AUXILIARY GROUP APPROACH FOR GROUP-SUBGROUP RELATED TRANSFORMATION MATRICES

J.N. Kotzev, M.K. Peev, M.I. Aroyo†
Department of Solid State Physics, University of Sofia
BG-1126 Sofia, 5 Anton Ivanov Blvd; Bulgaria

R. Dirl, P. Kasperkovitz
Institut für Theoretische Physik, TU Wien
A-1040 Wien, Karlsplatz 13; Austria

The *auxiliary group approach* is extended to a systematic calculation of subducing matrices and to relate various isoscalar matrices.

1. THE AUXILIARY GROUP APPROACH: Symmetries of subducing matrices, especially of Clebsch-Gordan matrices $\mathbf{C}^{k_1,k_2}$, have been studied for a long time. For instance the permutational properties of the latter matrices were used to correlate $\mathbf{C}^{k_1,k_2}$ and $\mathbf{C}^{k_2,k_1}$; later complex conjugation [1], [2] and other operations [3], [4], [5] were similarly employed. In a series of papers [6], [7], [8], [9] we developed a systematic approach combining all these operations by closing them in what we call the *auxiliary group* $\tilde{\mathcal{Q}}$. The elements of this group are bijective mappings q of the set of all unitary matrix (co)representations $\mathbf{D}(g)$ of some fixed group $\mathcal{G}$.

$$q = \begin{cases} a \in \mathcal{ASS} & (a_j\mathbf{D})(g) = \mathbf{D}^j(g) \otimes \mathbf{D}(g) \quad (\dim \mathbf{D}^j = 1) \\ b \in \mathcal{AUT} & (b\mathbf{D})(g) = \mathbf{D}(\beta^{-1}(g)) \quad \beta \in Aut(\mathcal{G}) \\ c \in \mathcal{CON} & (c\mathbf{D})(g) = \mathbf{D}(g)^* \end{cases}$$

$$q = (a,b,c): \qquad (q\mathbf{D})(g) = \mathbf{D}^j(g) \otimes \mathbf{D}(\beta^{-1}(g))^* \tag{1}$$

It follows from these definitions that $\tilde{\mathcal{Q}}$ has the structure

$$\tilde{\mathcal{Q}} = \{q\} = \mathcal{ASS}(\times(\mathcal{AUT} \times \mathcal{CON}) \tag{2}$$

The action of q on a given (co)rep $\mathbf{D}(g)$ transforms it either into an equivalent one or an inequivalent one. In the first case we obtain a *symmetry* relation while the second case leads to a *generating* relation. It is natural to introduce a subgroup of $\tilde{Q}$ leaving $\mathbf{D}(g)$ invariant up to equivalence transformations. Here we are primarily interested in (co)irreps $\mathbf{D}^k(g)$ of $\mathcal{G}$; the corresponding subgroups are denoted by $\tilde{Q}^k$.

$$\tilde{\mathcal{Q}}^k = \{q \in \tilde{\mathcal{Q}} | q\mathbf{D}^k \sim \mathbf{D}^k\} \tag{3}$$

† present adress: Institut für Kristallographie, Universität Karlsruhe D 75 Karlsruhe, Kaiserstaße 12; BRD

For each $\tilde{\mathcal{Q}}^k$ the group $\tilde{\mathcal{Q}}$ is decomposed with respect to this subgroup.

$$\tilde{\mathcal{Q}} = \bigcup_{\ell} q_\ell^{(k)} \tilde{\mathcal{Q}}^k \ , \qquad \mathcal{R}^k = \{q_\ell^{(k)}\} = \text{fixed set of coset representatives} \tag{4}$$

These decompositions allow us to define $\tilde{\mathcal{Q}}$-classes $[k]$ ($\tilde{\mathcal{Q}}$ orbits) consisting of (co)irreps that are linked by the transformations $q \in \tilde{\mathcal{Q}}$.

$$[k] = \{\mathbf{D}^\ell | \mathbf{D}^\ell \sim q\mathbf{D}^k, \quad q \in \tilde{\mathcal{Q}}\} \tag{5}$$

Staring from a fixed $\mathbf{D}^k$ *standard* (co)irreps are generated by the convention

$$\mathbf{D}^\ell = q_\ell^{(k)} \mathbf{D}^k, \quad q_\ell^{(k)} \in \mathcal{R}^k. \tag{6}$$

For $q \in \tilde{\mathcal{Q}}^k$ one also has to construct the matrices $\mathbf{U}^k(q)$ occurring on the RHS of the equation

$$q\mathbf{D}^k(g) = \mathbf{U}^k(q)^\dagger \mathbf{D}^k(g) \mathbf{U}^k(q)^{(g)}. \tag{7}$$

Having in mind (6) and (7) it is then easy to find for any $q \in \tilde{\mathcal{Q}}$ matrices $\mathbf{U}^{\ell',\ell}(q)$ relating $\mathbf{D}^\ell$ and $\mathbf{D}^{\ell'}$.

$$q\mathbf{D}^\ell = \mathbf{U}^{\ell',\ell}(q)^\dagger \mathbf{D}^{\ell'} \mathbf{U}^{\ell',\ell}(q), \quad q \in \tilde{\mathcal{Q}} \tag{8}$$

$$\mathbf{U}^{\ell',\ell}(q) = q_{\ell'}^{(k)} \mathbf{U}^k(q'), \quad q' = q_{\ell'}^{(k)^{-1}} q q_\ell^{(k)} \tag{9}$$

The next step in our procedure is to transfer the action of the operations q from the the (co)irrep $\mathbf{D}^k(g)$ to the reducing matrices. For this purpose we use the definition of the latter (see eq. (11) below) and the matrices $\mathbf{U}^{\ell',\ell}(q)$. This results in generating relations of the following form.

$$\mathbf{S}^k \xrightarrow{\mathbf{U}^{\ell,k}} \mathbf{S}^\ell \qquad (\mathbf{S} = \text{unitary matrix}),$$

$$\mathbf{C}^{k_1,k_2} \xrightarrow{\mathbf{U}^{\ell_1,k_1} \otimes \mathbf{U}^{\ell_2,k_2}} \mathbf{C}^{\ell_1,\ell_2} \qquad (\mathbf{C} = \text{Clebsch-Gordan matrix})$$

These relations will be discussed in more detail for the special case where the reducing matrix is a subducing one.

2. THE AUXILIARY GROUP APPROACH FOR SUBDUCING MATRICES: If a group $\mathcal{G}_A$ is restricted to a subgroup $\mathcal{G}_B$ a (co)irrep $\mathbf{D}_A^k$ of $\mathcal{G}_A$ becomes a (co)rep of $\mathcal{G}_B$ which is general reducible.

$$\mathbf{D}_A^k \downarrow \mathcal{G}_B = \mathbf{D}_A^{k\downarrow} = \{\mathbf{D}^k(g) : \quad g \in \mathcal{G}_B\} \tag{10}$$

The matrices reducing these (co)reps are the so-called *subducing matrices.*

$$\mathbf{S}^{k\dagger}\mathbf{D}_A^{k\downarrow}(g)\mathbf{S}^{k(g)} = \oplus_s \mathbf{E}(k|s) \otimes \mathbf{D}_B^s(g) \qquad g \in \mathcal{G}_B \tag{11}$$

$$\begin{aligned} \mathbf{E}(d) &= \text{unit matrix of dimension } d \\ (k|s) &= \text{subduction multiplicity of } \mathbf{D}_B^s \text{ in } \mathbf{D}_A^{k\downarrow} \\ \mathbf{D}_B^s &= \text{(co)irrep of } \mathcal{G}_B \end{aligned}$$

The matrices $\mathbf{S}^k$ can be calculated straightforwardly from a set of linear equations from (11) by varying g over $\mathcal{G}_B$. However our goal is to reduce and to systematize this calculation and to obtain a set of correlated subducing matrices in one run. For this purpose we have to adapt our approach for the case where (co)irreps of two groups $\mathcal{G}_A$ and $\mathcal{G}_B \subset \mathcal{G}_A$ are involved. In general there exists no direct relation between the auxiliary groups $\widetilde{\mathcal{Q}}_A$ and $\widetilde{\mathcal{Q}}_B$ introduced before. However if we restrict $\widetilde{\mathcal{Q}}_A$ to a subgroup $\mathcal{Q}_A$ by excluding all automorphisms of $\mathcal{G}_A$ that do not leave $\mathcal{G}_B$ invariant, then there exists a homomorphism Φ from $\mathcal{Q}_A$ ($\subset \widetilde{\mathcal{Q}}_A$) to a subgroup $\mathcal{Q}_B$ ($\subset \widetilde{\mathcal{Q}}_B$).

$$\Phi(\mathcal{Q}_A) = \mathcal{Q}_B \tag{12}$$

This homomorphism also relates subgroups of $\mathcal{Q}_A$ to subgroups of $\mathcal{Q}_B$ that are need in our approach.

$$\begin{array}{llllllll} \mathcal{G}_A: & \widetilde{\mathcal{Q}}_A & \supset & \mathcal{Q}_A & \supset & \mathcal{Q}_A^k & \supset & \mathcal{Q}_A^{k,t} \\ \cup & \updownarrow ? & & \downarrow \Phi & & \downarrow \Phi & & \downarrow \Phi \\ \mathcal{G}_B: & \widetilde{\mathcal{Q}}_B & \supset & \mathcal{Q}_B & \supset & \mathcal{Q}_B^{k\downarrow} & \supset & \mathcal{Q}_B^{k\downarrow,t} \end{array} \tag{13}$$

Here $\mathcal{Q}_A^k$ is the "stabilizer" of $\mathbf{D}_A^k$ (cf. (3)). Proceeding as before we decompose $\mathcal{Q}_A$ with respect to $\mathcal{Q}_A^k$, fix a set of coset representatives $\mathcal{R}_A^k$, and define standard representatives $\mathbf{D}_A^\ell$ of the $\mathcal{Q}_A$-classes $[k]_A$. Moreover we determine the subgroup $\mathcal{Q}_B^{k\downarrow,t}$ of $\mathcal{Q}_B^{k\downarrow} = \Phi(\mathcal{Q}_A^k)$ leaving invariant some of the (co)irreps $\mathbf{D}_B^t$ contained in $\mathbf{D}_A^{k\downarrow}$, and construct $\mathcal{Q}_B^{k\downarrow}$-classes $[t]_B^{k\downarrow}$ analogously. Employing the matrices $\mathbf{U}_A^{\ell',\ell}(q_A)$ and $\mathbf{U}_B^{t',t}(q_B)$ defined in complete analogy to (8), we arrive at the following general result.

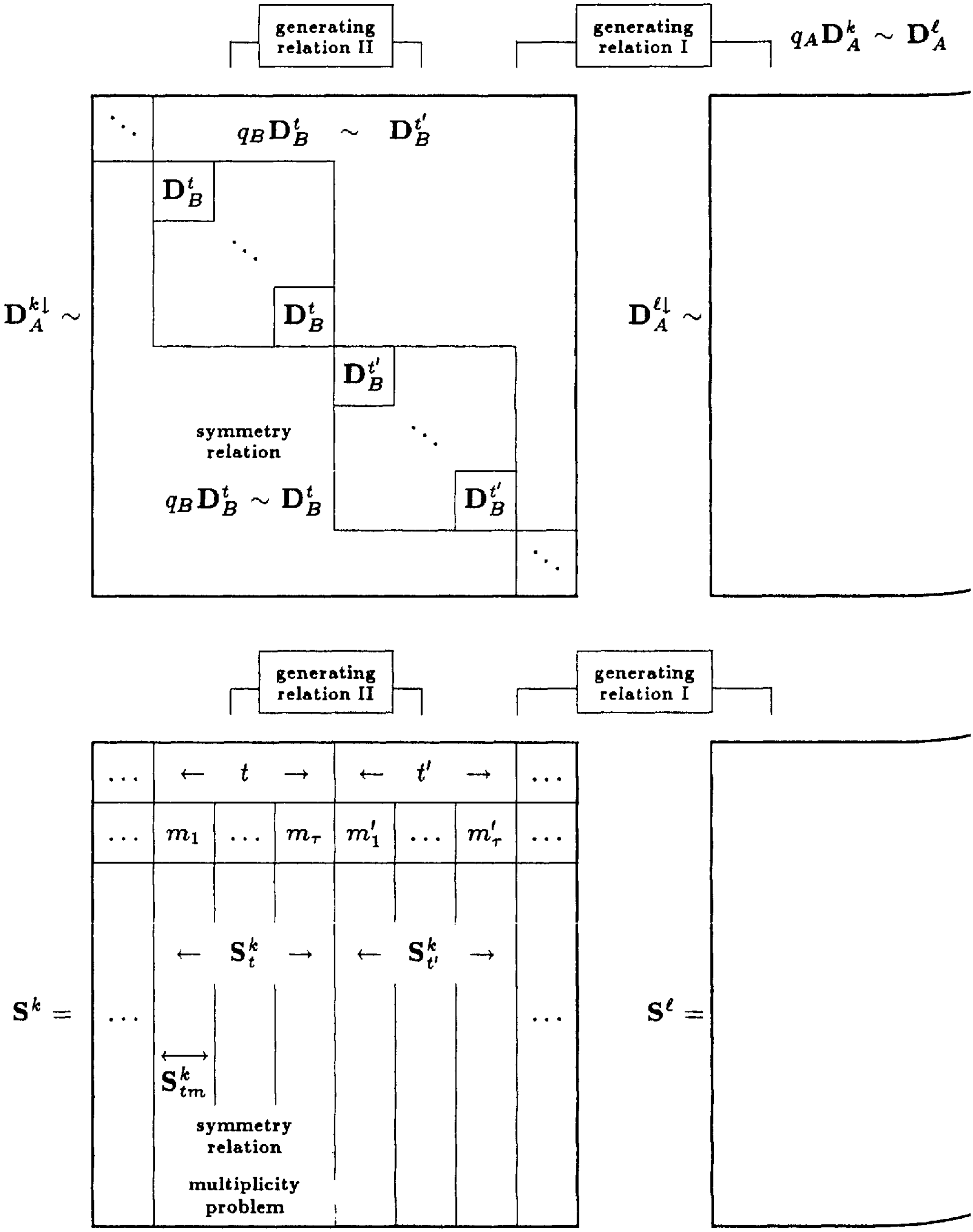
$$\mathbf{S}^{\ell'}\mathbf{M}_B^{\ell'}(q_B) = \mathbf{U}_A^{\ell',\ell}(q_A)(q_B\mathbf{S}^\ell)\mathbf{Z}_B^\ell(q_B)^\dagger \tag{14}$$

$$q_B = \Phi(q_A)$$

$$q_B\mathbf{S}^\ell = \begin{cases} \mathbf{S}^{\ell *} & \text{if } q_B \text{ contains } c \\ \mathbf{S}^\ell & \text{otherwise} \end{cases} \tag{15}$$

$$\mathbf{Z}_B^\ell(q_B) = \oplus_t \mathbf{E}(l|t) \otimes \mathbf{U}_B^{t',t}(q_B) \tag{16}$$

In (14) $\mathbf{M}_B^{\ell'}$ is a matrix commuting with the reduced form of $\mathbf{D}_A^{k\downarrow}$. Now a careful selection of q_A and the corresponding matrices $\mathbf{M}_B^{\ell'}(q_B)$ allows us to reduce the calculations and to establish generating relations for the subducing matrices. The various steps of our approach are visualized in the following figure.



The first kind of generating relations allows one to obtain the matrices $\mathbf{S}^\ell$, $\ell \in [k]_A$, from a single matrix $\mathbf{S}^k$. Setting $q_A \in \mathcal{R}_A^k$, $\mathbf{M}_B^{\ell,k}(q_A) = \mathbf{E}(n_\ell)$ in (14) we get

$$\mathbf{S}^\ell = (q_B \mathbf{S}^k)\mathbf{Z}_B^k(q_B)\,, \quad q_B = \Phi(q_A). \tag{17}$$

The second kind of generating relations shows that we need not have to calculate the whole matrix $\mathbf{S}^k$ but only some of its subblocks $\mathbf{S}_t^k$ because all $\mathbf{S}_{t'}^k$, $t' \in [t]_B^{k\downarrow}$ are related by suitable transformations. We take $q_A = \Phi^{-1}(q_B)$, $q_B \in \mathcal{R}_B^{k\downarrow}$, and split (14) into rectangular blocks $\mathbf{S}_t^k$ (see the Figure). Choosing the submatrix $\mathbf{M}_B^{k,t'}(q_B)$ which is a constituent of $\mathbf{M}_B^k(q_A)$ as unit matrix we arrive at the equation

$$\mathbf{S}_{t'}^k = \mathbf{U}_A^k(q_A)(q_B \mathbf{S}_t^k)\left[\mathbf{E}(k|s) \otimes \mathbf{U}_B^{t',t}(q_B)\right]^\dagger. \tag{18}$$

Apart from these generating relations we can also tackle the multiplicity problem, i.e. reduce the inherent ambiguity of the subblocks $\mathbf{S}_{t,m}^k$ (see the Figure). For this purpose we set $q_A \in \mathcal{Q}_A^{k,t}$ and split (18) into the above mentioned subblocks. The RHS of this equation may then be viewed as an (anti-)linear operator acting on the blocks $\mathbf{S}_{t,m}^k$.

$$T(q_A)\mathbf{S}_{t,m}^k = \mathbf{U}_A^k(q_A)(q_A \mathbf{S}_{t,m}^k)\mathbf{U}_B^t(q_B)^\dagger \tag{19}$$

$$T(q_A)\mathbf{S}_{t.m}^k = \sum_{m'} \mathbf{S}_{t,m'}^k \mathbf{L}^t(q_A)_{m',m} \tag{20}$$

Here $\mathbf{L}^t(q_A)$ is a matrix commuting with the (co)irrep $\mathbf{D}_B^t$. Thus it is obvious that a group of (anti)linear operators $T(q_A)$ is defined in the vector space spanned by the blocks $\mathbf{S}_{t,m}^k$ and that the matrices $\mathbf{L}^t(q_A)$ form a (co)representation. This space can be decomposed into its irreducible constituents. If none of them occurs more than once the multiplicity problem is completely solved, while in all other cases it is only reduced. The method outlined in this section generalizes the results for Kronecker products described in detail in Ref.[9].

3. GENERATING RELATIONS FOR ISOSCALAR MATRICES: Up to now the auxiliary group approach has always been utilized to simplify the calculation of reducing matrices. We now briefly discuss an extension where these methods are used to systematize the determination of matrices that relate pairs of *different* reducing matrices. To be specific the action of $\mathcal{Q}_A$ and $\mathcal{Q}_B$ is now transferred from the (co)representations to the *isoscalar matrices* $\mathbf{X}$ defined by the Racah Lemma [10], [11].

$$\left\{\mathbf{S}^{\underline{\ell}}\mathbf{P}^{\oplus\underline{t}}\left[\oplus_{\underline{t}}\mathbf{E}(\underline{\ell}|\underline{t}) \otimes \mathbf{C}_B^{\underline{t}}\right]\right\} = \left\{\mathbf{C}_A^{\underline{\ell}}\left[\oplus_\ell \mathbf{E}(\underline{\ell}|\ell) \otimes \mathbf{S}^\ell\right]\mathbf{P}^{\oplus\ell}\right\}\mathbf{X}_B^\ell \tag{21}$$

In this equation $\mathbf{C}_A^{\underline{\ell}} = \mathbf{C}_A^{\ell_1,\ell_2}$ and $\mathbf{C}_B^{\underline{t}} = \mathbf{C}_B^{t_1,t_2}$ are Clebsch-Gordan matrices for $\mathcal{G}_A$ and $\mathcal{G}_B \subset \mathcal{G}_A$ respectively. Moreover $\mathbf{S}^\ell$ is a subducing matrix and $\mathbf{S}^{\underline{\ell}}$

is a shorthand notation for $\mathbf{S}^{\ell_1} \otimes \mathbf{S}^{\ell_2}$. The symbol $(\underline{\ell}|\ell)$ denotes the Kronecker multiplicity of $\mathbf{D}_A^{\ell}$ in $\mathbf{D}_A^{\underline{\ell}}$ while $(\underline{\ell}|\underline{t})$ is the number of times $\mathbf{D}_B^{\underline{t}}$ occurs in $\mathbf{D}_B^{\underline{\ell}\downarrow}$ (subduction multiplicity). Moreover $\mathbf{P}^{\oplus\underline{\ell}}$ and $\mathbf{P}^{\oplus\ell}$ are well-defined permutational matrices. Finally, $\mathbf{X}_B^{\underline{\ell}}$ is the so-called *isoscalar matrix* that links the two matrices in curly brackets both of them reducing $\mathbf{D}_B^{\underline{\ell}\downarrow}$ into the direct sum $\oplus_t \mathbf{E}(\underline{\ell}|t) \otimes \mathbf{D}_B^t$.

In order to make use of the auxiliary group approach we need not only the groups $\mathcal{Q}_A$ and $\mathcal{Q}_B$ but also the corresponding groups for the Kronecker products, namely $\underline{\mathcal{Q}}_A$ and $\underline{\mathcal{Q}}_B$. Each of these groups, $\underline{\mathcal{Q}} = \{\underline{q}\}$, is isomorphic to an extension of the groups $\mathcal{Q} = \{q\}$ introduced previously for the factors of the Kronecker products (for details see [4]).

$$\underline{\mathcal{Q}} = (\mathcal{ASS} \times \mathcal{ASS})(\times(\mathcal{AUT} \times \mathcal{CON} \times \mathcal{S}_2) \tag{22}$$

The action of $\underline{\mathcal{Q}} \cap (\mathcal{Q} \times \mathcal{Q})$ is defined by

$$(\underline{q}\mathbf{D}^{\underline{\ell}})(g) = (q_1, q_2)\left(\mathbf{D}^{\ell_1} \otimes \mathbf{D}^{\ell_2}\right)(g) = \left(\left(q_1\mathbf{D}^{\ell_1}\right) \otimes \left(q_2\mathbf{D}^{\ell_2}\right)\right)(g) \tag{23}$$

and the effect of p_{12}, the only non-trivial element of $\mathcal{S}_2$, is simply

$$\left(p_{12}\mathbf{D}^{\underline{\ell}}\right)(g) = \left(p_{12}\left(\mathbf{D}^{\ell_1} \otimes \mathbf{D}^{\ell_2}\right)\right)(g) = \mathbf{D}^{\ell_2}(g) \otimes \mathbf{D}^{\ell_1}(g). \tag{24}$$

There exists a natural homomorphism mapping $\underline{\mathcal{Q}}$ onto $\mathcal{Q}$,

$$\mathcal{H} : \underline{\mathcal{Q}} \longrightarrow \mathcal{Q}\,, \tag{25}$$

and the homomorphism Φ introduced for single (co)representations in Section 2 can be extended to a homomorphisms $\underline{\Phi}$ defined for Kronecker products. The following commutative diagram of morphisms related to the group-subgroup pair $\mathcal{G}_B \subset \mathcal{G}_A$, shows the relations between the four homomorphisms $\mathcal{H}_A$, $\mathcal{H}_B$, Φ and $\underline{\Phi}$.

$$\begin{array}{cccc} \mathcal{G}_A : & \underline{\mathcal{Q}}_A & \xrightarrow{\mathcal{H}_A} & \mathcal{Q}_A \\ \cup & \underline{\Phi}\downarrow & & \downarrow\Phi \\ \mathcal{G}_B : & \underline{\mathcal{Q}}_B & \xrightarrow{\mathcal{H}_B} & \mathcal{Q}_B \end{array} \tag{26}$$

From these relations we may derive the following general result

$$\mathbf{M}_1^{\underline{\ell}'}(\underline{q}_A)^\dagger \mathbf{X}_B^{\underline{\ell}'} \mathbf{M}_2^{\underline{\ell}'}(\underline{q}_A) = \left[\oplus_t \mathbf{E}(\underline{\ell}|t) \otimes \mathbf{U}_B^{t',t}(q_B)\right] \mathbf{X}_B^{\underline{\ell}} \left[\oplus_t \mathbf{E}(\underline{\ell}|t) \otimes \mathbf{U}_B^{t',t}(q_B)\right]^\dagger \tag{27}$$

In this equation $q_B = \Phi(\mathcal{H}_A(\underline{q}_A)) = \mathcal{H}_B(\Phi(\underline{q}_A))$ and $\mathbf{M}_1^{\underline{\ell}'}(\underline{q}_A)$ and $\mathbf{M}_2^{\underline{\ell}'}(\underline{q}_A)$ are special matrices, each commuting with $\oplus_t \mathbf{E}(\underline{\ell}|t) \otimes \mathbf{D}_B^{t'}(g)$. A careful choice of transformations $\underline{q}_A$ and matrices $\mathbf{M}_j^{\ell'}(\underline{q}_A)$, $j = 1, 2$, leads us to generating relations for isoscalar matrices, symbolically written as

$$\mathbf{X}_B^{\ell} \xrightarrow{\underline{q}_A} \mathbf{X}_B^{\ell'}.$$

However, as this is a rather involved procedure the detailed formulas as well as some illustrative examples will be given elsewhere [12], [13].

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