^\vee \end{array}$$

The dualization of this diagram is

$$\begin{array}{ccc} L \otimes L & \xleftarrow{C} & L \\ {\scriptstyle A \circ (T \otimes T)} \uparrow & & \uparrow {\scriptstyle T} \\ L^\vee \otimes L^\vee & \xleftarrow[\delta]{} & L^\vee \end{array}$$

(4.36) then follows from the KLT relation, Proposition 4.8. □

The tensor product $L(A) \otimes L(A)$ can be regarded as a module for the adjoint representation of $L(A)$, where the left adjoint $[\alpha,\ ]$ acts as

$$[\alpha, \beta \otimes \gamma] = [\alpha, \beta] \otimes \gamma + \beta \otimes [\alpha, \gamma], \tag{4.38}$$

and likewise for the right-adjoint.

Lemma 4.14. (Tree splitting lemma.) *For $\alpha \in L^+(A)$ and $\beta \in L^+(B)$,*

$$C([\alpha, \beta]) = [C(\alpha), \beta] + [\alpha, C(\beta)] + s_{A,B}(\alpha \otimes \beta - \beta \otimes \alpha). \tag{4.39}$$

Proof. Write

$$[\alpha, \beta] = T(\{S(\alpha), S(\beta)\}).$$

Then (4.36) implies that

$$C([\alpha, \beta]) = A \circ (T \otimes T) \circ \delta(\{S(\alpha), S(\beta)\}).$$

Equation (4.41) then follows from Lemma 4.3. □

The identity, (4.39), is reminiscent of the following identity satisfied by the deshuffle coproduct on Lie $\langle A \rangle$,

$$\delta_{\sqcup\!\sqcup}([\alpha, \beta]) = [\delta_{\sqcup\!\sqcup}\alpha, \beta] + [\alpha, \delta_{\sqcup\!\sqcup}\beta], \tag{4.40}$$

a consequence of (3.7) in Section 3.1. The additional term appearing in (4.39) motivates the following definition.

Definition 4.15. *The tree splitting map $D : \mathcal{L}(A) \to \mathcal{L}(A) \wedge \mathcal{L}(A)$ is the linear map that acts on monomials $[\alpha, \beta] \in L(A \cup B)$ as*

$$D : [\alpha, \beta] \mapsto s_{A,B}\, (\alpha \otimes \beta - \beta \otimes \alpha), . \tag{4.41}$$

where $\alpha \in L^+(A)$ and $\beta \in L^+(B)$.

Lemma 4.14 can be adopted as a definition of C, and hence is an alternative definition of the S bracket. Nesting (4.39) gives a closed formula for the action of C on monomials $\alpha \in L(A)$. Recalling (3.29), about the propagator set $P(\alpha)$, it follows that $C(\alpha)$ may be written as the sum

$$C(\alpha) = \sum_{I \in P(\alpha)} \alpha[\alpha_I \to D(\alpha_I)], \tag{4.42}$$

where $\alpha[\alpha_I \to D(\alpha_I)]$ denotes the replacement, in α, of the sub-monomial α_I with $D(\alpha_I)$. For example,

$$C([1, [2, 3]]) = s_{1,23}(1 \otimes [2, 3] - [2, 3] \otimes 1) + s_{23}[1, 2 \otimes 3]. \tag{4.43}$$

4.A The conventional KLT formula

This section explicitly computes the KLT components $S(ia, ib)$, defined in section 4.2, to recover the formula conventionally adopted in the literature. Write the S bracket $\{a, i\}$ as

$$\{a, i\} \sim \sum_{a=bc} s_{i;b} bic. \tag{4.44}$$

Then the left bracketing $\ell(123...n)$ can be expanded

$$\begin{aligned}
[...[[1,2],3],...,n] &= s_{12}[...[[T(12),3],...],n] \\
&= s_{12}s_{1,3}[...[T(132),...],n] \\
&\quad + s_{12}s_{12,3}[...[T(123),...],n] \\
&= \text{etc.}
\end{aligned}$$

The result is the nested sum,

$$s_{12}\left(\sum_{12=a_2b_2} s_{2,a_2}\left(...\left(\sum_{a_{n-1}nb_{n-1}=a_nb_n} s_{n;a_n} T(a_n n b_n)\right)...\right)\right), \tag{4.45}$$

where, by convention, $s_{i,a} := 0$ if a is the empty word. All the words a_i appearing in the sum begin with the letter 1. It follows that the order of the summations can be reversed to give

$$\ell(12...n) = \sum_{a\in\mathfrak{S}(A-1)}\left(\prod_{i=2}^{n}\left(\sum_{c_i=a_iib_i} s_{i;a_i}\right)\right) T(1a), \tag{4.46}$$

where c_i is the word $1a$ with the letters $1, ..., i-1$ removed. Said another way, a_i is a word in $\mathfrak{S}(A_i)$, where A_i is the subset containing all letters j such that $j < i$ in the ordering $1a$, and $i < j$ in the ordering $12...n$. It follows that, for words $a, b \in \mathfrak{S}(A-1)$,

$$S(1a, 1b) = \prod_{i=2}^{n} s_{i;A_i(a,b)}, \tag{4.47}$$

where $A_i(a, b)$ is the subset of $\{1, 2, ..., n\}$ containing all letters that both precede i in $1b$, and are preceded by i in $1a$.

Chapter 5

The scattering equations

RNS type I string theory partial tree amplitudes (i.e. 'open string tree amplitudes') are conventionally written as an integral over a real simplex of the form

$$\mathcal{A}(123...n-1,n) = \int\limits_{\substack{z_i<z_{i+1}\\ z_1=0\\ z_{n-1}=1}} \mathrm{d}^{n-3}z\, f_s\, I, \tag{5.1}$$

where $\mathrm{d}^{n-3}z = \mathrm{d}z_2...\mathrm{d}z_{n-2}$, and the multi-valued Koba-Nielsen factor is

$$f_s = \prod_{i<j} z_{ij}^{\alpha' s_{ij}} = \exp\left(\sum_{i<j}^{n-1} \alpha' s_{ij} \log z_{ij}\right). \tag{5.2}$$

For example, for gluon partial tree amplitudes the integrand can be written compactly as a Berezinian integral, [45]

$$f_s I = \int \mathrm{d}\theta \mathrm{d}\phi \prod_{i<j} \exp\left(\frac{\theta_i\theta_j k_i\cdot k_j + \theta_i\phi_j k_i\cdot\epsilon_j + \phi_i\phi_j\epsilon_i\cdot\epsilon_j}{z_{ij}-\theta_i\theta_j}\right). \tag{5.3}$$

For large α', the saddle point equations for the integral, (5.1), are

$$E_i = 0, \tag{5.4}$$

where

$$E_i := \frac{\partial}{\partial z_i} f_s = \sum_{\substack{j=1 \\ j \neq i}}^{n-1} \frac{\alpha' s_{ij}}{z_{ij}}. \tag{5.5}$$

The equations $E_i = 0$ are called the *scattering equations*.

This chapter studies identities involving the scattering equations. The S bracket $\{\ ,\ \}$ and contact term map C both appear in this context. It is convenient to at first study the configuration space $\mathrm{Conf}_{n-1}(\mathbb{C})$. The open stratum of $\mathcal{M}_{0,n}$ is the quotient of this by the free action of $\mathbb{C} \ltimes \mathbb{C}$, that acts as $(a, b) : z \mapsto az + b$,

$$\mathcal{M}_{0,n}(\mathbb{C}) \simeq \mathrm{Conf}_{n-1}(\mathbb{C})/\mathbb{C} \ltimes \mathbb{C}.$$

5.1 Rational function identities

Fix a subset $A \subset \mathbb{N}$. Let W_A be the dimension $|A| - 1$ vector space, with coordinate functions z_i, for each $i \in A$, subject to

$$\sum_i z_i = 0. \tag{5.6}$$

Let H_A be the set of functions

$$z_{ij} := z_i - z_j,$$

for $i, j \in A$. The intersection of the hyperplanes $z_{ij} = 0$ is the big diagonal, $\Delta \subset W_A$, and the complement,

$$W_A^* := W_A - \Delta,$$

is configuration space. Let $k[W_A]$ be the ring of polynomial functions on W_A. Then

Definition 5.1. *The ring of rational functions on* W_A^* *is*

$$\mathcal{O}_A := H_A^{-1} k[W_A]. \tag{5.7}$$

Call a subset $h \subset H_A$ a *basis* if the functions in h are a basis for the dual, $W_A^\vee$.

Example 5.2. *Setting* $A = \{1, ..., n-1\}$*, the set* $h = \{z_{12}, z_{23}, ..., z_{n-2n-1}\}$ *is a basis. So is* $h = \{z_{12}, z_{13}, ..., z_{1n-1}\}$*.*

For a basis $h \subset H_A$, write

$$\frac{1}{z_h} := \prod_{z_{ij} \in h} \frac{1}{z_{ij}}. \tag{5.8}$$

The rational functions $1/z_h$ generate a $k[W_A]$-module, S_A.[1] For a word $a \in \mathfrak{S}(A)$, write $\mathrm{pt}(a)$ for the *broken Parke-Taylor function*:

$$\mathrm{pt}(a) := \prod_{i=1}^{|A|-1} \frac{1}{z_{a(i)a(i+1)}} \in S_A. \tag{5.9}$$

Note that, for $i \in A$ a letter,

$$\mathrm{pt}(aib) = \mathrm{pt}(ai)\mathrm{pt}(ib), \tag{5.10}$$

and also that the $\mathrm{pt}(a)$ functions also satisfy the circle relation,

$$\sum_{i=1}^{n} \mathrm{pt}(ii+1...n1...i-1) = 0. \tag{5.11}$$

A consequence of Proposition 5.6, below, is that S_A is spanned by the (broken) Parke-Taylor functions $\mathrm{pt}(a)$. This result essentially follows from dimension counting. But the rest of this section proves this explicitly, because the formulas arising in the explicit proof are used in Section 5.3.

Any subset $h \subset H_A$ defines an oriented graph with vertex set A, and an edge $i \to j$ associated to every $z_{ij} \in h$.

Lemma 5.3. *A set* $h \subset H_A$ *is a basis iff its associated graph is a spanning tree of the complete graph on* A*.*

Proof. Take $h \subset H_A$, and let G be the associated graph. The functions in h span the hyperplane $W_A^\vee$ iff G is connected and meets every vertex. The functions in h are linearly independent iff G is a tree. □

[1][14] define and study S_A for an arbitrary hyperplane arrangement.

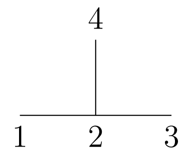


Figure 5.1: A tree associated to a basis of $W^\vee_{1234}$.

A given tree, T, is associated to a function $1/z_h$ that can be written as a product of Parke-Taylor functions. This product of Parke-Taylor functions is not unique. For example, the function $1/z_h$ associated to the tree in Figure 5.1 is

$$\mathrm{pt}(124)\mathrm{pt}(23) = \mathrm{pt}(123)\mathrm{pt}(24) = \frac{1}{z_{12}z_{23}z_{24}}. \tag{5.12}$$

Lemma 5.4. *For disjoint sets A and B, and words $a \in \mathfrak{S}(A)$ and $b \in \mathfrak{S}(B)$,*

$$\mathrm{pt}(a \sqcup\!\sqcup\, b) = 0. \tag{5.13}$$

Proof. Inducting on the length of a and b, the base case is

$$\mathrm{pt}(ijk) + \mathrm{pt}(jki) + \mathrm{pt}(jik) = 0. \tag{5.14}$$

(5.10) implies that

$$\begin{aligned} \mathrm{pt}((ai \sqcup\!\sqcup\, a'i')k) = \;& \mathrm{pt}((a \sqcup\!\sqcup\, a'i')i)\,(\mathrm{pt}(ik) - \mathrm{pt}(i'k)) \\ &+ \mathrm{pt}((ai) \sqcup\!\sqcup\, (a'i'))\,\mathrm{pt}(i'k). \end{aligned}$$

Using this to expand $\mathrm{pt}(aij \sqcup\!\sqcup\, a'i'j')$ gives the lemma. □

Corollary 5.5. (Kleiss-Kuijf.) *For a word $bjc \in W(A)$, with a letter $j \in A$,*

$$\mathrm{pt}(bjc) = \mathrm{pt}(j(\tilde{b} \sqcup\!\sqcup\, c)), \tag{5.15}$$

and so

$$\mathrm{pt}(aj)\mathrm{pt}(bjc) = \mathrm{pt}(aj(\tilde{b} \sqcup\!\sqcup\, c)). \tag{5.16}$$

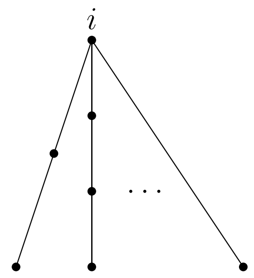


Figure 5.2: Branches in a tree.

For example, using Corollary 5.5, it follows that

$$\mathrm{pt}(12)\mathrm{pt}(13)\mathrm{pt}(14) = \mathrm{pt}(2314) + \mathrm{pt}(3214).$$

Let T be a spanning tree of $\{1, ..., n\}$. Orient T by making vertex 1 a sink, and demanding that each other vertex has only one outgoing edge. Then write h_T for the set of functions z_{ix_i}, $i = 2, ..., n$, where ix_i is the outgoing edge of i. Clearly h_T is a basis.

Proposition 5.6. *With h_T as above, the function $1/z_{h_T}$ has the following expansion*

$$\frac{1}{z_{h_T}} = \sum_{\substack{a\in\mathfrak{S}_{n-1} \\ x_i <_{1a} i}} \mathrm{pt}(1a), \tag{5.17}$$

where the sum is over all orderings a such that x_i precedes i (in the ordering $1a$) for every i.

Proof. This follows by repeated applications of (5.16). The orientation of T induces a partial order on the vertices, with 1 the smallest vertex. Let i be one of the largest vertices with valence greater than 1, and suppose that i has k incoming edges. All vertices greater than i have valence 1, so that the tree 'greater than i' is comprised of some number of 'branches', as in Figure 5.2. By (5.16),

$$\mathrm{pt}(ia)\mathrm{pt}(ib) = (-1)^{|b|}\mathrm{pt}(\bar{b}ia) = \mathrm{pt}(i(a \sqcup b)). \tag{5.18}$$

This can be used to combine the branches in the tree greater than i. By induction, a product of k branches is associated to the function

$$\text{pt}(ia_1)\text{pt}(ia_2)...\text{pt}(ia_k) = \text{pt}(i(a_1 \sqcup a_2 \sqcup ... \sqcup a_k)). \tag{5.19}$$

Moving 'down the tree', the branches encountered at each step can be combined using (5.19). □

5.2 The scattering equations and the S bracket

For disjoint subset I and J, write

$$E_{I,J} = \sum_{\substack{i \in I \\ j \in J}} \frac{s_{ij}}{z_{ij}}. \tag{5.20}$$

For $I \subset A$, write

$$E_{I;A} = E_{I,A \setminus I}. \tag{5.21}$$

Due to the antisymmetry of the z_{ij},

$$\sum_{i \in I} E_{i;A} = E_{I,J}. \tag{5.22}$$

Also,

$$\sum_{i \in A} z_i E_{i;A} = s_A. \tag{5.23}$$

When s_A is set to 0, the equations

$$E_{i;A} = 0$$

are called the *scattering equations* (as in the discussion above (5.5)). The following identity relates the scattering equations to the S bracket.

Lemma 5.7. *For disjoint nonempty words a and b,*

$$\text{pt}(a)\, E_{A,B}\, \text{pt}(b) = \text{pt}(\{a, b\}), \tag{5.24}$$

where $a \in \mathfrak{S}(A)$ and $b \in \mathfrak{S}(B)$.

Proof. By (5.16)

$$\mathrm{pt}(aib)\mathrm{pt}(ij)\mathrm{pt}(cjd) = \mathrm{pt}((a \sqcup \tilde{b})ij(\tilde{c} \sqcup d)). \tag{5.25}$$

The Lemma then follows from the definition of the S bracket (Definition 4.1). □

A special case of (5.24) is

$$E_{i,A}\,\mathrm{pt}(a) = \mathrm{pt}(\{i, a\}), \tag{5.26}$$

and this is used in [19] to prove the fundamental BCJ relation from a Grassmannian integral formula.

Corollary 5.8. *Fix $A = \{1, ..., n\}$, and fix the ordering $a = 123...n$. Then*

$$\prod_{i=2}^{n} E_{i,123...i-1} = \sum_{b \in \mathfrak{S}_{n-1}} S(12...n, 1b)\mathrm{pt}(1b). \tag{5.27}$$

Proof. Expanding in a basis gives

$$\mathrm{pt}(\ell\{a\}) = \sum_{b} S(a, ib)\mathrm{pt}(ib), \tag{5.28}$$

where $S(a, ib) = (\ell\{a\}, \ell(ib))$. But $\mathrm{pt}(\ell\{a\})$ can be computed explicitly, using

$$\mathrm{pt}(\ell\{ai\}) = \mathrm{pt}(\ell\{a\})E_{a,i}, \tag{5.29}$$

by Lemma 5.7. □

5.3 Matrix tree identities

This section proves two determinantal identities involving the $E_{I,J}$, which have consequences for CHY formulas, as explained in Section 6.2. It is convenient to first state two variants of the matrix tree theorem, adapted to the discussion in Section 5.1.

Proposition 5.9. (Kirkhoff.) *Let* $\alpha_{ij} = -\alpha_{ji}$ *be the components of an* $n \times n$ *antisymmetric matrix. Write* A *for the matrix with components*

$$A_{ij} = \begin{cases} \alpha_{ij} & i \neq j \\ -\sum_k \alpha_{ik} & i = j \end{cases} \tag{5.30}$$

Let $A[k]$ *be* A *with row* k *and column* k *removed. Then*

$$\det A[k] = \sum_{\substack{trees\\T}} \prod_{\substack{edges\\ i \to j\\ in\ T}} \alpha_{ij}, \tag{5.31}$$

where the sum is over trees T*, rooted at* k*, with the orientation chosen so that* k *is a source, and every other vertex has exactly one incoming edge.*

For a proof, see, e.g., [74]. The symmetric version of this theorem is more well known.

Proposition 5.10. (Symmetric Kirkhoff.) *Let* $\alpha_{ij} = \alpha_{ji}$ *be components of a symmetric matrix. Form* A *as in* (5.30)*. Then*

$$\det A[k] = \sum_{\substack{trees\\T}} \prod_{\substack{edges\\ i - j\\ in\ T}} \alpha_{ij}, \tag{5.32}$$

where the sum is over all trees, rooted at k*.*

Define the *matrix of scattering equations*, A, by

$$A_{ij} = \begin{cases} E_{i,j} & i \neq j \\ -\sum_{k \neq i} E_{i,k} & i = j \end{cases} \tag{5.33}$$

Lemma 5.11. *With* A *as in* (5.33)*, the reduced determinant of* A *is*

$$\det A[1] = \sum_{a \in \mathfrak{S}_{n-2}} \mathrm{pt}(1a) S(1a, 1a), \tag{5.34}$$

where $A[1]$ is the matrix A with column 1 and row 1 removed.

Proof. By Kirkhoff,

$$\det A[1] = \sum_{\text{trees } T} \prod_{z_{ij} \in h_T} \frac{s_{ij}}{z_{ij}}, \tag{5.35}$$

where h_T is defined as in the paragraph preceding Proposition 5.6. Fix 1 as the root of any tree T, and write z_{ix_i} for the functions in h_T. Then, by Proposition 5.6,

$$RHS = \sum_G \prod_{i=2}^n s_{ix_i} \sum_{a, x_i <_a i} \text{pt}(1a). \tag{5.36}$$

Reversing the ordering of the summations gives

$$\det A[1] = \sum_a \text{pt}(1a) \prod_{i=2}^n \sum_{x_i <_a i} s_{ix_i}, \tag{5.37}$$

which is (5.34). □

[75] introduced the following matrix,

$$\Psi_{ij} = \begin{cases} \frac{s_{ij}}{z_{ij} w_{ij}} & i \neq j \\ -\sum_{k \neq i} \Psi_{ik} & i = j \end{cases}, \tag{5.38}$$

where z_{ij} and w_{ij} are coordinates on the direct sum $W_A \oplus W_A$.

Lemma 5.12. *With Ψ as in (5.38), its reduced determinant is*

$$\det \Psi[1] = \sum_{a, b \in \mathfrak{S}_{n-2}} \text{pt}(1a)_z S(1a, 1b) \text{pt}(1b)_w. \tag{5.39}$$

Proof. By (5.8),

$$\sum_{a,b} \text{pt}(1a) S(a,b) \text{pt}(1b) = \sum_{a \in \mathfrak{S}_{n-2}} \text{pt}(1a) \prod_{i=2}^{n-1} \left(\sum_{j <_a i} E_{i,j} \right).$$

Any set z_{ix_i}, for $i = 2...n-1$, with $x_i <_a i$, is a basis. Conversely, any basis h can be written in this way, by the proof of Proposition 5.6. So, reversing the order of the summations,

$$RHS = \sum_{\substack{b_i \neq i \\ \text{the set } z_{ib_i} \text{ a basis}}} \left(\prod_{i=2}^{n-1} E_{i,b_i} \right) \sum_{\substack{a \\ b_i <_a i}} \text{pt}(1a).$$

(5.17) then gives the result. □

[20] prove (5.39) using an indirect residue argument.

5.4 Forms and residues

This section computes the residues at the boundaries of Conf_n of top forms on Conf_n. The relationship between these residues and trees will be used in Chapter 6. The graded signs discussed in Section 3.5 appear naturally here, because the symmetric group acts on top forms on Conf_n according to the sign representation. The aim of this section is to be very explicit about these signs, and the Poincaré residues of top log forms on Conf_n. [25] and [44] also discuss these residues, albeit less explicitly.

Write

$$\omega_{ij} = \frac{1}{2\pi i} \frac{\text{d}z_{ij}}{z_{ij}}. \tag{5.40}$$

Then the de Rham cohomology $H^i(W_A^*, \mathbb{C})$ is generated by the classes $[\omega_{ij}]$. In particular, $H^1(W_A^*, \mathbb{C})$ has dimension $|A|(|A|-1)/2$. The cohomology ring is isomorphic to the exterior algebra generated by the ω_{ij}. It follows from the results of [65] that the top component $H^{n-2}(W_A^*)$ is linearly generated by the forms

$$\omega_h = \omega_{x_1} \wedge ... \wedge \omega_{x_{n-2}}, \tag{5.41}$$

for ordered bases $(x_1, ..., x_{n-2})$. In other words,

$$H^{n-2}(W_A^*) \simeq \wedge^{n-2} W_A \otimes S_A. \tag{5.42}$$

Any top form in $\wedge^{n-2} W_A$ is proportional to

$$e_a = \text{d}z_{a(1)a(2)} \wedge ... \wedge \text{d}z_{a(n-1)a(n)}, \tag{5.43}$$

for some fixed choice of ordering a.

Lemma 5.13. (Signs.) *For a permutation $\sigma \in S_A$,*

$$e_a = (-1)^{\mathrm{sgn}(a)} e_{\sigma a}. \tag{5.44}$$

Proof. Expanding (5.43),

$$e_a = \sum_{k=1}^{n} (-1)^k \mathrm{d}z_{a(1)} \wedge \ldots \wedge \mathrm{d}\hat{z}_{a(k)} \wedge \ldots \wedge \mathrm{d}z_{a(n)}. \tag{5.45}$$

Consider a transposition, (ij). Each term in the sum, (5.45), changes by a minus sign under (ij). □

This Lemma means that $\wedge^{n-2} W_A$ can be identified with the sign representation of the symmetric group, S_A. Write

$$\omega(a) = e_a \, \mathrm{pt}(a). \tag{5.46}$$

These top forms span $H^{n-2}(W_A^*)$. They satisfy the following relations, analogous to Lemma 5.4.

Lemma 5.14. (Graded Kleiss-Kuijf.) *For disjoint words $a, b \in W(A)$,*

$$\omega(a \sqcup\!\sqcup_{\mathrm{gr}} b) = 0, \tag{5.47}$$

where $\sqcup\!\sqcup_s$ is the super shuffle product. In particular,

$$\omega(bic) = \omega(i(\tilde{b} \sqcup\!\sqcup_{\mathrm{gr}} c)), \tag{5.48}$$

for disjoint words b and c, not containing i.

Proof. This is a consequence of Lemma 5.4 and the sign Lemma, Lemma 5.13, together with the definition of $\sqcup\!\sqcup_{\mathrm{gr}}$ in Remark 3.12. □

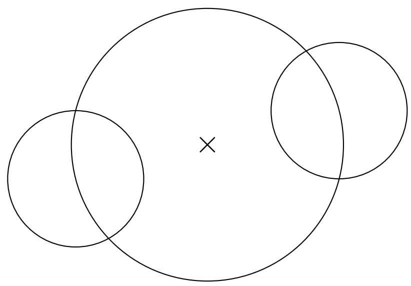


Figure 5.3: Orbit product.

Remark 5.15. *Write*

$$F_A = d\log f_A = \sum_{\{i,j\}\subset A} s_{ij}\, d\log z_{ij}, \tag{5.49}$$

for the $d\log$ *of the Koba-Nielsen function* f_A*. Then, for orderings* $a \in \mathfrak{S}(A)$ *and* $b \in \mathfrak{S}(B)$*,*

$$\omega(a) \wedge F_{A\cup B} \wedge \omega(b) = \omega(\{a, b\}_{gr}), \tag{5.50}$$

where $\{a, b\}_{gr}$ *is the super version of the S bracket,* $\{\ ,\ \}$*.*

To compute residues of the forms $\omega(a)$, appropriate contours can be constructed as follows. Write $C_{[i,j]}$ for the cycles

$$C_{[i,j]} : S^1 \to W^*_{ij} \tag{5.51}$$

that map S^1 to the circle $|z_{ij}| = 1$. Orient $C_{[i,j]}$ so that

$$\int_{C_{[i,j]}} \omega_{ij} = 1. \tag{5.52}$$

With the obvious inclusion, the classes $[C_{[i,j]}]$ give a basis for $H_1(W^*_A, \mathbb{C})$ that is dual to the basis $[\omega_{ij}]$ of $H^1(W^*_A, \mathbb{C})$, (5.40). Fix two disjoint $B, C \subset A$, and take two cycles,

$$C_1 : S^{|B|-1} \to W^*_B \qquad \text{and} \qquad C_2 : S^{|C|-1} \to W^*_C, \tag{5.53}$$

that are non-homotopic to zero, where S^r is the r-torus. There is an *orbit map* (a good review is [72])

$$\lambda_{B,C} : W_{12}^* \times W_B^* \times W_C^* \to W_{B\cup C}^*, \tag{5.54}$$

which sends $(z_1, z_2), (z_i)_{i\in B}, (z_j)_{j\in C}$ to $(rz_1 + z_i; rz_2 + z_j)$, for sufficiently large r.[2] Concatenating $C_{[1,2]} \times C_1 \times C_2$ with $\lambda_{B,C}$ gives a cycle

$$C_1 \times C_2 : S^{|B|+|C|+1} \to W_{B\cup C}^*. \tag{5.55}$$

The cycle $C_1 \times C_2$ is the orbit of C_1 and C_2 around each other. The following is proved in [33], and goes back to [15]:

Lemma 5.16. (Orbit map.) *The orbit map induces a product on homology,* $\times : H_i \otimes H_j \to H_{i+j+1}$, *which is graded symmetric,*

$$C_1 \times C_2 = (-1)^{|B||C|+1} C_2 \times C_1, \tag{5.56}$$

and satisfies the graded Jacobi identity.

For $\alpha \in SL^+(A)$ a super Lie monomial, write $[C_\alpha]$ for the homology class inductively defined by

$$[C_{[\alpha,\beta]}] = [C_\alpha] \times [C_\beta]. \tag{5.57}$$

Lemma 5.17. (Cohen.) *For a monomial* $\alpha \in SL^+(A)$ *and an ordering* $a \in \mathfrak{S}(A)$,

$$\int_{C_\alpha} \omega_a = (a, \alpha)_{\mathrm{gr}}. \tag{5.58}$$

In particular, the cycles

$$C_{\ell(ia)}, \tag{5.59}$$

give a basis of $H_{n-2}(W_A^*)$ *that is dual to the forms* ω_{ia}.

Proof. For super Lie monomials $\alpha \in SL^+(A)$ and $\beta \in SL^+(B)$, and orderings $a \in \mathfrak{S}(A)$ and $b \in \mathfrak{S}(B)$,

$$\int_{C_\alpha \times C_\beta} \omega_a \wedge \omega_{ij} \wedge \omega_b = \int_{C_\alpha} \omega_a \cdot \int_{C_\beta} \omega_b, \tag{5.60}$$

[2]For example, r could be taken to be $(\sum_{i\in B, j\in C} |z_{ij}|)/|z_{12}|$.

for any $i \in A$ and $j \in B$. (5.60) is well defined since

$$\omega_a \wedge \omega_{ij} \wedge \omega_b = (-1)^{(|A|+1)(|B|+1)+1} \omega_b \wedge \omega_{ij} \wedge \omega_a, \tag{5.61}$$

which agrees with the sign, (5.56). □

5.A $\mathcal{M}_{0,n}$

Let $\overline{\mathcal{M}}_{0,n}$ be the compactification of $\mathcal{M}_{0,n}$, with divisor D. The open stratum, $\mathcal{M}_{0,n}$, is

$$\mathcal{M}_{0,n} \simeq W_A^*/\mathbb{C} \ltimes \mathbb{C}, \tag{5.62}$$

where $\mathbb{C} \ltimes \mathbb{C}$ acts as $(a, b) : z \mapsto az + b$. Then

$$H^i(\overline{\mathcal{M}}_{0,n}, D) = H^i(W_A^*, \Delta)^{\mathbb{C} \ltimes \mathbb{C}}, \tag{5.63}$$

where $H^i(W_A^*, \Delta)^{\mathbb{C} \ltimes \mathbb{C}}$ denotes the $\mathbb{C} \ltimes \mathbb{C}$ invariant classes. Write

$$f_s = \prod_{i<j} z_{ij}^{s_{ij}}, \tag{5.64}$$

for the Koba-Nielsen function. Introduce coordinates on $H^1(W_A^*, \Delta)$ using the isomorphism

$$\pi : \tilde{K}_A \to H^1(W_A^*, \Delta) : (s_{ij}) \mapsto \mathrm{d} \log f_s. \tag{5.65}$$

The function f_s is invariant under $\mathbb{C} \ltimes \mathbb{C}$ iff

$$\sum_{i<j} s_{ij} = 0. \tag{5.66}$$

Write $K_A \subset \tilde{K}_A$ for the hyperplane defined by (5.66). Then $H^1(\overline{\mathcal{M}}_{0,n}, D)$ can be identified with the image of K_A under π. In fact, by dimension counting,

$$K_A \simeq H^1(\overline{\mathcal{M}}_{0,n}, D). \tag{5.67}$$

Top cohomology. The cohomology ring is described in detail by [16]. The top cohomology, $H^{n-3}(\overline{\mathcal{M}}_{0,n}, D)$, is in one lower degree, compared to $H^{n-2}(W_A^*, \mathbb{C})$, because the $\mathbb{C} \ltimes \mathbb{C}$ action can be used to fix two points. Fix $z_1 = 0$ and $z_{n-1} = 1$, say. Write

$$e_a = \mathrm{sgn}(a)\mathrm{d}z_2 \wedge ... \wedge \mathrm{d}z_{n-1}, \tag{5.68}$$

where $\mathrm{sgn}(a)$ is the sign of the permutation $123...n-1 \to a$. Then define the volume forms

$$\omega(a) = e_a \mathrm{pt}(a), \tag{5.69}$$

where $z_1 = 0$ and $z_{n-1} = 1$. These span $H^{n-3}(\overline{\mathcal{M}}_{0,n}, D)$ and, by Lemma 5.14, they satisfy $\omega(b \sqcup\!\sqcup_{\mathrm{gr}} c) = 0$, for nontrivial shuffles. Alternatively, an $SL_2\mathbb{C}$-covariant definition of $\omega(a)$ can be taken to be

$$\omega(a) = z_{1n-1}z_{n-1n}z_{n1}\, e_a \mathrm{PT}(an), \tag{5.70}$$

where

$$\mathrm{PT}(123...n) = \frac{1}{z_{12}z_{23}...z_{n-1n}z_{n1}}. \tag{5.71}$$

It is easily checked that (5.69) and (5.70) agree in the appropriate gauge-fixing.

Cross ratios. Write

$$(ij|kl) := \frac{z_{ik}z_{jl}}{z_{il}z_{jk}} \tag{5.72}$$

for the cross-ratios of $z_1, ..., z_n$, where $z_n = \infty$. Fix $a \in \mathfrak{S}(A)$. This defines a cyclic ordering of the points $1, ..., n$ around the disk. For each diagonal of this disk, $a(i) - a(j)$, write u_{ij} for the cross-ratio

$$u_{ij} = u_{ji} := (a(i)a(i-1)|a(j)a(j-1)). \tag{5.73}$$

For the diagonals $a(i) - n$,

$$u_{in} := (a(i)a(i-1)|na(n-1)) = \frac{z_{a(i-1)a(n-1)}}{z_{a(i)a(n-1)}}. \tag{5.74}$$

For disjoint subsets $A, B \subset \{1, ..., n\}$, write

$$u_{A,B} = \prod_{i \in A, j \in B} u_{ij}. \tag{5.75}$$

Choose any four points, increasing in the given cyclic ordering, $i < j < k < l$. Let $A \sqcup B \sqcup C \sqcup D$ be the associated partition of $\{1, ..., n\}$ into four subsets, where $A = \{i, ..., j-1\}$, and so on. It is an observation of [53] that

Lemma 5.18. (Koba-Nielsen.) *The products of the cross-ratios u_{ij} satisfy,*

$$u_{A,C} + u_{B,D} = 1, \tag{5.76}$$

for A, B, C, D as above.

Dihedral coordinates. Treating the u_{ij} as formal variables, the equations, (5.76), define an affine variety in $k[u_{ij}]$. Write $\mathcal{M}_{an}$ for this variety, and $\iota : \mathcal{M}_{an} \to \overline{\mathcal{M}_{0,n}}$ for its inclusion into $\overline{\mathcal{M}_{0,n}}$, induced by the map defined by evaluating the cross-ratios. For each triangulation of the n-gon, α, write u_I for the cross-ratios associated to the $n-3$ diagonals $I \in P(\alpha)$. For any choice of α, the associated u_I form a set of *dihedral coordinates*. For example, fix $a = 123...n-1$. Take $\alpha = \ell(123...n-1)$, then the associated dihedral coordinates are the cross-ratios u_{in}, for $i = 2, ..., n-2$. The other cross-ratios are rational expressions in the u_{in}, namely,

$$u_{ij} = \frac{w_{ij} w_{i-1j-1}}{w_{ij-1} w_{i-1j}}, \tag{5.77}$$

where $w_{ij} = 1 - u_{[ij],n}$.

Top forms in dihedral coordinates. Fixing $z_n = \infty$,

$$\frac{u_{in}}{1 - u_{in}} = \frac{z_{1i-1}}{z_{i-1i}}. \tag{5.78}$$

So the volume form ω_{1a} can be written

$$\omega_{1a} = \bigwedge_{i=3}^{n-1} \mathrm{d}\log \frac{u_{1i}}{1-u_{1i}}. \tag{5.79}$$

Alternatively,[52] [53] write this as

$$\omega_{1a} = \frac{\mathrm{d}u_{2n} \wedge ... \wedge \mathrm{d}u_{n-2n}}{w_{23}w_{34}...w_{n-3n-2}}. \tag{5.80}$$

Recall the definition of the cycles, C_α, in configuration space, W_A^*. Fix $a \in \mathfrak{S}(A)$. Let $C_{\alpha,a}$ be the image of C_α in $\mathcal{M}_{an}$. The cycle $C_{\alpha,a}$ the torus around the point D_α defined by the intersection of $u_I = 0$, for all $I \in P(\alpha)$. In other words, the Poincaré dual of $C_{\alpha,a}$ is the top degree form

$$e_{\alpha,a} = \bigwedge_{I \in P(\alpha)} \frac{1}{2\pi i} \mathrm{d}\log u_I, \tag{5.81}$$

where the overall sign is fixed by the orientation of $C_{\alpha,a}$.

5.B The contact term map

The contact term map, studied in Section 4.4, induces a map on homology. For a class $P \in H_{|B|+|C|-1}(W_A^*)$, define δP by

$$(P, \omega(\{b,c\})) = (\delta P, \omega(a) \otimes \omega(b)), \tag{5.82}$$

for all disdjoint orderings $b \in \mathfrak{S}(B)$ and $c \in \mathfrak{S}(C)$. The pairing, (,), is integration, extended to take values in Mandelstam variables. A consequence of Remark 5.15 is that

$$\int_{\delta C} \omega(a) \otimes \omega(b) = \int_C \omega(b) \wedge F_A \wedge \omega(c), \tag{5.83}$$

with $F_A = \mathrm{d}\log f_A$, as in (5.49).

The contact term map appears in superstring computations in [59], and in follow up articles. That reference does computations with superfields T_α, corresponding to

Lie monomials α. (In fact, only α of the form $\ell(a)$ are considered in [59]. There, they construct the $T_{\ell(a)}$ by by taking a nested iteration of OPEs. Such a nested series of OPEs is defined, up to sign, by a Lie monomial α.) The action of the BRST charge is experimentally found to be

$$QT_\alpha = \sum_{b,c} (\delta(\alpha), b \otimes c) T_{\ell(b)} T_{\ell(c)}, \tag{5.84}$$

where δ is the contact term map. The superfields T_α in [59] are defined in a complicated manner, but they are closely related to a nested residue. Ignoring the complications, write

$$T'_\alpha := \int_{C_\alpha} \Omega_A, \tag{5.85}$$

for such a residue, where Ω_A is a top log-form in $H^{|A|-1}(W_A^*, \mathbb{C})$. Let Ω_B and Ω_C be top forms on W_B^* and W_C^*, respectively. Then, by (5.83),

$$\int_{\delta C} \Omega_B \otimes \Omega_C = \int_C \Omega_B \wedge F_{B\cup C} \wedge \Omega_C. \tag{5.86}$$

So an identity amoung the residues T'_α of the form (5.84) could be written in the form

$$QT'_\alpha = \sum_{B,C} \int \Omega_B \wedge F_A \wedge \Omega_C. \tag{5.87}$$

This might explain the appearance of identities like (5.84).

Chapter 6

CHY formulas

Cachazo-He-Yuan [21] proposed string-like formulas for the tree partial amplitudes of several massless gauge theories (YM, NLSM, biadjoint scalar), and for the tree amplitudes of perturbative gravity (and also for gravity-like theories, such as the galileon and Dirac-Born-Infeld theories). In [21], these formulas are written as integrals of the form

$$\int \mathrm{d}\mu(a) \left(z_{ij} z_{jk} z_{ki} \prod\nolimits_{l \neq i,j,k} \delta(E_l) \right) I, \tag{6.1}$$

for functions I of appropriate weight, and where $E_i = E_{i;A}$ are the scattering equation functions. The $\mathcal{M}_{0,n}$ volume form $\mathrm{d}\mu(a)$ is defined by (5.69), or its $SL_2\mathbb{C}$-covariant version, (5.70). As written, (6.1) is not well defined. It is understood to denote a sum of residues at the solutions of $E_i = 0$. So (6.1) can be expanded as

$$\int \mathrm{d}\mu(a) \left(z_{ij} z_{jk} z_{ki} \prod\nolimits_{l \neq i,j,k} \delta(E_l) \right) I, = \sum_x \mathrm{Res}_x \, \frac{\mathrm{d}\mu(a) I}{\det(\partial_i E_j)}, \tag{6.2}$$

where the sum is over all solutions to the equations $E_i = 0$. Following [60], this sum of residues is sometimes written as,

$$\int \mathrm{d}\mu(a) \left(z_{ij} z_{jk} z_{ki} \bigwedge_{l \neq i,j,k} \overline{\partial} \left(\frac{1}{E_l} \right) \right) I. \tag{6.3}$$

Notice that sign choices are left implicit in these residue formulas. It is conventionally assumed that the residue contour at a solution to $E_1, ..., E_{n-3} = 0$ is oriented by the form $\mathrm{d}\log E_1 \wedge ... \wedge \mathrm{d}\log E_{n-3}$. This involves fixing a particular ordering of the E_i.

One example of a CHY formula is the formula for the biadjoint scalar partial tree amplitudes, $m(a, n|b, n)$.

$$m(a, n|b, n) = \int \mathrm{d}\mu(a) \left(z_{ij} z_{jk} z_{ki} \prod\nolimits_{l \neq i,j,k} \delta(E_l) \right) \mathrm{PT}(b, n), \tag{6.4}$$

where $\mathrm{PT}(bn)$ is the Parke-Taylor function, (5.71). This formula was proved via an unenlightening inductive argument in [28]. Appendix 6.A expresses the formula in dihedral coordinates, which clarifies the proof. A second example is the CHY formula for the Yang-Mills partial tree amplitudes,

$$A(a, n) = \int \mathrm{d}\mu(a) \left(z_{ij} z_{jk} z_{ki} \prod\nolimits_{l \neq i,j,k} \delta(E_l) \right) \mathrm{Pf}'\Psi, \tag{6.5}$$

where $\mathrm{Pf}'\Psi$ is the Pfaffian of the matrix that appears in (5.3). The CHY formula, (6.5), closely resembles the superstring gluon tree amplitude formula, (5.1). The existence of the CHY formula motivated work to derive it from string theory, as in [60] and [62]. The equations $E_i = 0$ are the saddle point equations for (5.1), but the CHY formula, (6.5), is not the saddle-point approximation of the superstring integral, (5.1), for large α'. The saddle point approximation to (5.1) would give a sum over solutions to $E_i = 0$ of the form

$$\sum_x \mathrm{Res}_x \frac{\mathrm{d}\mu(a)\, f_s\, I}{\det(\partial_i E_j)}. \tag{6.6}$$

The Koba-Nielsen function, f_s, has different values at different solutions, and so (6.6) is not equal to (6.5). No completely satisfying derivation of (6.5) has been given from the superstring formula. The ambitwistor string models [60] reproduce the CHY formulas at tree level, and there are several known links between the ambitwistor models and the full superstring models. [23]

This chapter applies the results of Chapter 5 to deduce facts about CHY integrals. The main result is a formula for non-linear sigma model (NLSM) tree amplitudes in 'BCJ form'.

This chapter is also a bridge between the results about trees discussed in Chapters 4 and 5, and the rest of the thesis. The biadjoint scalar partial amplitudes,

$m(a,n|b,n)$, have an 'ABHY presentation' in terms which are directly related to the residues in Conf_n discussed in Section 5.4. The ABHY presentation of the partial amplitudes is in terms of an associahedron in the vector space of Mandelstam variables, K_n. The construction was motivated by the CHY formulas, but the ABHY construction, unlike the CHY formulas, introduces ideas which clearly generalise to the full Feynman perturbation series.

6.1 CHY formulas

Fix any function I that transforms appropriately under Möbius transformations, and write

$$A(a,n) = \int \mathrm{d}\mu(a) \left(z_{ij} z_{jk} z_{ki} \prod\nolimits_{l \neq i,j,k} \delta(E_l) \right) I \tag{6.7}$$

for the associated CHY integrals.

Corollary 6.1. *For two disjoint orderings, $b \in \mathfrak{S}(B)$ and $c \in \mathfrak{S}(C)$, the CHY integrals satisfy*

$$A(\{b,c\},n) = 0. \tag{6.8}$$

Proof. The integral, (6.7), vanishes if the integrand is proportional to E_i. Moreover, any $E_{I,J}$ appearing in $\{b,c\}$ is a sum of E_i, by (5.22), and so $A(\{b,c\},n)$ vanishes by (5.24). □

Moreover, it follows from Corollary 5.8 that

Corollary 6.2. *Fix i. For any ordering $a \in \mathfrak{S}_{n-1}$, the CHY integrals satisfy*

$$\sum_{b \in \mathfrak{S}_{n-2}} S(a, ib) A(ib, n) = 0. \tag{6.9}$$

Lemma 5.11 showed that

$$\det A[1] = \sum_{a \in \mathfrak{S}_{n-2}} \mathrm{pt}(1a) S(1a, 1a), \tag{6.10}$$

where A is the matrix of scattering equations, with $A_{ij} = E_{i,j}$, as in (5.33). The tree partial amplitudes for the non-linear sigma model (NLSM) are given by a CHY formula with integrand

$$I = \det A[1], \tag{6.11}$$

so that

$$A_{\text{NLSM}}(a,n) = \int \mathrm{d}\mu(a) \left(\prod\nolimits_i' \delta(E_i)\right) \det A[1]. \tag{6.12}$$

Lemma 6.3. (NLSM numerators.) *The NLSM partial amplitudes have an expansion*

$$A(a,n) = \sum_\alpha \frac{(a,\alpha)N_\alpha}{s_\alpha}, \tag{6.13}$$

where the N_α are local BCJ numerators given by

$$N_\alpha = \sum_{a \in \mathfrak{S}_{n-2}} (1a,\alpha) S(1a,1a). \tag{6.14}$$

Proof. $A(a,n)$ may be written as

$$A(a,n) = S(1b,1b)m(1b,n|a,n). \tag{6.15}$$

It is clear from (6.14) that if $\alpha + \beta + \gamma = 0$, then

$$N_\alpha + N_\beta + N_\gamma = 0, \tag{6.16}$$

so that the N_α are local BCJ numerators (in the sense of Section 2.3), because N_α has no poles in the Mandelstam variables. □

A formula similar to (6.14) was conjectured in [22] and [56]. It is significant that these numerators can also be computed from the Feynman rules of the Lagrangian studied in [24] and [64]. There have been a number of other attempts to compute numerators for other theories from CHY formulas, including [10] and [32].

6.2 (Off-shell) ABHY hyerplanes

The ABHY presentation [2] of CHY formulas uses a natural set of *ABHY hyperplanes* in the vector space, K_A, of Mandelstam variables, as well as differential forms on K_A with logarithmic singularities. This section begins by making explicit how these hyperplanes and forms arise from the topology of configuration spaces and $\mathcal{M}_{0,n}$.

Fix $A = \{1, ..., n-1\}$. As in Chapter 5, write W_A^* for the configuration space of $n-1$ points, and write $\tilde{K}_A$ for the vector space of Mandelstam variables, with coordinates s_{ij}, $1 \leq i < j \leq n-1$. There is a map

$$\pi : s_{ij} \mapsto s_{ij}\omega_{ij}, \tag{6.17}$$

from $\tilde{K}_A$ to $H^1(W_A^*)$, and by dimension counting, this is an isomorphism. The map π induces a double fibration,

$$\begin{array}{ccc} & \tilde{K}_A \times W_A^* & \\ \swarrow & & \searrow^{\pi} \\ \tilde{K}_A & & T^*W_A^* \end{array} \tag{6.18}$$

Introduce coordinates τ_i on $T^*W_A^*$ by writing a generic 1-form at a point in W_A^* as

$$\sum \tau_i \mathrm{d}z_i. \tag{6.19}$$

Then the closed 2-form,

$$\omega = \sum_{i \in A} \mathrm{d}\tau_i \wedge \mathrm{d}z_i, \tag{6.20}$$

pulls back under π to give

$$\pi^*\omega = \sum_{i<j} \mathrm{d}s_{ij} \wedge \frac{\mathrm{d}z_{ij}}{z_{ij}}. \tag{6.21}$$

Lemma 6.4. *The top power of $\pi^*\omega$ is*

$$(\pi^*\omega)^{n-2} = \sum_h \bigwedge_{z_{ij} \in h} \left(ds_{ij} \wedge \frac{dz_{ij}}{z_{ij}} \right), \tag{6.22}$$

where the sum is over all bases, h, of $W_A^\vee$.

Proof. For any subset $h \subset H_A$ of size $n-2$, the form

$$\bigwedge_{z_{ij} \in h} \mathrm{d}z_{ij} \tag{6.23}$$

is nonvanishing iff h is a set of linearly independent functions. This implies (6.21). There is no ambiguity about the sign because it is a wedge product of 2-forms. □

Write

$$w_\alpha := \int_{C_\alpha} (\pi^* \omega)^{n-2} \tag{6.24}$$

for the residues at C_α of $(\pi^*\omega)^{n-2}$. These residues are $n-2$ forms on $\tilde{K}_A$.

Lemma 6.5. *The forms w_α are given explicitly by*

$$w_{[\alpha,\beta]} = \omega_\alpha \wedge ds_{A\cup B} \wedge \omega_\beta, \tag{6.25}$$

for α a Lie monomial in $SL^+(A)$ and β a Lie monomial in $SL^+(B)$. In particular, the forms w_α satisfy the graded Jacobi identity.

Proof. It follows from (5.60) that

$$w_{[\alpha,\beta]} = w_\alpha \wedge \mathrm{d}s_{A,B} \wedge w_\beta. \tag{6.26}$$

But

$$\mathrm{d}s_{A,B} = \mathrm{d}s_{A\cup B} - \mathrm{d}s_A - \mathrm{d}s_B, \tag{6.27}$$

so that (6.25) follows, inductively assuming $w_\alpha \wedge \mathrm{d}s_A = 0$. □

Let $H_a \subset K_A$ be the hyperplane defined by

$$s_{a(i)a(j)} = 0 \tag{6.28}$$

for all $i < j-1$, and oriented by the form

$$\mathrm{d}s_{a(1)a(2)} \wedge \ldots \wedge \mathrm{d}s_{a(n-2)a(n-1)}. \tag{6.29}$$

The next Lemma shows that these hyperplanes are dual to the forms w_α.

Lemma 6.6. (ABHY planes and forms.) *Let* (,) *be the pairing between forms and planes on* $\tilde{K}_A$*. Then*

$$(P_a, w_\alpha) = (a, \alpha)_{\mathrm{gr}}. \tag{6.30}$$

Proof. The restriction of $(\pi^*\omega)^{n-2}$ to P_a is

$$P_a \lrcorner \, (\pi^*\omega)^{n-2} = \omega_a. \tag{6.31}$$

So the Lemma is a consequence of Lemma 5.17. □

Combining the above results implies the following explicit formula for $(\pi^*\omega)^{n-2}$. Fix $i \in A$, then

$$(\pi^*\omega)^{n-2} = \sum_{a \in \mathfrak{S}_{n-2}} w_{\ell(ia)} \wedge \omega(ia). \tag{6.32}$$

Likewise, for any choice of dual bases, there is an expansion of $(\pi^*\omega)^{n-2}$ of this form.

6.3 The ABHY presentation

The 'off shell' ABHY hyperplanes and forms in the previous section have the following 'on shell' restriction to the momentum conserving hyperplane. As in Appendix 5.6.1, write $K_A \subset \tilde{K}_A$ for the momentum conserving hyperplane

$$\sum_{i<j} s_{ij} = 0. \tag{6.33}$$

Then the double fibration, (6.18), has the following restriction to $\mathcal{M}_{an}$,

$$\begin{array}{ccc} & K_A \times \mathcal{M}_{an} & \\ \swarrow & & \overset{\pi}{\searrow} \\ K_A & & T^*\mathcal{M}_{an} \end{array} \tag{6.34}$$

The projection, π, is

$$\pi : (s_{ij}) \mapsto \sum_{I \in \mathrm{Ord}_a} X_I \mathrm{d}\log u_I, \tag{6.35}$$

where the sum is over all diagonals of the n-gon. The map π is again an isomorphism, $K_A \simeq H^1(\mathcal{M}_{an}, D)$.

Let $v \in \tilde{K}_A$ be the vector pointing in the $(1, ..., 1)$ direction, so that the momentum conserving hyperplane is the orthogonal space, $K_A = v^\perp$. Then K_A is dual to the quotient $\tilde{K}_A/v$. The $n-2$ plane $\tilde{H}_A \subset \tilde{K}_A$ intersects the momentum conserving hyperplane K_A in a $n-3$ plane, $H_A \subset K_A$. Dually, the $n-2$ form $\tilde{w}_\alpha$ defines an $n-3$ form on $\tilde{K}_A/v$,

$$w_\alpha = v \lrcorner \tilde{w}_\alpha. \tag{6.36}$$

Then Lemma 6.6 implies

Lemma 6.7. (ABHY planes and forms.) *Let* (,) *be the pairing between forms and planes on* K_A*. Then*

$$(H_a, w_\alpha) = (a, \alpha)_{\mathrm{gr}}. \tag{6.37}$$

For each $b \in \mathfrak{S}(A)$, define a form on K_A with logarithmic poles on the hyperplanes $X_A = 0$,

$$\Omega_b = \sum_\alpha \frac{(b, \alpha)_{\mathrm{gr}}\, w_\alpha}{s_\alpha}. \tag{6.38}$$

Contracting this with the plane H_a gives

$$H_a \lrcorner \Omega_b = \sum_\alpha \frac{(a, \alpha)_{\mathrm{gr}}\, (b, \alpha)_{\mathrm{gr}}}{s_\alpha} = \sum_\alpha \frac{(a, \alpha)\, (b, \alpha)}{s_\alpha}. \tag{6.39}$$

This is the ABHY presentation of the CHY formula for the biadjoint scalar amplitude $m(a, n|b, n)$.

6.4 ABHY associahedra

Let $H_a \subset K_A$ be the hyperplane as above, and let K_A^+ be the positive orthant, defined by $s_{ij} > 0$. Write v for some integer vector in K_A^+, given by

$$s_{ij} = c_{ij}, \tag{6.40}$$

for some positive integers c_{ij}. Write $H_a + v$ for the translation of H_a by v. Then the intersection

$$P := (H_a + v) \cap K_A^+ \tag{6.41}$$

is a polytope. In principle, it could be an unbounded polytope, and its face structure could depend on the choice of vector, v. [2] found that

Proposition 6.8. *(ABHY.)* For any choice of $v \in K_A^+$, the polytope $P = (H_a + v) \cap K_A^+$ is the associahedron.

This is a consequence of the following two lemmas.

Lemma 6.9. (Incompatible diagonals.) *If ij and kl are crossing diagonals, then $H_{ij} \cap H_{kl}$ does not intersect P.*

Proof. Fix the ordering $a = 12...n-1$. Write

$$X_{ij} = \sum_{i \leq k < l < j} s_{kl}, \tag{6.42}$$

for $i < j \leq n$, and $(i,j) \neq (1,n)$. Consider any two diagonals, ij and kl that cross each other, with

$$1 \leq i < k < j < l \leq n. \tag{6.43}$$

Then

$$X_{ij} + X_{kl} - X_{il} - X_{kj} = \sum_{\substack{i \leq a < k \\ j \leq b < l}} s_{ab}. \tag{6.44}$$

But the sum on the RHS is strictly positive. It follows that $X_{ij} = 0$ and $X_{kl} = 0$ do not intersect in $(H_a + v) \cap K_A^+$. $\square$

When $n = 4$, P_{123} is the intersection of $s_{13} = c_{13}$ and K_A^+, which is a line segment. This is the A_1 associahedron. Suppose that it is known that P_a is an associahedron for $|a| < n-1$. Then, for $|a| = n-1$,

Lemma 6.10. (Factorization. [3]) *The face $H_{ij} \cap P_a$ is a polytope whose face lattice is the cross product of two associahedra.*

Proof. The X_{ij} form a complete set of coordinates for K_A. Write $K_{B,C} \subset K_A$ for the subspace spanned by the directions X_{ij} for $i \in B$ and $j \in C$. Now fix ij, a diagonal, with $i < j$. Write $B = \{1, ..., i-1\}$, $C = \{i+1, ..., j-1\}$ and $D = \{j+1, ..., n-1\}$. Also write $a = bicjd$, for orderings $b = 12...i-1$, $c = i+1...j-1$, and $d = j+1...n-1$. Lemma 6.9 implies that $X_{ij} \cap P_a$ does not intersect the boundary of the orthant K_A^+ on the subspace $K_{C,B\cup D} \subset K_A$. On the other hand, ij is compatible with K_{ijC} and K_{ijBD}. The intersection of $H_{ij} \cap P_a$ with the subspace K_{ijC} is an associahedron. By (6.44),

$$H_{ij} \cap P_a \cap K_{ijC}^+ = (H_{icj} + v_C) \cap K_{iCj}^+, \tag{6.45}$$

where v_C is the projection of v onto $K_{ijC} \subset K_A$. Likewise,

$$H_{ij} \cap P_a \cap K_{BijD}^+ = (H_{bijc} + v_{BijC} + v') \cap K_{BijD}^+, \tag{6.46}$$

where v_{BijC} is the projection of v onto K_{BijD}, and where v' is

$$\sum_{k \in B\cup D} \left(\sum_{l \in C} c_{kl} \right) e_{ik}, \tag{6.47}$$

where e_{ij} is the unit vector in the s_{ij} direction. □

The vertices of P_a are labelled by triangulations, α, of the n-gon. The vertex associated to α is the intersection of H_a with the hyperplanes $X_I = 0$, $I \in P(\alpha)$.

One interest of this particular family of polytopes, P_a, is their connection to the partial tree amplitudes, $m(a,n|b,n)$. Section 6.3 defined a form, Ω_a on K_A that has poles on the $X_I = 0$ hyperplanes in K_A. The restriction of Ω_b to the hyperplane H_a is the partial amplitude

$$H_a \lrcorner \Omega_b = m(a,n|b,n). \tag{6.48}$$

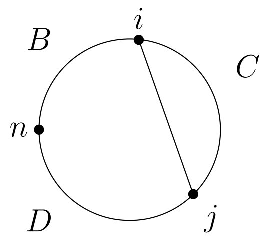


Figure 6.1: The sets B, C, D used in the discussion of the $X_{ij} = 0$ face.

$m(a, n|a, n) = H_a \lrcorner \Omega_a$ can be regarded as the volume of the dual polytope of P_a, as explained in the following remark.

Remark 6.11. (Volumes.) *Take P any simple polytope (i.e. each vertex has valence equal to the dimension), containing the origin, and let F be its normal fan in $V^\vee$. The rays of F are generated by some vectors $w_i \in V^\vee$. The polytope P can be defined by the intersection of the halfplanes $v \cdot w_i \leq 1$. The dual polytope P' cut out by the halfplanes $v_i \cdot w \leq 1$. For $w \in V^\vee$, define the 'volume' of P',*

$$Vol(P')(w) = \sum_{\text{vertices of } P'} \frac{|\det(v_1, ..., v_n)|}{\prod_i (v_i \cdot w)}. \tag{6.49}$$

Now consider the special situation in which, for every vertex of P, the determinant of the associated vectors, $\det v_1 ... v_n$, is ± 1. In this case, define the differential volume form

$$\Omega := \sum_{\text{vertices of } P'} \det[v_{a(1)} ... v_{a(n)}]^{-1} d\log(v_{a(1)} \cdot w) \wedge ... \wedge d\log(v_{a(n)} \cdot w). \tag{6.50}$$

This is equal to $Vol(P')$, multiplied by a top form $\pm d(v_1 \cdot w) \wedge ... \wedge d(v_n \cdot w)$, for some choice of vertex of P'. Ω_a, defined in (6.38) is such a volume form for the ABHY associahedron.

6.A Remark on the proof of CHY formulas

Fix two orderings $a, b \in \mathfrak{S}_n$. Let $\mathrm{d}\mu(a,b)$ be the Möbius invariant CHY measure,

$$\mathrm{d}\mu(a,b) := \frac{\mathrm{d}z_1 ... \mathrm{d}z_n}{\mathrm{d}z_r \mathrm{d}z_s \mathrm{d}z_t} \left(\prod_{i \neq r,s,t} \delta(E_i) \right) \mathrm{PT}(a)\mathrm{PT}(b). \tag{6.51}$$

CHY integrals are then of the form

$$\int \mathrm{d}\mu(a,b)\, I, \tag{6.52}$$

for Möbius invariant functions I. Under a Möbius transformation,

$$z \to \zeta = \frac{az+b}{cz+d}, \tag{6.53}$$

the functions E_i transform as

$$E_i(z) \mapsto E_i(\zeta) = \frac{(cz_i+d)^2}{ad-bc} E_i(z) + \frac{c(cz_i+d)}{ad-bc} s_{i,A}. \tag{6.54}$$

When $s_{i,A} = 0$,

$$\prod_{i \neq r,s,t} \delta(E_{i;A}) \to \prod_{i \neq r,s,t} \frac{ad-bc}{(cz_i+d)^2} \prod_{i \neq r,s,t} \delta(E_{i;A}), \tag{6.55}$$

and this shows that $\mathrm{d}\mu(a,b)$ is Möbius invariant. [28] prove that, appropriately understood as a sum of residues, the integral

$$\int \mathrm{d}\mu(a,b) \tag{6.56}$$

is the partial amplitude $m(a,n|b,n)$. Their proof studies the residues in simplicial coordinates, and is somewhat technical. The rest of this section studies CHY formulas in dihedral coordinates, in order to sketch a more direct proof of the CHY formulas.

Fix an ordering, $a \in \mathfrak{S}_{n-1}$. As in Appendix 5.6.1, let u_I be the cross-ratios associated to the diagonals of the n-gon, $I \in \mathrm{Ord}_a$. Let $\mathcal{M}_{an}$ be the affine variety defined by the non-crossing equations,

$$u_I + \prod_{J \text{ crosses } I} u_J = 1, \tag{6.57}$$

and write

$$f_s = \prod_{I \in \mathrm{Ord}_a} u_I^{X_I}, \tag{6.58}$$

for the Koba-Nielsen function. For any triangulation of the n-gon, α, the associated $n-3$ cross-ratios, u_I ($I \in P(\alpha)$), are a set of dihedral coordinates on $\mathcal{M}_{an}$. For some such choice of triangulation, write (for $I \in P(\alpha)$)

$$E_I := \frac{\partial}{\partial u_I} \log f_s = \frac{X_I}{u_I} + \sum \frac{X_J}{u_J} \frac{\partial u_J}{\partial u_I}, \tag{6.59}$$

for the $n-3$ *(dihedral) scattering equation functions.* Since dihedral coordinates are coordinates, it follows that the equations $E_I = 0$ have the same vanishing set as the equations $E_i = 0$. It is known that the equations $E_i = 0$ have $(n-3)!$ distinct point solutions if $X_I > 0$ for all $I \in \mathrm{Ord}_a$. [27]

For $I \in P(\alpha)$,

$$\lim_{u_I \to 0} u_I E_I = X_I. \tag{6.60}$$

Whereas, if J crosses I,

$$\lim_{u_J \to 0} \frac{1}{u_I E_I} = 0. \tag{6.61}$$

It follows that

$$\frac{1}{s_\alpha} = \int_{C_\alpha} \omega(a) \frac{1}{\prod_{I \in P(\alpha)} (u_I E_I)}. \tag{6.62}$$

Assume all X_I are strictly positive. Then the integrand has apparent singularities at the points D_α, for triangulations α of the n-gon, as well as at the $(n-3)!$ solutions to the equations $E_I = 0$. But by (6.61), the residue of the integrand at D_β vanishes for all triangulations $\beta \neq \alpha$. So, by the higher dimensional version of Cauchy's theorem,

$$\frac{1}{s_\alpha} = \sum_x \int_{C_x} \omega(a) \frac{1}{\prod_{I \in P(\alpha)} (u_I E_I)}, \tag{6.63}$$

where the sum is over solutions to the scattering equations, and the C_x are the residue contours around the points x. The orientations of the C_x can be fixed by the form $\bigwedge_{I\in P(\alpha)} \mathrm{d}\log E_I$, for some fixed ordering of $P(\alpha)$.

Remark 6.12. *[29] study* (6.62) *and* (6.63) *from expressions for the integrand given in simplicial coordinates, and prove these formulas by induction.*

The rest of this section sketches a 'reverse engineered' derivation of the CHY formula by summing (6.62) over all tree diagrams, α, compatible with two orderings. If b is an ordering compatible with α, then

$$\lim_{D_\alpha} \left(\bigwedge_{I\in P(\alpha)} \frac{\partial}{\partial u_I} \right) \lrcorner\, \omega(b) \to \frac{1}{\prod_{I\in P(\alpha)} u_I}. \tag{6.64}$$

Using this to rewrite (6.62), $m(a,n|b,n)$ can be written as the sum

$$m(a,n|b,n) = \sum_\alpha \int_{C_\alpha} \omega(a) \frac{1}{\prod_{I\in P(\alpha)} E_I} \left(\bigwedge_{I\in P(\alpha)} \frac{\partial}{\partial u_I} \right) \lrcorner\, \omega(b). \tag{6.65}$$

Since this is a sum of residues, the product of E_I in the denominator can be replaced inside the integral with

$$\frac{1}{\prod_{I\in P(\alpha)} E_I} = \det\left| \frac{\partial u_I}{\partial u_J} \right| \frac{1}{\prod_{J\in P(\beta)} E_J}, \tag{6.66}$$

where some fixed ordering has been assumed for $P(\alpha)$ and $P(\beta)$. The Jacobian is cancelled by by

$$\left(\bigwedge_{I\in P(\alpha)} \frac{\partial}{\partial u_I} \right) = \det\left| \frac{\partial u_J}{\partial u_I} \right| \left(\bigwedge_{J\in P(\beta)} \frac{\partial}{\partial u_J} \right). \tag{6.67}$$

Fixing some triangulation β, (6.65) can then be written as

$$m(a,n|b,n) = \sum_\alpha \int_{C_\alpha} \omega(a) \frac{1}{\prod_{I\in P(\beta)} E_I} \left(\bigwedge_{I\in P(\beta)} \frac{\partial}{\partial u_I} \right) \lrcorner\, \omega(b). \tag{6.68}$$

The integrand has apparent poles only at D_α and at the solutions to the $E_I = 0$

equations. So Cauchy's theorem gives

$$m(a,n|b,n) = \sum_x \int_{C_x} \omega(a) \, \frac{1}{\prod_I E_I} \left(\bigwedge_I \frac{\partial}{\partial u_I} \right) \lrcorner \, \omega(b), \tag{6.69}$$

where the sum is over solutions to the scattering equations, and C_x is oriented by $\prod \mathrm{d} \log E_I$, ordered in the same way as above.

Chapter 7

Quiver modules and the ABHY associahedra

The foregoing chapters study partial tree amplitudes, and prove the KLT relation and other properties of these amplitudes in an elementary way. An extension of the field theory KLT relation to the full gauge and gravity amplitudes is not presently known. Formulating such an extension will likely involve the properties of the biadjoint scalar partial amplitudes at higher order (see the Discussion). The ABHY associahedron, Section 6.4, which can be used to compute the biadjoint scalar theory tree amplitudes, has a generalisation to multi-trace and multi-loop partial amplitudes. Such a generalisation is suggested by the main result of [5], which shows that the ABHY associahedron arises naturally when studying the category of A_n quiver representations. This section generalises the approach of [5] in a setting that is independent of the relation to Feynman diagrams and surfaces. Section 7.1 collects basic notions about quiver representations. Section 7.2 discusses short exact sequences and functor categories. Sections 7.3 and 7.4 give the construction of generalized ABHY polytopes.

7.1 Quiver representations

Fix any quiver Q, with nodes $i \in A$ and arrows $i \to j$. A *representation* of Q, M, is a vector space, V_i, for each $i \in A$, together with linear maps $\phi_{ij} : V_i \to V_j$ for each arrow $i \to j$ (considered up to the action of $GL(V_i)$). For two representations, M and M', a morphism in $\mathrm{Hom}(M, M')$ is a collection of maps $f_i : V_i \to V_i'$ that commute with the ϕ_{ij}. Given two morphisms, $f, g : M \to M'$, any linear combination $\alpha f + \beta g$

is likewise a morphism. So $\mathrm{Hom}_{kQ}(M, M')$ is a k vector space. Write

$$\hom(M, M') := \dim_k \mathrm{Hom}(M, M'), \tag{7.1}$$

for its dimension.

Example 7.1. *Take Q to be the A_2 quiver.*

$$Q \ = \bullet \longrightarrow \bullet$$

Let M be the representation $\mathbb{C} \xrightarrow{0} 0$ and M' be the representation $\mathbb{C} \xrightarrow{1} \mathbb{C}$ (with the identity map). Then there are nontrivial morphisms $M' \to M$ corresponding to the commutative diagram

$$\begin{array}{ccc} \mathbb{C} & \overset{\lambda}{\dashrightarrow} & \mathbb{C} \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle 1} \\ 0 & \underset{0}{\dashrightarrow} & \mathbb{C} \end{array}$$

for $\lambda \in \mathbb{C}$. But there is no nontrivial morphism $M \to M'$. So $\hom(M, M') = 0$*, whereas* $\hom(M', M) = 1$.

The category of (finite dimensional) representations of Q, Rep-Q, is equivalent to the category of (finite dimensional) modules of an algebra called the *path algebra*, kQ. For each edge $i \to j$ of Q, define an associated formal variable e_{ji}. And introduce variables e_i for every $i \in A$. Then kQ is the algebra of all paths, i.e. products of the form $...e_{lk}e_{kj}e_{ji}$. Define also $e_ie_j = \delta_{ij}e_i$ and $e_ke_{ij} = e_{ij}\delta_{ik}$. If M is a module for kQ, then the submodules e_iM are vector spaces, with maps $e_{ji} : e_iM \to e_jM$. In this way, a kQ module M defines a representation of Q. Conversely, kQ acts on a representation of Q in the obvious way. Write mod$-kQ$ for the category of (finite dimensional) kQ modules. There is an equivalence of categories between mod$-kQ$ and Rep-Q. [4]

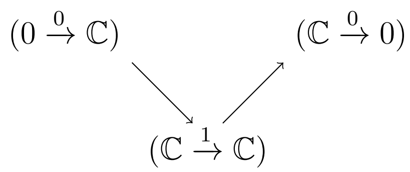
A module M in mod$-kQ$ is *indecomposable* if it is not the direct sum of two modules. M is *simple* if it contains no proper submodules. Write M_i for the simple module ke_i. Equivalently, M_i is the quiver representation that has $V_i = k$ and $V_j = 0$ for $j \neq i$. All other modules contain (at least one of) the M_i as proper submodules.

So the M_i, for $i \in A$, is the set of all simple modules. The following Lemma follows from, e.g., Proposition 1.4 of [4]. Say Q is *acyclic* if it contains no directed cycles.

Lemma 7.2. *If Q acyclic, the path algebra kQ is finite dimensional, and the Krull-Schmidt theorem implies that any finite dimensional kQ-module, M, has a unique direct sum decomposition into indecomposable modules.*

Write ind$-kQ$ for the set indecomposable modules, and also for the full subcategory of mod$-kQ$ induced by this set of modules. The Auslander-Reiten quiver is a way of describing the morphisms in mod$-kQ$. For two indecomposables, M and N, call a nonzero map $f : M \to N$ *indecomposable* if it cannot be written as the composition of two maps, $M \to E \to N$, for some module E. The AR quiver, Γ_Q, is the quiver with vertex set ind$-kQ$, and arrows for every indecomposable map.

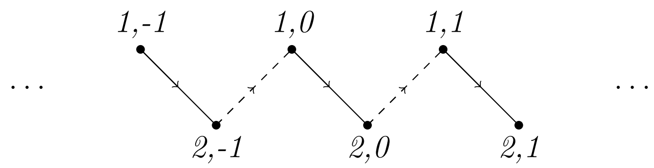
Example 7.3. *Let Q be the A_2, as above. There are three indecomposable modules, namely M and M' (as above), and also $M'' = \mathbb{C} \xrightarrow{0} 0$. The nontrivial indecomposable maps are $M \to M'$ and $M' \to M''$. So the AR quiver is*



(7.2)

When Q is an acyclic quiver, the AR quiver is related to the *translate quiver*, $\mathbb{Z}Q$. Define $\mathbb{Z}Q$ to be the quiver with vertices (i, a), for $i \in Q$ and $a \in \mathbb{Z}$. For every arrow $i \to j$ in Q, let $\mathbb{Z}Q$ have arrows $(i, a) \to (j, a)$ and $(j, a) \to (i, a + 1)$ for every a.

Example 7.4. *For $Q = A_2$, as above, $\mathbb{Z}Q$ is the following quiver.*

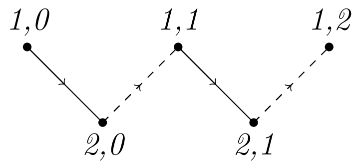


(7.3)

In particular, Γ_Q, described in (7.2), is isomorphic to the subquiver with vertices $(1, 0)$, $(2, 0)$ and $(1, 1)$ (or any translate of this). In fact, for any Dynkin quiver Q, it is known that Γ_Q is a subquiver of $\mathbb{Z}Q$. [4]

A *slice* of $\mathbb{Z}Q$ is a connected subquiver which has one vertex (i, a) for each $i \in Q$. Any such slice determines an orientation of Q. For Q Dynkin, all orientations of Q arise this way.

Example 7.5. (Slices.) *In the translate quiver* (7.3), *the slice with vertices* $(2, 0)$ *and* $(1, 1)$ *(or any of its translates) corresponds to Q with the opposite orientation. The slice with vertices* $(1, -1)$ *and* $(2, -1)$ *(or any of its translates) corresponds to Q with its original orientation. Moreover, consider the following subquiver of* $\mathbb{Z}Q$*:*



(7.4)

Write $M_{i,j}$ *for the module associated to* (i, j)*. The subquiver contains 5 slices:* $M_{1,0} - M_{2,0}$, $M_{2,0} - M_{1,1}$, $M_{1,1} - M_{2,1}$, *and* $M_{2,1} - M_{1,2}$.

7.2 Mesh relations

In this section and the next, $\mathcal{C}$ can be the category mod$-R$ of finite dimensional modules of an artinian algebra R. The intended applications are always to path algebras of a quiver.

Write $V := K_0(\mathcal{C}, 0)$ for the vector space generated by the vectors $[M]$ for all indecomposables $M \in \mathcal{C}$. Let I be the subspace spanned by

$$[\delta] = [A] - [E] + [B] \tag{7.5}$$

for all extensions

$$\delta : \qquad 0 \to A \to E \to B \to 0 \tag{7.6}$$

in $\text{Ext}^1(B, A)$. The Grothendieck group is

$$K_0(\mathcal{C}) = V/I. \tag{7.7}$$

In particular, if N is a submodule of M, then there is a short exact sequence

$$0 \to N \to M \to M/N \to 0, \tag{7.8}$$

so that $[M] = [N] + [M/N]$ in $K_0(\mathcal{C})$. It follows that a linear basis for $K_0(\mathcal{C})$ is given by the classes $[M_i]$, for $i \in Q$, where M_i are the simple modules (using here the descending chain condition). Introduce coordinates, X_N, on $K_0(\mathcal{C}, 0)$, so that a vector in $K_0(\mathcal{C}, 0)$ is given by

$$\sum_{N \in \text{ind}-\mathcal{C}} X_N\,[N]. \tag{7.9}$$

Then the subspace I is given by equations

$$X_A - X_E + X_B = 0, \tag{7.10}$$

where, if $E = \oplus N_i$ is the decomposition of E into indecomposables, $X_E = \sum_i X_{N_i}$. Call the equations (7.10) *mesh relations.*

Write $\text{Vect}^{\mathcal{C}}$ for the functor category, from $\mathcal{C}$ to vector spaces. For $M \in \mathcal{C}$, write

$$Y_M := \text{Hom}(\ , M) \qquad \text{and} \qquad {}_MY := \text{Hom}(M, \ ), \tag{7.11}$$

for the associated left and right Hom functors. And let E_M and ${}_ME$ be the left and right simple functors. E_M is the covariant functor that satisfies

$$E_M(N) = \begin{cases} 0 & \text{for } N \in \text{ind}-\mathcal{C} \text{ not } M, \\ k & \text{for } N = M, \end{cases} \tag{7.12}$$

and ${}_ME$ is the contravariant functor satisfying the same condition.[1] $\text{Vect}^{\mathcal{C}}$ inherits the notions of direct sums, quotients and subobjects from Vect. In particular, three functors, F, G and H, form a short exact sequence of functors if the sequences

$$0 \to F(N) \to G(N) \to H(N) \to 0 \tag{7.13}$$

[1]The functor E_M is defined, as in [43], as the quotient $E_M = Y_M/R$, where R is the maximal sub-functor of Y_M.

are exact for all $N \in \mathcal{C}$. Fix a SES in $\mathcal{C}$,

$$\delta : \qquad 0 \to A \to E \to B \to 0. \tag{7.14}$$

The failure for the Hom functors to be exact defines two functors, δ^* and δ_*, such that the sequences

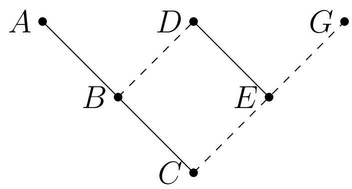
$$Y_A \to Y_B \to Y_C \to \delta^* \to 0 \tag{7.15}$$

and

$${}_CY \to {}_BY \to {}_AY \to \delta_* \to 0 \tag{7.16}$$

are exact in $\mathrm{Vect}^{\mathcal{C}}$. Finally, say that δ is a (left) Auslander-Reiten short exact sequence if $\delta^* \simeq E_C$. Likewise, δ is a (right) AR SES if $\delta_* \simeq {}_AE$. [43]

Example 7.6. *Take the AR quiver Γ_{A_3}, which is the following subquiver of $\mathbb{Z}A_3$,*



(7.17)

The letters labelling the quiver stand for some indecomposable kQ modules. Then

$$\delta_{AD} : \qquad A \to B \to D \tag{7.18}$$

is a non-split extension, and it is an AR SES because $\delta_{AD*} = {}_AE$. (It suffices to check that

$${}_AY(N) \to {}_BY(N) \to {}_DY(N) \tag{7.19}$$

is exact for indecomposables N that are not A.) On the other hand, there is an extension

$$\delta : \qquad A \to C \to G, \tag{7.20}$$

and this is not an AR SES. In fact, $[\delta_*] = [{}_AE] + [{}_BE] + [{}_DE]$, which can be checked by evaluating δ_* on indecomposables. The fact that δ is not an AR SES can also be *expressed by observing that the class*

$$[\delta] = [A] + [G] - [C], \tag{7.21}$$

can be written as the sum

$$[\delta] = [\delta_{AD}] + [\delta_{BE}] + [\delta_{DG}], \tag{7.22}$$

where $\delta_{AD}, \delta_{BE}, \delta_{DG}$ *are extensions associated to the pairs* AD, BE, *and* DG.

7.3 The deformed mesh relations

Write $W = K_0(\mathrm{Vect}^{\mathcal{C}}, 0)$ for the free vector space generated by vectors $[F]$, for indecomposable functors F. Let $J \subset W$ be the subspace spanned by the short exact sequences in $\mathrm{Vect}^{\mathcal{C}}$. The Grothendieck group is $K_0(\mathrm{Vect}^{\mathcal{C}}) = W/J$. A basis is given by the classes of the simple functors, $[E_M]$.

Lemma 7.7. *There is a pairing*

$$(\ ,\ ):\ K_0(\mathcal{C}, 0) \times K_0(\mathit{Vect}^{\mathcal{C}}) \to \mathbb{N}, \tag{7.23}$$

defined, for a module M *and a functor* F, *by*

$$([M], [F]) \mapsto [F(M)], \tag{7.24}$$

where $[F(M)]$ *is the vector space dimension* $\dim F(M)$.

Proof. Short exact sequences of functors in $\mathrm{Vect}^{\mathcal{C}}$ give exact sequences of vector spaces, as in the remark above (7.13). □

Fix some module $M \in \mathcal{C}$. Then M defines two vectors in $K_0(\mathcal{C}, 0)$,

$$v_M = \sum_{N \in \mathrm{ind}-\mathcal{C}} (N, Y_M)\,[N], \tag{7.25}$$

and

$$ {}_M v = \sum_{N \in \mathrm{ind}-\mathcal{C}} (N, {}_M Y)\,[N]. \tag{7.26} $$

Let $I \subset K_0(\mathcal{C}, 0)$ be the vector subspace of $K_0(\mathcal{C}, 0)$ spanned by all the short exact sequences, as in (7.5). The ABHY hyperplanes of $\mathcal{C}$ are

$$ H_M : I + {}_M v + v_M \subset K_0(\mathcal{C}, 0). \tag{7.27} $$

The dual description of H is in terms of coordinates on $V = K_0(\mathcal{C}, 0)$, as in (7.9):

Lemma 7.8. *The hyperplane H is given by the* deformed mesh relations*:*

$$ X_A - X_E + X_B = (M, [\delta_*] + [\delta^*]), \tag{7.28} $$

for every non-split extension δ, with $[\delta] = [A] - [E] + [B]$.

If R is finite dimensional, then (by Krull-Schmidt) there is a direct sum decomposition

$$ M = \bigoplus_{N \in \mathrm{ind}-\mathcal{C}} N^{\oplus c_N}, \tag{7.29} $$

for non-negative integers $c_N \geq 0$. More generally, assume such an M of this form, and write $Supp(M)$ for the set of indecomposables for which $c_N > 0$.

Lemma 7.9. *For a non-split extension δ between objects in $Supp(M)$, the integer $(M, \delta_*) + (M, \delta^*)$ is positive.*

Proof. The functors E_N give a linear basis $[E_N]$ for $K_0(\mathrm{Vect}^{\mathcal{C}})$, so the class $[\delta_*]$ can be expanded

$$ [\delta_*] = \sum_{N \in \mathrm{ind}-\mathcal{C}} (N, \delta_*)[E_N]. \tag{7.30} $$

The coefficients $(N, \delta_*) = \dim \delta_*(N)$ are nonnegative. If $(N, \delta_*) + (N, \delta^*)$ was zero for every N, then δ_* and δ^* would be zero. But that would mean that δ splits. So $(N, \delta_*) + (N, \delta^*)$ must be positive for at least one N. Then

$$ (M, \delta_*) + (M, \delta^*) = \sum_{N \in \mathrm{ind}-\mathcal{C}} ((N, \delta_*) + (N, \delta^*))c_N. \tag{7.31} $$

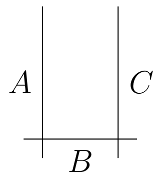
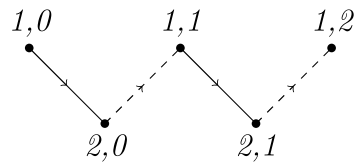


Figure 7.1: The polytope P in Example 7.11 is unbounded.

□

Example 7.10. *Consider again the following subquiver of* $\mathbb{Z}A_2$*, as in* (7.4)*:*



(7.32)

Write $M_{i,j}$ *for the module associated to* (i,j)*, and write* X_{ij} *for the coordinates on* $V = K_0(\mathcal{C}, 0)$*. Then*

$$X_{1,0} - X_{2,0} + X_{1,1} = c_{1,0} + c_{1,1} \tag{7.33}$$

$$X_{1,1} - X_{2,1} + X_{1,2} = c_{1,1} + c_{1,2} \tag{7.34}$$

$$X_{2,0} - X_{1,1} + X_{2,1} = c_{2,0} + c_{2,1}. \tag{7.35}$$

These are the ABHY equations *of Section 6.4, for* A_2*. To see the connection, write* $X_{1,0} = X_{13}$, $X_{2,0} = X_{14}$, $X_{1,1} = X_{24}$, $X_{2,1} = X_{25}$, $X_{1,2} = X_{35}$.

7.4 Faces and factorization

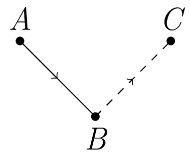
The previous section associated affine hyperplanes in $K_0(\mathcal{C}, 0)$ to modules $M \in \mathcal{C}$. Intersecting these hyperplanes with the positive orthant gives a (possibly unbounded) polytope, P. This section exhibits some generic facts about P, and its faces.

Let H_M be the hyperplane $I + v_M + {}_M v$, as above. Write V^+ for the positive orthant of $V = K_0(\mathcal{C}, 0)$, defined by $X_N > 0$ for all $N \in \text{ind}{-}\mathcal{C}$. Then call the polytope

$$P_{\mathcal{C},M} = H_M \cap V^+ \tag{7.36}$$

an 'ABHY polytope'. The ABHY polytope $P := P_{\mathcal{C},M}$ is not necessarily bounded, and it is not necessarily simple. The face lattice of P depends on the category, $\mathcal{C}$, being considered, and the support of M.

Example 7.11. (Unbounded example.) *Consider $\mathcal{C} = \text{mod}-A_2$, which has 3 indecomposables, and an AR quiver of the form:*



(7.37)

The hyperplane, H, has the equation

$$X_A - X_B + X_C = c_A + c_C, \tag{7.38}$$

where $M = A^{c_A} \oplus B^{c_B} \oplus C^{c_C}$ is the module. The polytope P is then the unbounded region $X_A > 0, X_B > 0, X_C > 0$ of this hyperplane, shown in Figure 7.1.

Write $H_A \subset V$ for the plane $X_A = 0$, orthogonal to $[A] \in V$. Assuming, as above, that P is nonempty, there exists some $A \in \text{ind}-\mathcal{C}$ so that $H_A \cap P$ is nonempty.

Remark 7.12. (Normal vectors of faces.) *For any $N \in \text{ind}-\mathcal{C}$, $[N] \in K_0(\mathcal{C})$ can be expanded*

$$[N] = \sum_{i=1}^{n} g_N(i)[P_i], \tag{7.39}$$

where the set of $[P_i]$ is some basis of $K_0(\mathcal{C})$. If $H_A \cap P$ is a face of P, then its normal vector is $[A]$. Using the X_{P_i} as coordinates on H, the normal vector $[A]$ has components $g_A(i)$.

Lemma 7.13. (Non-adjacent faces of P.) *If there is a non-split extension, δ, between two objects in $Supp(M)$, then $H_A \cap H_B \cap P$ is empty.*

Proof. Let δ be such an extension, with $[\delta] = [A] - [E] + [B]$. Then, in the dual description of H, the associated coordinates satisfy

$$X_A + X_B = X_E + (M, \delta_*) + (M, \delta^*). \tag{7.40}$$

But $X_E \geq 0$ in V^+, and $(M, \delta_*) + (M, \delta^*)$ is strictly positive. $\square$

This Lemma means that the face $H_A \cap P$ only intersects hyperplanes H_B for $B \in$ *Supp*(M) that have no extensions with A. This motivates the following definition, which leads to a description of the face lattice of P.

For $N \in \text{ind}-\mathcal{C}$, let $\mathcal{C}|_N$ be the full subcategory of $\mathcal{C}$ defined by the vanishing of both $\text{Ext}^1(\ ,N)$ and $\text{Ext}^1(N,\ )$. A full subcategory $\mathcal{D} \subset \mathcal{C}$ is *Serre* if, for every SES in $\mathcal{C}$,

$$\delta : \qquad 0 \to A \to E \to B \to 0, \tag{7.41}$$

$A, B \in \mathcal{D}$ implies that $E \in \mathcal{D}$. [73] If δ is non-split, the derived hom sequence gives, e.g.,

$$... \to \text{Ext}^1(N, A) \to \text{Ext}^1(N, E) \to \text{Ext}^1(N, B) \to ... \tag{7.42}$$

so that $\text{Ext}^1(N, A) = 0$ and $\text{Ext}^1(N, B) = 0$ implies $\text{Ext}^1(N, E) = 0$. This implies that:

Lemma 7.14. *The categories $\mathcal{C}|_N$, for $N \in \text{ind}-\mathcal{C}$, are Serre.*

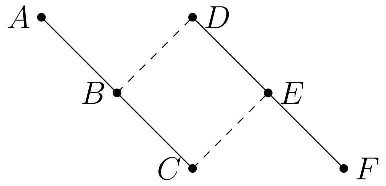
Remark 7.15. *In some applications, it will be necessary to define $\mathcal{C}[N]$, for an object N which is not indecomposable or which has self-extensions. $\mathcal{C}[N]$ is then the subcategory defined by removing the indecomposable subobjects of N, as well as the objects that arise in self-extensions of N. $\mathcal{C}[N]$ is not necessarily a Serre subcategory of $\mathcal{C}|_N$, or of $\mathcal{C}$.*

Theorem 7.16. (Faces of P.) *Fix a module M whose indecomposable subobjects do not have self-extensions. The face $[A]^\perp \cap P_\mathcal{C}$ is isomorphic to the ABHY polytope, $P_{\mathcal{C}|_A, M'}$, for a certain M' in $\mathcal{C}|_A$. This determines the face lattice of $P_\mathcal{C}$ inductively. In particular, the face lattice of P doesn't depend on the multiplicity of the indecomposable submodules of M.*

Proof. For $B \in Supp(M)$, $H_A \cap H_B \cap P$ is nonempty only if $B \in \text{ind}-\mathcal{C}[A]$ (Lemma 7.13). For any $B, C \in \text{ind}-\mathcal{C}[A]$, if there is an extension between B and C in $\mathcal{C}|_A$, say $B \hookrightarrow E \twoheadrightarrow C$, then $E \in \mathcal{C}|_A$, because $\mathcal{C}|_A$ is Serre (Lemma 7.14). So the deformed mesh relations involving $B \in \mathcal{C}|_A$ are independent of any X_C for $C \notin \mathcal{C}|_A$. In terms of the polytope, $H_A \cap P$ is isomorphic to $H_A \cap P_{\mathcal{C}|_A}$. But for any non-split extension in $\mathcal{C}|_A$, say $B \hookrightarrow E \twoheadrightarrow C$, A can only ever appear as a subobject of E. So the deformed mesh relations of $P_{\mathcal{C}|_A}$ are, after setting $X_A = 0$, the same as the deformed mesh relations of $P_{\mathcal{C}[A]}$. □

For $Supp(M)$ fixed, the polytope P does not depend on the choice of multiplicities $c_N > 0$. If one of the constants is set to zero, $c_N = 0$, thus changing $Supp(M)$ by removing N, then the resulting polytope will not have the same face lattice as P, as in the following:

Example 7.17. *Take a subcategory with AR quiver given by the following subquiver of $\mathbb{Z}A_3$:*



(7.43)

The associated hyperplane is given by the three deformed mesh relations:

$$X_A + X_D - X_B = c_A + c_D, \tag{7.44}$$

$$X_B + X_E - X_C - X_D = c_B + c_E, \tag{7.45}$$

$$X_C + X_F - X_E = c_C + c_F. \tag{7.46}$$

The polytope P is a cube. Setting $c_A + c_D = 0$ collapses the cube to a square.

Take a set of indecomposable objects, $\mathcal{D}$, which defines a full subcategory $\mathcal{D} \subset \mathcal{C}$. The (ambidextrous) *orthogonal subcategory*, $\mathcal{D}^{\perp}$, is the full subcategory of $\mathcal{C}$ with objects $B \in \mathcal{C}$ such that

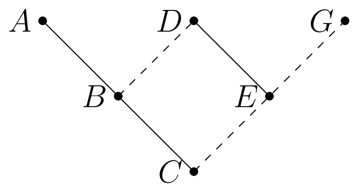
$$\text{hom}(A, B) = 0, \;\; \text{hom}(B, A) = 0, \;\; \text{ext}^1(A, B) = 0, \;\; \text{ext}^1(B, A) = 0, \tag{7.47}$$

for all $A \in \mathcal{D}$. A full subcategory $\mathcal{D} \subset \mathcal{C}$ that satisfies

$$\mathcal{D} = (\mathcal{D}^{\perp})^{\perp} \tag{7.48}$$

is called a 'left and right admissible' subcategory, or just *admissible*. If $\mathcal{D}$ is admissible, then so is $\mathcal{D}^{\perp}$, and so admissible subcategories come in pairs.

Example 7.18. *Take again the AR quiver Γ_{A_3}, as in (7.43):*



(7.49)

Then $C^{\perp}$ is D, and $D^{\perp}$ is C. So C and D are both admissible.

Lemma 7.19. *If $\mathcal{D} \subset \mathcal{C}$ is admissible, then $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are both Serre.*

Proof. For *any* subcategory $\mathcal{D}$, $\mathcal{D}^{\perp}$ is Serre. Indeed, for any non-split $A \hookrightarrow E \to B$, the associated LES, for $N \in \mathcal{D}$, is

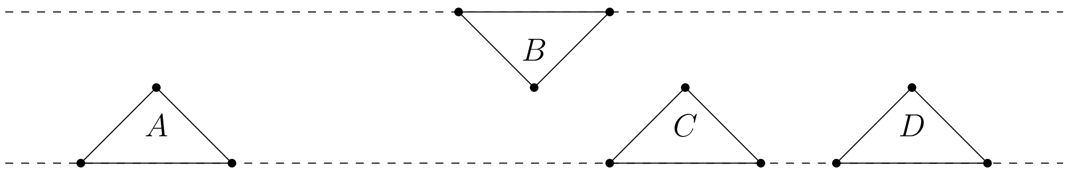
$$0 \to Hom(N, A) \to \mathrm{Hom}(N, E) \to \mathrm{Hom}(N, B) \to \\ \mathrm{Ext}^1(N, A) \to \mathrm{Ext}^1(N, E) \to \mathrm{Ext}^1(N, B), \tag{7.50}$$

(and the covariant analogue) so that if $A, B \in \mathcal{D}^{\perp}$, certainly $E \in \mathcal{D}^{\perp}$. But, if $\mathcal{D}$ is admissible, $\mathcal{D}$ is the orthogonal of something, and so is Serre. □

Say that an admissible subcategory $\mathcal{D} \subset \mathcal{C}$ is an *island of M* if the pair $\mathcal{D}$ and $\mathcal{D}^{\perp}$ partition $Supp(M)$. The admissible subcategories in Example 7.18 are not islands.

Example 7.20. *Consider a category with AR quiver $\mathbb{Z}A_n$, and a module M such that*

Supp(M) is given by the following subquiver with four connected components:



(7.51)

Then $B \cup C$ is an island, with $A \cup D$ the orthogonal. Likewise, A is an island, with $B \cup C \cup D$ the orthogonal. But, e.g., $C \cup D$ is not an island, because C has extensions with the modules in B, and so $C \cup D$ is not admissible.

Theorem 7.21. (Factorisation property.) *Suppose $\mathcal{D} \subset \mathcal{C}$ is an island of M. Then P is the cross product of the polytopes $P_{\mathcal{D},M_1}$ and $P_{\mathcal{D}^\perp,M_2}$ associated to $M_1 \in \mathcal{D}$ and $M_2 \in \mathcal{D}^\perp$ (with $M = M_1 \oplus M_2$).*

Proof. The indecomposables are partitioned by $\mathcal{D}$ and $\mathcal{D}^\perp$. Every non-split SES belongs to either $\mathcal{D}$ or $\mathcal{D}^\perp$, because they are admissible, and also therefore Serre. In particular, for a non-split SES δ in $\mathcal{D}$, δ_* and δ^* have support only in $\mathcal{D}$. The deformed mesh relations for $\mathcal{C}$ are of the form

$$X_A - X_E + X_B = (M', \delta_*) + (M', \delta^*), \tag{7.52}$$

for δ a SES in $\mathcal{D}$ (resp. $\mathcal{D}^\perp$), and M' the maximal submodule of M contained in $\mathcal{D}$ (resp. $\mathcal{D}^\perp$). □

Example 7.22. (Associahedron.) *Section 6.4 can now be understood in the following way. Form the quiver $\mathbb{Z}A_n$, and choose a slice of $\mathbb{Z}A_n$. The slice induces an orientation, Q, of the quiver A_n. Let α be any triangulation of the n-gon which is dual to Q. Label the slice of $\mathbb{Z}A_n$ by the diagonals of α. All other vertices of $\mathbb{Z}A_n$ can then be labelled by diagonals by demanding that a translation once to the right sends an arc I_{ij} to $I_{i+1,j+1}$.*

Let $\tilde{\Gamma}$ be a subquiver of $\mathbb{Z}A_n$ so that each diagonal appears exactly once in $\tilde{\Gamma}$. And let $\mathcal{C}$ be a category with AR quiver $\tilde{\Gamma}$ (see Remark 7.23). The mesh relations are of the form

$$[X_{ij}] + [X_{kl}] - [X_{il}] - [X_{jk}] \tag{7.53}$$

(with the cyclic ordering $i < k < j < l$ assumed). Introduce coordinates X_{ij}, so that $v \in V = K_0(\mathcal{C}, 0)$ is written

$$v = \sum_{i<j} X_{ij}[I_{ij}]. \tag{7.54}$$

The the deformed mesh relations corresponding to (7.53) *are the ABHY equations,* (6.44).

Let I be a diagonal, and H_I the hyperplane $X_I = 0$. Then the face $H_I \cap P$ is isomorphic to the polytope $P_{\mathcal{C}[I]}$ associated to the category $\mathcal{C}[I]$. $\mathcal{C}[I]$ contains all the diagonals J such that J doesn't cross I, $J \sim I$. These can be partitioned into two sets, $\mathcal{D}$ and $\mathcal{D}'$, which are the non-crossing diagonals on each of the two sides of I in the n-gon. Moreover, no $I \in \mathcal{D}$ crosses any $I' \in \mathcal{D}'$. In the language of Section 7.4, $\mathcal{D}$ is admissible, with $\mathcal{D}^\perp = \mathcal{D}'$. In $\mathcal{C}[I]$, $\mathcal{D}$ and $\mathcal{D}'$ are islands. It follows that the face $H_I \cap P$ is isomorphic to the cartesian product of $P_{\mathcal{D}}$ and $P_{\mathcal{D}'}$.

Remark 7.23. *$\tilde{\Gamma}$ in Example 7.22 is the AR quiver of the so-called cluster category of A_n. The cluster category of Q, $\mathcal{C}$, is a certain quotient category of $D^b(\text{mod}-kQ)$ introduced and studied in [18]. The indecomposables of $\mathcal{C}$ are in 1:1 correspondence with cluster variables: in the above example, there is one indecomposable for every diagonal of the n-gon. The slices of $\tilde{\Gamma}$ are in 1:1 correspondence with clusters. A slice of $\tilde{\Gamma}$ corresponds in $\mathcal{C}$ to a so-called* tilting object. *An object M which is the direct sum of n indecomposable objects is a tilting object if M has no self extensions, $Ext^1(M, M) = 0$.*

Chapter 8

The full perturbation series and surfaces

’t Hooft showed that the YM perturbation series implicitly contains a genus expansion, in addition to the expansion in the coupling constant, g_{YM}. [76] A more refined version of this statement is that A_{YM} has an expansion of the form

$$A_{\mathrm{YM}}(1,..,n) = \sum_{\Sigma} A_{\Sigma}, \tag{8.1}$$

where the A_Σ are contributions associated to punctured surfaces with boundary, Σ. The existence of an expansion of this form is folklore. This chapter considers the partial amplitude expansion of the biadjoint scalar theory amplitudes. The biadjoint scalar action is

$$S = \int \mathrm{d}^D x \left(\phi^{ab} \Box \phi^{ab} + i g f^{a_1 a_2 a_3} f^{b_1 b_2 b_3} \phi^{a_1 b_1} \phi^{a_2 b_2} \phi^{a_2 b_2} \right), \tag{8.2}$$

where the indexing and structure constants are explained in Section 8.2. The partial amplitude expansion of the biadjoint scalar theory is ‘universal’ in the sense that the partial amplitudes of any massless gauge theory has such an expansion. The reason for this is that any higher-valence interaction term in the action of a gauge theory must be an invariant trace of the gauge field and its derivatives.[1] This means that Feynman diagrams containing higher-valence vertices will have colour structures that are a sum of the colour structures of trivalent diagrams, as in the proof of the KK

[1]It is necessary to assume that the Lagrangian of the theory doesn’t have any multi-trace terms, of the form tr(...)tr(...)...tr(...). The partial amplitude expansion of such theories is more complicated than the one for single-trace Lagrangians.

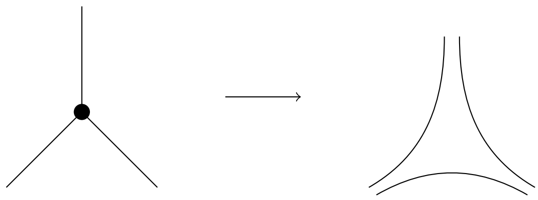


Figure 8.1: The thickening of a vertex in a ribbon graph.

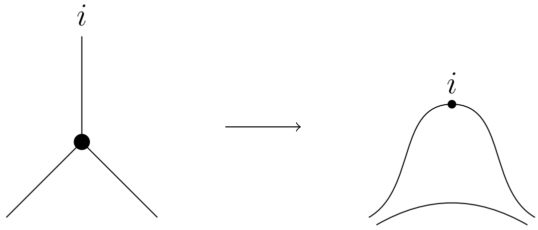


Figure 8.2: Boundary points of the surface, Fat_G, obtained by fattening G.

identity for YM in Section 2.1.

8.1 Ribbon graphs

A *ribbon graph* is a graph with a choice of cyclic ordering at each vertex. For a cubic ribbon graph drawn in the plane, denote the two possible cyclic orderings at a vertex by a black circle (for anti-clockwise) and a white circle (for clockwise). Let G be a cubic ribbon graph, with labelled external edges. Write Thin_G for the ordinary cubic graph obtained by forgetting about the orientations of the vertices. A standard procedure 'fattens' G to get a marked surface with boundary. [66] In this chapter, a 'surface' is a *marked surface with boundary*, which is a surface with boundary with labelled boundary points and labelled internal punctures. Only triangulable surfaces will be considered. A triangulable surface has at least one labelled boundary point on each boundary component, and the total number of labelled points on the surface

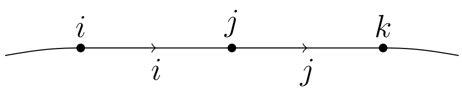


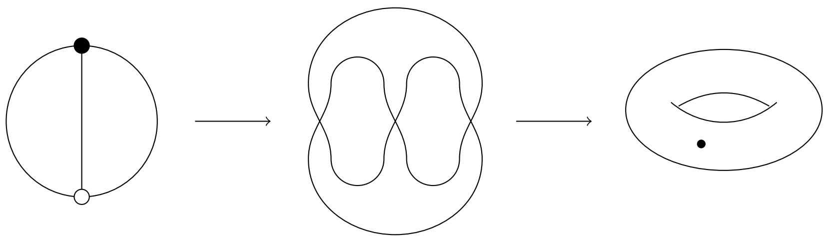
Figure 8.3: The induced orientation of the boundary of a marked surface Σ.

is at least $3 - 2g$, where g is the genus of the surface.

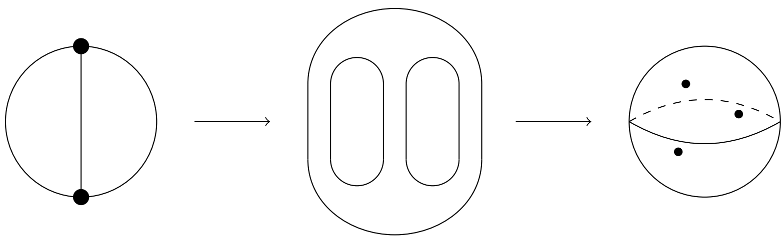
Definition 8.1. *Fix a ribbon graph G, and construct a surface,* Fat_G*, in the following way: (i) replace each vertex of G by an open surface, oriented according to the colour of the vertex, as in Figure 8.1; (ii) glue these surfaces together according to the edges of G, respecting orientations; (iii) replace each labelled external edge of G by a labelled boundary point, as in Figure 8.2; and (iv) contract any boundary with no marked points to create a puncture.*

The surface Fat_G is uniquely determined by G. Say that G 'fattens' to Fat_G. If $\overline{G}$ is the graph obtained by changing the orientations of all the vertices of G, then $\mathrm{Fat}_{\overline{G}}$ is the surface obtained from Fat_G by reversing the orientation.

Example 8.2. *Let T be the cubic graph with two vertices, and no external edges. Assigning orientations to each of the vertices produces two possible ribbon graphs, G, such that* $\mathrm{Thin}_G = T$*. One of these gives* Fat_G *a once punctured torus:*



The other choice of orientation gives Fat_G *a three punctured sphere:*



Fix a surface, Σ, with labelled boundary points and punctures. For any triangulation of Σ, α, there is a unique cubic ribbon graph G_α, embedded in Σ, that is dual to α. The external edges of G_α intersect the boundary segments of Σ. A labelling of the external edges of G_α is therefore the same as a labelling of the boundary segments of Σ. The orientation of Σ induces an orientation of each boundary component. Adopt the convention that a boundary segment is labelled by the boundary marked point at its 'left' end, as in Figure 8.3.

Lemma 8.3. *There is a 1:1 correspondence between triangulations of* Σ*, and labelled cubic ribbon graphs that fatten to* Σ*.*

Proof. With the conventions of the previous paragraph, it can be checked that G_α fattens to Σ. Conversely, suppose a labelled cubic ribbon graph G fattens to Σ. By Definition 8.1, there is an embedding of G in Σ such that the end points of G are at the boundary marked points of Σ. With respect to the orientation of the surface, rotate the end points of G to the 'right' on all boundary components. The resulting embedding of G is dual to some triangulation, α_G, of the surface. It can be checked that $\alpha \mapsto G_\alpha$ and $G \mapsto \alpha_G$ are inverses. □

This correspondence is not unique, because it depends on choosing a convention about labelling the boundary segments.

8.2 Feynman diagrams and ribbon graphs

The expansion of the action, (8.2), gives Feynman diagram contributions. The Feynman diagrams of the theory are cubic graphs. Each internal edge is associated with a propagator factor, given by

$$\frac{1}{k^2 + i\epsilon}, \tag{8.3}$$

in momentum space, with k^μ the momentum of the edge. The assignment of momenta to the edges and the integral associated to these propagator factors is discussed in Section 8.4. Each cubic vertex in the graph is associated with a factor of

$$igf^{a_1a_2a_3}\tilde{f}^{b_1b_2b_3}. \tag{8.4}$$

The indices here are understood to run over a basis, T^a, of the adjoint representation of $\mathfrak{g}$, $\mathrm{ad}_\mathfrak{g}$, regarded as a vector space. The structure constants are then defined by

$$[T^{a_1}, T^{a_2}] = f^{a_1a_2a_3}T^{a_3}. \tag{8.5}$$

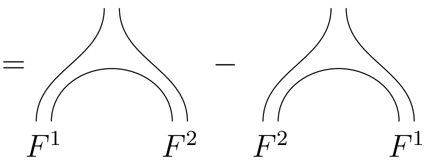
Choosing some basis for $\mathfrak{g}$, T^a can be regarded as a matrix in $\mathrm{End}(\mathfrak{g})$. Write F_{ij} for some matrix in $\mathrm{End}(\mathfrak{g})$. The Lie bracket on $\mathrm{ad}_{\mathfrak{g}}$ is the commutator,

$$[T^{a_1}, T^{a_2}]_{ij} = T^{a_1}_{ik} T^{a_2}_{kj} - T^{a_2}_{ik} T^{a_1}_{kj}. \tag{8.6}$$

Fix a Feynman diagram, D, with k internal vertices and n external lines. Let the external colour polarizations be written $F^a, \tilde{F}^a \in \mathrm{End}(\mathfrak{g})$, for $a = 1, ..., n$. The contribution of D to the amplitude is

$$A_D = (ig)^k \, C_D \tilde{C}_D I_D, \tag{8.7}$$

where C_D is the tensor contraction of the F^a, $\tilde{C}_D$ is the contraction of the $\tilde{F}^a$, and I_D is the Feynman integral associated to the internal edges. As a tensor diagram, the commutator bracket, (8.6), can be written as

$[F_1, F_2] =$ 

(8.8)

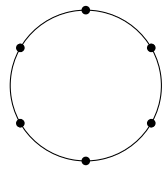
It follows that C_D can be expanded as a sum of 2^{k-1} terms, corresponding to 2^{k-1} choices of cyclic orientation to assign to each vertex of D. Each term corresponds to a ribbon graph, G, such that $\mathrm{Thin}_G = D$. Write c_G for the contraction of the F^i tensors according to the ribbon graph G. Then

$$C_D = \sum_{\substack{G \text{ s.t.} \\ \mathrm{Thin}_G = D}} (-1)^{|G|} \, c_G, \tag{8.9}$$

where $|G|$ is the number of white vertices of G, with the vertex connected to n fixed to be black, say. In fact, c_G depends only on the surface, Fat_G, that G fattens to. Each boundary component of $\Sigma = \mathrm{Fat}_G$ gives a trace. For instance, a disk with $1, ..., k$

arranged cyclically, gives a contribution

$$\mathrm{tr}(F^1 F^2 ... F^k) \quad = \tag{8.10}$$



A boundary component of Σ with no marked points gives a trace of the identity, which is

$$\mathrm{tr}(\mathrm{Id}) = N \tag{8.11}$$

in the case of $SU(N)$. It follows that

$$A_D = (ig)^k \, I_D \sum_{\substack{G,G' \text{ s.t.} \\ \mathrm{Thin}_G = D, \\ \mathrm{Thin}_{G'} = D}} (-1)^{|G|+|G'|} \, c_{\mathrm{Fat}_G} \, \tilde{c}_{\mathrm{Fat}_{G'}}. \tag{8.12}$$

It follows from Lemma 8.3 that a given surface, Σ, is the fattening of precisely those cubic ribbon graphs that are dual to the triangulations of Σ. Reversing the order of summations, the whole amplitude can therefore be written as

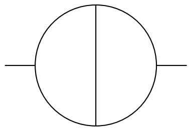
$$A_{\phi^3}(1, ..., n) = \sum_D A_D = \sum_{\Sigma, \Sigma'} c_\Sigma \, \tilde{c}_{\Sigma'} A(\Sigma, \Sigma'), \tag{8.13}$$

where the sum is over all pairs of surfaces with n boundary marked points, and where the partial amplitudes are

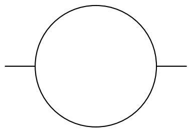
$$A(\Sigma, \tilde{\Sigma}) = (ig)^k \sum_{\substack{G \text{ s.t.} \\ \mathrm{Fat}_G = \Sigma}} \sum_{\substack{G' \text{ s.t.} \\ \mathrm{Fat}_{G'} = \Sigma', \\ \mathrm{Thin}_{G'} = \mathrm{Thin}_G}} (-1)^{|G|+|G'|} I_{\mathrm{Thin}_G}. \tag{8.14}$$

The number of triangles, k, in a triangulation of Σ is a topological invariant, so the factor of $(ig)^k$ has been put in front of the sum.

Remark 8.4. *If $\tilde{\Sigma}$ is obtained from Σ by permuting the boundary labels, the colour factors c_Σ and $\tilde{c}_{\tilde{\Sigma}}$ have the same 'trace structure'. But there are also nonzero partial amplitudes $A(\Sigma, \tilde{\Sigma})$ for surfaces Σ and $\tilde{\Sigma}$ that have different topologies and trace* *structures. For example, let Σ be the twice punctured disk, and $\tilde{\Sigma}$ be the torus with a disk cut out. The two surfaces appear in the YM expansion at different orders in $1/N$, but the partial amplitude $A(\Sigma, \tilde{\Sigma})$ is nonzero, due to the following Feynman graph*

$D_1 =$ 

Both Σ and $\tilde{\Sigma}$ have the same colour factors in this example. For an example involving different colour factors, let Σ be the once punctured disk with two boundary points, and let $\tilde{\Sigma}$ be the annulus with one boundary point on each boundary. Then $A(\Sigma, \tilde{\Sigma})$ is nonzero because the Feynman graph

$D_2 =$ 

can be fattened to both Σ and $\tilde{\Sigma}$.

8.3 Partial amplitudes and the $1/N$ expansion

Let

$$\Sigma_{p,g}^{n_1,\ldots,n_h} \tag{8.15}$$

be any marked surface with genus g, and p punctures, and h boundary components, each with n_i marked boundary points ($n_i > 0$). Write n for the total number of boundary points,

$$\sum_{i=1}^{h} n_i = n. \tag{8.16}$$

Fix α, any triangulation of $\Sigma_{p,g}^{n_1,\ldots,n_h}$. By Euler's theorem,

$$2(1-g) - h = p - e + k, \tag{8.17}$$

where e is the number of internal edges of α, and k is the number of triangular faces. α is dual to a ribbon graph G, which has e internal edges, n marked open ends, and k cubic vertices. Let ℓ be the number of 'loops' of G, given by

$$\ell = 1 - k + e. \tag{8.18}$$

By the Euler relation,

$$k = 2(\ell - 1) + n, \tag{8.19}$$

since $3k = 2e + n$.

Let $A_{p,g,h}$ be the sum of terms $A(\Sigma, \tilde{\Sigma})$, for genus g surfaces Σ with p punctures and h boundary components. Pulling out the factors of g and N from $A_{p,g,h}$, the Feynman perturbation series can be written as the sum

$$A_n = \sum_{\ell=0}^{\infty} (ig)^{2(\ell-1)+n} \sum_{h,p,g} N^{2p} A_{p,g,h}, \tag{8.20}$$

where the second sum is subject to the relation

$$p + 2g + h = \ell + 1. \tag{8.21}$$

For fixed ℓ, the largest possible value of p is ℓ. So it is natural to reorganize the terms in the above series in the following way, following 't Hooft,

$$A_n = (ig)^{n-2} \sum_{\ell=0}^{\infty} \lambda^{\ell} \sum_{h,p,g} \left(\frac{1}{N}\right)^{2g+h-1} A_{p,g,h}, \tag{8.22}$$

where $\lambda = g^2 N^2$. For large N and fixed λ, the leading contributions to the series are from surfaces with genus $g = 0$ and $h = 1$ boundary component.

8.4 Dual variables

Fix a marked surface Σ. For every arc, I, considered up to homotopy, introduce a variable X_I. For two fixed points, there may be multiple or infinitely many non-

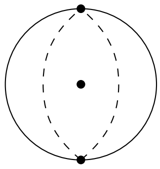


Figure 8.4: Two non-homotopic arcs with the same end points.

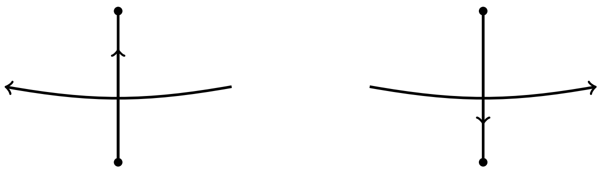


Figure 8.5: Orienting arcs in a triangulation dual to a 'circuit' diagram using the 'clockwise rule'.

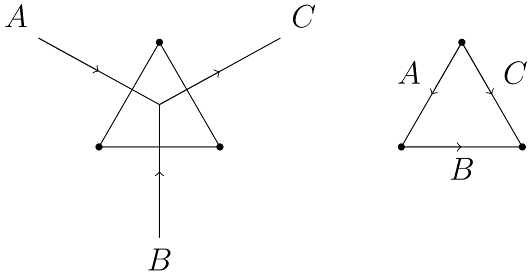


Figure 8.6: Verifying momentum conservation at each triangle in a triangulation: in both cases, $A + B = C$.

homotopic arcs with these endpoints, as in Figure 8.4. To every triangulation, α, associate the rational function, *the integrand*,

$$\frac{1}{s_\alpha} = \prod_{I \in P(\alpha)} \frac{1}{X_I}, \tag{8.23}$$

where $P(\alpha)$ is the set of arcs of α. This section is about relating the variables X_I to Mandelstam variables in a consistent way, which makes it possible to sum the integrands $1/s_\alpha$ before performing the loop integration.

Take a triangulation α dual to a ribbon graph G, with $\text{Thin}_G = D$. The Feynman integral of D is of the form

$$I_D = \int \mathrm{d}^D l_1 ... \mathrm{d}^D l_L \prod_{I \in P(D)} \frac{1}{P_I^2}, \tag{8.24}$$

where the l_i^μ are L loop momenta,[2] and P_I^μ is the momentum assigned to edge I in D. Fixing an orientation of the edges of D defines a *circuit* $\tilde{D}$. Fixing such an orientation, an 'assignment of momenta' is any choice of the P_I^μ so that momentum is conserved at each vertex. Write K_Σ for the $\mathbb{R}$ vector space with generators $k_1, ..., k_n$ and $\ell_1, ..., \ell_L$. Then an assignment of momenta to $\tilde{D}$ gives a vector in K_Σ for each (oriented) edge of $\tilde{D}$.

Fix an assignment of momenta to $\tilde{D}$. Now embed $\tilde{D}$ into Σ, using the fact that $\text{Thin}_G = D$. The orientation of the edges of $\tilde{D}$ induces an orientation of each arc in the triangulation of α: if the arrow e in $\tilde{D}$ crosses the arc I in α, then orient α with an arrow that points towards the endpoint of α around which e is moving "clockwise". Orienting the edges of α in this way, one obtains the *dual circuit* $\tilde{\alpha}$ of $\tilde{D}$. See Figure 8.5.

The momentum conservation condition at each vertex of $\tilde{D}$ is equivalent to a triangle sum condition at each triangle of $\tilde{\alpha}$, as shown in Figure 8.6. It follows from this that an assignment of momenta to $\tilde{D}$ induces a homomorphism

$$\mu : H_1(\Sigma, \mathbb{P}) \to K_\Sigma, \tag{8.25}$$

[2] L is given by $p + h + 2(g-2)$, as in the previous section.

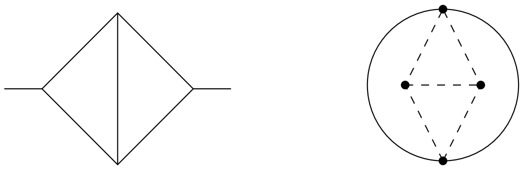


Figure 8.7: A two loop diagram, and the triangulation it is dual to.

where $\mathbb{P}$ is the set of marked points.

Lemma 8.5. *Given the homomorphism μ, as above, write*

$$X_I = \mu([I])^2, \tag{8.26}$$

for any arc, I, representing (when oriented) a class $[I] \in H_1(\Sigma, \mathbb{P})$. Then, for any triangulation of Σ, β, the associated Feynman integral is

$$I_\beta = \int \left(\prod_{i=1}^{L} d\ell_i \right) \prod_{I \in P(\beta)} \frac{1}{X_I}, \tag{8.27}$$

where the product is over all edges, I, of β.

Indeed, the homomorphism μ defines a consistent assignment of momenta to any Feynman graph that is dual to some triangulation of Σ.

Chapter 9

Surface-type cluster algebras

As explained in Chapter 8, the perturbation series organizes into a sum of partial amplitude contributions $A(\Sigma)$, for Σ a surface with boundary (or, for biadjoint scalar theory, into contributions $A(\Sigma, \Sigma')$, associated to two surfaces). $A(\Sigma)$ is a sum of terms associated to triangulations of Σ (or $A(\Sigma, \Sigma')$ is a sum of terms associated to triangulations of Σ that are compatible with Σ'). Fix α a triangulation of Σ. A triangulation β is called a *mutation* of α if it differs from α by one arc. The types of possible mutations are shown in Figures 9.2 to 9.5. Any triangulation of Σ can be obtained form α by applying a series of mutations. This fact motivated early work on hadronic amplitudes and string theory, including [53], which expressed the n particle tree level Veneziano amplitude as an integral over variables u_I, associated to the arcs, I, of the n-gon. In [53], the non-crossing relations were introduced:

$$u_I + \prod_{J \text{ crosses } I} u_J = 1. \tag{9.1}$$

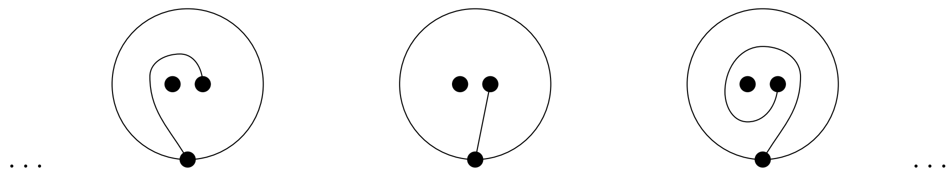


Figure 9.1: Infinitely many non-homotopic arcs between two points in the two punctured disk. The arcs are related to each other by Dehn twists of the surface.

Figure 9.2: The mutation of an arc inside a square.

Figure 9.3: The mutation of an arc inside a digon (left) is a self-folded triangle (right).

Figure 9.4: The mutation of an arc inside a monogon.

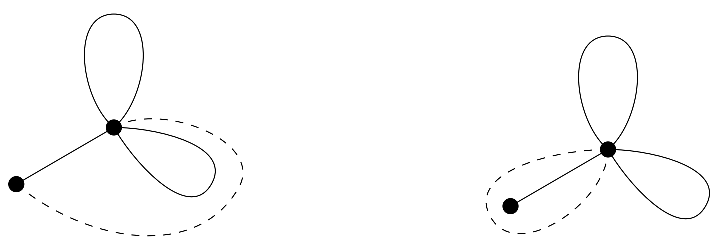


Figure 9.5: The mutation of an arc on a sphere.

The u_I variables, satisfying (9.1), were used to make it manifest that the leading order contributions to the Veneziano integral are from the points $(u_I = 0, ..., u_K = 0)$ associated to triangulations $\{I, J, ..., K\}$ of the n-gon, which are maximal sets of non-crossing arcs. The definition of the variables u_I is recalled in Appendix 5.A, and used in Chapter 6.

Attempts were made (e.g. in [49] and [50]) to generalize the approach of [53] beyond the n-gon, to arbitrary surfaces with boundary, i.e. to the loop and multi-trace contributions to the perturbation series. A variable u_I was defined for each homotopy class of arc, I, in a given marked surface, Σ. The non-crossing of two arcs, I and J, could then be imposed using equations of the form (9.1).This approach naively fails to give viable integrals, because there are often infinitely many non-homotopic arcs between two given marked points on a surface. Figure 9 shows one such family of arcs, for the two punctured disk. Chapter 10 will give one possible solution to this problem, based on the construction of ABHY polytopes.

Chapter 7 studied the ABHY polytopes associated to certain categories of modules. There is a general link between triangulations of surfaces and categories of quiver modules, because triangulations of surfaces are naturally associated to quivers that are 'dual' to the triangulation. The construction of [18] can be used to associate categories $\mathcal{C}$ to surfaces, called 'cluster categories', such that the indecomposable modules correspond (barring some complications) to arcs on the surface, and such that extensions between two modules in $\mathcal{C}$ correspond to mutations between arcs.

This chapter is a limited review of facts about cluster algebras and surfaces, and their relation to the quiver representation categories discussed in Chapter 7. The review is confined to the fewresults needed for Chapter 10, which gives examples of ABHY polytopes associated to surfaces. References include [36], [37], and [69].

9.1 Surfaces

Fix a marked surface Σ, as in Section 8.1, and a triangulation, α. Write $P(\alpha)$ for the set of arcs in α. Suppose that α has no self-folded triangles. Let $I \in P(\alpha)$ be an arc in α. Removing I from the triangulation, there is a unique arc $J \neq I$ that does not cross any of the arcs in $P(\alpha)\backslash I$. The *mutation* of α at I is the new triangulation, α_I,

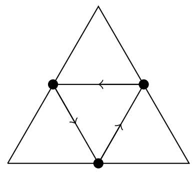


Figure 9.6: The construction of the quiver dual to a triangulation.

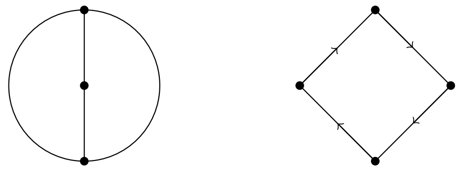


Figure 9.7: The digon, and its dual quiver.

with the arc I replaced by the arc J,

$$P(\alpha_I) = P(\alpha)\backslash I \cup \{J\}. \tag{9.2}$$

If I is mutable, then the two triangles in α that contain I must be one of the cases shown in Figures 9.2 to 9.5. If I is not contained in an unfolded square, as in Figure 9.2, then mutating I will produce a self-folded triangle, as in Figures 9.3 to 9.5.

Fix an orientation of Σ and take a triangulation, α, without self-folded triangles. There is an associated quiver Q_α, with vertex set $P(\alpha)$. The arrows of Q_α are defined by the following rule. For every triangle in α, with arcs (I, J, K), Q_α has an associated triangle of arrows:

$$I \to J \to K \to I, \qquad \text{or} \qquad I \leftarrow J \leftarrow K \leftarrow I, \tag{9.3}$$

where the choice of arrows is chosen to match the ordering induced by the orientation

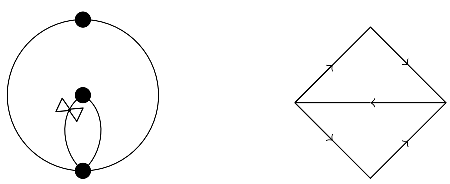
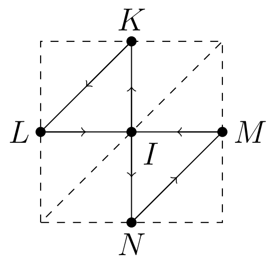


Figure 9.8: The tagged self-folded triangle, and its dual quiver.

of Σ. See Figure 9.6. Reversing the orientation of Σ leads to a triangulation $\overline{\alpha}$ of $\overline{\Sigma}$, and

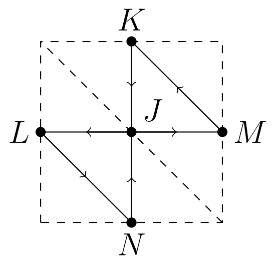
$$Q_{\overline{\alpha}} = (Q_\alpha)^{\mathrm{op}}. \tag{9.4}$$

Let $I \in P(\alpha)$ be a mutable arc, that is contained in an unfolded square (as in Figure 9.2). Then, near I, the quiver Q_α is:



(9.5)

There may be other arrows between K, L, M, N, induced by other triangles in α. Arrows pointing in opposite directions should be understood to cancel out. Mutating I, gives a new arc, J, and a new quiver, Q_{α_I}, with vertex set $P(\alpha_I)$. The arrows of Q_{α_I} near J are



(9.6)

This computation are summarised by the following lemma-definition.

Lemma 9.1. *Q_{α_I} has the following arrows:*

1. *$K \to L$ for every arrow $K \to L$ in $Q_\alpha \backslash I$,*

2. *$K \to L$ for every chain $K \to I \to L$ in Q_α,*

3. *$J \to K$ for every $K \to I$ in Q_α,*

4. *$K \to J$ for every $I \to K$ in Q_α.*

Arrows pointing in opposite directions cancel.

The rule in Lemma 9.1 can be taken as a definition of a *quiver mutation.* The correspondence between triangulations, α, and quivers Q_α ceases to hold if α has a self-folded triangle. This motivates the introduction of *tagged triangulations.* An arc in a tagged triangulation that ends at a puncture can be *tagged* with a $\bowtie$ sign, as in Figure 9.8.

Definition 9.2. *A tagged triangulation of Σ is a set of tagged and untagged arcs satisfying*

1. *If three or more arcs are incident on a puncture, the arcs are either all tagged with $\bowtie$ at that puncture, or all untagged.*
2. *If exactly two arcs are incident on a puncture, they are oppositely tagged at that puncture, and they have the same endpoints.*
3. *If all pairs of arcs as in (2) are replaced by a self folded triangle around that puncture, the resulting collection of arcs is a triangulation of Σ in the ordinary sense.*

The purpose of Rule 2 is to ensure that mutations of tagged triangulations correspond to mutations of the associated quivers. The quiver associated to the tagged self-folded triangle is given in Figure 9.8, and this completes the definition of the correspondence between tagged triangulations and the dual quivers:

Lemma 9.3. (Tagged triangulations. [36]) *Fix a surface Σ, with triangulation α, dual to a quiver Q. There is a 1:1 correspondence between the quivers obtained by mutating Q, and the tagged triangulations of Σ.*

9.2 Quiver cluster algebras/categories

Fix a quiver Q with vertex set $A \sqcup B$. The *quiver cluster algebra* associated to Q, $\mathcal{A}_Q$, is a subalgebra of the Laurent ring

$$L = k(x_i), \tag{9.7}$$

for some formal variables x_i, $i \in A \sqcup B$. The pair (Q, x), of a quiver decorated with variables $x_i \in L$ is called a *seed.* Let Q_i be the quiver obtained by mutating Q at some $i \in A$, using the rule in Lemma 9.1. Define the *mutated cluster variable*, $x_i' \in L$, by the relation

$$x_i x_i' = \prod_{i \to j} x_j + \prod_{j \to i} x_j. \tag{9.8}$$

Call the seed $(Q_i, x_1, ..., x_i', ..., x_n)$ the *mutation* of (Q, x) at vertex i. The quiver cluster algebra of Q, $\mathcal{A}_Q$, is the polynomial ring in L generated by all cluster variables obtained in this way, from arbitrary series of mutations of the vertices $i \in A$. The vertices $i \in B$ are not mutated and are called *frozen.*

Example 9.4. *Take Q to be the quiver in* (9.5)*, which mutates to the quiver in* (9.6)*. Take K, L, M, N to be frozen vertices. Then the cluster variables are*

$$x_K, x_L, x_M, x_N, x_I, x_J, \tag{9.9}$$

and they satisfy

$$x_I x_J = x_K x_N + x_M x_L. \tag{9.10}$$

The cluster algebra is therefore

$$\mathcal{A}_Q = k[x_K, x_L, x_M, x_N, x_I, x_J] / \langle x_I x_J - x_K x_N - x_M x_L \rangle\,. \tag{9.11}$$

This can be recognized as the coordinate ring of the Grassmannian $Gr(2,4)$, and (9.10) *is the Plücker relation.*

Rather than study the algebraic properties of $\mathcal{A}_Q$, the rest of this chapter reviews what is known about cluster mutations. Let Q have n unfrozen vertices. The cluster algebra $\mathcal{A}_Q$ can be associated with a fan, $\mathcal{F}_Q$, in $\mathbb{Z}^n$, called the *g-vector fan* of $\mathcal{A}_Q$. This fan has one cone for every cluster, a generator for every cluster variable, and two cones meet in codimension 1 iff the associated clusters are related by a mutation. For each cluster variable x, there is a distinguished generator, $g_x \in \mathbb{Z}^n$, called the g-vector of x. $\mathcal{F}_Q$ is independent of the frozen vertices of Q. It is possible to define $\mathcal{F}_Q$ using only the data above, as in [38], but this is cumbersome. In the case that Q

is dual to a triangulation of a surface, $\mathcal{F}_Q$ has a direct presentation in terms of the surface, given in Section 9.3, below. When Q is Dynkin, $\mathcal{F}_Q$ is a complete fan. It is not otherwise the case that $\mathcal{F}_Q$ is a complete fan, but $\mathcal{F}_Q$ has the following properties:

Proposition 9.5. ([38],[46].) *The g-vector cones are non-overlapping. If $g_1, ..., g_n$ are the g-vectors of a cluster, the determinant* $\det[g_1, ..., g_n]$ *is* ± 1*.*

The *cluster category* $\mathcal{C}$ of a quiver cluster algebra $\mathcal{A}_Q$ is a certain factor category of $D^b(\text{mod}-kQ)$, constructed in [18]. There is a correspondence between the indecomposable objects of the cluster category $\mathcal{C}$ and the cluster variables of $\mathcal{A}_Q$. The precise definition is not needed here. Let the classes $[P_i]$ be a basis of $K_0(\mathcal{C})$. For any cluster variable, $x \in L$, the class of its associated module, $N_x \in \text{ind}-\mathcal{C}$, can be written

$$[N_x] = \sum g_x(i)[P(i)], \tag{9.12}$$

for some integers $g_x(i)$. Equation (9.12) defines an integer vector $g_x \in \mathbb{Z}^n$. If the modules P_i are the projective indecomposable modules of $\mathcal{C}$, then g_x is the g-vector of x. Independent of any choice of basis, the g-vector fan $\mathcal{F}_Q$ is therefore naturally regarded as a fan in $K_0(\mathcal{C}) \otimes \mathbb{Z}$, with generators given by the classes $[N]$, $N \in \text{ind}-\mathcal{C}$. It is then a corollary of Remark 7.12 that:

Lemma 9.6. *If $\mathcal{C}$ is the cluster category for Q, the generators of the normal fan of the associated ABHY polytope are the g-vectors of $\mathcal{A}_Q$.*

Example 9.7. *For $Q = A_2$, the cluster category $\mathcal{C}$ has the AR quiver given in Example 7.5, above. It has five indecomposable objects: $M_{1,0}$, $M_{2,0}$, $M_{1,1}$, $M_{2,1}$, and $M_{1,2}$. The SESs in $\mathcal{C}$ give mesh relations,*

$$[M_{1,0}] + [M_{1,1}] - [M_{2,0}], \tag{9.13}$$

$$[M_{2,0}] + [M_{2,1}] - [M_{1,1}], \tag{9.14}$$

$$[M_{1,1}] + [M_{1,2}] - [M_{2,1}]. \tag{9.15}$$

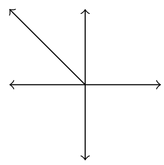
Take $[M_{1,0}]$, $[M_{2,0}]$ as a basis for $K_0(\mathcal{C})$. Then the other classes are given by

$$[M_{1,1}] = -[M_{1,0}] + [M_{2,0}], \tag{9.16}$$

$$[M_{2,1}] = -[M_{1,0}], \tag{9.17}$$

$$[M_{1,2}] = -[M_{2,0}]. \tag{9.18}$$

Drawing these as vectors gives the following g-vector fan for A_2:



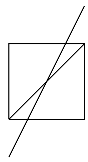
(9.19)

This fan is the normal fan of the ABHY polytope of $\mathcal{C}$, which is a pentagon.

9.3 Surface cluster algebras

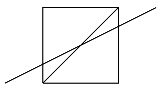
Fix $\mathcal{A}_Q$ a cluster algebra, with Q dual to a (tagged) triangulation α of a marked surface Σ. The marked surface Σ has finitely many triangulations, and, therefore, also finitely many tagged triangulations. It follows from Lemma 9.3 that Q mutates to finitely many quivers. But it is not necessarily the case that $\mathcal{A}_Q$ has finitely many clusters and cluster variables. Lemma 9.8, below, says that $\mathcal{A}_Q$ has as many cluster variables as there are (tagged) arcs on Σ (considered up to homotopy). Generically, there are infinitely many arcs on a surface, as in Figure 9.

The g-vector fan $\mathcal{F}_Q$ of $\mathcal{A}_Q$ has an elementary description in terms of the surface, Σ. Suppose that α has no self-folded triangles, and fix I any arc on Σ that does not end at a puncture. Let $e(I)$ be the curve on Σ obtained from I by moving the endpoints of I in the direction of the orientation of Σ, as in Section 8.1. And let $\overline{e}(I)$ be the curve obtained by doing the same with the opposite orientation. If I is an arc in the triangulation α, then $e(I)$ intersects I the triangulation in a 'Z' pattern:



(9.20)

Whereas $\overline{e}(I)$ cuts I in a 'S' pattern:



(9.21)

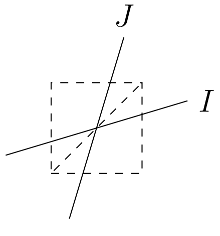
More generally, if an arc I intersects an arc $J \in P(\alpha)$, then say I intersects J *positively* if it is as in (9.20), and *negatively* if it is as in (9.21). Let $P(\alpha) = \{I_1, ..., I_n\}$ be the arcs of the initially chosen triangulation, α. Then define the g vector of the arc I:

$$g_I(i) = \begin{cases} +1 & e(I) \text{ cuts } I_i \text{ positively} \\ -1 & e(I) \text{ cuts } I_i \text{ negatively} \\ 0 & \text{else} \end{cases} \tag{9.22}$$

The g-vectors of the arcs in the triangulation are seen to be

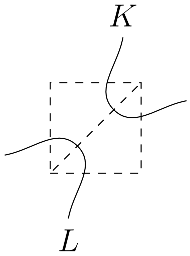
$$g_{I_i}(j) = \delta_{ij}. \tag{9.23}$$

Moreover, let I and J be two arcs on Σ that intersect once. Suppose their associated curves, $e(I)$ and $e(J)$, intersect in a square of the triangulation, α, (instead of in a punctured digon or twice puncture monogon).



(9.24)

Construct the following two new curves, K and L, that cut the arcs in the square the same number of times as I and J:



(9.25)

It is understood that K and L are the same outside the square as I and J. It then follows from (9.22) that

$$g_I + g_J = g_K + g_L. \tag{9.26}$$

When translated into statements about the dual quivers, equations (9.23) and (9.26) match the definition of g-vectors in [38], so that the vectors g_I are the g-vectors of cluster variables. This can then be used to show that

Lemma 9.8. ([36],[37]) *The cluster variables of $\mathcal{A}_Q$ are in 1:1 correspondence with (tagged) arcs of Σ.*

Remark 9.9. *It is explained above Lemma 9.6 that g-vectors can be regarded as classes $[N]$ in $K_0(\mathcal{C})$, independent of their expression in a fixed basis. It then follows from (9.26) that the modules associated to these curves satisfy*

$$[I] + [J] - [K] - [L] = 0 \tag{9.27}$$

in $K_0(\mathcal{C})$, where the same letter is used to denote both the arc on Σ and the module in $\mathcal{C}$ that corresponds to it.

Example 9.10. *The rules above can be used to find the g-vector fan $\mathcal{F}_Q$ associated to $Q = A_2$, computed above in Example 9.7. Take the pentagon, with vertices $1, ..., 5$, and arcs K_{ij}. Fix the triangulation with arcs K_{13}, K_{14}. Then, e.g., the curve $e(K_{24})$ cuts K_{13} negatively, and K_{14} positively, so that $g(K_{24}) = (-1, 1)$. Similarly, $g(K_{35}) = (0, -1)$ and $g(K_{25}) = (-1, 0)$. This reproduces the fan in Example 9.7.*

Chapter 10

Affine examples

This chapter studies two special families of ABHY polytopes (as in Chapter 7), that are related to the partial amplitude expansion in gauge theory (as in Chapter 8).

When Q is a Dynkin quiver, or an extended Dynkin quiver, the categories mod$-kQ$ are well studied, and so this chapter focusses on the extended Dynkin quivers. The $\tilde{A}_n$ quivers are dual to triangulations of the annulus. The $\tilde{D}_n$ quivers are dual to triangulations of the twice punctured disk. The associated cluster algebras $\mathcal{A}_Q$ have infinitely many clusters. However, by restricting to a finite set of arcs, the construction of Chapter 7 gives polytopes with finitely many faces. The normal fans of these polytopes are coarsenings of the g-vector fans, $\mathcal{F}_Q$. The face structure of the polytopes records the mutations between triangulations of the surfaces, analogous to the associahedron in the case of the n-gon.

10.1 Closed curves

Section 9.3 gave a rule for associating a g-vector g_I to an arc I on a triangulated surface. In applying this rule, the arc I is displaced, to give a curve, $e(I)$, which cuts arcs of the triangulation. More generally, if J is any curve on the surface, the rule can be used to find a vector g_J. If J is not $e(I)$, for some arc I, then g_J will not be a g-vector of a cluster variable, but the vector g_J will satisfy 'Skein relations', as in (9.26).

Let Σ be the annulus, or the twice punctured disk, and fix a triangulation. Write

Δ for the noncontractible closed curve on Σ.[1] Let I be an arc that intersects Δ once. Then notice that the Skein relation between I and Δ produces one arc, J (instead of two disconnected arcs). The associated g-vectors satisfy

$$g_I + g_\Delta - g_J = 0. \tag{10.1}$$

The arc J has the same endpoints as the arc I, but it differs from I by a twist. The direction of the twist (positive or negative) depends on both I and the choice of triangulation. The arc J intersects Δ just once, and so

$$g_J + g_\Delta - g_K = 0, \tag{10.2}$$

for some arc K related to J by a twist. Since K cannot be I, (this would imply that $g_\Delta = 0$), K must be twisted in the same direction as J. It follows that the vectors

$$g_I + kg_\Delta, \tag{10.3}$$

for $k \geq 1$, are all the g-vectors of arcs, and the arcs are related to I by performing twists. In [77], this is used to prove that

Lemma 10.1. ([77].) *For $Q = \tilde{A}_n$ or $Q = \tilde{D}_n$, the vector g_Δ is not contained in any of the cones in the fan $\mathcal{F}_Q \otimes \mathbb{R}$. But $\mathcal{F}_Q \otimes \mathbb{R}$ is dense in $\mathbb{R}^n$.*

Remark 10.2. *For $Q = \tilde{A}_n$ or $Q = \tilde{D}_n$, the AR translation on quiver modules is related to Dehn twists of the surface. Take $x \in \mathcal{A}_Q$ a cluster variable such that M_x is in the preprojective component, and M_x is not projective (i.e. x is not an initial cluster variable). Write I_x for the associated arc. The translation $\mathrm{Tr}DM_x$ is indecomposable, and so write $\mathrm{Tr}DM_x = M_y$, for y a cluster variable. The arc I_y is the half-Dehn twist of I_x.*

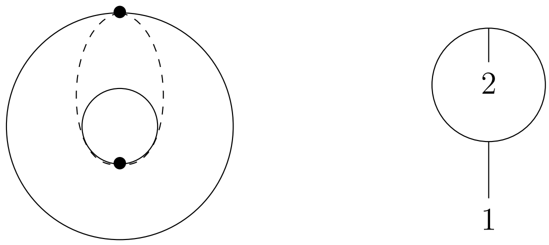


Figure 10.1: Left: any triangulation of the annulus is dual to the Kronecker quiver. Right: the Feynman ribbon graph that is dual to any of these triangulations.

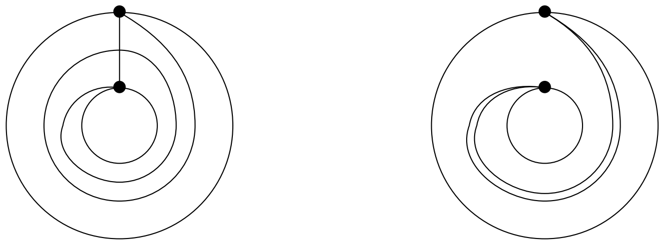


Figure 10.2: A Skein relation on the annulus: the sum of (classes of) the two crossing arcs on the left is equal to the sum of the two non-crossing arcs on the right, where the initial triangulation in Figure 10.2 has been assumed.

10.2 The Kronecker quiver

Let Q be the Kronecker quiver,

$$Q = \quad \bullet \rightrightarrows \bullet \tag{10.4}$$

This quiver is dual to triangulations of the annulus with one marked point on each of its boundaries, Σ. Mutations of Q result in one of two quivers: Q itself, and Q^{op}. The corresponding triangulations are both dual to the same Feynman diagram, shown in Figure 10.2.

The arcs on Σ all have the same end points: the two marked boundary points. Fix an arc between the two points, K_0, and write K_n for the arc obtained by twisting K_0 n times. Fix the triangulation of Σ, α, given by K_0 and K_1. The arcs K_0 and K_2

[1] In the case of the twice punctured disk, a closed curve that surrounds a single puncture is considered contractible, but not the closed curve that surrounds both punctures.

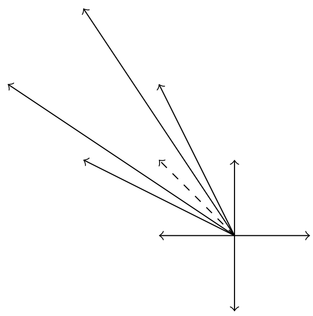


Figure 10.3: The g-vector fan, $\mathcal{F}_Q$, for the Kronecker quiver.

intersect once. The Skein relation with respect to α gives the mesh relation

$$[K_0] + [K_2] - 2[K_1] = 0. \tag{10.5}$$

See Figure 10.2. Likewise, for $n > 0$,

$$[K_n] + [K_{n+2}] - 2[K_{n+1}] = 0. \tag{10.6}$$

Taking $[K_0]$ and $[K_1]$ as the basis, it follows that the g-vectors for the arcs K_n are

$$[K_n] = -n[K_0] + (n+1)[K_1], \tag{10.7}$$

for $n \geq 2$. Likewise, the g-vectors for the negatively twisted arcs are

$$[K_{-n}] = -(n-1)[K_0] + (n-2)[K_1], \tag{10.8}$$

for $n \geq 1$. These g-vectors give the fan $\mathcal{F}_Q$ in Figure 10.3. The g-vector cones are dense in the plane, but do not include the direction $g_\Delta = (-1, 1)$.

The rest of this section derives equations that cut out polygons whose normal fans are coarsenings of this g-vector fan, and include the additional direction g_Δ. In principle, these equations can be deduced directly from the fan. This requires choosing constants c_i for each g-vector such that the halfplanes $g_i \cdot x \leq c_i$ intersect in the appropriate polygon. Notice that it is not enough to set each c_i to an arbitrary positive constant. The generalized ABHY construction of Chapter 7 solves the problem of finding appropriate constants automatically.

Write $\mathcal{C}_{n,m}$ for the subset of arcs containing K_{-m}, ..., K_0, K_1, ..., K_n and also Δ. Write $\mathcal{P}_n$ for the set of positive arcs, $K_0, ...K_n$, and write $\mathcal{I}_m$ for the set of negative ones. Let $\mathcal{F}_{m,n}$ be the subfan of $\mathcal{F}_Q$ that has only the vectors $[K_{-m}], ..., [K_n]$. And let $\overline{\mathcal{F}}_{m,n}$ be the fan obtained from $\mathcal{F}_{m,n}$ by adding $[\Delta]$, to create two additional cones.

Write K_k for the coordinate dual to the vector $[K_k]$, as well as for the arc K_k. Then the deformed mesh relations amoung the arcs $\mathcal{P}_n$ are, by (10.6),

$$K_k + K_{k+2} - 2K_{k+1} = \alpha_k, \tag{10.9}$$

for $k \geq 0$. Likewise, the deformed mesh relations amoung the arcs $\mathcal{I}_m$ are

$$K_{-k} + K_{-k+2} - 2K_{-k+1} = \alpha_{-k}, \tag{10.10}$$

for $k \geq 3$. Δ intersects each of the arcs N_k at one point. As discussed in Section 10.1, the Skein relation with Δ twists arcs. The Skein relations between Δ and $\mathcal{P}_m$ give

$$K_k + \Delta - K_{k+1} = \beta_k, \tag{10.11}$$

for $k \geq 0$, and the relations between Δ and $\mathcal{I}_m$ give

$$K_{-k} + \Delta - K_{-k-1} = \beta_{-k}, \tag{10.12}$$

for $k < 0$. The final case is Skein relations between arcs in $\mathcal{P}_n$ and arcs in $\mathcal{I}_m$. For $k \geq 1$ and $l \geq 2$,

$$K_k + K_{-l} - (k + l - 2)\Delta = \gamma_{k,l}. \tag{10.13}$$

There are linear dependencies between the relations, (10.9) – (10.13). Notice that

$$\beta_k + \alpha_{k-1} = \beta_{k-1}, \qquad \text{and} \qquad \beta_{-k} + \alpha_{-k-1} = \beta_{-k+1}. \tag{10.14}$$

Moreover,

$$\gamma_{k,l} + \beta_{k-1} = \gamma_{k-1,l}, \qquad \text{and} \qquad \gamma_{k,l} + \beta_{-l+1} = \gamma_{k,l-1}. \tag{10.15}$$

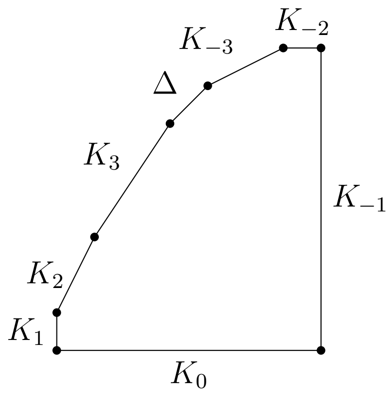


Figure 10.4: The octogon from Example 10.5.

A linearly independent set of deformed mesh relations is

$$\alpha_k = K_k + K_{k+2} - 2K_{k+1}, \tag{10.16}$$

$$\alpha_{-k} = K_{-k} + K_{-k+2} - 2K_{-k+1}, \tag{10.17}$$

$$\beta_{n-1} = K_{n-1} + \Delta - K_k, \tag{10.18}$$

$$\beta_{-m+1} = K_{-m+1} + \Delta - K_{-m}, \tag{10.19}$$

$$\gamma = K_n + K_{-m} - (n+m-2)\Delta. \tag{10.20}$$

These relations define an affine hyperplane, H, and a polygon, P. It follows from Theorem 7.16 that,

Proposition 10.3. *For any choice of positive constants ($\alpha_k, \beta_{n-1}, \beta_{-m+1}, \gamma$), the deformed mesh relations define a polytope with normal fan $\overline{\mathcal{F}}_{m,n}$.*

Example 10.4. *The minimal example in this case would be to take the set with just K_0, K_{-1}, and Δ. In this case, there is one deformed mesh equation,*

$$\gamma = K_0 + K_{-1} - \Delta, \tag{10.21}$$

and the associated polytope is a triangle.

Example 10.5. *Consider $\mathcal{C}_{3,3}$, the set with arcs K_{-3}, ... ,K_2 and Δ. The mesh equations involving the arcs K_i are*

$$c_2 = K_0 + K_2 - 2K_1 \tag{10.22}$$

$$c_3 = K_1 + K_3 - 2K_2 \tag{10.23}$$

$$c_{-1} = K_{-3} + K_{-1} - 2K_{-2}, \tag{10.24}$$

for some $c_i > 0$. The "mesh" equations involving Δ are

$$c_\Delta = K_2 + \Delta - K_3 \tag{10.25}$$

$$c_{-3} = K_3 + K_{-3} - 4\Delta \tag{10.26}$$

$$c_{-2} = \Delta + K_{-2} - K_{-3}. \tag{10.27}$$

Solving these equations in terms of K_0 and K_1 gives the following description of the hyperplane, H:

$$K_2 = -K_0 + 2K_1 + c_2 \tag{10.28}$$

$$K_3 = -2K_0 + 3K_1 + 2c_2 + c_3 \tag{10.29}$$

$$\Delta = -K_0 + K_1 + c_2 + c_3 + c_\Delta \tag{10.30}$$

$$K_{-3} = -2K_0 + K_1 + 2c_2 + 3c_3 + 4c_\Delta + c_{-3} \tag{10.31}$$

$$K_{-2} = -K_0 + c_2 + 2c_3 + 3c_\Delta + c_{-3} + c_{-2} \tag{10.32}$$

$$K_{-1} = -K_1 + c_3 + 2c_\Delta + c_{-3} + 2c_{-2} + c_{-1} \tag{10.33}$$

The intersection of this hyperplane with the positive orthant (i.e. $K_i > 0$, $\Delta > 0$)

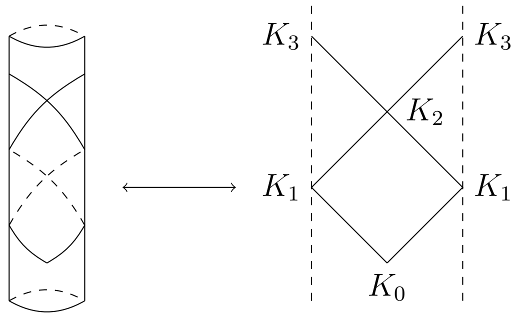


Figure 10.5: A segment of the preprojective component of the AR quiver for the Kronecker quiver. It can be drawn in a natural way on the cylinder.

gives an octogon with the following vertices:

$$
\begin{aligned}
(K_0, K_1) = \;& (0,0), (c_2, 0), (c_2 + 2c_3, c_3), \\
& (c_2 + 2c_3 + 3c_\Delta, c_3 + 2c_\Delta), \\
& (c_2 + 2c_3 + 3c_\Delta + c_{-3}, c_3 + 2c_\Delta + c_{-3}), \\
& (c_2 + 2c_3 + 3c_\Delta + c_{-3} + c_{-2}, c_3 + 2c_\Delta + c_{-3} + c_{-2}) \\
& (c_2 + 2c_3 + 3c_\Delta + c_{-3} + c_{-2}, c_3 + 2c_\Delta + c_{-3} + 2c_{-2} + c_{-1}) \\
& (0, c_3 + 2c_\Delta + c_{-3} + 2c_{-2} + c_{-1})
\end{aligned}
$$

Notice that the shape of the octogon is independent of the choice of positive constants, c_i *and* c_Δ*.*

Remark 10.6. (Quiver representations.) *The application of Theorem 7.16 to this example implicitly used the relation of the arcs on* Σ *to representations of the Kronecker quiver,* Q*. Choosing a subset of arcs amounts to choosing a subcategory of* $\mathrm{mod}{-}kQ$*, except that there is a subtlety because the arc* Δ *is associated to a family of modules. To explain this point, this remark recall the classification of the indecomposable modules of* $\mathrm{mod}{-}kQ$ *(which follows from AR theory). Write* M_1 *and* M_2 *for the simple modules,*

$$k \rightrightarrows 0 \qquad \textit{and} \qquad 0 \rightrightarrows k \tag{10.34}$$

M_1 is injective, and M_2 is projective. The other projective indecomposable is

$$P_1 : \qquad k \overset{a}{\underset{b}{\rightrightarrows}} k^2 \tag{10.35}$$

where a and b are any two linearly independent linear maps. The preprojective indecomposable modules are K_0, K_1, K_2, ..., with

$$K_n : \qquad k^n \overset{a}{\underset{b}{\rightrightarrows}} k^{n+1} \tag{10.36}$$

for any two linearly independent and injective linear maps, a and b. The preinjective indecomposable modules are $K_{-1}, K_{-2}, ...,$ with

$$K_{-n} : \qquad k^n \overset{a}{\underset{b}{\rightrightarrows}} k^{n-1} \tag{10.37}$$

for any two linearly independent and surjective linear maps, a and b. The classes of these modules, $[K_n] \in K_0(\mathrm{mod}{-}kQ)$, satisfy the relations (10.6) derived from the Skein relations in the surface. In fact, the action of the Auslander-Reiten translation is

$$\mathrm{TrD}K_n \simeq K_{n+2}, \qquad \mathrm{DTr}K_{-n} \simeq K_{-n-2}. \tag{10.38}$$

This means that, e.g.,

$$\delta : \qquad 0 \to K_n \to E \to K_{n+2} \to 0, \tag{10.39}$$

with $[E] = 2[K_{n+1}]$, is an 'AR SES' in the sense of Section 7.2. The associated functor, δ^*, is the simple functor E_{K_n}. The indecomposable modules of $\mathrm{mod}{-}kQ$ are the modules K_n, above, together with modules

$$R_n(z) : \qquad k^n \overset{J_a}{\underset{J_b^{tr}}{\rightrightarrows}} k^n \tag{10.40}$$

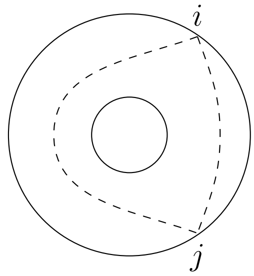


Figure 10.6: For two points i and j on the boundary of the annulus, there are two non-homotopic arcs, I_{ij} and I_{ji}.

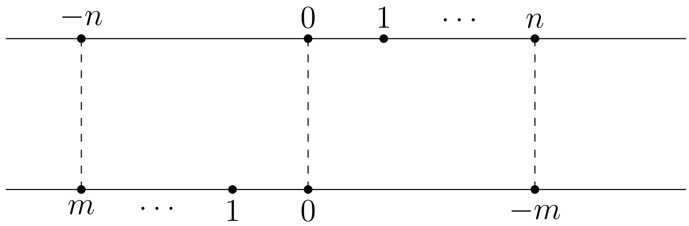


Figure 10.7: Arcs on the infinite strip become arcs on the annulus.

for every $z = [a : b] \in \mathbb{P}^1$, where J_a is the Jordan block, with a above the diagonal, in some fixed basis. The class of $R_n(z)$ is $[R_n(z)] = n[K_1] - n[K_0]$, and it satisfies the same mesh relations that were derived for Δ, equations (10.11) – (10.13)*. To apply Theorem 7.16 to the example in this section, it is necessary to pick some subcategory $\mathcal{C}$ of* mod$-kQ$ *that has the modules $K_{-n}, ..., K_m$, and also $R_1(z_0), ..., R_{n+m-2}(z_0)$, for some fixed $z_0 \in \mathbb{P}^1$. This is not a Serre subcategory, because $R_{n+m-2}(z_0)$ has self-extensions, and because it doesn't contain the extension $[K_m] + [\Delta] - [K_{m+1}]$.*

10.3 Double trace sector

The 'double trace' partial amplitudes are given by $A(\Sigma)$, for Σ with two boundary components. The leading order contribution is for Σ an unpunctured annulus. This section finds ABHY polytopes for the unpunctured annulus. The Kronecker quiver example, Section 10.2, is a special case. The general construction in Chapter 7 suffices to prove that the polytopes behave correctly. The main task is describe the deformed mesh relations explicitly in terms of the arcs on the annulus, as was done in the

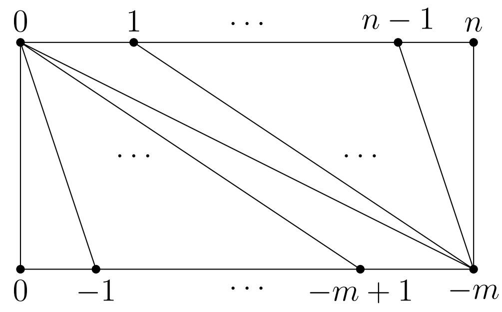


Figure 10.8: A triangulation of the annulus.

Kronecker example. To simplify this discussion, modules and arcs are conflated, as in the previous section.

Let Σ be the annulus with n marked points on the 'outside' boundary, and m marked points on the 'inside' boundary. For two points $i \neq j$ on the outside boundary, there are two non-homotopic arcs, I_{ij} and I_{ji}, from i to j, as in Figure 10.6. Likewise write J_{ij} for the 'in-in' arcs. To describe the 'out-in' arcs on Σ, consider an infinite strip, with points on the two boundaries labelled by the integers. Adopt the convention that the points on both boundaries are arranged in 'increasing' order with respect to the orientation, as in Figure 10.7. The annulus is recovered from this strip by identifying the points on the 'in' boundary mod m, and on the 'out' boundary mod n. Write K_{ij} for the arc on the infinite strip from point i on the outside to point j on the inside, for integers i and j. Then K_{ij} gives an 'out-in' arc on the annulus after the identification. For example, the arc $K_{0,2+5m}$ begins at point 0 on the outside, wraps anti-clockwise around the annulus 5 times, and then ends at point 2 on the inside. Notice that K_{ij} and $K_{i+n,j-m}$ give the same arcs on the annulus.

Lemma 10.7. (Arcs on Σ.) *The non-homotopic arcs on the surface Σ are given by the following list: n^2 'out-out' arcs, I_{ij}; m^2 'in-in' arcs, J_{ij}; an 'out-in' arc, K_{ij}, for each integer pair (i, j) modulo $(i, j) \sim (i + n, j - m)$; and the closed loop, Δ.*

The quotient of $\mathbb{Z}^2$ by the translations $(i, j) \sim (i + n, j - m)$ can be conveniently drawn as a lattice on a cylinder.

Figure 10.8 gives a triangulation of the annulus, α, that uses only 'in-out' arcs. α comprises the m arcs $K_{0,0}...K_{0,-m+1}$, and the n arcs $K_{0,-m}, ..., K_{n-1,-m}$. This

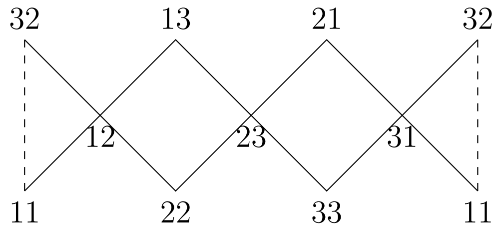


Figure 10.9: The in-in tube/cylinder for $n = 3$.

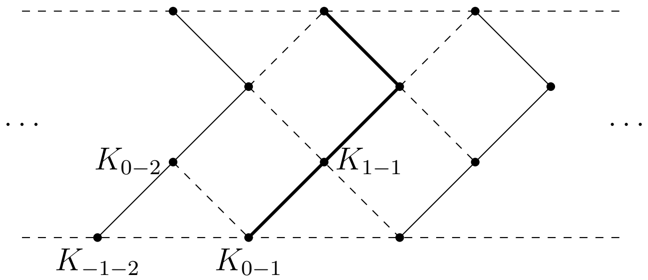
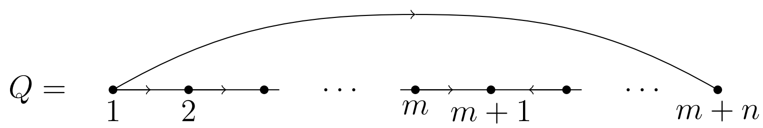


Figure 10.10: The 'in-out' arcs are arranged in a lattice on a cylinder, shown for $n = 2$, $m = 1$. The initial triangulation, corresponding to the bold line, cuts the cylinder into two halves.

triangulation is dual to the following $\tilde{A}_{m+n}$ quiver:



(10.41)

With the notation as above, it is possible to summarise the mesh relations that arise from the Skein relations between pairs of intersecting arcs, with respect to the triangulation α. All pairs of intersecting arcs on Σ belong to one of the following four cases.

Case 1. $I - I$**.** For two 'in-in' arcs that intersect, I_{ij} and I_{kl}, and choose the indices so that $i < k < j < l$. The Skein rule gives a mesh relation

$$[I_{ij}] + [I_{kl}] - [I_{il}] - [I_{jk}]. \tag{10.42}$$

All such classes are sums of 'small' mesh relations of the form

$$[I_{ij}] + [I_{i+1j+1}] - [I_{ij+1}] - [I_{ji+1}]. \tag{10.43}$$

The arcs I_{ij} can therefore be arranged into an AR diagram, in which each diamond is a mesh relation. An example is shown in Figure 10.9, in the case of the annulus with $n = 3$ points on the in boundary. The dashed lines are identified to create a cylinder, consistent with the identification of the labels mod n. The intersections of 'out-out' arcs are described analogously.

Case 2. $K - K$**.** All 'in-out' arcs, K_{ij}, can be obtained from the initial triangulation α by a series of mutations. Draw the cylinder which is the quotient of $\mathbb{Z}^2$ by the translations $(i, j) \sim (i + n, j - m)$. The initial triangulation, α, corresponds to a 'slice' of this cylinder. This 'slice' cuts the cylinder into two halves, as in Figure 10.10. Mutating Q on sources gives all arcs to the right of this slice (where 'right' is understood as in the figure). A source is of the form $K_{i+1j} \leftarrow K_{ij} \rightarrow K_{ij+1}$, for some i, j, and the 'small' mesh relation associated to mutating K_{ij} is

$$[K_{ij}] + [K_{i+1,j+1}] - [K_{i+1,j}] - [K_{i,j+1}]. \tag{10.44}$$

For two intersecting arcs in the same half of the cylinder, K_{ij} and K_{kl}, the associated mesh relation is a sum of these 'small' mesh relations. If K_{ij} and $K_{i-sn,j-sm}$ are arcs on opposite halves of the cylinder, then

$$[K_{ij}] + [K_{i-sn,j-sm}] - s[\Delta]. \tag{10.45}$$

The mesh relations between any two 'in-out' arcs can be obtained by adding relations of the form (10.44) and relations of the form (10.45).

Case 3. $I - K$**.** If K_{ij} is the on the 'right' half of the cylinder, then

$$[K_{ij}] + [I_{i-1i+1}] - [K_{i-1j}] - [K_{i+1j}]. \tag{10.46}$$

The case when K_{ij} intersects an 'out-out' arc is analogous.

Case 4. $\Delta - K$**.** If K_{ij} is the on the 'right' half of the cylinder, then

$$[K_{ij}] + [\Delta] - [K_{i+1j+1}]. \tag{10.47}$$

If it is on the 'left' half,

$$[K_{ij}] + [\Delta] - [K_{i-1j-1}]. \tag{10.48}$$

Remark 10.8. (AR SESs.) *The 'small' mesh relations,* (10.43) *and* (10.44)*, correspond, in the terminology of Section 7.2, to Auslander-Reiten SESs. It was observed in Example 7.6 that the classes of other extensions can be written as positive sums of the classes of AR SESs. This is captured by the 'tube' and 'cylinder' pictures for 'in-in' arcs and 'in-out' arcs, respectively: large diamonds in these pictures are the sums of the small diamonds that they contain.*

With the arcs described as above, let $\mathcal{C}$ be the set of arcs comprising: Δ; the 'in-in' and 'out-out' arcs, I_{ij} and J_{ij}; all K_{ij} between two slices of the 'in-out' cylinder, one slice on each of the two halves of the cylinder. For each of the SESs, $\delta : A \to E \to B$, appearing in equations (10.43) to (10.48), with $A, B, E \in \mathcal{C}$, form the deformed mesh relations,

$$X_A - X_E + X_B = \alpha_\delta, \tag{10.49}$$

for appropriate constants α_δ. These define the affine hyperplane, H, in $K_0(\mathcal{C}, 0)$, and the ABHY polytope, P, formed by intersecting H_M with the positive orthant. By Theorem 7.16,

Proposition 10.9. *P has vertices for every triangulation of Σ that can be formed from the arcs in $\mathcal{C}$.*

Some of the faces of P can be determined by performing cutting operations on Σ. If $K \in \mathcal{C}$ is an 'in-out' arc, then cutting Σ along K produces a disk. If $\mathcal{C}[M_I]$ contains every diagonal of this disk (it may contain fewer, depending on K and the choice of truncation), then the K face of P is the A_{m+n-1} associahedron. See Figure 10.11.

If $I \in \mathcal{C}$ is an 'in-in' arc, then cutting Σ along I produces a disk and an annulus. In the terminology of Section 7.4, $\mathcal{C}[M_I]$ contains two islands, corresponding to the disk

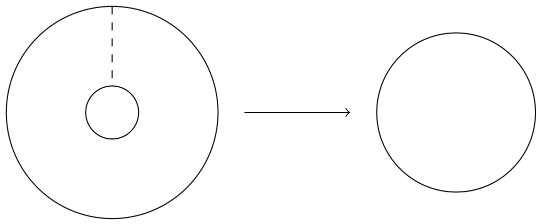


Figure 10.11: Factorization at an in-out arc.

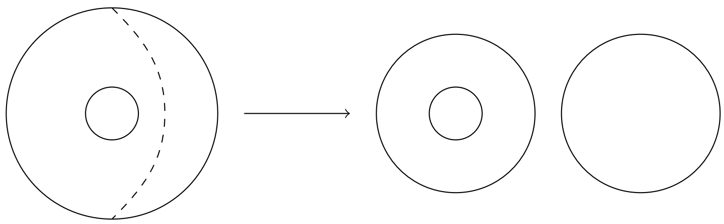


Figure 10.12: Factorization at an in-in arc.

and the annulus, respectively. Then, by Theorem 7.21, the I face of P is the cartesian product of an associahedron, and an ABHY polytope for the smaller annulus. See Figure 10.12.

Example 10.10. *Take the annulus with $n = 2$ and $m = 1$ marked points on each boundary. The triangulation with arcs K_{01}, K_{00}, and K_{10} is dual to an acyclic quiver Q. As is the triangulation with arcs K_{11}, K_{12} and K_{21}. Let $\mathcal{C}$ be given by these arcs, together with the in-in arcs, I_{00} and I_{11}, and the closed arc Δ. By drawing the arcs on the surface, it is possible to list all extensions. The Preprojective-Regular extensions*

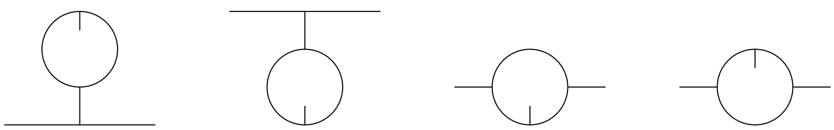


Figure 10.13: The 4 Feynman diagrams dual to triangulations of the $n = 2$, $m = 1$ annulus.

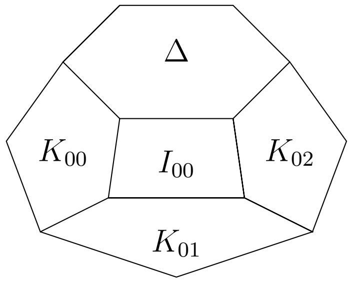


Figure 10.14: A view of one side of the ABHY polytope in Example 10.10, with faces labelled by the corresponding arcs.

are

$$\alpha_1 : K_{01} + I_{11} - K_{10} \tag{10.50}$$

$$\alpha_4 : K_{10} + I_{00} - K_{00} \tag{10.51}$$

$$\alpha_2 : K_{01} + \Delta - K_{00} \tag{10.52}$$

The Regular-Preinjective extensions are

$$\beta_2 : K_{11} + I_{00} - K_{02} \tag{10.53}$$

$$\beta_3 : K_{02} + I_{11} - K_{12} \tag{10.54}$$

$$\beta_1 : K_{11} + \Delta - K_{12} \tag{10.55}$$

The Preprojective-Preinjective extensions are

$$\gamma_2 : K_{00} + K_{02} - I_{00} \tag{10.56}$$

$$\gamma_5 : K_{10} + K_{12} - I_1 \tag{10.57}$$

$$\gamma_1 : K_{01} + K_{12} - K_{02} \tag{10.58}$$

$$\gamma_3 : K_{00} + K_{12} - \Delta \tag{10.59}$$

The Regular-Regular extension is

$$\delta : I_{00} + I_{11} - \Delta \tag{10.60}$$

There are 11 extensions in this list. But the following 3 classes are linear combinations of the others in the list: δ, γ_1, γ_3*. The remaining 8 classes satisfy the following 2 linear dependence relations:*

$$\gamma_5 + \beta_3 = \gamma_2 + \alpha_4, \tag{10.61}$$

$$\gamma_5 + \alpha_1 = \gamma_2 + \beta_2. \tag{10.62}$$

The deformed mesh relations span a 6-dimensional plane, H*, in the vector space* $K_0(\mathcal{C}, 0)$*. The polytope* P *has 14 vertices, which correspond to the maximal ways to combine compatible arcs from the set of 9 arcs in* $\mathcal{C}$*. 8 of these vertices are dual to the Feynman graphs in Figure 10.13. The remaining 6 vertices are on the* Δ *facet.*

10.A Facts from Auslander-Reiten theory

The context for the examples discussed in this chapter is the Auslander-Reiten theory for hereditary algebras. This appendix states without proof two Lemmas from the theory that 'explain' features of the analysis above.

If Q is extended Dynkin, the indecomposable projective modules of kQ have the following simple description. The arrows of Q induce a partial order on its vertices. For every vertex i in Q, let P_i be the module which has $V_j = k$ for $j > i$ and $V_j = 0$ for $j < i$. P_i is indecomposable by construction.

A left module for kQ is a right module for kQ^{op}. Given a left module for kQ^{op}, M, then $\text{Hom}_k(M, k)$ is a left module, DM, for kQ. For example, for $Q = A_2$, the representation $M = 0 \leftarrow k$ defines a left module for kQ^{op}, and DM is the left kQ module corresponding to the representation $0 \rightarrow k$.

The indecomposable projective modules of kQ^{op} can be constructed in the same way as for kQ, and write these as P_i^*. For any projective P in mod$-kQ$, take its decomposition $\oplus_i P_i^{\oplus c_i}$ and define the module P^* as $\oplus_i (P_i^*)^{\oplus c_i}$. Given M in mod$-kQ$, define its transpose, Tr$M \in$ mod$-kQ^{\mathrm{op}}$ as follows. Take M's projective resolution,

$$P' \to P \to M \to 0. \tag{10.63}$$

Then f induces a map $f^* : P^* \to P'^*$. Write Tr$M = $ cokerf^*. The *Auslander Reiten translation* of M is DTrM. If M is projective, DTr$M = 0$.

Example 10.11. *For $Q = A_2$, write $P_1 = k \to k$, $P_2 = 0 \to k$, and $M = k \to 0$. Then the projective resolution of M is*

$$P_2 \to P_1 \to M \to 0, \tag{10.64}$$

and TrM *is the kQ^{op} module $0 \leftarrow k$. Then D*Tr*$M = P_2$.*

Call a module $M \in$ ind$-kQ$ preprojective if $(D\mathrm{Tr})^n M$ is projective for some n. Likewise, call M preinjective if $(\mathrm{Tr}D)^n M$ is injective for some n. Otherwise, say that M is regular. If M is a regular module, then so is $(D\mathrm{Tr})^n M$ for all n, by definition.

Write $M \sim N$ if there exists an indecomposable morphism $M \to N$ or $N \to M$. This induces an equivalence relation. A component of ind$-kQ$ is a class under this relation. A regular component is called a *tube* if $(D\mathrm{Tr})^k M = M$ for all M in the component, and for some fixed k. A regular component is a *brick* if, for every M in the component, $(D\mathrm{Tr})^k M$ is never isomorphic to M, for all k. The main structure theorem in AR theory for hereditary algebras implies, when Q is acyclic, that [4][26]

Lemma 10.12. (AR Components.) *The preprojective kQ modules form one component. The preinjective modules form one component. Each regular component is either a tube or a brick.*

In the case of an annulus of twice punctured disk, the AR translation, in terms of arcs, corresponds to twists of the surface. The 'small meshes' that appear in the main text are then explained by [43]

Lemma 10.13. (AR SESs.) *Extensions of the form D*Tr*$B \to E \to B$ and $A \to E \to$* Tr*DA are Auslander-Reiten SESs.*

Chapter 11

Discussion

Throughout the thesis, the partial amplitude expansion of perturbative gauge theory has been central to the results. Gravity amplitudes do not have a partial amplitude expansion because gravity theories do not have colour. Instead, the coupling constant organizes the perturbative gravity amplitude (for n gravitons, say) into a sum

$$M = \sum_{\ell=0}^{\infty} g^{n-2+2\ell} M_\ell, \tag{11.1}$$

where M_ℓ is the contribution due to diagrams with ℓ loops. As is well known, graphs with n external lines and ℓ loops correspond to maximal degenerations of the n-punctured, genus ℓ Riemann surface, $\Sigma_{g,n}$. In physics, this is the closed string picture for gravity, and gravitational string amplitudes are associated to integrals over $\overline{\mathcal{M}}_{g,n}$. Each maximal degeneration of $\Sigma_{g,n}$ corresponds to a choice of $n-2+2\ell$ homotopically distinct closed arcs on the surface which define a pair of pants decomposition of $\Sigma_{g,n}$.

Fix a pair of pants decomposition of $\Sigma_{g,n}$ associated to some Feynman diagram D, and consider the union Σ^* of two pairs of pants that are joined at just one hole. There are four closed cycles $I_1, I_2, I_3, I_4 \in H_1(\Sigma_{g,n} - \mathbb{P})$ (where $\mathbb{P}$ is the set of marked points) associated to the open trouser holes of Σ^*. There are also three cycles I_{12}, I_{23}, I_{31} that separate two of the trouser holes from the other two. These cycles are naturally associated to momenta. Indeed, for a certain choice of orientations,

$$I_{12} = I_1 + I_2, \quad I_{23} = I_2 + I_3, \quad I_{31} = I_3 + I_1, \quad I_4 = I_1 + I_2 + I_3, \tag{11.2}$$

in homology, which are identical to the momentum conservation or 'Kirchhoff' relations for momenta associated to cubic graphs D. So, following the same reasoning as in Chapter 8, an assignment of momenta to the (oriented) edges of a given Feynman diagram D defines a homomorphism,

$$\mu : H_1(\Sigma_{g,n} - \mathbb{P}) \to \mathcal{K}, \tag{11.3}$$

for some $\mathbb{R}$ vector space of momenta, $\mathcal{K}$. Any closed arc Δ can then be associated to the Mandelstam variable $s_\Delta = (\mu(\Delta))^2$. In the case of the two pairs of pants, Σ^*, the cycles described above satisfy (11.2), which in turn implies the identity (the 'lantern relation'), equation (3.36),

$$s_{12} + s_{23} + s_{31} - s_1 - s_2 - s_3 + s_4 = 0, \tag{11.4}$$

for the associated Mandelstam variables.

In the examples of ABHY polytopes for affine quivers, Chapter 10, there is a facet corresponding to the one closed arc, Δ: in the case of the annulus, Δ is the arc that separates the two boundary components. More generally, an ABHY polytope will have many such Δ facets. For example, consider the ℓ-loop vacuum gauge theory partial amplitude at genus g, $A(\Sigma_{\ell,0,g})$, which is associated to the genus g surface $\Sigma_{p,h,g}$ with $p = \ell$ punctures, $h = 0$ boundaries. Up to homotopy, there are (except for $p = 3$, $g = 0$) infintely many homotopy classes of Δ arcs on $\Sigma_{p,h,g}$. Every pair of pants decomposition of $\Sigma_{p,h,g}$ can be viewed (as above) as arising from some set of $\ell + 2g - 2$ non-intersecting Δ arcs, and so every g-loop cubic gravity diagram D arises as a certain facet of the ABHY polytope of $\Sigma_{p,h,g}$. This is an avatar of the fact that Teichmuller space (see below) has boundaries associated to both ordinary arcs and closed Δ arcs. But, as in string theory, this suggests that there is an intimate relation between gravity and gauge amplitudes at higher orders in the perturbation series.

The relation between gauge and gravity tree amplitudes is the quadratic KLT relation studied in Chapters 3 and 4. It is natural to ask whether a quadratic relation of this kind exists at higher orders. The loop order is the only topological invariant that appears in both the gravity and the gauge theory perturbation series. In fact, all diagrams D that contribute to a given gauge theory partial amplitude $A(\Sigma)$ have the same loop number, $\ell = p + h + 2g - 2 = -\chi(\Sigma)$. So one might speculate that

$$M_\ell(1, ..., n) = \sum_{\Sigma_1, \Sigma_2} A(\Sigma_1) S(\Sigma_1, \Sigma_2) A(\Sigma_2)? \tag{11.5}$$

The sum in (11.5) is over all surfaces Σ_1, Σ_2 with n boundary marked points, and constrained by $\chi(\Sigma) = -\ell$. This speculation is further motivated by the picture of glueing ribbon graphs to each other to obtain a closed surface: one obtains a genus ℓ surface from such a choice of glueing if both ribbon graphs have loop number ℓ. But there are many possible ways to glue pairs of such ribbon graphs together, and it is not obvious that any relation exists of the form (11.5).

The main focus of Chapter 4 was not the KLT relation, per se, but a slightly 'off shell' extension of it obtained by taking one leg, say n, to be off shell. This partially off-shell KLT relation is better behaved than its on-shell limit. Chapter 4 derived the off-shell KLT relation simply by finding an inverse to a linear map. The on-shell KLT relation cannot be derived in this simple fashion directly, even though it can be obtained as a limit of the off-shell KLT relation. Given the importance of taking n to be off shell in Chapter 4, this might be similarly important for formulating a higher order KLT relation. To make this point more explicit, suppose particle n is chosen to be off-shell, then it is natural to regard the ribbon graphs G as 'rooted' at n. As done in Chapter 8, such a choice of root can be used to fix the sign of the colour factor of G, c_G, which can in turn be expanded as a sum (with signs) of colour factors $C(\Sigma)$. One can therefore regard the linear space of *rooted* ribbon graphs as a subspace in the linear space spanned by marked surfaces Σ, in analogy to the embedding of Lie(A) into the free associative algebra (Chapter 3). In this way, a choice of root also makes it possible to consistently define signs (G, Σ), which reverse if any of the vertices of G is 'flipped'. These signs then define a pairing which is the generalization to higher orders of the pairing between $L(A)$ and $L(A)^\vee$ introduced in Chapter 3.

Chapters 5 and 6 showed that there are 'kinematic BCJ numerators' N_α for the NLSM tree amplitudes, and as reviewed in Chapter 2, BCJ numerators are thought to exist for several gauge theories, including Yang-Mills. There have been attempts to find similar such numerators for gauge theory partial amplitudes at higher orders in the perturbation series, with an emphasis on the Jacobi identity. But it is no longer the case, beyond tree level, that the colour factors c_G can be treated as Lie polynomials. The importance of choosing the root when defining the sign of c_G suggests that taking n off-shell might be important also for finding higher order numerators N_α.

The derivations in Chapters 5 and 6 made heavy use of the scattering equations, and CHY formulas. The scattering equations can be regarded as imposing the vanishing of $\mathrm{d}\log f_s$, where $f_s = \prod (z_{ij})^{-\alpha' s_{ij}}$. But the scattering equations can alternatively be regarded as inducing map from the positive real part of $\mathcal{M}_{a,n}$ to the ABHY associahedron. Here, $\mathcal{M}_{a,n}$ is the affine 'dihedral variety' for the ordering a (Appendix 5.A), and it can be described by the non-crossing equations,

$$u_I + \prod_{J \text{ crosses } I} u_J = 1, \tag{11.6}$$

for diagonals of the n-gon, I. $\mathcal{M}_{a,n}$ is a positive variety in the sense of Fock-Goncharov, [34] which means that it has a positive part, $\mathcal{M}_{a,n}(\mathbb{R}_+)$ whose points are the solutions to 11.6 with all u_I real and positive. The positive part has a natural compactification by allowing $u_I \geq 0$, and the boundaries of $\mathcal{M}_{a,n}(\mathbb{R}_\geq)$ have the same lattice structure as the face lattice of the associahedron. One might say that $\mathcal{M}_{a,n}(\mathbb{R}_+)$ is the interior of a 'curvy associahedron', and that the scattering equations provide a diffeomorphism from the curvy associahedron to the ABHY associahedron, much as the moment map is a diffeomorphism from the positive part of a toric variety, to its moment polytope. This aspect of the ABHY associahedron should generalize to all ABHY polytopes.

For a marked surface, Σ, the associated positive variety is the Fock-Goncharhov X-type Teichmuller space $T^x(\Sigma)$. [35] Recall that the classical Fuchsian approach to hyperbolic surfaces is to present the surface as the quotient of the upper half plane $\mathbb{H}$ by a finitely generated Fuchsian subgroup $\Gamma \leq PSL_2\mathbb{R}$, with Γ fixed up to conjugation. Choosing generators of Γ, one obtains an isomorphism $\pi_1(\Sigma) \simeq \Gamma$, or, equivalently, a $PSL_2\mathbb{R}$ local system on Σ. $T^x(\Sigma)$ is the moduli space of such local systems on the surface $\Sigma_\circ$ obtained by opening up punctures to boundary components. Note that the hyperbolic geodesic length of these new boundary components in $\Sigma_\circ$ is not fixed. So, for example, the T^x space of the punctured torus has dimension 3, whereas the classical Teichmuller space of the punctured torus (which is $\mathbb{H}$) has dimension 2.

A generalization of the scattering equations ought to provide a *family* of maps from $T^x(\Sigma)(\mathbb{R}_+)$ into different realizations of the ABHY polytope associated to Σ. The homological algebra approach of Chapter 7 was introduced in order to solve the problem of finding all realizations of the ABHY polytope (i.e. of finding the constraints on the constants c_i such that the half-spaces $f_i \geq c_i$ intersect correctly). Generalizing the scattering equations to the X-type Teichmuller spaces might lead to a purely geometric solution to this same problem.

Appendix A

Berends-Giele recursion and Lie polynomials

The conventional computation of $A_{\mathrm{YM}}(1,...,n)$ gives the tree amplitude as a sum of Feynman tree graph contributions. The purpose of this appendix is to write $A_{\mathrm{YM}}(1,...,n)$ in the form

$$A_{\mathrm{YM}}(1,...,n) = \sum_{\alpha} \frac{n_\alpha c_\alpha}{s_\alpha}, \tag{A.1}$$

where the numerators n_α do not have any poles in the Mandelstam variables s_I. See Chapter 2 for review and motivation. This appendix constructs the numerators n_α using a recursion which is 'dual' to the Berends-Giele recursion relation. This idea is implicit in a lot of literature on BCJ numerators, especially in the string theory literature, but is not carried out explicitly. Because of its importance to the main discussion, this appendix gives the construction of n_α in full detail, and only using conventional Feynman rules.

Fix n, the number of particles, and label the particles $1,...,n$, with momenta k_i^μ, colour polarisations $t_i \in \mathrm{End}(\mathfrak{su}_N)$, and polarisations ϵ_i^μ. By a property of tree graphs, the edges of a Feynman tree graph G can be oriented in a unique way, such that all edges flow 'towards' n. Adopting this orientation, each vertex in G has just one outgoing edge. With this convention for the edge orientations, momentum conservation reads

$$\sum_{i=1}^{n-1} k_i^\mu = k_n^\mu. \tag{A.2}$$

The interaction terms in the YM Lagrangian are

$$g\,\mathrm{tr}\left(\partial_\mu A_\nu [A^\mu, A^\nu]\right), \tag{A.3}$$

and

$$g^2\,\mathrm{tr}\left([A_\mu, A_\nu][A^\mu, A^\nu]\right). \tag{A.4}$$

The cubic interaction, (A.3), associates to a cubic vertex in a Feynman graph the following contribution,

$$V(1,2)^\mu := igf^{a_1a_2a_3}\left(k_2^\mu g_{\mu_1\mu_2} + (2k_1+k_2)_{\mu_2} g^\mu_{\mu_1} - (12)\right), \tag{A.5}$$

where the colour indices a_i, the Lorentz indices μ_i, and the momenta are associated to the vertex as shown in figure A.1. Vertex 3 is the outgoing edge, and vertex 1 and 2 are the incoming edges. The quartic interaction, (A.4), gives the contribution

$$V(1,2,3)^\mu := ig^2 f^{a_1a_2c} f^{ca_3a_4}\left(g_{\mu_1\mu_3} g_{\mu_2}{}^\mu - g_{\mu_1}{}^\mu g_{\mu_2\mu_3}\right) + (123), \tag{A.6}$$

where the labels are shown in Figure A.1. The factor $f^{a_1a_2c}f^{ca_3a_4}$ is the same factor that occurs in the 3-vertex for the leftmost cubic Feynman diagram in Figure A.2. As remarked in Section 2.1, this observation implies that the sum of Feynman diagram contributions, A_G, for Feynman diagrams G, may be rewritten as a sum over Lie monomials $\alpha \in L_{n-1}$,

$$\sum_G A_G = \sum_\alpha A_\alpha c_\alpha, \tag{A.7}$$

with colour factors

$$c_\alpha := ig^{n-2}\mathrm{tr}(\alpha[t^{a_1}, ..., t^{a_{n-1}}], t^{a_n}). \tag{A.8}$$

The next two sections compute the coefficients A_α in full detail.

Remark A.1. (i's and ϵ's.) *The convention here is that* $f^{abc} := 2\mathrm{tr}([t^a, t^b]t^c)$. *The cubic vertex has a factor of* ig*, whereas the quartic vertex has a factor of* ig^2*, which is not equal to* $(ig)^2$*. The reason for this is that the gluon propagator is*

$$-i\delta^{ab}\frac{1}{p^2}\left(g_{\mu\nu} + \frac{ap_\mu p_\nu}{p^2}\right), \tag{A.9}$$

where the $i\epsilon$*'s in the denominators are suppressed. By convention, all factors of* $-i$ *appearing in the propagators have been included in the definition of* c_α*,* (A.8)*. The*

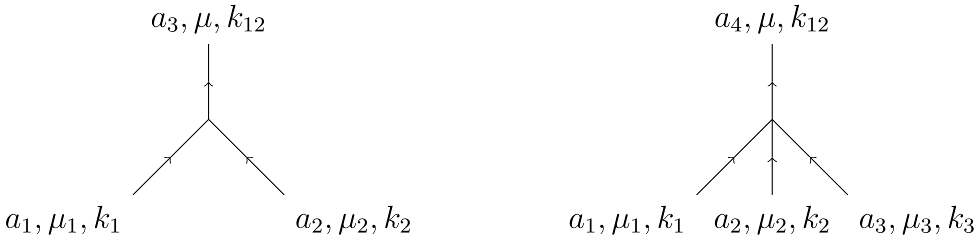


Figure A.1: Yang-Mills vertex labels.

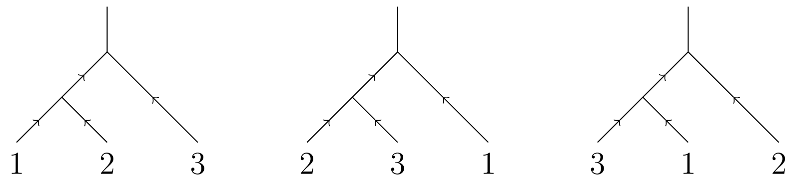


Figure A.2: Cubic diagrams.

overall factor of i *is given by* $(-i)^{n-3}(ig)^{n-2} = ig^{n-2}$.

A.1 The Berends-Giele bracket

Fix a cubic Feynman graph, G. The amplitude contribution of G, A_G, has a factor of c_α, for a Lie monomial $\alpha \in L_{n-1}$. Moreover, the product of propagators of G gives a factor of

$$\frac{1}{s_\alpha}, \tag{A.10}$$

where the Feynman $i\epsilon$'s are suppressed. The full contribution may then be written as

$$A_G = \frac{c_\alpha l_\alpha}{s_\alpha}, \tag{A.11}$$

where l_α contains all remaining factors from the cubic vertices, (A.5), and the gluon propagators, (A.9). l_α can be computed iteratively in the following way. Berends-Giele [7] define the following bracket operation,

$$[\epsilon_1, \epsilon_2]^\mu := k_1^\mu \epsilon_1 \cdot \epsilon_2 + 2\epsilon_2^\mu \epsilon_1 \cdot k_2 + \epsilon_2^\mu \epsilon_1 \cdot k_1 - (12), \tag{A.12}$$

so that the cubic vertex contribution may be written

$$V(1,2)^\mu = igf^{a_1 a_2 a_3}[\epsilon_1, \epsilon_2]^\mu. \tag{A.13}$$

To include the factors from the gluon propagators, (A.9), define also

$$[\epsilon_1, \epsilon_2]_a^\mu := [\epsilon_1, \epsilon_2]^\nu \left(g^{\mu\nu} + \frac{a k_{12}^\mu k_{12}^\nu}{k_{12}^2} \right). \tag{A.14}$$

If a is chosen as $a = -1$ ('Feynman gauge'), this new bracket operation preserves Lorenz gauge, in the sense that

$$k_{12} \cdot [\epsilon_1, \epsilon_2] = 0. \tag{A.15}$$

If instead $a = 0$, then

$$k_{12} \cdot [\epsilon_1, \epsilon_2] = (s_2 - s_1)\epsilon_1 \cdot \epsilon_2. \tag{A.16}$$

For any choice of a, the factor l_α can now be expressed as

$$l_\alpha := \alpha[\epsilon_1, ..., \epsilon_{n-1}] \cdot \epsilon_n, \tag{A.17}$$

where $\alpha[\epsilon_1, ..., \epsilon_{n-1}]$ is the nested bracketing of $\epsilon_1, ..., \epsilon_{n-1}$, using the bracket (A.14), induced by the Lie monomial $\alpha \in L_{n-1}$. (The dot in (A.17) is an abbreviated notation for contraction by $g_{\mu\nu}$.)

The BG bracket is not a Lie bracket because it does not satisfy the Jacobi relation. But its failure to satisfy the Jacobi relation can be described in the following way. Define the *dual BG 3-bracket*,

$$[\epsilon_1, \epsilon_2; \epsilon_3]^\mu := \left(\epsilon_2^\mu\, \epsilon_1 \cdot \epsilon_3 - \epsilon_1^\mu\, \epsilon_2 \cdot \epsilon_3\right), \tag{A.18}$$

which is an operation antisymmetric in 1 and 2. The quartic vertex contribution, (A.6), may then be written

$$V(1,2,3)^\mu := ig^2 f^{a_1 a_2 c} f^{c a_3 a_4}\, [\epsilon_1, \epsilon_2; \epsilon_3]^\mu + (123). \tag{A.19}$$

In terms of (A.18), the *BG 3-bracket* defined in [1988BG] is

$$[\epsilon_1, \epsilon_2, \epsilon_3]^\mu := [\epsilon_1, \epsilon_2; \epsilon_3]^\mu - [\epsilon_3, \epsilon_1; \epsilon_2]^\mu. \tag{A.20}$$

The failure for $[\ ,\ ]^\mu$ to be Lie is then described by the following identity.

Lemma A.2. *For $\epsilon_1, \epsilon_2, \epsilon_3$ in Lorenz gauge, the $a = 0$ gauge BG bracket satisfies*

$$[[\epsilon_1, \epsilon_2], \epsilon_3]^\mu + (123) = s_{12}[\epsilon_1, \epsilon_2; \epsilon_3]^\mu + (123), \tag{A.21}$$

after projecting out a term proportional to k_{123}^μ.

Proof. Write

$$[[\epsilon_1, \epsilon_2], \epsilon_3]^\mu + (123) = A^\mu_{[[1,2],3]} + B^\mu_{[[1,2],3]} + C^\mu_{[[1,2],3]}, \tag{A.22}$$

where A contains all terms of the form $(\epsilon \cdot \epsilon)\epsilon^\mu$, B contains terms of the form $(\epsilon \cdot k)(\epsilon \cdot k)\epsilon^\mu$, and C contains terms of the form $(\epsilon \cdot \epsilon)(\epsilon \cdot k)k^\mu$. Then

$$A_{[[1,2],3]} = \epsilon_3^\mu \epsilon_1 \cdot \epsilon_2 (s_{12} - s_{23}) + (123), \tag{A.23}$$

$$B_{[[1,2],3]} = \epsilon_3^\mu \left[2\epsilon_1 \cdot k_1 \epsilon_2 \cdot k_3 - \epsilon_2 \cdot k_2 \epsilon_1 \cdot (k_2 + 2k_3)\right] + (123), \tag{A.24}$$

$$C_{[[1,2],3]} = k_{123}^\mu \left[\epsilon_1 \cdot \epsilon_2 (2k_{12} + k_3) \cdot \epsilon_3 + (123)\right]. \tag{A.25}$$

C^μ is proportional to k_{123}^μ, and so vanishes after the Lorenz gauge projection. B^μ vanishes by the Lorenz condition, $k_i \cdot \epsilon_i = 0$. □

Remark A.3. *The equivalent computation for other gauge choices ($a \neq 0$) can be found by adding to* (A.21) *the contribution of the non-identity part of the projection, i.e.*

$$\frac{a\, k_{12} \cdot [\epsilon_1, \epsilon_2]}{s_{12}} [k_{12}, \epsilon_3]^\mu.$$

A.2 Numerators

This section computes the coefficients, A_α, appearing in the following expansion of the full tree amplitude:

$$A_{\mathrm{YM}}(1, ..., n) = \sum_\alpha A_\alpha c_\alpha.$$

A_α receives contributions from all Feynman graphs G for which c_α appears in a summand of A_G.

An arbitrary YM Feynman tree graph G contains quartic and cubic vertices. For such a tree, G, with k quartic vertices, the contribution to the amplitude, A_G, can be written as a sum of 3^k terms, each corresponding to a distinct binary tree, α. These binary trees are obtained from G by replacing each quartic vertex in G by one of the 3 possible cubic subgraphs in Figure A.2.

Conversely, fixing a binary tree α, with propagator set $P(\alpha)$, graphs with quartic vertices may be obtained by contracting edges $I \in P(\alpha)$. Call a set of edges $S \subset P(\alpha)$ a *separated set of edges* if no two edges, $I, J \in S$, meet at a vertex. Write $\mathrm{Sep}(\alpha) \subset \mathcal{P}P(\alpha)$ for the set of separated sets of edges of α. For example,

$$\mathrm{Sep}([[1, 2], 3]) = \{\emptyset, \{12\}\}. \tag{A.26}$$

The next Lemma gives an algorithm for computing $\mathrm{Sep}(\alpha)$. For two $P_1, P_2 \subset \mathcal{P}\mathbb{N}$, let

$$P_1 \cdot P_2 := \{Y_1 \cup Y_2 | Y_1 \in P_1, Y_2 \in P_2\} \subset \mathcal{P}\mathbb{N} \tag{A.27}$$

denote the set of pairwise unions of sets in P_1 and P_2.

Lemma A.4. (Enumerating separated sets of edges.) *Let* $\alpha = [\alpha_l, \alpha_r] \in L(A)$*, and* $\beta = [\beta_l, \beta_r] \in L(B)$*, for disjoint* $A, B \subset \mathbb{N}$*. Write*

$$\overline{\mathrm{Sep}}(\alpha) := \mathrm{Sep}(\alpha_l) \cdot \mathrm{Sep}(\alpha_r) \cdot \{A\}. \tag{A.28}$$

Then $\mathrm{Sep}([\alpha, \beta])$ *is the disjoint union*

$$(\mathrm{Sep}(\alpha) \cdot \mathrm{Sep}(\beta)) \sqcup (\overline{\mathrm{Sep}}(\alpha) \cdot \mathrm{Sep}(\beta)) \sqcup (\mathrm{Sep}(\alpha) \cdot \overline{\mathrm{Sep}}(\beta)). \tag{A.29}$$

Proof. Fix any $P \in \mathrm{Sep}(\alpha)$. Two propagators, $I, J \in P(\alpha)$, meet at a vertex iff their symmetric difference is a propagator, $I \triangle J \in P(\alpha)$ (see Section 3.3). It follows that either: (i) $A \in P$; or (ii) $B \in P$; or (iii) $A \notin P$ and $B \notin P$. Moreover, if $A \in P$, then $A_l \notin P$ and $A_r \notin P$. Likewise for $B \in P$. □

Remark A.5. (Counting separated sets of edges.) *For fixed n, the size of* $\mathrm{Sep}(\alpha)$ *depends on the choice of Lie monomial $\alpha \in L_n$. For a comb graph, $\ell(12...n) \in L_n$, Lemma A.4 implies that the size of* $\mathrm{Sep}(\ell(12...n))$ *satisfies,*

$$|\mathrm{Sep}(\ell(12...n))| = |\mathrm{Sep}(\ell(12...n-1))| + |\mathrm{Sep}(\ell(12...n-2))|, \tag{A.30}$$

so that $|\mathrm{Sep}(\ell(1...n))|$ *is the n^{th} Fibonacci number, and this is a lower bound on* $|\mathrm{Sep}(\alpha)|$.

It is now possible to return to the computation of the coefficients A_α in (A.7). For $I \in P(\alpha)$, let the associated sub-tree of α be

$$\alpha_I = [[\alpha_{I_1}, \alpha_{I_2}], \alpha_{I_3}],$$

where $I_1, I_2, I_3 \in P(\alpha)$ and $I_1 \cup I_2 \cup I_3 = I$. Then let $L(\alpha, \{I\})$ be the result obtained by replacing the

$$[[\epsilon_{I_1}, \epsilon_{I_2}], \epsilon_{I_3}]^\mu \tag{A.31}$$

that appears in $L(\alpha)$, with instead

$$s_{I_1 I_2}[\epsilon_{I_1}, \epsilon_{I_2}; \epsilon_{I_3}]^\mu. \tag{A.32}$$

This is the contribution of the non-colour part of the quartic vertex, (A.6). Likewise, for $P \subset P(\alpha)$ a separated set of edges, let $L(\alpha, P)$ be the result of substituting the 4-vertex factor, (A.32), into $L(\alpha)$, for every $I \in P$.

With these definitions, the full YM tree amplitude can be written

$$A_{\mathrm{YM}}(1, ..., n) = \sum_G A_G = \sum_\alpha \frac{c_\alpha N_\alpha}{s_\alpha}, \tag{A.33}$$

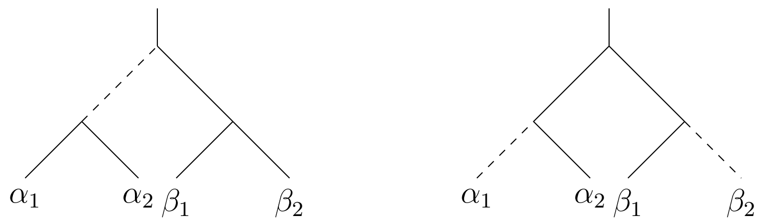


Figure A.3: Examples of separated sets of edges.

where

$$N_\alpha := \sum_{P \in \mathrm{Sep}(\alpha)} L(\alpha, P). \tag{A.34}$$

The next section shows that N_α can be computed using an iterative recursion, dual to the Berends-Giele recursion.

A.3 A perturbinor-like 'dual BG' recursion

Lemma A.4 suggests an iterative way to compute the numerators, N_α. The interest of this recursive relation is that it leads to proofs of the BCJ conjectures. The recursive relation in this section is the 'dual' of the conventional Berends-Giele relation, as explained in Section A.4, below.

Recall the BG bracket, $[\ ,\ ]^\mu_a$, defined in (A.14), on vectors ϵ_i associated to momenta k_i in momentum space.

Definition A.6. *The dual BG bracket is an extension of $[\ ,\ ]^\mu$, given in momentum space by:*

$$[J_1, J_2] = \left[\begin{pmatrix} M_1 \\ L_1 \end{pmatrix}, \begin{pmatrix} M_2 \\ L_2 \end{pmatrix} \right] := \begin{pmatrix} [M_1, M_2]^\mu - L_1^{\mu\nu} M_2^\nu + L_2^{\mu\nu} M_1^\nu \\ s_{12}\,(M_1^\mu M_2^\nu - M_1^\nu M_2^\mu) \end{pmatrix}, \tag{A.35}$$

where the bracket acts on pairs in $V \oplus \bigwedge^2 V$ of vectors $M^\mu \in V$ and anti-symmetric tensors $L^{\mu\nu} \in \bigwedge^2 V$:

$$J := \begin{pmatrix} M^\mu \\ L^{\mu\nu} \end{pmatrix}. \tag{A.36}$$

Definition A.7. (Dual BG currents). *Given external particle data* (k_i, ϵ_i) *for all* $i \in A$*, with* $k_i \cdot \epsilon_i = 0$*, write*

$$J_i := \begin{pmatrix} \epsilon_i^\mu \\ \\ 0 \end{pmatrix}. \tag{A.37}$$

For a Lie monomial $[\alpha, \beta] \in L(A)$*, the associated dual BG current is given by*

$$J_{[\alpha,\beta]} = [J_\alpha, J_\beta], \tag{A.38}$$

and this inductively defines a current J_α *for all monomials* $\alpha \in L(A)$*. A current* J_β *with* $\beta \in L^+(B)$ *is associated to the momentum* $k_B^\mu = \sum_{i\in B} k_i^\mu$*.*

Lemma A.8. (Numerators from dual BG.) *The numerators* N_α*, computed from Feynman rules in* (A.34)*, are related to the dual BG current* J_α *by:*

$$N_\alpha = M_\alpha^\mu \, \epsilon_n^\mu, \tag{A.39}$$

where $J_\alpha = (M_\alpha^\mu, L_\alpha^{\mu\nu})$*.*

Proof. Let $\alpha = [\alpha_l, \alpha_r] \in L(A)$, and $\beta = [\beta_l, \beta_r] \in L(B)$, for disjoint $A, B \subset \mathbb{N}$. By Lemma A.4,

$$N_{[\alpha,\beta]} = \sum_{P \in \mathrm{Sep}(\alpha)\mathrm{Sep}(\beta)} l_P + \sum_{P \in \overline{\mathrm{Sep}}(\alpha)\mathrm{Sep}(\beta)} l_P + \sum_{P \in \mathrm{Sep}(\alpha)\overline{\mathrm{Sep}}(\beta)} l_P. \tag{A.40}$$

Likewise, $J^\mu_{[\alpha,\beta]}$ is the sum of three terms,

$$J^\mu_{[\alpha,\beta]} = [M_\alpha, M_\beta]^\mu - s_A[M_{\alpha_l}, M_{\alpha_r}; M_\beta]^\mu + s_B\,[M_{\beta_l}, M_{\beta_r}; M_\alpha]^\mu. \tag{A.41}$$

These terms correspond, respectively, to Feynman graph contributions in which the top vertex is: (i) cubic, (ii) quartic, by contracting propagator A, and (iii) quartic, by contracting propagator B. The Lemma then follows by induction. □

Remark A.9. *For a tree,* $[[\alpha_1, \alpha_2], \alpha_3]$*, consider the Jacobi sum,*

$$J_{Jacobi} := J^\mu_{[[\alpha_1,\alpha_2],\alpha_3]} + (cyclic\ 123), \tag{A.42}$$

for dual BG currents J_{α_1}, J_{α_2}, J_{α_3}. Write $J_{\alpha_i} = (M_i, L_i)$. By Lemma A.2, the following sum vanishes identically:

$$[[M_1, M_2], M_3]^\mu + s_{12}\,[M_1, M_2; M_3]^\mu + (\mathit{cyclic}\ 123).$$

It follows that

$$J_{Jacobi} = [[L_3; M_1], M_2] - [[L_3; M_2], M_1] - [L_3; [M_1, M_2]] \\ + [L_3; [L_2; M_1]] - [L_3; [L_1; M_2]] + (123).$$

When the $L_i^{\mu\nu} = 0$, $J^\mu_{Jacobi} = 0$. This is a restatement of the observation made in [30] about the 4-point gluon amplitude. When one or more of the $L_i^{\mu\nu}$ is nonzero, the sum does not vanish.

A.4 Relation to Berends-Giele recursion

For a word $a \in \mathfrak{S}(A)$, define

$$J(a)^\mu := \sum_\alpha \frac{(a,\alpha) J^\mu_\alpha}{\tilde{s}_\alpha}, \tag{A.43}$$

where the sum is over Lie monomials in $L^+(A)$. For fixed α, write $\alpha = [[\beta,\beta'],[\gamma,\gamma']]$, so that

$$J^\mu_\alpha = [J_{[\beta,\beta']}, J_{[\gamma,\gamma']}]^\mu + s_{\beta\beta'} J^{[\mu}_\beta J^{\nu]}_{\beta'} J^\nu_\gamma - s_{\gamma\gamma'} J^\nu_\beta J^{[\mu}_\gamma J^{\nu]}_{\gamma'}. \tag{A.44}$$

The deconcatenation identity,

$$(a, [\alpha,\beta]) = \sum_{a=bc} (b,\alpha)(c,\beta) - (bc), \tag{A.45}$$

implies that (A.43) can be expanded as

$$J(a)^\mu := \frac{1}{s_a} \sum_{a=bc} [J(b), J(c)]^\mu + \frac{1}{s_a} \sum_{a=bcd} [J(b), J(c), J(d)]^\mu, \tag{A.46}$$

where

$$[J(b), J(c), J(d)]^{\mu} := [J(b) \cdot J(d) J(c)^{\mu} - (c \leftrightarrow d)] + (b \leftrightarrow d). \tag{A.47}$$

Equation (A.46) is the conventional BG recursion relation. Since $J(i) = J_i$, it follows that the $J(a)^{\mu}$ defined by (A.43) are the same as those in [7].

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